

COBORDISM MAPS AND ATIYAH–FLOER ISOMORPHISMS FOR FRAMED INSTANTON HOMOLOGY I

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ABSTRACT. We prove that isomorphisms of the type constructed by Daemi, Fukaya and Lipyanskiy, between framed instanton homology and the Lagrangian Floer homology of a Heegaard splitting, intertwine the cobordism maps of Kronheimer and Mrowka with maps defined by holomorphic curves alone, over the field of two elements. This holds for every connected cobordism between connected three-manifolds carrying a framed arc between its ends and an $\mathrm{SO}(3)$ bundle trivialized along the ends and the arc. The symplectic maps count holomorphic triangles in moduli spaces of flat connections on surfaces, with the correspondence of a compression body as boundary condition for one- and three-handles. Consequently these maps are independent of the handle decomposition and of the Heegaard data, and form a functor isomorphic to framed instanton homology; where both are defined, our isomorphisms agree with those of Daemi, Fukaya and Lipyanskiy modulo two. The proof rests on two analytic results for the mixed anti-self-duality and holomorphic curve equation: an adiabatic limit in which a compression body collapses next to a holomorphic strip and becomes the Lagrangian boundary condition of its flat connections, and an excision theorem along a torus.

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1. INTRODUCTION

1.1. The Atiyah–Floer isomorphism. Let M be a closed, connected, oriented three-manifold. Its framed instanton homology $I^\#(M)$ is the instanton Floer homology of the connected sum $M \# T^3$ with the $\mathrm{SO}(3)$ bundle that is trivial over M and nontrivial over T^3 , with the gauge group enlarged by the automorphism of the torus summand [Sca15, Section 7]; Kronheimer and Mrowka define it equivalently through a Hopf link [KM11b]. A Heegaard splitting $M = H_0 \cup_\Sigma H_1$ of genus g gives a second invariant. Take the boundary sum of each handlebody with $[0, 1] \times T^2$. The two resulting compression bodies C_0, C_1 have boundary $\Sigma_h \sqcup T^2$ with $h = g + 1$. Let $\mathcal{M}_h$ denote the moduli space of flat connections on the nontrivial $\mathrm{SO}(3)$ bundle over Σ_h . The flat connections that extend over C_i form an embedded Lagrangian $L(C_i) \subset \mathcal{M}_h$. The Lagrangian Floer homology $HF(L(C_0), L(C_1))$ is the symplectic framed Floer homology of [DFL21], introduced by Wehrheim and Woodward as torus-summed Floer homology [WW20, Definition 4.4.1]; they show by Cerf theory and geometric composition that it is an invariant of M . Atiyah conjectured that, for an integral homology sphere, instanton Floer homology [Flo88a] is recovered in this way from the $\mathrm{SU}(2)$ character varieties of a Heegaard splitting [Ati88]; for the version for admissible $\mathrm{SO}(3)$ bundles adopted here see [DF18, Section 10]. Daemi, Fukaya and Lipyanskiy proved that the two groups are isomorphic [DFL21, Theorem 3]. Their isomorphism counts solutions of a *mixed equation*, introduced by Lipyanskiy [Lip14]: the anti-self-duality equation on part of a four-dimensional domain, coupled along a matching surface to the holomorphic curve equation in $\mathcal{M}_h$ on the rest [DFL21, DFL24].

Both groups carry cobordism maps. On the gauge side a cobordism W with a path between its ends induces a map $I^\#(W): I^\#(M_0) \rightarrow I^\#(M_1)$, and these maps form a functor [KM11b, Section 5], [Sca15, Section 7]. On the symplectic side a handle decomposition of W can be assigned a composite of counts of holomorphic polygons and quilts, using the quilted Floer theory of Wehrheim and Woodward [WW10, WW12]. For the symplectic instanton homology of Manolescu and Woodward [MW12], such handle maps were defined and shown to be independent of the decomposition by Cazassus [Caz19], using Cerf theory, and Horton proved the corresponding functoriality for his theory built from traceless character varieties [Hor16], [Hor17, Theorems 1.1–1.2]. For the symplectic framed theory used here no maps for four-dimensional cobordisms were available, and in none of these settings had the symplectic maps been compared with the gauge cobordism maps. The comparison closest in spirit is the open–closed isomorphism of Salamon and Wehrheim [SW25, Theorem 5.8], which intertwines relative invariants of four-manifolds, with gauge theory on both sides. This paper compares the two families of maps.

1.2. Main result. We work over $\mathbb{F}_2$ throughout. Objects of the connected framed cobordism category $\mathbf{B}$ are closed, connected, oriented three-manifolds M with a parametrized ball, which fixes where the torus is attached (Definition 2.1). A morphism $(W, E, \nu): M_0 \rightarrow M_1$ is an isomorphism class of connected oriented cobordisms W with a framed path tube ν joining the two balls, and an $\mathrm{SO}(3)$ bundle E on W trivialized over M_0, M_1 and along the tube. The bundle is arbitrary in the interior; its class relative to the ends lies in $H^2(W, \partial W; \mathbb{F}_2)$. Composition is gluing.

Framed instanton homology is a functor $I^\#: \mathbf{B} \rightarrow \mathrm{Vect}_{\mathbb{F}_2}$ (Proposition 2.4). For each object M and each admissible presentation p (a Heegaard splitting adapted to the ball, together with the analytic data of Section 3) there are the gauge complex $C_G(M, p)$, the symplectic complex $C_S(M, p) = CF(L(C_0), L(C_1))$ and a mixed map $N_p: C_G(M, p) \rightarrow C_S(M, p)$. It is a quasi-isomorphism by the triangularity argument of [DFL21, Theorem 3.29] (Theorem 3.8). The

Lagrangians $L(C_i)$ are moved by *primary* perturbations, holonomy perturbations supported inside C_i ; the mixed equation is regularized by *secondary* perturbations in the interior of its domain, and the gauge complex by an *end* perturbation. All of them are taken invariant under the *lifting group* $\phi_M \cong \mathbb{Z}/2$, the extra automorphism of the torus summand [KM11b, Section 5.1].

For a morphism we define symplectic maps by counting holomorphic curves in the Heegaard moduli spaces, without reference to gauge theory:

- the trace of surgery on a framed link is sent to the triangle map $\mu_S(\cdot \otimes \theta_S)$ of a subordinate Heegaard triple $(\Sigma; H_0, H_1, H_2)$, where θ_S is the generator of maximal relative grading (the maximum of a height function) of the Floer complex of the Lagrangians of H_1 and H_2 (Section 9);
- a k -fold one-handle is sent to a compression cup, defined by the triangle map of (A_0, L_C, A_1) in $\mathbf{M}_h^- \times \mathbf{M}_{h+k}$. Here L_C is the correspondence of the compression body, and A_i is the product of a reversed handlebody Lagrangian of M and a handlebody Lagrangian of M^+ . The inputs are the generators of maximal grading for (A_0, L_C) and (L_C, A_1) , the maxima of explicit Morse functions (Section 10.1). Unfolding gives the quilted triangle with seam L_C (Remark 11.6, Figure 8). A three-handle is sent to the transposed compression cap (Section 11);
- a change of trivialization $u: M \rightarrow \mathrm{SO}(3)$ of the bundle (a *frame change*) is sent to the pushforward by the induced symplectomorphism of $\mathbf{M}_h$;
- a change of presentation is sent to the corresponding word of continuations, cups, caps and triangles (Section 12).

An *admissible decomposition* d of (W, E, ν) is a factorization into presentation changes, frame changes, single one- and three-handles and one two-handle cobordism, from (M_0, p_0) to (M_1, p_1) (Section 13.2); it gives a composite map $S_d(W, E, \nu)$ from $HF(M_0, p_0) = H(C_S(M_0, p_0))$ to $HF(M_1, p_1)$. A presentation is *admissible* if its primary holonomy perturbations lie in a fixed small open neighbourhood $\mathcal{U}(F)$ of zero in the span of a finite family F of holonomy perturbations, depending only on the Heegaard splitting, the metric data and F (Definitions 3.3 and 11.12). Within $\mathcal{U}(F)$ we prove that the mixed maps are natural under changes of primary data and that the transpose of the reflected mixed map inverts N_p (Propositions 11.10 and 11.11); this is used in the one-handle comparison.

Theorem A. *For every morphism $(W, E, \nu): M_0 \rightarrow M_1$ of $\mathbf{B}$, all admissible presentations p_0 of M_0 and p_1 of M_1 , and every admissible decomposition d ,*

$$H(N_{p_1}) \circ I^\#(W, E, \nu) = S_d(W, E, \nu) \circ H(N_{p_0}).$$

Consequently $S_d(W, E, \nu)$ depends only on the morphism and the end presentations. The groups $HF(M, p)$ form a transitive system over the presentations of M , and with its colimit $HF^\#(M)$ the maps $S(W, E, \nu)$ form a functor $HF^\#: \mathbf{B} \rightarrow \mathrm{Vect}_{\mathbb{F}_2}$. The maps $H(N_p)$ descend to a natural isomorphism $I^\# \Rightarrow HF^\#$. The same statements hold in the homotopy category of finite chain complexes over $\mathbb{F}_2$, with the colimit replaced by a choice of representative of the transitive system of homotopy equivalences.

The symplectic side of Theorem A contains no gauge theory. In particular, independence of the decomposition and functoriality of the symplectic maps are consequences, not hypotheses. The argument uses no Cerf theory on the symplectic side, in contrast with the direct constructions of [Caz19, Hor17] for other symplectic models, and no geometric composition isomorphism: the embedded compositions of the compression correspondence with handlebody Lagrangians enter only as sets, to identify generators and gradings. The quilted

description of the compression cup, which rests on the composition theorems of Wehrheim–Woodward and Lekili–Lipyanskiy, is recorded in Remark 11.6 and is not used.

Our mixed maps N_p are built by the method of [DFL21], with one change: Chern–Simons trajectories are regularized by an additional holonomy perturbation f of $M^\#$ rather than by the primary perturbations (Remark 3.10). When the primary perturbations already regularize them, as in [DFL21, Lemma 2.56], one may take $f = 0$; with the almost complex structures of [DFL21] and secondary perturbations invariant under the lifting group, N_p is then the isomorphism of [DFL21, Theorem 3.29] reduced modulo two (Proposition 3.11). Such primary data exist at every Heegaard splitting and all metric data, without the rescaling of the metric used in [DFL21, Proposition 6.8] (Theorem 3.6, proved in Appendix C). Hence Theorem A holds for the isomorphism of Daemi, Fukaya and Lipyanskiy itself, reduced modulo two, which on homology is independent of the choices (Corollary 13.4), and the mixed map at any admissible presentation is, on homology, that isomorphism composed with gauge and symplectic continuation maps (Section 13). Two points at which we depart from [DFL21] are recorded in the text: the inverse function statement [DFL21, Lemma 4.1(iii)] is used with an L^q -ball, $q > 2$, since it fails with an L^2 -ball (Remark 5.2); and for the inductive step of [DFL21, Proposition 6.8] we could not deduce the independence of a cokernel element from the translation vector on one piece of the splitting, which Appendix C supplies for separated base data (Remark 3.7).

1.3. The collapse theorem. Every elementary comparison in Theorem A reduces to one degeneration, stated here informally. Let X be a domain for the mixed equation containing a *compression rectangle*: a product $I \times C$ of an interval with a compression body, attached to a holomorphic strip along the matching surface $I \times \Sigma$, where Σ is the boundary component of C of larger genus (Figure 2). Scale the metric of C by ε^2 . After a conformal change each endpoint of I becomes a cylinder over (C, g_C) of length $\log(1/\varepsilon)$, and ε enters only through explicit weights. As $\varepsilon \rightarrow 0$ the rectangle disappears. In the limit the strip meets I along the Lagrangian boundary condition $L(C)$, and two mixed ends appear at the endpoints.

Let X_0 denote the limiting domain, in which the compression rectangle is replaced by the boundary condition $L(C)$, and X_ε the domain at scale ε .

Theorem B. *Assume Hypothesis 6.5 of Section 6 (near the rectangle the compression body is unperturbed and the almost complex structure on the holomorphic part is $\ast_\Sigma$; all other perturbations are supported away from it), that all data on X_ε outside a fixed neighbourhood of the rectangle are independent of ε , and that the moduli spaces on X_0 of index at most zero are regular. Then the following hold.*

- (i) *If on X_0 the Fredholm index and the topological energy κ , normalized as in [DFL21, Proposition 3.20], satisfy $\text{ind} = c + 8\kappa$ for all configurations with fixed asymptotic labels, then the same relation, with the same constant c , holds on X_ε for every $\varepsilon > 0$.*
- (ii) *For every sequence of solutions of index d on X_{ε_ν} , with $\varepsilon_\nu \rightarrow 0$, a subsequence converges to a solution on X_0 of index d_0 and broken trajectories of indices d_j , and $d - d_0 - \sum_j d_j = 4n$ for an integer $n \geq 0$.*
- (iii) *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ gluing the constant flat solution on the rectangle defines a bijection $\mathcal{G}_\varepsilon: \mathcal{M}^0(X_0) \rightarrow \mathcal{M}^0(X_\varepsilon)$ between the index-zero moduli spaces, and the solutions at scale ε are regular.*

The precise statement is Theorem 6.6. The limit is of the type proposed in the programme of Salamon and Wehrheim [Sal95, Conjecture 3.4], [Weh05, Section 4], in which the metric on a handlebody degenerates onto its spine and the handlebody is replaced by the Lagrangian

boundary condition of its flat connections. (The second degeneration of that programme, shrinking Σ in $[0, 1] \times \Sigma$, is the adiabatic limit of Dostoglou and Salamon [DS94]; Fukaya [Fuk98] coupled the anti-self-duality equation to holomorphic curves through a degenerate metric.) Here the collapsing body is a compression body whose boundary is matched, along part of it, to a holomorphic strip that does not collapse, and two mixed ends appear. To our knowledge the handlebody degeneration of [Sal95, Weh05] has not been carried out; Salamon and Wehrheim prove their open–closed isomorphism by a different route [SW25]. After a conformal change the collapse becomes neck stretching along cylinders over C . In the compactness theorem of Xu [Xu22, Theorem 1.4], for instantons on $\mathbb{R} \times M$ with M a three-manifold with a cylindrical end over Σ , energy escaping along the end produces holomorphic curves in the moduli space of Σ with the Lagrangian boundary condition of M . Theorem B adds to compactness the index–energy law at every scale and the bijection of index-zero counts. The symplectic analogue is strip shrinking in quilted Floer theory [WW12, BW18]. Three features of the proof are specific to this situation. First, the model linear estimates (Section 7) lose no power of $\log(1/\varepsilon)$; the patched inverse on the whole domain loses exactly one factor $1 + \log(1/\varepsilon)$, which is sharp, and the surjectivity argument is arranged so that this factor is always multiplied by a positive power of ε . The reason for the first point is that the slow harmonic directions of C are tied, pointwise in time, to the boundary value of the strip and are controlled by the strip’s unweighted norm. This depends on the orientation convention 3.1; with the other sign the estimate is false. Second, compactness requires no classification of bubbles. Energy loss is excluded by the affine law (i), which holds at every scale, together with nonnegativity of the limiting indices. Third, surjectivity of the gluing map is proved without entering a logarithmically shrinking uniqueness ball directly. A solution is first brought within a power of ε of the glued solution, and only then compared with it.

1.4. Excision along a torus. The one-handle comparison needs a second new result. Let M have a Heegaard splitting and let $M^+ = M \#^k(S^1 \times S^2)$ have the splitting obtained by adding k trivial handles. Cutting $(-M)^\#$ and $(M^+)^\#$ along torus fibres and regluing gives the closed three-manifold $Z \cong (-M) \# M^+ \# T^3$. The pair of compression bodies (D_0, D_1) splitting Z defines a Lagrangian pair (A_0, A_1) in $\mathbf{M}_h^- \times \mathbf{M}_{h+k}$, the *folded pair* (Section 10.1). Let N_Z be its mixed map, N_- the mixed map of the reflected presentation of $-M$ (valued in the dual Floer complex), N_+ that of M^+ , $\mathbf{E}$ Floer’s excision map [BD95, KM10], [KM11b, Theorem 5.6] from the instanton complex of Z to the tensor product of those of $(-M)^\#$ and $(M^+)^\#$, and $\mathcal{K}$ the folding isomorphism of Lagrangian Floer complexes for a split almost complex structure.

Theorem C. *With N_Z , $N_\pm$, $\mathbf{E}$ and $\mathcal{K}$ as above, $[\mathcal{K}][N_Z] = ([N_-] \otimes [N_+])[\mathbf{E}]$ on homology over $\mathbb{F}_2$.*

This is Theorem 10.2. It is not an instance of neck stretching along a closed hypersurface: the torus region runs into the mixed ends, so no closed three-manifold separates the two sides. Near the torus the domain is a product of T^2 with a planar region (Figure 5). We insert a flat region $T^2 \times [-\lambda, \lambda] \times \mathbb{R}$ and, on its boundary, impose only flatness of the restriction to each torus fibre; since the moduli space of flat connections on the torus with $w_2 \neq 0$ (the *odd torus*) is a point, this is the whole matching condition there. Two facts drive the proof. On the plane, the half-plane and straight strips, the linearized operator at the odd flat torus has an inverse bounded by μ_T^{-1} , independently of the strip width (Proposition 10.3); and this boundary condition admits small-energy regularity (Theorem 10.5). The proof compares the

two sides at a fixed width with Floer’s excision, varies the width, and lets it tend to infinity (Section 10).

1.5. Strategy. Theorem A is deduced in Section 13 from four inputs by a formal argument (Lemma 13.1):

- (1) $I^\#$ is a functor on $\mathbf{B}$ (Section 2);
- (2) N_p is an isomorphism on homology, and its homology class is independent of the end and secondary data (Theorem 3.8(ii)) and, within the admissible neighbourhoods, of the primary data (Proposition 11.10);
- (3) every morphism is a composite of elementary arrows: presentation moves, frame changes, one-handles, a single two-handle cobordism and three-handles (Section 4);
- (4) every elementary arrow e satisfies the square $H(N)I^\#([e]) = S_e H(N)$, where N denotes the mixed map N_p of the chosen presentation at each end (Sections 9–12).

The maps $H(N_p)$ are invertible. Hence (3) and (4) determine every symplectic word as $H(N)I^\#(W)H(N)^{-1}$. Independence of the word and functoriality are then inherited from the gauge side.

The squares for two-handles are proved as follows. The gauge two-handle map is a triangle map with an auxiliary input (Section 9). A mixed triangle degenerates in two ways. Stretching one family of necks produces the gauge triangle followed by N . Collapsing a compression rectangle by Theorem B and then stretching ordinary necks produces $N \otimes N$ followed by the symplectic triangle. A compact family of mixed triangles joins the two (Section 5).

For a k -fold one-handle the symplectic cup is a triangle count in the product $M_h^- \times M_{h+k}$ of two moduli spaces, with the correspondence of the compression body as middle Lagrangian and top classes of two auxiliary pairs as inputs (Section 11). Folding the triangle along the diagonal and applying the same two degenerations identifies the cup with the mixed map of a folded pair composed with a gauge triangle, and Theorem C splits off the extra three-torus summand. The result is the *cup formula*

$$U_C = N_{C \circ p} \circ I^\#(W_k) \circ N_p^\dagger,$$

where W_k is the one-handle cobordism and $N_p^\dagger$ is the transpose of the mixed map $N_{\bar{p}}$ of the reflected presentation. It remains to show that $N_p^\dagger = N_p^{-1}$ on homology, and this is done without any composition theorem for Lagrangian correspondences. For $k = 0$ the cup is a continuation map. At a common auxiliary presentation A , the cup formula for the reflections of two presentations with primary data P and Q reads (indexing mixed maps by primary data, and using $N_{\bar{P}}^\dagger = N_P^\vee$)

$$\text{cont}_S(\bar{P}, A) = N_A \circ \text{cont}_G(\bar{P}, A) \circ N_P^\vee, \quad \text{cont}_S(\bar{Q}, A) = N_A \circ \text{cont}_G(\bar{Q}, A) \circ N_Q^\vee.$$

Eliminating the invertible N_A gives the naturality of N under small changes of primary data (Proposition 11.10). Applying the $k = 0$ cup formula once more, with P itself as the lower presentation, and comparing with that naturality gives $N_P \circ N_P^\dagger = 1$ (Proposition 11.11). The three-handle squares follow by reflection.

1.6. Scope. The theorem is stated over $\mathbb{F}_2$. Over $\mathbb{Z}$ the symplectic counts require coherent orientations of the polygon and quilt moduli spaces, together with a comparison of those orientations with the homology orientations of the gauge side. Neither is addressed here. Integer coefficients are the subject of a sequel [Wue26b], whose aim is Theorem A over $\mathbb{Z}$ for the maps of this paper, up to one sign for each cobordism. The objects carry the trivial bundle on the original three-manifold. Disconnected objects, the units and counits of a

field theory, and closed pairings are not treated. The mixed maps N_p are constructed by the method of [DFL21], with the invariant perturbations of Section 3. Their agreement with the isomorphism of [DFL21] modulo two, at the common data described above, is Proposition 3.11; existence of such data at fixed metric data is Theorem 3.6. The maps of the transitive system of Theorem A are symplectic words; their independence of the word is proved through the gauge side, and we do not compare them with the Cerf-theoretic identifications of [WW20].

Organization. Section 2 defines $\mathbf{B}$ and the gauge functor. Section 3 recalls the Heegaard moduli spaces and the mixed isomorphism and fixes conventions. Section 4 contains the topology of handle decompositions and bundles. Section 5 treats neck-stretching, families and regularity in the invariant perturbation class. Sections 6–8 prove Theorem B. Section 9 proves the two-handle square. Section 10 proves the torus excision theorem (Theorem C), which the one-handle comparison requires. Section 11 proves the one- and three-handle squares, and Section 12 treats changes of presentation and frame. Section 13 proves Theorem A. Appendix C proves that primary perturbations regularizing the gauge complex, as required in [DFL21], exist at the metric data of a presentation. A table of notation follows.

Notation.

$M^\#, E_M^\#$	torus-summed object and its bundle (Section 2)
ϕ_M	lifting group, the subgroup ϕ of [KM11b, Section 5.1]
$\Sigma_h, \mathbf{M}_h$	Heegaard surface of genus h and its moduli space of odd flat connections
$C_0, C_1, L(C)$	compression bodies of a presentation and their Lagrangians; in Sections 6–8 also $P = L(C)$
p, N_p	presentation and its mixed map
$P, N_P, N_P^\dagger$	primary data of p , $N_P = N_p$, and $N_P^\dagger = N_P^\vee$
$C_G(M, p), C_S(M, p)$	gauge and symplectic complexes
$\mathfrak{h}_i, f, \eta$	primary perturbations, end perturbation, secondary perturbation
$\kappa, \mathfrak{c}$	topological energy and the index constants of [DFL21, Proposition 2.53]
$\text{cont}_S, \text{cont}_G$	symplectic and gauge continuation maps
X_0, X_ε	collapsing family (Section 6.2)
I, N_I	collapsing arc and its fixed neighbourhood
$\mathfrak{s}, \mathbf{A}$	compression scale, and $\mathbf{A} = 1 + \log(1/\varepsilon)$
$W_R, W_\infty, W_\lambda^J$	collared, completed and joined domains (Definition 10.1)
μ_T	spectral gap of the odd flat connection on the torus
$R_i, \tilde{P}_i, L_C$	height Lagrangians, lower Lagrangians, correspondence of $C_{h,g}$
$\mathbb{X}, A_0, A_1$	product target $\mathbf{M}_h^- \times \mathbf{M}_g$ and the folded pair in it
$\mathcal{K}, \mathbf{E}$	folding isomorphism, Floer's excision map
U_0, U_1	in Sections 10–11, compression bodies of the folded splitting
$P_L(V), \mathcal{G}_L(V)$	preglued configuration and glued solution at torus length L
$\mathcal{U}(F)$	admissible neighbourhood of primary data (Definition 11.12)
U_C, V_C	compression cup and cap (Definition 11.4)

2. THE FRAMED COBORDISM CATEGORY AND THE GAUGE FUNCTOR

2.1. The torus attachment. Write $T^3 = S^1 \times T^2$ and let $P_T \rightarrow T^3$ be the pullback of the $\text{SO}(3)$ bundle on T^2 with $w_2 \neq 0$. Fix once and for all a ball $B_T \subset T^3$ disjoint from a fibre $\{\theta_0\} \times T^2$, a parametrization of B_T by the unit ball B^3 , a trivialization of $P_T|_{B_T}$ and a metric on T^3 that is standard near B_T and flat on a neighbourhood of the fibre $\{\theta_0\} \times T^2$;

the flat metric it induces on T^2 is used for every torus collar below, in all presentations. Put $T^\circ = T^3 \setminus \text{int } B_T$. Let

$$(2.1) \quad \alpha_T = \text{pr}_{S^1}^*(\text{generator of } H^1(S^1; \mathbb{F}_2)) \in H^1(T^3; \mathbb{F}_2).$$

Then $\langle w_2(P_T), [T^2] \rangle = 1$, $\alpha_T|_{T^2} = 0$ and $\langle \alpha_T \smile w_2(P_T), [T^3] \rangle = 1$.

Definition 2.1. An *object* of $\mathbf{B}$ is a closed, connected, oriented three-manifold M together with an orientation-preserving embedding $b: B^3 \rightarrow M$. The *torus-summed object* is

$$M^\# = (M \setminus b(\text{int } B^3)) \cup_{S^2} T^\circ, \quad E_M^\# = \mathbb{R}^3 \cup P_T|_{T^\circ},$$

glued along the boundary spheres by the fixed parametrizations and trivializations.

A *decorated cobordism* $(W, E, \nu): (M_0, b_0) \rightarrow (M_1, b_1)$ consists of:

- a compact, connected, oriented four-manifold W with $\partial W = -M_0 \sqcup M_1$ and product collars at both ends;
- an embedding $\nu: B^3 \times [0, 1] \rightarrow W$ with $\nu(\cdot, i) = b_i$, a product in the collars (the *tube*);
- an $\text{SO}(3)$ bundle $E \rightarrow W$ with trivializations over M_0 , M_1 and over the image of ν , which agree on the overlaps.

An *isomorphism* of decorated cobordisms is an orientation-preserving diffeomorphism covered by a bundle isomorphism, both respecting the boundary identifications, the tubes and the trivializations. The morphisms of $\mathbf{B}$ are isomorphism classes of decorated cobordisms. Composition glues along the ends using the collars, tubes and trivializations; the identity of (M, b) is the product $[0, 1] \times M$ with the vertical tube and the product bundle.

Associativity and the unit laws hold on isomorphism classes, by the usual collar-reparametrization isomorphisms. A product $[0, 1] \times M$ whose bundle identifications at the two ends differ by a nontrivial automorphism is not the identity. Its image under the functor below is the corresponding bundle automorphism map.

For a decorated cobordism put $W^\circ = W \setminus \nu(\text{int } B^3 \times [0, 1])$. Then

$$(2.2) \quad W^\# = W^\circ \cup_{S^2 \times [0, 1]} (T^\circ \times [0, 1]), \quad E^\# = E|_{W^\circ} \cup (P_T|_{T^\circ} \times [0, 1]),$$

glued by the tube parametrization and the trivializations. Its ends are $-M_0^\# \sqcup M_1^\#$ with the bundles $E_{M_i}^\#$.

Lemma 2.2. *Restriction gives an isomorphism $H^1(W^\#; \mathbb{F}_2) \rightarrow H^1(W; \mathbb{F}_2) \oplus H^1(T^3; \mathbb{F}_2)$.*

Proof. W° is connected, since a path in the four-manifold W can be displaced off the core arc of the tube and then off its tubular neighbourhood. The pieces W° and $T^\circ \times [0, 1]$ are connected, their intersection $S^2 \times [0, 1]$ is connected and has vanishing H^1 , so Mayer–Vietoris gives $H^1(W^\#) = H^1(W^\circ) \oplus H^1(T^\circ)$. The same argument applied to $W = W^\circ \cup (B^3 \times [0, 1])$ and $T^3 = T^\circ \cup B_T$ identifies $H^1(W^\circ) = H^1(W)$ and $H^1(T^\circ) = H^1(T^3)$. $\square$

Let $\alpha_W \in H^1(W^\#; \mathbb{F}_2)$ correspond to $(0, \alpha_T)$ and put $\phi_W = \{0, \alpha_W\}$. On each end $M_i^\#$ the class α_W restricts to the class α_{M_i} corresponding to $(0, \alpha_T)$, and $\phi_{M_i} = \{0, \alpha_{M_i}\}$. The class α_W is realized by a bundle automorphism of $E^\#$: take the standard automorphism of P_T with lifting class α_T (the gauge transformation g of [Sca15, Section 7.1]), deform it to be the identity over B_T , use it constantly along $T^\circ \times [0, 1]$, and extend by the identity over W° .

2.2. The gauge functor. Recall from [KM11b, Section 5.1] that for an $\mathrm{SO}(3)$ bundle on a three-manifold the quotient of the automorphism group by the image of the determinant-one gauge group is $H^1(\cdot; \mathbb{F}_2)$, so a subgroup $\phi \subset H^1$ acts on the configuration space of connections modulo determinant-one gauge transformations. The class in $H^1(\cdot; \mathbb{F}_2)$ of an automorphism g is its *lifting class* $c(g)$, the obstruction to lifting g to a determinant-one gauge transformation; g is *even* if its lifting class vanishes. A subgroup ϕ of lifting classes used in this way is the *lifting group* of the problem; it is the subgroup ϕ of [KM11b, Section 5.1]. A triple satisfies the *ϕ -non-integral condition* if it contains a surface on which w_2 is odd and ϕ vanishes [KM11b, Definition 5.1]. For $M^\#$ the torus $\{\theta_0\} \times T^2$ is such a surface for ϕ_M . Under this condition ϕ acts freely on the critical set, the instanton homology of the quotient $I(M^\#, E_M^\#)^{\phi_M}$ is defined, and a cobordism with a subgroup $\phi \subset H^1(W; \mathbb{F}_2)$ restricting to the given subgroups at the ends induces a map $I(W^\#, E^\#)^\phi$ by counting points in the quotient moduli spaces [KM11b, pp. 154–155]. Over $\mathbb{F}_2$ no orientation data is needed.

Definition 2.3. For an object (M, b) put $I^\#(M) = I(M^\#, E_M^\#; \mathbb{F}_2)^{\phi_M}$. For a decorated cobordism put $I^\#(W, E, \nu) = I(W^\#, E^\#; \mathbb{F}_2)^{\phi_W} : I^\#(M_0) \rightarrow I^\#(M_1)$.

Here $I(M^\#, E_M^\#; \mathbb{F}_2)^{\phi_M}$ denotes a single vector space: the Floer groups for different metrics and perturbations are identified by continuation maps, which form a transitive system, and $I^\#(M)$ is its limit, that is, any one of the groups together with these identifications. The group $I^\#(M)$ is the framed instanton homology of [KM11b, Sca15] with $\mathbb{F}_2$ coefficients. Since $\langle \alpha_T \smile w_2(P_T), [T^3] \rangle = 1$, the nonzero element of ϕ_M shifts the grading of the determinant-one (unquotiented) theory by four (the proof of [KM11b, Proposition 5.2(a)]), and the determinant-one Floer homology of $(M^\#, E_M^\#)$ is two copies of $I^\#(M)$.

Proposition 2.4. $I^\#$ is a functor $\mathbf{B} \rightarrow \mathrm{Vect}_{\mathbb{F}_2}$.

Proof. Well defined. Isomorphic decorated cobordisms give isomorphic data $(W^\#, E^\#, \phi_W)$, hence equal maps by the usual continuation argument for metrics and perturbations [KM11b, Section 3], [Don02, Section 5.3].

Composition. Let $W_{01} : M_0 \rightarrow M_1$ and $W_{12} : M_1 \rightarrow M_2$ be composable, with composite W_{02} . Then $W_{02}^\# = W_{01}^\# \cup_{M_1^\#} W_{12}^\#$ as bundles. By [KM11b, Proposition 5.4], $I(W_{02}^\#)^\phi = I(W_{12}^\#)^{\phi_{12}} \circ I(W_{01}^\#)^{\phi_{01}}$ provided that ϕ contains the image ψ of the Mayer–Vietoris connecting map $H^0(M_1^\#; \mathbb{F}_2) \rightarrow H^1(W_{02}^\#; \mathbb{F}_2)$, and ϕ is the largest subgroup restricting to ϕ_{01} and ϕ_{12} on the pieces. Since $M_1^\#$ is connected, the restriction $H^0(W_{01}^\#) \oplus H^0(W_{12}^\#) \rightarrow H^0(M_1^\#)$ is onto, so $\psi = 0$. The class $\alpha_{W_{02}}$ restricts to $\alpha_{W_{01}}$ and $\alpha_{W_{12}}$. A class restricting to 0 or to α on both pieces is determined by its restrictions, because $H^1(W_{02}^\#) \rightarrow H^1(W_{01}^\#) \oplus H^1(W_{12}^\#)$ is injective when $\psi = 0$. Hence the largest such subgroup is $\phi_{W_{02}}$.

Unit. For the product $[0, 1] \times M$ with translation-invariant regular data the index-zero solutions are the constant trajectories. The group ϕ_M acts freely on the critical set. A trajectory from a critical point β to $\alpha_M \cdot \beta$ has index in the determinant-one theory congruent to 4 modulo 8 by the proof of [KM11b, Proposition 5.2(a)], so it does not contribute in index zero. The map is therefore the identity on the chain level for such data, and on homology for all data. $\square$

Remark 2.5. The functor uses the enlarged gauge group of [KM11b, Section 5] with the subgroup ϕ_W . It is not the projective functor of [KM11b, Section 4.1], which uses determinant-one gauge transformations and yields two copies of $I^\#$. Scaduto enlarges the gauge group on $W^\#$ by the transformations that restrict to g on the torus [Sca15, Section 7.2]; restricted

to classes compatible with the ends, this is ϕ_W . The subgroup ϕ_W vanishes on the original piece W° . Classes of $H^1(W)$ that vanish on ∂W are not added.

Remark 2.6. With the $\mathbb{Z}/4$ grading of [Sca15, Proposition 7.1], $I^\#(W, E, \nu)$ is homogeneous of degree $-\frac{3}{2}(\chi(W) + \sigma(W)) + \frac{1}{2}(b_1(M_1) - b_1(M_0)) + 2P(E)$ modulo 4, where $P(E)$ is the mod-two self-intersection of a surface in the interior of W dual to $w_2(E)$ relative to the trivialized ends.

3. HEEGAARD DATA, THE TWO COMPLEXES AND THE MIXED MAP

3.1. The moduli space of a surface. For $h \geq 1$ let Σ_h be a closed oriented surface of genus h and $F_h \rightarrow \Sigma_h$ the $\mathrm{SO}(3)$ bundle with $w_2 \neq 0$. Let $\mathcal{M}_h$ be the moduli space of flat connections on F_h modulo determinant-one automorphisms. It is a closed, simply connected, monotone symplectic manifold of dimension $6h - 6$. For $h \geq 2$ it has $\pi_2(\mathcal{M}_h) = \mathbb{Z}$ and minimal Chern number 2; for $h = 1$ it is a point [AB83, New72]. Its tangent space at $[\alpha]$ is $H^1(\Sigma_h; \mathrm{ad} \alpha)$.

Convention 3.1 (orientation). Let C be an oriented three-manifold and $\Sigma \subset \partial C$ a boundary component with the outward-normal-first orientation. On the moduli space of Σ viewed as a boundary component of C we use

$$\omega(v, w) = \int_{\Sigma} \langle v \wedge w \rangle, \quad J = *_{\Sigma},$$

where $\langle \cdot, \cdot \rangle$ is the inner product $\langle a, b \rangle = -\mathrm{tr}(ab)$ on $\mathrm{ad} F \cong \mathfrak{su}(2)$, used for all L^2 norms. In the notation of [DFL21], whose bilinear form tr is $-\frac{1}{2}$ of the Euclidean inner product on $\mathbb{R}^3 \cong \mathfrak{so}(3)$, the matching surface is $\Sigma_D = -\partial C$, with $\Omega_D = -\omega$ and $J_D = -J$; the two descriptions define the same mixed equation. Here Σ is the fibre of $U_- \times \Sigma$ in the four-manifold of [DFL21]: its upper arm is $[2, \infty)_{\theta} \times Y$ with ∂Y at $s = 0$ and outward normal $+\partial_s$, so the orientation is $d\theta \wedge ds \wedge \mathrm{vol}_{\partial Y} = -ds \wedge d\theta \wedge \mathrm{vol}_{\partial Y}$ and the fibre is $-\partial Y$.

Convention 3.1 is not a matter of taste. It is the sign for which the Green identity of the compression body cancels the Green identity of the adjacent strip (Lemma 6.8). With the opposite sign the uniform estimate of Proposition 7.4 fails, already for constant scale (Remark 7.5).

3.2. Compression bodies and their Lagrangians. A *torus compression body* of genus h is a compact oriented three-manifold C with $\partial C = \Sigma_h \sqcup T^2$ obtained from a handlebody H of genus $h - 1$ by boundary sum with $[0, 1] \times T^2$, together with the $\mathrm{SO}(3)$ bundle that is pulled back from the odd bundle on T^2 . More generally we consider compression bodies C from Σ_h to $\Sigma_{h'}$ with the odd bundle. Every flat connection on C is irreducible with $H^0(C; \mathrm{ad}) = H^2(C; \mathrm{ad}) = 0$, the flat moduli space $\mathcal{R}(C)$ is a compact manifold, and restriction to the boundary is a Lagrangian embedding into the product of the moduli spaces of the boundary components with the orientation of ∂C [DFL21, Proposition 2.21]. For a torus compression body the moduli space of the torus boundary is a point, so restriction to Σ_h is a Lagrangian embedding

$$L(C) \subset \mathcal{M}_h,$$

with $L(C) \cong (S^3)^{h-1}$, simply connected with $\pi_2 = 0$. The pair $(L(C_0), L(C_1))$ of two torus compression bodies is monotone with minimal Maslov number 4 [DFL21, Theorem 1]. For a compression body from Σ_h to $\Sigma_{h'}$ with $h' \geq 2$ the Lagrangian lies in $\mathcal{M}_h^- \times \mathcal{M}_{h'}$ (Definition 3.2 and Section 10.1).

The properties used in the collapse theorem are the following.

Definition 3.2. Let C be a compact oriented three-manifold with a metric g_C that is a product near ∂C , and an $\mathrm{SO}(3)$ bundle $\mathbb{E} \rightarrow C$. We impose the following conditions:

- (C1) $w_2(\mathbb{E})$ is nonzero on every component of ∂C ;
- (C2) every flat connection B on $\mathbb{E}$ has $H^0(C; \mathrm{ad} B) = H^2(C; \mathrm{ad} B) = 0$;
- (C3) the restriction $H^1(C; \mathrm{ad} B) \rightarrow H^1(\partial C; \mathrm{ad} B)$ is injective, and the flat moduli space of C restricts to an embedded Lagrangian $L(C)$ in the product of the moduli spaces of the boundary components with the signs of Convention 3.1.

The compression bodies of Sections 3.2 and 6.1 satisfy these conditions. Beyond (C1)–(C3), Sections 6–8 use only that $L(C)$ is compact and connected, not the structure of a compression body.

For $v \in T_q L(C)$ let h_v denote the harmonic one-form on C with values in $\mathrm{ad} B_q$, with vanishing normal component and boundary cohomology class v . The map $v \mapsto h_v$ is well defined because $H^1(C; \mathrm{ad}) \rightarrow H^1(\Sigma_h; \mathrm{ad})$ is injective.

Primary perturbations. Following [DFL21, Section 2.3], a *primary perturbation* of C is a holonomy perturbation $\mathfrak{h}$ supported in the interior of C° , the part of C outside the unit collar on which [DFL21, (2.1) and p. 9] require the perturbations to vanish (the special quintuple and the torus analysis of Section 10 use this collar unperturbed), small enough that the perturbed flat connections still form a compact manifold whose restriction $L_{\mathfrak{h}}(C) \subset \mathcal{M}_h$ is an embedded Lagrangian. It is Hamiltonian isotopic to $L(C)$: for $\mathfrak{h}$ small the implicit function theorem makes $L_{\mathfrak{h}}(C)$ C^1 -close to $L(C)$, hence the graph of a closed one-form in a Weinstein neighbourhood, and the one-form is exact because $L(C)$ is simply connected. ([DFL21] call these perturbations cylinder functions. In their notation C_0 and C_1 are the halves $Y_1 = Y_0 \cup [0, 1] \times \Sigma$ and Y'_1 of $Y_\#$ cut along $\{0\} \times \Sigma$ [DFL21, p. 79], and C_i° is Y_0, Y_0' ; Section 10 writes $(Y_\pm)_0$.) The universal restriction map from pairs (perturbation, flat connection) to $\mathcal{M}_h$ is a submersion [DFL21, Section 2.3], so finitely many primary perturbations make any finite collection of such Lagrangians pairwise transverse with empty triple intersections.

Trace functions. The determinant connection is fixed, so the holonomy of B around a loop ℓ is $\lambda_\ell g_\ell(B)$ with $g_\ell(B) \in \mathrm{SU}(2)$ and $\lambda_\ell \in U(1)$ independent of B [Don02, Section 5.6, p. 147]; λ_ℓ is determined up to sign. Over the loops of one embedding below we choose it continuously; this is possible because they are parametrized by a disk, and the choice is unique up to one sign. For an embedding $\gamma: S^1 \times D^2 \rightarrow \mathrm{int} C^\circ$ and a two-form μ supported in the interior of D^2 with $\int \mu = 1$, the *averaged trace function* is $\tau_\gamma(B) = \int_{D^2} \tau_z(B) \mu$, where $\tau_z(B) = \mathrm{tr} g_\ell(B)$ for the loop $\ell = \gamma(S^1 \times \{z\})$; a cylinder function is $G(\tau_{\gamma_1}, \dots, \tau_{\gamma_n})$ with G smooth, the γ_j being allowed to be immersions [DFL21, Section 6.1]. Daemi, Fukaya and Lipyanskiy use the trace of the holonomy itself; since $\mathrm{tr} g_\ell = \lambda_\ell^{-1} \mathrm{tr} \mathrm{Hol}_\ell$ with λ_ℓ independent of B , the results of [DFL21, Section 6] used below hold for these functions with the same proofs. On flat connections τ_γ is $\mathrm{tr} g_\ell$ for the core $\ell = \gamma(S^1 \times \{0\})$, and, with λ transported along a free homotopy, it depends only on the free homotopy class of the core. The transport is ambiguous only along a self-homotopy sweeping a torus on which w_2 of the bundle is nonzero, and there $\mathrm{tr} g_\ell$ vanishes on flat connections, because the lifts of the two generators of that torus anticommute. A finite family F of primary perturbations of (C_0, C_1) is *separated* if there are finitely many embeddings $\gamma_\alpha: S^1 \times D^2 \rightarrow \mathrm{int} C_0^\circ \sqcup \mathrm{int} C_1^\circ$ with pairwise disjoint images such that every member of F is a real affine combination of averaged trace functions of the γ_α . The height functions of Sections 9 and 10 enter the arguments only through their restrictions to flat connections. We fix, once for each Heegaard splitting, a realization of them as affine combinations of averaged trace functions of embedded solid tori, all the solid tori used in one compression body being pairwise disjoint, with parallel copies where two cores

are freely homotopic; for instance $h_Q = \sum_j (1 - \frac{1}{2}\tau_{\gamma_j})$ in Section 9, where the signs of the λ are chosen so that $\tau_{\gamma_j} = \text{tr } b_j$ on flat connections, with the lifts b_j fixed there. We also fix, once for each Heegaard splitting, a finite separated family F_* of primary perturbations of (C_0, C_1) containing the height functions of Section 10, such that the universal restriction map of $\text{span } F_*$ is a submersion at every flat connection of C_0 and of C_1 ; this is possible by compactness and [DFL21, Section 2.3], and Lemma C.14(i) provides a separated one. Every finite family of primary perturbations considered in Sections 11–13 contains F_* , and the generic perturbations of the auxiliary height Lagrangians of Section 11.6 are taken from $\text{span } F_*$, so that they do not depend on the family. The height functions of Section 9 belong to the families of the presentations compared there, which may be enlarged (Definition 11.12).

3.3. Admissible presentations.

Definition 3.3. Let (M, b) be an object of **B**. *Presentation data* p for M consist of:

- (i) a Heegaard splitting $M = H_0 \cup_{\Sigma} H_1$ of genus $g \geq 1$ adapted to b ;
- (ii) the torus compression bodies $C_i = H_i \natural ([0, 1] \times T^2)$, the boundary sums taken inside $b(B^3)$ through the equatorial disk, so that $C_0 \cup C_1$ is the torus-summed object $M^{\#}$ of Definition 2.1, with $h = g + 1 \geq 2$;
- (iii) identifications of collars of $\partial C_0, \partial C_1$ as in [DFL21, Section 2.1], and metrics on C_0, C_1 that are products on these collars and agree on Σ_h and on the collar of T^2 , which carries the flat metric fixed in Section 2 (the same for all presentations) (*metric data*); the product collar $I \times \Sigma_h$ of Σ_h in $M^{\#}$ given by these identifications is compatible with the gluing sphere $S = b(\partial B^3)$ of Definition 2.1: it meets S in $I \times \partial D$, where D is the equatorial disk of (ii), so that, with $T^{\circ} = T^3 \setminus \text{int } B_T$ as in Section 2, every collar line through a point of $\Sigma_h \cap T^{\circ}$ lies in T° and every collar line through a point of $\Sigma_h \setminus T^{\circ}$ lies in $M \setminus b(\text{int } B^3)$;
- (iv) primary perturbations $\mathfrak{h}_0, \mathfrak{h}_1$ with $L_{\mathfrak{h}_0}(C_0) \natural L_{\mathfrak{h}_1}(C_1)$ (*primary data*);
- (v) the regularizing data of Section 3.4, with the end perturbation f and the secondary perturbation below the thresholds of Theorem 3.8.

They form an *admissible presentation* if, for some finite family $F \supseteq F_*$ of primary perturbations, with F_* the family fixed above, the primary data lie in a fixed open neighbourhood $\mathcal{U}(F)$ of zero in the span of F , depending only on the splitting, the metric data and F (Definition 11.12): the primary data are joined to zero by a path in $\text{span } F$ along which the Lagrangians of the primary data, and those of their reflections, stay in a C^1 -neighbourhood of the unperturbed pair that is small relative to the heights of the auxiliary height functions of Section 10.1, which are fixed with the splitting.

Theorem 3.8 and Sections 5–10 apply to all presentation data, and more generally to the pairs of Remark 3.9; admissibility is used only in Sections 11–13.

3.4. The two complexes. Let p be presentation data and put $P_i = L_{\mathfrak{h}_i}(C_i)$.

The symplectic complex. $C_S(M, p) = CF(P_0, P_1)$ is the Floer complex over $\mathbb{F}_2$ generated by $P_0 \cap P_1$, with the differential counting index-one holomorphic strips for a generic t -dependent compatible almost complex structure equal near the boundary to $J_D = -J$ in the strip coordinates (s, θ) , that is, to J in the coordinates $(t, r) = (\theta, s)$ of Convention 3.1 [Oh93, DFL21]. Monotonicity and minimal Maslov number 4 exclude disk bubbling in dimensions at most one.

The gauge complex. Glue C_0 and $-C_1$ along $\Sigma_h \natural T^2$ to obtain $M^{\#}$ with its bundle. The primary perturbations define a perturbed Chern–Simons functional $\text{CS}_{\mathfrak{h}}$ on $M^{\#}$, whose

critical orbits correspond bijectively to $P_0 \cap P_1$ and are nondegenerate [DFL21, Lemmas 2.45, 2.47]. The functional $\text{CS}_\mathfrak{h}$ is invariant under α_M : since α_M vanishes on each C_i , the $\text{SU}(2)$ traces of loops in C_i° are unchanged [DFL21, p. 19]. The nonzero element of ϕ_M acts freely on the critical set. Choose an *end perturbation* f : a finite holonomy perturbation of the Chern–Simons functional on $M^\#$, invariant under ϕ_M , vanishing on a fixed neighbourhood of the critical set, and such that all nonconstant trajectories of $\text{CS}_\mathfrak{h} + f$ of index at most seven in the determinant-one theory are regular. Such f exist, may be taken arbitrarily small in the Banach norm of holonomy perturbations, and do not change the critical set or the Hessians (Proposition 3.4). Regularity is achieved by f rather than by the primary perturbations because in the two-handle comparison the primary perturbation of the middle compression body must vanish (Remark 3.10). If the trajectories of $\text{CS}_\mathfrak{h}$ of index at most seven are already regular, $f = 0$ is allowed; we then call the primary data *DFL-regular*. These are the primary data required in [DFL21, Lemma 2.56] and Theorem 3.29 there; they exist at the metric data of every presentation (Theorem 3.6). The gauge complex $C_G(M, p)$ is the Floer complex of $\text{CS}_\mathfrak{h} + f$ modulo ϕ_M . Its homology is canonically $I^\#(M)$.

Both complexes have the same generator set $P_0 \cap P_1$. This is an identification of generators, not of differentials.

Proposition 3.4. *For every choice of primary data there are end perturbations f as above, arbitrarily small in the Banach norm of holonomy perturbations [Don02, Section 5.5] and vanishing on a fixed neighbourhood of the critical set.*

Proof. This is the standard transversality argument for holonomy perturbations [Don02, Section 5.5], carried out with perturbations that vanish near the critical set; we indicate the three points to check.

The perturbations reach every nonconstant trajectory. The critical set is finite and nondegenerate. Let γ be a nonconstant trajectory of index at most seven in the determinant-one theory. Since γ is nonconstant, there is a time s_0 at which $\gamma(s_0)$ lies outside a fixed neighbourhood of the critical set, and there the perturbation f may be nonzero.

The perturbations detect cokernel elements. Let ψ be a nonzero element of the cokernel of the linearization at γ . By unique continuation for the trajectory equation, forward and backward in time [SW08, Proposition 8.6 and Theorem C.1], both ψ and $\partial_s \gamma$ are nonzero at s_0 , which we may choose generic. They are moreover linearly independent there: $\partial_s \gamma$ lies in the kernel of the linearization and ψ in the kernel of its adjoint, so $\frac{d}{ds} \langle \psi(s), \partial_s \gamma(s) \rangle = 0$ because the Hessian is symmetric, and the pairing vanishes by decay at the ends. This rules out a cokernel element proportional to $\partial_s \gamma$ near s_0 , which no perturbation localized near $\gamma(s_0)$ could detect. Holonomy functions separate $\gamma(s_0)$ from the rest of γ , from its translate $\alpha_M \cdot \gamma$ and from the critical set, and their gradients at $\gamma(s_0)$ span a dense subspace of the tangent space of the configuration space (compare the finite embedding argument of [Don02, Section 5.5, Lemma 5.16]). So some holonomy direction pairs nontrivially with ψ , and the Sard–Smale theorem makes all such trajectories regular for generic f .

Invariance and finiteness. Averaging over ϕ_M preserves the construction: a cylinder function whose gradient is supported, in the configuration space, near $\gamma(s_0)$ has its α_M -translate supported near $\alpha_M \cdot \gamma(s_0)$, and these neighbourhoods are disjoint because ϕ_M acts freely. There are finitely many energy levels below the index-seven bound, and the choice is made by induction on the energy. $\square$

Existence of DFL-regular primary data. Let $\mathfrak{h}$ be primary data with transverse Lagrangians. For $i \in \{0, 1\}$ let $\mathcal{B}(C_i^\circ)$ be the space of connections on C_i° modulo all determinant-one gauge transformations of C_i° , with no boundary condition, and let $X_i \subset \mathcal{B}(C_i^\circ)$ be the finite set of restrictions of the critical points of $\text{CS}_{\mathfrak{h}}$. A *split addition* is a finite sum $g = g_0 + g_1$ of cylinder functions, g_i with loops in $\text{int } C_i^\circ$ and vanishing on a neighbourhood of X_i . For an irreducible connection B on C_i° let π_B^i be the L^2 -orthogonal projection of $L^2(C_i^\circ; \Lambda^1 \otimes \text{ad})$ onto the orthogonal complement of the closed subspace $d_B L_1^2(C_i^\circ)$. One-forms on $M^\#$ are *linearly independent modulo gauge on C_i° at B* if the images of their restrictions under π_B^i are linearly independent. Trajectories are taken in temporal gauge, in the determinant-one theory, between lifts of critical points; the cokernel of the linearization at a trajectory $\gamma = (A_s)_{s \in \mathbb{R}}$ is represented by the decaying solutions ψ of the adjoint equation, and $\psi(s)$ is a one-form on $M^\#$ (Lemma C.2).

Lemma 3.5. *Let $\mathfrak{h}$ be primary data in the span of a separated family, with transverse Lagrangians, and let g be a split addition so small that $\text{CS}_{\mathfrak{h}+g}$ and $\text{CS}_{\mathfrak{h}}$ have the same critical points. Let $\gamma = (A_s)_{s \in \mathbb{R}}$ be a nonconstant trajectory of $\text{CS}_{\mathfrak{h}+g}$, of any index and in any path class, and $\psi \neq 0$ an element of the cokernel of its linearization. Then for each $i \in \{0, 1\}$:*

- (i) *the path $s \mapsto [A_s|_{C_i^\circ}]$ in $\mathcal{B}(C_i^\circ)$ is injective and does not meet X_i ;*
- (ii) *there is T such that for every $s_0 < -T$ the restriction $A_{s_0}|_{C_i^\circ}$ is irreducible, and $\partial_s \gamma(s_0)$ and $\psi(s_0)$ are linearly independent modulo gauge on C_i° ;*
- (iii) *there is a cylinder function g'_i with loops in $\text{int } C_i^\circ$, vanishing on a neighbourhood of X_i , such that $\int_{\mathbb{R}} \langle \nabla g'_i(A_s), \psi(s) \rangle ds \neq 0$.*

Theorem 3.6. *Fix a Heegaard splitting and metric data as in Definition 3.3(i)–(iii). There are families F as in Definition 3.3 that are separated, and for each of them generic primary data in a neighbourhood of zero in $\text{span } F$ have transverse Lagrangians. Let F be a family as in Definition 3.3 that is separated, and $\mathfrak{h} \in \mathcal{U}(F)$ primary data with transverse Lagrangians. Then there are arbitrarily small split additions g , each lying in the span of a finite family G of cylinder functions with loops in $\text{int } C_0^\circ \sqcup \text{int } C_1^\circ$, such that $\text{CS}_{\mathfrak{h}+g}$ has the same critical points as $\text{CS}_{\mathfrak{h}}$, the primary data $\mathfrak{h} + g$ are DFL-regular, and $\mathfrak{h} + g \in \mathcal{U}(F \cup G)$; presentation data with primary data $\mathfrak{h} + g$ are therefore admissible. The family G may depend on the required smallness of g , and its loops may be taken in $\text{int } C_0^\circ$ alone. The metric data are not changed.*

Lemma 3.5 and Theorem 3.6 are proved in Appendix C. Near $s = -\infty$ both $\partial_s \gamma$ and ψ are governed by eigenvectors of the Hessian at the limiting critical point, with eigenvalues of opposite signs. The Hessian acts on each piece separately, so eigenvectors with distinct eigenvalues stay independent modulo gauge on each piece, provided no eigenvector vanishes on the unperturbed part of the neck. That last property is where separatedness is used (Proposition C.8).

Remark 3.7. Daemi, Fukaya and Lipyanskiy obtain DFL-regular primary data after a constant rescaling of the metric of $M^\#$ [DFL21, Lemma 2.56 and Proposition 6.8]. The cylinder functions added in the inductive step of their proof are split, with loops in the interior of one of the two pieces, and the step on [DFL21, p. 80] passes from the linear independence of ψ and $\partial_s \gamma$ on $M^\# \setminus \Sigma$ to split cylinder functions that pair nontrivially with ψ . For split functions the condition needed concerns one piece: independence of ψ and $\partial_s \gamma$ modulo gauge there, together with injectivity and irreducibility of the restricted path. We could not deduce independence on one piece from the L^2 -orthogonality of ψ and $\partial_s \gamma$ on $M^\#$: $\psi = c_i \partial_s \gamma$ modulo gauge on C_i , with constants $c_0 \neq c_1$, is compatible with it, and for such ψ every cylinder function with loops in one piece pairs to zero. The isomorphism of [DFL21, Theorem 3.29]

is not affected, since Theorem 3.6 supplies the regular data it requires. Lemma 3.5 provides these conditions on each piece when the base data lie in a separated family. Theorem 3.6 is therefore an alternative proof of the existence statement of [DFL21, Proposition 6.8] for base data in a separated family, with the added perturbations supported in $\text{int } C_0^\circ \sqcup \text{int } C_1^\circ$ and without rescaling. For a general base perturbation in the Banach space of [DFL21, Section 6.1] the argument would need the conclusion of Proposition C.8, which we prove only for separated data. The rescaling preserves the regularity of the trajectories, which it only reparametrizes in time, and the Lagrangians, since $*_3 \nabla h$ does not change under a constant rescaling of the metric; it changes the flat metric of the torus collar, which Definition 3.3(iii) fixes.

3.5. The mixed map. Let q_s be the special quintuple of [DFL21, Section 3.1] associated to the compression bodies C_0, C_1 : a four-manifold X containing $\mathbb{R} \times C_0^\circ, \mathbb{R} \times (-C_1^\circ)$ and $\mathbb{U}_- \times \Sigma_h$, with a gauge-theoretic end $(-\infty, -3] \times M^\#$, and a surface $S = \mathbb{U}_+$ with a symplectic end $[3, \infty) \times [-1, 1]$ and two boundary components labelled by P_0 and P_1 . Here $\mathbb{C}$ is cut by the matching line $\mathbb{U}_\partial = i\mathbb{R}$ into $\mathbb{U}_-$ and $\mathbb{U}_+$; the planar region $\mathbb{U}_-$ contains the input arm $(-\infty, -3] \times [-1, 1]$, two arms $[-2, 0] \times [2, \infty)$ and $[-2, 0] \times (-\infty, -2]$ and the square $[-1, 0] \times [-2, 2]$, and $\mathbb{R} \times C_0^\circ$ and $\mathbb{R} \times (-C_1^\circ)$ are glued to $\mathbb{U}_- \times \Sigma_h$ along the two boundary curves of $\mathbb{U}_-$ other than $\mathbb{U}_\partial$ [DFL21, Figure 1]. The ends of the two arms are the mixed ends of the quintuple. We call a domain of this type, for any pair covered by the results below, a *DFL domain*. A *DFL disk* is a DFL domain obtained by cutting a larger domain along ordinary strip ends.

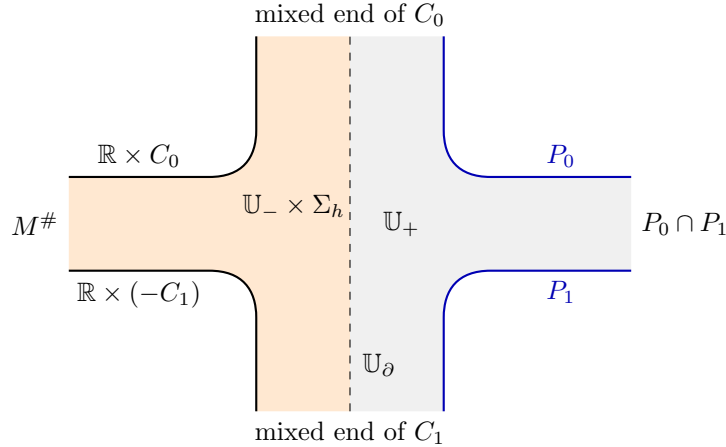


FIGURE 1. The planar base of the special quintuple of [DFL21, Section 3.1], after [DFL21, Figure 1]. Over the region $\mathbb{U}_-$ (shaded) the gauge domain is $\mathbb{U}_- \times \Sigma_h$, with $\mathbb{R} \times C_0$ and $\mathbb{R} \times (-C_1)$ glued along the two boundary curves; the section is defined over $\mathbb{U}_+$, with boundary conditions P_0 and P_1 ; the two are matched along the line $\mathbb{U}_\partial$. The ends are the gauge input $M^\#$, the strip output at $P_0 \cap P_1$, and two mixed ends along the matching line.

The torus T^2 and its point moduli space are carried along; the matching condition on $\mathbb{U}_\partial \times T^2$ imposes flatness of the torus restriction. On the gauge-theoretic end we add χf , where χ is a cutoff equal to 1 near $-\infty$ and supported before the matching region, and we choose a secondary perturbation η : a finite holonomy perturbation supported in a compact subset of

X away from the matching line and invariant under ϕ_M , together with a compact deformation of the almost complex structure away from the matching line. The mixed equation is

$$(3.1) \quad F_A^+ + (*_3 \nabla \mathfrak{h})^+ + \chi(*_3 \nabla f)^+ + \eta(A) = 0, \quad \partial_s u + J_{s,\theta}(u) \partial_\theta u = 0,$$

with the matching condition along $\mathbb{U}_\partial \times \Sigma_h$ and the Lagrangian boundary conditions on S . Here $J_{s,\theta}$ is a family of compatible almost complex structures equal to $J_D = -J$ near the matching line and near $\theta = \pm 1$, in the coordinates (s, θ) of [DFL21, Section 3.1]. For generators a, x let $\mathcal{M}^d(a, x)$ be the space of finite-energy solutions of index d asymptotic to a at the gauge end and to x at the symplectic end, modulo the gauge group enlarged by ϕ_M .

Theorem 3.8. *(i) For f sufficiently small and invariant secondary perturbations η that are sufficiently small, including the deformation of the almost complex structure, and generic, every $\mathcal{M}^d(a, x)$ with $d \leq 1$ is regular, the zero-dimensional spaces are finite, and*

$$N_p(a) = \sum_x \# \mathcal{M}^0(a, x) x$$

is a chain map $C_G(M, p) \rightarrow C_S(M, p)$. With respect to the filtration by the constants $\mathfrak{c}$ of [DFL21, Proposition 2.53], which normalize the index formula, $N_p(a) = a +$ (terms of lower filtration); in particular N_p is a chain isomorphism.

(ii) The chain homotopy class of N_p , composed with the gauge continuation map between two choices of f , is independent of f , χ and η .

We call N_p the *mixed map* at p . In Section 12 the presentations are the vertices of a graph whose edges are changes of presentation.

Proof. Energy and index. Let κ be the topological energy normalized as in [DFL21, Proposition 3.20] (Section 3.6). For a solution of (3.1) in the homotopy class of (A, u) , the analytic energy equals

$$8\pi^2 \kappa(A, u) + 2 \|\eta(A)\|_{L^2(X)}^2 + 2 \int \chi'(s) f(A_s) ds,$$

as in [DFL21, (3.24)] with the additional end term, whose absolute value is at most $2 \sup |f|$, since $\int |\chi'| = 1$ and f vanishes at the critical point at the gauge end. Indeed, if $K = F_A + *_3 \nabla \mathfrak{h}$ and $P = \chi *_3 \nabla f$, then $P \wedge P = 0$ and, with the inner product $-\text{tr}$, the cross term is $2 \int \text{tr}(K \wedge P) = -2 \int \chi \frac{d}{ds} f(A_s) ds = 2 \int \chi' f(A_s) ds$. The index is $\text{ind } D_{(A,u)} = 8\kappa(A, u) + \mathfrak{c}(x) - \mathfrak{c}(a)$; the *asymptotic labels* of a configuration are the critical points, intersection points or components of critical manifolds to which it converges at its ends. For two configurations with the same labels, $2(\kappa - \kappa') \in \mathbb{Z}$ [DFL21, Lemma 3.15], and $\mathfrak{c}$ is the function on generators of [DFL21, Proposition 2.53], determined up to an additive constant by this formula.

Regularity. Transversality for secondary perturbations is obtained by DFL's induction over finite-dimensional families of perturbations and the Sard–Smale theorem [DFL21, Section 6.2]. The perturbations can be chosen invariant under ϕ_M by Lemma 5.5: holonomies along loops with vanishing lifting class span the relevant test fields. Regularity in index at most one is achieved by a finite choice, proceeding upward from the lowest index allowed by the energy bound.

Chain map. The one-dimensional spaces compactify with boundary the broken configurations of [DFL21, Proposition 3.26]: a gauge trajectory followed by an index-zero mixed solution, or an index-zero mixed solution followed by a holomorphic strip. Sphere and disk bubbles and energy concentration are excluded in dimension at most one by the index-energy relation and monotonicity [DFL21, Section 5.4]. Hence $\partial_S N_p = N_p \partial_G$.

Triangularity. Fix $\kappa_+ > 0$ below every positive value of $|\mathfrak{c}(a) - \mathfrak{c}(x)|/8$. An index-zero solution from a to x has $\kappa = (\mathfrak{c}(a) - \mathfrak{c}(x))/8$ by the index formula, and since its analytic energy is nonnegative, $8\pi^2\kappa \geq -2\sup|f| - 2\sup_A \|\eta(A)\|_{L^2(X)}^2$. So if $2\sup|f| + 2\sup_A \|\eta(A)\|_{L^2(X)}^2 < 8\pi^2\kappa_+$, then $\kappa \geq 0$, hence $\mathfrak{c}(a) \geq \mathfrak{c}(x)$, with equality only if $\kappa = 0$. The constant pairs of [DFL21, Section 3.2] are regular by [DFL21, Proposition 3.19], so by the implicit function theorem, for small (f, η) , a fixed neighbourhood of each constant pair contains exactly one solution, of index zero and with $\kappa = 0$. The second threshold, below which these are all the index-zero solutions with $\kappa = 0$, is obtained by contradiction. Suppose that for data $(f_n, \eta_n) \rightarrow 0$ there are index-zero solutions c_n with $\kappa = 0$ outside these neighbourhoods. Their analytic energies are at most $2\sup|f_n| + 2\sup_A \|\eta_n(A)\|_{L^2(X)}^2 \rightarrow 0$. A nonconstant trajectory of $\text{CS}_h + f_n$ between critical points, or a nonconstant holomorphic strip, has energy bounded below independently of n , because f_n vanishes near the critical set; so no energy escapes along the ends, and a subsequence of (c_n) converges to a configuration of zero energy with the same asymptotic labels, that is, to a constant pair; breaking along the two mixed ends is excluded by the energy quantum and exponential decay there [DFL21, Proposition 3.6 and Section 5.4]. The decay of the c_n along the ends is uniform in n , again because f_n vanishes near the critical set, so the convergence holds in the topology in which the neighbourhoods above are open, and c_n lies in one of them for large n , a contradiction. Below both thresholds, therefore, the index-zero solutions with $\kappa = 0$ are exactly one for each generator, from the generator to itself. This is the argument of [DFL21, Theorem 3.29].

Independence. At fixed f , a path of choices (χ_t, η_t) has parameter derivative supported in a compact core, and defines a parametrized moduli space whose index -1 elements occur at isolated parameters and are glued using the parametrized operator $(\xi, t) \mapsto D\xi + t \partial_\lambda F$ (Theorem 5.4). Its one-dimensional part has boundary $N^1 - N^0$ and the broken configurations, giving a chain homotopy $N^1 \simeq N^0$. A change of f changes the equation on the whole gauge end and is not a compactly supported variation, so it is treated separately: insert a long gauge cylinder carrying the continuation from f_0 to f_1 at the gauge end. By Theorem 5.1 applied at the closed cut $M^\#$, for long necks the count on the glued domain is $N_{f_1} \circ \text{cont}_G(f_0, f_1)$, while the glued domain is a compactly supported deformation of the domain for f_0 , so the previous argument identifies the count with N_{f_0} up to chain homotopy. $\square$

Remark 3.9. Theorem 3.8 and its proof hold verbatim for any pair of three-manifolds satisfying Definition 3.2, with common boundary, possibly disconnected, product target and transverse primary Lagrangians, including boundary markings twisted by a frame transition: in particular for the pairs (C_0, C_j) with the ρ_ζ -marking and (C_1, C_2) of Section 9 (whose critical sets are described by Lemma 9.8), and for the pairs (D_i, C) and (D_0, D_1) of Section 10.1, whose matching surface is $\Sigma_h \sqcup \Sigma_g$ and target $M_h^- \times M_g$. The inputs [DFL21, Lemma 3.15, Proposition 3.20] and the triangularity argument use only embeddedness and monotonicity of the Lagrangians, and [DFL24] allows a disconnected odd matching surface. The two further inputs of the triangularity step also hold: regularity of the constant pairs [DFL21, Proposition 3.19] reduces to transversality of the pair at the corresponding intersection point, and each listed pair is an admissible splitting in the sense of [DFL21], with two odd boundary components, so [DFL21, Propositions 2.53 and 3.20] apply to it directly.

Remark 3.10. The map N_p is defined by the construction of [DFL21, Section 3.2], with two changes: the gauge complex may be regularized by an end perturbation instead of by the primary perturbations, and the secondary perturbations are invariant under ϕ_M . The first is needed in the two-handle comparison, where the primary perturbation of the middle compression body must vanish, so that [DFL21, Lemma 2.56] is not available there. The second

makes the counts independent of the lift of a generator to the determinant-one configuration space: DFL count solutions asymptotic to a fixed representative of each generator [DFL21, p. 27], and pass to the quotient by the involution ι of [DFL21, p. 18], which is α_M . Where both constructions apply they agree (Proposition 3.11).

Proposition 3.11 (comparison with the DFL isomorphism). *Let p be presentation data, or more generally a pair of torus compression bodies as in Remark 3.9 with collars and metrics as in Definition 3.3(iii) but an arbitrary flat metric on the torus collar, with DFL-regular primary data. Take $f = 0$; a family of almost complex structures as in [DFL21, Section 3.1 and Lemma 2.35], equal near $\theta = \pm 1$ to DFL’s standard structure, which is $J_D = -J$ in DFL’s coordinates (s, θ) , so that in the coordinates $(t, r) = (\theta, s)$ of Sections 6–7 the equation is the J -holomorphic equation; and a ϕ_M -invariant secondary perturbation η , sufficiently small and supported, with the deformation of the almost complex structure, in a compact set away from the matching line as in [DFL21, Proposition 3.23(i)], for which the mixed moduli spaces of expected dimension at most three are regular. Such η exist, the isomorphism N^{DFL} of [DFL21, Theorem 3.29] is defined at these data, and*

$$N_p = N^{\text{DFL}} \otimes \mathbb{F}_2: C_G(M, p) \rightarrow C_S(M, p)$$

as chain maps.

Proof. The complexes. DFL’s gauge complex is generated by the critical points of the perturbed functional modulo determinant-one gauge transformations and the involution ι given by the class Poincaré dual to a component of the matching surface [DFL21, pp. 18–19, Lemmas 2.43 and 2.45]. In the framed splitting this component may be taken to be the torus, so $\iota = \alpha_M$ and DFL’s generators are ours. With $f = 0$ the gauge equations agree, so the differentials agree mod two; the strip complexes agree because the Lagrangians and almost complex structures do. Regular families equal to the standard structure near $\theta = \pm 1$ exist, since the transversality perturbations for nonconstant strips can be supported in the interior of the strip.

The equations. With $f = 0$, equation (3.1) is [DFL21, (3.22)] on the same special quintuple; Convention 3.1 records that the two sign conventions define the same equation.

The counts. DFL’s zero-dimensional space $\mathcal{M}_\eta(\tilde{a}, x)_0$ consists of solutions asymptotic to a fixed determinant-one representative $\tilde{a}$ of a , modulo determinant-one gauge transformations. The automorphism α_M extends over the special quintuple [DFL21, Remark 3.7], it maps solutions for η to solutions for $\alpha_M^* \eta$, and $\alpha_M^* \eta = \eta$ by invariance. So it identifies the spaces for the two representatives $\tilde{a}$, $\alpha_M \tilde{a}$, and $\mathcal{M}_\eta(\tilde{a}, x)_0$ maps bijectively onto our $\mathcal{M}^0(a, x)$, the space of solutions modulo the gauge group enlarged by ϕ_M (which acts freely by Theorem 5.6(i)). Hence the counts agree mod two.

Existence of η and DFL’s hypotheses. By Theorem 5.6(ii) the linearized operator together with the ϕ_M -invariant holonomy directions supported in a compact set away from the matching line is onto at every solution, and the Sard–Smale theorem, applied finitely many dimensions at a time, gives η , small and compactly supported away from the matching line, with all moduli spaces of expected dimension at most three regular. These are the requirements of [DFL21, Propositions 3.23 and 3.26], whose remaining conclusions (nonnegative energy, the unique solution in $\mathcal{M}_\eta(a, a)_0$, compactness and the boundary description) follow from smallness and compact support as in [DFL21, Section 5]. $\square$

By Theorem 3.6, at every Heegaard splitting and all metric data there are admissible presentations with DFL-regular primary data $\mathfrak{h} + g$, where $\mathfrak{h} \in \mathcal{U}(F)$, for a family F as in

Definition 3.3 that is separated, is any primary data with transverse Lagrangians and g is an arbitrarily small split addition; Proposition 3.11 applies at these presentations.

3.6. Energy, gradings and filtrations. We use DFL’s normalization throughout: the topological energy κ of a mixed pair is the Chern–Weil integral of the gauge part plus the symplectic area of the section, divided so that $\text{ind} = 8\kappa + \text{const}$ [DFL21, Proposition 3.20], and the analytic energy of a solution is $8\pi^2\kappa$ plus the secondary terms [DFL21, (3.24)]. For two configurations with the same asymptotic labels $2(\kappa - \kappa') \in \mathbb{Z}$ [DFL21, Lemma 3.15]; we write κ_0 for the resulting coset representative. Thus every nonzero index difference between classes with the same labels has absolute value at least 4. We use only this divisibility, without asserting that every allowed set of classes realizes the smallest possible period. In this normalization a change $\Delta\kappa = \frac{1}{2}$ changes $8\pi^2\kappa$ by $4\pi^2$. For a pure strip the topological energy is $\kappa(u) = \frac{1}{4\pi^2} \int u^* \omega$, since its full Dirichlet energy is twice its symplectic area [DFL21, Proposition 4.69]. In particular, a positive generator sphere, when $h \geq 2$, contributes $\Delta\kappa = \frac{1}{2}$ and index 4 already in the determinant-one surface moduli space. No enlargement of the gauge group is needed for this sphere contribution.

4. HANDLE DECOMPOSITIONS AND BUNDLES

This section contains the topology needed to write every morphism of $\mathbf{B}$ as a composite of elementary pieces. All handle decompositions are relative to the tube: handles are attached away from ν , and the tube is a product throughout.

4.1. Decompositions relative to the tube.

Proposition 4.1. *Let $(W, E, \nu): M_0 \rightarrow M_1$ be a morphism of $\mathbf{B}$. Then W has a handle decomposition relative to $M_0 \times [0, 1] \cup \nu(B^3 \times [0, 1])$ with handles of index 1, 2 and 3 only, attached in this order and away from the tube, such that every regular level is connected. All two-handles can be attached along one framed link in a single level. Consequently*

$$W = W_3 \circ W_2 \circ W_1,$$

where $W_1: M_0 \rightarrow Z_0$ consists of one-handles, $W_2: Z_0 \rightarrow Z_1$ is the trace of surgery on a framed link, and $W_3: Z_1 \rightarrow M_1$ consists of three-handles, with Z_0, Z_1 connected.

Proof. Let W° be the complement of the open tube; it is connected (proof of Lemma 2.2). Choose a Morse function $f: W^\circ \rightarrow [0, 1]$ that equals the $[0, 1]$ coordinate on a collar $S^2 \times [0, \delta] \times [0, 1]$ of the vertical boundary and near the ends, and a gradient-like vector field equal to ∂_t on the collar. The collar is a union of flow lines, so every stable and unstable manifold of an interior critical point, and every modification below, lies outside it.

Order of indices. If a k -handle is attached after a j -handle with $k \leq j$, the attaching sphere of dimension $k - 1$ and the belt sphere of dimension $3 - j$ in a three-dimensional level satisfy $(k - 1) + (3 - j) < 3$. An isotopy supported off the collar separates them, and the critical values can be interchanged [Mil65, Section 4], [GS99, Propositions 4.2.7 and 4.2.13].

No zero- or four-handles. After ordering, the union of the incoming collar with the zero- and one-handles is connected, because later handles have connected attaching regions and cannot join components, while W° is connected. Choose a spanning tree of the graph whose vertices are the zero-handles and the incoming collar and whose edges are the one-handles. A leaf zero-handle meets its tree one-handle in one point of the attaching sphere, and the pair cancels [Mil65, Theorem 5.4]. Proceeding from the leaves removes all zero-handles. Four-handles are removed by the same argument applied to $1 - f$.

Connected levels. Attaching a one-handle to a connected level gives a connected level. Surgery on a circle in a connected three-manifold gives a connected manifold, since the complement of a circle is connected. Surgery on a two-sphere changes the number of components by 0 or +1. Starting from the connected M_0 , all levels before the three-handles are connected. Since there are no four-handles, the number of components does not decrease from the first three-handle on, and it equals one at M_1 . Hence all levels are connected.

One level of two-handles. A two-handle can be attached below the belt circle of an earlier two-handle, since $1 + 1 < 3$. Pushing all attaching circles into the level just above the one-handles gives a single framed link. $\square$

Proposition 4.2. *In the decomposition of Proposition 4.1, the trivialization of E over M_0 and the tube extends over W_1 , and the trivialization over M_1 and the tube extends over W_3 . In particular $E|_{Z_0}$ and $E|_{Z_1}$ are trivial, and (W, E, ν) is the composite of $(W_1, \mathbb{R}^3)$, $(W_2, E|_{W_2})$ with the induced frames at Z_0 , Z_1 , and $(W_3, \mathbb{R}^3)$.*

Proof. Up to homotopy equivalence relative to $M_0 \cup \nu$, the pair $(W_1, M_0 \cup \nu)$ is $(M_0 \times [0, 1], M_0 \times 0 \cup b_0 \times [0, 1])$ with relative one-cells attached along pairs of points. The trivialization extends over the product. Over a one-handle $D^1 \times D^3$ the two transition functions on the attaching balls are null-homotopic maps $D^3 \rightarrow \mathrm{SO}(3)$, and a path in the connected group $\mathrm{SO}(3)$ joining their values extends the frame across the handle. The statement for W_3 is the same with f replaced by $1 - f$. The decorated pieces glue to (W, E, ν) by construction. $\square$

The extension in Proposition 4.2 is not unique: frames on Z_0 may differ by an element of $\pi_1(\mathrm{SO}(3))$ for each one-handle. This ambiguity is absorbed by the frame-change arrows of Section 4.5, and it is invisible in the final statement, because the composite morphism does not depend on it.

4.2. One- and three-handles and adapted presentations. A Heegaard splitting $M = H_0 \cup_\Sigma H_1$ is *adapted to b* if $\Sigma \cap b(B^3)$ is the equatorial disk D and Σ is standard on a collar of $b(B^3)$. The attaching balls of a one-handle can be moved, by an isotopy of the level fixed near $b(B^3)$, into a neighbourhood of a Heegaard surface meeting $b(B^3)$ in D , because the complement of a ball in a connected three-manifold is connected. The isotopy extends to the cobordism by a diffeomorphism fixed on the tube, so the decorated morphism is unchanged. Reading W_3 from M_1 , each three-handle becomes a one-handle attached to $-M_1$, and its attaching sphere is the belt sphere of that one-handle; by Proposition 4.1 it is nonseparating in its level.

4.3. Subordinate Heegaard triples and the tube. Let $L \subset Z$ be a framed link of n components in a connected three-manifold with a ball b disjoint from L , and let $W(Z, L)$ be the trace of surgery. By [OS06, Lemma 4.5] there is a Heegaard triple $(\Sigma; H_0, H_1, H_2)$ of genus g subordinate to a bouquet for L , in the sense of [OS06, Definition 4.2]. After an isotopy in the complement of the bouquet, Σ meets $b(B^3)$ in the equatorial disk. Then $Y_{01} = H_0 \cup H_1 = Z$, $Y_{02} = H_0 \cup H_2 = Z(L)$ and $Y_{12} = H_1 \cup H_2 \cong \#^r(S^1 \times S^2)$ with $r = g - n$. Let X be the four-manifold

$$X = (\Delta \times \Sigma) \cup (e_0 \times H_0) \cup (e_1 \times H_1) \cup (e_2 \times H_2)$$

of [OS06, Section 2.2], where Δ is a triangle with edges e_0, e_1, e_2 , and let $H_r = \natural^r(S^1 \times D^3)$.

Proposition 4.3. *There is a diffeomorphism $\Phi: X \cup_{Y_{12}} H_r \rightarrow W(Z, L)$ that is the identity on Z and on $Z(L)$, and that restricts to the identity from a tube $D_z \times [-1, 1] \times [-1, 1]$ onto $b(B^3) \times [-1, 1] \subset Z \times [-1, 1]$, where $D_z \subset \Sigma$ is a disk around the basepoint missing all*

attaching curves. The core of the tube is carried to the product arc with its product normal framing. The same holds for any other identification of the filling with H_r , after an isotopy relative to the boundary.

Proof. Let V be a regular neighbourhood of the bouquet, a handlebody of genus n containing L . By [OS06, Definition 4.2], $Z \setminus \text{int } V = H_0 \cup_\Sigma \mathcal{K}$, where $\mathcal{K}$ is the compression body obtained from $\Sigma \times [0, 1]$ by attaching two-handles along the curves $\beta_{n+1}, \dots, \beta_g$; moreover $H_1 = \mathcal{K} \cup V$ and $H_2 = \mathcal{K} \cup V(L)$, where $V(L)$ is V surgered along L . Fix a bicollar $\Sigma \times [-1, 1]_s \subset H_0 \cup \mathcal{K}$ and put $B = D_z \times [-1, 1]_s$, a ball with $B \cap \Sigma = D_z$, disjoint from L ; we choose the bicollar so that $B = b(B^3)$ as parametrized balls.

A product model of X . For a planar hexagon Q whose sides alternate between edge sides $\partial_0, \partial_1, \partial_2$ and vertex sides, set $X(Q) = (\Sigma \times Q) \cup \bigcup_i H_i^0 \times \partial_i Q$, where H_i^0 is H_i minus a collar, with corners smoothed. A diffeomorphism of hexagons matching labelled sides, and a product near the edge sides, induces a diffeomorphism of the corresponding spaces. Take $Q' = [-1, 1]_s \times [-1, 1]_t \setminus ((0, 1] \times (-\epsilon, \epsilon))$ with $\partial_0 = \{s = -1\}$, $\partial_1 = \{s = 1, t \leq -\epsilon\}$ and $\partial_2 = \{s = 1, t \geq \epsilon\}$. Then

$$X \cong Z \times [-1, -\epsilon] \cup H_0 \times [-\epsilon, \epsilon] \cup Z(L) \times [\epsilon, 1],$$

and near the basepoint X contains the product $D_z \times Q'$.

The filling. The complement of X in $(Z \setminus V) \times [-1, 1]$ is $\mathcal{K}^{s \geq 0} \times (-\epsilon, \epsilon)$. Let Q_V be a four-ball bounding the three-sphere $V \times \{-\epsilon\} \cup \partial V \times [-\epsilon, \epsilon] \cup V(L) \times \{\epsilon\}$, a Heegaard splitting of S^3 of genus n . Put $H = (\mathcal{K}^{s \geq 0} \times [-\epsilon, \epsilon]) \cup Q_V$. Since $\mathcal{K} \times I$ is $\partial V \times I \times I$ with $g - n$ four-dimensional one-handles, $H \cong \natural^r(S^1 \times D^3)$ with $\partial H = Y_{12}$. Near the basepoint H contains the half-ball $Q_z = D_z \times [0, 1]_s \times [-\epsilon, \epsilon]$.

The diffeomorphism. Now $X \cup H = (Z \setminus V) \times [-1, 1] \cup W'_V$ with $W'_V = V \times [-1, -\epsilon] \cup Q_V \cup V(L) \times [\epsilon, 1]$, while $W(Z, L) = (Z \setminus V) \times [-1, 1] \cup W_V$ with W_V the union of $V \times [-1, 1]$ and the two-handles along $L \times \{1\}$. Both W_V and W'_V are four-balls: in W_V each two-handle cancels a distinct one-handle of $V \times I$. Their boundaries are identified by the identity on $V \times \{-1\} \cup \partial V \times [-1, 1]$ and the tautological identification of the surgered handlebody at the top. By Cerf's theorem [Cer68] this boundary diffeomorphism extends over the balls. Define Φ to be the identity on $(Z \setminus V) \times [-1, 1]$ and this extension on W'_V .

The tube. The tube $(D_z \times Q') \cup Q_z = B \times [-1, 1]$ lies in $(Z \setminus V) \times [-1, 1]$, where Φ is the identity. Its core and its normal frame $(\partial_x, \partial_y, \partial_s)$ are carried to themselves. For another filling H'_r , a diffeomorphism $H'_r \rightarrow H$ can be chosen to be the identity on the boundary by Laudenbach–Poénaru [LP72]. Since Q_z is a regular neighbourhood of a boundary ball inside a collar, and collars are unique up to isotopy, the diffeomorphism may be isotoped relative to the boundary to preserve Q_z . $\square$

4.4. Bundles on a two-handle cobordism. Let $V = H^1(\Sigma; \mathbb{F}_2)$, and let $\Lambda_i \subset V$ be the image of $H^1(H_i; \mathbb{F}_2)$, a Lagrangian subspace of dimension g . Put $W_2 = X \cup_{Y_{12}} H_r$.

Proposition 4.4. *Let $\delta: V^3 \rightarrow H^2(X; \mathbb{F}_2)$ be the Mayer–Vietoris coboundary for the cover of X by $\Delta \times \Sigma$ and $e_i \times H_i$. Then $x \mapsto \delta(x, 0, 0)$ induces an isomorphism*

$$V / (\Lambda_0 + (\Lambda_1 \cap \Lambda_2)) \xrightarrow{\cong} H^2(W_2; \mathbb{F}_2).$$

The class $\delta(x, 0, 0)$ restricts to x modulo $\Lambda_0 + \Lambda_1$ on Y_{01} and to x modulo $\Lambda_0 + \Lambda_2$ on Y_{02} . In particular every $\text{SO}(3)$ bundle on W_2 that is trivial over both ends has $w_2 = \delta(x, 0, 0)$ with $x \in (\Lambda_0 + \Lambda_1) \cap (\Lambda_0 + \Lambda_2)$.

Proof. The pieces $e_i \times H_i$ are pairwise disjoint and meet $\Delta \times \Sigma$ in copies of $e_i \times \Sigma$. Mayer–Vietoris reads

$$V \oplus \bigoplus_i H^1(H_i) \xrightarrow{\rho} V^3 \xrightarrow{\delta} H^2(X) \rightarrow H^2(\Delta \times \Sigma) \rightarrow H^2(\Sigma)^3,$$

with $\rho(v, u_0, u_1, u_2) = (v - u_0, v - u_1, v - u_2)$. The last map is the diagonal $\mathbb{F}_2 \rightarrow \mathbb{F}_2^3$ and is injective, so $H^2(X) = (V/\Lambda_0 \oplus V/\Lambda_1 \oplus V/\Lambda_2)/\Delta(V)$. The same cover of Y_{ij} gives $H^2(Y_{ij}) = V/(\Lambda_i + \Lambda_j)$, with restriction $[x_0, x_1, x_2] \mapsto x_i - x_j$. Since $H^1(H_r) \rightarrow H^1(Y_{12})$ is onto and $H^2(H_r) = 0$, Mayer–Vietoris for $W_2 = X \cup H_r$ identifies $H^2(W_2)$ with the kernel of $H^2(X) \rightarrow H^2(Y_{12})$. For a class in this kernel, $x_1 - x_2 \in \Lambda_1 + \Lambda_2$. Subtracting $\Delta(x_2)$ and then $\Delta(a_2)$ for suitable $a_2 \in \Lambda_2$ brings it to the form $[x, 0, 0]$. Two such classes agree exactly when $x - x' \in \Lambda_0 + (\Lambda_1 \cap \Lambda_2)$. $\square$

Example 4.5. For the (-1) -framed unknot in S^3 one may take $g = 1$, $\alpha = \lambda$, $\beta = \mu$, $\gamma = \lambda - \mu$. Then $r = 0$, $\Lambda_1 \cap \Lambda_2 = 0$ and $H^2(W_2; \mathbb{F}_2) = \mathbb{F}_2^2/\langle \lambda \rangle \cong \mathbb{F}_2$, the class of the nontrivial bundle on $\overline{\mathbb{C}P}^2$ minus two balls. Its restrictions to both ends vanish.

Definition 4.6. Let E be a bundle on W_2 trivial over both ends, with $w_2(E) = \delta(\zeta, 0, 0)$. A *filling frame* for E consists of a trivialization of E over the central product $\Delta \times \Sigma$, over $e_1 \times H_1$, $e_2 \times H_2$ and H_r with identity transitions, and over $e_0 \times H_0$ with transition a map $\rho_\zeta: \Sigma \rightarrow \text{SO}(3)$ of lifting class ζ , equal to the identity near D_z ; the frame agrees with the given trivialization of E along the tube.

Filling frames exist, since $[\Sigma/D_z, \text{SO}(3)] = H^1(\Sigma; \mathbb{F}_2)$ (the map $\text{SO}(3) \rightarrow \mathbb{R}P^\infty$ is 3-connected), and the agreement along the tube can be arranged because the tube is contractible. A filling frame is not a trivialization of E over $Z_0 = H_0 \cup (-H_1)$: its two handlebody frames differ along Σ by ρ_ζ . In the surface coordinates of the frame on $\Delta \times \Sigma$, the boundary conditions of the triangle are the three Lagrangians $(\tau_\zeta L(C_0), L(C_1), L(C_2))$ of Section 9.1, where τ_ζ is the symplectomorphism of the surface moduli space induced by ρ_ζ .

Global frames from the filling. Since $E|_{Z_0}$ is trivial, $\zeta \in \Lambda_0 + \Lambda_1$ (Proposition 4.4); write $\zeta = a_0 + a_1$ with $a_i \in \Lambda_i$. Choose $w_1: H_1 \rightarrow \text{SO}(3)$, equal to 1 near the ball, with lifting class a_1 . The map $w_1 \rho_\zeta^{-1}|_\Sigma$ has lifting class $a_0 \in \Lambda_0$, hence extends, after a homotopy absorbed in a collar, to $w_0: H_0 \rightarrow \text{SO}(3)$. Then $\phi_0^{\text{gl}} = (w_0 \cdot \text{frame on } H_0) \cup (w_1 \cdot \text{frame on } H_1)$ is a trivialization of $E|_{Z_0}$; in the surface coordinates it acts by τ_{a_1} , which carries $(\tau_\zeta L(C_0), L(C_1))$ to $(L(C_0), L(C_1))$, because $\tau_{a_1} \tau_\zeta = \tau_{a_0}$ and τ_{a_i} preserves $L(C_i)$. For a perturbed Lagrangian $L_{h_i}(C_i)$ the frame carries h_i to the transported perturbation $w_i^* h_i$, since trace functions change sign on loops with $a_i(\ell) = 1$; this is a change of primary data, and the presentations p_i^ϕ of Section 9 carry the transported data. The same construction at Z_1 , with $\zeta = a'_0 + a'_2$, gives ϕ_1^{gl} .

4.5. Frame changes. A *frame change* of an object M is a map $u: M \rightarrow \text{SO}(3)$ equal to 1 near $b(B^3)$. It defines a morphism $[u]: M \rightarrow M$ of $\mathbf{B}$: the product $[0, 1] \times M$ with the vertical tube and the product bundle, whose trivialization at the outgoing end is changed by u . Frame changes compose by pointwise multiplication, $[v] \circ [u] = [vu]$, and $[u]$ depends only on the homotopy class of u relative to the ball.

Lemma 4.7. Let $(W_2, E; \sigma_0, \sigma_1): Z_0 \rightarrow Z_1$ be the two-handle cobordism with the trivializations σ_i induced by Proposition 4.2, and put $u_i = \sigma_i(\phi_i^{\text{gl}})^{-1}$. Then in $\mathbf{B}$

$$(4.1) \quad (W_2, E; \sigma_0, \sigma_1) = [u_1] \circ (W_2, E; \phi_0^{\text{gl}}, \phi_1^{\text{gl}}) \circ [u_0]^{-1}.$$

If W_2 has no two-handles, $E|_{W_2}$ is trivial and the morphism $(W_2, E; \sigma_0, \sigma_1)$ is the single frame change $[\sigma_1 \tilde{\sigma}_1^{-1}]$, where $\tilde{\sigma}$ extends σ_0 as a product.

Proof. Both sides have the same underlying cobordism and bundle; the trivializations at the ends agree after composing with the product cobordisms $[u_i]$, since $u_i = 1$ on the ball. $\square$

5. NECKS, FAMILIES AND INVARIANT REGULARITY

This section collects three statements used repeatedly: gluing along necks with invertible cross-sections, one-parameter families, and regularity within the class of perturbations invariant under the lifting group. They are standard in structure, following [Don02, Sections 3–5] and [DFL21, Section 4]; we state them in the form needed. The points specific to the present setting are the extended weighted spaces at the mixed ends, with the weight below the stationary gap, and invariance under the lifting group; the proofs are written out mainly for these.

5.1. Pieces, ends and hypotheses. A *piece* is a pure gauge cobordism with cylindrical ends, a DFL mixed domain [DFL21, Section 3.1], possibly with several gauge inputs, or a holomorphic polygon domain with strip-like ends on pairwise transverse Lagrangians. Ends are *closed* ($\mathbb{R}_+ \times Y$ with a nondegenerate irreducible critical set), *ordinary* (strip ends at transverse intersections) or *mixed* (DFL mixed ends, whose stationary solutions form a Morse–Bott family, treated in extended weighted spaces). Only closed and ordinary ends are joined. We assume:

- (H1) *Lattice.* For fixed labels every finite-energy configuration on a piece, a joined problem or a family fibre satisfies $\text{ind} = c + 8\kappa$ with $2\kappa \in \mathbb{Z} + 2\kappa_0$.
- (H2) *Component regularity.* Each piece is regular in index at most zero (at most one when chain identities are needed), and every nonconstant trajectory on a joined end has index at least one.
- (H3) *Supports.* The autonomous primary and end perturbations are fixed on the ends and on the corresponding product necks. The additional secondary perturbations, the nonautonomous cutoff changes, and the parameter derivatives have their output and entire loop-dependence regions in a common compact set away from the inserted necks. This compact set is fixed after the finite selection of secondary directions and before neck lengths vary.
- (H4) *Energy.* On solutions, analytic energy equals topological energy plus terms bounded independently of neck lengths and family parameters.
- (H5) *Lifting group.* A finite group Φ of lifting classes vanishing on every matching surface, satisfying the ϕ -non-integral condition of [KM11b, Definition 5.1] on every component of every cut (a surface with $w_2 \neq 0$ on which Φ vanishes), acts freely on the solution spaces; cuts are connected, so the group of the joined problem is the fibre product of the groups of the pieces [KM11b, Proposition 5.4].

5.2. Neck gluing. Function spaces. Fix an integer $l \geq 3$. On a piece, a joined problem or a family fibre, connections and sections are of class $L^2_{l,\text{loc}}$ and gauge transformations of class $L^2_{l+1,\text{loc}}$. On closed and ordinary ends, and in particular on every neck, the norms are unweighted: L^2_l on configurations and tangent vectors, L^2_{l-1} on the target of the equations. On the mixed ends we use the spaces of [DFL21, Definitions 3.8–3.9, (4.5) and (4.8)]. The weight is $e^{\delta\theta}$ in the end coordinate θ of a mixed end and 1 elsewhere; a tangent vector is the cut-off pull-back of a tangent vector to the limiting Lagrangian plus a remainder in $L^2_{l,\delta}$;

the target is $L_{l-1,\delta}^2$. Configurations are compared in the corresponding distance: the L_l^2 distance off the mixed ends, the $L_{l,\delta}^2$ distance of the decaying parts on the mixed ends, and the distance of the limits. The weight $\delta > 0$ is smaller than the constant δ_0 of [DFL21, (4.24)], so that the linearized operators are Fredholm, and smaller than the exponential rate of [DFL21, Proposition 5.29], so that solutions lie in these spaces. Moreover

$$(5.1) \quad \delta < \mu_*,$$

where μ_* is defined as follows. For a mixed end with three-manifold Y and a connection B on Y representing a point of the limiting Lagrangian, let $\lambda_1(B)$ be the lowest eigenvalue of $d_B^* d_B$ on $\Omega^0(Y; \text{ad } E)$ with the Neumann condition at ∂Y . It is positive: an element of the kernel is parallel, and $B|_{\partial Y}$ is flat on a bundle with $w_2 \neq 0$ on every component of ∂Y , so it has no nonzero parallel section. The eigenvalue depends continuously on B and only on its gauge class, and the limiting Lagrangians are compact [DFL21, paragraph after Proposition 2.16]. Hence μ_* , the minimum of $\lambda_1(B)^{1/2}$ over the mixed ends and the points of their limiting Lagrangians, is positive. All three conditions on δ are upper bounds. Theorem 5.1 is stated and proved in these spaces, and regularity means surjectivity of the linearized operator between them.

Theorem 5.1 (neck gluing). *Let pieces $X_1, \dots, X_k$ be joined along a tree of closed or ordinary ends with lengths $R = (R_e)$, under (H1)–(H5), in the function spaces just fixed. There is R_0 such that for $\min_e R_e \geq R_0$:*

- (i) *the joined problem has no solutions of negative index;*
- (ii) *gluing is a Φ -equivariant bijection from the union, over matched intermediate labels, of products of index-zero component spaces onto the index-zero space of the joined problem, whose elements are regular;*
- (iii) *the joined count is the composite of the component counts over $\mathbb{F}_2$;*
- (iv) *the same holds with one component an index-one trajectory on an end of the joined domain, modulo translation, and with one component an index -1 piece of a family together with the parametrized operator.*

The chart at a preglued configuration. Let $p = p_R$ be the preglued configuration of a matched tuple of component solutions. Its connection lives on the four-dimensional part X_R of the joined domain and its section on the two-dimensional part. The boundary of X_R is the union S of the matching surfaces $\mathbb{U}_\partial \times \Sigma$ of the pieces, with collars $[-2, 0]_s \times \mathbb{R}_\theta \times \Sigma$ and outward normal ∂_s ; we write ι_n for the normal component at S . Let B_p be the space of [DFL21, Section 4.1] formed on the joined domain: pairs $w = (\zeta, \nu)$ in the function spaces above that satisfy the linearized matching, class-matching and Lagrangian conditions (iii)–(iv) there, with the norm $|w|$ of [DFL21, (4.5)]. Let $E_p \subset B_p$ be the subspace cut out by $\iota_n \zeta|_S = 0$, and F_p the target [DFL21, (4.8)]. On a piece of DFL type the chart P_p is that of [DFL21, Section 4.1]. On the joined domain it is its analogue: $P_p(w)$ is obtained from $p + w$ by three corrections, all supported in the pieces. On the collar of every component of S , including a component of genus one, where the moduli space of flat connections on that component is a point and the matching condition is the flatness of the slices, the slice $A_{s,\theta} + \zeta_{s,\theta}$ is replaced by

$$A_{s,\theta} + \zeta_{s,\theta} + \rho(s)(E_{\tau(0,\theta)}(A_{s,\theta}, \zeta_{s,\theta}) - A_{s,\theta} - \zeta_{s,\theta}),$$

with the maps E_t of [DFL21, Lemma 4.1 and the paragraph after Lemma 4.4] and DFL's cut-off functions ρ and τ . On the mixed ends, DFL's finite-dimensional term moves the limits onto fixed families B_b, B'_b of connections representing points of the limiting Lagrangians. The

section is $e_\tau(\nu)$, fibrewise over u_p , with the maps e_t of [DFL21, Lemma 4.4]. The families are normalized by $\frac{d}{db}B_b|_{b=0} = b$ and by the vanishing of the normal component of $B_b - B$ on ∂Y , and likewise for B'_b . The chart is used with end gauge 1. Put

$$\mathcal{G}_p(w) := (\mathcal{F}(P_p(w)), d_p^* \zeta),$$

where $\mathcal{F}$ is the perturbed mixed equation, with its Cauchy–Riemann component regarded as a map into $\mathbb{R}^N$ for a fixed embedding of the target symplectic manifold into $\mathbb{R}^N$. The zeros of $\mathcal{G}_p$ on E_p are the points w of the slice [DFL21, (4.7)] for which $P_p(w)$ is a solution.

Remark 5.2 (the radius in [DFL21, Lemma 4.1]). The chart applies E_t to the slices $\zeta_{s,\theta}$, and we use [DFL21, Lemma 4.1(iii)] with its radius measured in L^q for a fixed $q > 2$. Let $\mathcal{B} := L^2_{l-1}(\Sigma; \Lambda^1 \otimes \text{ad } F)$ and $\mathcal{B}^{(q)}_{<\varepsilon} := \{c \in \mathcal{B} : \|c\|_{L^q} < \varepsilon\}$. The statement used is the following: there is $\varepsilon_q > 0$ such that for every connection σ of class L^2_{l-1} on F , every $t \in [-1, 1]$ and every $s \in [0, 1]$, the map $c \mapsto c + s(E_t(\sigma, c) - \sigma - c)$ is a diffeomorphism of $\mathcal{B}^{(q)}_{<\varepsilon_q}$ onto an open neighbourhood of 0 in $\mathcal{B}$, and, for σ of class L^2_k with $k \geq l-1$, restricts to a diffeomorphism at the Sobolev level k . For $s = 1$ this is (iii) of [DFL21, Lemma 4.1] with the L^2 -ball $\mathcal{B}_{<\varepsilon}$ replaced by $\mathcal{B}^{(q)}_{<\varepsilon_q}$; this ball is gauge invariant, so (i) of that lemma keeps its meaning. The proof is DFL’s, made quantitative in L^q as follows. Write $c = c_0 + d_\alpha \xi + *d_\alpha \xi'$ with respect to the flat connection α of [DFL21, Lemma 4.2]. Then L^q elliptic estimates, Kato’s inequality and Morrey’s inequality give $\sup_\Sigma |\xi| + \|d_\alpha \xi\|_{L^q} \leq C\|c\|_{L^q}$, with C independent of α by gauge invariance and the compactness of the moduli space. So on a ball of fixed radius the derivative of the map differs from the identity by at most $\frac{1}{2}$ in the L^q operator norm; the map is injective there, and its derivative is injective at every level $k \geq l-1$ and, being a compact perturbation of an invertible operator, an isomorphism.

We use the statement with an L^q -ball, $q > 2$; the following example shows that an L^2 -ball does not suffice. By (ii) and (iv) of [DFL21, Lemma 4.1], $E(\alpha, d_\alpha \xi)$ depends on ξ only through $\exp \xi$. Let ψ be a cut-off function equal to 1 on a disc, and let $e \neq e'$ be sections of $\text{ad } F$ with eigenvalues $\pm i$ that agree where $\psi < 1$, with e parallel on $\text{supp } \psi$. Then $2\pi\psi e$ and $2\pi\psi e'$ have the same exponential, while $d_\alpha(2\pi\psi e) \neq d_\alpha(2\pi\psi e')$. Choose $e' = \text{Ad}_{\exp \eta} e$ with a nonzero bump η supported where $\psi = 1$ and with arbitrarily small $\|d\eta\|_{L^2}$; then $\|d_\alpha(2\pi\psi e')\|_{L^2} \leq 2\pi\|d\psi\|_{L^2} + C\|d\eta\|_{L^2}$, and a logarithmic cutoff makes the first term small as well. So the two distinct forms lie in any L^2 ball. By the Sobolev embedding, an L^2_{l-1} -ball of fixed radius lies in $\mathcal{B}^{(q)}_{<\varepsilon_q}$.

Lemma 5.3 (the chart at a preglued configuration). *There are $R_0, \kappa_* > 0, r_* > 0$ and C , independent of R with $\min_e R_e \geq R_0$ and of the matched tuple, such that:*

- (a) P_p is defined and injective on $\{w \in B_p : |w| < \kappa_*\}$ and takes values in the mixed configuration space;
- (b) on that ball $\mathcal{G}_p$ is twice continuously differentiable with $\|d^2 \mathcal{G}_p\| \leq C$, and the same holds in a family for $(t, w) \mapsto (\mathcal{F}^{\lambda_*+t}(P_p(w)), d_p^* \zeta)$ with $|t| < \kappa_*$, where λ_* is the base parameter;
- (c) every configuration x with $\|x - p\| < r_*$ whose limits on the mixed ends lie in the families B_b, B'_b is $P_p(w)$ for some w with $|w| < \kappa_*$, and $|w| \leq C\|x - p\|$.

Proof. The necks lie in closed and ordinary ends, so they meet neither S , nor its collar, nor the mixed ends. Hence the collars, the slices of p on them and the mixed ends are those of the finitely many component solutions and do not depend on R , and every unit patch of the

joined domain is isometric to one of finitely many models, on which p is bounded in every C^k norm.

(a) By the trace theorem on unit patches, $\sup_{s,\theta} \|\zeta_{s,\theta}\|_{L^q(\Sigma)} + \sup |\nu| \leq C|w|$. For $|w|$ small the slices therefore lie in the ball of Remark 5.2, and ν lies in a fixed neighbourhood of the zero section on which the maps e_t are fibrewise diffeomorphisms. $P_p(w)$ satisfies the matching, class-matching and Lagrangian conditions by [DFL21, Lemma 4.1(iv), Lemma 4.4(iv) and the paragraph after it]. It determines w : the end parameters through its limits, ν through the injectivity of the e_t , and ζ slice by slice through Remark 5.2.

(b) The difference $P_p(w) - p - w$ is the sum of the collar terms $\mathcal{N}_i(\zeta)$ of the pieces, which depend only on ζ on the collar of the piece i and on its slices; the finite-dimensional end term; and a term on the section given pointwise by the family e_t . The last two are smooth, with bounds uniform over unit patches. Each $\mathcal{N}_i$ is smooth, with $\mathcal{N}_i(0) = 0$ and $d\mathcal{N}_i(0) = 0$, on a ball of the space of forms on the collar that are sums of cut-off pull-backs of profiles in $L^2_l([-2, 0] \times \Sigma)$ at $\theta \rightarrow \pm\infty$ and a remainder in $L^2_{l,\delta}$. Indeed, where the slice has small curvature, DFL’s construction expresses $E_t(\sigma, c)$ through the flat connection α of [DFL21, Lemma 4.2], the Hodge decomposition of c with respect to α , the exponential of the potential of its exact part, and a horizontal lift of the path $r \mapsto e_t([\alpha], rc_0)$ of its harmonic part c_0 . These ingredients depend smoothly on α in every Sobolev norm on Σ , by the inverse function theorem applied at each Sobolev level to the map $(\alpha, \eta) \mapsto \alpha + *d_\alpha \eta$ of [DFL21, Lemma 4.2] and by Hodge theory for the irreducible flat connection α . Along the collar, $\alpha = \alpha_{s,\theta}$ is smooth in (s, θ) , with bounded derivatives that converge exponentially along the mixed ends. So $\mathcal{N}_i$ is assembled from three kinds of terms: slice-wise linear operators with bounded (s, θ) -derivatives, which act boundedly on L^2_l of the collar; entire functions of $\text{ad } \xi$, which are smooth because L^2_l of a four-manifold of bounded geometry is an algebra for $l \geq 3$, also with the weight on one factor; and a finite-rank term, smooth into every Sobolev space on Σ , to which the composition estimates of Moser type apply [Tay11, Chapter 13, Proposition 3.9]; their proof holds verbatim for maps from a finite-dimensional space into a Hilbert space. The equation $\mathcal{F}$ is quadratic in the connection up to the perturbation terms, and acts pointwise on the section, with bounds uniform over unit patches; the fixed autonomous primary and end terms have uniform bounds on unit patches. The additional secondary terms and nonautonomous parameter variations have compact dependence regions as in (H3). Hence the second derivative of $\mathcal{G}_p$ is bounded independently of R , the norms adding over a partition into pieces and necks.

(c) The inverse is constructed slice by slice. The end parameters are read from the limits of x , the families B_b being injective immersions near $b = 0$ by the normalization. The section coordinate ν is e_τ^{-1} of the section of x , pointwise. On each collar, ζ solves $(\text{id} + \mathcal{N}_i)(\zeta) = a$, where a is $x - p$ minus the end term; since $d\mathcal{N}_i(0) = 0$, the inverse function theorem solves this uniquely on a ball of fixed radius. The resulting w lies in B_p . At $s = 0$ the slice of x is $E_t(A_\theta, \zeta_\theta)$ with A_θ flat, and it is flat; since $E_t(A_\theta, \cdot)$ maps the small d_{A_θ} -closed forms onto the flat connections near A_θ and is injective, $d_{A_\theta} \zeta_\theta = 0$. The class matching follows from $[E_t(\alpha, c)] = e_t([\alpha], [c])$ for d_α -closed c and the injectivity of e_t , and the Lagrangian condition from the choice of e_1 as the exponential map of a metric in which the Lagrangian is totally geodesic [DFL21, Lemma 4.4 and its proof]. The constants depend only on the pieces. $\square$

Proof of Theorem 5.1. The proof is the standard neck-stretching argument of [Don02, Chapters 3–5]. The two points that differ for the mixed equation are the chart in which the contraction is run and the gauge slice into which nearby solutions are brought; Section 10.10 uses the same steps. Monotonicity is used only through compactness.

Compactness. By (H4) and (H1) the energy is bounded at fixed index. By the compactness theorems for the pieces [DFL21, Theorems 5.1–5.2], [DFL24, Theorem 3], [Don02, Theorem 5.4] and Gromov compactness, a sequence with $\min_e R_e \rightarrow \infty$ converges to broken configurations on the pieces. The energy lost in the limit is nonnegative, and the lattice turns it into an index loss $4N$ with $N \geq 0$; with (H2) this excludes negative indices and, at index zero, all breaking except at the joined necks.

Uniform inverse. Each neck has an invertible cross-section: on closed necks the extended Hessian has a spectral gap, the critical points being nondegenerate and irreducible, and on ordinary necks the intersections are transverse. Hence the linearized operator $D_p: E_p \rightarrow F_p$ at the preglued configuration p (on pieces of DFL type, the operator of [DFL21, (4.8)] with the gauge-fixing component d_p^*) has a right inverse Q_p with $\|Q_p\| \leq C_Q$ independent of the neck lengths, by the patching argument of [Don02, Proposition 3.9], which takes place on the unweighted necks.

Existence and local uniqueness. The matching, class-matching and Lagrangian conditions are nonlinear, and an affine contraction does not preserve them. We therefore run the contraction for the map $\mathcal{G}_p$ on E_p , in the chart P_p of Lemma 5.3 with end gauge 1 (on a piece of DFL type, the chart of [DFL21, Section 4.1] on the Coulomb slice of [DFL21, Proposition 4.6]). Its Cauchy–Riemann component is a section along the section of $P_p(w)$, not along u_p ; let $\hat{\mathcal{G}}_p$ be obtained by projecting it pointwise onto the tangent spaces along u_p and, in the setting of (iv) with a family, the self-dual component onto the self-dual forms at the base parameter. These projections are injective nearby, so near 0 the two maps have the same zeros. By the normalization $\frac{d}{db}B_b|_{b=0} = b$, the derivative of $\hat{\mathcal{G}}_p$ at 0 differs from D_p by a zeroth-order term bounded by the error of p , so for $\min_e R_e$ large it has a right inverse Q'_p with $\|Q'_p\| \leq 2C_Q$ and $\text{im } Q'_p = \text{im } Q_p$; the normalization of the normal component of $B_b - B$ makes the chart preserve the normal condition at S . The component solutions converge exponentially to nondegenerate limits, so $\|\hat{\mathcal{G}}_p(0)\|_{F_p} \leq C_e e^{-\delta_e \min_e R_e}$ for some $\delta_e > 0$, and by Lemma 5.3 the second derivative of $\hat{\mathcal{G}}_p$ is bounded by some C_2 on a ball whose radius does not depend on the neck lengths. Write $\hat{\mathcal{G}}_p(z) = \hat{\mathcal{G}}_p(0) + D\hat{\mathcal{G}}_p(0)z + \mathcal{R}_p(z)$, so that $\|\mathcal{R}_p(z) - \mathcal{R}_p(z')\| \leq C_2(|z| + |z'|)|z - z'|$. The zeros of $\hat{\mathcal{G}}_p$ in $\text{im } Q_p$ are the fixed points of $z \mapsto -Q'_p(\hat{\mathcal{G}}_p(0) + \mathcal{R}_p(z))$, which is a contraction with constant at most $\frac{1}{2}$ on the ball of $\text{im } Q_p$ of radius $\rho_0 = (8C_Q C_2)^{-1}$ (reduced, if necessary, to the radius of the chart and to a radius on which the projections are injective). For $\min_e R_e$ large it maps the ball of radius $4C_Q C_e e^{-\delta_e \min_e R_e}$ into itself, so the contraction mapping principle gives a zero $z_R \in \text{im } Q_p$ with $|z_R| \leq 4C_Q C_e e^{-\delta_e \min_e R_e}$, the only zero in the ball of radius ρ_0 ; $P_p(z_R)$ is the glued solution, and at index zero $\text{im } Q_p = E_p$. The derivative of $\hat{\mathcal{G}}_p$ at z_R differs from that at 0 by at most $C_2|z_R|$, so it is onto; it is the linearization at $P_p(z_R)$ in the chart followed by the projections, which are isomorphisms there, and the slice is transverse to the gauge orbits, so $P_p(z_R)$ is regular. In the setting of (iv) with a family the contraction is run in $(t, z) \in \mathbb{R} \oplus E_p$ with the parametrized operator $(t, z) \mapsto D_p z + t \partial_\lambda \mathcal{F}$.

Surjectivity. Let y_n be solutions with $\min_e R_{e,n} \rightarrow \infty$ in a fixed class of index zero. By the compactness step, after passing to a subsequence and applying gauge transformations, y_n converges on compact subsets of the pieces to a matched tuple c of index-zero component solutions, with no energy lost and no trajectory breaking off, so its energy on the necks and on the ends outside a fixed compact set tends to zero. Exponential decay for solutions of small energy then controls y_n there, with constants independent of the neck lengths: on closed necks and gauge ends by [Don02, Propositions 4.3–4.4 and Section 5.5]; on ordinary necks and

ends by [Sim14, Proposition 2.3.1], after the Hamiltonian term is removed by its flow, with all derivatives controlled by [RS01, Lemma C.3] and the linear equation for the time derivative; and on the mixed ends by [DFL21, Proposition 5.29]. Patching gauge transformations over bands on which the limits are irreducible gives g_n with $\|g_n^* y_n - p_n\| \rightarrow 0$, where p_n is the preglued configuration of c .

To bring $g_n^* y_n$ into the slice at p_n with control independent of the neck lengths, let $\Delta_p \xi = (d_p^* d_p \xi, \iota_n d_p \xi|_S)$ act on infinitesimal gauge transformations of class L_{l+1}^2 off the mixed ends that are asymptotic on each mixed end to the pull-back of a section of $\text{ad } E$ over Y , with a remainder in $L_{l+1,\delta}^2$. Its asymptotic operators are positive: on closed necks and gauge ends because the critical points are irreducible, and on the mixed ends because $\lambda_1(B) \geq \mu_*^2$. Consequently $\|\xi\|_{L^2} \leq C_P \|d_p \xi\|_{L^2}$ for every $\xi \in L_1^2(X_R)$, with no boundary condition and with C_P independent of the neck lengths. If this failed, a sequence with $\|\xi_n\|_{L^2} = 1$ and $\|d_{p_n} \xi_n\|_{L^2} \rightarrow 0$ would, by the Poincaré inequalities on the cross-sections (on Y with no boundary condition, by the variational characterization of $\lambda_1(B)$), have L^2 norm tending to zero on the necks and ends; on the compact parts of the pieces it would converge to a parallel section vanishing on an open set, hence to zero, every component of a piece having an end. In addition to (5.1), take $\delta C_P \leq \frac{1}{2}$, and let τ be a smoothing of the end coordinate of the mixed ends, extended by zero, with $|d\tau| \leq 1$ and depending only on θ near S . For $U = e^{\delta\tau} \xi$ the weak Neumann form becomes $\langle d_p U - \delta d\tau U, d_p V + \delta d\tau V \rangle_{L^2}$, whose diagonal $\|d_p U\|^2 - \delta^2 \|d\tau U\|^2 \geq \frac{3}{4} \|d_p U\|^2$ is coercive by the Poincaré inequality. Since $\partial_n \tau = 0$, the Lax–Milgram lemma and local Neumann regularity on unit patches invert Δ_p on the weighted spaces and, with the Neumann problem on Y for the asymptotic values, on the spaces with asymptotic values, with bounds independent of the neck lengths. After the mixed-end limits are brought into the families B_b, B'_b by a gauge transformation on Y, Y' (Coulomb gauge with the Neumann condition at B and [DFL21, Proposition 2.10]), the inverse function theorem for the action of the decaying gauge group in the chart, whose linearization $(\xi, w) \mapsto w - d_p \xi$ is inverted by means of Δ_p^{-1} , writes every configuration within a distance r_C of p as $g^* P_p(w)$, with w in the slice [DFL21, (4.7)] and $|w|$ bounded by a fixed multiple of that distance; by Lemma 5.3, r_C and the multiple do not depend on the neck lengths. This is [DFL21, Proposition 4.6] at p with uniform radii.

Hence y_n is gauge equivalent to $P_{p_n}(w_n)$, with w_n a zero of $\widehat{\mathcal{G}}_{p_n}$ in E_{p_n} and $|w_n| \rightarrow 0$. At index zero $|w_n| < \rho_0$ for n large, and by uniqueness $P_{p_n}(w_n)$ is the glued solution. At augmented index zero in a family the same argument applies to (t, w) , with the compactness step of Theorem 5.4 and the slice at the base parameter. With one index-one trajectory, the zeros of $\widehat{\mathcal{G}}_p$ in a ball of radius independent of the neck length form an arc through the glued point, a graph over $\ker D_p$ with values in $\text{im } Q_p$, along which the glued point moves with speed bounded below as the gluing parameter varies; so every nearby solution is glued. The gluing map is continuous, locally injective, and leaves every compact subset of the one-dimensional moduli space as the parameter tends to infinity; its image lies in a noncompact component, an interval, on which such a map is monotone, hence injective.

Injectivity. Distinct component tuples give distinct glued solutions for large neck lengths: if the glued solutions of c and c' were gauge equivalent along a sequence of neck lengths tending to infinity, both would converge in L_l^2 on compact subsets of the pieces, to c_i and c'_i ; the relating gauge transformations would be bounded in L_{l+1}^2 there, a diagonal subsequence would converge, and its limit would relate c_i to c'_i [Don02, Section 4.4]. Equivariance and the fibre-product description of the framed quotient follow from (H5). $\square$

5.3. Families.

Theorem 5.4 (families). *Let F_λ , $\lambda \in [0, 1]$, be a smooth family of data on one domain, constant near $\lambda = 0, 1$, with ends and critical data independent of λ , $\partial_\lambda F$ supported in a compact region, all scales in a compact positive range, and Φ -invariant. Assume both endpoint fibres are regular in index at most zero. After adding a generic small Φ -invariant perturbation from a finite-dimensional space, vanishing near the endpoints, with supports and loop regions in a compact set (Theorem 5.6(iii)):*

- (i) *there are no solutions of fibre index less than -1 ; the index -1 solutions occur at finitely many parameters, and at each the augmented operator $(\xi, t) \mapsto D\xi + t \partial_\lambda F$ is invertible;*
- (ii) *for each such solution and each index-one trajectory at an closed or ordinary end there is exactly one boundary branch of the parametrized index-zero space;*
- (iii) *the compactified parametrized index-zero space is a compact one-manifold whose boundary consists of the two endpoint index-zero spaces and the branches of (ii); breaking at the mixed ends does not occur, since a nonconstant mixed-end trajectory has $\kappa \in \frac{1}{2}\mathbb{Z}_{>0}$ and hence index at least four;*
- (iv) *consequently the two endpoint counts P_0, P_1 satisfy $P_0 + P_1 = dK + Kd$ over $\mathbb{F}_2$, where K counts the index -1 solutions.*

Proof. Compactness in the family. The compactness theorem [DFL24, Theorem 3] is stated for the fixed collar geometry $ds^2 + d\theta^2 + g_\Sigma$ of [DFL24, (1.1)], and only its interior estimate [DFL24, Lemma 4.50] is formulated for a C^k -compact family of metrics. We explain why it holds uniformly over the family. Near the seams, the metrics of the family are of the form $e^{2f_\lambda}(ds^2 + d\theta^2) + c_\lambda^2 g_\Sigma$ with f_λ in a C^k -compact set and scales c_λ in a compact positive range. In every family used in this paper (the path of Lemma 9.12, the stages in the proof of Theorem 11.8, and the families of Lemma 9.7, a class closed under convex combinations) one has $e^{-f_\lambda} c_\lambda \equiv 1$, and we use the statement only in that case: the conformal change below then gives exactly the product metric [DFL24, (1.1)], to which [DFL24, Theorem 4.2 and Lemma 4.25] apply verbatim. For completeness we indicate the argument for general warped families. The four-dimensional conformal change by e^{-2f_λ} turns this into the warped product $ds^2 + d\theta^2 + (e^{-f_\lambda} c_\lambda)^2 g_\Sigma$, whose fibre scale has logarithmic derivatives bounded uniformly in λ . On a unit patch, after rescaling (s, θ) by the fibre scale at its centre, the metric is a C^k -small perturbation of the fixed product $ds^2 + d\theta^2 + g_\Sigma$, with smallness controlled by the patch size. The anti-self-duality equation and the curvature energy are invariant under four-dimensional conformal changes, the Cauchy–Riemann equation and the energy of the section depend only on the conformal structure of the two-dimensional factor, the matching condition does not involve the metric, and the holonomy perturbations vanish on the collars. The mean-value inequality [DFL24, Theorem 4.2] and the seam derivative bounds [DFL24, Lemma 4.25] are proved by elliptic estimates whose constants depend continuously on the metric in C^k , so they hold on these patches with uniform constants, and so does seam regularity. Away from the seams, Uhlenbeck compactness and [DFL24, Lemma 4.50] apply to compact families of metrics, and the perturbations depend smoothly on λ .

Consequently a sequence of solutions U_ν of fibre index at most one and bounded energy at parameters $\lambda_\nu \rightarrow \lambda$ has a subsequence converging, after gauge transformations, in C^∞ on compact subsets of the complement of a finite set Z and modulo breaking at the ends, to a solution at λ , and exactly one of the following holds: $Z = \emptyset$ and no energy is lost, in which case the convergence is C^∞ on compact sets up to breaking; or $Z \neq \emptyset$ and at least one quantum of energy is lost at Z , in which case convergence fails in C^∞ near Z and holds only

in L_1^p on compact subsets of the complement. In the second case (H1) converts the energy loss into an index loss of at least four, so the limit has fibre index at most -3 , which the regularity of the family ((H2) and (i)) excludes. Hence in index at most one only the first alternative occurs. Regularity of the parametrized problem is Theorem 5.6 applied with the parameter. Gluing at an index -1 piece uses the augmented operator [Don02, Section 5.3, (5.6)]. The remaining statements are the standard boundary count. $\square$

5.4. Invariant regularity. We use holonomy perturbations in the sense of [DFL21, Section 6.2, (6.13)–(6.14)]: finite sums of terms $\pi_{\text{su}(2)} \text{Hol}_\gamma(\tilde{A}) \otimes \omega$, where γ is a family of loops, $\tilde{A}$ a local unitary lift and ω a compactly supported form, with every loop having zero value under every element of Φ .

Lemma 5.5. *Let Σ be a closed oriented surface of genus at least one, $U \rightarrow \Sigma$ a rank-two Hermitian bundle of odd degree with fixed determinant connection, and $\tilde{B}$ a unitary connection inducing it whose associated $\text{PU}(2)$ connection is flat. Let $\mathcal{H} \subset \text{U}(2)$ be its based holonomy group. Then $\mathcal{H}$ acts irreducibly on $\mathbb{C}^2$, $\mathcal{H}$ contains $\text{U}(1) \cdot \text{Id}$, and the real span of $\mathcal{H}$ is $M_2(\mathbb{C})$. The same holds for holonomies defined only up to sign.*

Proof. An invariant line would carry half the curvature of the determinant, hence have half-integer degree. Small discs where the central part of the curvature is nonzero give a continuous arc of scalar holonomies, which generates $\text{U}(1)$. By Burnside’s theorem the complex span of $\mathcal{H}$ is $M_2(\mathbb{C})$, and since $i\mathcal{H} \subset \mathcal{H}$ its real span is the same. A sign change does not change a real linear span. $\square$

For a holonomy perturbation term $\pi \text{Hol}_\gamma \otimes \omega$ call $\text{supp } \omega$ its *support* and the union of the images of its loops its *loop region*. On each gauge component choose an open set O containing the supports of all fixed perturbation derivatives; it may extend down original ends. Choose an open set Λ such that every component of O is joined through Λ to an original end whose limiting connection is flat on a collar of an odd surface, and on which the lifting classes of loops vanish. On a joined torus domain take the union of these sets on the two sides; the loops need not cross the inserted torus bridge. The compact set containing the supports and loop regions of the finitely many perturbations eventually chosen is fixed afterwards.

Theorem 5.6 (invariant regularity). *(i) Let Y be a closed connected three-manifold containing a surface Σ with $w_2(E)|_\Sigma \neq 0$, and g an automorphism of the $\text{SO}(3)$ bundle E whose lifting class vanishes on Σ . If g fixes a connection whose restriction to Σ is flat, then $g = 1$. In particular Φ acts freely on critical points and on finite-energy solutions.*

(ii) Let D be the linearized operator at a solution of a gauge or mixed problem, with O and Λ as above. Then D , augmented by the derivatives of the Φ -invariant holonomy perturbations with support in O and loop region in $O \cup \Lambda$, is onto. On purely symplectic components the same holds with the usual almost complex and Hamiltonian variations in place of holonomy perturbations.

(iii) Under the compactness and gluing hypotheses of Theorems 5.1 and 5.4, generic small Φ -invariant perturbations from a finite-dimensional space of such directions, with supports and loop regions in a compact set fixed before the neck lengths are varied, make all spaces of index at most one regular, relative to strata that are already regular.

Proof. (i) A fixed automorphism commutes with the holonomy of the connection on Σ , which is irreducible by Lemma 5.5, the bundle being odd on Σ ; so g is the identity on Σ , and hence everywhere by unique continuation of parallel sections along paths, Y being connected. The

hypotheses cannot be dropped, as the example of T^3 with holonomy the Klein four-group shows.

(ii) Write the full gauge-fixed derivative at a solution as

$$D(a, v) = (d_A^* a, (d_A^+ + K_A) a, D_u v),$$

where K_A includes all primary and secondary holonomy derivatives. Let (μ, ξ, z) annihilate the range of this operator and of all perturbation directions as in (ii). We show $\mu = 0$, then $\xi = 0$, then $z = 0$. A Neumann gauge variation $(d_A f, 0)$ is an admissible source; its surface harmonic class is zero. Gauge covariance at a solution gives $(d_A^+ + K_A) d_A f = 0$, so $\langle \mu, d_A^* d_A f \rangle = 0$. The scalar Neumann operator is onto at each such solution. Indeed its end operators have positive scalar gaps; on a completed torus end use the odd-fibre gap. The compact-remainder argument therefore makes it Fredholm at weight zero. Integration by parts identifies its kernel with decaying parallel sections, which vanish by irreducibility. Self-adjointness gives index zero, and sufficiently small weights preserve that index and introduce no kernel. Hence $\mu = 0$.

At the limiting odd reference surface, Lemma 5.5 provides finitely many holonomies spanning the adjoint fibre over $\mathbb{R}$. Move their loops slightly into the unperturbed flat collar and then to a sufficiently far, finite slice of the original end. Convergence to the limiting connection preserves this span. Conjugate the loops by a common path to any ball in O . The path and its reverse cancel in every lifting class, so these loops still have zero lifting class. Thicken them smoothly to holonomy families. Their self-dual output forms can be chosen arbitrarily in the ball; the real coefficients expressing a target vector in the spanning holonomies are computed at this solution and fixed as part of the perturbation. Pairing with all such directions therefore gives $\xi|_O = 0$.

All fixed nonlocal derivatives have output in O . Consequently, for every test variation a , $\langle K_A^* \xi, a \rangle = \langle \xi, K_A a \rangle = 0$. Thus the interior adjoint equation is local: $d_A^{+,*} \xi = 0$. Applying d_A^+ gives an elliptic second-order equation with scalar principal symbol. Unique continuation [Aro57] implies $\xi = 0$ on each connected gauge component.

On an open matching arc choose an arbitrary section trace, represent its components by harmonic forms on the matching surfaces, and extend them independently into the gauge collars with zero normal component. These are admissible coupled variations. Since both gauge cokernel components vanish, their Green pairing forces the section adjoint to have zero boundary trace on that arc. Mixed adjoint regularity supplies this trace. Extension by zero across a smaller arc, followed by unique continuation for the real Cauchy–Riemann equation, gives $z = 0$. Purely symplectic components are regularized separately by their usual J and Hamiltonian variations [WW15b, Theorem 3.9] and use no holonomy perturbations. The pure gauge pieces have framed reference ends with the same unperturbed odd collar. For the closed gauge factors in the excision theorem we use the regularity results of [KM11b, Section 5.3].

(iii) At a fixed solution finitely many directions suffice because the cokernel is finite dimensional. At each index, remove neighbourhoods of the already regular broken strata and of fixed regular parameter faces. Compactness of the remaining locus gives a finite collection of directions by a finite cover. Parameter cutoffs implement relative choices. Sard–Smale then gives a generic small regular choice. Only now take a compact set containing the output supports and the entire loop regions of these directions, and place the original-end truncations farther out. This set is independent of subsequently inserted lengths. Each direction is invariant, so no averaging or loss of span occurs. $\square$

6. THE COLLAPSE THEOREM: GEOMETRY, INDEX AND COMPACTNESS

This section and the next two prove Theorem B. The present section fixes the geometry, states the theorem, proves the index-energy law at every scale and proves compactness with the loss formula. Section 7 contains the uniform linear estimates, and Section 8 the nonlinear gluing and the surjectivity of the gluing map.

6.1. Geometric hypotheses. Let $(C, g_C, \mathbb{E})$ satisfy Definition 3.2, and write $n = \dim L(C)$ and $P = L(C)$; Σ denotes the union of the boundary components matched to the section and $\mathbf{M}$ their moduli space.

The two instances used in this paper are the torus compression body $C = C_1$ with $\partial C = \Sigma_h \sqcup T^2$, where the torus has a one-point moduli space, $L(C) \subset \mathbf{M}_h$ and $n = 3h - 3$; and the compression body $C_{h,g}$ from Σ_h to Σ_g used in the folded geometry of Section 10, with $L(C_{h,g}) \subset \mathbf{M}_h^- \times \mathbf{M}_g$ and $n = 3(h + g - 2)$. Conditions (C1)–(C3) hold for both: irreducibility, forced by the odd bundle on a boundary component, gives $H^0 = 0$; since $H^0(\partial C; \text{ad}) = 0$, duality gives $H^2(C; \text{ad}) \cong H^1(C, \partial C; \text{ad})^*$, which vanishes exactly when $H^1(C; \text{ad}) \rightarrow H^1(\partial C; \text{ad})$ is injective. Injectivity holds because the inclusion of the large boundary surface $\partial_+ C$ is π_1 -surjective: first cohomology with local coefficients is group cohomology of π_1 , and inflation $H^1(\pi_1 C; \text{ad}) \rightarrow H^1(\pi_1 \partial_+ C; \text{ad})$ is injective for a surjective homomorphism. The same surjectivity makes the image an embedded Lagrangian [DFL21, Proposition 2.21].

For $v \in T_q L(C)$ let h_v be the harmonic $\text{ad } B_q$ -valued one-form on C with vanishing normal component and boundary class v ; it is unique by (C3).

Lemma 6.1 (distance to the Lagrangian). *There is c_C such that every smooth connection B on $\mathbb{E}$ with flat boundary restriction satisfies $\text{dist}([B|_{\partial C}], L(C)) \leq c_C \|F_B\|_{L^2(C)}$.*

Proof. Since the target is compact, it suffices to treat small curvature. If $\|F_{B_i}\|_{L^2} \rightarrow 0$, weak Uhlenbeck compactness on the compact three-manifold with boundary [Weh04, Theorem A] gives gauges in which $B_i \rightarrow B_\infty$ in $W^{1,2}$ and $B_i \rightarrow B_\infty$ in L^4 , with B_∞ flat; the relative Coulomb slice, with vanishing normal component, and the normal Hodge estimate $\|a\|_{W^{1,2}} \leq C(\|d_{B_\infty} a\| + \|d_{B_\infty}^* a\| + \|a\|)$ upgrade this to strong $W^{1,2}$ convergence. In a flat chart about B_q , fix the gauge so that $a = B - B_q$ is in Coulomb gauge with vanishing normal component and L^2 -orthogonal to the harmonic fields, adjusting q ; this is possible by the implicit function theorem because the gauge and flat directions are transverse. Then $\|a\|_{W^{1,2}} \leq C\|d_{B_q} a\| \leq C(\|F_B\| + \|a\|_{L^4}^2)$, so $\|a\|_{W^{1,2}} \leq C\|F_B\|$, and the trace inequality, read in the $H^{1/2}$ surface Coulomb chart of the flat connections (proof of Lemma 6.15), bounds the distance of the boundary class from $q \in L(C)$. $\square$

 6.2. The collapsing family X_ε .

Definition 6.2. Let X_0 be a domain for the mixed equation of [DFL21]: a four-manifold with a bundle, a surface S_0 with boundary arcs carrying Lagrangians (the surface on which the holomorphic part u of a configuration, called the *section* as in [DFL21], is defined), matching seams, and ends of three kinds: gauge-theoretic cylindrical ends, strip ends, and mixed ends of Morse–Bott type. Assume that S_0 has a boundary arc I carrying $L(C)$ between two boundary punctures $z_\pm$, and that at $z_\pm$ the domain has the standard mixed end of C : in polar coordinates $z = z_+ + xe^{i\theta}$ at z_+ and $z = z_- - xe^{-i\theta}$ at z_- , $0 < x < x_0$, $0 \leq \theta \leq \pi$, with $x_0 < |I|/2$ so that the two ends do not overlap, the gauge piece $\{0 < x < x_0\} \times C$ is matched to the section along the ray $\theta = 0$, and the ray $\theta = \pi$ lies in I . The ends of X_0 other than the two mixed ends at $z_\pm$ are its *original ends*. For $\varepsilon \in (0, x_0/4)$ the domain X_ε is obtained as follows.

- (i) The section surface of X_ε is S_0 with I and $z_\pm$ restored: near I it is the upper half-plane chart with coordinate z , and I becomes part of a single matching seam $(-2, 2) \subset \mathbb{R}$. Its complex structure j_ε is the push-forward under F_ε of the conformal structure of the collar, extended scale-covariantly so that it is self-similar under $z \mapsto (z - z_\pm)/\varepsilon$; it agrees with that of S_0 outside two half-disks of radius $c_{\text{core}}\varepsilon$ about $z_\pm$, for a constant $c_{\text{core}} \geq 2$ fixed with the insertion.
- (ii) The gauge piece over the seam is $(-2, 2) \times C$ with the metric $g_\varepsilon = dt^2 + \lambda_\varepsilon(t)^2 g_C$, where $\lambda_\varepsilon = \varepsilon \ell(x/\varepsilon)$ near the endpoints (x the distance to the nearer endpoint, positive outside I), $\lambda_\varepsilon = \varepsilon$ on I , and $\lambda_\varepsilon = 1$ near $t = \pm 2$; here $\ell: \mathbb{R} \rightarrow [1, \infty)$ is smooth with $\ell(s) = 1$ for $s \leq 1$ and $\ell(s) = s$ for $s \geq 2$. For $x \geq 2\varepsilon$, where $\lambda_\varepsilon = x$, this piece is the exterior gauge piece of X_0 at $z_\pm$.
- (iii) The matching seam is $t \mapsto t$ along $(-2, 2)$, the boundary of the section collar embedded by $F_\varepsilon(t, \theta) = t + \lambda_\varepsilon(t) [\sigma b_\varepsilon(t)(\cos \theta - 1) + i((1 - b_\varepsilon(t))\theta + b_\varepsilon(t) \sin \theta)]$ for $0 \leq \theta < \theta_0$, a fixed small angle, with $\sigma = \pm 1$ at the right and left endpoints and b_ε a cutoff equal to 0 for $x \leq \varepsilon$ and 1 for $x \geq 2\varepsilon$. On I it is $t + i\varepsilon\theta$; for $x \geq 2\varepsilon$ it is the polar collar $z_\pm \pm x e^{\pm i\theta}$. For θ in this range F_ε is an embedding.
- (iv) Outside a fixed neighbourhood N_I of I and its endpoints, X_ε coincides with X_0 ; N_I is taken to contain the patches $\mathcal{H}_{r_0}(b)$ of Lemma 6.15, for the radius r_0 fixed there, about every seam point b of I and of the endpoint necks $\{x \leq x_0\}$.

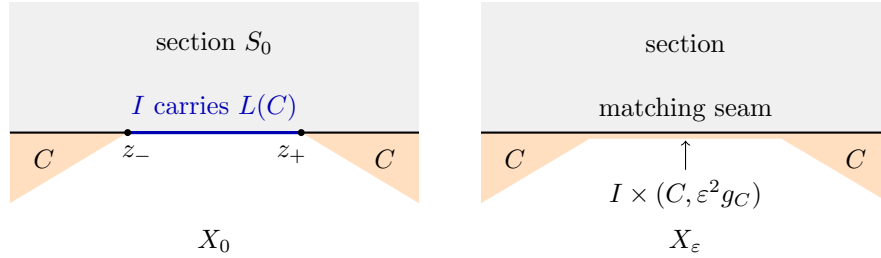


FIGURE 2. The collapsing family of Definition 6.2 near the arc I . The shaded region below the seam is the gauge piece, drawn with thickness equal to the scale of the metric on C : it is proportional to the distance to $z_\pm$ outside I , and equal to ε over I on X_ε . On X_0 (left) the arc I carries the Lagrangian boundary condition $L(C)$, and after the conformal change of Lemma 6.3 the two points $z_\pm$ become mixed ends. On X_ε (right) the whole segment is a matching seam, and the thin part is the compression rectangle.

- Lemma 6.3.** (i) On the collar, $\lambda_\varepsilon^{-2} g_\varepsilon = ds^2 + g_C$ with $ds = dt/\lambda_\varepsilon$, and the section metric is conformal to $ds^2 + d\theta^2$. The matching conditions on ∂C therefore have the product form of [DFL21, Section 3.1] at every scale, with the fixed surface metric.
- (ii) Near each endpoint, for $c_{\text{core}}\varepsilon < x < x_0$, $s = -\log x$ and polar coordinates on S_0 , the domain is conformal to a cylinder of length $\log(x_0/2\varepsilon)$ with slice

$$(C, g_C) \sqcup [0, \pi] \sqcup (-C, \mathfrak{s}^2 g_C), \quad \mathfrak{s} = \varepsilon e^s = \varepsilon/x,$$

in which the first copy is the exterior gauge piece at the point $z_\pm \pm x$, matched at $\theta = 0$, and the last is the rectangle at the point $z_\pm \mp x \in I$, matched at $\theta = \pi$. The two copies carry opposite orientations, so that the two seam terms of Lemma 6.8 cancel: with the time chosen so that (time, θ) is positively oriented, namely $\rho = L - s$ at z_+ and

s at z_- , where θ runs clockwise, the copy at $\theta = \pi$ is the reversed one. At z_- the tangential operator is conjugate to $-A_{\mathfrak{s}}$, and every estimate below is invariant under this sign change.

- (iii) Along I , at distance $r \geq 2\varepsilon$ from the nearer endpoint, the conformal change by r^{-2} gives the compression the metric $dT^2 + (\varepsilon/r)^2 g_C$ with $dT = dt/r$; so $\mathfrak{s} = \varepsilon/r$ along I .

The anti-self-duality equation, the Cauchy–Riemann equation and their energies are conformally invariant.

Proof. (i) and (iii) are direct computations. For (ii), $dx^2 + x^2 g_C = x^2(ds^2 + g_C)$, and the collar embedding is the polar map. Conformal invariance holds in dimensions four and two respectively. $\square$

The gauge-fixing (Coulomb) condition is not conformally invariant. In the conformal coordinates of Lemma 6.3 it acquires a zeroth-order term $2\varsigma \Pi_f$ on the collapsing component, where $\varsigma = \partial_s \log \mathfrak{s}$ and Π_f is the projection onto the scalar component.

Definition 6.4. A *configuration* on X_ε ($\varepsilon \geq 0$) with given asymptotic labels is a pair (A, u) of a connection on the bundle and a section map, of class $L^2_{k,\text{loc}}$ for a fixed large k , such that: the restriction of A to each matching surface is flat and represents the value of u on the seam; u takes values in the given Lagrangians on the Lagrangian arcs (in $L(C)$ on I when $\varepsilon = 0$); and at every end (A, u) converges exponentially, at the fixed weights below the gaps, to the labelled critical data, with free limits in the critical manifolds at mixed ends. Configurations are considered up to the determinant-one gauge group enlarged by the lifting group, and carry their relative homotopy class and bundle marking.

6.3. Data and hypotheses.

Hypothesis 6.5. The data on X_ε , $\varepsilon \geq 0$, satisfy:

- (X1) *Structures near the collapse.* On N_I the primary perturbation of C vanishes, the section almost complex structure is the complex structure $*_\Sigma$ of Convention 3.1 with no inhomogeneous term, and all metrics are products near the seams.
- (X2) *Perturbations.* The end perturbations are the fixed autonomous perturbations of the original gauge ends, as in Proposition 3.4, vanishing near the critical set and switched on by monotone cutoffs along the end coordinates; their outputs (the supports of their forms), their dependence regions (the unions of their loops) and their cutoff bands lie in the original ends, outside N_I . All secondary perturbations, including the nonautonomous holonomy terms χ_{gg} of the regularizations, have outputs and dependence regions in a compact set disjoint from N_I , with $\sup \|\eta\|_{L^2} \leq b_\eta$ and $\sum \|g\|_{C^0} \text{TV}(\chi_g) \leq b_\eta$.
- (X3) *Ends.* The original ends are nondegenerate gauge ends, transverse strip ends or mixed ends with fixed weights below their gaps. Every nonconstant autonomous trajectory on an original end that can occur has index at least one, where at mixed ends the index is taken with fixed incoming and free outgoing limit.
- (X4) *Regularity at scale zero.* For every choice of asymptotic labels, there are no solutions on X_0 of index less than zero, and the index-zero solutions form a finite set of regular points, the index being computed in the extended weighted spaces at the two new ends.
- (X5) *Index-energy law at scale zero.* For fixed labels, $\text{ind } D_V - 8\kappa(V)$ is the same constant c for all configurations V on X_0 , and $2\kappa(V) - 2\kappa_0 \in \mathbb{Z}$ for a fixed κ_0 .

The data outside N_I , including the neck lengths of any ordinary necks, are chosen before ε .

Hypothesis (X5) is verified for the domains used in this paper in Lemma 6.7.

6.4. Statement.

Theorem 6.6 (collapse theorem). *Let $(C, g_C, \mathbb{E})$ satisfy Definition 3.2 and $\{X_\varepsilon\}_{\varepsilon \geq 0}$ the domains of Definition 6.2, with data satisfying Hypothesis 6.5. Fix asymptotic labels.*

- (i) *Index-energy law. Every configuration U with these labels on X_ε , for any $\varepsilon > 0$, has $\text{ind } D_U = c + 8\kappa(U)$ with $2\kappa(U) - 2\kappa_0 \in \mathbb{Z}$, with the constants of (X5). Every solution of index d has analytic energy at most $\pi^2(d - c) + B$ in the normalization of Section 3.6, with $B = B(b_\eta, \sum \|f\|_{C^0})$ independent of ε .*
- (ii) *Loss formula. For every sequence $\varepsilon_\nu \rightarrow 0$ and solutions U_ν on X_{ε_ν} of index d , a subsequence converges in the sense of Proposition 6.16 to a solution V on X_0 and broken trajectories Γ_j on the original ends, with*

$$d - \text{ind}(V) - \sum_j \text{ind}(\Gamma_j) = 4n, \quad n \in \mathbb{Z}_{\geq 0},$$

the indices of Γ_j taken as in (X3). If $d \leq 3$, $\text{ind}(V) \geq 0$ and every $\text{ind}(\Gamma_j) \geq 1$, then $n = 0$ and no energy is lost.

- (iii) *Counts. There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there are no solutions on X_ε of negative index, and gluing (the construction of Section 8) defines a bijection*

$$\mathcal{G}_\varepsilon: \mathcal{M}^0(X_0) \longrightarrow \mathcal{M}^0(X_\varepsilon)$$

onto a set of regular points, preserving asymptotic labels and bundle markings.

Lemma 6.7. *Hypothesis (X5) holds when X_0 is obtained from DFL domains of compression pairs and a polygon domain with pairwise transverse Lagrangians by joining along ordinary strip ends, as in Sections 9 and 11, in the target $\mathbf{M}_h$ or in the product target $\mathbf{M}_h^- \times \mathbf{M}_g$ with labels among products of handlebody Lagrangians and the correspondence L_C .*

Proof. For a DFL domain, $\text{ind} = 8\kappa + \mathfrak{c}(x) - \mathfrak{c}(a)$ [DFL21, Proposition 3.20], and two configurations with the same labels have $2(\kappa - \kappa') \in \mathbb{Z}$ [DFL21, Lemma 3.15]. For a polygon domain with simply connected Lagrangians with $\pi_2 = 0$ in the simply connected $\mathbf{M}_h$, relative classes with the same corners form a torsor over $\pi_2(\mathbf{M}_h) = \mathbb{Z}$, on which the positive generator changes κ by $\frac{1}{2}$ in the normalization of Section 3.6, and the index by $2c_1 = 4$; this gives the law with its own constant. In the product target $\mathbf{M}_h^- \times \mathbf{M}_g$ of Section 11, with Lagrangians among products of simply connected Lagrangians with $\pi_2 = 0$ and the correspondence L_C , relative classes with the same corners form a torsor over $\pi_2(\mathbf{M}_h^- \times \mathbf{M}_g) = \mathbb{Z}^2$ modulo the image of $\pi_2(L_C) = \mathbb{Z}$, which is generated by spheres of zero area and zero Chern number and maps to $\mathbb{Z}(1, 1)$. The quotient is $\mathbb{Z}$, and its generator changes κ by $\frac{1}{2}$ and the index by 4, so the law holds with its own constant. Along an ordinary strip end with transverse intersection the cross-section is invertible, so the index is additive [Don02, Proposition 3.9] and the topological energy is additive by locality. The constants add. $\square$

6.5. Fredholm theory and excision at a fixed scale.

Lemma 6.8 (Green identity). *Let B be a flat connection on C , and let $x = (p, b)$, $y = (\pi, c)$ be pairs of an $\text{ad } B$ -valued function and one-form on C with $b(n) = c(n) = 0$ on ∂C . Put $\mathcal{B}(p, b) = (d_B^* b, - * d_B b + d_B p)$. Then*

$$\langle \mathcal{B}x, y \rangle_{L^2(C)} - \langle x, \mathcal{B}y \rangle_{L^2(C)} = - \int_{\partial C} \langle b \wedge c \rangle.$$

If the traces $b|_{\partial C}$, $c|_{\partial C}$ are closed, the right side equals $-\omega(\tau b, \tau c)$, where τ denotes the harmonic part of the trace and ω is as in Convention 3.1. For a section variation ν on a strip $[0, 1]_r$ with J -compatible metric, $\langle -J\partial_r \nu, \mu \rangle - \langle \nu, -J\partial_r \mu \rangle = -[\omega(\nu, \mu)]_{r=0}^{r=1}$.

Proof. The scalar terms cancel because the normal components vanish. For the one-form terms, $\langle - * db, c \rangle - \langle b, - * dc \rangle = - \int_C (db \wedge c - b \wedge dc) = - \int_{\partial C} b \wedge c$ by Stokes. Exact parts of closed traces pair to zero with closed forms, and on a torus boundary component every closed trace is exact. The strip identity is integration by parts in r . $\square$

With the matching condition and Convention 3.1, the seam contributions of Lemma 6.8 from the compression and from the section cancel. This makes the mixed tangential operators symmetric, and it is the mechanism behind every uniform estimate below.

At a mixed end the stationary solutions form a Morse–Bott family, and we use *extended weighted spaces*, as in [DFL21, Definitions 3.8–3.9]: the exponentially weighted Sobolev space of decaying fields plus the span of the stationary fields $c_e(v)$, v in the tangent space of the critical family, cut off to the end. The coefficient v is the *asymptotic value* of the field. This is the extended L^2 space of Atiyah, Patodi and Singer [APS75] in the weighted form.

Proposition 6.9. *For every $\varepsilon > 0$ and every configuration U on X_ε , the linearized mixed operator D_U , with its gauge-fixing component and on the extended weighted spaces at the mixed ends, is Fredholm [DFL21, Proposition 3.18]. The same holds on X_0 , where the two new ends are mixed ends with asymptotic values in $T_{q\pm} L(C)$. The index is locally constant along continuous paths of configurations with fixed asymptotic data.*

Proof. At fixed $\varepsilon > 0$ the region N_I is compact with a smooth metric, and by Lemma 6.3(i) its seam is a product mixed seam after a conformal change that does not affect the Fredholm property. The local estimates on the compact part are [DFL21, Theorem 4.17 and Remark 4.19]; the end parametrices are those of [DFL21, Section 4.2.1]. On X_0 the new ends are mixed ends of the same type. Paths of configurations are identified with paths of operators on fixed spaces by local matching charts, and the index is continuous. $\square$

Lemma 6.10 (excision). *Let D_1, D_2, D_3, D_4 be mixed operators, Fredholm on weighted spaces, on domains $X_1 = P \cup F$, $X_2 = P' \cup F'$, $X_3 = P \cup F'$, $X_4 = P' \cup F$, glued along a common collar $K = [-1, 1] \times \Gamma$, where Γ is a slice of the form $(\pm C) \sqcup [0, \pi] \sqcup (\pm C)$ with the boundary conditions of Definition 6.4. Suppose the operators agree on the pieces P , P' , F , F' as indicated and on K . Then $\text{ind } D_1 + \text{ind } D_2 = \text{ind } D_3 + \text{ind } D_4$.*

Proof. Let ϕ, ψ be functions of the collar coordinate with $\phi^2 + \psi^2 = 1$, $\phi = 1$ on the P and P' sides and $\psi = 1$ on the F and F' sides. The map $R(x_1, x_2) = (\phi x_1 + \psi x_2, -\psi x_1 + \phi x_2)$, where x_1 is read on X_1 and x_2 on X_2 and the products are transplanted to X_3 and X_4 through the common collar, is an isomorphism between the domains of $D_1 \oplus D_2$ and $D_3 \oplus D_4$: multiplication by a function of the collar coordinate alone preserves the vanishing of the normal component, the closedness of the traces and the harmonic matching, because these conditions are linear and pointwise in the collar coordinate. Then $(D_3 \oplus D_4)R - R(D_1 \oplus D_2)$ is multiplication by $(\partial\phi, \partial\psi)$ on the compact collar, which is compact from the first-order domain to L^2 by the Rellich lemma on the compact domain K , whose corners, where the seams meet the cut, are those of a product of an interval with a manifold with boundary. Hence the two Fredholm operators have the same index (compare the addition property of [Don02, Section 3.3]). $\square$

6.6. The index–energy law at positive scale.

Lemma 6.11 (flattening). *Let U be a configuration on X_ε , $\varepsilon > 0$, and $q_* \in L(C)$. There is a continuous path of configurations, with fixed ends and fixed data outside a compact neighbourhood K_2 of the rectangle, from U to a configuration U^b equal to the constant flat pair (B_{q_*}, q_*) on a smaller neighbourhood K_0 of the rectangle. The topological energy is constant along the path, and $\kappa(U) = \kappa_0(V)$, where V is the configuration on X_0 obtained from U^b by removing the rectangle and completing the two new ends by (B_{q_*}, q_*) .*

Proof. The section. In a star-shaped half-disk chart about the rectangle, contract the section to its value $u(0)$ near the rectangle by $u_s = u \circ ((1 - s\chi)z)$, with χ a cutoff; then move $u(0)$ to q_* along a path c in the connected moduli space, $u_{1+s} = c(s\rho)$, with ρ a smaller cutoff. The seam carries no Lagrangian condition at positive scale, so the path may leave $L(C)$.

Flat surface traces. Lift the seam values to actual flat connections on the matching surfaces by path lifting in the principal bundle of flat connections over the moduli space, with structure group the determinant-one gauge group modulo its centre, keeping the lift constant where the base path is constant. At the end the trace over a central time interval differs from B_{q_*} by a path h in the gauge group. The correction needed is a path of surface gauge transformations over the interval, equal to h in the middle and to the identity at both ends; it exists because the determinant-one gauge group of the surface is connected. No loop of gauge transformations arises, so the correction introduces no lifting class. Do the same on the other boundary components.

Connections. Let β_s be the resulting boundary connections. Choose an auxiliary connection equal to B_{q_*} near the rectangle and to A outside K_2 , interpolate affinely, and add a collar extension of the difference between β_s and the trace of the interpolant. The result has the prescribed flat traces at every parameter and equals the flat product near the rectangle at the end.

Energy. The variation of the topological energy along a compactly supported path is a boundary integral; on the matching surfaces the gauge term cancels the section term by the flat matching condition and Convention 3.1, exact parts integrate to zero, and on a torus face the variations are exact because $H^1(T^2; \text{ad}) = 0$. So the energy is constant. The removed region is flat with constant section, and the added tails have zero energy density, so $\kappa(U^b) = \kappa_0(V)$. $\square$

Let $F_\varepsilon(q)$ denote the linearized operator of the *flat cap at scale ε* : the section is a half-disk bounded by the seam over $(-2, 2)$, the gauge piece is $(-2, 2) \times C$ with the metric g_ε , the two lateral interfaces are completed by the cylinder with slice $(-C) \sqcup [0, \pi] \sqcup C$, and the configuration is the constant flat pair (B_q, q) . Let $F_0(q)$ be the zero-scale operator: the section is $S_0 = \overline{\mathbb{H}} \setminus \{\pm 1\}$ with boundary arcs $\ell_- = (-\infty, -1)$, $I = (-1, 1)$, $\ell_+ = (1, \infty)$, the gauge pieces are $Z_\pm = \ell_\pm \times C$ matched along $\ell_\pm$, the condition $L(C)$ is imposed on I , and there are three ends: the mixed ends $E_{\pm 1}$ with slice $C \sqcup [0, \pi]$ and the end E_∞ with slice $C \sqcup [0, \pi] \sqcup C$. All operators act on extended weighted spaces, with one asymptotic value at each end, represented by the cut-off stationary fields $c_e(v) = (0, h_v, v)$ at $E_{\pm 1}$ and (h_v, v, h_v) at E_∞ , for $v \in T_q L(C)$.

Proposition 6.12. *For every $q \in L(C)$, $\ker F_\varepsilon(q) \cong \ker F_0(q) \cong T_q L(C)$ and $\text{coker } F_\varepsilon(q) = \text{coker } F_0(q) = 0$. In particular $\text{ind } F_\varepsilon(q) = \text{ind } F_0(q) = n$.*

Proof. We give the proof for $F_0(q)$. The positive-scale case is the same with one end and no Lagrangian arc; there the kernel lies in $T_q L(C)$ because the constant section value must be the boundary class of a closed form on all of C , namely the harmonic field h_v . Use the weight δ below all end gaps and below the Coulomb gap of the Neumann Laplacian on C .

Energy identity. For $(\zeta, \nu), (\zeta', \nu')$ in the domain,

$$\int_{Z_+ \sqcup Z_-} \text{tr}(d\zeta \wedge d\zeta') + \int_{S_0} \omega(d\nu \wedge d\nu') = 0.$$

Indeed both integrands are exact because B_q is flat. On $\ell_\pm \times T^2$ the boundary terms vanish since closed traces are exact; on $\ell_\pm \times \Sigma$ the gauge term is $\pm \int \omega(\nu, \partial_t \nu') dt$ and cancels the seam term of the section by Convention 3.1; on I the term vanishes because $T_q L(C)$ is Lagrangian; at the ends the cross terms between a stationary field and a decaying remainder form the Green pairing of the self-adjoint end operator and vanish, and the remaining terms tend to zero.

Kernel. For (ζ, ν) in the kernel the identity gives $\|d\zeta\|^2 + \|d\nu\|^2 = 0$, so $\nu \equiv v$ with $v \in T_q L(C)$ by the condition on I . On $Z_+ = \mathbb{R} \times C$ write $\zeta = b_t + \varphi_t dt$; $d\zeta = 0$ gives $d_C b_t = 0$ and $\partial_t b_t = d_C \varphi_t$, so the class of b_t is constant with boundary value v , and $b_t = h_v + d_C \psi_t$ with $\partial_t \psi_t = \varphi_t$. Hence $\zeta = \pi^* h_v + d\psi$. The pullback $\pi^* h_v$ is closed, coclosed and has vanishing normal component for any metric conformal to $dt^2 + g_C$ by a factor depending on t , so the gauge-fixing equation gives $\Delta \psi = 0$ with Neumann condition and decay, and $\psi = 0$ by the maximum principle. So the kernel is $\{(\pi^* h_v, v)\} \cong T_q L(C)$.

Adjoint domain. Let $w = (\xi, \mu, v)$ in the dual weighted space annihilate the range. Testing with $(d\chi, 0)$, χ with Neumann condition, and using that the Neumann Laplacian is onto on the weighted spaces, gives $\xi = 0$. Interior tests give $d\mu = 0$ with μ self-dual and $dv = 0$ for the section $(0, 1)$ -form. Near $\ell \times \partial C$ write $\mu = dt \wedge m + *_C m$. Boundary tests give $m(n) = 0$, closed restrictions of m to Σ and to T^2 , and, by Convention 3.1, $[m|_\Sigma] = v(\partial_t)$ on the seams; on I , $v(\tau) \in T_q L(C)$, since a Lagrangian subspace is its own symplectic complement. These are the conditions of the original domain under $\mu \mapsto m$.

Decay. On each end w solves the translation-invariant adjoint equation, so $w = w_0 + O(e^{-\gamma \rho})$ with w_0 in the kernel of the end operator. Pair with $F_0(q)(\beta_e(\cdot - R)c_e(v)) = \beta'_e(\cdot - R)c_e(v)$, where β_e is a cutoff; letting $R \rightarrow \infty$ removes the contribution of the decaying part and gives $\langle c_e(v), w_0 \rangle = 0$ for all v , so $w_0 = 0$ and w decays at a rate exceeding δ .

Vanishing. The section surface S_0 is contractible and v is closed and decaying, so $v = d\nu'$ with ν' converging at each puncture; normalize $\nu'(1) = 0$; on I , $\nu' \in T_q L(C)$. On Z_+ put $\zeta'_+(t) = \pi^* h_{\nu'(1)} + \int_{-\infty}^t m dt'$; then $d\zeta'_+ = \mu$, $\zeta'_+(n) = 0$, its traces are closed, and by the seam condition its boundary class is $\nu'(t)$. At the other end ζ'_+ converges to $h_{\nu'(1)} + M$ with $M = \int m$ closed, coclosed and normal-free, hence harmonic with boundary class $\nu'(\infty) - \nu'(1)$; so $\nu'(\infty) \in T_q L(C)$ and $\zeta'_+ \rightarrow h_{\nu'(\infty)}$. The same on Z_- . Thus (ζ', ν') lies in the domain, with asymptotic values at the three ends, and $0 = \langle F_0(q)(\zeta', \nu'), w \rangle = \|\mu\|^2 + \|v\|^2$.

The primitive $\zeta'_\pm$ need not decay, since M may be nonzero; it is asymptotically constant, and this is exactly what the extended spaces allow. In the positive-scale case normalize instead $\nu'(\infty) = 0$ and put $\zeta' = \int_{-\infty}^t m dt'$: both seam ends tend to the same end, so $M = \int m$ is harmonic with zero boundary class and vanishes by (C3). The index is therefore $\dim \ker = n$. $\square$

Proof of Theorem 6.6(i). Let U be a configuration on X_ε . By Lemma 6.11 and Proposition 6.9, $\text{ind } D_U = \text{ind } D_{U^b}$ and $\kappa(U) = \kappa_0(V)$. The four operators D_{U^b} , $F_0(q_*)$, D_V and $F_\varepsilon(q_*)$ agree on a fixed collar around the cut with slice $(-C) \sqcup [0, \pi] \sqcup C$: on the U^b and F_ε side it is the rectangle at positive scale near its lateral interfaces, and on the V and F_0 side the corresponding part of the new mixed ends, and all four configurations are the flat pair there. Lemma 6.10 gives $\text{ind } D_{U^b} + \text{ind } F_0(q_*) = \text{ind } D_V + \text{ind } F_\varepsilon(q_*)$. By Proposition 6.12 the cap indices are equal, so $\text{ind } D_U = \text{ind } D_V = c + 8\kappa_0(V) = c + 8\kappa(U)$ by (X5).

For a solution, the analytic energy equals $8\pi^2\kappa$ plus $2\|\eta\|^2 \leq 2b_\eta^2$ plus the terms $2\int \chi' f$ of the end perturbations, bounded by twice the sum of their C^0 norms, plus the analogous terms of the nonautonomous holonomy perturbations, bounded by $2b_\eta$; the first two terms are [DFL21, (3.24)], and the end terms are estimated as in the proof of Theorem 3.8. The bound follows. $\square$

6.7. Regularity along the collapsing seam. We will use the following quantitative consequence of fixed-geometry mixed regularity. The energy is measured on a larger patch than the one on which derivatives are estimated.

Lemma 6.13 (Quantitative local seam estimate). *Consider the unperturbed mixed equation in the product geometry of [DFL24, (1.1)], with fixed surface metric and its induced complex structure. Let Q_2 be a gauge half-disk times the matching surface, together with the section half-disk, and let Q_1 be the concentric patch of half the radius. There are $e_* > 0$ and C_k such that every solution of energy $e \leq e_*$ on Q_2 satisfies*

$$\sup_{Q_1} (|\nabla_A^k F_A| + |\nabla^k du|) \leq C_k \sqrt{e}.$$

The constants are uniform on compact families of these product data.

Proof. Choose nested intermediate patches. Mixed small-energy regularity and compactness give gauges in which solutions of energy below a fixed concentration threshold have bounded smooth coefficients on these patches [DFL24, Theorem 4.2, Lemma 4.25 and pp. 38–39]. As the energy tends to zero their limits are flat connections and constant sections. The possible flat surface classes form a compact space, so finitely many gauge and target charts suffice. These are qualitative coefficient bounds; the energy rate is obtained next.

Let t be the coordinate tangent to the seam and set $\zeta = \iota_{\partial_t} F_A$, $\xi = \partial_t u$. Covariant differentiation of the unperturbed mixed equation in this product coordinate shows that (ζ, ξ) satisfies its linearized equation. The gauge row is zero by the Yang–Mills identity $d_A^* F_A = 0$. Its normal component on the seam is zero, since anti-self-duality identifies F_{ts} with the curvature of the flat surface restriction. Differentiating that restriction and its class gives the closed-trace and harmonic-matching conditions. Thus (ζ, ξ) lies in the local mixed boundary domain, including the gauge row, as in the translation-field argument of [DFL21, Proposition 5.29]. Use matched scalar cutoffs on the gauge and section patches, with identical seam values and supports away from the artificial edges. Their commutators are zeroth order. The local all-order boundary estimates of [DFL24, Theorems 5 and 5.9, Remarks 5.73 and 5.76], with the uniformity of [DFL21, Remark 4.19], give

$$\|\zeta\|_{H^{j+1}(Q_1)} + \|\xi\|_{H^{j+1}(Q_1)} \leq C_j (\|\zeta\|_{L^2(Q')} + \|\xi\|_{L^2(Q')}) \leq C'_j \sqrt{e},$$

where $Q_1 \Subset Q' \Subset Q_2$. One can equivalently trivialize the moving matching domain by its smooth chart Q and commute $Q\partial_t Q^{-1}$, rather than raw tangential differentiation, before recovering normal derivatives from the equation. Use successively smaller patches. The qualitative smooth bounds control every coefficient and its commutators at each fixed order. They also give uniform constants over the finite chart cover.

For an anti-self-dual two-form on this product, contraction with ∂_t determines all its components and $|F_A|^2 = 2|\zeta|^2$. The Cauchy–Riemann equation similarly determines du from ξ . These identities and Sobolev embedding prove the claimed bounds. The threshold is chosen once below concentration; only C_k depends on the derivative order. $\square$

Away from the seam we use the corresponding interior estimates [DFL24, Lemma 4.50].

The following lemma is the source of all uniform local control near the collapse. For a seam point b (on I , or on an endpoint neck) let $\mathfrak{s}_b$ be the local compression scale ($\mathfrak{s}_b = \varepsilon$ on I , and the value of Lemma 6.3(ii) on the necks), and let $\mathcal{H}_r(b)$ be the half-disk of radius r in the section about b together with the gauge region over $[b - r, b + r]$. Let $E(b, r)$ be the energy of a solution on $\mathcal{H}_r(b)$, and $r_0 > 0$ fixed.

In the following lemma $\Sigma \subset \partial C$ is the union of the boundary components matched to the section and $\mathbf{M}$ their moduli space, the remaining components carrying the flatness condition, as in Section 7.1. The topological energy $\mathcal{Q}$ of a configuration is that of Section 3.6: the Chern–Weil integral of the gauge part plus the symplectic area of the section, normalized so that it equals the analytic energy of a solution of the unperturbed equations.

Lemma 6.14 (quantization of capped configurations). *There is $\hbar > 0$, depending only on $(C, g_C, \mathbb{E})$, with the following property. Let $x \in P$, and let (A, u) be a continuous, piecewise smooth configuration on $(\mathbb{R} \times C) \cup \overline{\mathbb{H}}$: A is a connection on $\mathbb{R} \times \mathbb{E}$ whose restriction to every slice $\{t\} \times \partial C$ is flat, and $u: \overline{\mathbb{H}} \rightarrow \mathbf{M}$ satisfies $u(t) = [A|_{\{t\} \times \Sigma}]$ for $t \in \mathbb{R} = \partial \mathbb{H}$. Suppose that (A, u) equals the constant pair (B_x, x) outside a compact set. Then $\mathcal{Q}(A, u) \in \hbar \mathbb{Z}$.*

Proof. (1) *Invariance.* $\mathcal{Q}$ is unchanged by gauge transformations of $\mathbb{R} \times \mathbb{E}$, whose action on the integrands is pointwise, and by homotopies through configurations of the same kind that are constant outside a fixed compact set. For the latter, Bianchi’s identity and the closedness of ω reduce the variation of $\mathcal{Q}$ to boundary integrals over $\mathbb{R} \times \partial C$ and $\partial \mathbb{H}$, which cancel by the matching condition and Convention 3.1 exactly as in the energy paragraph of the proof of Lemma 6.11; on the torus faces the variations are exact. The slice contributions at the interfaces of the pieces cancel by continuity.

(2) *Removing the section.* Choose T_0 such that $(A, u) = (B_x, x)$ for $|t| \geq T_0$ and outside the half-disk of radius T_0 . For $s \in [0, 1]$ put $u_s(z) = u(z + isT_0)$ and $q_s(t) = u_s(t)$ for $t \in \mathbb{R}$; then $u_1 \equiv x$ and $q_s(\pm T_0) = x$. By (C1) the flat connections on ∂C form a principal bundle over $\mathbf{M}$ whose structure group is the determinant-one gauge group of ∂C modulo its centre, as in the proof of Lemma 6.11. The map $(s, t) \mapsto q_s(t)$ on $[0, 1] \times [-T_0, T_0]$ has the prescribed lift $\alpha(t) = A|_{\{t\} \times \partial C}$ on $\{0\} \times [-T_0, T_0]$ and the constant lift $B_x|_{\partial C}$ on $[0, 1] \times \{\pm T_0\}$; the union of these three edges is a deformation retract of the square, so the lift extends to flat connections $\alpha_s(t)$ with $[\alpha_s(t)]|_{\Sigma} = q_s(t)$. With a bounded extension operator $\mathcal{E}$ from tangential one-forms on ∂C to one-forms on C , supported in a collar, the connections $A_s = B(t) + \mathcal{E}(\alpha_s(t) - \alpha(t)) + \Psi dt$, where $A = B(t) + \Psi dt$, together with u_s , form a homotopy as in (1). Hence $\mathcal{Q}(A, u) = \mathcal{Q}(A_1, x)$, and the latter is the Chern–Weil term alone.

(3) *Reduction to a gauge transformation.* The restrictions of A_1 to the slices of $\mathbb{R} \times \partial C$ are flat of class x , so $\alpha_1(t) = h(t)^*(B_x|_{\partial C})$ for a path h in the determinant-one gauge group of ∂C with $h(t) = 1$ for $t \leq -T_0$; for $t \geq T_0$ the path is constant and central, the stabilizer of an irreducible flat connection being the centre. Restriction to ∂C is a fibration of gauge groups, by the homotopy extension property of $(C, \partial C)$, so h lifts to a path $\hat{h}$ of determinant-one gauge transformations of C with $\hat{h}(t) = 1$ for $t \leq -T_0$ and $\hat{h}$ constant for $t \geq T_0$. The gauge transformation $(t, y) \mapsto \hat{h}(t)(y)^{-1}$ carries A_1 to a connection A_2 whose restriction to every slice of $\mathbb{R} \times \partial C$ is $B_x|_{\partial C}$, equal to B_x for $t \leq -T_0$ and to g^*B_x for $t \geq T_0$, where $g = \hat{h}(T_0)^{-1}$ is central on ∂C . By (1), $\mathcal{Q}(A, u) = \mathcal{Q}(A_2, x)$.

(4) *The period.* Let $\mathcal{G}_\partial$ be the group of determinant-one gauge transformations of C that are central on ∂C . For $g \in \mathcal{G}_\partial$ let $\Xi(g) = \mathcal{Q}(A', x)$ for any connection A' as in (3), with slice restrictions $B_x|_{\partial C}$, equal to B_x for $t \ll 0$ and to g^*B_x for $t \gg 0$. This does not depend on A' : two choices are joined by the affine homotopy, which keeps the slice restrictions fixed,

and (1) applies. Concatenation and gauge invariance give $\Xi(g_1 g_2) = \Xi(g_1) + \Xi(g_2)$, and Ξ is constant on path components: a path $\lambda \mapsto g_\lambda$ in $\mathcal{G}_\partial$ gives the connection $g_{\lambda(t)}^* B_x$, which is flat on every slice, so its Chern–Weil integrand $F \wedge F$ vanishes, and whose slice restrictions are $B_x|_{\partial C}$. Thus Ξ is a homomorphism from $\pi_0(\mathcal{G}_\partial)$ to $\mathbb{R}$. Let $\mathcal{G}_1 \subset \mathcal{G}_\partial$ be the normal subgroup of transformations equal to 1 on ∂C . Its index $N = [\mathcal{G}_\partial : \mathcal{G}_1]$ is finite, since $\mathcal{G}_\partial/\mathcal{G}_1$ embeds in the locally constant central-valued maps on ∂C , and its group of components is $H^3(C, \partial C; \mathbb{Z}) \cong \mathbb{Z}$ by obstruction theory, the fibre S^3 of the group bundle being 2-connected. For $g \in \mathcal{G}_1$ we may, after a homotopy in $\mathcal{G}_\partial$, take $g = 1$ near ∂C , and then $\Xi(g)$ is the Chern–Simons difference of $g^* B_x$ and B_x , whose boundary term vanishes, so $\Xi(g) = c_\Xi \deg g$ with c_Ξ independent of x . For $g \in \mathcal{G}_\partial$ the power g^N lies in $\mathcal{G}_1$, so $N \Xi(g) = \Xi(g^N) \in c_\Xi \mathbb{Z}$. All values of Ξ , for every $x \in P$ and on every component of P , therefore lie in $(c_\Xi/N)\mathbb{Z}$. Take $\hbar = |c_\Xi|/N$ if $c_\Xi \neq 0$, and $\hbar = 1$ if $c_\Xi = 0$, in which case Ξ vanishes identically. Thus $\hbar$ depends only on C . $\square$

Lemma 6.15 (seam regularity). *There are $e_0, \alpha, C > 0$, independent of ε and of the seam point, such that for every solution $U = (A, u)$ on X_ε and every seam point b with $\text{dist}(b, z_\pm) \geq c_{\text{core}}\varepsilon$ and $E(b, r_0) \leq e_0$, with c_{core} as in Definition 6.2(i):*

- (i) $E(b, r) \leq C E(b, r_0) (r/r_0)^\alpha$ for $\mathfrak{s}_b \leq r \leq r_0$;
- (ii) $|du(z)| \leq C \sqrt{E(b, r_0)} (y + \mathfrak{s}_b)^{\alpha/2-1}$ at distance $y \leq r_0/2$ from the seam, and $d(u(z), u(z')) \leq C \sqrt{E(b, r_0)} |z - z'|^{\alpha/2}$ on $\mathcal{H}_{r_0/2}(b)$;
- (iii) $\text{dist}(u(b), L(C)) \leq C \sqrt{E(b, r_0)} \mathfrak{s}_b^{\alpha/2}$; in the rescaled coordinates on the patch of size $\mathfrak{s}_b$ about b the curvature, the section derivative and all their covariant derivatives are bounded by $C_k \sqrt{E(b, r_0)} \mathfrak{s}_b^{\alpha/2}$; and in gauges on the compression slices over $[b - \mathfrak{s}_b, b + \mathfrak{s}_b]$, $\|B(t) - B_{q(t)}\|_{C^k(C)} \leq C_k \sqrt{E(b, r_0)} \mathfrak{s}_b^{\alpha/2}$ with $q(t) \in L(C)$.

Proof. Rescaled geometry. In the coordinate $w \mapsto (w - b)/\mathfrak{s}_b$ (on I) or Φ_b with $\Phi'_b = 1/\mathfrak{s}$ (on the necks), the collapsing compression and the section have exactly the product mixed geometry of [DFL24, (1.1)] with the fixed surface metric, and unit patches correspond to regions of size $C\mathfrak{s}_b$. Energy is conformally invariant. So Lemma 6.13 in that fixed product geometry and the interior estimate [DFL24, Lemma 4.50], which allows a C^k -compact family of metrics, apply on every unit patch with constants independent of ε and b : on a patch of energy at most e_0 the curvature, the section derivative and all their covariant derivatives are bounded by $C\sqrt{e}$ in the rescaled coordinates.

Decay above the local scale. Fix r with $\mathfrak{s}_b \leq r \leq r_0$ and write $E(b, r) = G(b, r) + S(b, r)$, with G the gauge part and S the section part; the gauge region of $\mathcal{H}_r(b)$ is $[b - r, b + r] \times C$, and its section region is a half-disk D_r with diameter $[b - r, b + r]$ on the seam.

The endpoint slices. The gauge energy density on the compression is $\mathfrak{s}^{-1} \|F_B\|_{L^2(C)}^2$ per unit time, so the connections $B_\pm$ on the slices $t = b \pm r$ satisfy $\|F_{B_\pm}\|^2 \leq C\lambda_\varepsilon(b \pm r) \partial_r G(b, r) \leq Cr \partial_r G(b, r)$, where λ_ε is the compression scale. When this is small, the proof of Lemma 6.1 gives gauge transformations $g_\pm$ and $q_\pm \in P$ with $B_\pm = g_\pm^*(B_{q_\pm} + \beta_\pm)$, where $\beta_\pm$ lies in the Neumann slice orthogonal to the harmonic fields and $\|\beta_\pm\|_{H^1(C)} \leq C \|F_{B_\pm}\|$; the representatives B_q, q near x , are taken from a smooth local section of flat connections through B_x . The image γ of the semicircle $\partial D_r \setminus [b - r, b + r]$ satisfies $\text{len}(\gamma)^2 \leq Cr \partial_r S(b, r)$ by Cauchy–Schwarz. Its endpoints $u(b \pm r) = [B_\pm|_\Sigma]$ lie within $C \|\beta_\pm\|_{H^1}$ of $q_\pm$, so with $x = q_-$,

$$d(q_+, x) \leq C\delta_b, \quad \delta_b = \text{len}(\gamma) + \|\beta_-\|_{H^1} + \|\beta_+\|_{H^1}.$$

Paths with flat boundary. Let B be a connection on C with flat boundary restriction α , and $\|B - B_x\|_{H^1(C)} = \delta$ small; write $\alpha_x = B_x|_{\partial C}$. Every component of ∂C carries only

irreducible flat connections, so the surface Coulomb slice, in which the gauge parameter lies in $H^{3/2}(\partial C)$ (an algebra acting on $H^{1/2}$), and the harmonic chart of the flat connections give $\alpha = \exp(\zeta)^*(\alpha_x + h_\nu + \sigma(\nu))$, where h_ν is the harmonic representative of $\nu \in H^1(\Sigma; \text{ad } \alpha_x)$, σ the quadratic correction of the chart, and $\|\zeta\|_{H^{3/2}} + |\nu| \leq C\delta$. Put $\alpha^\lambda = \exp(\lambda\zeta)^*(\alpha_x + h_{\lambda\nu} + \sigma(\lambda\nu))$ and

$$B^\lambda = B_x + \lambda(B - B_x) + \mathcal{E}(\alpha^\lambda - \alpha_x - \lambda(\alpha - \alpha_x)), \quad 0 \leq \lambda \leq 1,$$

with $\mathcal{E}: H^{1/2}(\partial C) \rightarrow H^1(C)$ a bounded tangential extension. Then $B^0 = B_x$, $B^1 = B$, every B^λ has the flat boundary restriction α^λ , $\sup_\lambda (\|B^\lambda - B_x\|_{H^1} + \|\partial_\lambda B^\lambda\|_{H^1}) \leq C\delta$, and the boundary classes of B^λ form a path of length at most $C\delta$. By the Sobolev embedding $H^1 \subset L^4$ on C , $\|F_{B^\lambda}\|_{L^2} \leq C\delta$, so the path, regarded as a connection on $[0, 1] \times C$ with zero temporal component, has Chern–Weil integral at most $C \int_0^1 \|\partial_\lambda B^\lambda\| \|F_{B^\lambda}\| d\lambda \leq C\delta^2$ in absolute value. A temporal component Ψ does not change this integral, since $\int_C \langle d_B \Psi \wedge F_B \rangle = \int_{\partial C} \langle \Psi, F_B|_{\partial C} \rangle = 0$ by Bianchi’s identity and the flatness of the boundary restriction; so Ψ can be chosen to match the temporal component of the solution at the junctions. We use this with $B = B_{q_-} + \beta_-$, for which $\delta = \|\beta_-\|_{H^1}$, and with $B = B_{q_+} + \beta_+$, for which $\delta \leq C\delta_b$, the second path traversed backwards.

The cap. Composing with a fixed representative of the component class in $H^1(C; \mathbb{Z}/2)$, which acts on flat connections of C and preserves $L(C)$, we may take $g_\pm$ of determinant one. The determinant-one gauge group of C is connected, because $H^3(C; \mathbb{Z}) = 0$ for $\partial C \neq \emptyset$ and the fibre S^3 is 2-connected; choose a path $g(t)$, $t \in [b-r, b+r]$, from g_- to g_+ . The gauge transformation $(t, y) \mapsto g(t)(y)^{-1}$ changes neither the topological energy of the solution on $[b-r, b+r] \times C$ nor its boundary classes, keeps all its slice restrictions flat, the torus faces included, and makes its end slices equal to $B_{q_\pm} + \beta_\pm$. Concatenate in t : B_x on $(-\infty, b-r-1]$, the path from B_x to $B_{q_-} + \beta_-$ on $[b-r-1, b-r]$, the transformed solution on $[b-r, b+r]$, the path from $B_{q_+} + \beta_+$ to B_x on $[b+r, b+r+1]$, and B_x on $[b+r+1, \infty)$. On the section keep u on D_r ; on the half-annulus $\{r \leq |z-b| \leq r+1, \text{Im } z \geq 0\}$, whose boundary loop consists of γ , of the boundary class paths of the two flat-boundary paths on the seam intervals $[b-r-1, b-r]$ and $[b+r, b+r+1]$, and of the outer semicircle mapped to x , use the geodesic cone from x over this loop; and put $u = x$ elsewhere. The loop has length at most $C\delta_b$ and lies in the ball of radius $C\delta_b$ about x , so the cone has area at most $C\delta_b^2$. The result is a configuration as in Lemma 6.14. Its topological energy is $E(b, r) + Q_{\text{cap}}$, since the solution satisfies the unperturbed equations near the seam by Hypothesis 6.5(X1)–(X2), and

$$|Q_{\text{cap}}| \leq C\delta_b^2 \leq C(\text{len}(\gamma)^2 + \|\beta_-\|_{H^1}^2 + \|\beta_+\|_{H^1}^2) \leq Cr\partial_r E(b, r).$$

By Lemma 6.14, $E(b, r) + Q_{\text{cap}} \in \hbar\mathbb{Z}$.

Conclusion. Take $e_0 < \hbar/2$. There is $c_2 > 0$ such that $r\partial_r E(b, r) \leq c_2$ ensures the smallness required above and $|Q_{\text{cap}}| < \hbar/2$; then $E(b, r) + Q_{\text{cap}}$ lies in $\hbar\mathbb{Z} \cap (-\hbar, \hbar) = \{0\}$, and $E(b, r) \leq |Q_{\text{cap}}| \leq Cr\partial_r E(b, r)$. If $r\partial_r E(b, r) > c_2$, then $E(b, r) \leq e_0 \leq (e_0/c_2)r\partial_r E(b, r)$. Hence $E \leq C_0 r\partial_r E$ for $\mathfrak{s}_b \leq r \leq r_0$, and integration gives (i) with $\alpha = 1/C_0$.

Derivative and Hölder bounds. At distance $y \leq c\mathfrak{s}_b$ from the seam apply the rescaled estimates on patches of radius $C\mathfrak{s}_b$, whose energy is $O(E\mathfrak{s}_b^\alpha)$ by (i); for $y \geq \mathfrak{s}_b$ apply the interior estimate for holomorphic curves on disks of radius $y/2$, whose energy is $O(Ey^\alpha)$. Since $\alpha > 0$ the bound is integrable in y , which gives the Hölder bound. For (iii), part (i) gives energy at most $CE(b, r_0)\mathfrak{s}_b^\alpha$ on the patch of size $R\mathfrak{s}_b$ about b , for a fixed R ; the rescaled small-energy estimates on that patch give the bounds with the factor $\mathfrak{s}_b^{\alpha/2}$, and the

spatial Neumann slice about the nearest flat connection, with Lemma 6.1, gives the gauge statement. $\square$

6.8. Compactness and the loss formula.

Proposition 6.16. *Let $\varepsilon_\nu \rightarrow 0$ and $U_\nu = (A_\nu, u_\nu)$ be solutions on X_{ε_ν} with bounded energy. After passing to a subsequence and gauge transformations there are a finite set $Z \subset X_0$ and a solution V on X_0 such that:*

- (i) $U_\nu \rightarrow V$ in C^∞ on compact subsets of $X_0 \setminus (Z \cup I \cup \{z_\pm\})$; here X_{ε_ν} agrees with X_0 on every compact subset of the complement of $I \cup \{z_\pm\}$ for large ν ;
- (ii) on compact subsets of $\overline{S_0}$ away from Z and $z_\pm$, including points of I , the sections converge in C^0 and are uniformly Hölder, and the limit takes values in $L(C)$ on I ;
- (iii) V is smooth up to I , the points of Z are removable, and V extends to a finite-energy solution on X_0 whose tails at the new ends converge exponentially to points $q_\pm \in L(C)$;
- (iv) the original ends break into finitely many trajectories Γ_j .

Proof. (i) On the complement of N_I and on the part of N_I at distance at least $c_{\text{core}}\varepsilon_\nu$ from I and its endpoints the geometry is fixed, the secondary perturbations have compact support away from N_I , and the end perturbations are autonomous beyond a fixed cutoff band on the original ends, so this is the compactness theorem for the mixed equation [DFL21, Theorem 5.2], [DFL24, Theorem 3 and Theorem 4.44], together with Uhlenbeck and Gromov compactness, with a finite concentration set Z outside of which energy densities converge. Holonomy perturbation terms converge because the loop families avoid the finite concentration set for almost every output point.

(ii) Energy concentration along I occurs at finitely many points, since each concentration point carries at least a fixed quantum: at points of I the threshold e_0 of Lemma 6.15, elsewhere the thresholds of the fixed-geometry theory. Away from them, $E(b, r_0) \leq e_0$ for ν large and r_0 small, and Lemma 6.15 gives uniform Hölder bounds for u_ν up to the seam and $\text{dist}(u_\nu(b), L(C)) \leq C\varepsilon_\nu^{\alpha/2} \rightarrow 0$ on I .

(iii) By (ii), near a point of I away from Z the limit is a continuous J -holomorphic map, J being the complex structure $*_\Sigma$ there by (X1), with boundary values in the embedded Lagrangian $L(C)$ and finite energy. It is of class $W^{1,2}$ and continuous up to I . In coordinates in which $L(C) = \mathbb{R}^n$ and $J|_{L(C)} = J_0$, reflection across $L(C)$ and the small oscillation of $J(u)$ give, by the Calderón–Zygmund inequality, $W^{1,p}$ regularity for every $p < \infty$, and boundary regularity for pseudo-holomorphic curves with Lagrangian boundary conditions [MS12, Theorem B.4.1] gives smoothness. Removal of singularities at interior, seam and boundary points of Z follows from [DFL24, Theorem 4.44], Uhlenbeck’s theorem and [MS12, Theorem 4.1.2]. At the new ends the energy of V is finite; after the conformal rescaling of the end to the normalization of [DFL21], in which g_C is scaled together with the strip width, and passing to temporal gauge, [DFL21, Proposition 5.29] gives exponential decay.

(iv) The chains on the original ends are extracted as in [DFL21, Section 5.4]. $\square$

Proof of Theorem 6.6(ii). The analytic energy of U_ν is $8\pi^2\kappa(U_\nu)$ plus terms that converge along the subsequence. Energy is lower semicontinuous under the convergence of Proposition 6.16, including at the points of Z and along the necks of the original ends, where the action is constant on each connected critical family. Hence the topological energy lost is nonnegative. Glue V and the Γ_j into a configuration with the labels of U_ν ; at a mixed end the incoming limit of Γ_j equals the outgoing limit of the previous piece, and with the convention of (X3) (fixed incoming, free outgoing limit) the index is additive. The topological

energy is additive by locality. By part (i) applied on X_{ε_ν} and (X5) on X_0 , both with the constant c , the lost topological energy lies in $\frac{1}{2}\mathbb{Z}_{\geq 0}$ and the lost index is eight times it. This is $d - \text{ind}(V) - \sum_j \text{ind}(\Gamma_j) = 4n$ with $n \geq 0$. If $d \leq 3$, $\text{ind}(V) \geq 0$ and $\text{ind}(\Gamma_j) \geq 1$, then $4n \leq 3$ forces $n = 0$; energy is then preserved and there is no concentration. $\square$

At index zero, Hypothesis 6.5(X3)–(X4) and the loss formula give $\text{ind}(V) = 0$, no broken trajectories, $n = 0$, and energy exhaustion: $Z = \emptyset$, and every compact part of the rescaled endpoint regions and of the rescaled neighbourhood of I carries energy tending to zero.

Corollary 6.17 (convergence on the middle compression). *In this situation fix $r_0 > 0$ so small that the energy of V on every half-disk $\mathcal{H}_{r_0}(b)$ with $b \in I$ is less than $e_0/2$. Then for ν large, uniformly for $b \in I$ at distance at least r_0 from the endpoints and on rescaled unit patches of size ε_ν about b : the curvature and all its derivatives tend to zero; in the spatial Neumann slice the compression connection tends in C^N to the flat lift $B_{q_V(t)}$ of the boundary values q_V of V on I ; and the rescaled time derivatives of the harmonic coefficient v (the component along the harmonic fields h_v , $v \in T_q L(C)$) tend to zero. The same smallness holds on the necks where $x/\varepsilon_\nu \rightarrow \infty$ and on the rescaled cores at $z_\pm$.*

Proof. By energy exhaustion $E(b, r_0) \leq e_0$ for ν large, uniformly in b . By Lemma 6.15(iii) the rescaled quantities are bounded by $C_N \sqrt{e_0} \varepsilon_\nu^{\alpha/2} \rightarrow 0$, and $q(t)$ is within $C \varepsilon_\nu^{\alpha/2}$ of the section value, which converges uniformly to q_V by Proposition 6.16(ii). Near the endpoints, on the necks where $x/\varepsilon_\nu \rightarrow \infty$, Lemma 8.5 gives smallness $C \sqrt{e_0} (\varepsilon_\nu/x)^{\alpha/2} \rightarrow 0$, and on the rescaled cores the convergence to zero comes from the convergence of the rescaled transitions to the flat pairs at $q_\pm$ (Section 8.3). $\square$

7. THE COLLAPSE THEOREM: UNIFORM LINEAR ESTIMATES

This section proves the linear estimates on which the gluing and surjectivity arguments of Section 8 rest. The basic estimates are on a model: a family of compressions at varying small scale coupled to a holomorphic strip along the whole of one boundary line (Sections 7.1–7.4). The endpoint transitions are treated in Section 7.5, and the patched inverse on the actual domain in Section 7.7.

The mechanism is the following. On the collapsing compression the operator splits into a fast part, controlled by the Hodge theory of the compression body at scale $\mathfrak{s}$, and a slow part along the harmonic fields, which is coupled to the strip through the matching condition. The uniform Gårding estimate (Proposition 7.4) controls both with no loss in $\mathfrak{s}$: the seam terms of the gauge and section parts cancel by Convention 3.1, and the slow coefficient is controlled by the boundary trace (Lemma 7.2). The only loss is the factor $A = 1 + \log(1/\varepsilon)$, which comes from the stationary coefficients on the two endpoint necks of length $\log(1/\varepsilon)$ (Proposition 7.10). In the gluing argument of Section 8 it always multiplies a positive power of ε , so it does no harm.

7.1. The model collapsing strip. Let $(C, g_C, \mathbb{E})$ satisfy Definition 3.2, with a boundary component $\Sigma \subset \partial C$ carrying the section, and let $P = L(C)$ (for simplicity of notation we suppress the other boundary components, whose moduli spaces are points or enter as fixed factors). Fix a compact Lagrangian K in the moduli space $\mathbf{M}$ of Σ . Fix:

- a path $q: \mathbb{R} \rightarrow P$ with $Q = \sup |\dot{q}| < \infty$, flat connections $B(t)$ representing $q(t)$, and $\Psi(t)$ with $\partial_t B - d_B \Psi = h_{\dot{q}}$;
- a map $u: S = \mathbb{R} \times [0, 1] \rightarrow \mathbf{M}$ with $u(t, 0) = q(t)$, $u(t, 1) \in K$, bounded in C^1 , and a compatible $J(t, r)$;

- a scale $\mathfrak{s} : \mathbb{R} \rightarrow (0, \bar{\mathfrak{s}}]$ with $|\mathfrak{s}'/\mathfrak{s}| \leq H$; write $\varsigma = \mathfrak{s}'/\mathfrak{s}$.

For $U = (p, b)$, a pair of $\text{ad } B(t)$ -valued function and one-form on C depending on t , put $D_t = \partial_t + [\Psi, \cdot]$, $\mathcal{B}_t(p, b) = (d_B^* b, - * d_B b + d_B p)$ and $\Pi_f(p, b) = (p, 0)$, and define

$$D_{\mathfrak{s}}(U, \nu) = (D_t U - \mathfrak{s}^{-1} \mathcal{B}_t U + 2\varsigma \Pi_f U, D_u \nu),$$

where D_u is the linearized Cauchy–Riemann operator. The domain $\mathcal{D}$ consists of (U, ν) with $b(n) = 0$ on ∂C , closed traces on every component of ∂C , harmonic part of $b|_{\Sigma}$ equal to $\nu(t, 0)$, and $\nu(t, 1) \in T_{u(t, 1)} K$. Norms:

$$\begin{aligned} \|(U, \nu)\|_X^2 &= \int (\mathfrak{s} \|U\|^2 + \mathfrak{s} \|D_t U\|^2 + \mathfrak{s}^{-1} \|\mathcal{B}_t U\|^2) dt + \|\nu\|_{H^1(S)}^2, \\ \|(f, g)\|_Y^2 &= \int \mathfrak{s} \|f\|^2 dt + \|g\|_{L^2(S)}^2, \end{aligned}$$

all norms on C in the fixed metric g_C . This operator is the linearization of the mixed equation at a configuration that is flat on the compression, in the conformal coordinates of Lemma 6.3; the weight $\mathfrak{s}$ is the Jacobian of the conformal change.

7.2. Hodge theory on a compression body. Let $\mathbb{H}_t = \{h_v : v \in T_{q(t)} P\}$ be the space of absolute harmonic fields and Π_t the $L^2(C)$ -projection onto $\{(0, h) : h \in \mathbb{H}_t\}$. Constants depend only on $(C, g_C, \mathbb{E})$ and on P , which is compact.

Lemma 7.1. *For $U = (p, b)$ in the static domain:*

- (i) $\|\mathcal{B}_t U\|^2 = \|d^* b\|^2 + \|db\|^2 + \|dp\|^2$;
- (ii) $\|U - \Pi_t U\|_{H^1(C)} \leq c \|\mathcal{B}_t U\|$;
- (iii) $\ker \mathcal{B}_t = \{(0, h_v) : v \in T_q P\}$.

Proof. (i) $\langle *db, dp \rangle = \int_C dp \wedge db = \int_{\partial C} p d(b|_{\partial C}) = 0$ since the traces are closed and B is flat. (ii) is the Gaffney–Friedrichs inequality for one-forms with vanishing normal component [Sch95, Corollary 2.1.6 and Theorem 2.1.7], whose kernel is the space of absolute harmonic fields, together with the Poincaré inequality for p , which holds because $H^0(C; \text{ad}) = 0$. (iii) follows from (i), (ii) and (C3). $\square$

We call the harmonic fields h_v the *slow directions*: they span the kernel of the fast operator $\mathfrak{s}^{-1} \mathcal{B}_t$, and their only dynamics is inherited from the boundary value of the strip. The corresponding coefficient is the *slow coefficient*.

Lemma 7.2 (slow directions are tied to the trace). *For U in the static domain write $\Pi_t U = (0, h_w)$ and let ν_0 be the harmonic part of $b|_{\Sigma}$. Then*

$$|\nu_0 - w| \leq c \|\mathcal{B}_t U\|, \quad \|U\|_{L^2} + \|b\|_{H^1(C)} \leq c (\|\mathcal{B}_t U\| + |\nu_0|).$$

Proof. The harmonic part of $h_w|_{\Sigma}$ is w , and the harmonic projection of the trace of $b - h_w$ is bounded by $c \|b - h_w\|_{H^1} \leq c \|\mathcal{B}_t U\|$ by Lemma 7.1(ii). The second inequality follows, since $|w| \leq |\nu_0| + c \|\mathcal{B}_t U\|$ and $\|h_w\| \leq c |w|$. $\square$

Lemma 7.3 (the seam term). *For $U(t)$ smooth in t in the static domains, write $b|_{\Sigma} = \beta + d\xi$ with β harmonic, and $b|_T = d\xi_T$ on the other boundary components. Then*

$$\int_{\partial C} \langle b \wedge D_t b \rangle = \int_{\Sigma} \langle \beta \wedge D_t \beta \rangle + R, \quad |R| \leq c |\dot{q}| \|b\|_{H^1(C)}^2.$$

Proof. On Σ , $D_t(d\xi) = d(D_t\xi) + [a_\Sigma, \xi]$ with $a = h_{\dot{q}}$. Expanding, the terms $\int \beta \wedge d(\cdot)$ and $\int d\xi \wedge dD\xi$ vanish by Stokes and flatness, and $\int d\xi \wedge D\beta = \int \xi[a_\Sigma \wedge \beta]$. The remaining terms are bounded by $c|a|_\infty(|\beta| + \|d\xi\|)\|\xi\|$, and $\|\xi\| \leq c\|d\xi\|$ because $H^0(\Sigma; \text{ad}) = 0$. The other components have $H^0 = H^1 = 0$ and are treated the same way. The trace $H^1(C) \rightarrow L^2(\partial C)$ is bounded, and $|a| \leq c|\dot{q}|$. $\square$

7.3. The Gårding estimate. Fix symplectic frames along u as follows: transport a symplectic frame along $t \mapsto u(t, 0)$ and then along each segment $r \mapsto u(t, r)$ by a symplectic connection, and write $\nu = \Phi\xi$ with $\xi: S \rightarrow \mathbb{R}^{2m}$, $\xi_0 = \xi(\cdot, 0)$. Then $D_u\nu = \Phi(\partial_t\xi + \tilde{J}\partial_r\xi + \tilde{S}\xi)$ with $\tilde{J} = \Phi^{-1}J\Phi$ and $\tilde{S}$ bounded by the C^1 norms of u and J ; the derivatives of $\tilde{J}$ contribute zeroth-order terms of the same size. Let $e_i(t)$ be the harmonic representatives of $\Phi(t, 0)\varepsilon_i$. Under Convention 3.1, $\omega(e_i, e_j) = \omega_0(\varepsilon_i, \varepsilon_j)$, and

$$(7.1) \quad \int_\Sigma \langle \beta \wedge D_t\beta \rangle = \omega_0(\xi_0, \partial_t\xi_0) + R', \quad |R'| \leq c(K_u + Q)|\xi_0|^2,$$

where K_u bounds the connection coefficients of the frame. At $r = 1$, $|\omega_0(\xi_1, \partial_t\xi_1)| \leq cK_u|\xi_1|^2$ because K is Lagrangian.

Proposition 7.4 (uniform Gårding estimate). *There are $\mathfrak{s}_1 > 0$ and C , depending on $(C, g_C, \mathbb{E})$, on Q , H , on the Lagrangian K and on the C^1 norms of u and J , but not on $\mathfrak{s}$, $\inf \mathfrak{s}$ or the support of z , such that for $\sup \mathfrak{s} \leq \mathfrak{s}_1$ and every smooth compactly supported $z = (U, \nu) \in \mathcal{D}$,*

$$\|z\|_X \leq C(\|D_{\mathfrak{s}}z\|_Y + \|\nu\|_{L^2(S)}).$$

The same holds for the formal adjoint in the measure $\mathfrak{s} dt dV_C$, whose gauge-fixing component is $-D_t - \mathfrak{s}^{-1}\mathcal{B}_t - \varsigma + 2\varsigma\Pi_f$, on the same domain.

Proof. The energy identity. Let $\mathcal{A}_t = [D_t, \mathcal{B}_t]$, a zeroth-order operator with $|\mathcal{A}_t U| \leq c|\dot{q}||U|$. By Lemma 6.8 (without the closedness of the traces),

$$\frac{d}{dt} \langle U, \mathcal{B}U \rangle = 2\langle DU, \mathcal{B}U \rangle + \langle U, \mathcal{A}U \rangle + \int_{\partial C} \langle b \wedge D_t b \rangle.$$

For a gauge-fixing component $f = DU - \mathfrak{s}^{-1}\mathcal{B}U + ZU$ with a pointwise endomorphism $|Z_t| \leq z_0 H$ (here $Z = 2\varsigma\Pi_f$), expanding $\mathfrak{s}|f|^2$ and integrating the total derivative gives

$$(7.2) \quad \int \mathfrak{s}\|f\|^2 = \int (\mathfrak{s}\|DU + ZU\|^2 + \mathfrak{s}^{-1}\|\mathcal{B}U\|^2) - 2\int \langle ZU, \mathcal{B}U \rangle + \int \langle U, \mathcal{A}U \rangle + \int \int_{\partial C} \langle b \wedge D_t b \rangle.$$

No factor $\mathfrak{s}$ multiplies the last three terms, and no derivative of $\mathfrak{s}$ appears: the weight $\mathfrak{s}$ cancels the $\mathfrak{s}^{-1}$ of $\mathcal{B}$ before any integration by parts in t .

For the strip, with $D_0 = \partial_t + \tilde{J}\partial_r$,

$$(7.3) \quad \|D_0\xi\|^2 = \|\partial_t\xi\|^2 + \|\partial_r\xi\|^2 + \int \omega_0(\xi, \partial_t\xi)|_{r=1} dt - \int \omega_0(\xi, \partial_t\xi)|_{r=0} dt,$$

from $-2\omega_0(\partial_t\xi, \partial_r\xi) = \partial_r\omega_0(\xi, \partial_t\xi) - \partial_t\omega_0(\xi, \partial_r\xi)$, and $\|D_u\nu\|^2 \geq \|D_0\xi\|^2 - 2\|\tilde{S}\|\|D_0\xi\|\|\xi\|$.

Cancellation. By Lemma 7.3 and (7.1), the last term of (7.2) is $\int \omega_0(\xi_0, \partial_t\xi_0) dt$ up to an error bounded by $cQ \int \|\mathcal{B}U\|^2 + c(Q + K_u) \int (|\xi_0|^2 + |\xi_1|^2)$; the coefficient K_u multiplies only the trace term. Adding (7.2) and (7.3), the two terms $\pm \int \omega_0(\xi_0, \partial_t\xi_0)$ cancel identically. This is the only point where Convention 3.1 is used.

Absorption. Put $\mathcal{E} = \int \|\mathcal{B}U\|^2 \leq \mathfrak{s}_1 \int \mathfrak{s}^{-1}\|\mathcal{B}U\|^2$ and $\mathcal{J} = \int (|\xi_0|^2 + |\xi_1|^2)$. By Lemma 7.2, $\|U\|$ and the slow coefficient w are bounded pointwise in t by $\|\mathcal{B}U\| + |\nu_0|$, so $|\langle U, \mathcal{A}U \rangle| \leq cQ(\|\mathcal{B}U\|^2 + |\nu_0|^2)$, $2|\langle ZU, \mathcal{B}U \rangle| \leq \frac{1}{4}\mathfrak{s}^{-1}\|\mathcal{B}U\|^2 + cH^2\mathfrak{s}(\|\mathcal{B}U\|^2 + |\nu_0|^2)$,

and $\mathfrak{s}\|U\|^2 \leq c\mathfrak{s}_1(\|\mathcal{B}U\|^2 + |\nu_0|^2)$. The one-dimensional trace inequality $|g(0)|^2 \leq \theta\|g'\|^2 + (1 + \theta^{-1})\|g\|^2$ on $[0, 1]$ bounds $\mathcal{J}$ by a small multiple of $\|\partial_r \xi\|^2$ plus a multiple of $\|\xi\|^2$. Choosing $\mathfrak{s}_1 = c(1 + Q + H)^{-2}$ gives

$$\|z\|_X^2 \leq C(\|D_{\mathfrak{s}}z\|_Y^2 + \|\nu\|_{L^2(S)}^2), \quad C = c(1 + Q + K_u + H)^4.$$

Adjoint. By Lemma 6.8 and the bilinear form of (7.3), $\langle D_{\mathfrak{s}}z, z' \rangle_Y - \langle z, D_{\mathfrak{s}}^*z' \rangle_Y = \int \omega(\nu_0, \mu_0) - \int \omega(\nu_0, \mu_0) = 0$ for $z, z' \in \mathcal{D}$, the term at $r = 1$ vanishing because K is Lagrangian. So the adjoint has the same domain. The reflection $t \mapsto -t$ carries $D_{\mathfrak{s}}^*$ to an operator of the same form with $Z = \mp(2\zeta\Pi_f - \zeta)$, and the same argument applies. $\square$

Remark 7.5. With the opposite sign in Convention 3.1 the estimate fails already for constant $\mathfrak{s}$ and constant q . For a unit vector v and $\zeta = v - iJv$, the pair $b = \chi \operatorname{Re}(e^{i\tau t + s\tau(s-1)}\zeta)$ near the seam, where s is the normal coordinate of C with the boundary at $s = 1$, $\nu = \chi \operatorname{Re}(e^{i\tau t - \tau r}\zeta)$ on the strip, with cutoffs χ , lies in the domain and decays away from the seam on both sides. Its ratio $\|z\|_X^2/(\|D_{\mathfrak{s}}z\|_Y^2 + \|\nu\|^2)$ grows like τ^2 . This is a boundary layer at the seam, where a $\bar{\partial}$ -operator would be glued to a ∂ -operator; it is unrelated to the slow directions.

Remark 7.6 (why no logarithm appears). The slow component $w(t)$ is never estimated through the gauge part. A bound through $\int \mathfrak{s}(|\dot{w}|^2 + |w|^2)$ would require a weighted Hardy inequality, which for $\mathfrak{s} = \bar{\mathfrak{s}}e^{-H|t|}$ costs a factor $\log(1/\inf \mathfrak{s})$. Here w is controlled pointwise in t by ν_0 through Lemma 7.2, and ν_0 by the unweighted strip norm.

7.4. The uniform inverse at a regular limit. Assume now that u is J -holomorphic, converges exponentially to points $x_{\pm} \in P \cap K$, and that J is independent of t near the ends. Assume $D_u: \{\nu \in H^1 : \nu(\cdot, 0) \in TP, \nu(\cdot, 1) \in TK\} \rightarrow L^2$ is onto with kernel $\mathcal{K}$ of dimension d , and fix d smooth functionals Γ of ν with compact support such that $\Gamma|_{\mathcal{K}}$ is an isomorphism.

Lemma 7.7 (ends). *There are $T_0, \mathfrak{s}_1, C$ such that for $\sup \mathfrak{s} \leq \mathfrak{s}_1$ and $z \in \mathcal{D}$ supported in $\{|t| \geq T_0\}$, $\|z\|_X \leq C\|D_{\mathfrak{s}}z\|_Y$, and similarly for $D_{\mathfrak{s}}^*$.*

Proof. For $|t| \geq T_0$ the Lagrangian frames $\Phi^{-1}TP$ and $\Phi^{-1}TK$ at $r = 0, 1$ are close to transverse limits, so $\sup_r |\xi(t, r)| \leq C(\|\partial_r \xi(t, \cdot)\| + \|\mathcal{B}_t U\|)$, using Lemma 7.2 for the distance of ν_0 from TP . Squaring and integrating over t ,

$$\mathcal{J} + \|\xi\|^2 \leq C_P \left(\|\partial_r \xi\|^2 + \int \|\mathcal{B}_t U\|^2 dt \right),$$

where C_P is fixed by the transverse limiting pair. It need not be small. Return to the energy identities (7.2) and (7.3), before replacing the boundary terms by an unweighted L^2 remainder. Put

$$\mathcal{E}_0 = \|\partial_t \xi\|^2 + \|\partial_r \xi\|^2 + \int (\mathfrak{s}\|D_t U\|^2 + \mathfrak{s}^{-1}\|\mathcal{B}_t U\|^2).$$

The coefficients involving $\dot{q}$, the derivatives of the strip, and the moving Lagrangian frames tend to zero on the ends. Their contributions, including the lower-order term of the strip operator, are at most $o_{T_0}(1)(\mathcal{E}_0 + \mathcal{J} + \|\xi\|^2)$ after Young's inequality. The bounded scale derivative contributes at most a fixed multiple of $H^2 \int \mathfrak{s}\|U\|^2$, after absorbing one quarter of the $\mathcal{B}_t U$ term. Lemma 7.2 and the preceding Poincaré bound give

$$\int \mathfrak{s}\|U\|^2 + \mathcal{J} + \|\xi\|^2 \leq C(1 + C_P)\mathcal{E}_0, \quad \int \mathfrak{s}\|U\|^2 \leq C\mathfrak{s}_1(1 + C_P)\mathcal{E}_0.$$

Consequently

$$\mathcal{E}_0 \leq C\|D_{\mathfrak{s}}z\|_Y^2 + (o_{T_0}(1)(1 + C_P) + CH^2\mathfrak{s}_1(1 + C_P))\mathcal{E}_0.$$

Choose T_0 first and then $\mathfrak{s}_1$ to absorb the last term. The complementary spatial estimate and the same Poincaré bound recover the full X norm from $\mathcal{E}_0$. For the adjoint the extra scalar endomorphism is bounded by CH and is treated by the identical small-scale estimate. This proves both assertions without a small Poincaré constant. $\square$

Combining Proposition 7.4 on $[-T_0 - 1, T_0 + 1]$ with Lemma 7.7 by a partition of unity in t gives

$$(7.4) \quad \|z\|_X \leq C(\|D_{\mathfrak{s}} z\|_Y + \|\nu\|_{L^2([-T_1, T_1] \times [0, 1])}),$$

with $C, T_1, \mathfrak{s}_1$ independent of $\mathfrak{s}$, and the same for $D_{\mathfrak{s}}^*$.

Proposition 7.8 (uniform inverse without logarithmic loss). *There are $\mathfrak{s}_1, C$ such that for every scale with $\sup \mathfrak{s} \leq \mathfrak{s}_1$ and $|\mathfrak{s}'/\mathfrak{s}| \leq H$, the operator $(D_{\mathfrak{s}}, \Gamma): X \rightarrow Y \oplus \mathbb{R}^d$ is an isomorphism with $\|(D_{\mathfrak{s}}, \Gamma)^{-1}\| \leq C$. Moreover, if $J_{\mathfrak{s}} k = ((0, h_{k(t,0)}), k)$ for $k \in \mathcal{K}$ and $G_{\mathfrak{s}} k$ is the kernel element of $D_{\mathfrak{s}}$ with $\Gamma G_{\mathfrak{s}} k = \Gamma k$, then $\|G_{\mathfrak{s}} k - J_{\mathfrak{s}} k\|_X \leq C\sqrt{\sup \mathfrak{s}} \|k\|$.*

Proof. Injectivity estimate. Suppose $\|z_j\|_X = 1$ and $\|D_{\mathfrak{s}_j} z_j\|_Y + |\Gamma z_j| \rightarrow 0$ with $\sup \mathfrak{s}_j \rightarrow 0$. By (7.4), $\liminf \|\nu_j\|_{L^2([-T_1, T_1] \times [0, 1])} > 0$. A subsequence of ν_j converges weakly in H^1 and strongly in L^2_{loc} to ν_{∞} with $D_u \nu_{\infty} = 0$, $\nu_{\infty}(\cdot, 1) \in TK$, and $\nu_{\infty}(\cdot, 0) \in TP$, the last because $\int \text{dist}(\nu_j(t, 0), TP)^2 \leq c \int \|\mathcal{B}U_j\|^2 \leq c \sup \mathfrak{s}_j$ by Lemma 7.2. Thus $\nu_{\infty} \in \mathcal{K}$ and $\Gamma \nu_{\infty} = 0$, so $\nu_{\infty} = 0$, a contradiction. For the adjoint the limit satisfies $D_u^* \mu_{\infty} = 0$ with Lagrangian boundary values, hence is orthogonal to the image of D_u , which is everything. The scales $\mathfrak{s}_j$ play no role in the limit, which lives on the strip alone: no convergent subsequence of scales and no lower bound on $\mathfrak{s}_j$ is used.

Surjectivity. The same contradiction argument, applied to $D_{\mathfrak{s}}^*$ (whose limit object is an element of the kernel of D_u^* with Lagrangian boundary values, hence zero), gives $\|y\|_X \leq C\|D_{\mathfrak{s}}^* y\|_Y$. Coercivity of $\langle D_{\mathfrak{s}}^* y, D_{\mathfrak{s}}^* y' \rangle$ and the Lax–Milgram lemma then give, for $f \in Y$, a weak solution $z = D_{\mathfrak{s}}^* y$ of $D_{\mathfrak{s}} z = f$ with $\|z\|_Y \leq C\|f\|_Y$; write $R_{\mathfrak{s}} f = z$, a right inverse of $D_{\mathfrak{s}}$. At fixed $\mathfrak{s}$ weak solutions are strong: the slice operator $(\mathfrak{s}^{-1} \mathcal{B}_t, -J\partial_r)$ with the domain of Section 7.1 is symmetric by Lemma 6.8 and the seam cancellation. It is self-adjoint: an element of the domain of the adjoint is a weak solution, with L^2 right-hand side, of the same mixed boundary problem (vanishing normal component, closed traces, harmonic matching), and the fixed-geometry regularity for exactly this mixed boundary condition [DFL21, Theorem 4.17 and Remark 4.19] places it in the domain. That theorem is stated for the four-dimensional cylinder with one compression body; it applies to a slice operator after extending an element w of the adjoint domain as $\beta(t)w$ on an interval times the slice, to slices with two gauge branches after localizing near each seam and reflecting $\theta \mapsto \pi - \theta$ at the collapsing seam, and the finite-rank harmonic matching does not affect regularity. After a bounded trivialization of the t -dependent domain, regularity of weak solutions is that of [RS95]. Applying (7.4) to cut-off versions of z gives $z \in X$ with $\|z\|_X \leq C\|f\|_Y$.

Kernel. $J_{\mathfrak{s}} k$ lies in the domain, its strip part is in $\ker D_u$, and $D_{\mathfrak{s}} J_{\mathfrak{s}} k = ((0, D_t h_{k(t,0)}), 0)$ has Y -norm at most $C\sqrt{\sup \mathfrak{s}} \|k\|$ since k decays exponentially. Hence $z_k = J_{\mathfrak{s}} k - R_{\mathfrak{s}} D_{\mathfrak{s}} J_{\mathfrak{s}} k$ lies in the kernel, Γ is onto on the kernel for small $\sup \mathfrak{s}$, and $(D_{\mathfrak{s}}, \Gamma)$ is an isomorphism. The estimate for $G_{\mathfrak{s}} k - J_{\mathfrak{s}} k$ follows from the uniform bound. $\square$

7.5. The endpoint transition. In this subsection we work with the scale $\mathfrak{s}$ of the collapsing branch of a single slice; on the endpoint neck of X_{ε} it is $\mathfrak{s} = \varepsilon e^s$. Near an endpoint, with $L = \log(1/\varepsilon)$, in the coordinate $\rho = L - s = \log(1/\mathfrak{s})$ (with s as in Lemma 6.3(ii)), the slice at scale $\mathfrak{s} = e^{-\rho}$ is $(C, g_C) \sqcup [0, \pi] \sqcup (-C, \mathfrak{s}^2 g_C)$, the copy at $\theta = \pi$ carrying the reversed

orientation in the conventions of Lemma 6.3(ii). On the slice Hilbert space $\mathcal{H}_s$ with norm $\|z\|_s^2 = \|U_f\|^2 + s\|U_c\|^2 + \|\nu\|^2$, the tangential operator is

$$A_s(U_f, U_c, \nu) = (\mathcal{B}U_f, s^{-1}\mathcal{B}U_c, -J\partial_\theta\nu)$$

on the s -independent domain $\mathcal{W}$ given by vanishing normal components, closed traces and harmonic matching at both ends of $[0, \pi]$. The quadratic form $\langle z, A_s z \rangle_s$ does not depend on s . By Lemma 6.8 and Convention 3.1 the seam terms cancel, and A_s is symmetric. It is self-adjoint by the same argument: with $I_s = \text{diag}(1, s, 1)$, an element of the domain of the adjoint of $I_s A_s$ is a weak solution of the mixed boundary problem, hence in the domain by [DFL21, Theorem 4.17 and Remark 4.19].

Proposition 7.9. *There are $\gamma > 0$ and $s_0 > 0$ such that for $s \leq s_0$ and $q \in P$, $\ker A_s = \{k_v = (h_v, h_v, v) : v \in T_q P\}$ and the spectrum of A_s meets $(-\gamma, \gamma)$ only in 0. Moreover $\|U_c - h_{\Pi\nu(\pi)}\|_{H^1} \leq C\sqrt{s} \|A_s z\|_s$, where Π is the projection to $T_q P$.*

Proof. Kernel. If $A_s z = 0$, then $\mathcal{B}U_f = \mathcal{B}U_c = 0$, so by Lemma 7.1 $U_f = (0, h_{v_f})$ and $U_c = (0, h_{v_c})$; and ν is constant, so the matching gives $v_f = \nu = v_c =: v$, with $v \in T_q P$ because h_v is defined only there.

The complementary operator. The orthogonal complement of $T_q P$ in $T_q M$ is $J T_q P$, since $T_q P$ is Lagrangian and J is compatible, so $1 - \Pi$ projects onto $J T_q P$. Let $\mathcal{W}_q^\perp$ be the space of $U = (p, b)$ on C with $b(n) = 0$, closed traces, and harmonic part of $b|_\Sigma$ in $J T_q P$, and let $\mathcal{B}^\perp$ be $\mathcal{B}$ with domain $\mathcal{W}_q^\perp$. By Lemma 6.8 it is symmetric, $J T_q P$ being Lagrangian. It satisfies

$$(7.5) \quad \|U\|_{H^1(C)} \leq c_1 \|\mathcal{B}U\| \quad (U \in \mathcal{W}_q^\perp) :$$

if $w \in T_q P$ is the bulk harmonic coefficient of U and $\nu_0 \in J T_q P$ the harmonic part of its trace, Lemma 7.2 gives $|w| \leq |\nu_0 - w| \leq c \|\mathcal{B}U\|$, because $\nu_0 \perp w$, and Lemma 7.1(ii) gives the rest. Hence $\mathcal{B}^\perp$ is injective with closed range; it is self-adjoint by the local regularity used above for A_s , so it is an isomorphism $\mathcal{W}_q^\perp \rightarrow L^2(C)$ with inverse bounded by c_1 into H^1 (this is the complementary problem of (7.12)), and $\mathcal{B}^\perp - \mu$ is invertible with $\|(\mathcal{B}^\perp - \mu)^{-1}\|_{L^2 \rightarrow H^1} \leq 2c_1$ for $|\mu| \leq 1/(2c_1)$.

Elimination of the collapsed component. Let A_0 be the tangential operator of the mixed end, on $\mathcal{H}_0 = L^2(C) \oplus L^2([0, \pi])$ with the domain $\mathcal{W}_0$ of pairs (U_f, ν) with $b_f(n) = 0$, closed traces, harmonic part of $b_f|_\Sigma$ equal to $\nu(0)$, and $\nu(\pi) \in T_q P$. It is self-adjoint and its spectrum meets $(-2\gamma, 2\gamma)$ only in 0, uniformly in q [DFL21, (4.20)–(4.24)]; its kernel is $\{(h_v, v) : v \in T_q P\}$. Let $|\lambda| = \gamma$, $s\gamma \leq 1/(2c_1)$, and consider $(A_s - \lambda)z = f$ with $z \in \mathcal{W}$ and $f = (f_f, f_c, f_\nu) \in \mathcal{H}_s$. Put $\tilde{v} = \Pi\nu(\pi)$ and $\tilde{U}_c = U_c - h_{\tilde{v}}$. The harmonic part of the trace of U_c is $\nu(\pi)$, so that of $\tilde{U}_c$ is $(1 - \Pi)\nu(\pi) \in J T_q P$ and $\tilde{U}_c \in \mathcal{W}_q^\perp$; since $\mathcal{B}h_{\tilde{v}} = 0$, the collapsing equation $\mathcal{B}U_c - s\lambda U_c = s f_c$ becomes $(\mathcal{B}^\perp - s\lambda)\tilde{U}_c = s f_c + s\lambda h_{\tilde{v}}$. Thus

$$\tilde{U}_c = s(\mathcal{B}^\perp - s\lambda)^{-1}(f_c + \lambda h_{\tilde{v}}), \quad \|\tilde{U}_c\|_{H^1} \leq C s(\|f_c\| + |\tilde{v}|),$$

and the matching condition at $\theta = \pi$ becomes the perturbed Lagrangian condition

$$(7.6) \quad (1 - \Pi)\nu(\pi) = s\lambda \Lambda_{s\lambda} \tilde{v} + s m_f, \quad \Lambda_\mu v = \tau((\mathcal{B}^\perp - \mu)^{-1} h_v), \quad m_f = \tau((\mathcal{B}^\perp - s\lambda)^{-1} f_c),$$

where τ is the harmonic part of the trace on Σ . Here $\Lambda_\mu : T_q P \rightarrow J T_q P$ is bounded uniformly, and $|s m_f| \leq C s \|f_c\| \leq C \sqrt{s} \|f\|_s$ because $s \|f_c\|^2 \leq \|f\|_s^2$. Conversely, a pair (U_f, ν) satisfying the fixed-branch and section equations and (7.6), completed by $U_c = h_{\tilde{v}} + \tilde{U}_c$, is a solution.

Reduction to A_0 and the Neumann series. Fix a cutoff χ_π on $[0, \pi]$, equal to 1 near π and to 0 on $[0, \pi/2]$, and for (U_f, ν) satisfying (7.6) put $\tilde{\nu} = \nu - \chi_\pi(s\lambda \Lambda_{s\lambda} \tilde{v} + s m_f)$. Then

$\tilde{\nu}(\pi) = \tilde{v} \in T_q P$ and $\Pi\tilde{\nu}(\pi) = \tilde{v}$, so $y = (U_f, \tilde{\nu}) \in \mathcal{W}_0$, and the fixed-branch and section equations become

$$(A_0 - \lambda)y = (f_f, f_\nu) + \Theta_\lambda y + \mathfrak{s}F_\lambda f_c,$$

with $\Theta_\lambda y = (0, (J\partial_\theta + \lambda)(\chi_\pi \mathfrak{s} \Lambda_{\mathfrak{s}\lambda} \Pi\tilde{\nu}(\pi)))$ and $\mathfrak{s}F_\lambda f_c = (0, (J\partial_\theta + \lambda)(\chi_\pi \mathfrak{s} m_f))$. Since the trace at π is bounded by the graph norm, $\|\Theta_\lambda y\|_{\mathcal{H}_0} \leq C\mathfrak{s}\|y\|_{\mathcal{W}_0}$, and $\|\mathfrak{s}F_\lambda f_c\| \leq C\sqrt{\mathfrak{s}}\|f\|_{\mathfrak{s}}$. On $|\lambda| = \gamma$ the resolvent $(A_0 - \lambda)^{-1}: \mathcal{H}_0 \rightarrow \mathcal{W}_0$ is bounded by C_γ in the graph norm, so $\|(A_0 - \lambda)^{-1}\Theta_\lambda\|_{\mathcal{W}_0 \rightarrow \mathcal{W}_0} \leq \frac{1}{2}$ for $\mathfrak{s} \leq \mathfrak{s}_0$, and

$$(7.7) \quad y = \sum_{j \geq 0} ((A_0 - \lambda)^{-1}\Theta_\lambda)^j (A_0 - \lambda)^{-1}((f_f, f_\nu) + \mathfrak{s}F_\lambda f_c).$$

Conversely (7.7), followed by $\nu = \tilde{\nu} + \chi_\pi(\mathfrak{s}\lambda\Lambda_{\mathfrak{s}\lambda}\tilde{v} + \mathfrak{s}m_f)$ and $U_c = h_{\tilde{v}} + \tilde{U}_c$, defines a solution $z \in \mathcal{W}$; for $f = 0$ it gives $y = 0$, then $\tilde{v} = 0$, $\nu = 0$ and $U_c = 0$. Hence the circle $|\lambda| = \gamma$ lies in the resolvent set of $A_{\mathfrak{s}}$. Let $\varpi(U_f, U_c, \nu) = (U_f, \nu)$ and $\iota(U_f, \nu) = (U_f, 0, \nu)$; then $\varpi\iota = 1$, $\iota\varpi = 1 - P_c$, ι is an isometry $\mathcal{H}_0 \rightarrow \mathcal{H}_{\mathfrak{s}}$ and $\varpi = \iota^*$. From (7.7) and the bounds above,

$$\|\varpi(A_{\mathfrak{s}} - \lambda)^{-1}\iota - (A_0 - \lambda)^{-1}\| \leq C\mathfrak{s}, \quad \|P_c(A_{\mathfrak{s}} - \lambda)^{-1}\| + \|(A_{\mathfrak{s}} - \lambda)^{-1}P_c\| \leq C\sqrt{\mathfrak{s}}$$

on $|\lambda| = \gamma$: for the first, $f = \iota g$ has $f_c = 0$ and the series differs from its first term by $O(\mathfrak{s})$, as does ν from $\tilde{\nu}$; for the second, the collapsed component of $(A_{\mathfrak{s}} - \lambda)^{-1}f$ has $\mathcal{H}_{\mathfrak{s}}$ -norm $\sqrt{\mathfrak{s}}\|U_c\| \leq \sqrt{\mathfrak{s}}(C\|\tilde{v}\| + \|\tilde{U}_c\|) \leq C\sqrt{\mathfrak{s}}\|f\|_{\mathfrak{s}}$, and a source $f = P_c f$ has $\|f\|_{\mathfrak{s}} = \sqrt{\mathfrak{s}}\|f_c\|$, so its solution is $O(\mathfrak{s}\|f_c\|) = O(\sqrt{\mathfrak{s}}\|f\|_{\mathfrak{s}})$.

Riesz projections. Let $P_{\mathfrak{s}}$ and P_0 be the Riesz projections of $A_{\mathfrak{s}}$ and A_0 for the circle $|\lambda| = \gamma$. Integrating the two estimates, $\|\varpi P_{\mathfrak{s}}\iota - P_0\| \leq C\mathfrak{s}$ and $\|P_c P_{\mathfrak{s}}\| + \|P_{\mathfrak{s}} P_c\| \leq C\sqrt{\mathfrak{s}}$. Since $P_{\mathfrak{s}} - (1 - P_c)P_{\mathfrak{s}}(1 - P_c) = P_c P_{\mathfrak{s}} + P_{\mathfrak{s}} P_c - P_c P_{\mathfrak{s}} P_c$,

$$\|P_{\mathfrak{s}} - \iota P_0 \varpi\|_{\mathcal{H}_{\mathfrak{s}}} \leq \|P_c P_{\mathfrak{s}}\| + \|P_{\mathfrak{s}} P_c\| + \|P_c P_{\mathfrak{s}} P_c\| + \|\iota(\varpi P_{\mathfrak{s}}\iota - P_0)\varpi\| \leq C\sqrt{\mathfrak{s}}.$$

The operator $\iota P_0 \varpi$ is an orthogonal projection on $\mathcal{H}_{\mathfrak{s}}$ of rank $\text{rank } P_0 = n$, and two projections at distance less than one have the same rank; so $P_{\mathfrak{s}}$ has rank n for $\mathfrak{s}$ small. Its range contains the n -dimensional kernel found above. As $A_{\mathfrak{s}}$ is self-adjoint and $\pm\gamma$ are not in its spectrum, $P_{\mathfrak{s}}$ is the spectral projection of $(-\gamma, \gamma)$, and the spectrum of $A_{\mathfrak{s}}$ in $(-\gamma, \gamma)$ is $\{0\}$. Uniformity over q holds because P is compact and the gap of A_0 is uniform [DFL21, (4.24)].

Spatial estimate. Write $U_c = h_{\Pi\nu(\pi)} + \tilde{U}_c$ with $\tilde{U}_c \in \mathcal{W}_q^\perp$, as above. By (7.5), $\|\tilde{U}_c\|_{H^1} \leq c_1\|\mathcal{B}\tilde{U}_c\| = c_1\|\mathcal{B}U_c\| \leq c_1\sqrt{\mathfrak{s}}\|A_{\mathfrak{s}}z\|_{\mathfrak{s}}$, since $\|A_{\mathfrak{s}}z\|_{\mathfrak{s}}^2 \geq \mathfrak{s}^{-1}\|\mathcal{B}U_c\|^2$. $\square$

Along the endpoint neck the scale varies, $\mathfrak{s}(\rho) = \mathfrak{s}_b e^{\pm\rho}$. The linearized operator is $\mathcal{D} = \partial_\rho - A_{\mathfrak{s}(\rho)} + 2(\log \mathfrak{s})'\Pi_f$, which is $\partial_\rho - A_{\mathfrak{s}(\rho)} - 2\Pi_f$ for $\mathfrak{s} = \mathfrak{s}_b e^{-\rho}$; here Π_f projects to the scalar of the collapsing component, acting on sections of the bundle of Hilbert spaces $\mathcal{H}_{\mathfrak{s}(\rho)}$. Since the quadratic form of $A_{\mathfrak{s}}$ is independent of $\mathfrak{s}$, the cross term in $\|(\partial_\rho - A)z\|^2$ is $-2\int \langle z', Az \rangle_{\mathfrak{s}} = -\int \partial_\rho \langle z, Az \rangle_{\mathfrak{s}} d\rho = 0$ for compactly supported z , so

$$\int \|(\partial_\rho - A)z\|_{\mathfrak{s}}^2 d\rho = \int (\|z'\|_{\mathfrak{s}}^2 + \|Az\|_{\mathfrak{s}}^2) d\rho,$$

with no derivative of $\mathfrak{s}$. The metric derivative enters only through $\partial_\rho \|z\|_{\mathfrak{s}}^2 = 2\langle z', z \rangle_{\mathfrak{s}} + (\log \mathfrak{s})'\|P_c z\|_{\mathfrak{s}}^2$, where P_c projects to the collapsing component, and hence only in weighted estimates, where it is absorbed by Proposition 7.9.

Proposition 7.10. (i) (*Tail estimate.*) Let the scale satisfy $|(\log \mathfrak{s})'| \leq 1$, and put $\|y\|_X^2 = \int (\|y\|_{\mathfrak{s}}^2 + \|\partial_\rho y\|_{\mathfrak{s}}^2 + \|A_{\mathfrak{s}} y\|_{\mathfrak{s}}^2) d\rho$ and $\|y\|_Y^2 = \int \|y\|_{\mathfrak{s}}^2 d\rho$. For $0 < |\delta| < \gamma/2$

there is $\mathfrak{s}_0(\delta) > 0$ such that for z supported where $\mathfrak{s}(\rho) \leq \mathfrak{s}_0(\delta)$,

$$\|e^{\delta\rho}z\|_X \leq C_\delta \|e^{\delta\rho}\mathcal{D}z\|_Y, \quad \|e^{\delta\rho}z\|_X \leq C_\delta \|e^{\delta\rho}\mathcal{D}^*z\|_Y,$$

where $\mathcal{D}^* = -\partial_\rho - A_\mathfrak{s} - (\log \mathfrak{s})'P_c + 2(\log \mathfrak{s})'\Pi_f$ is the formal adjoint.

- (ii) (Conserved pairing.) For h in the kernel of the adjoint and $v \in T_qP$, $\langle k_v, h \rangle_{\mathfrak{s}(\rho)}$ is independent of ρ .
- (iii) (Transition.) Let $\mathcal{T}$ be the complete flat transition: the domain consisting of the rescaled endpoint core (the half-disk of radius $c_{\text{core}}\varepsilon$ about $z_\pm$, c_{core} as in Definition 6.2(i), rescaled by ε^{-1} , with its fixed conformal structure), continued by the endpoint neck to $\rho = \infty$, with the flat pair (B_q, q) and the scale $\mathfrak{s} \rightarrow 0$ along the neck. With the weight $e^{\delta\rho}$, $0 < \delta < \gamma/2$, and fixed limit 0, $\mathcal{D}$ is an isomorphism; with a free limit in T_qP it is onto with kernel $\{k_v\}$ and index n . For $0 < \delta \leq \delta_0$ the fixed-limit inverse satisfies $\|\mathcal{D}^{-1}\| \leq C/\delta$, uniformly in $q \in P$.

Proof. (i) Let $y = e^{\delta\rho}z$, so that $e^{\delta\rho}\mathcal{D}z = (\partial_\rho - A_\mathfrak{s} - \delta + 2(\log \mathfrak{s})'\Pi_f)y$. Since the quadratic form of $A_\mathfrak{s}$ does not depend on $\mathfrak{s}$, and $\partial_\rho\|y\|_\mathfrak{s}^2 = 2\langle \partial_\rho y, y \rangle_\mathfrak{s} + (\log \mathfrak{s})'\|P_c y\|_\mathfrak{s}^2$,

$$\int \|(\partial_\rho - A_\mathfrak{s} - \delta)y\|_\mathfrak{s}^2 d\rho = \int (\|\partial_\rho y\|_\mathfrak{s}^2 + \|(A_\mathfrak{s} + \delta)y\|_\mathfrak{s}^2 + \delta(\log \mathfrak{s})'\|P_c y\|_\mathfrak{s}^2) d\rho.$$

Because $0 < |\delta| < \gamma/2$, Proposition 7.9 gives $\|(A_\mathfrak{s} + \delta)y\|_\mathfrak{s} \geq c_\delta(\|y\|_\mathfrak{s} + \|A_\mathfrak{s}y\|_\mathfrak{s})$ with $c_\delta > 0$. Its spatial estimate, together with $\|h_{\Pi\nu(\pi)}\| \leq C|\nu(\pi)| \leq C(\|y\|_\mathfrak{s} + \|A_\mathfrak{s}y\|_\mathfrak{s})$, gives $\|P_c y\|_\mathfrak{s} \leq C\sqrt{\mathfrak{s}}(\|y\|_\mathfrak{s} + \|A_\mathfrak{s}y\|_\mathfrak{s})$, and $\|\Pi_f y\|_\mathfrak{s} \leq C\mathfrak{s}\|A_\mathfrak{s}y\|_\mathfrak{s}$ by Lemma 7.1(i) and the Poincaré inequality. For $\mathfrak{s} \leq \mathfrak{s}_0(\delta)$ the last term and the term $2(\log \mathfrak{s})'\Pi_f$ are therefore absorbed. For $\mathcal{D}^*$ the conjugated operator is $-\partial_\rho - (A_\mathfrak{s} - \delta) - (\log \mathfrak{s})'P_c + 2(\log \mathfrak{s})'\Pi_f$, the same computation with $-\partial_\rho$ gives the identity with $A_\mathfrak{s} - \delta$ in place of $A_\mathfrak{s} + \delta$, and the extra term $(\log \mathfrak{s})'P_c y$ is absorbed in the same way.

(ii) For h with $\mathcal{D}^*h = -h' - Ah + P_ch - 2\Pi_fh = 0$, differentiate $\langle k_v, h \rangle_\mathfrak{s}$: the term P_ch in the equation cancels the metric derivative, and $Ak_v = \Pi_fk_v = 0$.

(iii) *Kernel with fixed limit.* If $\mathcal{D}z = 0$ with decay, the linear energy identity on the core and neck, whose seam terms cancel by Convention 3.1, gives z flat and constant, hence 0. *Cokernel.* A cokernel element h of the free operator pairs trivially with $\mathcal{D}(\beta k_v) = \beta'k_v$ for a cutoff β , so by (ii) its stationary limit is orthogonal to the kernel and h decays; it then vanishes by the argument of Proposition 6.12.

The bound. We use the norms of (i) on the neck, weighted by $e^{\delta\rho}$, together with the fixed H^1 and L^2 norms on the core, and write X_δ, Y_δ . Fix $\bar{\delta} \in (0, \gamma/2)$. By what has been shown, $\mathcal{D}: X_{\bar{\delta}} \rightarrow Y_{\bar{\delta}}$ with fixed limit 0 is an isomorphism; write $\mathcal{D}_{\bar{\delta}}^{-1}$ for its inverse, which is bounded uniformly in q .

Stationary splitting. In the component coordinates the map $k: T_qP \rightarrow \mathcal{H}_\mathfrak{s}$, $v \mapsto k_v$, does not depend on ρ . Let P_ρ^{st} be the $\mathcal{H}_{\mathfrak{s}(\rho)}$ -orthogonal projection onto $\ker A_{\mathfrak{s}(\rho)} = k(T_qP)$, let $P_\rho^n = 1 - P_\rho^{\text{st}}$, and write $P_\rho^{\text{st}}y = k_{\mathfrak{a}(y)}$. With $I_\mathfrak{s} = \text{diag}(1, \mathfrak{s}, 1)$ and the Gram matrix $G_\mathfrak{s} = k^*I_\mathfrak{s}k$, which is uniformly positive, $P^{\text{st}} = kG_\mathfrak{s}^{-1}k^*I_\mathfrak{s}$ and $\mathfrak{a} = G_\mathfrak{s}^{-1}k^*I_\mathfrak{s}$. Since $\partial_\rho I_\mathfrak{s} = (\log \mathfrak{s})'I_\mathfrak{s}P_c$,

$$(7.8) \quad \partial_\rho P^{\text{st}} = (\log \mathfrak{s})'P^{\text{st}}P_cP^n, \quad \|\partial_\rho P^{\text{st}}\| \leq C|(\log \mathfrak{s})'|\sqrt{\mathfrak{s}},$$

the bound because the collapsed component of k_v has $\mathcal{H}_\mathfrak{s}$ -norm $\sqrt{\mathfrak{s}}\|h_v\|$. Three facts give the structure of the equation: $\mathcal{D}k_a = k_{a'}$ for a function a with values in T_qP , since k does not depend on ρ and $A_\mathfrak{s}k_v = \Pi_fk_v = 0$; $P^{\text{st}}A_\mathfrak{s} = 0$ and $P^{\text{st}}\Pi_f = 0$, since $A_\mathfrak{s}$ is symmetric and annihilates k_v , and k_v has no scalar component; and $P^{\text{st}}\partial_\rho y = -(\partial_\rho P^{\text{st}})y$ when $P^{\text{st}}y = 0$. Consequently, for $z = k_a + z^n$ with $a = \mathfrak{a}(z)$ and $z^n = P^n z$, the equation $\mathcal{D}z = f$ is equivalent

to the triangular system

$$(7.9) \quad \mathcal{D}^n z^n = P^n f, \quad a' = a(f) - \mathbf{b}_s z^n,$$

where $\mathcal{D}^n = P^n \partial_\rho - A_s + 2(\log \mathfrak{s})' \Pi_f$ acts on P^n -valued fields and $\mathbf{b}_s y = -(\log \mathfrak{s})' G_s^{-1} k^* I_s P_c y$ satisfies $|\mathbf{b}_s y| \leq C |(\log \mathfrak{s})'| \sqrt{\mathfrak{s}} \|y\|_s$. The first equation does not involve a .

The normal equation. Let $\mathcal{Y}$ be the complete line obtained by continuing the neck to $\rho \rightarrow -\infty$ with the slice $(C, g_C) \sqcup [0, \pi] \sqcup (-C, \mathfrak{s}_Y^2 g_C)$ and the flat pair (B_q, q) , where $\mathfrak{s}_Y$ is smooth and nonincreasing, equals $\mathfrak{s}(\rho)$ for $\rho \geq \rho_1 + 2$, is constant for $\rho \leq \rho_1$, and satisfies $|(\log \mathfrak{s}_Y)'| \leq 1$; here ρ_1 is fixed below, independently of δ and q . All of the above holds on $\mathcal{Y}$ with $\mathfrak{s}_Y$ in place of $\mathfrak{s}$. For a compactly supported P^n -valued field y in the domain, $P^n \partial_\rho y = \partial_\rho y + (\partial_\rho P^{\text{st}})y$. By the identity displayed before the proposition, the gap $(\|y\|_s \leq \gamma^{-1} \|A_s y\|_s$ for y orthogonal to $\ker A_s$), (7.8) and $\|\Pi_f y\|_s \leq C \mathfrak{s} \|A_s y\|_s$,

$$\|\mathcal{D}^n y\|_Y \geq (\|\partial_\rho y\|_Y^2 + \|A_s y\|_Y^2)^{1/2} - C \mathfrak{s}(\rho_1)^{1/2} \|y\|_X \geq c \|y\|_X$$

once $\mathfrak{s}(\rho_1)$ is small. Conjugation by $e^{\sigma\rho}$ replaces $\mathcal{D}^n$ by $\mathcal{D}^n - \sigma$, so the same bound holds in the norms weighted by $e^{\sigma\rho}$ for $|\sigma| \leq \sigma_0 := c/2$, with a constant independent of σ . The formal adjoint $\mathcal{D}^*$ maps P^n -valued fields to P^n -valued fields: for $P^{\text{st}} y = 0$, $P^{\text{st}} \mathcal{D}^* y = (\partial_\rho P^{\text{st}})y - (\log \mathfrak{s})' P^{\text{st}} P_c y = 0$ by (7.8). The same computation therefore bounds $\|y\|_X$ by the norm of the normal adjoint applied to y , and, weak solutions being strong by the local regularity used above for A_s , $\mathcal{D}^n$ is an isomorphism between the P^n -valued parts of the weighted spaces on $\mathcal{Y}$:

$$(7.10) \quad \|e^{\sigma\rho} y\|_{X(\mathcal{Y})} \leq C_n \|e^{\sigma\rho} \mathcal{D}^n y\|_{Y(\mathcal{Y})}, \quad |\sigma| \leq \sigma_0,$$

with C_n independent of σ and q .

The stationary equation. For $0 < \delta \leq \sigma_0$ and $f \in Y_\delta(\mathcal{Y})$ let $z^n = (\mathcal{D}^n)^{-1} P^n f$, $g = a(f) - \mathbf{b}_s z^n$ and $a(\rho) = -\int_\rho^\infty g$. By (7.10), $\|e^{\delta\rho} g\|_{L^2} \leq C \|f\|_{Y_\delta}$. Since $e^{\delta\rho} a(\rho) = -\int_0^\infty e^{-\delta u} (e^{\delta\rho} g)(\rho + u) du$ is the convolution of $e^{\delta\rho} g$ with a one-sided kernel of L^1 norm δ^{-1} and L^2 norm $(2\delta)^{-1/2}$,

$$(7.11) \quad \|e^{\delta\rho} a\|_{L^2} \leq \delta^{-1} \|e^{\delta\rho} g\|_{L^2}, \quad \sup_\rho e^{\delta\rho} |a(\rho)| \leq (2\delta)^{-1/2} \|e^{\delta\rho} g\|_{L^2}, \quad a' = g.$$

As $A_s k_a = 0$, $\partial_\rho k_a = k_{a'}$ and $\|k_v\|_s \leq C|v|$, the field $S_\delta f = k_a + z^n$ solves $\mathcal{D} S_\delta f = f$ on $\mathcal{Y}$, tends to zero, and satisfies $\|S_\delta f\|_{X_\delta(\mathcal{Y})} \leq C \delta^{-1} \|f\|_{Y_\delta(\mathcal{Y})}$. The factor δ^{-1} arises only in the undifferentiated stationary coefficient; the normal part is solved with no loss.

The core. Fix $\rho_2 \geq \rho_1 + 2$ and cutoffs χ_1, χ_0 of ρ with $\chi_1 = 0$ for $\rho \leq \rho_2$, $\chi_1 = 1$ for $\rho \geq \rho_2 + 1$, $\chi_0 = 0$ for $\rho \leq \rho_2 + 2$ and $\chi_0 = 1$ for $\rho \geq \rho_2 + 3$; on the support of χ_1 the operators of $\mathcal{T}$ and of $\mathcal{Y}$ agree. For $0 < \delta \leq \delta_0 := \min(\sigma_0, \bar{\delta})$ and $f \in Y_\delta$, regard $\chi_0 f$ as a source on $\mathcal{Y}$ and put

$$z_1 = \chi_1 S_\delta(\chi_0 f), \quad r = f - \mathcal{D} z_1 = (1 - \chi_0) f - \chi_1' S_\delta(\chi_0 f).$$

The source r is supported in the fixed compact region $\{\rho \leq \rho_2 + 3\}$ of $\mathcal{T}$, where $e^{\delta\rho}$ and $e^{\bar{\delta}\rho}$ are comparable with constants independent of δ . By (7.11) the stationary part of $S_\delta(\chi_0 f)$ is bounded there by $C \delta^{-1/2} \|f\|_{Y_\delta}$, so $\|r\|_{Y_\delta} \leq C \delta^{-1/2} \|f\|_{Y_\delta}$. Then $z = z_1 + \mathcal{D}_\delta^{-1} r$ solves $\mathcal{D} z = f$ with limit zero, and since $\|\cdot\|_{X_\delta} \leq C \|\cdot\|_{X_{\bar{\delta}}}$ for $\delta \leq \bar{\delta}$,

$$\|z\|_{X_\delta} \leq C \delta^{-1} \|f\|_{Y_\delta},$$

while on every fixed compact subset of $\mathcal{T}$ the field z is bounded by $C \delta^{-1/2} \|f\|_{Y_\delta}$. The fixed-limit kernel is zero at every positive weight, so $z = \mathcal{D}^{-1} f$. The constants are uniform in q ,

because P is compact and γ , C_n and $\mathcal{D}_\delta^{-1}$ are uniform. The bound is sharp: $z_\delta = k_v \beta(\delta(\rho - \rho_0))$ for a fixed cutoff β has $\|z_\delta\|/\|\mathcal{D}z_\delta\| \geq c/\delta$. $\square$

7.6. Transport along a path in $L(C)$. Let $q(t)$ be a smooth path in $P = L(C)$ with bounded derivatives of every order; in Section 8 it is the path $q_R(t)$ of section boundary values of the approximate solution W_ε . Write $\mathcal{L}_t = T_{q(t)}P$, $j_tv = (0, h_v)$ for the harmonic extension, $\ell_t = (j_t^* j_t)^{-1} j_t^*$ and $\Pi_t = j_t \ell_t$, all bulk operators on $L^2(C)$. A variation $U = (p, b)$, $p = \mathfrak{s}\psi$, splits as $U = j_tv + U_\perp$ with $v = \ell_t U$ and $U_\perp = (1 - \Pi_t)U$. On the domain (vanishing normal component, closed traces) the complementary estimate

$$(7.12) \quad \|U_\perp\|_{H^{r+1}(C)} \leq C_r \|\mathcal{B}_t U\|_{H^r(C)} \quad (r \geq 0)$$

holds: solve the complementary problem with harmonic trace in $J\mathcal{L}_t$, use its elliptic regularity, and remove the finite harmonic kernel by the bulk projection.

Lemma 7.11 (transport). *There is a connection $\nabla_t = D_t + C_t$ along $q(t)$ on the bundle of spatial domains, with $\|C_t\|_{H^r \rightarrow H^r} \leq C_r$ and the same bounds for its time derivatives, which preserves the vanishing of the normal component, the closedness of the traces and the harmonic seam matching, and satisfies*

$$\nabla_t(j_tv) = j_t \nabla_t^\mathcal{L} v, \quad \nabla_t^\mathcal{L}(\ell_t U) = \ell_t \nabla_t U, \quad [\nabla_t, \Pi_t] = 0, \quad [\nabla_t, \Pi_f] = 0.$$

Consequently the iterated commutators $\mathcal{B}_t^{[0]} = \mathcal{B}_t$, $\mathcal{B}_t^{[j+1]} = [\nabla_t, \mathcal{B}_t^{[j]}]$ satisfy $\mathcal{B}_t^{[j]} j_t = 0$ and $\|\mathcal{B}_t^{[j]} U\|_{H^r} \leq C_{j,r} \|\mathcal{B}_t U\|_{H^r}$. The connection is the derivative of transports $\tilde{T}$ from the domain at q_0 to the domain at q , defined for q near q_0 and bounded on every $H^r(C)$, with $\tilde{T}j_0 = j_q K$, $\ell_q \tilde{T} = K \ell_0$ and $\Pi_q \tilde{T} = \tilde{T} \Pi_0$, where $K: T_{q_0}M \rightarrow T_q M$ is a smooth family of isomorphisms carrying $\mathcal{L}_{q_0}$ to $\mathcal{L}_q$, used also on the section collar.

Proof. For q near q_0 , choose a smooth family of isomorphisms $K: T_{q_0}M \rightarrow T_q M$ carrying $\mathcal{L}_{q_0}$ to $\mathcal{L}_q$; domain elements have arbitrary harmonic trace, and the same K transports the section values on the collar, so the harmonic seam matching is preserved. On every parallel surface of the boundary collars, identify the exact, coexact and harmonic summands of the Hodge decompositions for the two flat restrictions, blended to the identity in the interior, leaving scalars and normal components unchanged; this gives T^0 , which identifies the domains and their harmonic boundary values by K and is bounded on every $H^r(C)$. Correct it on the harmonic fields: $T^1 = T^0 + (j_q K - T^0 j_0) \ell_0$, which changes no boundary condition and satisfies $T^1 j_0 = j_q K$. For an interior cutoff χ close to one the matrix $M_q = \ell_q \chi j_q$ is invertible; with $e_q = \chi j_q M_q^{-1}$, which has zero boundary trace, put $\tilde{T} = T^1 + e_q(K \ell_0 - \ell_q T^1)$. Then $\tilde{T}j_0 = j_q K$, $\ell_q \tilde{T} = K \ell_0$ and $\Pi_q \tilde{T} = \tilde{T} \Pi_0$. Differentiating these transports along $q(t)$ defines ∇_t , and the identities follow. Since $\mathcal{B}_t j_t = 0$ and ∇_t preserves harmonic extension, induction gives $\mathcal{B}_t^{[j]} j_t = 0$; the commutators have spatial order at most one, so $\|\mathcal{B}_t^{[j]} U\|_{H^r} \leq C \|U_\perp\|_{H^{r+1}}$, and (7.12) gives the bound. $\square$

7.7. The inverse on the glued domain. Let V be a regular index-zero solution on X_0 with limits $q_\pm$ at the new ends and $\ker D_V = 0$ (on the extended weighted spaces), and let W_ε be the approximate solution on X_ε of Section 8: V truncated at radius $R = \varepsilon^{1/4}$ about each endpoint, the flat transitions at $q_\pm$ inserted, and the flat lift of the section boundary values over $I \times C$. Put $L = \log(1/\varepsilon)$ and $A = 1 + L$.

Norms. On X_ε use the complete conformal representatives: on the middle compression $dt^2 + (\varepsilon/r)^2 g_C$ with $dt = dx/r$; on the endpoint necks $ds^2 + g_C$ and the collapsing branch at

scale $\mathfrak{s} = \varepsilon e^s$; on the rescaled endpoint cores the fixed transition metric; on the section the corresponding fixed and logarithmic strip metrics. On a small compression with $U = (p, b)$, $p = \mathfrak{s}\psi$, the norms X_ε and Y_ε are

$$\int (\mathfrak{s}|U|^2 + \mathfrak{s}|\mathbf{D}_t U|^2 + \mathfrak{s}^{-1}|\mathcal{B}_t U|^2) dt, \quad \int \mathfrak{s}|f|^2 dt,$$

on the section and the fixed branch the H^1 and L^2 norms, on the cores the fixed mixed H^1 and L^2 norms, patched by uniformly bounded cutoffs, and on the original ends the fixed weights with the free limiting vectors. The norm includes the L^2 part of a stationary field on every finite neck: for a constant stationary vector on a region of length $O(L)$,

$$(7.13) \quad \|k_v\|_{X_\varepsilon(\text{region})} \leq C\sqrt{A}|v|.$$

Components. The two *transitions* are the complete flat transitions $\mathcal{T}_\pm$ of Proposition 7.10 at $q_\pm$. The *auxiliary exterior* is the following complete domain.

Definition 7.12 (auxiliary exterior). Let r be a smooth positive function on I equal to the distance to the nearer endpoint where that distance is at most $x_0/2$, and let t be the complete coordinate on I with $dt = dx/r$, where x is arclength; near $z_\pm$ the coordinate t agrees, up to sign and an additive constant, with $s = -\log r$ of Lemma 6.3(ii). Fix a smooth $F: (0, \infty) \rightarrow (0, 1]$ with $F(y) = y$ for $y \leq \frac{1}{2}$, $F(y) = 1$ for $y \geq 2$ and $0 \leq yF'(y)/F(y) \leq 1$, and put

$$\mathfrak{s}_E = \sqrt{\varepsilon} F(\sqrt{\varepsilon}/r).$$

The *auxiliary exterior* E_ε is X_0 with the gauge piece $I \times C$, carrying the metric $dt^2 + \mathfrak{s}_E^2 g_C$, attached along I : the Lagrangian condition $L(C)$ on I is replaced by the matching condition of Definition 6.4. Its configuration W_ε^E is the truncation V_R of V at $R = \varepsilon^{1/4}$ of Section 8 on X_0 , and on $I \times C$ the flat lift $B_{q_R(t)}$ of the boundary values q_R of V_R on I , with temporal component in the covariant Neumann slice, $\partial_t B - d_B \Psi = h_{\dot{q}_R}$. The operator D_E is the linearized mixed operator at W_ε^E , with the gauge-fixing component of the metric $dt^2 + \mathfrak{s}_E^2 g_C$ on the compression and of the complete metrics of X_0 elsewhere, as for D_ε in the conformal representative above, on the domain given along I by the conditions of Section 7.1 (vanishing normal component, closed traces, harmonic part of $b|_\Sigma$ equal to the section value) and by the original conditions elsewhere, on the extended weighted spaces at the original mixed ends and at the two new ends.

The scale satisfies $\sup \mathfrak{s}_E \leq \sqrt{\varepsilon}$ and $|\partial_t \log \mathfrak{s}_E| \leq H$ with H independent of ε ; moreover $\mathfrak{s}_E = \varepsilon/r$ where $r \geq 2\sqrt{\varepsilon}$ and $\mathfrak{s}_E = \sqrt{\varepsilon}$ where $r \leq \sqrt{\varepsilon}/2$. Near $z_\pm$, in the coordinate s , the domain E_ε is a half-cylinder with slice $(C, g_C) \sqcup [0, \pi] \sqcup (-C, \mathfrak{s}_E(s)^2 g_C)$, where $\mathfrak{s}_E(s) = \varepsilon e^s$ for $s \leq L/2 - \log 2$ and $\mathfrak{s}_E(s) = \sqrt{\varepsilon}$ for $s \geq L/2 + \log 2$. There is a fixed c_R such that W_ε^E coincides with V and its flat lift for $s \leq L/4 - c_R$, and with the flat pair $(B_{q_\pm}, q_\pm)$ on all three branches for $s \geq L/4 + c_R$. On the region $\{r \geq 2\sqrt{\varepsilon}\}$, which contains the whole exterior side of $s = L/2 - \log 2$ and in particular $J_\pm$, the domain, the configuration and the operator coincide with X_ε , W_ε and D_ε in the conformal representative above.

On the new ends of the components, which are not present on X_ε , use the weight $\delta_L = c_0/(1+L)$, bounded above and below on the parts of the components used in the gluing. Each transition overlaps the actual domain on $J_\pm = [\alpha_L, \beta_L]$ with $\alpha_L = L/4 + b_0$, with a fixed $b_0 \geq c_R + 1$ so that W_ε^E is flat on $J_\pm$, $\beta_L = L/2 - \log 2$, of length $\mathbb{T} \sim L/4$, on which the three operators agree under $\rho = L - s$ (Figure 3).

Norms and asymptotic values. Let ϕ_E be a smooth function on E_ε , equal to $\delta_* s$ on the new ends, to $\delta_a s$ on the original end a (the fixed weights of (X3)) and to 0 on a fixed compact core,

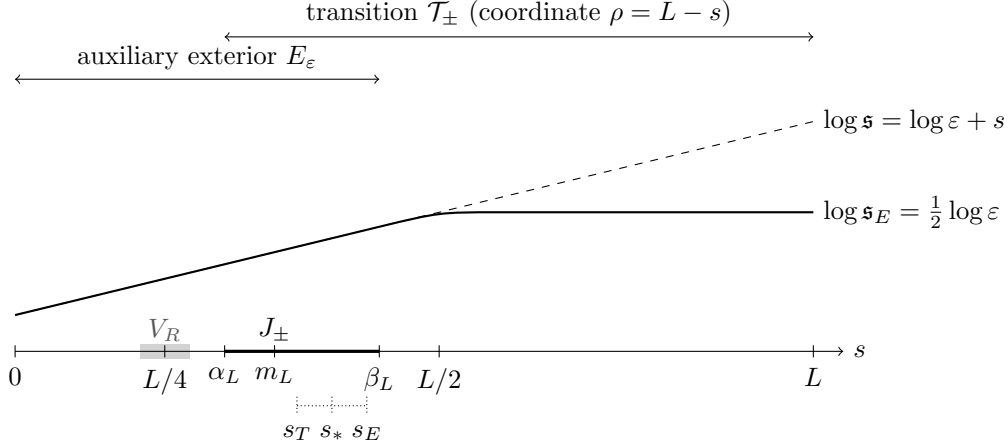


FIGURE 3. An endpoint neck in the exterior coordinate $s \in [0, L]$, $L = \log(1/\varepsilon)$, with $r = e^{-s}$ (not to scale). The dashed line is $\log \mathfrak{s}$ for the scale $\mathfrak{s} = \varepsilon e^s$ of X_ε ; the solid line is the scale $\mathfrak{s}_E$ of the auxiliary exterior, equal to $\mathfrak{s}$ for $s \leq L/2 - \log 2$ and to $\sqrt{\varepsilon}$ for $s \geq L/2 + \log 2$. The shaded interval about $L/4$ ($r = \varepsilon^{1/4}$) is where the truncation V_R of the limit is joined to the flat pair. The two components are used over the arrows and overlap the glued domain on $J_\pm = [\alpha_L, \beta_L]$, with joining point m_L (Section 7.7). The dotted marks are the cuts $s_T < s_E$ of the proof of Proposition 8.12, at distance $C_* + D$ and $C_* - D$ from $L/2$, and their common midpoint $s_* = L/2 - C_*$; for large L they lie between m_L and β_L .

and depending on the compression only through t . Let $\mathbb{V}$ be the direct sum of $T_{q_-}P \oplus T_{q_+}P$ and of the tangent spaces of the critical families at the original mixed ends. Let $\mathfrak{k}_0(v)$, $v \in \mathbb{V}$, be the cut-off stationary fields of D_V of Proposition 6.9: a fixed cutoff of the end coordinate times the stationary field of the end, in the end chart of V ; at the new ends this is (h_v, v) on the slice $C \sqcup [0, \pi]$. Let $\mathfrak{k}_0^\varepsilon(v)$ be the same fields built with the end charts of V_R ; they coincide with $\mathfrak{k}_0(v)$ where $s \leq L/4 - c_R$ and differ from it by $O(e^{-\beta s}|v|)$ elsewhere. On E_ε put $\mathfrak{k}_E(v) = \mathfrak{k}_0^\varepsilon(v)$ off the compression and $\mathfrak{k}_E(v) = (0, h_{\nu(t)})$ on $I \times C$, where $\nu(t) \in T_{q_R(t)}P$ is the section component of $\mathfrak{k}_0^\varepsilon(v)$ on I . Then $\mathfrak{k}_E(v)$ lies in the domain of D_E . Every element of X_E is $z = \bar{z} + \mathfrak{k}_E(v)$ with $\bar{z}$ decaying, and

$$\begin{aligned} \|z\|_{X_E}^2 &= \|e^{\phi_E} \bar{z}\|_{H^1}^2 + \int_I e^{2\phi_E} (\mathfrak{s}_E \|\bar{U}_c\|^2 + \mathfrak{s}_E \|\mathbf{D}_t \bar{U}_c\|^2 + \mathfrak{s}_E^{-1} \|\mathcal{B}_t \bar{U}_c\|^2) dt + |v|^2, \\ \|f\|_{Y_E}^2 &= \|e^{\phi_E} f\|_{L^2}^2 + \int_I e^{2\phi_E} \mathfrak{s}_E \|f_c\|^2 dt, \end{aligned}$$

where the first terms are the norms of the section and of the gauge fields of X_0 , and the compression integrals use the fixed metric g_C on C . On $J_\pm$ these are the norms $X_\varepsilon, Y_\varepsilon$ of this section multiplied by the weight. Since $\mathcal{B}_t$ and Π_f annihilate harmonic fields, the compression component of $D_E \mathfrak{k}_E(v)$ is $\mathbf{D}_t(0, h_{\nu(t)})$; as $\mathfrak{s}_E \leq \sqrt{\varepsilon}$, and $\partial_t \nu$ and $\dot{q}_R$ decay at the rate $\beta > \delta_*$ along the ends,

$$(7.14) \quad \|(D_E \mathfrak{k}_E(v))_c\|_{Y_E} \leq C\varepsilon^{1/4}|v|, \quad \|D_E \mathfrak{k}_E(v)\|_{Y_E} \leq C|v|,$$

and $e^{\phi_E}(\mathfrak{k}_0^\varepsilon(v) - \mathfrak{k}_0(v))$, $e^{\phi_E}(D_{V_R} \mathfrak{k}_0^\varepsilon(v) - D_V \mathfrak{k}_0(v))$ have H^1 and L^2 norms at most $C\varepsilon^{(\beta - \delta_*)/4}|v|$.

Proposition 7.13 (uniform inverse on the auxiliary exterior). *Fix δ_* with $0 < \delta_* < \min(\beta, \gamma/2)$, where β is the decay rate of V and γ the gap of Proposition 7.9. There are $\varepsilon_1 > 0$ and C such that for $0 < \varepsilon < \varepsilon_1$ the operator $D_E: X_E \rightarrow Y_E$ is an isomorphism and*

$$\|z\|_{X_E} \leq C \|D_E z\|_{Y_E} \quad (z \in X_E).$$

In particular the limiting values $v_\pm$ of z at the new ends satisfy $|v_-| + |v_+| \leq C \|D_E z\|_{Y_E}$.

At the weight δ_* , which lies below the gaps of the new mixed ends of X_0 and below β , the operator D_V on the extended weighted spaces is an isomorphism by Hypothesis 6.5(X4), V being regular of index zero.

Proof. Step 1: a graph estimate with compact remainder. Fix s_1 large, to be chosen in (b) independently of ε , and put $r_1 = e^{-s_1}$. We first derive a consequence of Lemma 7.2: at every t , $\mathfrak{s}_E \|U_c\|^2 \leq c \mathfrak{s}_E (\|\mathcal{B}_t U_c\|^2 + |\nu|_I^2) \leq c \sup \mathfrak{s}_E (\mathfrak{s}_E^{-1} \|\mathcal{B}_t U_c\|^2 + |\nu|_I^2)$, and the trace inequality on a collar of I in the section gives

$$(7.15) \quad \int_I e^{2\phi_E} \mathfrak{s}_E \|\bar{U}_c\|^2 dt \leq C \varepsilon^{1/2} \|\bar{z}\|_{X_E}^2.$$

(a) *The middle of I .* On $\{r \geq r_1/4\}$ use conformal coordinates $t' + i\theta'$ on a collar of I in S_0 with $I = \{\theta' = 0\}$, for instance $t' + i\theta' = \log((z - z_-)/(z_+ - z))$ in the half-plane chart, and let $\theta_1 > 0$ be such that $\{r \geq r_1/4, 0 \leq \theta' \leq \theta_1\}$ lies in the collar. There dt'/dt is bounded above and below with all its derivatives, independently of ε . In the time coordinate t' the compression metric is $(dt/dt')^2(dt'^2 + \mathfrak{s}'^2 g_C)$ with $\mathfrak{s}' = \mathfrak{s}_E dt'/dt$. The anti-self-duality equation is conformally invariant, and in dimension four the gauge-fixing component of a metric changed by a conformal factor depending on t' alone differs from the original one by a positive factor and by a zeroth-order term in the temporal component. Hence, on fields supported in $\{r \geq r_1/2\}$ whose section component vanishes for $\theta' \geq \theta_1/2$, $\Upsilon D_E = D_{\mathfrak{s}'} + 2h''\Pi_f$, where Υ multiplies the gauge-fixing component by a positive function of t' bounded above and below with its derivatives, $D_{\mathfrak{s}'}$ is the operator of Section 7.1 with time t' , the strip $\mathbb{R} \times [0, \theta_1]$, the path q_R and the section of V_R , and h'' is bounded independently of ε ; a term $2h''\Pi_f$ is allowed in the proof of Proposition 7.4 as part of the endomorphism Z . Here $\sup \mathfrak{s}' \leq C\varepsilon/r_1$, while $|\partial_{t'} \log \mathfrak{s}'|$, $\sup |\partial_{t'} q_R|$ and the C^1 norms of the section are bounded independently of ε . The Lagrangian condition at the far edge of the strip enters the proof of Proposition 7.4 only through the boundary term $\omega_0(\xi_1, \partial_t \xi_1)$ and the trace term $|\xi_1|^2$, which vanish for these fields, and the proof is unchanged for a strip of fixed width θ_1 . The norms of Section 7.1 and those of X_E, Y_E are uniformly equivalent on this region. Hence, for such fields and ε small,

$$\|\bar{z}\|_{X_E} \leq C (\|D_E \bar{z}\|_{Y_E} + \|\nu\|_{L^2(\{r \geq r_1/2, \theta' \leq \theta_1/2\})}).$$

(b) *The new ends.* For $s \geq s_1 - 1$ the configuration W_ε^E is within $Ce^{-\beta s}$ of the flat pair $(B_{q_\pm}, q_\pm)$ in every C^k norm, with C independent of ε , by the exponential convergence of V (Proposition 6.16(iii)) and the construction of V_R . Identify the slice domains of W_ε^E with those of the flat pair by the map equal on the collapsing branch to the transport $\tilde{T}$ from $q_\pm$ to $q_R(s)$ of Lemma 7.11, and on the fixed branch and the section to an end chart of V_R compatible with the identification K of $\tilde{T}$ at the seam; since $\tilde{T}$ carries h_v to h_{Kv} and harmonic boundary values by K , this map preserves the domain. In this identification $D_E = \mathcal{D}_\pm + \mathfrak{E}$, where $\mathcal{D}_\pm$ is the neck operator of Section 7.5 at the flat pair with scale $\mathfrak{s}_E(s)$, written in the coordinate s as on $J_\pm$, and $\|\mathfrak{E}y\|_{Y_E} \leq Ce^{-\beta s_1} \|y\|_{X_E}$ for y supported in $\{s \geq s_1 - 1\}$. Indeed the only coefficient of $\mathfrak{E}$ carrying $\mathfrak{s}_E^{-1}$ is $\mathfrak{s}_E^{-1}(\mathcal{B}_{q_R(s)} \tilde{T} - \tilde{T} \mathcal{B}_{q_\pm})$; it vanishes on harmonic fields, and it is a first-order operator whose coefficients are $O(e^{-\beta s})$, so by Lemma 7.1(ii)

it is bounded by $Ce^{-\beta s} \mathfrak{s}_E^{-1} \|\mathcal{B}_{q_\pm} U_c\|$. By Proposition 7.10(i), which uses of the scale only $|\partial \log \mathfrak{s}| \leq 1$ and a small upper bound, and applies here because $0 \leq \partial_s \log \mathfrak{s}_E \leq 1$ and $\mathfrak{s}_E \leq \sqrt{\varepsilon}$, we have $\|e^{\delta_* s} y\|_X \leq C \|e^{\delta_* s} \mathcal{D}_\pm y\|_Y$ for decaying y supported in $\{s \geq s_1 - 1\}$, with C independent of ε ; by Lemma 7.1 these norms are uniformly equivalent to X_E and Y_E on the ends. Choosing s_1 large, independently of ε , to absorb $\mathfrak{E}$,

$$\|\bar{z}\|_{X_E} \leq C \|D_E \bar{z}\|_{Y_E} \quad \text{for decaying } \bar{z} \text{ supported in } \{s \geq s_1 - 1\}.$$

(c) *The remaining regions.* On the original ends, beyond a fixed start, the weighted end estimates of [DFL21, Section 4.2.1] at the fixed weights give $\|\bar{z}\|_{X_E} \leq C \|D_E \bar{z}\|_{Y_E}$ for decaying $\bar{z}$ supported there; on the gauge ends these are the estimates for the trajectories of $\text{CS}_\mathfrak{h} + f$, whose Hessian at the critical points is that of $\text{CS}_\mathfrak{h}$ because f vanishes near the critical set. The rest of E_ε is a fixed compact subset of X_0 at positive distance from I and from all ends, on which $W_\varepsilon^E = V$ for ε small; there the local mixed estimates [DFL21, Theorem 4.17 and Remark 4.19] give $\|\bar{z}\|_{H^1} \leq C(\|D_E \bar{z}\|_{L^2} + \|\bar{z}\|_{L^2})$ for $\bar{z}$ supported in it.

(d) *Patching.* Choose a finite partition $\sum_\alpha \chi_\alpha^2 = 1$ subordinate to the regions of (a)–(c), by cutoffs that are functions of t alone on the compression, take the same value on the two sides of the seam over I (on the section the cutoff of (a) is $\chi(t')\psi(\theta')$ with $\psi = 1$ near 0 and $\psi = 0$ for $\theta' \geq \theta_1/2$), and have gradients supported in a fixed compact set. These cutoffs preserve the domain of D_E , whose conditions are linear and pointwise in t , and $[D_E, \chi_\alpha]$ is multiplication by derivatives of χ_α . Its compression component has Y_E -norm at most $C(\int_I e^{2\phi_E} \mathfrak{s}_E \|\bar{U}_c\|^2 dt)^{1/2} \leq C\varepsilon^{1/4} \|\bar{z}\|_{X_E}$ by (7.15); its other components are bounded by the L^2 norm of $\bar{z}$ over a fixed compact set $\Omega_0 \subset \overline{X_0}$ containing these gradients and the section region of (a). Squaring and summing the estimates of (a)–(c) for $\chi_\alpha \bar{z}$, and absorbing the compression term for ε small, gives

$$(7.16) \quad \|\bar{z}\|_{X_E} \leq C(\|D_E \bar{z}\|_{Y_E} + \|\bar{z}\|_{L^2(\Omega_0)}),$$

where $\|\cdot\|_{L^2(\Omega_0)}$ is the L^2 norm of the section and of the gauge fields of X_0 over Ω_0 , and C and Ω_0 do not depend on ε . The same argument proves (7.16) for the formal adjoint $\widehat{D}_E^*$ of the conjugated operator $\widehat{D}_E = e^{\phi_E} D_E e^{-\phi_E}$ in the unweighted spaces: Proposition 7.4 contains its adjoint, Proposition 7.10(i) contains $\mathcal{D}^*$ with weights of either sign, and the fixed estimates of (c) hold for adjoints.

Step 2: the injectivity estimate. We claim that $\|z\|_{X_E} \leq C \|D_E z\|_{Y_E}$ for ε small. Otherwise there are $\varepsilon_i \rightarrow 0$ and $z_i = \bar{z}_i + \mathfrak{k}_E(v_i)$ on E_{ε_i} with $\|z_i\|_{X_E} = 1$ and $\|D_E z_i\|_{Y_E} \rightarrow 0$. After passing to a subsequence, $v_i \rightarrow v$, and the restrictions of $e^{\phi_E} \bar{z}_i$ to the section and to the gauge pieces of X_0 , which form a fixed space, converge weakly in H^1 and strongly in L^2 on compact subsets of $\overline{X_0}$, with traces converging strongly in L^2 on compact subarcs of I ; let $\bar{z}_0$ be the limit and $z_0 = \bar{z}_0 + \mathfrak{k}_0(v)$. Lemma 7.2 gives $\text{dist}(\nu_i(t), T_{q_{R_i}(t)} P) \leq c \|\mathcal{B}_t U_{i,c}(t)\|$ at every t , hence $\int_I e^{2\phi_E} \text{dist}(\nu_i, T_{q_{R_i}} P)^2 dt \leq c \sup \mathfrak{s}_E \rightarrow 0$; since $q_{R_i} \rightarrow q_V$ uniformly on compact subarcs, the section component of z_0 has its trace on I in $T_{q_V} L(C)$. The other boundary conditions are closed linear conditions on traces and pass to the limit. On every compact subset of X_0 the coefficients of D_E off the compression equal those of D_V for large i , so $D_V z_0 = 0$ in the interior and up to the boundary. By Fatou's lemma, the convergence of $\mathfrak{k}_0^{\varepsilon_i}(v_i)$ to $\mathfrak{k}_0(v)$ and (7.14), the field z_0 lies in the domain of D_V on the extended weighted space at weight δ_* , with limiting vector v . Since this operator is injective, $z_0 = 0$ and $v = 0$. Then $\|\bar{z}_i\|_{L^2(\Omega_0)} \rightarrow 0$, and $\|D_E \bar{z}_i\|_{Y_E} \leq \|D_E z_i\|_{Y_E} + C|v_i| \rightarrow 0$ by (7.14), so (7.16) gives $\|\bar{z}_i\|_{X_E} \rightarrow 0$; with $|v_i| \rightarrow 0$ this contradicts $\|z_i\|_{X_E} = 1$.

Step 3: surjectivity. By Step 2, D_E is injective with closed range for ε small. In the conjugated coordinates $\hat{z} = e^{\phi_E} \bar{z}$ the operator is $\hat{D}_E$ on the decaying part, and the asymptotic values act by $\hat{F}v = e^{\phi_E} D_E \mathfrak{k}_E(v)$. Suppose that for $\varepsilon_i \rightarrow 0$ there are f_i of unit norm orthogonal to the range. Then $\langle \hat{D}_E \hat{z}, f_i \rangle = 0$ for all decaying $\hat{z}$ in the domain and $\langle \hat{F}v, f_i \rangle = 0$ for all $v \in \mathbb{V}$. The first identity says that f_i is a weak solution of $\hat{D}_E^* f_i = 0$. At the fixed scale ε_i the local regularity of [DFL21, Theorem 4.17 and Remark 4.19], applied with cutoffs, places f_i in the domain, and (7.16) for $\hat{D}_E^*$ bounds f_i in the graph norm by $C\|f_i\|_{L^2(\Omega_0)} \leq C$. After passing to a subsequence, f_i converges weakly on X_0 , and strongly in $L^2(\Omega_0)$, to some f , while its compression component satisfies $\int_I \mathfrak{s}_E \|f_{i,c}\|^2 dt \leq C\varepsilon_i^{1/2}$, by the argument of (7.15). Let ζ be a smooth compactly supported decaying field in the domain of $\hat{D}_V$, and lift it to E_{ε_i} by $\zeta_i = (\zeta, (0, h_{\nu_\zeta(t)}))$, where $\nu_\zeta(t) \in T_{q_V(t)}P$ is the section component of ζ on I . For large i the support of ζ lies where $q_{R_i} = q_V$, so ζ_i lies in the domain of $\hat{D}_E$; off the compression $\hat{D}_E \zeta_i = \hat{D}_V \zeta$, and its compression component $e^{\phi_E} D_t(0, h_{\nu_\zeta})$ has norm $O(\varepsilon_i^{1/4})$. Hence $0 = \langle \hat{D}_E \zeta_i, f_i \rangle \rightarrow \langle \hat{D}_V \zeta, f \rangle$. Likewise $0 = \langle \hat{F}v, f_i \rangle \rightarrow \langle e^{\phi_E} D_V \mathfrak{k}_0(v), f \rangle$, because $\hat{F}v$ converges strongly by (7.14). Smooth compactly supported fields are dense in the decaying domain, so f annihilates the range of the conjugated operator D_V on the extended weighted spaces, which is the whole space; hence $f = 0$. Then $\|f_i\|_{L^2(\Omega_0)} \rightarrow 0$, and the graph bound gives $\|f_i\| \rightarrow 0$, a contradiction. Thus D_E is onto for ε small, and the bound of Step 2 is the bound of the proposition. $\square$

Lemma 7.14. *The auxiliary exterior operator D_E on the extended weighted spaces at the new ends is an isomorphism at weight δ_L on the new ends, and*

$$\|z_E\|_{X_E} \leq CA\|D_E z_E\|, \quad |v_-| + |v_+| \leq C\sqrt{A}\|D_E z_E\|,$$

where $v_\pm$ are the limiting values of z_E at the new ends. A solution of a source supported in $\{s \leq s'\}$ on a new end, with $s' \geq L/4 + c_R$ so that W_ε^E is flat beyond s' , differs from its stationary limit, a distance $\mathbb{T}$ beyond s' , by $O(e^{-\sigma_0 \mathbb{T}})$ times the bound, with σ_0 the rate of (7.10).

Proof. At the fixed weight δ_* the operator is an isomorphism with a bound independent of ε by Proposition 7.13. Passing from δ_* to δ_L on the two new ends is the argument of the proof of Proposition 7.10(iii) on each end, with the inverse of Proposition 7.13 in place of the fixed-weight transition inverse: the normal part is solved with no loss, and the stationary coefficient costs $\delta_L^{-1} = CA$ in the weighted L^2 norm, which gives the first bound, and $\delta_L^{-1/2} = C\sqrt{A}$ in the weighted supremum, which bounds the limiting values. On the part $s \leq L/4 + c_R$ of an end the configuration W_ε^E is not flat; in the identification of Step 1(b) of the proof of Proposition 7.13 its deviation adds to both equations of the triangular system (7.9) terms bounded by $Ce^{-\beta s}(|a| + \|z^n\|)$, and the scale adds the term bz^n with coefficient $O(\mathfrak{s}_E^{1/2}|\partial_s \log \mathfrak{s}_E|)$. The tail is solved with fixed limit 0, as S_δ is in Proposition 7.10(iii), and the limiting vector comes from the δ_* -inverse applied to the residual supported near s_1 . Measuring a by $\sup_s e^{\delta_L s}|a(s)|$ and z^n in the weighted graph norm, these terms enter the normal equation and the stationary integral with coefficients whose L^1 and L^2 norms over the end are $O(e^{-\beta s_1} + \varepsilon^{1/4})$, and they are absorbed with no further loss in δ_L for s_1 fixed large and ε small; their contribution to the weighted L^2 norm of a is at most $C(1 + \varepsilon^{1/4}L^{1/2})$ times the weighted supremum, hence $O(\delta_L^{-1/2})$. The homogeneous-tail statement is the exponential decay of the normal part and the integration of the stationary equation from the source-free region. $\square$

Proposition 7.15. *For ε small, the linearization D_ε of the mixed equation at W_ε is invertible, with $\|D_\varepsilon^{-1}f\|_{X_\varepsilon} \leq CA\|f\|_{Y_\varepsilon}$, and $\text{ind } D_\varepsilon = 0$.*

Proof. Source removal. On each overlap put $u = (s - \alpha_L)/T$ and fix cutoffs θ, ω of u with $\theta\omega = \omega$, θ supported in $(1/8, 7/8)$ and equal to one on $[1/4, 3/4]$, ω supported in $(1/4, 3/4)$ and equal to one on $[1/3, 2/3]$. Let S be the inverse of the stationary line problem on the overlap, extended to a complete line of constant scale beyond it, at weight $e^{\delta_L(s - \alpha_L)}$. The line estimate for S_δ in the proof of Proposition 7.10(iii) holds for every scale profile with $|(\log \mathfrak{s})'| \leq 1$ and small $\sup \mathfrak{s}$, in particular for this line of constant scale on both sides, and gives $\|S(\omega f)\| \leq CA\|f\|$. Put $z_{\text{loc}} = \sum_{\pm} \theta S(\omega f)$ and $f' = f - D_\varepsilon z_{\text{loc}} = (1 - \omega)f - [D_\varepsilon, \theta]S(\omega f)$. Since the derivative of θ is $O(T^{-1}) = O(A^{-1})$, $\|f'\| \leq C\|f\|$, and $f' = 0$ on $1/3 \leq u \leq 2/3$.

Components. Split f' into sources f_E on the exterior and $f_\pm$ on the transitions, with the partition changing inside the band $1/3 \leq u \leq 2/3$ on which $f' = 0$. Solve $z_E = D_E^{-1}f_E$ by Lemma 7.14, and on each transition $z_\pm = k_{v_\pm} + \mathcal{D}^{-1}f_\pm$ with the limiting values $v_\pm$ of z_E . This imposes two separate equalities between the exterior and each transition; no equality between v_- and v_+ , and no rank condition on the exterior evaluation, is imposed. Both component solutions are homogeneous a distance at least $T/6 - O(1)$ before the joining point $m_L = (\alpha_L + \beta_L)/2$. On the exterior Lemma 7.14 applies; on the transitions the triangular system (7.9) shows that the normal part decays at rate σ_0 while the stationary part changes only by $\int \mathbf{b}z^n$. Hence, on a fixed collar of m_L , $\|z_E - z_\pm\| \leq CAe^{-\sigma_0 T/6}\|f\|$; the stationary parts cancel because the limiting values agree.

Patching. With cutoffs β_E and $\beta_\pm = 1 - \beta_E$ of fixed slope at m_L , put $\mathcal{P}(f) = z_{\text{loc}} + \beta_E z_E + \sum \beta_\pm z_\pm$. The boundary conditions are linear, so $\mathcal{P}(f)$ lies in the domain, and $D_\varepsilon \mathcal{P}(f) = f + \sum [D_\varepsilon, \beta_E](z_E - z_\pm)$, whose error has norm at most $CAe^{-\sigma_0 T/6}\|f\|$. Converting the component norms to X_ε with (7.13), the stationary pieces cost $C\sqrt{A}(|v_-| + |v_+|) \leq CA\|f\|$, so $\|\mathcal{P}(f)\| \leq CA\|f\|$. For large L the Neumann series gives a right inverse $\mathcal{Q}_\varepsilon = \mathcal{P}(1 + \mathcal{E}_\mathcal{P})^{-1}$, where $\mathcal{E}_\mathcal{P} = D_\varepsilon \mathcal{P} - 1$ has norm at most $CAe^{-\sigma_0 T/6}$, with $\|\mathcal{Q}_\varepsilon\| \leq CA$.

Injectivity. Let $D_\varepsilon z = 0$. On each overlap write $z = k_a + w$ with w normal; w decays exponentially toward m_L in terms of its collar norms, and $|a(m_L)| \leq C\|z\|$. Cut z into component fields $z_E^{\text{cut}}, z_\pm^{\text{cut}}$ equal to z on $\text{supp } \beta_E$, respectively $\text{supp } \beta_\pm$, and completed beyond by the stationary value $k_{a(m_L)}$, so that $z = \beta_E z_E^{\text{cut}} + \sum \beta_\pm z_\pm^{\text{cut}}$; their sources are cutoff derivatives times $z - k_{a(m_L)}$, of norm $Ce^{-\sigma_0 T/6}\|z\|$. The component inverses and (7.13) give $\|z\| \leq CAe^{-\sigma_0 T/6}\|z\|$, so $z = 0$ for large L , since $T \sim L/4$ and $A = 1 + L$.

Index. D_ε is invertible, so its index is 0, in agreement with Theorem 6.6(i), since W_ε is a configuration in the class of V . $\square$

The loss of one factor A is sharp in these norms: a stationary field k_v multiplied by a cutoff of slope δ_L has norm ratio c/δ_L . It is harmless because the approximation error of W_ε is a positive power of ε (Section 8).

7.8. The scalar Laplacian on the glued domain. Let g_ε be the complete conformal metric on the gauge domain of X_ε used in Section 7.7. Choose a smooth $\lambda_\varepsilon > 0$ equal to $\mathfrak{s}$ on the compression, equal to one on the unchanged pieces and away from the ends of the transition cores, interpolated on fixed buffers and constant in the normal direction at the boundary, and put $\widehat{g}_\varepsilon = \lambda_\varepsilon^{-2}g_\varepsilon$, $d\mu_\varepsilon = \lambda_\varepsilon^2 dV_{\widehat{g}_\varepsilon}$. On the compression $\widehat{g}_\varepsilon = dT^2 + g_C$ with $dT = dt/\mathfrak{s}$; the rescaled metric has uniformly bounded local geometry, and its large parts are product cylinders with compact cross-section joined through finitely many fixed

cores. In dimension four the physical norm of a one-form is $\|a\|_{L^2(g_\varepsilon)}^2 = \int |a|_{\hat{g}_\varepsilon}^2 d\mu_\varepsilon$, which is $\int |\mathfrak{s}(p, b)|^2 dt dV_C$ on the compression, with $p = \mathfrak{s}\psi$.

Write $H_{\text{uloc}}^j(\hat{g}_\varepsilon)$ for the supremum of the H^j norms over a uniform cover by unit patches in the rescaled metric, with covariant derivatives taken using the reference connection.

Lemma 7.16 (scalar inverse). *Let $\mathcal{L}_\varepsilon = \lambda_\varepsilon^2 d_{\mathbb{A}}^{*, g_\varepsilon} d_{\mathbb{A}}$ with Neumann condition, for a reference $\mathbb{A}$ within a fixed small C^k -distance of W_ε in the rescaled metric. Then $\|\chi\|_{L^2(\mu_\varepsilon)} \leq C_P \|d_{\mathbb{A}}\chi\|_{L^2(\mu_\varepsilon)}$ and $\|\mathcal{L}_\varepsilon^{-1}f\|_{H_{\text{uloc}}^{m+1}(\hat{g}_\varepsilon)} \leq C_m \|f\|_{H_{\text{uloc}}^{m-1}(\hat{g}_\varepsilon)}$, with constants independent of ε ; the same holds with the weights of the original ends.*

Proof. The quadratic form of $\mathcal{L}_\varepsilon$ is $\int |d_{\mathbb{A}}\chi|^2 d\mu_\varepsilon$. On each product piece of the compression the Poincaré inequality is the spatial Neumann gap, uniform over P because every flat connection on C is irreducible, integrated in time. The transition cores have compact Neumann inequalities once enlarged to contain an irreducible slice (on each core λ_ε is comparable with one), the tails of the original ends have their spatial gaps, and a fixed compact core of the unchanged gauge domain, enlarged to contain an irreducible slice, has its Neumann inequality. Each of these regions R_k therefore carries an inequality $\|\chi\|_{L^2(R_k)} \leq C \|d_{\mathbb{A}}\chi\|_{L^2(R_k)}$ with no mean-value term, since $\mathbb{A}$ is irreducible on every slice, and the regions cover with overlap at most N_0 , so

$$\|\chi\|_{L^2(\mu_\varepsilon)}^2 \leq \sum_k \|\chi\|_{L^2(R_k)}^2 \leq C \sum_k \|d_{\mathbb{A}}\chi\|_{L^2(R_k)}^2 \leq CN_0 \|d_{\mathbb{A}}\chi\|_{L^2(\mu_\varepsilon)}^2.$$

For the inverse, cover by unit rescaled patches Q_j ; the number of patches in a distance shell grows polynomially. Conjugating the form by $e^{\gamma d_j}$, with d_j a smoothed distance from Q_j and γ small, and using the variational inverse and local elliptic regularity gives

$$\|\mathcal{L}_\varepsilon^{-1}f_j\|_{H^{m+1}(Q_\ell)} \leq C_m e^{-\gamma d(Q_\ell, Q_j)} \frac{\lambda_\varepsilon(Q_j)}{\lambda_\varepsilon(Q_\ell)} \|f_j\|_{H^{m-1}(Q_j)}$$

for f_j supported in Q_j . Since $\partial_T(\mathfrak{s}^{-1}) = -h$, the function λ_ε^{-1} is uniformly Lipschitz in the rescaled distance and bounded below, so the ratio is at most $C(1 + d(Q_\ell, Q_j))$, which is absorbed by half the exponential. Writing $f = \sum_j f_j$ with f_j supported in Q_j and $\|f_j\|_{H^{m-1}} \leq C\|f\|_{H^{m-1}(Q'_j)}$ for a fixed enlargement Q'_j ,

$$\begin{aligned} \|\mathcal{L}_\varepsilon^{-1}f\|_{H^{m+1}(Q_\ell)} &\leq C_m \sum_j e^{-\gamma d(Q_\ell, Q_j)/2} \|f_j\|_{H^{m-1}(Q_j)} \\ &\leq C_m \left(\sup_{n \geq 0} \#\{j : n \leq d(Q_\ell, Q_j) < n+1\} e^{-\gamma n/4} \right) \sum_{n \geq 0} e^{-\gamma n/4} \|f\|_{H_{\text{uloc}}^{m-1}}, \end{aligned}$$

and the supremum is finite, uniformly in ℓ and ε , because the shell counts grow polynomially in n . This is the inverse bound; uniqueness in the uniformly local space follows from the quadratic identity with a small decreasing exponential weight. The weights of the original ends, whose logarithmic gradients are smaller than γ , are handled in the same way. $\square$

8. THE COLLAPSE THEOREM: GLUING AND SURJECTIVITY

Throughout, V is a regular index-zero solution on X_0 with limits $q_\pm$ at the new ends, $L = \log(1/\varepsilon)$ and $\mathbf{A} = 1 + L$. Constants C, c depend on the fixed data and on the finite set $\mathcal{M}^0(X_0)$ but not on ε , on neck lengths, or on the minimum of the compression scale.

The section has two parts. Existence (Section 8.1 and Proposition 8.3): the approximate solution W_ε is corrected by a contraction in strong norms, whose inverse loses the factor $\mathbf{A}^{m+1}$,

so the solution is unique only in a ball of radius shrinking like a power of $1/A$. Surjectivity (Proposition 8.12): an arbitrary solution near V is first shown, by cutting it along the necks and comparing the pieces with the components, to lie within a positive power of ε of W_ε , by estimates that lose no logarithm; only then does it enter the uniqueness ball.

8.1. Strong norms and the strong inverse. *Strong norms.* On the small compression, with w the fixed end weight (equal to one on the region $N_I \cap X_\varepsilon$, which contains the part where X_ε differs from X_0 , Definition 6.2(iv)), put

$$\|f\|_{Y_\varepsilon^m}^2 = \sum_{i+r \leq m} \|\nabla_t^i (w s^{1/2} f)\|_{L_t^2 H^r(C)}^2,$$

$$\|z\|_{X_\varepsilon^m}^2 = \|wv\|_{H_t^m}^2 + \sum_{i=0}^m \|w s^{-1/2} \nabla_t^i U_\perp\|_{L_t^2 H^{m+1-i}(C)}^2 + \|w s^{1/2} \nabla_t^{m+1} U\|_{L_t^2 L^2(C)}^2,$$

and on the other gauge regions and the section weighted H^m and H^{m+1} , patched by cutoffs of bounded overlap, with the free limiting vectors at the original ends added. For $m = 0$ this is equivalent to X_ε . On the region $N_I \cap X_\varepsilon$

$$(8.1) \quad \|U\|_{H^m} + \|s^{1/2} U\|_{H^{m+1}} + \|s^{-1/2} U_\perp\|_{H^m} + \|s^{-1/2} \nabla_C U_\perp\|_{H^m} \leq C_m \|z\|_{X_\varepsilon^m},$$

and Sobolev embedding on unit patches gives unscaled C^1 bounds for the spatial coefficients, p and the section variation.

Proposition 8.1 (strong inverse). $D_\varepsilon: X_\varepsilon^m \rightarrow Y_\varepsilon^m$ is invertible with $\|D_\varepsilon^{-1}\| \leq C_m A^{m+1}$.

Proof. Let $\mathcal{V}$ be a global tangential differential operator equal to ∇_t where $s \leq s_0/2$ on the compression, extended over the adjacent section collar by the corresponding boundary derivative, cut off where $s \geq s_0$ and in the collar interior, and vanishing on the other regions and on the supports of all perturbations. By Lemma 7.11, $\mathcal{V}$ preserves the linear domain. We prove by induction on $j \leq m$ that

$$(8.2) \quad \|\mathcal{V}^j z\|_{X_\varepsilon} \leq C_j A^{j+1} \|D_\varepsilon z\|_{Y_\varepsilon^j}.$$

For $j = 0$ this is Proposition 7.15. At fixed ε the region where the geometry changes is compact, so $\mathcal{V}^j D_\varepsilon^{-1} f$ lies in X_ε by fixed-geometry regularity [DFL21, Theorem 4.17 and Remark 4.19], or by difference quotients; the estimates below then bound it. Commuting D_ε through $\mathcal{V}^j$ gives

$$D_\varepsilon(\mathcal{V}^j z) = \mathcal{V}^j D_\varepsilon z + \sum_{i < j} \sum_{k=0}^{j-i} s^{-1} a_{ijk}(t) \mathcal{B}_t^{[k]} \mathcal{V}^i U + \sum_{i < j} b_{ij} \mathcal{V}^i z,$$

where $\mathcal{B}_t^{[0]} = \mathcal{B}_t$, all coefficients are bounded, and b_{ij} collects the remaining terms: those of order zero, and the first-order terms in ν produced by the section-collar cutoff in $\mathcal{V}$, which are bounded by $\|\mathcal{V}^i \nu\|_{H^1} \leq \|\mathcal{V}^i z\|_X$. Differentiating the scale produces the terms with $k = 0$, even when the background is constant. By Lemma 7.11, for every $k \geq 0$ the operator $\mathcal{B}_t^{[k]}$ annihilates the harmonic fields and $\|\mathcal{B}_t^{[k]} U\| \leq C \|\mathcal{B}_t U\|$, so no factor s^{-1} multiplies a slow term and

$$\|s^{-1} a_{ijk} \mathcal{B}_t^{[k]} \mathcal{V}^i U\|_{Y_\varepsilon} \leq C \|s^{-1/2} \mathcal{B}_t \mathcal{V}^i U\| \leq C \|\mathcal{V}^i z\|_{X_\varepsilon}.$$

Applying Proposition 7.15 to $\mathcal{V}^j z$, which costs one factor $\mathbf{A}$, and the induction hypothesis to the terms with $i < j$,

$$\|\mathcal{V}^j z\|_{X_\varepsilon} \leq C\mathbf{A} \left(\|D_\varepsilon z\|_{Y_\varepsilon^j} + \sum_{i < j} C_i \mathbf{A}^{i+1} \|D_\varepsilon z\|_{Y_\varepsilon^i} \right) \leq C_j \mathbf{A}^{j+1} \|D_\varepsilon z\|_{Y_\varepsilon^j},$$

which is (8.2). The spatial derivatives cost nothing: on the compression they are recovered from the equation by (7.12); for the section, the derivative in the direction normal to the seam is recovered from the Cauchy–Riemann equation and the tangential derivative, and the seam trace of $\nabla_t^j \nu$ is controlled through the harmonic matching, which ∇_t preserves by Lemma 7.11; both bounds are uniform in $\mathfrak{s}$. In the regions where the geometry is uniformly nondegenerate, ordinary mixed elliptic estimates give H^{m+1} control by $\|D_\varepsilon z\|_{Y^m} + \|z\|_{X_\varepsilon}$. Summing gives the bound with $j = m$. $\square$

The exact matching chart. Along $q(t)$, let $B_t(v)$ be the flat connection in the spatial Neumann Coulomb slice of $B_t = B_{q(t)}$ with bulk harmonic coordinate v : $F_{B_t(v)} = 0$, $d_{B_t}^*(B_t(v) - B_t) = 0$, vanishing normal component, $\ell_t(0, B_t(v) - B_t) = v$, $D_v B_t(0) = j_t$. Let e_t be a section chart whose restriction to $\mathcal{L}_t$ is the boundary class of $B_t(v)$. For $U = j_t v + U_\perp$ with $U_\perp = (p, w)$ put

$$\Phi_t(U) = B_t + (B_t(v) - B_t + T_{t,v}w + \mathcal{R}_t(v, w)) + (\Psi_t + p/\mathfrak{s}) dt,$$

where $T_{t,v}$ is the transport of the complementary field to $B_t(v)$ by the surface Hodge decompositions on the parallel surfaces of a fixed collar of ∂C , blended to the identity in the interior and leaving the normal component unchanged, with the harmonic summands identified so that the class of the trace $(T_{t,v}w)|_\Sigma$ is $de_t(v) \tau_t w$; and $\mathcal{R}_t(v, w)$ is the surface correction defined below, which is tangential, is supported in the collar, makes the trace flat with class $e_t(v + \tau_t w)$, and satisfies $\mathcal{R}_t(v, 0) = 0$ and $D_w \mathcal{R}_t(v, 0) = 0$. On the section use e_t at the seam extended into a collar. In this chart the matching and flatness conditions hold exactly, and the mixed operator $\mathfrak{B}_t(v, U_\perp)$ satisfies

$$(8.3) \quad \mathfrak{B}_t(v, 0) = 0, \quad D_{U_\perp} \mathfrak{B}_t(0, 0) = \mathcal{B}_t,$$

so the quadratic part vanishes on the slow directions.

The surface correction. Let $[0, r_1) \times \partial C$ be a product collar of ∂C with normal coordinate r , write $\Sigma_r = \{r\} \times \partial C$, and let χ be a cutoff of r equal to 1 near 0 and to 0 for $r \geq r_1/2$. For given t, v and r let $\alpha = \alpha_{t,v,r}$ be the restriction of $B_t(v)$ to Σ_r . It is flat and, by (C1), irreducible on every component, so $\Delta_\alpha = d_\alpha d_\alpha^*$ is invertible on two-forms and $\ker d_\alpha^*$ on one-forms is the sum of the harmonic space $\mathcal{H}_\alpha$ and $\text{im } d_\alpha^*$. Let $\mathfrak{i}_{t,v,r}: T_{q(t)}\mathbf{M} \rightarrow \mathcal{H}_\alpha$ be the isomorphism given by $de_t(v)$ and the identification of the cohomology of Σ_r with that of Σ by the flat connection $B_t(v)$ on the collar (on the torus faces $\mathcal{H}_\alpha = 0$). For a tangential one-form ϕ on Σ_r small in $H^{1/2}$ and $\vartheta \in T_{q(t)}\mathbf{M}$ small, let $\zeta = \mathcal{S}_\alpha(\phi, \vartheta)$ be the small solution in $\ker d_\alpha^*$ of the *surface flatness equation*

$$(8.4) \quad d_\alpha \zeta + \frac{1}{2}[(\phi + \zeta) \wedge (\phi + \zeta)] = 0, \quad \text{harmonic part of } \zeta = \mathfrak{i}_{t,v,r} \vartheta.$$

Equivalently $\zeta = \mathfrak{i}_{t,v,r} \vartheta + \mathbf{K}_\alpha(\phi + \zeta, \phi + \zeta)$ with $\mathbf{K}_\alpha(a, b) = -\frac{1}{2} d_\alpha^* \Delta_\alpha^{-1} [a \wedge b]$, a bilinear operator of order -1 on Σ_r ; since $[\cdot \wedge \cdot]$ maps $H^{1/2} \times H^{1/2}$ to L^2 on a surface, the contraction principle gives $\mathcal{S}_\alpha$, smooth in (ϕ, ϑ) and in the parameters, with values in H^1 and $\mathcal{S}_\alpha(0, 0) = 0$. By construction $F_{\alpha+\phi+\zeta} = d_\alpha \phi$; in particular $\alpha + \phi + \mathcal{S}_\alpha(\phi, \vartheta)$ is flat whenever $d_\alpha \phi = 0$. Let $\phi_{t,r}$ be the tangential part of $T_{t,v}w$ on Σ_r . On $\Sigma_0 = \partial C$ it is closed, because w has closed traces

and $T_{t,v}$ preserves closedness. Let $\vartheta = \vartheta_t(v, w)$ be the small solution of the finite-dimensional equation

$$(8.5) \quad [\alpha_{t,v,0} + \phi_{t,0} + \mathcal{S}_{\alpha_{t,v,0}}(\phi_{t,0}, \vartheta)] = e_t(v + \tau_t w),$$

which exists by the implicit function theorem, since $\vartheta \mapsto [\alpha_{t,v,0} + \mathcal{S}_{\alpha_{t,v,0}}(0, \vartheta)]$ is the harmonic chart of $\mathbf{M}$ at $e_t(v)$, with derivative $de_t(v)$ at 0; the class map is smooth on flat connections of class $H^{1/2}$ near $\alpha_{t,v,0}$. At $w = 0$ the derivative of the left side in the class equation is $de_t(v)(\tau_t \dot{w} + \dot{\vartheta})$, whereas that of the right side is $de_t(v)\tau_t \dot{w}$. Thus $\vartheta_t(v, 0) = 0$ and $D_w \vartheta_t(v, 0) = 0$ for every v .

Extension in the normal direction. The harmonic parameter is a function of the boundary trace of w . Its extension must respect both the time and the normal derivatives in X_ε^m . Choose a nonnegative $\rho \in C_c^\infty((-1, 1))$ with integral one. At a fixed scale $s \in (0, 1]$, in a smooth frame of $T_{q(t)}\mathbf{M}$, set

$$(\mathcal{E}_s \vartheta)(t, r) = \int \rho(u) \vartheta(t - sru) du, \quad r \geq 0.$$

Its boundary value is $\vartheta(t)$. For a varying scale, choose unit time intervals, a subordinate partition $\sum \psi_j = 1$, and larger intervals on which $\mathfrak{s}/\mathfrak{s}(j)$ and its logarithmic derivatives are bounded. Extend $\psi_j \vartheta$ by $\mathcal{E}_{\mathfrak{s}(j)} \vartheta$ and sum, with a time cutoff equal to one on the slightly enlarged support of each summand. The collar radius is fixed small enough that all translates used in this formula remain in the larger interval. Denote the result by $\mathcal{E}_s \vartheta$. The overlap and cutoff constants are independent of the minimum scale and of the number of intervals. Define the exact correction by

$$(8.6) \quad \Theta(t, r) = (\mathcal{E}_s \vartheta(v, w))(t, r), \quad \mathcal{R}(v, w)(t, r) = \chi(r) \mathcal{S}_{\alpha_{t,v,r}}(\phi_{t,r}, \Theta(t, r)).$$

At $r = 0$ this is the solution of (8.5). Thus the trace is flat with the prescribed class. The correction and its first derivative in w vanish at $w = 0$, so (8.3) and the linearized operator are unchanged.

Here is the estimate for this extension. On a constant-scale patch, write $\mathcal{X}_s^m$ for the complementary part of X_ε^m , with end weight one. The trace norm $\mathcal{T}_s^m$ is given, in local time Fourier coordinates and surface spectral coordinates, by

$$\|f\|_{\mathcal{T}_s^m}^2 = \int \sum_\lambda s^{-1} (1 + \xi^2 + \lambda^2)^m (1 + \lambda^2 + s^2 \xi^2)^{1/2} |\widehat{f}(\xi, \lambda)|^2 d\xi.$$

Equivalent norms are obtained with a finite surface atlas and smooth cutoffs. For a finite-dimensional harmonic coefficient omit λ . The trace and extension estimates are

$$(8.7) \quad \|w|_{\partial C}\|_{\mathcal{T}_s^m} \leq C_m \|w\|_{\mathcal{X}_s^m}, \quad \|\chi \mathcal{E}_s \vartheta\|_{\mathcal{X}_s^m} \leq C_m \|\vartheta\|_{\mathcal{T}_s^m}.$$

For the first, apply the normal one-dimensional trace inequality with parameter $(1 + \lambda^2 + s^2 \xi^2)^{1/2}$ to the time and surface transform. Tangential differentiation gives the factor $(1 + \xi^2 + \lambda^2)^m$. The resulting bulk norm is equivalent to the displayed complementary norm in X_ε^m : terms containing $s^2 \xi^2$ and a spatial derivative are dominated by its spatial sum, and the remaining highest time term is $s \|\partial_t^{m+1} w\|^2$. For the second estimate the multiplier is $\widehat{\rho}(sr\xi)$. Its derivatives decrease rapidly for $sr|\xi| \gg 1$. Integrating each normal derivative in r gives

$$\int_0^{r_1} |\partial_r^j (\chi(r) \widehat{\rho}(sr\xi))|^2 dr \leq C_j (1 + s|\xi|)^{2j-1},$$

with the same assertion for $j = 0$. Since $s \leq 1$, inserting the time derivatives and summing proves (8.7). In particular the top time derivative is estimated through a half-order boundary norm, not an L^2 trace of that derivative.

Lemma 8.2 (quadratic estimates in the matching chart). *Fix $m \geq 4$ and a compact smooth family of reference paths q . In the chart (8.6), write $\mathcal{F}_\varepsilon(W_\varepsilon + z) = f_\varepsilon + D_\varepsilon z + N_\varepsilon(z)$. For z, z' in a sufficiently small ball of X_ε^m , the strong ball, with constants independent of ε ,*

$$(8.8) \quad \|N_\varepsilon(z) - N_\varepsilon(z')\|_{Y_\varepsilon^m} \leq C_m(\|z\|_{X_\varepsilon^m} + \|z'\|_{X_\varepsilon^m})\|z - z'\|_{X_\varepsilon^m}.$$

Let $\mathbf{d}(z)$ be the supremum of $\|v\|_{C^0} + \|U_\perp\|_{C^0}$ on compression patches, of $\|a\|_{C^0}$ on other gauge patches, and of $\|\nu\|_{C^0}$ on section patches. On a small smooth coefficient neighbourhood one also has

$$(8.9) \quad \|N_\varepsilon(z)\|_{Y_\varepsilon} \leq C\mathbf{d}(z)\|z\|_{X_\varepsilon}.$$

If z_* is in the strong ball and z is smooth with small coefficient size, then

$$(8.10) \quad \|N_\varepsilon(z) - N_\varepsilon(z_*)\|_{Y_\varepsilon} \leq C(\mathbf{d}(z) + \|z_*\|_{X_\varepsilon^m})\|z - z_*\|_{X_\varepsilon}.$$

The same local assertions hold on the component and exterior domains.

Proof. We first describe the surface estimate that is needed in addition to (8.7). Freeze t, v and use the harmonic frame $\mathbf{i}_{t,v,0}$. Thus the prescribed linear class is $h = \mathbf{i}_{t,v,0}^{-1} H_\alpha \phi = \tau_t w$, and $A(v, h)$ is the flat slice representative of $e_t(v + h)$ relative to $\alpha = B_t(v)|_\Sigma$. We write $A(h)$ while v is fixed. For differentiation, use the following gauge equation also for nonclosed ϕ , with $h = \mathbf{i}_{t,v,0}^{-1} H_\alpha \phi$. It defines an extension of the harmonic class parameter off the closed-form subspace; on that subspace it equals the parameter in (8.5). The class equation is implemented by solving

$$d_\alpha^*(g^*A(h) - \alpha - \phi) = 0, \quad g = \exp \eta.$$

For a fixed $p > 2$ this is an equation from $W^{1,p}$ to $W^{-1,p}$, with invertible derivative $d_\alpha^* d_\alpha$ at zero. Small C^0 norm of ϕ gives a small $W^{1,p}$ solution η ; hence η is small in C^0 . The differentiated equation, viewed from H^1 to H^{-1} , gives

$$\|D\eta[\dot{\phi}]\|_{H^1} \leq C\|\dot{\phi}\|_{L^2}.$$

Indeed its leading coefficient is close in C^0 to that of the covariant Laplacian, and its remaining coefficients are small in L^p . Hölder's inequality and the surface Sobolev inequality absorb those terms using the irreducible scalar gap.

Now $\vartheta = H_\alpha(g^*A(h) - \alpha - \phi)$ in the chosen harmonic frame. Harmonic projection kills the exact part of $D\eta$, and the linear terms $\dot{h} - H_\alpha \dot{\phi}$ cancel. The remaining terms contain ϕ , $g - 1$, or the nonlinear part of the flat slice. Pairing with smooth harmonic forms and using the preceding estimate gives

$$(8.11) \quad |D_\phi \vartheta[\dot{\phi}]| \leq C\|\phi\|_{C^0}\|\dot{\phi}\|_{L^2}.$$

For example, the commutator term is bounded by $C\|g^*A(h) - \alpha\|_{L^2}\|D\eta[\dot{\phi}]\|_{L^2}$. The quantitative differentiated estimates, in the fixed frame, are

$$(8.12) \quad \begin{aligned} |D_\phi^2 \vartheta[\psi_1, \psi_2]| &\leq C\|\psi_1\|_2\|\psi_2\|_2, \\ |D_v D_\phi \vartheta[b, \psi]| &\leq C|b|\|\phi\|_2\|\psi\|_2, \\ |D_v \vartheta[b]| &\leq C|b|\|\phi\|_2^2. \end{aligned}$$

To see these bounds, differentiate the weak gauge equation, retaining it in divergence form. Each differentiated gauge term has at most one derivative of a gauge parameter. A first variation lies in H^1 ; products of an undifferentiated H^1 variation and an L^2 differentiated variation lie in L^r for every $r < 2$. The inverse weak Laplacian gives $W^{1,r}$ for the next gauge variation. Surface Sobolev embedding and Hölder's inequality then bound its harmonic pairings in L^1 . Repeating this argument at any fixed finite order gives bounded multilinear harmonic pairings, choosing the intermediate exponents sufficiently close to two. No derivative of ϕ is used in these estimates. Differentiating the smooth finite-dimensional flat slice and harmonic projection adds only terms with the same bounds. The identities $\vartheta(v, 0) = D_\phi \vartheta(v, 0) = 0$ eliminate the constant and linear terms in every v -derivative. Integrating the bounded $D_v D_\phi^2 \vartheta$ gives the last two estimates in (8.12), and integrating these differential bounds gives their difference versions.

Here is an explicit strong trace bound. On a constant-scale patch put $Z = \|z\|_{X_s^m}$. The trace theorem and the linear matching identity give

$$\|\phi\|_{H_t^m L_\Sigma^2} \leq C\sqrt{s} Z, \quad \|\phi\|_{H_t^{m+1/2} L_\Sigma^2} + \|v\|_{H_t^{m+1/2}} \leq CZ.$$

For the second bound on v , use $v = \Pi\nu|_\Sigma - \Pi\tau_t w$: the section has H^{m+1} regularity and its trace has $H^{m+1/2}$ regularity, while the harmonic part of the complementary trace is bounded by (8.7). The transport $T_{t,v}$ obeys the same estimates by the product rule. The one-dimensional tame product inequalities, applied to the differentiated harmonic pairings above, now give

$$\begin{aligned} \|\vartheta\|_{H_t^m} &\leq C\|\phi\|_{H_t^m L_\Sigma^2}^2 (1 + \|v\|_{H_t^m}), \\ \|\vartheta\|_{H_t^{m+1/2}} &\leq C\|\phi\|_{H_t^m L_\Sigma^2} (\|\phi\|_{H_t^{m+1/2} L_\Sigma^2} + \|\phi\|_{H_t^m L_\Sigma^2} \|v\|_{H_t^{m+1/2}}). \end{aligned}$$

Since $\|\vartheta\|_{\mathcal{T}_s^m}$ is equivalent to $s^{-1/2}\|\vartheta\|_{H_t^m} + \|\vartheta\|_{\dot{H}_t^{m+1/2}}$, these prove $\|\vartheta\|_{\mathcal{T}_s^m} \leq C\sqrt{s} Z^2$ in a fixed small ball. The differentiated inequalities give

$$\|\vartheta(z) - \vartheta(z')\|_{\mathcal{T}_s^m} \leq C\sqrt{s} (\|z\|_{X_s^m} + \|z'\|_{X_s^m}) \|z - z'\|_{X_s^m}.$$

These are the weighted class-parameter estimates used below.

The spatial part of $\mathcal{S}_\alpha$ is controlled by its formula $i\Theta + K_\alpha(\phi + \zeta, \phi + \zeta)$. The operator K_α gains one surface derivative. Time and normal derivatives commute through it with bounded lower-order commutators; the coefficients depend smoothly on the fixed family and on v . For $m \geq 4$, the bulk H^m algebra inequality and the boundary $H^{m-1/2}$ algebra inequality control all products below the highest order. At the highest time order, use (8.11), its differentiated equation, and (8.7). The half-order time seminorm is estimated by differences before differentiating the extension. This gives the tame quadratic bound for $\mathcal{R}$ in $\mathcal{X}_s^m$. The same argument applies to the difference of two corrections. No extra top-order boundary trace is taken.

At first order this argument uses only small coefficients. For clarity, the harmonic trace norm with $m = 0$ is equivalent to $s^{-1}\|\vartheta\|_2^2 + \|\vartheta\|_{\dot{H}_t^{1/2}}^2$. Apply (8.11) to the differences in the second term. Terms involving a difference of v are treated by splitting the difference integral at $|t - u| = s$. On the near part, $|v(t) - v(u)|^2 \leq |t - u| \int_u^t |v'|^2$ supplies the weighted quantity $s\|v'\|_2^2$; on the far part use $2\|v\|_\infty$ and the $s^{-1}\|\phi\|_2^2$ term. The two factors of ϕ in a v -derivative ensure that the undifferentiated factor is a small coefficient. This proves the absolute first-order estimate for the boundary correction. Expanding about a strong reference puts its differentiated coefficients on that reference and treats the unknown remainder quadratically; it gives the asymmetric version by the same difference-integral argument. More

explicitly, write $y = z - z_*$ and use the integral Taylor formula at z_* . Terms quadratic in its complementary remainder are bounded by $Cd(y)\|y\|_{X_s}$. In the linear and mixed terms, $\|\phi_*\|_{C_t^1 C_\Sigma^0} \leq C\sqrt{s}a$ and $\|v_*\|_{C^1} \leq Ca$, where $a = \|z_*\|_{X_s^m}$. Multiplication by these coefficients is bounded on the half-order time norm. A term linear in the slow remainder has coefficient $O(sa^2)$ by (8.12); this cancels the $s^{-1/2}$ in the low-frequency trace norm, and the high-frequency part is controlled by the matching bound $\|v_y\|_{H^{1/2}} \leq C\|y\|_{X_s}$. The near/far split treats the remaining mixed differences. Consequently

$$\|\vartheta(z)\|_{\mathcal{T}_s^0} \leq Cd(z)\|z\|_{X_s}, \quad \|\vartheta(z) - \vartheta(z_*)\|_{\mathcal{T}_s^0} \leq C(d(z) + a)\|z - z_*\|_{X_s}.$$

The other chart terms have the same estimates directly. In the surface Hodge transport, the principal symbols do not change with the reference flat connection. Thus $T_{t,v} - 1$ has order at most -1 on the fixed surface and is bounded on C^0 , with norm $O(|v|)$; its finite-dimensional harmonic part has the same bound. The spatial operator vanishes on the exact flat family $B_t(v)$. Every singular remaining term has a complementary factor, measured with $s^{-1/2}$ in X_s . A differentiated slow factor in the temporal terms carries $s^{1/2}$: in particular $s^{1/2}|v'w|$ is bounded by $\|w\|_\infty$ times the weighted temporal norm of v . The terms from $T_{t,v} - 1$, the scalar row, and the local bracket are bounded in this way. The pure section operator has the usual quadratic estimate in its fixed chart. These estimates prove all three inequalities at constant scale. Finally freeze the weights on the unit patches used above and sum. Bounded logarithmic derivatives, bounded overlap and the support of ρ give the same constants for the varying-scale extension. $\square$

The chart is locally invertible also in uniformly local rescaled Hölder norms: use $T = t/s$ on a constant-scale patch. The extension is then $\int \rho(u)\vartheta(T - ru)du$, bounded in C^α ; the surface constructions and harmonic projections are smooth in these norms, and the correction has zero derivative at zero. The inverse function theorem gives a uniform small neighbourhood. Smooth fields with small C^0 coefficients and uniformly bounded higher rescaled derivatives enter it by interpolation. This is the neighbourhood used for the unknown solutions in the proof of surjectivity. First differentiate the inverse equation in time and in the surface directions. The leading operator remains a small perturbation of the identity on the local Hölder space. Then recover normal regularity from the explicit extension: each normal derivative of $\int \rho(u)\vartheta(T - ru)du$ is an integral of a time derivative of ϑ , already controlled by the first step. This triangular induction preserves every fixed higher rescaled local bound; a normal derivative of the extension is not a lower-order commutator. At a fixed positive ε the changed region is compact, so these smooth local bounds imply finite first-order norm there. On the original ends subtract the free stationary limits first. The extension has uniformly bounded time support and is bounded on the fixed weighted spaces; the trace estimates and the small-coefficient inverse give the same weighted bounds for the remainders. Thus the inverse preserves the end limits and finite X_ε norm; this assertion uses the trace estimates as well as local Hölder invertibility.

8.2. The approximate solution and existence. Let $R = \varepsilon^{1/4}$. Near each endpoint, for $x \leq R$, replace V by its limiting flat pair $(B_{q_\pm}, q_\pm)$ using a cutoff of fixed slope in the logarithmic coordinate, performed in the exact matching chart about the flat pair, so that the matching slices remain flat; since V converges to its limits at rate $\beta > 0$ (Proposition 6.16(iii)), this changes V by $O(R^\beta)$ in every C^k norm. Over $I \times C$ use the flat lift $B_{q_R(t)}$ in the covariant Neumann slice of the boundary values $q_R(t)$ of the truncated section, with temporal component chosen so that $\partial_t B - d_B \Psi = h_{\dot{q}_R}$. On the necks and transitions use the flat constant pair. The only errors are the truncation error and the horizontal velocity of the flat lift, whose

weighted size is $\mathfrak{s}^{1/2}|\dot{q}_R| \leq C\varepsilon^{1/2}r^{\beta-1/2}$ on the compression; since ∇_t preserves the harmonic extension, the same bound holds for all its derivatives. Integrating,

$$(8.13) \quad \|\mathcal{F}_\varepsilon(W_\varepsilon)\|_{Y_\varepsilon^m} \leq C_m e_\varepsilon, \quad e_\varepsilon = R^\beta + \sqrt{\varepsilon K_\beta(R)}, \quad K_\beta(R) = 1 + \int_R^{r_0} r^{2\beta-2} dr,$$

and $e_\varepsilon \leq C\varepsilon^{\sigma_1}$ for some $\sigma_1 > 0$.

Proposition 8.3 (existence). *Let $\varrho \leq r_0$ be a radius below which the summed quadratic and difference estimates hold, and with $\varrho \leq 1/(4C_q)$, where C_q is the constant of the quadratic estimate. For ε small there is a unique solution $\mathcal{G}_\varepsilon(V) = W_\varepsilon + z^*$ with $\|z^*\|_{X^m} \leq \varrho/(C_m A^{m+1})$, and in fact $\|z^*\|_{X^m} \leq C_m A^{m+1} e_\varepsilon$. It is regular.*

Proof. In the exact matching chart the equation is $\mathcal{F}_\varepsilon(W_\varepsilon) + D_\varepsilon z + N_\varepsilon(z) = 0$. By Proposition 8.1, the quadratic estimate and (8.13), the map $z \mapsto -D_\varepsilon^{-1}(\mathcal{F}_\varepsilon(W_\varepsilon) + N_\varepsilon(z))$ is a contraction of the ball of radius $\varrho/(C_m A^{m+1})$ for small ε : its Lipschitz constant is at most $2C_q \varrho \leq \frac{1}{2}$, and it maps the ball to itself since $A^{2m+2} e_\varepsilon \rightarrow 0$. For regularity, the asymmetric quadratic bound gives $\|DN_\varepsilon(z^*)\|_{X \rightarrow Y} \leq C a$ with $a = \|z^*\|_{X^m} \leq C A^{m+1} e_\varepsilon$, and $\|D_\varepsilon^{-1}\| \leq C A$, so $A a \leq C A^{m+2} e_\varepsilon \rightarrow 0$ and the linearization at $W_\varepsilon + z^*$ is invertible. $\square$

8.3. The endpoint necks. On each endpoint neck, at a time s , the restriction of a configuration to the slice is a triple $c = (B_f, B_c, \nu)$ of a connection on the fixed copy of C , a connection on the collapsing copy, and a path in the moduli space of Σ with $\nu(0)$ and $\nu(\pi)$ the restrictions of B_f and B_c . The critical family is $\mathcal{F} = \{(B_q, B_q, q) : q \in P\}$. Gluing the triple gives one connection on the double of C along Σ with a collar of length π inserted, and the mixed action becomes the Chern–Simons functional of that three-manifold. For $C = C_1$ it has two torus boundary components carrying the flat condition; since the flat connection on the odd bundle over T^2 is isolated with $H^1(T^2; \text{ad}) = 0$, the functional is still single valued on a gauge-invariant neighbourhood of $\mathcal{F}$, and its critical set there is $\mathcal{F}$, of Morse–Bott type. Since $\mathcal{F}$ is connected, the action is constant on it, and we normalize it to vanish there.

Proposition 8.4 (neck decay). *(i) There are $\delta, C, \mathfrak{s}_2 > 0$ such that for $\mathfrak{s} \in (0, \mathfrak{s}_2]$ and all triples within δ of $\mathcal{F}$ in the H^1 norm of the fixed metric g_C on both branches, $|a(c)| \leq C \|\nabla_{\mathfrak{s}} a(c)\|_{\mathfrak{s}}^2$, where the gradient is taken in the metric weighting the collapsing component by $\mathfrak{s}$.*

(ii) Let $c(s)$, $s \in [s_0, s_1]$, solve the neck equation with $\mathfrak{s}(s) = \mathfrak{s}_b e^{\pm s} \leq \mathfrak{s}_2$, all slices within δ of $\mathcal{F}$, and energy e . Let $p(s) \in P$ be the projection determined by the flat projection of the fixed branch B_f . Then the energy on $[s_0 + r, s_1 - r]$ is at most $e e^{-\gamma r}$ and

$$\text{length}_P(p|_{[s_0+r, s_1-r]}) \leq C \sqrt{e} e^{-\gamma r/2},$$

with C, γ independent of $s_1 - s_0$ and of $\mathfrak{s}_b$.

Proof. (i) Let $\mathfrak{s}_2$ be the constant $\mathfrak{s}_0$ of Proposition 7.9. The action $a(c)$ does not depend on $\mathfrak{s}$, and the squared gradient norm $\|F_{B_f}\|^2 + \mathfrak{s}^{-1}\|F_{B_c}\|^2 + \|\partial_\theta \nu\|^2$ (up to the slice terms) is nonincreasing in $\mathfrak{s}$. The neighbourhood in the statement does not depend on $\mathfrak{s}$, so the inequality at $\mathfrak{s} = \mathfrak{s}_2$ implies it for all $\mathfrak{s} \leq \mathfrak{s}_2$. At $\mathfrak{s} = \mathfrak{s}_2$ write c as the image of x under the exact matching chart (8.3) centred at a point $c_0 \in \mathcal{F}$, with x in the Coulomb slice and in the linear domain of $A_{\mathfrak{s}_2}$, and orthogonal to $T_{c_0} \mathcal{F}$; in this chart the matching and flatness conditions hold exactly, and the decomposition exists for δ small because P is compact. The Hessian at c_0 is $A_{\mathfrak{s}_2}$, whose kernel is $T_{c_0} \mathcal{F}$ and whose spectrum meets $(-\gamma_0, \gamma_0)$ only in 0, with γ_0 the gap of Proposition 7.9; with the elliptic estimate at the fixed scale $\mathfrak{s}_2$ this gives

$\|x\|_{H^1} \leq C\|A_{\mathfrak{s}_2}x\|$. Taylor expansion, with the multiplication $H^1 \times H^1 \rightarrow L^2$ in dimension three and the chart bounds of Lemma 8.2, gives

$$|a(c) - \tfrac{1}{2}\langle A_{\mathfrak{s}_2}x, x \rangle| \leq C\|x\|_{H^1}^3, \quad \|\nabla_{\mathfrak{s}_2}a(c) - A_{\mathfrak{s}_2}x\| \leq C\|x\|_{H^1}^2.$$

Hence $\|x\|_{H^1} \leq C\|\nabla_{\mathfrak{s}_2}a(c)\| + C\|x\|_{H^1}^2$, so $\|x\|_{H^1} \leq 2C\|\nabla_{\mathfrak{s}_2}a(c)\|$ once $\|x\|_{H^1} \leq 1/(2C)$, and

$$|a(c)| \leq C\|x\|_{H^1}^2 \leq C'\|\nabla_{\mathfrak{s}_2}a(c)\|^2.$$

(ii) Along a solution $a' = -\sigma^2$ with $\sigma^2 = \|\xi_f\|^2 + \mathfrak{s}\|\xi_c\|^2 + \|\partial_s\nu\|^2$, the kinetic energy density. By (i), $a' \leq -a/C$ from both ends of the neck, which gives the exponential decay of the energy. The projection p determined by the fixed branch satisfies $|p'| \leq C\sigma$, because the fixed branch is measured in the unweighted metric; integrating $\sigma \leq \sqrt{-a'}$ over dyadic intervals gives the length bound with no factor depending on the length of the neck. $\square$

The choice of projection matters: with the nearest-point projection of the collapsing branch in the unscaled metric, the method only gives the bound multiplied by $(\inf \mathfrak{s})^{-1/2}$.

8.4. Convergence to the flat family on the necks. Let U_ν be index-zero solutions on X_{ε_ν} converging to V as in Proposition 6.16. By Theorem 6.6(ii) there is no energy loss and no breaking, so the energy e_ν of U_ν on the part of each endpoint neck at distance at least s_0 from the exterior tends to the tail energy of V , which is small for s_0 large, and on windows $[S, L - S]$ with $S \rightarrow \infty$ it tends to zero.

Lemma 8.5 (convergence to the flat family). *Fix $e_0 > 0$ small. Let J be an interval in the neck and let J^+ contain its closed three-unit neighbourhood in the complete neck coordinate. Include in J^+ the full fixed-radius seam half-disks used below. If the energy e on this larger region is at most e_0 , then, for ν large, the following hold on J , uniformly in its position and in ν :*

- (i) *the spatial curvatures satisfy $\sup_s \|F_{B_j(s)}\|_{L^2(C)} \leq C\sqrt{e}$, $j = f, c$, and there are gauges in which $B_j(s) = B_{q_j(s)} + \beta_j(s)$ with β_j in the Neumann slice orthogonal to the harmonic fields and $\|\beta_j(s)\|_{H^1} \leq C\sqrt{e}$;*
- (ii) *on the collapsing seam the conclusions of Lemma 6.15 hold with $E(b, r_0) \leq e$; in particular the section is uniformly Hölder on a fixed collar, and $\sup_\theta d(u(s, \theta), q_f(s)) + d_P(q_c(s), q_f(s)) \leq C\sqrt{e}$.*

Proof. (i) Near a seam center b use the holomorphic coordinate Φ_b with $\Phi'_b(s) = 1/\mathfrak{s}(s)$; in it the collapsing compression and the section have exactly the fixed product mixed geometry, and fixed patches correspond to regions of size $C\mathfrak{s}_b$. Choose their inner patches to cover J and their larger patches inside J^+ , decreasing the fixed patch radius if necessary. The energy on each larger patch is at most e , by conformal invariance. The mixed mean-value inequality [DFL24, Theorem 4.2], the interior estimate [DFL24, Lemma 4.50] and the derivative bounds of [DFL24, Lemma 4.25] give the curvature bound after integration over a finite cover; the spatial slice and a Hodge bootstrap give the gauges.

(ii) The hypothesis of Lemma 6.15 is the energy bound $E(b, r_0) \leq e \leq e_0$. Its part (ii) gives the Hölder bound, and integrating the derivative bound across the section interval from the fixed branch, together with its part (iii), gives the last statement. $\square$

Lemma 8.5 is applied with a fixed small bound e_0 rather than with vanishing energy. Apply it to $[s_0, L - s_0]$ using the energy on $[s_0 - 3, L - s_0 + 3]$ and its seam patches, choosing s_0 larger first. It gives a uniform Hölder bound for the section on unit patches of the complete section geometry, including the region between the exterior and the windows. With vanishing

energy on the windows $[S, L - S]$ it gives convergence into any prescribed flat neighbourhood there, and by Proposition 8.4 and Proposition 6.16, the rescaled transitions converge to the flat pairs at $q_{\pm}$.

8.5. First-order estimates on the necks and cuts. On a neck interval $[a, b]$ with midpoint s_* , let q_m be the fixed-branch projection at s_* and $B_* = (B_{q_m}, B_{q_m}, q_m)$ the flat pair at q_m , with temporal coefficient zero. Let $\|z\|_s^2 = \|U_f\|^2 + \mathfrak{s}\|U_c\|^2 + \|\nu\|^2$ and A_s be as in Section 7.5, and define $\|z\|_{X^0(J)}^2 = \int_J (\|z\|_s^2 + \|z_s\|_s^2 + \|A_s z\|_s^2)$. Put $d(s) = \min(s - a, b - s)$ and $w = e^{cd(s)}$ with $2c < \gamma$.

Lemma 8.6 (first-order neck estimate). *For e and the neighbourhood of Lemma 8.5 small, there is a reference Coulomb gauge on each branch of the neck at B_* in which the exact matching coordinate z (Section 8.1) satisfies*

$$\|wz\|_{X^0([a+3, b-3])} \leq C\sqrt{e}, \quad \|z\|_{X^0([a+r, b-r])} \leq C\sqrt{e}e^{-cr}.$$

Proof. Gauge. Start from the gauges of Lemma 8.5(i), in which $B_j(s) = B_{q_j(s)} + \beta_j(s)$ with β_j in the spatial Neumann slice. Differentiating the slice condition in s gives $d_B^* dB\psi_j = d_B^*(\partial_s B_j)$ for the temporal coefficients, so the spatial Neumann gap gives $\|\psi_j(s)\| \leq C\sigma(s)$, where σ is the kinetic energy density of Proposition 8.4(ii). Since the energy on $[a + r, b - r]$ is at most $e e^{-\gamma r}$ and $2c < \gamma$, summation over unit intervals gives $\|w\psi\| \leq C\sqrt{e}$ in the physical measure $\int (\|\psi_f\|^2 + \mathfrak{s}\|\psi_c\|^2) ds$, and likewise $\sup_s w d_P(q_j(s), q_m) \leq C\sqrt{e}$ by the length bound. For the bound on ψ_j : in the Coulomb form of the local section $q \mapsto B_q$, $d_B^* \partial_s B_j = O(|\beta_j|(|q'_j| + \|\partial_s \beta_j\|))$, and $|q'_j| + \|\partial_s \beta_j\| \leq C(\sigma + \|\psi_j\|_{H^1})$, so the term is absorbed by the smallness of β_j . Let χ solve the reference Coulomb equation $d_{B_*}^*((\exp \chi)^* A - B_*) = 0$ on each branch, in the physical metric, with Neumann conditions on ∂C and at $s = a, b$; it exists and is small on the finite neck by the Neumann gap and the implicit function theorem. Test it against $w^2 \chi$ in the physical one-form measure. The spatial slice parts pair trivially with $d_C \chi$ up to the difference of the references $B_{q_j(s)} - B_*$, which is bounded by $C d_P(q_j(s), q_m)$, and the remaining linear source is the temporal coefficient. Since $|\partial_T \log w| \leq c$ in the conformal time, absorption gives $\|w d\chi\| \leq C\sqrt{e}$.

Equation. In the exact matching chart the equation is $D_s z + N_s(z) = 0$ with $D_s = \partial_s - A_s + 2\zeta \Pi_f$ and $\|N_s(z)\|_s \leq C d(z)(\|z_s\|_s + \|A_s z\|_s)$, where the coefficient size $d(z)$ bounds the spatial coefficients and the C^0 norm of the section. For compactly supported v , $\|(\partial_s - A_s)v\|^2 = \|v_s\|^2 + \|A_s v\|^2$, since $I_s A_s$ is independent of s and the Green pairings cancel; hence $\|v\|_{X^0} \leq C(\|D_s v\| + \|v\|)$, the L^2 term being kept for the stationary kernel. Apply this to $v = w\zeta z$ with a cutoff ζ , and use $\|wz\|_{L^2} \leq C\sqrt{e}$, which follows by applying Lemma 8.5 on unit windows with the local energy $e e^{-\gamma d(s)}$ of Proposition 8.4(ii), since $2c < \gamma$, together with the gauge estimate. $\square$

Cut positions (Figure 3). Fix D larger than all cutoff buffers and choose $C_* > D + 1 + \log 2$. On each neck, of length L in the exterior cylindrical coordinate s , cut the exterior piece near $s_E = L/2 - C_* + D$ and the transition piece near $s_T = L/2 - C_* - D$, and use the same midpoint $s_* = L/2 - C_*$ to define the stationary parameter of both pieces. The parts of U_ν used for the two pieces therefore overlap on a band of fixed width $2D$ on which both representatives are the actual solution, and both cuts, with their cutoff bands of unit width, lie in $\{s \leq L/2 - \log 2\}$, where E_ε coincides with X_ε . By Lemma 8.6, applied on the interval $[s_0, L - 2C_* - s_0]$, whose midpoint is s_* , each exact matching cut has first-order trace of size $C\sqrt{e}e^{-c(L/2 - O(1))}$ at the cut, since each cut lies at distance $L/2 - O(1)$ from both ends of

that interval, where $L = \log(1/\varepsilon)$. With component end weights $\delta \leq c/4$ on the new ends, the cut error in the weighted first-order norms is

$$(8.14) \quad \tau_{\text{cut}} \leq C\sqrt{e}e^{-cL/2}e^{\delta L/2} \leq C\varepsilon^{c/4}.$$

8.6. Comparison on the transitions and the exterior. The comparison estimates are proved in Coulomb gauges on the two kinds of components. These gauges are constructed with the scalar inverse of Lemma 7.16.

Let U_ν^E be the *cut exterior piece* of U_ν : equal to U_ν on the exterior side of the exterior cut s_E on each neck, and completed for $s \geq s_E + 1$ by the flat pair at the midpoint value q_m of that neck, by a cutoff of fixed slope in s applied in the gauge of Lemma 8.6. Likewise the *cut transition* is U_ν on the transition side of s_T , completed by the flat pair at q_m .

Lemma 8.7 (component Coulomb gauges). *Lemma 7.16 holds, with the same proof, on the auxiliary exterior E_ε with the metric of Definition 7.12, the weight e^{ϕ_E} and λ equal to $\mathfrak{s}_E$ on the compression, and on the complete flat transition $\mathcal{T}$ with the weight $e^{\delta\rho}$. For ν large there is a gauge transformation of U_ν^E , trivial at the original ends in the limit, after which $U_\nu^E = W_\varepsilon^E + \alpha_E$ with*

$$\lambda^2 d_{W_\varepsilon^E}^* \alpha_E = 0, \quad \alpha_E(n) = 0, \quad \|\alpha_E\|_{H_{\text{uloc}}^N} \rightarrow 0,$$

and $\alpha_E - \mathfrak{b}_E(v_*)$ decaying at the new ends. Here $v_* = (v_-, v_+)$ records the midpoint values in the Neumann Coulomb coordinates at $q_\pm$, and $\mathfrak{b}_E(v_*)$ is a cutoff extension of the actual stationary difference: its connection component on each new end is $B_{q_\pm}(v_\pm) - B_{q_\pm}$, using the flat family of Section 8.1, and its section component has the corresponding limit at q_m . In the exact matching coordinate z , the remainder $z - \mathfrak{k}_E(v_*)$ decays. The same holds for the cut transition relative to the flat pair at q_m , with fixed limit zero.

Proof. The rescaled geometry of E_ε and of $\mathcal{T}$ is again a union of product cylinders with compact cross-section joined through finitely many fixed cores, λ^{-1} is Lipschitz in the rescaled distance because $|\partial_T \mathfrak{s}_E^{-1}| = |\partial_t \log \mathfrak{s}_E| \leq H$, and every slice is irreducible, so the proof of Lemma 7.16 applies verbatim. Take the local gauges of Proposition 6.16(i) on compact subsets of X_0 , of Corollary 6.17 on the middle compression and of Lemma 8.6 on the necks. At each new end choose the midpoint flat representative in the Neumann Coulomb slice of $B_{q_\pm}$ given by Section 8.1. On an overlap of two of them the two representatives are related by a gauge transformation g ; on each slice both are close to the same irreducible flat connection, so after a choice of central sign $g = \exp \chi$ with χ small, and the gauge equation bounds χ by the two connection differences. Interpolate by $\exp(\theta\chi)$ with a fixed cutoff θ whose derivative in the product time is bounded; the overlaps form a tree, so the central signs can be chosen consistently. This gives a representative $W_\varepsilon^E + \alpha$ with $\|\alpha\|_{H_{\text{uloc}}^N}$ small. The reference Coulomb equation $\lambda^2 d_{W_\varepsilon^E}^* ((\exp \chi)^*(W_\varepsilon^E + \alpha) - W_\varepsilon^E) = 0$, $\nabla_n \chi = 0$, has derivative $\lambda^2 d^* d$ at $\chi = 0$, which is invertible on uniformly local spaces by the first part; the implicit function theorem gives a small solution. On each new end the nonlinear stationary difference $B_{q_\pm}(v_\pm) - B_{q_\pm}$ is time-independent, coclosed relative to $B_{q_\pm}$ and normal-free by its Neumann Coulomb-slice construction. It therefore contributes no Coulomb source, and the gauge parameter decays. $\square$

Lemma 8.8 (transition comparison). *In the Coulomb gauge of Lemma 8.7, which retains the midpoint value q_m as a free stationary parameter, the cut transition differs from the flat pair at q_m by at most $C\tau_{\text{cut}}$ in the weighted first-order norm.*

Proof. The flat pair at q_m is a solution. Write the cut transition as the flat pair plus z in the exact matching chart. Then

$$\mathcal{D}z = -N(z) + e_{\text{cut}}, \quad \|N(z)\|_{Y_\delta} \leq C\mathbf{d}(z)\|z\|_{X_\delta}, \quad \|e_{\text{cut}}\|_{Y_\delta} \leq C\tau_{\text{cut}},$$

where e_{cut} is the error of the cutoff to the flat pair and $\mathbf{d}(z)$, the size of the coefficients, tends to zero; the gauge-fixing component carries no source because z is in Coulomb gauge. The fixed-limit inverse of Proposition 7.10(iii) at weight δ gives $\|z\|_{X_\delta} \leq C\delta^{-1}(\mathbf{d}(z)\|z\|_{X_\delta} + C\tau_{\text{cut}})$, and for $\mathbf{d}(z)$ small the first term is absorbed. $\square$

Lemma 8.9 (exterior comparison). *Let z be the exact matching coordinate (Section 8.1) of $W_\varepsilon^E + \alpha_E$, the cut exterior piece U_ν^E (Section 8.6) in the Coulomb gauge of Lemma 8.7, relative to the configuration W_ε^E of the auxiliary exterior (Definition 7.12), with a weight $\delta_* \leq c/4$ on the new ends. Then*

$$\|z\|_{X_E} \leq C(\tau_{\text{cut}} + \varepsilon^{1/4} + \varepsilon^{(\beta-\delta_*)/4}),$$

where X_E is the norm of Proposition 7.13, and the free end coordinates of z , which are the midpoint parameters of the two necks, obey the same bound.

Proof. Write $\mathcal{F}_E(z) = f_E + D_E z + \mathcal{N}_E(z)$. The reference error is $\|f_E\|_{Y_E} \leq C(\varepsilon^{1/4} + \varepsilon^{(\beta-\delta_*)/4})$: the first term is the horizontal velocity $\mathfrak{s}_E^{1/2}|\dot{q}_R|$ of the flat lift, with $\mathfrak{s}_E \leq \sqrt{\varepsilon}$; the second is the truncation of V at $R = \varepsilon^{1/4}$, which is supported where $s = L/4 + O(1)$, has size $O(R^\beta)$ there, and carries the weight $e^{\delta_* s} = O(R^{-\delta_*})$. The nonlinear term with comparison coordinate zero satisfies $\|\mathcal{N}_E(z)\|_{Y_E} \leq C\mathbf{d}(z)\|z\|_{X_E}$, because the spatial remainder vanishes on the exact flat family, the temporal coefficient derivative is quadratic, and the stationary fields are exact flat members. Here $\mathbf{d}(z) \rightarrow 0$ is the coefficient size, which tends to zero by Proposition 6.16, Corollary 6.17 and Lemma 8.5. The operator D_E is inverted from Y_E to X_E with a bound independent of ε by Proposition 7.13; the gauge-fixing component of z vanishes by the choice of gauge, and the cut error, which comes from the cutoff to the flat pair at q_m and is bounded by (8.14), is measured in the same norms. Absorption gives the bound. The end coordinate of z is the midpoint parameter because the same midpoint is used for both cuts. $\square$

8.7. The global gauge and the uniform estimate on the glued domain. *Zeroth-order norm.* We use the rescaled metric $\widehat{g}_\varepsilon$ and measure μ_ε of Section 7.8. A variation x on the glued domain is written $x = \bar{x} + \mathfrak{e}_\varepsilon(v_{\text{orig}})$, where v_{orig} is the vector of limiting values at the original ends and $\mathfrak{e}_\varepsilon$ a fixed extension supported in the tails of the original ends, with zero temporal and normal components and $\|wD_\varepsilon \mathfrak{e}_\varepsilon(v)\| \leq C|v|$. Put

$$\|x\|_{Z_\varepsilon} = \|w\bar{x}\|_{L^2(\mu_\varepsilon)} + \|w\bar{x}_{\text{sec}}\|_{L^2} + |v_{\text{orig}}|,$$

the physical zeroth-order norm of gauge field and section, and use the same expression for nonlinear configurations relative to W_ε . It contains the stationary fields along the entire finite necks, which on a neck of length comparable to L contribute up to $\sqrt{L}$ times their pointwise size.

Lemma 8.10 (global gauge). *In the situation of Proposition 8.12, for large ν there is a gauge transformation of U_ν on the glued domain, trivial at the original ends in the limit, after which $U_\nu = W_\varepsilon + x$ with x in the reference Coulomb gauge of W_ε , x of finite X_ε norm, and*

$$\|x\|_{Z_\varepsilon} \leq C\sqrt{A}(\tau_{\text{cut}} + \varepsilon^{1/4} + R^{\beta-\delta_*}), \quad \|x\|_{H_{\text{uloc}}^N(\widehat{g}_\varepsilon)} \rightarrow 0.$$

Proof. Components. By Lemmas 8.8 and 8.9 the cut transitions and the cut exterior differ from the pieces of W_ε by at most $C(\tau_{\text{cut}} + \varepsilon^{1/4} + R^{\beta-\delta_*})$ in first-order norms, and the change from V to its truncation costs CR^β . A stationary parameter of size ρ on a neck of length $O(L)$ has physical L^2 norm $C\sqrt{A}\rho$; the compression contribution is no larger. This gives the Z_ε bound on the part of each piece that is used. The qualitative H_{uloc}^N bound follows from Proposition 6.16, Corollary 6.17 and Lemma 8.5.

Joining. On an overlap band the two representatives represent the same solution, $A_T = g^*A_E$. On each slice both are close to the same irreducible flat connection, so g may be chosen near the identity, $g = \exp \chi$, and the spatial gauge equation with the Neumann gap gives $\|\chi(T)\|_{H^1(C)} \leq C(\|(A_E - B_q)_{\text{sp}}\| + \|(A_T - B_q)_{\text{sp}}\|)$; the temporal part of the gauge equation controls $\partial_T \chi$ in the same way. Interpolate by $\exp(\theta \chi)$ with a fixed cylindrical cutoff θ , whose product derivative on the compression is $\mathfrak{s}\theta'$ and hence bounded. The result is an actual gauge-equivalent representative $W_\varepsilon + \alpha$ with the two stated bounds. The pieces overlap along a tree of bands, so the choices of central sign on the bands can be made consistently with the marked bundle.

Coulomb gauge. The reference Coulomb equation $\lambda_\varepsilon^2 d_{A^0}^{*,g_\varepsilon}((\exp \chi)^*(A^0 + \alpha) - A^0) = 0$, $\nabla_n \chi = 0$, with A^0 the connection of W_ε , has derivative $\mathcal{L}_\varepsilon$ at $\chi = 0$; by Lemma 7.16 and the small H_{uloc}^N norm of α it has a small solution with $\|\chi\|_{H_{\text{uloc}}^{m+1}} \leq C\|\alpha\|_{H_{\text{uloc}}^m}$, which belongs to the weighted space at the original ends. To estimate it in the physical norm, write $(\exp \chi)^*(A^0 + \alpha) - A^0 = J(\chi)d_0\chi + \text{Ad}_{e^{-\chi}}\alpha$ with $J(0) = 1$, and test the equation against $w^2\chi$:

$$0 = \int \langle d_0(w^2\chi), J(\chi)d_0\chi + \text{Ad}_{e^{-\chi}}\alpha \rangle d\mu_\varepsilon.$$

The main term is at least $(1 - o(1))\|wd_0\chi\|^2$; the weight terms are of relative size δ , the maximal weight exponent of the original ends. The stationary part $\mathfrak{e}_\varepsilon(v_{\text{orig}})$ of α is integrated by parts onto its divergence, bounded by $C|v_{\text{orig}}|$, and $\text{Ad}_{e^{-\chi}}\alpha - \alpha$ is small times χ . Absorption gives $\|wd_0\chi\| \leq C(\|w\tilde{\alpha}\| + |v_{\text{orig}}|)$, and the same bound for the change of the connection. The section is unchanged. Finite X_ε norm holds for each fixed ν since the region $N_I \cap X_\varepsilon$ is compact. $\square$

Proposition 8.11 (uniform estimate on the glued domain). *Uniformly for small ε , $\|x\|_{X_\varepsilon} \leq C(\|D_\varepsilon x\|_{Y_\varepsilon} + \|x\|_{Z_\varepsilon})$ for all x in the domain, with no normalization of the asymptotic values.*

Proof. Write $D_\varepsilon = P_\varepsilon + K_\varepsilon$, with P_ε the differential part and K_ε the perturbation term. Let $\mathfrak{s}_1$ be the threshold of Proposition 7.4. On a unit patch of $N_I \cap X_\varepsilon$ on which $\mathfrak{s} \leq \mathfrak{s}_1$, the configuration W_ε is the flat lift of the path $q(t)$ and P_ε is the operator of Proposition 7.4 (linearization at the flat family along $q(t)$, scale with $|\mathfrak{s}'/\mathfrak{s}| \leq H$); since $\|\nu\|_{L^2(S)} \leq \|z\|_{L^2}$, that proposition gives

$$(8.15) \quad \|z\|_{X_\varepsilon}^2 \leq C(\|P_\varepsilon z\|_{Y_\varepsilon}^2 + \|z\|_{L^2}^2) \quad (\text{supp } z \text{ in one patch}).$$

On unit patches of the endpoint necks the slice is $C \sqcup [0, \pi] \sqcup (-C, \mathfrak{s}^2 g_C)$, not the model of Proposition 7.4, whose far edge is Lagrangian; there (8.15) follows instead from the neck identity $\|(\partial_\rho - A_\mathfrak{s})z\|^2 = \|z'\|^2 + \|A_\mathfrak{s}z\|^2$ of Section 7.5, the term $2\zeta\Pi_f$ being absorbed into the L^2 term. The identity is exact on the flat part $s \geq L/4 + c_R$; on the remaining patches the seam and $\dot{q}$ remainders are absorbed by trace interpolation, as in the proof of Proposition 7.4. On patches where $\mathfrak{s} \geq \mathfrak{s}_1/2$ the rescaled geometry lies in a fixed compact family, and the ordinary local mixed estimates give (8.15) there as well.

Use a partition $\sum \chi_\alpha^2 = 1$ of bounded overlap by cutoffs that depend on the complete time coordinate only on the compression (constant on each fibre C), share the same value

on the two sides of every seam, and are extended constantly across a fixed section collar; a cutoff in $\log(\mathfrak{s}/\mathfrak{s}_1)$ has derivative bounded by $C|\varsigma|$. A spatial cutoff on C would have commutator $\mathfrak{s}^{-1}d_C\chi$, which is not bounded uniformly; these cutoffs preserve the domain and have localized commutator bounds $\|[P_\varepsilon, \chi_\alpha]z\|_Y \leq C\|z\|_{L^2(\text{supp } d\chi_\alpha)}$. Applying (8.15) to each $\chi_\alpha z$ and summing,

$$\|z\|_{X_\varepsilon}^2 \leq C \sum_{\alpha} \|\chi_\alpha z\|_{X_\varepsilon}^2 \leq C \sum_{\alpha} (\|\chi_\alpha P_\varepsilon z\|_{Y_\varepsilon}^2 + \|[P_\varepsilon, \chi_\alpha]z\|_{Y_\varepsilon}^2 + \|\chi_\alpha z\|_{L^2}^2) \leq C(\|P_\varepsilon z\|_{Y_\varepsilon}^2 + \|z\|_{L^2}^2),$$

the first inequality because the cutoffs have bounded derivatives and are constant on the fibres, and the last because the supports of the χ_α and of the $d\chi_\alpha$ have bounded overlap. This is the claim for the differential part P_ε , with constants depending on the overlap and not on the number of patches. The perturbation term has output and dependence region in the unchanged geometry and is bounded on L^2 . Conjugation by the weights of the original ends adds bounded zeroth-order terms, and subtracting $\mathfrak{e}_\varepsilon(v_{\text{orig}})$ reduces to decaying fields. $\square$

8.8. Surjectivity of the gluing map.

Proposition 8.12 (surjectivity of gluing). *Let $\varepsilon_\nu \rightarrow 0$ and let U_ν be index-zero solutions on X_{ε_ν} converging, in the sense of Proposition 6.16, to the regular index-zero solution V . Then $U_\nu = \mathcal{G}_{\varepsilon_\nu}(V)$ for all large ν .*

Proof. We write $\varepsilon = \varepsilon_\nu$; the sequence enters only through thresholds.

Step 1: global gauge. By Lemma 8.10, after a gauge transformation $U_\nu = W_\varepsilon + x$ with x of finite X_ε norm, $\|x\|_{Z_\varepsilon} \leq C\sqrt{\mathbf{A}}(\tau_{\text{cut}} + \varepsilon^{1/4} + R^{\beta-\delta_*})$ and coefficient size $\mathbf{d}(x) \rightarrow 0$.

Coordinates. Write a_{aff} for the affine difference produced by Lemma 8.10. Its coefficients tend to zero, and its higher rescaled local norms are bounded. The rescaled Hölder inverse of the chart therefore applies. Write x for the resulting exact matching coordinate; it is smooth, has finite X_ε norm, and $\mathbf{d}(x) \rightarrow 0$. We need only

$$(8.16) \quad \|x\|_{Z_\varepsilon} \leq C\|a_{\text{aff}}\|_{Z_\varepsilon} + C\mathbf{d}(x)\|x\|_{X_\varepsilon}.$$

Indeed $a_{\text{aff}} - x$ is the sum of the quadratic term in the harmonic coordinate, the transport correction, and $\mathcal{R}(v, w)$, with the analogous section correction. The first two are bounded in Z_ε by $C\mathbf{d}(x)\|x\|_{X_\varepsilon}$. For the third use the trace and extension estimates (8.7) at order zero and the surface estimate (8.11). Its Z_ε norm has the factor $\mathfrak{s}^{1/2}$, so the bound follows from the first-order correction estimate. The section correction has the ordinary C^0 quadratic bound. This proves (8.16). The gauge-fixing row is the affine Coulomb condition composed with the chart.

Step 2: a priori estimate in the first-order norm. Write $\mathcal{F}_\varepsilon(W_\varepsilon + x) = f_\varepsilon + D_\varepsilon x + N_\varepsilon(x)$ with $\|f_\varepsilon\| \leq Ce_\varepsilon$ by (8.13) and $\|N_\varepsilon(x)\|_{Y_\varepsilon} \leq C\mathbf{d}(x)\|x\|_{X_\varepsilon}$: the pure harmonic spatial term vanishes on the exact flat family by (8.3), every remaining singular term contains a complementary field, which carries the weight $\mathfrak{s}^{-1}$ of the norm, and the section uses only C^0 smallness. Proposition 8.11, (8.16), and (8.9) give $\|x\|_{X_\varepsilon} \leq C(e_\varepsilon + \|a_{\text{aff}}\|_{Z_\varepsilon}) + C\mathbf{d}(x)\|x\|_{X_\varepsilon}$. Absorbing the last term gives

$$\|x\|_{X_\varepsilon} \leq C(e_\varepsilon + \sqrt{\mathbf{A}}(\tau_{\text{cut}} + \varepsilon^{1/4} + R^{\beta-\delta_*})) \leq C\varepsilon^{\sigma_c}$$

for some $\sigma_c > 0$, since τ_{cut} , e_ε and $R^{\beta-\delta_*}$ are positive powers of ε by (8.14) and (8.13).

Step 3: interpolation. On four-dimensional unit patches of the conformal geometry, which are copies of one fixed compact manifold, interpolation between the local H^N bounds, which hold uniformly by the convergence of Proposition 6.16, Corollary 6.17 and Lemma 8.5 and are preserved by the gauges of Step 1, and the first-order bound of Step 2 gives $\|U_\perp\|_{C^k} \leq C\varepsilon^{\sigma_c\theta_1}$

for the complementary gauge field, using $\int \mathfrak{s}^{-1} \|U_\perp(t)\|_{H^1}^2 dt = \int \|U_\perp(T)\|_{H^1}^2 dT$ with $dt = \mathfrak{s} dT$, so that no power of the minimal scale is lost. The slow coefficient is not interpolated on product patches; its supremum is obtained from the section through the matching identity $v = \Pi\nu|_{\text{seam}} - \Pi\tau_t(U_\perp|_\Sigma)$. The section is interpolated on two-dimensional patches between the uniform Hölder bound of Lemma 8.5 and the L^2 bound. Hence the coefficient size satisfies $d(x) \leq C\varepsilon^\sigma$ for some $\sigma > 0$.

Step 4: uniqueness. The norm X_ε on the glued domain carries only the original end weights; the artificial weights of the component problems do not enter. Let $z^* = \mathcal{G}_\varepsilon(V) - W_\varepsilon$, so that $\|z^*\|_{X^M} \leq a := C_M \mathbf{A}^{M+1} e_\varepsilon$ by Proposition 8.3. The difference $y = x - z^*$ satisfies $D_\varepsilon y = N_\varepsilon(z^*) - N_\varepsilon(x)$. The difference estimate holds in the asymmetric form

$$\|N_\varepsilon(x) - N_\varepsilon(z^*)\|_{Y_\varepsilon} \leq C(d(x) + a)\|x - z^*\|_{X_\varepsilon},$$

where $d(x)$ is the unweighted coefficient size of x from Step 3 and the weighted suprema $\mathfrak{s}^{-1/2}|p_{z^*}|$, $\mathfrak{s}^{-1/2}|\nabla_C(U_\perp)_{z^*}|$ needed by the singular terms, such as $\mathfrak{s}^{-1}[p_{z^*}, h_{v_x - v_{z^*}}]$, are placed on the factor z^* , where they are bounded by a through (8.1). The inverse of Proposition 7.15 then gives

$$\|y\| \leq CA(d(x) + a)\|y\|.$$

Both $\mathbf{A}d(x)$ and $\mathbf{A}a$ are positive powers of ε multiplied by powers of $\mathbf{A}$, hence tend to zero, and $y = 0$ for large ν . $\square$

The unknown solution never has to enter the logarithmically shrinking ball of Proposition 8.3 directly. It first reaches a positive power of ε by estimates free of logarithms, and only then meets the single factor $\mathbf{A}$ of the inverse on the glued domain.

8.9. Proof of Theorem 6.6(iii). The order of choices is: the data of the pieces and the bound b_η ; the neck lengths of the ordinary necks, for which Hypothesis 6.5(X4) holds, and then N_I , which may meet the necks; the finite set $\mathcal{M}^0(X_0)$, which fixes the decay rates β_V and the weights; and finally ε . Negative indices: if solutions of negative index existed for a sequence $\varepsilon_\nu \rightarrow 0$, a subsequence would converge by Theorem 6.6(ii); the energy bound of part (i) leaves finitely many negative indices, and the loss formula with Hypothesis 6.5(X4) excludes each. For index zero, $\mathcal{G}_\varepsilon$ is defined by Proposition 8.3 and is injective for small ε : $\mathcal{M}^0(X_0)$ is finite, and $\mathcal{G}_\varepsilon(V) \rightarrow V$ as $\varepsilon \rightarrow 0$ in the sense of Proposition 6.16, since $\|\mathcal{G}_\varepsilon(V) - W_\varepsilon\|_{X^M} \rightarrow 0$ and W_ε agrees with V away from shrinking neighbourhoods of the rectangle. If it were not surjective for a sequence $\varepsilon_\nu \rightarrow 0$, a sequence of solutions outside its image would converge to some $V \in \mathcal{M}^0(X_0)$ by Theorem 6.6(ii), contradicting Proposition 8.12. Regularity of the image is part of Proposition 8.3. $\square$

9. TWO-HANDLES

9.1. Heegaard triples and the three Lagrangians. Let $(W_2, E, \nu): Z_0 \rightarrow Z_1$ be a two-handle cobordism along a framed link L of n components, with E trivial at both ends (Proposition 4.2), with a filling frame (Definition 4.6) of lifting class ζ . Choose a subordinate Heegaard triple $(\Sigma; H_0, H_1, H_2)$ of genus g adapted to the ball (Section 4.3), and take the boundary sum with the fixed torus: $C_i = H_i \natural ([0, 1] \times T^2)$, $h = g + 1 \geq 2$ (after a stabilization in a small ball of the standard collar, away from the bouquet, if necessary). Put $r = g - n$. The transition ρ_ζ of the filling frame on Σ induces a symplectomorphism τ_ζ of $\mathbf{M}_h$, and the three Lagrangians are

$$(9.1) \quad (\tau_\zeta L_{\mathfrak{h}_0}(C_0), L(C_1), L_{\mathfrak{h}_2}(C_2)) =: (P_0, P_1, P_2),$$

where the primary perturbation of C_1 is zero, that of C_2 is the height function th_Q of the next subsection, and that of C_0 is small and chosen so that P_0 is transverse to P_1 and P_2 and misses $P_1 \cap P_2$. Since ρ_ζ is an automorphism of the $\mathrm{SO}(3)$ bundle, it conjugates every operator in the problem; in the trivialization of the filling frame over $e_0 \times H_0$ the boundary problem of C_0 is literally the untwisted one, and ζ enters only through the global topology of the triangle.

9.2. The auxiliary pair. The manifold $Y_{12} = H_1 \cup H_2$ is $\#^r(S^1 \times S^2)$ and, for vanishing primary perturbations of C_1 and C_2 , the pair $(L(C_1), L(C_2))$ intersects cleanly in $Q \cong \mathrm{SU}(2)^r$, with $TQ = TL(C_1) \cap TL(C_2) = H^1(Y_{12}^\#; \mathrm{ad})$. Here the flat connections of $Y_{12}^\#$ are determined by the holonomies $b_j \in \mathrm{SU}(2)$ of the r free generators of π_1 , with lifts fixed by the global frame of $H_1 \cup H_2$; the torus summand contributes the fixed quaternion pair, and the nonzero element of ϕ_M identifies its two sign choices. The primary perturbation of C_2 is the height function th_Q , $h_Q = \sum_j (1 - \frac{1}{2} \mathrm{tr} b_j)$, with $t > 0$ small, and the primary perturbation of C_1 is zero; transversality of (P_0, P_2) and empty triple intersection are obtained by perturbing C_0 alone. The critical points are the 2^r points a_S with $b_j = -1$ for $j \in S$ and $b_j = 1$ otherwise; a_S has Morse index $m(a_S) = 3|S|$, and there is a unique maximum $q_* = a_{\{1, \dots, r\}}$.

For these auxiliary data we distinguish the relative class from the total perturbed energy. For a gauge, mixed or strip configuration from a to x , put

$$(9.2) \quad \kappa_{\mathrm{aux}} = \kappa - \frac{t}{4\pi^2} (h_Q(a) - h_Q(x)).$$

Here κ is the topological energy of Section 3.6. Thus, for a strip, $4\pi^2 \kappa_{\mathrm{aux}} = \int u^* \omega - t(h_Q(a) - h_Q(x))$. The relative class vanishes on a reference path in the clean intersection Q and takes values in $\frac{1}{2}\mathbb{Z}$. For a filling with outgoing limit a , we instead normalize κ_{aux} to vanish on its flat disk-extendible reference with that limit. The following proof verifies these normalizations and the associated index formulas.

Lemma 9.1. *Use the conventions under which the anti-self-duality equation on an outgoing end $[0, \infty) \times Y$ is the downward gradient flow of the Chern–Simons functional perturbed by $+th_Q$ with $t > 0$.*

- (i) *For the gauge filling $F_r = \natural^r(S^1 \times D^3) \natural(T^2 \times D^2)$ of $Y_{12}^\#$ with outgoing limit a , on the component of the flat fillings, $\mathrm{ind} = 3r - m(a)$; other components have index $3r - m(a) + 8\kappa_{\mathrm{aux}}$.*
- (ii) *For a gauge, mixed or ordinary configuration from a to x , with the relative class just defined, $\mathrm{ind} = m(a) - m(x) + 8\kappa_{\mathrm{aux}}$, and the full analytic energy is $8\pi^2 \kappa_{\mathrm{aux}} + 2t(h_Q(a) - h_Q(x))$ plus secondary terms.*
- (iii) *If $2t \max h_Q + 2\|f\|_\infty + 2 \sup_A \|\eta(A)\|_2^2 < 4\pi^2$ on each counted domain, every counted configuration has $\kappa_{\mathrm{aux}} \geq 0$.*

Reversing one of the three conventions (the sign of the flow, the sign of t , outgoing versus incoming end) replaces $3r - m(a)$ by $m(a)$ and the maximum by the minimum throughout; reversing two restores the statement.

Proof. (i) At a flat connection A on F_r with the unperturbed end, the operator $-d_A^* \oplus d_A^+$ on decaying spaces is injective and surjective: a decaying kernel element is closed and represents a class in the image of $H_c^1(F_r; \mathrm{ad}) \cong H_3(F_r; \mathrm{ad}) = 0$, hence is exact, $a = d_A f$, and $f = 0$ by $H^0(Y_{12}^\#; \mathrm{ad}) = 0$; a cokernel element has vanishing function part by $H^0(F_r; \mathrm{ad}) = 0$ and a self-dual part that is exact by $H^2(F_r; \mathrm{ad}) = H^2(T^2; \mathrm{ad}) = 0$, hence decays and vanishes by $\|\omega\|^2 = \int \omega \wedge \omega = 0$. So the decaying index is 0. The asymptotic operator on the tangent

space TQ of the critical manifold moves from $-\delta$ to $t \operatorname{Hess}_a h_Q$, which has $m(a)$ negative and $3r - m(a)$ positive eigenvalues; for an outgoing end each eigenvalue crossing from negative to positive increases the index by one. Hence $\operatorname{ind} = 3r - m(a)$. This is a spectral-flow calculation; it does not identify solutions of the perturbed four-dimensional equation with finite-dimensional gradient trajectories in Q . Relative to the flat reference, a point insertion changes κ_{aux} by 1 and the index by 8. The possible nonextending meridional lifting class changes these by a half-integer and an integer congruent to 4 modulo 8, respectively. Indeed, the filling retracts onto $T^2 \vee \bigvee^r S^1$, and the quotient of the boundary lifting classes by those extending over it is generated by the torus meridian. With that character fixed, relative H^4 records the point insertion. The Pontryagin-square congruence gives the stated half-integral lattice and the formula in (i).

(ii) Join a to x by a path of flat classes in Q , constant near its endpoints. Before the height perturbation, the cylinder with positive weights at both clean ends has index $-3r$. The physical incoming and outgoing conditions admit $m(a)$ and $3r - m(x)$ small modes, respectively. Hence its perturbed index is $m(a) - m(x)$. Formula (2.52) of [DFL21] gives its topological energy $t(h_Q(a) - h_Q(x))/(4\pi^2)$. Changing the relative class changes energy and index by $k/2$ and $4k$, with $k \in \mathbb{Z}$. Gluing this reference to the constant mixed pair at x and using [DFL21, Lemma 4.62, Proposition 4.65] proves the mixed formula. Gluing the constant mixed pair at a to a strip and using [DFL21, Proposition 4.69] proves the strip formula. In particular, the same integer function m serves on both sides; it is the relative class κ_{aux} , rather than the total perturbed energy κ , which occurs in this formula. The full energy identity follows from (9.2) and the identity in the proof of Theorem 3.8.

(iii) On the outgoing filling collar, let β and ρ be increasing cutoffs switching on th_Q and f , respectively. Integration by parts gives the full energy

$$8\pi^2 \kappa_{\text{aux}} - 2th_Q(a) + 2t \int \beta' h_Q(A_s) ds + 2 \int \rho' f(A_s) ds + 2\|\eta(A)\|_2^2.$$

Since $0 \leq h_Q \leq \max h_Q$ and both cutoffs have total variation one, the absolute value of the height correction is at most $2t \max h_Q$. The same bound applies to the endpoint term in (ii). The remaining corrections are bounded by $2\|f\|_\infty + 2 \sup_A \|\eta(A)\|_2^2$. Nonnegative analytic energy and $\kappa_{\text{aux}} \in \frac{1}{2}\mathbb{Z}$ therefore exclude every negative relative class under the stated inequality. $\square$

With Scaduto's $\mathbb{Z}/4$ grading, $\operatorname{gr} = m - r$ modulo four on every generator [Sca15, Section 7.3]; in particular q_* has degree $2r$, the degree of the relative invariant of $\natural^r(S^1 \times D^3)$ [Sca15, Sections 7.7–7.8].

Lemma 9.2 (top generator). *Let C, C' be finite complexes over $\mathbb{F}_2$ with bases indexed by the same set, graded by m , and let $N: C \rightarrow C'$ be an isomorphism whose matrix entries $N_{a,x}$ vanish unless $m(x) = m(a) + 4k$ with $k \geq 0$. Then N maps the unique generator of maximal m to itself.*

Proof. If a has maximal m , then $N(a)$ is a multiple of a , and the multiple is nonzero because N is injective. $\square$

The hypothesis $k \geq 0$ cannot be replaced by a congruence modulo four: for $r = 4$ the isomorphism $N = I + E_{\{1,2,3,4\}, \emptyset}$, which adds the minimum to the image of the maximum, satisfies the congruence and violates the conclusion.

Definition 9.3. On $CF(P_1, P_2)$ define the relative grading m by $m(a) - m(x) = \operatorname{ind}(u) - 8\kappa_{\text{aux}}(u)$ for any strip class u from a to x ; it is independent of u by monotonicity, and for t

small it is the Morse index of h_Q . Let $\theta_S \in CF(P_1, P_2)$ be the unique generator of maximal m ; it is the maximum q_* of h_Q . Let $\theta_G \in C_G(Y_{12})$ be the relative invariant of the filling F_r : the count of index-zero solutions on F_r with cylindrical end $Y_{12}^\#$, with lifts by even gauge transformations, projected to the framed complex.

Proposition 9.4. $\theta_G = q_*$ and $\theta_S = q_*$ as chains; both are cycles, their classes are nonzero, and $N_{12}\theta_G = \theta_S$.

Proof. By Lemma 9.1 an index-zero filling has $\kappa_{\text{aux}} = 0$ and $m(a) = 3r$, so $a = q_*$. First set $f = \eta = 0$. The filling energy identity in that lemma then gives $E = -2t \max h_Q + 2t \int \beta' h_Q(A_s) ds \leq 0$. Thus $E = 0$. In temporal gauge the end is stationary at the flat maximum, and the connection on the unperturbed core is flat. The boundary fundamental group surjects onto that of the filling, so this boundary class has exactly one flat extension. Its fixed-limit operator is invertible. Part (i) gives index zero; for injectivity use the triangular splitting of Section 7.5 on the end. The stationary coefficient satisfies $a' = -t\beta \text{Hess}(h_Q)a + O(t^2)a$ with $\text{Hess}(h_Q) \leq -c$ at the maximum q_* , so $\frac{d}{ds}|a| \geq t\beta(c|a| - Ct \sup |a|)$, which excludes an interior maximum of $|a|$; since $a \rightarrow 0$, this forces $a \equiv 0$; the remainder lies in the decaying space, where part (i) gives an inverse, and the $O(t)$ perturbation is absorbed. For sufficiently small secondary regularizations this is still the complete count: any additional index-zero solutions along data tending to zero would have energy tending to zero, hence no bubbles or nonconstant end trajectories, and would enter the inverse-function neighbourhood of that solution. Therefore $\theta_G = q_*$. (This agrees with [Sca15, Sections 7.7–7.8], where the relative invariant of $\mathfrak{h}^r(S^1 \times D^3)$ is a basis element of the free group $I^\#(\#^r(S^1 \times S^2))$ of rank 2^r , by the boundary-sum identity [Sca15, Section 7.2] and [KM11b, Proposition 5.4] at the connected cut T^3 .) A trajectory from q_* has index $3r - m(x) + 8\kappa_{\text{aux}}$ with $\kappa_{\text{aux}} \geq 0$, which is never one, so q_* is a cycle on both sides. By Lemma 9.1 the map N_{12} of Theorem 3.8 satisfies the hypothesis of Lemma 9.2, so $N_{12}q_* = q_*$. The filtrations on the two sides are defined by the same function m , so the gauge filling class and the direction of triangularity of N_{12} cannot come apart under a change of conventions. $\square$

9.3. The torus-summed triangle and its lifting group. Let $X^\#$ be the torus-summed triangle: the union of $\Delta \times \Sigma_h$ and $\Delta \times T^2$ (the two surface sheets) with the edge pieces $e_i \times C_i$, where each torus compression body C_i is glued to both sheets along $e_i \times (\Sigma_h \sqcup T^2)$. Its ends are $Y_{ij}^\# = C_i \cup (-C_j)$. By Proposition 4.3 and the construction of Section 2, removing the tube from W_2 and inserting $T^\circ \times [-1, 1]$ gives $W_2^\# = X^\# \cup_{Y_{12}^\#} F_r$ with its bundle. The classes of $H^1(X^\#; \mathbb{F}_2)$ that vanish on both surface sheets restrict to zero on each compression piece, since $H^1(C_i) \rightarrow H^1(\Sigma_h)$ is injective, so they come from the incidence graph of the torus region, a pair of pants with three boundary circles: this group is $\Phi = \mathbb{F}_2^3 / \langle (1, 1, 1) \rangle$, whose restriction to the end $Y_{ij}^\#$ is $s_{ij} = t_i + t_j$. Each class is realized by an automorphism supported on the torus region and equal to the identity near its joining tree.

9.4. The gauge two-handle map as a triangle map. Let $E^\#$ be the bundle on the torus-summed triangle $X^\#$ of Section 9.3, and $\Phi = \mathbb{F}_2^3 / \langle (1, 1, 1) \rangle$ its lifting group, with $s_{ij} = t_i + t_j$. Let μ_G^E be the Φ -framed gauge triangle map $C_G(Y_{01}) \otimes C_G(Y_{12}) \rightarrow C_G(Y_{02})$.

Proposition 9.5. Let $F_G(W_2, E)$ be the map of [KM11b, Section 5] of $(W_2^\#, E^\#)$ with the subgroup ϕ_{W_2} , between the complexes of $(Z_i^\#, E^\#|_{Z_i^\#})$ with the markings of the filling frame. Then $F_G(W_2, E) = [\mu_G^E(\cdot \otimes \theta_G)]$ on homology.

Proof. By Proposition 4.3, $W_2^\# = X^\# \cup_{Y_{12}^\#} F_r$, where F_r is the gauge filling of Lemma 9.1(i), with the tube in the product region. Write $W_2^\# = X^\# \circ W'$ with $W' = (Y_{01}^\# \times [0, 1]) \sqcup F_r$, cut along the disconnected manifold $Y_{01}^\# \sqcup Y_{12}^\#$. The restriction $H^0(W') \oplus H^0(X^\#) = \mathbb{F}_2^2 \oplus \mathbb{F}_2 \rightarrow H^0(Y_{01}^\# \sqcup Y_{12}^\#) = \mathbb{F}_2^2$ is $(a, b, c) \mapsto (a + c, b + c)$, which is onto, so the connecting image ψ of [KM11b, Proposition 5.4] vanishes. The subgroup of $W_2^\#$ restricts to the index-two subgroup $\{s_{12} = 0\} \subset \Phi$ on $X^\#$ and to 0 on F_r , since the torus meridian class does not extend over $T^2 \times D^2$. By [KM11b, Proposition 5.4] and the Künneth isomorphism of [KM11b, Section 5.2], $F_G(W_2, E) = \mu^{\{s_{12}=0\}}(\cdot \otimes \tilde{\theta}_G)$, where $\tilde{\theta}_G$ is the relative invariant of F_r (as in the definition of θ_G), computed modulo even gauge transformations only. Finally let $g \in \Phi$ with $s_{12} = 1$, let ι be its action on the critical points of $Y_{12}^\#$, and let π be the projection to the Φ -quotient complex. With Φ -invariant data, g maps solutions with Y_{12} -limit b bijectively to those with limit ιb , preserving the orbits at the other two ends, whose restriction of $\{s_{12} = 0\}$ is already onto. So g acts freely on the lifts at Y_{12} and trivially on the orbits at the other ends, and each Φ -orbit of solutions contains exactly one solution for each lift of b . Hence $\mu^{\{s_{12}=0\}}(a \otimes b) = \mu^\Phi(a \otimes \pi b)$ with no factor two, and $\mu^{\{s_{12}=0\}}(\cdot \otimes \tilde{\theta}_G) = \mu_G^E(\cdot \otimes \theta_G)$. Nontrivial E is covered by [KM11b, Section 5.1], which allows an arbitrary bundle. $\square$

9.5. The mixed triangle.

Definition 9.6. The *mixed triangle* is the following domain for the mixed equation. Let Δ be a triangle with incoming vertices v_{01}, v_{12} , outgoing vertex v_{02} and edges e_i , and let γ be a proper arc from the interior of e_0 to the interior of e_2 separating v_{02} from the incoming vertices, dividing Δ into G (containing v_{01}, v_{12}) and S . The gauge part is the restriction of $X^\#$ over G : the two surface sheets over G and the edge pieces over $e_i \cap G$. The section is a map $S \rightarrow M_h$, with boundary conditions P_0 on $e_0 \cap S$ and P_2 on $e_2 \cap S$, matched along γ to both sheets (the torus sheet matching the one-point moduli space). At $\gamma \cap e_0$ and $\gamma \cap e_2$ the domain has the mixed ends of C_0 and C_2 . The ends are the gauge inputs $Y_{01}^\#, Y_{12}^\#$, the strip output at $P_0 \cap P_2$, and the two mixed ends. Metrics are products near the seams, and the section almost complex structure is the complex structure $*_\Sigma$ near γ . Let $B \subset G$ be the rectangle joining a compact subinterval I of the middle edge e_1 to an arc $J \subset \gamma$; the collapse of Theorem 6.6 takes place over B .

Lemma 9.7 (vertex identification). *Let D be any domain of DFL type for a pair (P, P') covered by Theorem 3.8 and Remark 3.9, with the same end types as the special quintuple of [DFL21, Section 3.1], and regular invariant data of the following form: near each seam the metric is of the form used in the proof of Theorem 5.4, conformal near each seam to the product metric [DFL24, (1.1)], that is, with $e^{-f}c \equiv 1$ in the notation there, and the section almost complex structure is $*_\Sigma$; on each mixed end the metric is conformal to a product $d\theta^2 + g_{\text{slice}}$. The translation-invariant data on the mixed ends may differ from those of the quintuple. Then its index-zero count is chain homotopic to the vertex map of Theorem 3.8 for that pair.*

Proof. First give each mixed end its cylindrical conformal representative. Conformal invariance preserves the solution space and surjectivity modulo gauge; the Coulomb row and weighted realization are chosen for this representative. The space of these domains and data, with fixed end types, is connected: the conformal structure of the section region, the placement of the seam, the metrics near the mixed ends and the neck lengths vary in convex sets, and the changes are compactly supported. Theorem 5.4 gives a chain homotopy along a path,

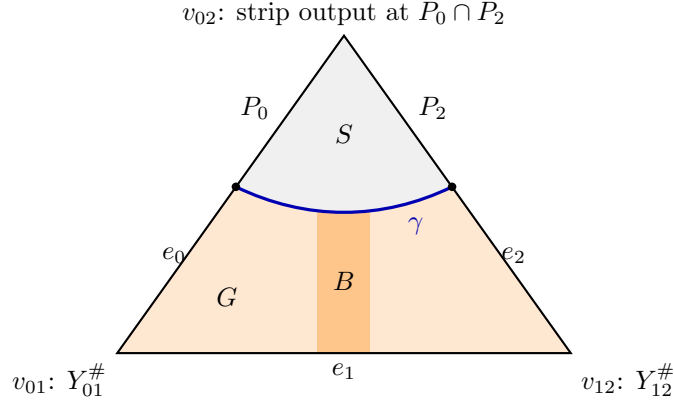


FIGURE 4. The mixed triangle of Definition 9.6. The gauge part lies over G and the section over S ; they are matched along the arc γ , whose endpoints are the mixed ends of C_0 and C_2 . The rectangle B joins a subinterval of the middle edge e_1 to γ . Collapsing B (Theorem 6.6) and stretching the ordinary necks gives the symplectic triangle composed with $N_{01} \otimes N_{12}$; stretching along the closed three-manifold $Y_{02}^{\#}$, which by Lemma 9.11 passes through B , gives N_{02} composed with the gauge triangle.

and Theorem 3.8(ii) handles the choices of f , χ and η . A change of the translation-invariant data on a mixed end (the polar geometry of Lemma 6.3(ii) against the product geometry of the quintuple, matched along the seam) is a family whose parameter derivative is independent of the end coordinate there; the flat references on C and the constant sections do not depend on the end metric, and the mixed ends have a uniform gap on the complement of the stationary kernel. Lemma 10.15 then gives the chain homotopy once its hypothesis of uniform energy decay on the mixed ends is verified. By conformal invariance of the two equations and their energies (Lemma 6.3) we compute in the product metric on each mixed end. We choose the path within the data of the statement, with the complex structure of the section on each mixed end equal to the standard one pulled back by $(\sigma, \theta) \mapsto (\varphi(\sigma), \theta)$, where φ is a diffeomorphism of the section interval equal to the identity near the seam and to a translation near the Lagrangian arc; changes of the width of the interval are made by such maps. Along a compact path of these data the three steps of the proof of Proposition 10.8(i) hold with constants independent of the parameter. First, local compactness and the seam estimates hold with constants uniform over C^∞ -compact families of such data (the compactness paragraph of the proof of Theorem 5.4), and Gromov compactness is uniform for J in a compact family; so there is a threshold, independent of the parameter, below which the kinetic energy on an interval of fixed length forces the slices into a fixed neighbourhood of the critical family. Second, the local action on that neighbourhood, the Chern–Simons functional with the primary function plus the symplectic action of the section, involves neither the metric nor J , so its critical set does not depend on the parameter. The Coulomb–Neumann slices and the charts in which it is expanded depend continuously on the parameter, so the remainder bounds of the expansion hold with uniform constants; the gradient is taken in the L^2 inner product of the parameter, and these inner products are uniformly equivalent. With the uniform gap this gives $|\mathcal{F} - \mathcal{F}_{\text{crit}}| \leq C \|\nabla \mathcal{F}\|^2$ with C independent of the parameter. Third, the product metric and the form of j on the end give $\partial_\theta \mathcal{F} = -\|\nabla \mathcal{F}\|^2$. Hence the kinetic energy beyond θ

decays at a rate c independent of the parameter. With the weight δ of Lemma 10.15 chosen below c , this is its hypothesis. $\square$

The DFL pieces produced by the degenerations below are such domains, so the maps they count are the mixed maps up to chain homotopy.

9.6. Critical points on the twisted inputs and the gauge triangle.

Lemma 9.8. *Let $ij \in \{01, 12, 02\}$, with the ρ_ζ -marking on C_0 when $i = 0$, and let α be a critical point of the primary equation $*F_B + \nabla \mathfrak{h}_{ij}(B) = 0$ on $Y_{ij}^\#$. Then α is flat on a collar of a copy of Σ_h disjoint from the primary dependence regions, w_2 is nonzero on Σ_h , α is irreducible, an automorphism fixing α whose lifting class vanishes on Σ_h is the identity, the nonzero element of the lifting group acts freely on the critical set, and the critical set corresponds bijectively to $P_i \cap P_j$ and is nondegenerate. Moreover end perturbations as in Proposition 3.4 can be chosen so that the critical set of $\text{CS}_{\mathfrak{h}_{ij}} + f_{ij}$ equals that of $\text{CS}_{\mathfrak{h}_{ij}}$.*

Proof. The pair (C_0, C_j) with the ρ_ζ -composed boundary marking is a pair of the type of [DFL21, Section 2.3], and its Lagrangian is $\tau_\zeta L(C_0)$. Flatness near Σ_h , irreducibility and the correspondence with $P_i \cap P_j$ with nondegeneracy are [DFL21, Lemmas 2.43, 2.45, 2.47], whose proofs use only flatness and irreducibility on the overlap and Mayer–Vietoris. The restriction of $E_{ij}^\#$ to a push-off of Σ_h into C_j is the odd surface bundle, unchanged by ρ_ζ . Freeness follows from Theorem 5.6(i). For the last statement, the twisted-marked pairs are not presentations, so Proposition 3.4 is not applied as stated; its proof is repeated for each pair ij , as follows. If a sequence f_n with vanishing C^0 norms of gradients and of their derivatives on fixed balls had critical points B_n not gauge equivalent to critical points of $\text{CS}_{\mathfrak{h}}$, then Uhlenbeck compactness on the closed three-manifold and elliptic regularity would give a limit critical point α of $\text{CS}_{\mathfrak{h}}$, near which the inverse function theorem for the gauge-fixed critical equation, with invertible extended Hessian, allows only the zero α itself, since f_n vanishes near α . $\square$

Proposition 9.9. *On $X^\#$ with the bundle $E^\#$ and the three closed ends of Lemma 9.8, arbitrarily small Φ -invariant secondary perturbations give: no solutions of negative index, a finite regular index-zero space, and regular one-dimensional spaces whose boundary strata are an index-zero solution with one index-one trajectory at one end. The count μ_G^E is a chain map.*

Proof. The ends are nondegenerate and irreducible by Lemma 9.8, so the gauge-fixed operator is Fredholm, the gauge-fixing component having positive gaps. The index-energy law $\text{ind}(A) = 8\kappa(A) + c(\alpha_0, \alpha_1, \alpha_2)$ holds on the pure gauge triangle: for two connections A, A' with the same limits, glue to both the same connections on compact fillings of the three ends (every $\text{SO}(3)$ bundle over a closed three-manifold extends over some compact four-manifold); since the ends are acyclic, additivity of the index [Don02, Proposition 3.9] reduces $\text{ind}(A) - \text{ind}(A')$ to the difference on the closed manifold, where the Atiyah–Singer formula gives $8(\kappa(A) - \kappa(A'))$. Compactness is Uhlenbeck compactness with point concentrations of energy $8\pi^2$ in the normalization of Section 3.6 and index loss eight; a small three-sphere about a concentration point carries no nonzero lifting class, so the limiting bundle is identified with $E^\#$ without any unitary lift. Surjectivity in the invariant class is Theorem 5.6(ii), with the loops supplied by an incoming end: on $Y_{12}^\#$ the bundle admits a $\text{U}(2)$ lift because H^3 of a closed oriented three-manifold is torsion free, the critical limit is flat on a collar of Σ_h with odd bundle, and loops on that surface, which have zero lifting class in Φ , are transported into the interior along a path; holonomies are conjugated by parallel transport, which preserves

a basis. Finite selection after compactness at each index and freeness of Φ complete the proof. $\square$

9.7. The triangle identity.

Theorem 9.10. *With the mixed maps N_{01} , N_{12} , N_{02} of Theorem 3.8 for the three pairs of (9.1) and the symplectic triangle map μ_S^E , which counts holomorphic triangles only,*

$$N_{02} \circ \mu_G^E \simeq \mu_S^E \circ (N_{01} \otimes N_{12}) \quad \text{over } \mathbb{F}_2.$$

The proof compares two degenerations of the mixed triangle of Definition 9.6, with the regions G , S , the arc γ and the rectangle B fixed there.

Lemma 9.11. *Every proper arc in G from the interior of e_0 to the interior of e_2 meets B . Consequently the closed cut $Y_{02}^\#$ passes through the collapsing rectangle, and a product collar of the cut exists only when the compression scale over I is of order one.*

Proof. B meets the frontier of G only in I and J , and $G \setminus B$ has two components, one containing each input vertex. The arcs $e_0 \cap G$ and $e_2 \cap G$ lie in different components, so an arc joining them meets B . Over B the metric is $dt^2 + \lambda_\varepsilon^2(d\rho^2 + g_\Sigma)$, whose surface factor is scaled by ε , so no neighbourhood of the cut is a product with a fixed closed metric until λ is raised to one. $\square$

Lemma 9.12. *There is a smooth path F_τ , $\tau \in [0, 1]$, of data on one marked domain with the bundle E , with the following properties.*

- (i) *Near $\tau = 0$ it is the collapse datum of Theorem 6.6 at small scale ε , with the ordinary necks of Theorem 5.1, satisfying Hypothesis 6.5. Near $\tau = 1$ it is the datum with a long product collar of the closed cut $Y_{02}^\#$ inserted.*
- (ii) *It satisfies the hypotheses of Theorem 5.4: (H1) and (H3)–(H5) on every fibre, and (H2) at the endpoints, as verified in the proof of Theorem 9.13. The inserted necks of (H3) are the ordinary necks near $\tau = 0$ and the cut collar near $\tau = 1$.*
- (iii) *The scales stay in a compact positive range: the compression scale over I is at least ε , and the cut collar has bounded length.*
- (iv) *The holonomy loop families are fixed. Only their forms vary, with support away from the seam collars and the ends; the loops themselves may pass through these regions. The derivative $\partial_\tau F_\tau$ has compact support disjoint from all ends.*
- (v) *Every F_τ is invariant under Φ .*

Proof. The path has five stages, each inside a convex set of such data: (A) raise the scale over I to one by the convex combination $\mu_s^2 = (1-s)\lambda_\varepsilon^2 + s$ of metrics of the form $dt^2 + \mu^2 g_C$ and of the matching section metrics; (B) interpolate linearly to a metric which is a product $a^2 du^2 + g_{02}$ on a collar of the cut, the two metrics agreeing with every prescription on every overlap because on the compression pieces the collar coordinate is the edge coordinate and g_{02} restricts to the compression metrics, and on B it is the stage (A) metric; (C) lengthen the collar; (D) switch on the end perturbation of $Y_{02}^\#$ on the collar, by a cutoff equal to one on the middle of the collar and monotone on its two end bands, which become the cutoff bands of the adjacent ends; during this stage the term is a compactly supported nonautonomous perturbation, and it is fixed from then on; (E) interpolate the remaining metrics, section structures and forms to the data with the collar of the cut inserted. Compatible almost complex structures are joined by $J_k = (-A_k^2)^{-1/2} A_k$ for the convex combination k of the associated metrics, where $\omega(u, v) = k(A_k u, v)$; this equals the Hodge structure wherever

both ends do. Forms are replaced by their self-dual projections in the varying metric. The bundle over the collar is a product because all transitions are constant along the edges, so stretching the collar lifts to E with no clutching. Invariance holds because every class of Φ vanishes on the surface sheets and hence, by injectivity of $H^1(C_i) \rightarrow H^1(\Sigma_h)$, on the compression pieces, and every loop has zero lifting class in Φ . $\square$

Proof of Theorem 9.10. Endpoint 1: the outgoing cut. Stretch along the closed three-manifold $Y_{02}^\#$ separating the gauge triangle from the DFL domain for the ρ_ζ -marked pair (P_0, P_2) . We verify the hypotheses of Theorem 5.1 for these two pieces and this connected cut. (H1): on the gauge piece the index-energy law is part of Proposition 9.9; on the DFL piece it is [DFL21, Proposition 3.20, Lemma 3.15] for the marked pair, and on the joined problem it follows by additivity across the cut, the constant pair at each output generator having index zero, so that the constant of the joined problem is the sum of the two constants. (H2): μ_G^E is regular by Proposition 9.9, and N_{02} is regular in the invariant class by Theorem 5.6, with the loops supplied from its incoming end $Y_{02}^\#$, whose critical limits are flat on a collar of an odd copy of Σ_h by Lemma 9.8; the section part of a cokernel element is killed by harmonic traces on the matching arc, and the two compression ends are conjugate to the untwisted ones by τ_ζ . The cross-section at the cut is invertible because the critical points of $Y_{02}^\#$ are nondegenerate and irreducible (Lemma 9.8), so the extended Hessian, including the gauge-fixing component, has a spectral gap. (H3)–(H5) hold by construction. Theorem 5.1 gives the count $N_{02} \circ \mu_G^E$ for long cut collars, the DFL piece being identified with the mixed map by Lemma 9.7, and a finite change of collar length is a compact family (Theorem 5.4). The lifting groups glue by the fibre product: restriction $\Phi \rightarrow \mathbb{F}_2^2$ to the two incoming ends is an isomorphism, the output group of the DFL piece restricts isomorphically to the lifting group of the cut, and the connecting H^0 term at the connected cut vanishes.

Endpoint 0: the interval collapse. Collapsing B by Theorem 6.6, the limit domain is a disk with a physical P_1 boundary interval and two new mixed ends; cutting off two DFL disks along ordinary strip ends leaves a triangle domain with boundary (P_0, P_1, P_2) . The triangle piece is regular for a generic domain-dependent almost complex structure supported away from the neighbourhood N_I of the collapsed arc, where Hypothesis 6.5(X1) fixes the complex structure $*_\Sigma$: there are no constant triangles because the triple intersection is empty, so every solution is nonconstant on an open set away from N_I by unique continuation, and the standard transversality argument applies there. Choose the ordinary neck lengths first: by Theorem 5.1 the zero-scale domain is then regular in index at most zero with index-zero count $\mu_S^E(N_{01} \otimes N_{12})$, the two DFL pieces being identified with the mixed maps by Lemma 9.7; this is Hypothesis 6.5(X4), and (X5) holds by Lemma 6.7. Then choose ε small; by Theorem 6.6(iii) the count at scale ε equals the zero-scale count. Hypotheses (X1)–(X3) hold because the middle primary perturbation vanishes, the end perturbations are the fixed autonomous perturbations of the original gauge ends, with outputs and loops in those ends and cutoff bands placed beyond the compact set N_I , the secondary perturbations have outputs and loop regions in a compact set disjoint from N_I , and the original ends are regular; at the original mixed ends this holds because a nonconstant mixed-end trajectory has $\kappa \geq \frac{1}{2}$, the action being constant on the connected critical family, and hence index at least four. The zero-scale gauge region consists of two DFL pieces (Lemma 9.11), with lifting group $\mathbb{F}_2 \times \mathbb{F}_2$; restriction $\Phi \rightarrow \mathbb{F}_2^2$, $t \mapsto (s_{01}, s_{12})$, is an isomorphism, so the marking-preserving bijection of Theorem 6.6(iii) compares Φ -framed counts with no factor two.

The family. Apply Theorem 5.4 to the path of Lemma 9.12. Its hypotheses: the index-energy law on every fibre with the constant of the twisted endpoint, since a reference configuration on one fibre is a reference on every fibre and the topological energy involves no metric (so the index of a fixed reference is constant in τ by continuity of the Fredholm family, and the law at $\tau = 0$ is Theorem 6.6(i)); an energy bound independent of τ , the terms $2 \int \chi' f$ being bounded by twice the C^0 norms of the end perturbations times the total variation of their cutoffs, which is one for the original ends and at most two for the switch of stage (D), whose cutoff rises and falls once; family compactness, with weak limits identified with the prescribed bundle E by modifying bundle isomorphisms only on small balls around the concentration points joined by arcs to an incoming slice, which introduces no lifting class; family regularity in the invariant class by Theorem 5.6, with holonomy terms built from local unitary lifts, which exist on neighbourhoods of the loops and are unique up to tensoring with a flat real line, so that each term is defined up to sign and the span is unaffected; and no internal neck breaks, since the scales stay in a compact positive range. The breaks of the parametrized one-dimensional space therefore occur only at the original ends: the two closed inputs and the ordinary output. No break occurs at the two original mixed ends at $\gamma \cap e_0$ and $\gamma \cap e_2$: a nonconstant mixed-end trajectory has index at least four, as noted at Endpoint 0. At an incoming break the index -1 piece may carry the other even lift of that input; a unique class of Φ aligns it, and by invariance of the whole family the aligned piece is already counted. The boundary count gives the chain homotopy. $\square$

9.8. The two-handle square. Let p_i^ϕ be the admissible presentations of $Z_0 = Y_{01}$ and $Z_1 = Y_{02}$ given by the splittings (H_0, H_1) and (H_0, H_2) with the global frames ϕ_i^{gl} of Section 4.4, whose primary data are those transported by the frames. These presentations are admissible: $\text{span } F$ contains each trace function and the constants, so both transports act on it linearly, and the shared perturbation on C_0 is taken small enough that both transported data lie in the open neighbourhood $\mathcal{U}(F)$. Let $P_{\phi_i}^G$ and $P_{\phi_i}^S$ be the transport of complexes from the filling frame to the global frame: on the gauge side the pushforward by the bundle isomorphism, and on the symplectic side the pushforward by the symplectomorphism τ_{a_1} (respectively $\tau_{a'_2}$) of $\mathbf{M}_h$. By naturality of every construction under bundle isomorphisms, $N_{p_i^\phi} P_{\phi_i}^G = P_{\phi_i}^S N_{0,i+1}$ on chains, and by the definition of the morphisms of [KM11b], which include the identifications at the ends, $I^\#(W_2, E; \phi^{\text{gl}}) = P_{\phi_1}^G F_G(W_2, E)(P_{\phi_0}^G)^{-1}$.

Theorem 9.13. *Define the symplectic map*

$$S_{W_2, E} = P_{\phi_1}^S \mu_S^E(\cdot \otimes \theta_S) (P_{\phi_0}^S)^{-1} : HF(Z_0, p_0^\phi) \rightarrow HF(Z_1, p_1^\phi).$$

Then $H(N_{p_1^\phi}) \circ I^\#(W_2, E; \phi_0^{\text{gl}}, \phi_1^{\text{gl}}) = S_{W_2, E} \circ H(N_{p_0^\phi})$.

Proof. Insert $x \otimes \theta_G$ into the chain homotopy of Theorem 9.10. Since θ_G is a cycle, $N_{02} \mu_G^E(x \otimes \theta_G)$ is homologous to $\mu_S^E(N_{01} x \otimes N_{12} \theta_G) = \mu_S^E(N_{01} x \otimes \theta_S)$ by Proposition 9.4. By Proposition 9.5, $N_{02} F_G(W_2, E) = \mu_S^E(\cdot \otimes \theta_S) N_{01}$ on homology. Conjugating by the transports gives the claim. $\square$

The map $S_{W_2, E}$ is defined by two pushforwards by symplectomorphisms and a triangle count; it does not involve gauge theory.

Remark 9.14. For E nontrivial the gauge map can vanish for grading reasons: for the (-1) -framed unknot with the nontrivial bundle, $\chi = 1$, $\sigma = -1$, $P(E) = 1$, and the map $I^\#(S^3) \rightarrow$

$I^\#(S^3)$ has degree two by Remark 2.6, hence is zero [Sca15, Section 7.6]. The theorem then says that the symplectic triangle map $S_{W_2, E}$ vanishes on homology.

10. EXCISION ALONG A TORUS FOR THE MIXED EQUATION

The one-handle comparison of Section 11 passes through a mixed domain whose gauge input is a connected sum along a torus. On the gauge side such a manifold is split by Floer’s excision theorem [BD95, KM10], in the form of [KM11b, Theorem 5.6]. This section proves the corresponding mixed statement, Theorem 10.2.

10.1. The folded splittings. Let M have presentation data with torus compression bodies U_0, U_1 of genus $h \geq 2$, and let $C = C_{h, g}$, $g = h + k$, be a compression body from Σ_h to Σ_g adding k handles. Put $V_i = C \circ U_i$, the enlarged torus compression bodies. Then $V_0 \cup (-V_1)$ is the torus-summed object $(M^+)^\#$ of $M^+ = M \#^k(S^1 \times S^2)$: every added meridian disk occurs on both sides, and the two disks form a nonseparating sphere. Let $L_C \subset M_h^- \times M_g$ be the Lagrangian correspondence of C . For a Lagrangian $P \subset M_h$ transverse to the lower projection, the composition $P \circ L_C$ is embedded, because the lower projection of L_C is a submersion and its upper projection an embedding: L_C is the flat moduli space of C , which fibres over the moduli space of Σ_h with fibre $\mathrm{SU}(2)^k$, and restriction to Σ_g is injective since $\pi_1(\Sigma_g) \rightarrow \pi_1(C)$ is onto [DFL21, Proposition 2.21].

Height Lagrangians. For $t > 0$ small, perturb the primary data of V_i by t times a height function whose restriction to the clean intersection $L(V_i) \cap (L(U_i) \circ L_C) \cong \mathrm{SU}(2)^{g-1}$ has the 2^{g-1} nondegenerate critical points of Section 9; call the resulting upper Lagrangians $R_i \subset M_g$ the *height Lagrangians* at height t . The top generator q_i^* of each auxiliary pair below is defined intrinsically, by the grading m of Section 11.3; at unperturbed lower data and uncorrected height data it is the maximum or the minimum of the height function, according to whether the perturbed Lagrangian is the first or the second entry of the pair, and in general it is the continuation of that critical point (Section 11.6; Lemma 11.7(i) for $k \geq 1$).

Lower Lagrangians. Then choose small primary data $\tilde{P}_i$ of U_i , in a neighbourhood of $L(U_i)$ that is small relative to t : for $k = 0$ lower data in the set $\mathcal{U}_F(t)$ of Definition 11.12, whose Lagrangians lie in $\mathcal{N}(t)$, and in general the neighbourhoods $\mathcal{N}_{C, R}$ of Lemma 11.7(i). In $\mathbb{X} = M_h^- \times M_g$ put

$$A_0 = \tilde{P}_0^- \times R_0, \quad A_1 = \tilde{P}_1^- \times R_1,$$

consider the triple (A_0, L_C, A_1) , and require $A_0 \pitchfork L_C$, $L_C \pitchfork A_1$, $A_0 \pitchfork A_1$ and $A_0 \cap L_C \cap A_1 = \emptyset$. These are generic conditions: the triple intersection has expected dimension $3d - 2 \cdot 2d = -d < 0$, where $d = (3h - 3) + (3g - 3)$ is the dimension of each of the three Lagrangians in the manifold $\mathbb{X}$ of dimension $2d$, and the universal restriction map for primary perturbations is a submersion [DFL21, Section 2.3]. (With unperturbed data the triple intersection would contain the fibres of L_C over $L(U_0) \cap L(U_1)$.) The pairs (A_0, L_C) and (L_C, A_1) have intersection generators identified with those of $(R_0, \tilde{P}_0 \circ L_C)$ and $(\tilde{P}_1 \circ L_C, R_1)$, hence, by continuation, with the critical sets of the height functions. Their strip problems remain the product-target problems in $\mathbb{X}$; exact unfolding is used only for the diagonal case $k = 0$.

The gauge pieces are $D_i = (-U_i) \cup_{T^2} V_i$, with $\partial D_i = (-\Sigma_h) \sqcup \Sigma_g$, and C . Flat connections on D_i restrict bijectively to pairs of flat connections on U_i and V_i : across the odd torus the gluing is unique, since the flat connection on the odd torus has central determinant-one stabilizer and $H^*(T^2; \mathrm{ad}) = 0$; so $L(D_i) = L(U_i)^- \times L(V_i)$, perturbed to A_i . The closed cut

of the pair (A_0, A_1) is

$$Z = D_0 \cup (-D_1) \cong (-M) \# M^+ \# T^3,$$

with one torus factor. For a split almost complex structure $J_- \times J_+$ on $\mathbb{X}$, folding gives an isomorphism of complexes $\mathcal{K}: CF_{\mathbb{X}}(A_0, A_1) \rightarrow CF(\tilde{P}_0, \tilde{P}_1)^\vee \otimes CF(R_0, R_1)$, and a cycle $\sum x_i^\vee \otimes y_i$ on the right is the chain map $v \mapsto \sum x_i^\vee(v) y_i$.

The metrics are chosen as follows: U_0, U_1 share their metric near Σ_h , V_0, V_1 near Σ_g , and all torus collars carry the flat metric induced from the fixed metric on T^3 , with the fixed odd bundle.

10.2. The torus region and the domains. Let $T = \mathbb{R}^2/\Lambda$ be the flat torus, P_T the odd bundle and α its flat connection, unique up to gauge. The adjoint bundle splits into three flat real line bundles with nontrivial characters, so $H^*(T; \text{ad } \alpha) = 0$, the twisted Laplacian satisfies $\Delta_\alpha \geq \mu_T^2$ in every degree with $\mu_T = \pi \min\{|\xi| : \xi \in \Lambda^* \setminus 2\Lambda^*\}$, and the determinant-one stabilizer of α is the centre.

Consider the special quintuple X_D of [DFL21, Section 3.1] for a pair of torus compression bodies (Y, Y') with $\partial Y = \Sigma' \sqcup T$. Its gauge part contains $\mathbb{U}_- \times T$, where $\mathbb{U}_-$ is a planar region with one gauge input arm, two mixed-end arms $\theta \rightarrow \pm\infty$ and the matching line $\mathbb{U}_\partial = \{0\} \times \mathbb{R}_\theta$; the metric on $\mathbb{U}_-$ is flat near the matching line. Hence X_D contains the flat product region

$$N_X = T \times [-1, 0]_s \times \mathbb{R}_\theta,$$

whose face $\{s = 0\}$ is the torus component of the matching surface, and for $|\theta| \geq 2$ this region lies in the mixed ends. Since the moduli space of the odd torus is a point, the matching condition there says only that the restriction of the connection to each fibre $T \times \{(0, \theta)\}$ is flat; we call it the *flat-fibre condition*. Its linearization is $d_\alpha \zeta_T = 0$ on the face, ζ_T the component along the torus; together with the Neumann gauge condition $\zeta(\partial_s) = 0$ this gives the boundary conditions used below, the longitudinal component $\zeta(\partial_\theta)$ being free.

Definition 10.1. (a) For $0 \leq R < \infty$, the *collared domain* is $W_R = X_D \cup (T \times [0, R]_r \times \mathbb{R}_\theta)$, glued along the torus face, with the flat product metric and no perturbation on the collar and the flat-fibre condition at $r = R$. Its mixed-end slices are $Y_R = Y \cup ([0, R] \times T)$ and Y'_R . It is diffeomorphic to X_D by a map ψ_R that stretches $s \in [-3/4, -1/4]$ and is the identity outside N_X , hence on the supports of all perturbations, on the gauge and strip ends, near the other matching surface and on the section.

(b) The *completed domain* is $W_\infty = X_D \cup (T \times [0, \infty)_r \times \mathbb{R}_\theta)$, with no torus face and mixed slices Y_∞, Y'_∞ .

(c) For the two sides $X_{D,-}$ (the reflected presentation of the lower end M of the one-handle cobordism) and $X_{D,+}$ (the enlarged presentation of M^+) and $\lambda > 0$, the *joined domain* W_λ^J is $W_{-, \lambda} \cup W_{+, \lambda}$ glued along their torus faces by the marked bundle isometry reversing the torus orientation together with the normal coordinate. It has the bridge $T \times [-\lambda, \lambda] \times \mathbb{R}_\theta$, two gauge inputs $(-M)^\#$ and $(M^+)^\#$, a section $\mathbb{U}_+ \rightarrow \mathbf{M}_h^- \times \mathbf{M}_g$ with the split structure $J_- \times J_+$ on all of $\mathbb{U}_+$, strip output at $A_0 \cap A_1$, and mixed-end slices $D_{0, \lambda} = Y_- \cup_{[-\lambda, \lambda] \times T} Y_+$ and $D_{1, \lambda} = Y'_- \cup_{[-\lambda, \lambda] \times T} Y'_+$: on each arm the slices of the two sides are joined through the bridge.

Weights are functions $\tau \geq 0$ with $|d\tau| \leq 1$ and $|\nabla d\tau| \leq 1$, equal to the end coordinate on the ends and, on the torus region, to a function $\tau_0(\theta)$ of θ alone. The domains $W_{\pm, R}$ and $W_{\pm, \infty}$ are called *separate*, as opposed to the *joined* domains W_λ^J ; their torus collars, and the mixed maps defined on them, are called separate as well.

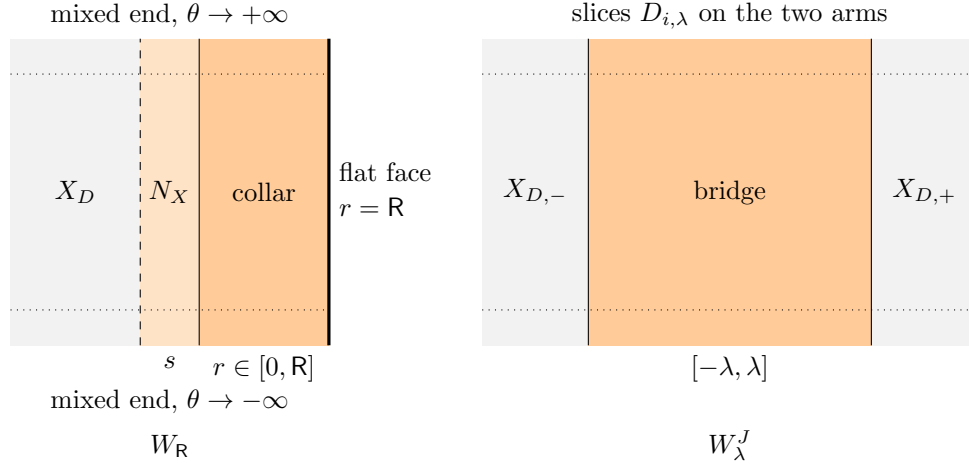


FIGURE 5. The torus region, drawn in the base coordinates (normal coordinate horizontally, the end coordinate θ vertically); every point carries a torus T with the flat odd connection. Left: the collared domain W_R of Definition 10.1(a), with the flat product region N_X of X_D , the collar and the flat face at $r = R$; for $|\theta| \geq 2$ (dotted lines) the region lies in the mixed ends. The completed domain W_∞ has $R = \infty$. Right: the joined domain W_λ^J , in which the collars of the two sides are glued along their flat faces into a bridge of width 2λ . No closed hypersurface separates the two sides, since the torus region runs into the mixed ends.

The critical manifold of the joined mixed ends is $P_i \times Q_i$ for the two separate critical manifolds, because the two flat connections glue uniquely across the odd torus. The lifting group of W_λ^J is $\Phi_J \cong \mathbb{F}_2^2$, restricting isomorphically to $\Phi_- \oplus \Phi_+$, with lifting class $s_Z = s_- + s_+$ at the cut Z ; its elements are lifting classes that vanish on every piece (they come from the graph with a vertex for each surface sheet and each compression piece and an edge for each contact region), since restriction of H^1 from each compression body to its full boundary is injective.

Counting index-zero solutions defines: the separate maps $N_\pm^{(R)}$ on W_R , which for $R = 0$ are $N_\pm$; the completed maps $N_{\pm, \infty}$ on W_∞ ; and the joined maps $N_\lambda^J: C_G((-M)^\#) \otimes C_G((M^+)^\#) \rightarrow CF_{\mathbb{X}}(A_0, A_1)$.

10.3. Statement. Let $N_-: C_G((-M)^\#) \rightarrow CF(\tilde{P}_0, \tilde{P}_1)^\vee$ be the mixed map of the reflected presentation of the lower end M (Section 11.2), $N_+: C_G((M^+)^\#) \rightarrow CF(R_0, R_1)$ that of the enlarged presentation with the height Lagrangians, and $N_Z: C_G(Z) \rightarrow CF_{\mathbb{X}}(A_0, A_1)$ the mixed map of the pair (A_0, A_1) (Remark 3.9) with the data at width λ_0 fixed in Section 10.9. Let $E: C_G(Z) \rightarrow C_G((-M)^\#) \otimes C_G((M^+)^\#)$ be the excision map of [KM11b, Theorem 5.6], with the end-perturbation data f_Z of the same data on Z , and $\bar{E}$ the reverse excision map. For other data on Z both sides change by the same gauge continuation map.

Theorem 10.2 (mixed torus excision). *On homology over $\mathbb{F}_2$,*

$$[\mathcal{K}][N_Z] = ([N_-] \otimes [N_+]) [E].$$

The gauge half. Z contains two parallel tori separating the two summands; cutting along them and regluing crosswise gives $Z' = (-M)^\# \sqcup (M^+)^\#$. The bundle is odd on these tori

and every component of Z and Z' meets one. The excision cobordism $\mathcal{E}: Z \rightarrow Z'$ is $T^2 \times P$, P a pair of pants with input c and outputs $c_\pm$, with the products of the punctured summands inserted along two arcs from c to $c_\pm$; its lifting group is $\Phi_{\mathcal{E}} = \{(s, s_-, s_+) : s = s_- + s_+\}$, dual to the four hypersurfaces of the construction of [KM11b, Section 5.3]. By [KM11b, Theorem 5.6], $[\mathcal{E}]$ is an isomorphism with inverse $[\bar{\mathcal{E}}]$. The theorem is stated over $\mathbb{Z}$; the maps $\mathcal{E}$, $\bar{\mathcal{E}}$ and the chain homotopies in its proof are defined over $\mathbb{Z}$, so the conclusion holds after reduction to $\mathbb{F}_2$, and the Künneth identification for Z' is over the field $\mathbb{F}_2$.

Plan of the proof. Only the mixed statement is new. It is not an instance of neck-stretching along a closed hypersurface: the torus region runs into the mixed ends, so no closed three-manifold separates the two sides. The proof has three steps:

- (a) (closed cut) $\mathcal{K}N_{\lambda_0}^J \simeq \mathcal{K}N_Z \bar{\mathcal{E}}$ at a fixed width λ_0 (Proposition 10.21, including the data change of Lemma 10.20);
- (b) (width) $N_b^J \simeq N_a^J$ for $\lambda_* \leq a \leq b < \infty$ (Proposition 10.17);
- (c) (completion) for large λ and large R , $\mathcal{K}N_\lambda^J = N_{-, \infty} \otimes N_{+, \infty}$ and $N_\pm^{(R)} = N_{\pm, \infty}$ as matrices, and $N_\pm^{(R)} \simeq N_\pm$ (Theorem 10.32, Proposition 10.16).

Then $[\mathcal{K}N_Z][\bar{\mathcal{E}}] = [N_- \otimes N_+]$, and since $[\bar{\mathcal{E}}] = [\mathcal{E}]^{-1}$ the theorem follows.

10.4. The linearized operator at the odd flat torus. The constant $\mu_T > 0$ of Section 10.2, the square root of the lowest eigenvalue of the twisted Laplacian of α , is the spectral gap that makes the following operators invertible.

Proposition 10.3 (the operator at the odd flat torus). *Let B be the Euclidean plane, a half-plane, or a straight strip, with the product metric on $T \times B$. Let $D = (d_\alpha^*, \sqrt{2}d_\alpha^+)$ act on ad-valued one-forms, with $a(\nu) = 0$ and $d_\alpha(a|_T) = 0$ on the boundary. Both D and its adjoint are invertible and $\|D^{-1}\|_{L^2 \rightarrow L^2} \leq \mu_T^{-1}$, independently of the strip width. Conjugating by $e^{\delta\tau}$, where $|\delta\tau| \leq 1$ and τ is constant on torus fibres, gives the bound $(\mu_T - |\delta|)^{-1}$ for $|\delta| < \mu_T$.*

Proof. The operator and its boundary conditions. Complexify the flat adjoint bundle and decompose into its twisted Fourier modes. There is no zero mode. On a mode of torus frequency $k \geq \mu_T$, rotate the orthonormal torus frame so that its covector is $k dx$ and write $a = f ds + g dt + p dx + q dy$. In the isometric identification of self-dual forms with three components, the four components of Da are

$$-f_s - g_t - ikp, \quad g_s - f_t + ikq, \quad p_s - ikf - q_t, \quad q_s + p_t - ikg.$$

At a straight boundary the conditions are $f = q = 0$: the latter is exactly the closedness of the torus restriction for this mode. Integration by parts cancels all cross terms and gives

$$\|Da\|^2 = \|\partial_s a\|^2 + \|\partial_t a\|^2 + k^2 \|a\|^2.$$

The boundary Green form pairs the free components g, p with the second and third output components. Thus those output components vanish in the adjoint domain. In invariant notation these conditions are $\omega(\nu, \tau_B) = 0$ and $d_\alpha^*((\iota_\nu \omega)|_T) = 0$; the scalar output is free. The same calculation for the adjoint gives the identical coercive bound. Summing over the torus modes proves the estimates on the plane, half-plane and strip. The closed-range theorem and the vanishing adjoint kernel give surjectivity. The argument uses no constant depending on the distance between the two strip boundaries. Conjugation by the weight adds a zeroth-order principal-symbol term of norm at most $|\delta|$; a Neumann series proves the weighted claim. $\square$

Corollary 10.4 (the stationary operator at the odd flat torus). *Let $I \subset \mathbb{R}$ be a compact interval, a closed half-line or $\mathbb{R}$, with coordinate r . On pairs $\zeta = (\varphi, b)$ of an ad-valued function and one-form on $T \times I$, with the product metric and the pulled-back flat connection α , let*

$$\mathcal{A}_\alpha(\varphi, b) = (d_\alpha^* b, *d_\alpha b + d_\alpha \varphi),$$

with the domain given by $b(\partial_r) = 0$ and $d_\alpha(b|_{T \times \{r\}}) = 0$ at every endpoint r of I , and no condition on φ . Then $\mathcal{A}_\alpha$ is self-adjoint and $\|\mathcal{A}_\alpha \zeta\| \geq \mu_T \|\zeta\|$. If $\tau = \tau(r)$ satisfies $|\tau'| \leq 1$ and $|\delta| < \mu_T$, then $\|e^{\delta\tau} \mathcal{A}_\alpha \zeta\| \geq (\mu_T - |\delta|) \|e^{\delta\tau} \zeta\|$.

Proof. Let $\zeta = (\varphi, b)$ lie in the domain and have compact support, and let $\beta \in C_c^\infty(\mathbb{R})$ be a nonzero cutoff. On $T \times B$ with $B = \mathbb{R}_t \times I$, a plane, a half-plane or a strip, put $a_N = \beta(t/N)(-\varphi dt + b)$. It satisfies the boundary conditions of Proposition 10.3: $a_N(\partial_r) = \beta(t/N)b(\partial_r) = 0$, and its restriction to each boundary torus fibre is closed. For a one-form $a = \psi dt + c$ on $T \times B$ one has $d^*a = -\partial_t \psi + d^*c$ and $\sqrt{2}d^+a$ corresponds, under the identification of Λ^+ with the one-forms of the three-dimensional slice used in [DFL21, (4.16)], to $\partial_t c + *d_\alpha c - d_\alpha \psi$, isometrically. Hence $\|Da_N\| = \|(\partial_t + \mathcal{A}_\alpha)(\beta(t/N)\zeta)\|$, and

$$\|Da_N\|^2 = N\|\beta\|^2\|\mathcal{A}_\alpha \zeta\|^2 + N^{-1}\|\beta'\|^2\|\zeta\|^2,$$

because the cross term is a multiple of $\int (\beta^2)' = 0$. Proposition 10.3 gives $\|Da_N\|^2 \geq \mu_T^2 \|a_N\|^2 = \mu_T^2 N \|\beta\|^2 \|\zeta\|^2$. Dividing by N and letting $N \rightarrow \infty$ gives the inequality for compactly supported ζ , and the general case follows by approximation. The weight $\tau(r)$ is constant on the torus fibres and independent of t , so the weighted version of Proposition 10.3 gives the weighted inequality in the same way. For ζ_1, ζ_2 in the domain, Green's formula leaves the boundary terms $\pm \int_{T \times \partial I} \langle b_1 \wedge b_2 \rangle$ and terms containing $b_i(\partial_r)$; the latter vanish, and the former vanish because the torus restrictions are closed, hence exact since $H^1(T; \text{ad } \alpha) = 0$, and $\int_T \langle d_\alpha u \wedge d_\alpha v \rangle = \int_T d \langle u, d_\alpha v \rangle = 0$. The boundary condition is elliptic in the sense of [BB16], the closed and the exact conditions on T coinciding, so $\mathcal{A}_\alpha$ is self-adjoint; with the lower bound it is invertible. $\square$

Theorem 10.5 (regularity at the flat torus boundary). *For $\rho > 0$ let $D_\rho^+ = \{(s, t) : s \geq 0, s^2 + t^2 < \rho^2\}$ and $Q_\rho = T \times D_\rho^+$. Let A be an L_1^p connection on Q_2 , $p > 2$, anti-self-dual, with weakly flat restriction to $T \times \{z\}$ for z on the boundary diameter.*

- (a) *A is gauge equivalent to a connection smooth up to the boundary.*
- (b) *There are $\hbar > 0$ and C_k , depending only on T and k , such that if $\epsilon^2 = \int_{Q_2} |F_A|^2 \leq \hbar$ then $\sup_{Q_1} |\nabla_A^k F_A| \leq C_k \epsilon$, and in a Coulomb–Neumann gauge $A = \alpha + a$ with $\|a\|_{C^k(Q_1)} \leq C_k \epsilon$.*
- (c) *Suppose that A satisfies the flat-fibre condition and has finite energy on the half-space $T \times \{s \geq 0\}$. Then*

$$\sup_{T \times D_1(z)} |\nabla_A^k F_A| \longrightarrow 0 \quad (|z| \rightarrow \infty).$$

Proof. (a) is the regularity theorem of [DFL24, Theorem 1] with the torus as matching surface; its proof uses the Lagrangian boundary condition only through local charts, and applies to the one-point moduli space. We prove (b) in three steps.

A fixed Neumann slice. Choose rounded contractible domains $\Omega_1 \Subset \Omega_2 \Subset D_2^+$, with relative closure along the physical diameter, such that $D_1^+ \Subset \Omega_1$. The qualitative compactness argument of [DFL24, Theorem 4.2, Lemma 4.25 and pp. 38–39], applied with the one-point torus moduli space, implies that connections whose energy tends to zero converge smoothly

in gauges on $T \times \Omega_2$ to a flat connection. That connection is gauge equivalent to α : the base is contractible and the odd torus has a single flat class. On a slightly smaller rounded domain, still denoted by Ω_2 , impose

$$d_\alpha^* a = 0, \quad a(\nu) = 0 \quad \text{on the whole boundary of } T \times \Omega_2.$$

There is no prescribed gauge on the artificial boundary. The scalar Neumann linearization is invertible since $H^0(T; \text{ad } \alpha) = 0$, so the slice follows from the implicit function theorem. In this slice a tends to zero smoothly as the energy tends to zero. The absolute twisted cohomology in degree one vanishes on $T \times \Omega_2$. The Hodge estimate and four-dimensional Sobolev embedding therefore give

$$\|a\|_{H^1} \leq C \|d_\alpha a\|_2 \leq C(\epsilon + \|a\|_4^2) \leq C\epsilon + C'\|a\|_{H^1}^2.$$

Qualitative smallness absorbs the last term. Thus $\|a\|_{H^1(T \times \Omega_2)} \leq C\epsilon$.

The boundary estimate. On the physical face let P_{coex} denote fibrewise coexact projection and set $Q_T = d_\alpha^*(d_\alpha d_\alpha^*)^{-1}$ on torus two-forms. Flatness is equivalent to

$$P_{\text{coex}} a_T = -\frac{1}{2} Q_T [a_T \wedge a_T].$$

The operator Q_T is bounded on each isotropic Sobolev space on $T \times \mathbb{R}_t$; no gain in the t direction is asserted. For a form w supported in a coordinate half-disk and with zero normal component, the linear estimate is

$$\|w\|_{H^{j+1}} \leq C_j (\|Dw\|_{H^j} + \|P_{\text{coex}} w_T\|_{H^{j+1/2}} + \|w\|_2).$$

For zero coexact trace, its H^1 version is the Fourier calculation in Proposition 10.3. Tangential derivatives commute with D and preserve the boundary condition; the equation recovers normal derivatives. For nonzero trace, subtract its Poisson extension, which maps $H^{j+1/2}$ to H^{j+1} and has zero normal component. This proves the displayed estimate at every integer $j \geq 0$.

Choose a scalar base cutoff χ and apply the estimate to $w = \chi a$. Its equation and boundary value are

$$Dw = -\sqrt{2}\chi(a \wedge a)^+ + [D, \chi]a, \quad P_{\text{coex}} w_T = -\frac{1}{2} Q_T [a_T \wedge w_T].$$

At each fixed order, qualitative smooth smallness of a makes multiplication by a , both in the interior and on the boundary, small in the displayed norms. The trace theorem therefore absorbs the terms containing the highest derivatives of w into the same H^{j+1} norm on the left. The cutoff commutator contains only j derivatives of a . Induction on nested half-disks, starting with the H^1 estimate, proves the H^{j+1} bound $C_j \epsilon$. Interior patches have the same argument without the trace term. Sobolev embedding then gives the asserted connection and curvature bounds at each fixed order, initially below an order-dependent threshold $\hbar_k$.

A common energy threshold. Fix $\hbar$ below the concentration threshold in the cited compactness theorem and small enough to enter the single low-order Neumann slice on $T \times \Omega_2$. The family of energy at most $\hbar$ is smoothly precompact modulo gauge on smaller domains: any failure would give concentration, whose energy is bounded below by that threshold. The normalized Neumann representatives are consequently smoothly precompact on $T \times \Omega_1$; the slice correction is unique near the identity and elliptic regularity controls it at every higher order. Let M_k be a bound for their C^k connection and curvature norms. For energies $E \in [\hbar_k, \hbar]$ use $M_k \leq (M_k/\sqrt{\hbar_k})\sqrt{E}$, and for $E < \hbar_k$ use the linear estimate just proved. Increasing C_k gives (b) with one $\hbar$ for all k .

(c) Given $\epsilon > 0$, choose a compact set outside which the energy is less than $\min(\epsilon^2, \hbar)$; for $|z|$ large $T \times D_2(z)$ lies outside it, and (b), or its interior version (Remark 10.6), gives the bound. $\square$

Remark 10.6. Part (b) of Theorem 10.5 holds, with constants depending only on T and k , on interior boxes $T \times D_\rho(z)$ disjoint from the face and with no boundary condition. Below the threshold the curvature bounds are Uhlenbeck’s interior estimates [Don02, Section 2.3], applied on balls of a fixed radius, and the gauge statement follows from the contradiction argument of (b), since a flat connection on $T \times D_2(z)$ is gauge equivalent to α . The local gauges may be taken even: the stabilizer of α is the Klein four-group, whose three nontrivial elements are odd and realize the three nonzero classes of $H^1(T; \mathbb{F}_2)$, so composing with one of them makes any local gauge even without moving α . Below, *small-energy regularity* refers to (b) and to this interior version.

10.5. The stationary problems and uniform decay.

Proposition 10.7 (uniform stationary gap). *Let $q \in P$ and $R \in [0, \infty]$. The stationary operator $\mathcal{A}_{q,R}(\varphi, b, \nu) = (d^*b, (*d_B + \text{Hess } h)b + d\varphi, J\partial_\sigma\nu)$ on Y_R and the section interval, with the matching conditions at the surface, the Lagrangian condition at the end of the interval and the flat-fibre condition at $r = R < \infty$, is self-adjoint with kernel T_qP , and its nonzero spectrum avoids $(-\gamma, \gamma)$ for a $\gamma > 0$ independent of q and R . The same holds for the joined slices $D_{i,\lambda}$, with kernel $T(P_i \times Q_i)$, uniformly in $\lambda \geq \lambda_*$, for a width threshold $\lambda_* > 0$. The stationary representatives B_q may be taken equal to α on the collar.*

Proof. Self-adjointness uses the Green cancellation at the matching surface (Convention 3.1) and the elliptic boundary theory of [BB16]. The kernel is computed as in Proposition 7.9. For uniformity, suppose that there are $q_n \in P$, $R_n \in [0, \infty]$ and ζ_n orthogonal to the kernel with $\|\zeta_n\| = 1$ and $\|\mathcal{A}_{q_n, R_n}\zeta_n\| \rightarrow 0$. Let χ be a cutoff of r , equal to 0 for $r \leq 1$ and to 1 for $r \geq 2$; it commutes with the primary Hessian, which is supported away from the collar. On the collar the operator is the stationary operator $\mathcal{A}_\alpha$ at the odd flat torus on $T \times [0, R_n]$, so Corollary 10.4 gives

$$\mu_T \|\chi\zeta_n\| \leq \|\mathcal{A}_\alpha(\chi\zeta_n)\| \leq \|\mathcal{A}_{q_n, R_n}\zeta_n\| + \|\chi'\zeta_n\|_{L^2(1 \leq r \leq 2)}.$$

Hence $\|\zeta_n\|_{L^2(K)} \geq 1/C$ for the fixed compact region $K = \{r \leq 2\}$ of the slice together with the section interval. Local elliptic estimates give a bound in $H^1(K)$, and after passing to a subsequence $q_n \rightarrow q$, $R_n \rightarrow R \in [0, \infty]$ (for a finite limit the slices Y_{R_n} are transported to Y_R by stretching maps as for ψ_R), and ζ_n converges weakly to a limit $\zeta \neq 0$, strongly in $L^2(K)$, with $\mathcal{A}_{q,R}\zeta = 0$ and the limiting boundary conditions. The kernel elements satisfy $\mathcal{A}_\alpha\zeta = 0$ on the collar and therefore decay in weighted L^2 like $e^{-\delta r}$ for every $\delta < \mu_T$, by Corollary 10.4 applied with the truncated weights $\min(r, M)$, uniformly in q ; so the kernels converge strongly and orthogonality to the kernel passes to the limit; ζ is a nonzero kernel element orthogonal to the kernel, a contradiction. $\square$

Proposition 10.8 (uniform decay on growing mixed ends). *There are $e_0, c > 0$ and, for every $m \geq 0$, numbers $e'_m \in (0, e_0]$ and C_m , all independent of $R \in [0, \infty]$ and of $\lambda \geq \lambda_*$, with the following properties. Let U be a finite-energy solution of the autonomous mixed equation on a half-cylinder $[\theta_0, \infty)$ over Y_R (or $D_{i,\lambda}$), and let $E(\theta)$ be its kinetic energy beyond θ .*

- (i) *If $E(\theta_0) < e_0$, then $E(\theta) \leq e^{4c}E(\theta_0)e^{-2c(\theta-\theta_0)}$ for $\theta \geq \theta_0$, and U converges to a stationary solution (B_q, q) .*

(ii) (*A priori estimate.*) If moreover U is smooth and $E(\theta_0) < e'_m$, then the limit of U in a temporal gauge is smooth, and in the temporal gauge normalized by a θ -independent even gauge transformation so that the limit is exactly π^*B_q ,

$$(10.1) \quad \|U - \pi^*B_q\|_{H^m([\theta, \theta+1] \times Y_R)} \leq C_m E(\theta_0)^{1/2} e^{-c(\theta - \theta_0)} \quad (\theta \geq \theta_0 + 2),$$

where the section minus the constant q is included in the norm (and Y_R is replaced by $D_{i,\lambda}$ in the joined case).

Proof. In a temporal gauge the solution is a path $\theta \mapsto (B(\theta), \nu(\theta))$ of slice configurations, a connection on the slice and a path in M across the section interval, subject to the matching condition at the surface, the Lagrangian condition at the end of the interval and, for $R < \infty$, the flat-fibre condition at $r = R$. It is the downward gradient flow of the local action $\mathcal{F}$, the Chern–Simons functional with the primary function plus the symplectic action of ν . Let Y' be the fixed compact part of the slice consisting of Y (of the two genus pieces, in the joined case) and the part $r \leq 3$ of the collar (of the bridge); it contains the loops of the primary perturbation. On the rest of the slice no perturbation acts, and anti-self-duality on the product gives $|F_A|^2 = 2|\iota_{\partial_\theta} F_A|^2$ there.

The neighbourhood. For $\epsilon > 0$ let $\mathcal{N}_\epsilon$ be the set of slice configurations that, after an even gauge transformation of the slice, differ from a stationary pair (B_q, q) by a field ξ with $\|\xi\|_{L^2_1} + \|\xi\|_{C^1} \leq \epsilon$, the norms being taken over the whole slice with the covariant derivative of B_q . For every $\epsilon > 0$ there is $e_0 > 0$, independent of R and λ , such that a solution whose kinetic energy on $[\theta - 2, \theta + 2]$ is less than e_0 has its slice at θ in $\mathcal{N}_\epsilon$. On Y' this follows by contradiction from the local compactness theorem [DFL21, Theorem 5.2], [DFL24, Theorem 3], in its local form: a sequence of solutions on $[\theta - 2, \theta + 2] \times Y''$, where $Y'' = Y \cup \{r \leq 4\}$, whose kinetic energy tends to zero converge, after gauge transformations, in C^∞ on $[\theta - 1, \theta + 1] \times Y'$ to a stationary pair. For $R \in [0, 4]$ use ψ_R and Theorem 10.5 at the physical face; compactness of this parameter interval makes the bounds uniform. Part (i) is gauge invariant. On the unperturbed collar interior regularity and Theorem 10.5(a) give a smooth representative, with local gauges patched as in Step 1 of Lemma 10.11. Apply the torus slice lemma to this representative. On the collar, small-energy regularity (Theorem 10.5(b), Remark 10.6) at orders at most three on the boxes $T \times D_2(z)$, and Lemma 10.9 below on the base $[1, R] \times [\theta - 1, \theta + 1]$, with (10.3) for $k \leq 2$, write the connection as $\alpha + b + \phi_1 dr + \phi_2 d\theta$ with b, ϕ and their first derivatives bounded pointwise by the fibre norms of F_A and its covariant derivatives of order at most three. These are bounded by the L^2 norms of F_A on the boxes; summing over a cover of bounded overlap bounds the L^2_1 and C^1 norms of the slice field $b + \phi_1 dr$ on the collar by $CE(\theta - 2)^{1/2}$, with C independent of R and λ . On the overlap $1 \leq r \leq 3$ the two gauges differ by a gauge transformation which maps a configuration close to α to one close to α ; it is close to the stabilizer of α , the Klein four-group of constant parallel elements. If it is close to a nontrivial element v , replace the collar gauge by its composite with v , which fixes $\alpha = B_q|_{\text{collar}}$. The transition is then close to the even stabilizer, which is trivial, so it is $\exp \varphi$ with φ small, and a cutoff of φ in r joins the two gauges. The Coulomb–Neumann implicit function theorem below is uniform in R and λ because the Neumann scalar Laplacian has a uniform gap on Y_R and $D_{i,\lambda}$, by the cutoff argument of Proposition 10.7 and $\Delta_\alpha \geq \mu_T^2$ on the collar.

Gradient inequality. Fix ϵ small. A configuration in $\mathcal{N}_\epsilon$ is written as $(B_q, q) + Q_q(\xi)$, where ξ satisfies $d_{B_q}^* \xi = 0$, has vanishing normal component on the boundary of the slice, satisfies the linearized matching, Lagrangian and flat-fibre conditions and is L^2 -orthogonal to the tangent space of the critical family at q ; here q is chosen by the implicit function theorem,

the critical family being compact modulo gauge and Morse–Bott, and $Q_q(\xi) = \xi + O(|\xi|^2)$ is the chart of [DFL21, Section 4.1] (the maps E_τ on a collar of the surface, and a metric in which the Lagrangian is totally geodesic on the section interval), together with the fibrewise correction of Lemma 10.28(a) at the flat face, used here on the single torus at the three-dimensional face of the slice in the $L_1^2 \cap C^1$ topology: the contraction is fibrewise, and $\|fg\|_{H^{1/2}} \leq C\|f\|_{C^1}\|g\|_{H^{1/2}}$ gives $\|c(b)\|_{H^{3/2}(T)} \leq C\|b\|_{C^1}\|b\|_{H^{1/2}}$. In this chart

$$\begin{aligned} \mathcal{F} - \mathcal{F}_{\text{crit}} &= \frac{1}{2}\langle \mathcal{A}_q \xi, \xi \rangle + R_3, & \nabla \mathcal{F} &= \mathcal{A}_q \xi + R_2, \\ |R_3| &\leq C\|\xi\|_{C^1}\|\xi\|_{L_1^2}^2, & \|R_2\|_{L^2} &\leq C\|\xi\|_{C^1}\|\xi\|_{L_1^2}, \end{aligned}$$

where $\mathcal{A}_q = \mathcal{A}_{q,R}$ is the operator of Proposition 10.7 and the gradient is identified with a vector at ξ through the derivative of Q_q , which differs from the inclusion by $O(\|\xi\|_{C^1})$. The remainders are the cubic terms of the Chern–Simons functional and of the symplectic action, the second-order terms of the chart, and those of the primary function, which is smooth [DFL21, Section 6] and supported in Y ; C is independent of R and λ because the slice is covered by unit patches of finitely many isometry types. The flat-fibre condition removes the boundary term at $r = R$ in the Hessian. By Proposition 10.7, $\|\mathcal{A}_q \xi\| \geq \gamma\|\xi\|$, and the elliptic estimate $\|\xi\|_{L_1^2} \leq C(\|\mathcal{A}_q \xi\| + \|\xi\|)$, uniform because it is local and the slice has bounded geometry (as in model (E) of the proof of Proposition 10.27), gives $\|\xi\|_{L_1^2} \leq C\|\mathcal{A}_q \xi\|$. For ϵ small, $\|\nabla \mathcal{F}\| \geq \frac{1}{2}\|\mathcal{A}_q \xi\|$ and

$$|\mathcal{F} - \mathcal{F}_{\text{crit}}| \leq \frac{1}{2}\|\mathcal{A}_q \xi\|\|\xi\| + C\epsilon\|\xi\|_{L_1^2}^2 \leq C'\|\mathcal{A}_q \xi\|^2 \leq 4C'\|\nabla \mathcal{F}\|^2,$$

with C' depending only on γ and on the constants above.

Decay. For $\theta \geq \theta_0 + 2$ the slice lies in $\mathcal{N}_\epsilon$, and along the solution $\partial_\theta \mathcal{F} = -\|\nabla \mathcal{F}\|^2 \leq -(\mathcal{F} - \mathcal{F}_{\text{crit}})/(4C')$. So the kinetic energy $E(\theta) = \mathcal{F}(\theta) - \mathcal{F}_{\text{crit}}$ decays at the rate $2c = 1/(4C')$ beyond $\theta_0 + 2$; since E is nonincreasing, this gives the bound of (i). The length of the path of slices over $[\theta, \infty)$ is at most the sum over unit intervals of the square roots of their kinetic energies, which decays at the rate c ; so the solution converges to a stationary solution (B_q, q) .

The a priori estimate. Let U be smooth, and let $\zeta = (\iota_{\partial_\theta} F_A, \partial_\theta u)$ be its gauge-covariant θ -derivative, whose L^2 norm on $[\theta, \infty)$ is $E(\theta)^{1/2}$. On the collar, small-energy regularity at order k gives $\sup_{T \times D_1(z)} |\nabla_A^k \zeta| \leq C_k \|F_A\|_{L^2(T \times D_2(z))}$ once the energy of $T \times D_2(z)$ is below the threshold at order k , and that energy is at most $2E(\theta(z) - 2)$. On Y' , ζ satisfies the linearized equation by translation invariance. In temporal gauge its gauge row vanishes: $d_A^* \zeta = -d_B^*(F_B + \nabla h(B)) = 0$, by Bianchi and gauge invariance of the cylinder function [DFL21, Proposition 2.10]. At the seam its normal component is $F_A(\partial_\theta, \partial_s) = 0$: anti-self-duality identifies this component with the curvature on the flat surface slice. Thus it belongs to the boundary domain of the gauge-fixed linearization. The local estimates of [DFL21, Proposition 5.27 and Remark 4.19], stated there over the whole slice, hold on the larger region Y'' , which has an artificial boundary at $r = 4$: their proof by contradiction uses only interior estimates near $r = 4$. They give $\sup_{[\theta, \theta+1] \times Y'} |\nabla_A^k \zeta| \leq C_k \|\zeta\|_{L^2([\theta-1, \theta+2] \times Y'')}$ when the kinetic energy is below a threshold depending on k . Choose $e'_m \leq e_0$ below these thresholds and below half the thresholds of small-energy regularity, for $k \leq m+1$. Summing the collar bounds over a cover of bounded overlap, for $\theta \geq \theta_0 + 2$ and $k \leq m+1$,

$$(10.2) \quad \sup_{[\theta, \theta+1] \times Y_R} |\nabla_A^k \zeta| + \|\nabla_A^k \zeta\|_{L^2([\theta, \theta+1] \times Y_R)} \leq C_k E(\theta - 2)^{1/2},$$

with C_k independent of R , λ and U . In a temporal gauge, which is smooth, $\partial_\theta A = \iota_{\partial_\theta} F_A$. Expanding the covariant derivative of $A(\theta)$ about that of $A(\theta_0 + 2)$ and inducting on k ,

(10.2) and integration in θ bound the derivatives of $A(\theta) - A(\theta_0 + 2)$ uniformly in θ , and $A(\theta)$ converges in every C^k to a limit B , which is therefore smooth. By (i), B and B_q represent the same point of the critical manifold, a quotient by the determinant-one group, so $B = g^* B_q$ for an even gauge transformation g of the slice, smooth because $dg = gB - B_q g$. In the normalized temporal gauge $A^N = (g^{-1})^* A$ the limit is B_q , and $A^N(\theta) - B_q = -\int_{\theta}^{\infty} \iota_{\partial_{\theta}} F_{A^N} d\theta'$. Now $\nabla_{B_q} = \nabla_{A^N} - [A^N - B_q, \cdot]$, so $\nabla_{B_q}^k \zeta$ is $\nabla_{A^N}^k \zeta$ plus terms that are polynomial in the derivatives of order less than k of $A^N - B_q$ and linear in those of ζ . Inducting on k and integrating from infinity, (10.2) and (i) give

$$\begin{aligned} \sup_{Y_R} |\nabla_{B_q}^k (A^N(\theta) - B_q)| + \|\nabla_{B_q}^k (A^N(\theta) - B_q)\|_{L^2(Y_R)} \\ \leq C_k \sum_{n \geq 0} E(\theta + n - 2)^{1/2} \leq C'_k E(\theta_0)^{1/2} e^{-c(\theta - \theta_0)} \end{aligned}$$

for $k \leq m$, the lower-order coefficients being bounded at each stage by the previous one. The θ -derivatives of $A^N - \pi^* B_q$ are the covariant θ -derivatives of ζ , bounded by (10.2), and the section is treated in the same way with $\partial_{\theta} u$. This proves (10.1). $\square$

10.6. Decay along the infinite torus collar. A gauge transformation of the odd bundle over a region of $T \times \mathbb{R}^2$ is *even* if it lifts to the determinant-one group. The even stabilizer of α is trivial (Section 10.2); the full stabilizer is the Klein four-group, whose three nontrivial elements are odd. For a connection A on $T \times \Omega$, $\Omega \subset \mathbb{R}^2$ with coordinates $z = (z_1, z_2)$, write $F_{jT} = (\iota_{\partial_{z_j}} F_A)|_{T \times \{z\}}$ and F_T for the restriction of F_A to a torus fibre.

Lemma 10.9 (torus slice). *There are $e_T > 0$ and, for each $k \geq 0$, constants C_k depending only on T , with the following property. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and A a smooth connection on $T \times \Omega$ such that every $z \in \Omega$ has a neighbourhood N and an even gauge transformation on $T \times N$ after which $\|A - \alpha\|_{C^1(T \times N)} \leq e_T$. Then there is a unique even gauge transformation g of $T \times \Omega$ with*

$$g^* A = \alpha + b + \phi_1 dz_1 + \phi_2 dz_2, \quad d_{\alpha}^* b(z) = 0, \quad \|b(z)\|_{C^0(T)} \leq C_0 e_T \quad (z \in \Omega),$$

where $b(z)$ is a one-form and the $\phi_j(z)$ are sections of ad , both on the fibre T . If moreover $\|F_A\|_{C_A^{k+1}(T \times \Omega)} \leq e_T$, then for every $z \in \Omega$

$$(10.3) \quad \sum_{|\beta|+j \leq k+1} (\|\partial_z^{\beta} b(z)\|_{H^j(T)} + \|\partial_z^{\beta} \phi(z)\|_{H^j(T)}) \leq C_k \sum_{i \leq k+1} \|(\nabla_A^i F_A)|_{T \times \{z\}}\|_{L^2(T)}.$$

The fibrewise equations gain derivatives only in the torus directions, so the top base derivatives of ϕ require the curvature derivative of order $k+1$. We call $g^* A$ the torus slice of A .

Proof. The fibre slice. Consider the map $(\xi, b) \mapsto \exp(\xi)^*(\alpha + b)$ from $\Omega^0(T; \text{ad}) \times \ker d_{\alpha}^*$ to connections on the odd bundle over T , in $C^{1,\beta} \times C^{0,\beta}$. Its derivative at $(0, 0)$ is $(\xi, b) \mapsto d_{\alpha} \xi + b$, an isomorphism onto $C^{0,\beta}$ one-forms, because $\Omega^1(T; \text{ad } \alpha) = \text{im } d_{\alpha} \oplus \ker d_{\alpha}^*$ and d_{α} is injective ($H^0 = H^1 = 0$). By the implicit function theorem every connection C^1 -close to α is $\exp(\xi)^*(\alpha + b)$ with $d_{\alpha}^* b = 0$ and ξ, b small. The decomposition is unique among even gauge transformations: if g is even and $g^*(\alpha + b) = \alpha + b'$ with b, b' small in C^0 , then $dg = g(\alpha + b') - (\alpha + b)g$ bounds g in C^1 , and as $b, b' \rightarrow 0$ every limit of g lies in the stabilizer of α and is even, hence is the identity; so for e_T small g is close to the identity and equals the gauge transformation given by the implicit function theorem.

The global gauge. By hypothesis and the fibre slice, each fibre $T \times \{z\}$ carries a unique even $g(z)$ putting $A|_{T \times \{z\}}$ into the slice; by uniqueness and the implicit function theorem

with parameters, $g(z)$ depends smoothly on z . The resulting family is fibrewise even, and it lifts to the determinant-one group over $T \times \Omega$ because the obstruction lies in $H^1(T \times \Omega; \mathbb{F}_2) = H^1(T; \mathbb{F}_2)$ and vanishes on a fibre. Uniqueness of g follows from the fibrewise uniqueness.

Estimates. On a fibre, $F_T = d_\alpha b + \frac{1}{2}[b \wedge b]$. Since $H^1(T; \text{ad } \alpha) = 0$, the elliptic estimate for $d_\alpha + d_\alpha^*$ gives $\|b\|_{H^{j+1}(T)} \leq C\|d_\alpha b\|_{H^j(T)} \leq C(\|F_T\|_{H^j(T)} + \|b\|_{C^0}\|b\|_{H^{j+1}(T)})$, and the last term is absorbed. For the base components, $F_{jT} = \partial_{z_j} b - d_{\alpha+b}\phi_j$. Applying d_α^* and using $d_\alpha^* \partial_{z_j} b = \partial_{z_j} d_\alpha^* b = 0$ gives

$$\Delta_\alpha \phi_j + d_\alpha^*[b, \phi_j] = -d_\alpha^* F_{jT},$$

and $\Delta_\alpha \geq \mu_T^2$ yields $\|\phi_j\|_{H^{j'+1}(T)} \leq C\|F_{jT}\|_{H^{j'}(T)}$. Then $\partial_{z_j} b = F_{jT} + d_{\alpha+b}\phi_j$ controls the first base derivatives of b . Differentiating these identities in z , and writing $\nabla_A = \nabla_\alpha + [b + \phi dz, \cdot]$, an induction on $|\beta|$ gives (10.3): for $|\beta| = k + 1$ the equation for $\partial_z^\beta \phi_j$ has source $d_\alpha^* \partial_z^\beta F_{jT}$, a curvature derivative of order $k + 1$, and the lower-order products are absorbed because $\|F_A\|_{C_A^{k+1}} \leq e_T$. $\square$

Proposition 10.10 (decay along the completed tail). *Let $V = (A, u)$ be a finite-energy solution on W_∞ (or on $W_{\pm, \infty}$). Let $\gamma_T = \mu_T/4$ and $0 \leq \delta \leq \delta_T$, where*

$$\delta_T = \min\{\mu_T/4, \mu_T^2/8, c/2\},$$

c is the rate of Proposition 10.8, and the weight on the torus region is $\tau_0(\theta)$ with $|\tau'_0| \leq 1$ and $|\tau''_0| \leq 1$. There is $S_0 \geq 3$ such that:

- (i) *for every k there is C_k with $\sup_{T \times D_1(z)} |\nabla_A^k F_A| \leq C_k e^{-\gamma_T(r(z) - S_0)}$ for every z with $r(z) \geq S_0 + 2$, uniformly in $\theta(z)$;*
- (ii) *Lemma 10.9 applies on $\Omega = (S_0, \infty) \times \mathbb{R}_\theta$, and for every m there are $S_m \geq S_0$ and C_m such that in the torus slice $A = \alpha + a_V$*

$$\|a_V\|_{H_\delta^m(T \times [S, \infty) \times \mathbb{R})} \leq C_m e^{-\gamma_T(S - S_0)} \quad (S \geq S_m + 2),$$

where H_δ^m is weighted by $e^{\delta\tau_0(\theta)}$.

Proof. Smallness. The energy of V on $T \times [S, \infty) \times \mathbb{R}$ tends to zero as $S \rightarrow \infty$, since these regions decrease to the empty set and carry no perturbation. By small-energy regularity (Remark 10.6) applied on the boxes $T \times D_2(z)$ with $r(z) \geq S_0 - 1$, for S_0 large we have $\sup_{r \geq S_0 - 1} |\nabla_A^k F_A| \leq e_T$ for $k \leq 2$, together with local even gauges in which $\|A - \alpha\|_{C^1} \leq e_T$. So Lemma 10.9 applies on $(S_0 - 1, \infty) \times \mathbb{R}$, and on each fibre $A = \alpha + b$ with $\|b\|_{C^0} \leq C e_T$. This fixes S_0 , independently of k and m .

A differential inequality. Put $e(z) = \int_{T \times \{z\}} |F_A|^2$. On the flat product the anti-self-dual curvature satisfies the Weitzenböck identity $\nabla_A^* \nabla_A F_A = -\{F_A, F_A\}$, with no curvature term and $|\langle \{F, F\}, F \rangle| \leq c_W |F|^3$. Summing $\partial_i^2 |F|^2$ over the four coordinate directions and integrating over T , where the torus directions integrate to zero,

$$(\partial_{z_1}^2 + \partial_{z_2}^2)e \geq 2 \int_T |\nabla_{A,T} F_A|^2 - 2c_W \sup |F_A| e.$$

The components of F_A in the parallel frame of $\Lambda^2(\mathbb{R}^4)$ are sections of ad over the fibre, and $\int_T |\nabla_T^\alpha w|^2 = \langle \Delta_\alpha w, w \rangle \geq \mu_T^2 \|w\|^2$ for such sections. With $A|_T = \alpha + b$ and $\|b\|_{C^0}$ small this gives $\int_T |\nabla_{A,T} F_A|^2 \geq \frac{3}{4} \mu_T^2 e$ for e_T small, and therefore

$$(10.4) \quad (\partial_{z_1}^2 + \partial_{z_2}^2)e \geq \mu_T^2 e \quad \text{on } r \geq S_0 - 1.$$

Integrability in θ . Let $E_{\pm}(\theta)$ be the kinetic energy of V on the mixed end $\pm\theta \rightarrow \infty$ beyond $|\theta|$, and let $\Theta_* \geq 4$ be such that $E_{\pm}(\Theta_* - 2)$ is less than e_0 and than half the threshold of small-energy regularity at order two. For $r(z) \geq S_0 - 1$ and $|\theta(z)| \geq \Theta_*$ the box $T \times D_2(z)$ lies in the unperturbed collar of a mixed end, where $|F_A|^2 = 2|\iota_{\partial_{\theta}} F_A|^2$ by anti-self-duality; so its energy is at most $2E_{\pm}(|\theta(z)| - 2) \leq Ce^{-2c|\theta(z)|}$, by Proposition 10.8(i). Small-energy regularity on the box (Remark 10.6) gives $\sup_{T \times D_1(z)} |\nabla_A^i F_A| \leq Ce^{-c|\theta(z)|}$ for $i \leq 2$, uniformly in r . The function e and its first two base derivatives are integrals over T of expressions quadratic in F_A and its covariant derivatives of order at most two, so $e + |\partial e| + |\partial^2 e| \leq Ce^{-2c|\theta|}$ for $|\theta| \geq \Theta_*$, uniformly in $r \geq S_0 - 1$, where ∂ is any base derivative; for $|\theta| \leq \Theta_*$ these quantities are bounded by the smallness step. Since $2\delta \leq c < 2c$, the function $f(r) = \int_{\mathbb{R}} e^{2\delta\tau_0(\theta)} e(r, \theta) d\theta$ is finite and of class C^2 for $r \geq S_0 - 1$, and the integrations by parts below have no boundary terms. Finally, for $|\theta| \geq \Theta_*$ the same identity gives $\int_{\theta}^{\theta+1} \int_{S_0}^{\infty} e dr d\theta' \leq 2E_{\pm}(|\theta|) \leq Ce^{-2c|\theta|}$; with $e^{2\delta\tau_0} \leq Ce^{2\delta|\theta|}$ and $2\delta < 2c$ the sum over unit intervals converges, and the range $|\theta| \leq \Theta_*$ contributes at most a constant times the energy of V . Hence $\int_{S_0}^{\infty} f(r) dr < \infty$.

Decay. By (10.4),

$$f'' = \int e^{2\delta\tau_0} ((\partial_r^2 + \partial_{\theta}^2)e) d\theta - \int (e^{2\delta\tau_0})'' e d\theta \geq (\mu_T^2 - 4\delta^2 - 2\delta)f \geq \frac{1}{2}\mu_T^2 f,$$

using $(e^{2\delta\tau_0})'' = (4\delta^2\tau_0'^2 + 2\delta\tau_0'')e^{2\delta\tau_0}$ and $\delta \leq \delta_T$. So f is convex, nonnegative and integrable on $[S_0, \infty)$, hence nonincreasing with limit 0. Put $\gamma'_T = \mu_T/\sqrt{2}$ and $h(r) = f(r) - f(S_0)e^{-\gamma'_T(r-S_0)}$. Then $h'' \geq \gamma_T'^2 h$, $h(S_0) = 0$ and $h \rightarrow 0$; at a positive interior maximum $h'' \leq 0 < \gamma_T'^2 h$, which is impossible. Hence

$$f(r) \leq f(S_0)e^{-\gamma'_T(r-S_0)} \quad (r \geq S_0).$$

Higher norms. (i) Fix k . The energy of $T \times D_2(z)$ is at most $\int_{r(z)-2}^{r(z)+2} f$, because $e^{2\delta\tau_0} \geq 1$; let $S'_k \geq S_0 + 2$ be such that this is below the threshold of small-energy regularity at order k for $r(z) \geq S'_k$. For such z , small-energy regularity bounds $\sup_{T \times D_1(z)} |\nabla_A^k F_A|$ by $C_k(\int_{r(z)-2}^{r(z)+2} f)^{1/2} \leq C'_k e^{-\gamma_T(r(z)-S_0)}$, since $\gamma'_T/2 > \gamma_T$. On the strip $S_0 + 2 \leq r(z) \leq S'_k$ the same supremum is bounded: for $|\theta(z)|$ large the box lies in the unperturbed collar of a mixed end with energy below that threshold, by Proposition 10.8(i), and small-energy regularity applies; the rest of the strip is compact, and there V is locally gauge equivalent to a smooth connection by interior regularity of the anti-self-duality equation, no perturbation acting on the torus region. Enlarging C'_k gives (i). (ii) Fix m , and let $S_m \geq S'_m + 2$ be such that, by (i), $\|F_A\|_{C_A^m} \leq e_T$ on $r \geq S_m$. By small-energy regularity, which is linear in the square root of the local energy, cover each unit square Q by a fixed finite collection of disks $D_1(z)$ whose larger disks $D_2(z)$ lie in $\{r \geq S'_m\}$. Apply the estimate on these larger disks for every $i \leq m$, taking the minimum of the finitely many energy thresholds. The weight varies by a bounded factor on each disk. Summing over a cover with bounded overlap,

$$\sum_{i \leq m} \int_{r \geq S} e^{2\delta\tau_0} \|\nabla_A^i F_A\|_{L^2(T)}^2 dz \leq C \int_{S-2}^{\infty} f(r) dr \leq C f(S_0) e^{-\gamma'_T(S-2-S_0)} \quad (S \geq S_m + 2).$$

On $(S_m, \infty) \times \mathbb{R}$ Lemma 10.9 applies with $\|F_A\|_{C_A^m} \leq e_T$, and by fibrewise uniqueness its torus slice is the restriction of that on $(S_0, \infty) \times \mathbb{R}$. Integrating (10.3) with $k = m - 1$ against the weight gives (ii), since $\gamma'_T/2 > \gamma_T$. $\square$

For each mixed end we use the stationary representatives B_q of Proposition 10.7, equal to α on the torus collar. For q near a given point of the critical manifold we fix a smooth local

family $q \mapsto B_q$ with this property, and in temporal gauge on product collars of the genus boundary surfaces: the flat restrictions to a product collar of T inside the compression body are gauge equivalent to α by gauge transformations depending smoothly on q , and these are extended over the compression body by cutoffs of their logarithms; the primary perturbation is supported away from the collars, where the connections are flat, so the temporal gauge along a collar is available. On Y_R and on $D_{i,\lambda}$ the family is independent of R and λ away from the collar, and equal to α on the collar (on the bridge in the joined case). Its tangent vectors are supported away from the torus collar and have vanishing normal components at the genus boundary. More is true: every B_q is in temporal gauge on the product collars of the genus boundary surfaces, so the normal component of $B_q - B_{q'}$ vanishes on these collars for all q, q' in the domain of the family, not only to first order. Such a family is defined near a base point. Where finitely many base points occur, as for the limits of the finite completed index-zero sets in Section 10.10, we fix one family near each distinct base point, on pairwise disjoint neighbourhoods in the critical manifold, and solutions with the same limit use the same family.

From now on we fix a level $m \geq 4$.

Lemma 10.11 (adapted representatives). *Let V be a finite-energy solution on W_∞ with limits q_e at the mixed ends e . There are $S_V \geq S_m + 3$, Θ_V and a representative $V^{\text{ad}} = (A^{\text{ad}}, u)$ of V , obtained from V by an even gauge transformation of class H_{loc}^{m+1} , which is smooth on W_∞ up to the genus matching faces and the Lagrangian arcs, such that:*

- (i) *on $T \times [S_V, \infty) \times \mathbb{R}_\theta$, $A^{\text{ad}} = \alpha + a_V$ is the torus slice, so Proposition 10.10(ii) applies to a_V ; moreover a_V has bounded derivatives of every order there, and for every k there are $S(k)$ and C_k with $\sup_{T \times D_1(z)} |\nabla_\alpha^k a_V| \leq C_k e^{-\gamma T(r(z) - S_0)}$ for $r(z) \geq S(k)$;*
- (ii) *on each mixed end e , with outward end coordinate θ , for every k there is C_k with*

$$\|(A^{\text{ad}} - \pi^* B_{q_e}, u - q_e)\|_{H^k([\theta, \theta+1] \times Y_{e,\infty})} \leq C_k e^{-c\theta} \quad (\theta \geq \Theta_V),$$

on the whole slice, its torus part included; in particular, since $\delta < c$, $V^{\text{ad}} - \pi^* B_{q_e}$ (and the section minus the constant q_e) lies in $H_\delta^m(\{\theta \geq \Theta_V\})$, and its norm on $\{\theta \geq \Theta\}$ tends to 0 as $\Theta \rightarrow \infty$;

- (iii) *on the gauge end the limit is exactly the fixed representative of the critical point, and on the gauge and strip ends V^{ad} converges to its limit in every C^k at every exponential rate below the spectral gap of the asymptotic operator [Don02, Section 4.1], [RS01].*

The same holds on $W_{\pm,\infty}$ and for pairs on $W_{-,\infty} \sqcup W_{+,\infty}$. Solutions on the finite domains are treated in Lemma 10.12.

Proof. Step 1: a smooth representative. Every point of W_∞ has a neighbourhood on which V is gauge equivalent, by a gauge transformation of class H^{m+1} , to a smooth solution: near the genus matching faces, on charts $D_-(\rho) \times \Sigma'$ together with the half-disks $D_+(\rho)$ of the section, by the seam regularity theorem [DFL24, Theorem 1] in a local Coulomb gauge, as in the proof of [DFL21, Proposition 3.6]; for the section, by elliptic regularity for pseudo-holomorphic maps with Lagrangian boundary conditions [MS12, Theorem B.4.1]; at the other points, by elliptic regularity of the perturbed anti-self-duality equation in a local Coulomb gauge. The holonomy terms are not local, so the last step is carried out level by level on nested locally finite covers. In each initial ball use radii $\rho_i(1 + 2^{-k})$ at level k , with the balls of radius ρ_i still covering. Fix the local Coulomb gauges at level m and shrink one step for each interior elliptic estimate. If the chart representatives are of class L_k^2 and the transition functions of class L_{k+1}^2 , then every holonomy term is of class L_k^2 on every chart: the holonomies

along the loops, based at the points of the chart, are products of parallel transports along subarcs contained in single charts and of transition functions at the switching points, and the terms are obtained from them by smooth functions and by integration against smooth compactly supported densities [Kro05, Proposition 3.1 and Section 3.2]. The elliptic estimate for (d^*, d^+) with a source of class L_k^2 then gives class L_{k+1}^2 . No holonomy term acts near the genus matching faces, on the torus region or on the section (the primary loops lie in the interiors of the compression pieces, and the secondary perturbations are supported in the core away from the matching lines), so the seam charts do not enter the induction. By induction the chart representatives are smooth, and so are the transition functions, since $dt_{ij} = t_{ij}A_j - A_it_{ij}$ with smooth A_i, A_j . Hence V is a smooth connection on a smooth bundle P' with a continuous bundle isomorphism $\Phi: P' \rightarrow P$. A smooth isomorphism Ψ that is C^0 -close to Φ exists, and $g = \Psi\Phi^{-1}$ is C^0 -close to the identity, hence even; $V^{\text{sm}} = g \cdot V$ is smooth.

Step 2: the ends. (a) Let $K_e = Y_{e,\infty} \cap \{r \leq S_V + 3\}$, a compact part of the slice containing Y , and let Θ_0 be such that on each mixed end the kinetic energy of V beyond Θ_0 is below e'_0 . By Proposition 10.8(ii) the temporal gauge of V^{sm} on $\{\theta \geq \Theta_0\}$ has a smooth limit, and after a θ -independent even gauge transformation, smooth because it relates two smooth stationary connections, the limit is $\pi^*B_{q_e}$; call the result V^N . The normalization is even because the critical manifolds are quotients by the determinant-one group. For every k , the C^k norm of $V^N - \pi^*B_{q_e}$ on $[\theta, \theta + 1] \times K_e$, section minus q_e included, is at most $C_k e^{-c\theta}$ for $\theta \geq \Theta_0$: for large θ by (10.1) with $m = k + 3$, applied beyond a θ at which the kinetic energy is below e'_{k+3} , and Sobolev embedding; on the remaining finite range by compactness. (b) By the smallness step of the proof of Proposition 10.10, Lemma 10.9 applies to V^{sm} on $(S_V - 1, \infty) \times \mathbb{R}$; the fibrewise implicit function theorem with parameters, at smooth data, gives a smooth gauge, so the torus slice $V^T = \alpha + a_V$ is smooth. (c) Let $\Theta_1 \geq \Theta_0$ be large. On $\{\theta \geq \Theta_1, S_V - 1 \leq r \leq S_V\}$ the representatives V^N and V^T are related by an even gauge transformation h , smooth because it relates smooth connections. It maps a connection close to α to one close to α (the first because $B_{q_e} = \alpha$ on the collar), so it is close to the even stabilizer of α , which is trivial, and $h = \exp \varphi$ with φ small. From $\nabla_\alpha h = h a_V - (V^N - \alpha)h$ and the injectivity of d_α on ad-valued functions on each fibre,

$$\|\varphi\|_{C^{k+1}(T \times D_1(z))} \leq C_k (\|a_V\|_{C^k(T \times D_2(z))} + \|V^N - \alpha\|_{C^k(T \times D_2(z))}).$$

Both terms are $O(e^{-c\theta(z)})$: the second by (a); the first by (10.3) on $\{r > S_V - 1, \theta > \Theta\}$ for large Θ , whose torus slice is the restriction of V^T by fibrewise uniqueness, because small-energy regularity and Proposition 10.8(i) give $\sup_{T \times D_1(z)} |\nabla_A^i F_A| \leq C_i E(\theta(z) - 2)^{1/2} \leq C'_i e^{-c\theta(z)}$, uniformly in r . (d) Put $V^E = \exp(\chi(r)\varphi)^* V^N$, with $\chi(r) = 0$ for $r \leq S_V - 1$ and $\chi(r) = 1$ for $r \geq S_V$. It agrees with V^T for $r \geq S_V$ and with V^N for $r \leq S_V - 1$. (e) On the gauge and strip ends, V^{sm} converges in a temporal gauge, in every C^k and at every rate below the spectral gap, to a nondegenerate critical point, respectively to a transverse intersection point [Don02, Section 4.1], [RS01]; a smooth normalization independent of the end coordinate makes the gauge-end limit the fixed representative.

Step 3: assembly. Let $\Theta_V \geq \Theta_1$. On the outer region $O = \{r \geq S_V\} \cup \{|\theta| \geq \Theta_V\}$, together with the gauge and strip ends, Step 2 gives representatives related to V^{sm} by smooth even gauge transformations that agree on the overlaps. They lift to the determinant-one group, so the obstruction to extending them over the core K lies in $H^4(K, K \cap O; \mathbb{Z})$, which vanishes because K meets the free genus matching faces. A continuous even extension therefore exists;

it can be taken smooth relative to a neighbourhood of $K \cap O$ within its homotopy class, hence even. The result is V^{ad} , which is smooth.

Step 4: (i). Given k , let $S(k) \geq S_0 + 3$ be such that the bounds of Proposition 10.10(i) at orders at most $k + 2$ are at most e_T on $\{r \geq S(k) - 1\}$. On $(S(k) - 1, \infty) \times \mathbb{R}$ Lemma 10.9 applies with $\|F_A\|_{C_A^{k+2}} \leq e_T$, its torus slice is the restriction of V^{ad} by fibrewise uniqueness, and (10.3) at order $k + 1$, with $H^2(T) \subset C^0(T)$, bounds $\sup_{T \times D_1(z)} |\nabla_\alpha^k a_V|$ by $C_k e^{-\gamma_T(r(z) - S_0)}$ for $r(z) \geq S(k)$. On $\{S_V \leq r \leq S(k)\}$ the derivatives of a_V of order k are bounded by Step 2(c) for $|\theta|$ large and by compactness otherwise. Since $S_V \geq S_m + 3$, Proposition 10.10(ii) applies to a_V .

Step 5: (ii). For $\theta \geq \Theta_V$ split the slice into three parts. On $\{r \leq S_V - 1\}$, $V^{\text{ad}} = V^N$ and (a) applies. On the band $S_V - 1 \leq r \leq S_V$, $V^{\text{ad}} - \alpha$ is obtained from $V^N - \alpha$ by the gauge transformation $\exp(\chi\varphi)$, and it is $O(e^{-c\theta})$ in every C^k by (a) and (c). On $\{r \geq S_V\}$, $V^{\text{ad}} - \pi^* B_{q_e} = a_V$; there, for θ beyond the threshold at which $\|F_A\|_{C_A^{k+1}} \leq e_T$ on $\{r > S_V - 1, \theta' > \theta - 1\}$, integrating the square of (10.3) at order k gives

$$\begin{aligned} \|a_V\|_{H^{k+1}([\theta, \theta+1] \times \{r \geq S_V\})} &\leq C_k \sum_{i \leq k+1} \|\nabla_A^i F_A\|_{L^2([\theta, \theta+1] \times \{r \geq S_V\})} \\ &\leq C'_k \|F_A\|_{L^2([\theta-1, \theta+2] \times \{r \geq S_V-1\})} \leq C''_k E(\theta - 1)^{1/2}, \end{aligned}$$

by small-energy regularity summed over unit squares and $|F_A|^2 = 2|\iota_{\partial_\theta} F_A|^2$ on the collar, and this is $O(e^{-c\theta})$ by Proposition 10.8(i). On the finite range of θ below that threshold the norm is finite by Step 4. Gauge transformations do not change the section, and (a) bounds $u - q_e$ on the section interval. Since $\delta < c$, (ii) follows.

Step 6: (iii) is Step 2(e), which Step 3 does not change. $\square$

Lemma 10.12 (regularity on the torus domains). *Let U be a finite-energy solution of class H_{loc}^m on $W_{\pm, \mathbb{R}}$ with $\mathbb{R} < \infty$, or on W_λ^J with $\lambda \geq \lambda_*$, for the data of this section (the primary data, end perturbations and almost complex structures of Section 10.1, with secondary perturbations supported in the cores away from the torus region and the matching lines), or for data that differ from these by compactly supported changes of the metric, cylinder-function terms, four-dimensional holonomy terms and deformations of the almost complex structure, all vanishing near the genus matching lines and, when present, the flat torus face. Then there is an even gauge transformation of class H_{loc}^{m+1} after which U is smooth up to the genus matching faces, the flat face and the Lagrangian arcs; its gauge-end limits are the fixed representatives; it converges on the gauge and strip ends in every C^k at every rate below the spectral gaps; and on each mixed end its difference from its stationary limit, section included, has C^k norm on $[\theta, \theta + 1]$ times the compact slice at most $C_k e^{-c\theta}$, for every k .*

Proof. Steps 1, 2(e), 3 and 6 of the proof of Lemma 10.11 apply on the domain at hand, with three changes. At the flat face the local smooth gauges are given by Theorem 10.5(a); no holonomy term acts there. The mixed-end slices are compact, so Proposition 10.8(ii) applies on the whole slice and no torus slice is needed: the normalized temporal gauge serves on the whole end. The additional terms are compactly supported and of the kinds handled in Step 1, the cylinder-function terms living on a compact cylinder over a closed three-manifold [KM11b, Sections 3.4–3.5]; they vanish near the genus matching lines, where Step 1 uses the seam regularity theorem. This covers the closed-cut data $\mathfrak{d}_Z(\ell)$ and the interpolation of Lemma 10.20: the secondary data of N_Z are supported away from the matching lines (Section 3.5 and Remark 3.9); those of $\bar{E}$ lie in the piece (II), which does not contain the matching lines (Lemma 10.18(ii)); the further perturbation of Proposition 10.21 is chosen

away from the matching lines; and the neck with f_Z and the change of metric lie in a collar of Z_{cut} , which meets the genus parts in $\{|\operatorname{Re} + 3| < 1\} \cup \{|\operatorname{Re} - 2\lambda - 3| < 1\}$, at distance of least two from the matching lines. $\square$

The moduli spaces on $W_{\pm, R}$ and W_{λ}^J used in this section therefore consist of classes of smooth solutions with these asymptotics, as on the domains of [DFL21, Proposition 3.6].

10.7. Energy, index and the loss formula.

Theorem 10.13. *(i) For every $R \in [0, \infty]$ and every configuration c on W_R with labels $\mathbf{a} = (a, x)$, the linearized operator on the extended weighted spaces is Fredholm and $\operatorname{ind} D_c = 8\kappa(c) + \mathfrak{c}(x) - \mathfrak{c}(a)$, with $2\kappa(c) - 2\kappa_0(\mathbf{a}) \in \mathbb{Z}$, the constants being those of [DFL21, Proposition 3.20] for X_D . On each mixed end e the stationary kernel is represented by the Coulomb fields $h_e(v) = (h_e^A(v), v)$, with v in the tangent space at q_e of the critical family of the end e and $h_e^A(v)$ in the kernel of the stationary operator $\mathcal{A}_{q_e}$, with zero normal component; the operator realized with these representatives on the extended weighted spaces, the extended Coulomb realization, is onto at every regular solution.*

(ii) For every $\lambda \geq \lambda_*$, the width threshold of Proposition 10.7, and every configuration c on W_{λ}^J with labels $(\mathbf{a}_-, \mathbf{a}_+)$, the same holds with the sum of the two constants, and $\operatorname{ind} D_c = \operatorname{ind} D_{c_-} + \operatorname{ind} D_{c_+}$, where $c_{\pm}$ are obtained by first deforming c , with fixed labels and hence fixed κ and index, to a configuration equal to α on a slice $T \times \{r_0\} \times \mathbb{R}$ of the bridge, then cutting there and completing by α .

(iii) The constants depend only on the labels, not on R or λ .

Proof. (i) For $R < \infty$, pull back by ψ_R : the pulled-back configuration is a DFL configuration on X_D with the metric changed on $T \times [-\frac{3}{4}, -\frac{1}{4}] \times \mathbb{R}_{\theta}$, a region that runs into both mixed ends and changes their slices from Y to Y_R . Along the family of metrics g_s , $s \in [0, R]$, the operator on the extended weighted spaces stays Fredholm, because Proposition 10.7 gives a gap uniform in the collar length; so the index is constant along the family and equals the index on X_D , given by [DFL21, Proposition 3.20 and Lemma 3.15]. The topological energy is independent of the metric, and the boundary term in Stokes' theorem at the flat face vanishes, the fibre restriction being flat and $H^1(T; \operatorname{ad}) = 0$. For $R = \infty$, deform c to equal α far out on the tail and compare W_{∞} with W_R by excision across a band of the collar on which all operators are the operator of Proposition 10.3. The excision is run on the unextended δ -weighted spaces, so that no extension vector crosses the cut, and the same dimension correction is added on both sides: the operators $D_{W_{\infty}} \oplus M_R$ and $D_{W_R} \oplus M_{\infty}$, where M_R is the invertible model of Proposition 10.3 on $T \times (-\infty, R] \times \mathbb{R}$ with the flat face and M_{∞} the model on $T \times \mathbb{R}^2$, have equal index once the band is longer than a constant depending only on the Fredholm estimate of $D_{W_{\infty}}$; that estimate, for $D_{W_{\infty}}$ and its adjoint, is the compact-remainder estimate of Step 2 of the proof of Proposition 10.27 with an infinite collar. The tail contributes nothing to the topological energy. The extended Coulomb realization is onto at a regular solution: its kernel and that of the realization of [DFL21] both equal H^1 of the deformation complex, and their indices agree by the finite-rank comparison of Remark 10.24. (ii) The energy is local and the section splits, so $\kappa(c) = \kappa(c_-) + \kappa(c_+)$; the index splits by the same excision with two bands, the plane models having index zero, and the asymptotic values add. (iii) holds because every constant was fixed on X_D . $\square$

Proposition 10.14 (loss formula). *Let U_j be solutions of index d on W_{R_j} with $R_j \rightarrow \infty$, or on $W_{\lambda_j}^J$ with $\lambda_j \rightarrow \infty$. After passing to a subsequence there are solutions V_{β} on the completed*

domains and broken trajectories Γ_ℓ on the original ends with $d - \sum_\beta \text{ind}(V_\beta) - \sum_\ell \text{ind}(\Gamma_\ell) = 4n$, $n \in \mathbb{Z}_{\geq 0}$. If all $\text{ind}(V_\beta) \geq 0$, all $\text{ind}(\Gamma_\ell) \geq 1$ and $d \leq 3$, then $n = 0$ and no energy is lost: after gauge transformations U_j converges to the limit pieces in C^∞ on compact subsets of each component of the broken limit. For every $\varsigma > 0$ there are compact sets in all its components such that, for large j , the energy outside their images in the finite domain is less than ς ; trajectory components are translated by their breaking parameters. If no trajectory breaks, these are fixed compact sets in the completed core domains. In particular this is the case when $d = 0$ under the stated index assumptions. The same statements hold for sequences of solutions on a fixed completed domain W_∞ or $W_{-, \infty} \sqcup W_{+, \infty}$.

Proof. Energy can be lost through interior instantons, through finite-energy instantons on $T \times \mathbb{R}^2$ or on T times a half-plane with flat-fibre boundary, through concentration on a boundary torus fibre, through seam and section bubbles, and through escape along the mixed ends, possibly spread along the lengthening slices. No classification is needed. Extract limits V_β on compact subsets of the completed domains, using fixed-geometry compactness and Theorem 10.5 at the flat face, and chains Γ_ℓ at the original ends. Build a reference configuration $\mathcal{R}_j$ on the same finite geometry as U_j , with the labels of U_j , as follows. First make each V_β and each Γ_ℓ exactly stationary near the attaching regions on the original ends, by cutoffs along paths of configurations with fixed limits; this changes neither the topological energy nor the index. For a separate sequence, pass to the adapted representative $V^{\text{ad}} = \alpha + a_V$ on $r \geq S_V$ (Lemma 10.11) and truncate its torus tail: put $\alpha + \chi_j(r)a_V$ on $T \times [S_V, R_j] \times \mathbb{R}$, with $\chi_j = 1$ for $r \leq R_j/2$, $\chi_j = 0$ for $r \geq R_j/2 + 1$ and $|\chi'_j| \leq 2$. The result equals α near $r = R_j$, so it satisfies the flat-fibre condition and is a configuration on W_{R_j} . For a joined sequence truncate the two tails in the same way at $r_\pm = \lambda_j/2$ and join the flat parts by the bridge identification, which carries α to α . Finally attach the chains Γ_ℓ at the original ends by the cutoff construction of [DFL21, Propositions 4.65 and 4.69]. The path $s \mapsto \alpha + (1 - s + s\chi_j)a_V$ consists of configurations on W_∞ with fixed labels, so by the Stokes argument in the proof of Theorem 10.13 the topological energy is constant along it, and the truncated configuration has the same energy on W_{R_j} because it is flat beyond $r = R_j/2 + 1$. Hence $\kappa(\mathcal{R}_j) = \sum \kappa(V_\beta) + \sum \kappa(\Gamma_\ell)$ exactly, and by Theorem 10.13(i)–(iii), with additivity of the constants $\mathfrak{c}$ along the original ends, $\text{ind } D\mathcal{R}_j = I = \sum \text{ind } V_\beta + \sum \text{ind } \Gamma_\ell$. By Theorem 10.13, $d - I = 4n$ and $\kappa(U_j) - \kappa(\mathcal{R}_j) = \frac{1}{2}n$. The analytic energy is $8\pi^2\kappa$ plus a correction from the perturbations that converges, by strong L^p convergence of the holonomy sources away from the concentration set; lower semicontinuity of the analytic energy gives $n \geq 0$. If $n = 0$, the limit of the analytic energies equals the sum of the analytic energies of the limit pieces. The energy on a compact set converges to that of the limit on the same set, by the smooth convergence away from the concentration set, and every concentration point carries a fixed quantum; so there is no concentration and the energy outside the union of compact sets in the core and translated trajectory components is small when those sets are large. When there is no breaking, only the core compact sets are required. For a fixed completed domain the same argument applies with $\mathcal{R}_j$ replaced by a fixed truncation. $\square$

10.8. Families with stationary ends.

Lemma 10.15. *Let F_λ , $\lambda \in [0, 1]$, satisfy the hypotheses of Theorem 5.4 except compact support of $\partial_\lambda F$, and instead: on each mixed end the critical set is connected, and it and a smooth family of stationary references R_q are independent of λ , with $F_\lambda(R_q) = 0$; there $\partial_\lambda F_\lambda$ is independent of the end coordinate, and it vanishes on the gauge and ordinary ends; and the stationary operators have a gap and the energy decay of Proposition 10.8(i) holds along*

the family, with constants e_0 and c independent of λ and with c larger than the weight δ of the weighted spaces. Then the conclusions of Theorem 5.4 hold.

Proof. On a mixed end, in the chart $R_q + z$ with z in the weighted domain (weight $e^{\delta\tau}$ in the end coordinate τ), differentiating $F_\lambda(R_q) = 0$ in λ gives $\partial_\lambda F_\lambda(R_q) = 0$, since R_q does not depend on λ . Hence

$$\begin{aligned} \partial_\lambda F_\lambda(R_q + z) &= \int_0^1 D(\partial_\lambda F_\lambda)(R_q + sz) z \, ds, \\ \|e^{\delta\tau} \partial_\lambda F_\lambda(R_q + z)\|_{L^2} &\leq \sup_{s \in [0,1]} \|D(\partial_\lambda F_\lambda)(R_q + sz)\| \|e^{\delta\tau} z\|_{L_1^2}, \end{aligned}$$

and the operator norm is bounded uniformly along the end because $\partial_\lambda F_\lambda$ is independent of the end coordinate and $\|z\|_{C^0}$ is small there. So the parameter derivative lies in the weighted target, $\lambda \mapsto F_\lambda$ is C^1 as a map into it, and the augmented operator $(\dot{\lambda}, z) \mapsto \dot{\lambda} \partial_\lambda F_\lambda + D_\lambda z$ is a rank-one extension of a Fredholm operator. Breaks occur only at the gauge and ordinary ends. A finite-energy trajectory on a mixed end runs from the connected critical family to itself. Its energy is a period of the action; if it is zero the trajectory is constant, and otherwise $\kappa > 0$, so by the lattice condition (H1) its index is at least four. Such a trajectory cannot split off a family of fibre index at most one whose other components, being regular, have index at least -1 . The preglued configuration agrees with the core on the mixed end, so the parameter derivative vanishes on the joined end and neck. The solutions are smooth up to gauge ([DFL21, Proposition 3.6], Lemma 10.12), and the argument of the proof of Proposition 10.8(ii), applied to each solution at its own parameter, shows that it decays at the rate $c > \delta$ in every H^k of its mixed-end slices, so it lies in the weighted domain. Family compactness uses the uniform energy decay and the λ -independent constants of Theorem 10.13. $\square$

Proposition 10.16. *Changing the length of a separate torus collar from ℓ_0 to ℓ_1 , with the outer flat-fibre boundary present throughout, changes the regular separate map by a chain homotopy.*

Proof. Apply Lemma 10.15; its hypothesis that the critical set is connected holds, the critical sets being the perturbed $L(U_i)$ and R_i , diffeomorphic to $\mathrm{SU}(2)^{h-1}$ and $\mathrm{SU}(2)^{g-1}$. The stationary references are perturbed-flat and equal to α on the collar; the perturbed flatness equation involves the metric only on the support of the perturbation, so the references are independent of the collar length; the pulled-back reference satisfies the equation on the collar because it is flat there; and the variation of the metric is supported in a region of the collar independent of the end coordinate. $\square$

Proposition 10.17 (width homotopy). *For $\lambda_* \leq a \leq b < \infty$, with λ_* the width threshold of Proposition 10.7, $N_b^J \simeq N_a^J$.*

Proof. Change the bridge length by a family of metrics supported on the bridge. The exact stationary references $R_{\lambda,q}$ on the joined mixed ends, flat and equal to α on the bridge, depend on the width only through the length of the flat region; differentiating $F_\lambda(R_{\lambda,q}) = 0$ shows that the nondecaying part of the parameter derivative vanishes. After pulling back by width-changing diffeomorphisms that are the identity outside the bridge, the hypotheses of Lemma 10.15 hold; on the compact range $[a, b]$ each slice has a gap, depending continuously on the width, so the gap is uniform along the family, and so is the energy decay, by the argument of the proof of Proposition 10.8(i). $\square$

10.9. The closed cut. We describe the gauge part of W_λ^J in the coordinates of the special quintuple [DFL21, Section 3.1]. For each side, $\mathbb{U}_- \subset \mathbb{C}$ is the region of [DFL21, Figure 1]: it contains the input arm $(-\infty, -3] \times [-1, 1]$, the arms $[-2, 0] \times [2, \infty)$ and $[-2, 0] \times (-\infty, -2]$ and the square $[-1, 0] \times [-2, 2]$, its boundary consists of the matching line $\mathbb{U}_\partial = i\mathbb{R}$ and two curves η_- , η'_- , and the gauge domain is $\mathbb{R} \times Y_0 \cup \mathbb{U}_- \times \Sigma \cup \mathbb{R} \times (-Y'_0)$, where $\mathbb{R} \times \partial Y_0$ is glued to $\eta_- \times \Sigma$ through the parameter t of η_- and the collar coordinate of a neighbourhood $\mathbb{R} \times (\frac{1}{2}, 1]$ of η_- , and similarly for η'_- . Along the input arm, $t = \text{Re}$, η_- is the line $\text{Im} = 1$ and η'_- the line $\text{Im} = -1$. For the two sides of W_λ^J write $Y_\pm, Y'_\pm$ for the compression bodies ($Y_- = -U_0$, $Y'_- = -U_1$, $Y_+ = V_0$, $Y'_+ = V_1$), $\Sigma_- = \Sigma_h$, $\Sigma_+ = \Sigma_g$, and $\mathbb{U}_\pm^{(\pm)}$, $\eta_\pm^{(\pm)}$, $\eta'_\pm^{(\pm)}$ for the planar data. The torus part of the gauge domain of W_λ^J is $T \times \tilde{\mathbb{U}}_\lambda$ with

$$\tilde{\mathbb{U}}_\lambda = \mathbb{U}_-^{(-)} \cup ([0, 2\lambda] \times \mathbb{R}) \cup \varrho_\lambda(\mathbb{U}_-^{(+)}), \quad \varrho_\lambda(z) = 2\lambda - \bar{z},$$

where the bridge coordinate of Definition 10.1 is $\text{Re } z - \lambda$, and the torus factors of the two sides are identified by the marked bundle isometry. The genus parts are $\mathbb{U}_-^{(-)} \times \Sigma_h$ and $\mathbb{U}_-^{(+)} \times \Sigma_g$.

The cut. Let $\gamma_0 \subset \tilde{\mathbb{U}}_\lambda$ be the arc consisting of the segment $\{-3\} \times [\frac{1}{2}, 1]$, the segment $[-3, 2\lambda + 3] \times \{\frac{1}{2}\}$ and the segment $\{2\lambda + 3\} \times [\frac{1}{2}, 1]$, with its two corners rounded, and γ_1 its mirror image in the real axis. The endpoints of γ_0 are the points of $\eta_-^{(-)}$ and $\varrho_\lambda(\eta_-^{(+)})$ with parameter $t = -3$, and the arc leaves them along the collar coordinate; similarly for γ_1 . Let $\mathbf{a}_\pm = \{-3\} \times [-1, 1]$ in the input arm of $\mathbb{U}_\pm^{(\pm)}$. Put

$$Z_{\text{cut}} = \bigcup_{\pm} (\{-3\} \times (Y_\pm)_0 \cup \{-3\} \times (-Y'_\pm)_0 \cup \mathbf{a}_\pm \times \Sigma_\pm) \cup (\gamma_0 \sqcup \gamma_1) \times T.$$

Let $\tilde{\mathbb{U}}_{\text{up}}$, $\tilde{\mathbb{U}}_{\text{mid}}$, $\tilde{\mathbb{U}}_{\text{down}}$ be the closures of the components of $\tilde{\mathbb{U}}_\lambda \setminus (\gamma_0 \cup \gamma_1)$ containing the upper arm, the two input arms, and the lower arm.

- Lemma 10.18** (the closed cut in the joined domain). *(i) Z_{cut} is a closed connected hypersurface in the gauge part of W_λ^J , disjoint from the matching lines, the section and the supports of the secondary perturbations, and there is an orientation-preserving diffeomorphism $Z_{\text{cut}} \cong Z = D_0 \cup (-D_1)$, covered by a bundle isomorphism, which is the identity on the pieces $(Y_\pm)_0$, $(-Y'_\pm)_0$ and the genus collars, and carries $\gamma_i \times T$ onto the torus neck of D_i .*
- (ii) Z_{cut} separates the gauge part into two pieces. The piece (I) containing the matching lines, together with the section $\mathbb{U}_+$ and a cylindrical end $(-\infty, 0] \times Z$ attached along Z_{cut} , is a domain of DFL type for the pair (A_0, A_1) with compression bodies D_0, D_1 , with the end types of the special quintuple of (D_0, D_1) : its genus parts are $(\mathbb{U}_-^{(\pm)} \cap \{\text{Re} \geq -3\}) \times \Sigma_\pm$, its compression pieces $[-3, \infty) \times (Y_\pm)_0$ and $[-3, \infty) \times (-Y'_\pm)_0$ together with $(\tilde{\mathbb{U}}_{\text{up}} \sqcup \tilde{\mathbb{U}}_{\text{down}}) \times T$, its gauge input slice is Z_{cut} , and its mixed-end slices are $D_{0,\lambda}$ and $D_{1,\lambda}$, which are D_0 and D_1 with torus necks of length $2\lambda + 4$, the slices including their arm collars $[-2, 0]$ on both sides of the interval $[-\lambda, \lambda]$ of Definition 10.1(c).*
- (iii) The other piece (II), with a cylindrical end attached along Z_{cut} , is the reverse excision cobordism $\bar{W}: Z' \rightarrow Z$ of [KM11b, Section 5.3], with $Z' = (-M)^\# \sqcup (M^+)^\#$ and cylindrical input ends. Here the two tori of Theorem 5.6 of [KM11b] are $T_a = T \times \{x_a\}$ and $T_b = T \times \{x_b\}$ for points $x_a \in \gamma_0$, $x_b \in \gamma_1$, identified through the product structure*

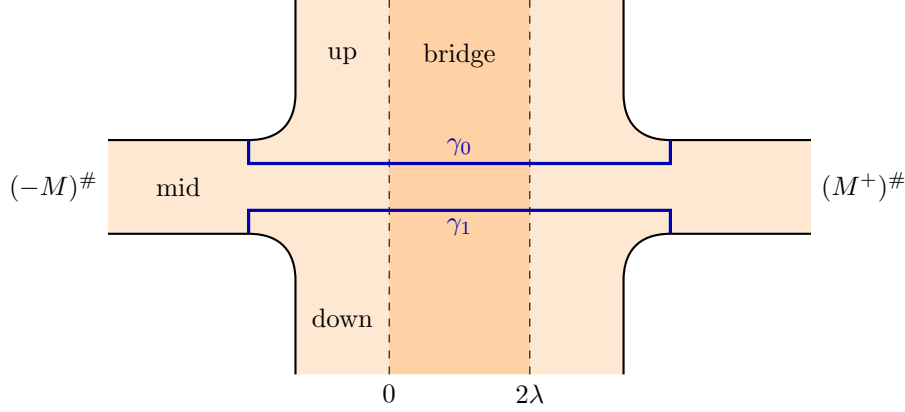


FIGURE 6. The torus base $\tilde{\mathbb{U}}_\lambda$ of the joined domain W_λ^J (Section 10.9): the planar region $\mathbb{U}_-$ of the lower side on the left, its reflection for the upper side on the right, and the bridge between the two matching lines (dashed). The arcs γ_0 and γ_1 run from the boundary curves of one input arm to those of the other. The closed cut Z_{cut} consists of $(\gamma_0 \sqcup \gamma_1) \times T$ together with the slices of the input arms at $t = -3$; it separates the part containing the matching lines (regions up and down, a domain for the pair (A_0, A_1)) from the part containing the two gauge inputs and the middle of the bridge (region mid, Kronheimer and Mrowka's reverse excision cobordism).

of $T \times \tilde{\mathbb{U}}_\lambda$; the product part $T \times U$ of [KM11b, Figure 4] is $T \times \tilde{\mathbb{U}}_{\text{mid}}$, truncated on the input arms; and the bundle is odd on T_a and T_b .

- (iv) Let $d_1, d_2, d_3, d_4 \in H^1(W_\lambda^J; \mathbb{F}_2)$ be dual to the hypersurfaces $\eta_-^{(-)} \times T$, $\eta_-'^{(-)} \times T$, $\varrho_\lambda(\eta_-^{(+)}) \times T$, $\varrho_\lambda(\eta_-'^{(+)}) \times T$. Then $d_1 = d_2$, $d_3 = d_4$ and $\Phi_J = \langle d_1, d_3 \rangle$. Restricted to (II), Φ_J is the subgroup $\phi_{\overline{W}}$ of [KM11b, Section 5.3] generated by the duals of the four copies of an interval times T over the dotted edges of U , and its image at the ends is $\Phi_{\overline{E}} = \{(s, s_-, s_+) : s = s_- + s_+\}$. Restricted to (I), Φ_J is the lifting group of the domain of the pair (A_0, A_1) , generated by the class that restricts to s_Z on Z ; and Φ_J is the fibre product of the two restrictions over $\phi_Z = \langle s_Z \rangle$.
- (v) After a change of the metric of $\tilde{\mathbb{U}}_\lambda$ supported in a compact neighbourhood of $\gamma_0 \cup \gamma_1$ inside the regions $\mathbb{U}^{(\pm)}$, disjoint from the bridge ends, the arms and the matching lines, Z_{cut} has a neighbourhood $(-1, 1) \times Z_{\text{cut}}$ with the product metric.

Proof. (i) Near the input arms the gauge domain is the product of $t = \text{Re}$ with the slice $\{t\} \times (Y_\pm)_0 \cup \{t\} \times [-1, 1] \times \Sigma_\pm \cup \{t\} \times (-Y'_\pm)_0$ in the genus part and with the collar segments $\{t\} \times (\frac{1}{2}, 1]$ in the torus part. Reading Z_{cut} across: the arc γ_0 joins the torus boundary of $(Y_-)_0$ at $\eta_-^{(-)}(-3)$ to that of $(Y_+)_0$ at $\varrho_\lambda(\eta_-^{(+)})(-3)$, so $(Y_-)_0 \cup \gamma_0 \times T \cup (Y_+)_0$ is $Y_- \cup_T Y_+ = (-U_0) \cup_T V_0 = D_0$ with a torus neck of length equal to the length of γ_0 ; likewise $(-Y'_-)_0 \cup \gamma_1 \times T \cup (-Y'_+)_0 = -D_1$; and $\mathbf{a}_- \times \Sigma_h$, $\mathbf{a}_+ \times \Sigma_g$ glue D_0 to $-D_1$ along Σ_h and Σ_g . So $Z_{\text{cut}} \cong D_0 \cup (-D_1) = Z$. The bundle is the given one on each piece and the pulled-back odd bundle on $\gamma_i \times T$, so the identification is a bundle isomorphism. The primary perturbations of the pieces $\mathbb{R} \times (Y_\pm)_0$ and $\mathbb{R} \times (-Y'_\pm)_0$ are independent of t and restrict on Z_{cut} to the primary perturbations of D_0 and D_1 . Connectedness is clear. The matching lines lie in $\text{Re} \in \{0, 2\lambda\}$, the section lies in $\{t > -3\}$, and the secondary supports of the data $\mathfrak{d}_Z(\ell)$

avoid Z_{cut} by their choice; the closed-cut comparison (Lemma 10.20) does not use this for the joined data $\mathfrak{d}_J$. The end-perturbation terms $\chi_{\pm} f_{\pm}$ on the two gauge inputs are chosen with $\text{supp } \chi_{\pm} \subset \{t \leq -4\}$, so that the transition lies in the piece (II), whose interior data are arbitrary, and the neck along Z_{cut} carries only the end perturbation f_Z .

(ii) The component of the complement containing the matching lines consists of the pieces listed. The region $\tilde{U}_{\text{up}}$ is bounded by γ_0 , by the parts $t \geq -3$ of $\eta_-^{(-)}$ and $\varrho_{\lambda}(\eta_-^{(+)})$, and by the end of the upper arm, whose cross-section is $[-2, 2\lambda + 2]$; it is diffeomorphic to $[-3, \infty) \times [0, 1]$ by a diffeomorphism that is an isometry onto a product for large t and that matches the collar coordinates at the two η -curves. Hence $[-3, \infty) \times (Y_-)_0 \sqcup \tilde{U}_{\text{up}} \times T \sqcup [-3, \infty) \times (Y_+)_0$ is diffeomorphic to $[-3, \infty) \times (D_0)_0$, and similarly for D_1 . With the genus parts this is the special quintuple of (D_0, D_1) restricted to $t \geq -3$, except that the metric of the compression pieces is induced from $\tilde{U}_{\lambda}$: on $\{t \leq t_1\}$ for a fixed t_1 it differs from a product in a compact set, and for $t \geq t_1$ it is the product with the slices $D_{i,\lambda}$, whose torus necks have length $2\lambda + 4$ rather than the length of γ_i . Its section is $\mathbb{U}_+$ with the Lagrangians A_0, A_1 , the perturbed Lagrangians of D_0, D_1 being $\tilde{P}_i^- \times R_i = A_i$ (Section 10.1).

(iii) The other component is

$$(-\infty, -3] \times ((Y_-)_0 \sqcup (-Y'_-)_0 \sqcup (Y_+)_0 \sqcup (-Y'_+)_0) \\ \cup (\{\text{Re} \leq -3\}\text{-parts of the input arms}) \times \Sigma_{\pm} \cup \tilde{U}_{\text{mid}} \times T.$$

The first two parts form $(-\infty, -3] \times Z'_{\text{cut}}$, where $Z'_{\text{cut}} = (Y_- \cup_{\Sigma_h} (-Y'_-)) \sqcup (Y_+ \cup_{\Sigma_g} (-Y'_+))$ is $Z' = (-M)^{\#} \sqcup (M^+)^{\#}$ cut along its two tori, with four boundary tori. Truncate the input arms at $\text{Re} = -T_1$ and $\text{Re} = 2\lambda + T_1$. Then $U := \tilde{U}_{\text{mid}} \cap \{-T_1 \leq \text{Re} \leq 2\lambda + T_1\}$ is a disk with eight corners, whose edges in cyclic order are: γ_0 ; the part $t \in [-T_1, -3]$ of $\varrho_{\lambda}(\eta_-^{(+)})$; the end of the right input arm; the corresponding part of $\varrho_{\lambda}(\eta_-'^{+})$; γ_1 ; the parts of $\eta_-'^{(-)}$ and, after the end of the left input arm, of $\eta_-^{(-)}$. The four η -edges (the dotted edges of [KM11b, Figure 4]) times T are glued to $[-T_1, -3] \times \partial Z'_{\text{cut}}$; the two arm ends times T are two-sided collars of the tori of Z' ; and $\gamma_0 \times T, \gamma_1 \times T$ are two-sided collars of T_a, T_b in Z . This is the description of the standard cobordism in the proof of [KM11b, Theorem 5.6]: the complement of $T \times U$ is a product of an interval with $Z'_{\text{cut}} = Z_{\text{cut}} \setminus (\text{collars of } T_a \cup T_b)$, and $T \times U$ is a product. The cobordism runs from Z' at $\text{Re} = -T_1$ to Z , so it is the reverse cobordism $\bar{W}$. The bundle is the product of the odd bundle with U on $T \times U$ and the product bundle of Z'_{cut} elsewhere; the identification of T_a with T_b required in [KM11b, Section 5.3] is the product structure.

(iv) The hypersurfaces $\eta_-^{(-)} \times T$ and $\eta_-'^{(-)} \times T$, together with the matching face $U_{\partial}^{(-)} \times \Sigma_h$ and parts of the ends, bound the region $\mathbb{R} \times (Y_-)_0 \cup U_-^{(-)} \times \Sigma_h \cup \mathbb{R} \times (-Y'_-)_0$; so their duals agree in $H^1(W_{\lambda}^J; \mathbb{F}_2)$, which is Lefschetz dual to H_3 relative to the boundary and the ends. The same holds on the $+$ side. The class d_1 restricts to the dual of the torus in $(-M)^{\#}$ and to zero in $(M^+)^{\#}$, and d_3 the other way round, so $\langle d_1, d_3 \rangle$ maps isomorphically onto $\Phi_- \oplus \Phi_+$, which is Φ_J . On Z , both d_1 and d_3 restrict to the dual of T_a , which equals that of T_b since the two tori cobound Z'_{cut} -pieces; this is s_Z . The restrictions to (II) are the duals of the dotted-edge hypersurfaces of (iii), which generate $\phi_{\bar{W}}$, with image $\{s = s_- + s_+\}$ at the ends. In (I) the two hypersurfaces $\eta_-^{(-)}|_{t \geq -3} \times T$ and $\varrho_{\lambda}(\eta_-^{(+)})|_{t \geq -3} \times T$ cobound $\tilde{U}_{\text{up}} \times T$ relative to the boundary, so d_1 and d_3 restrict to the same class, which vanishes on the matching surfaces and restricts to s_Z on Z . Since Z is connected, restriction $H^1(W_{\lambda}^J) \rightarrow H^1(\text{I}) \oplus H^1(\text{II})$ is injective with image the pairs agreeing on Z .

(v) Choose coordinates $(u, s) \in (-\frac{1}{4}, \frac{1}{4}) \times \gamma_i$ near γ_i , so that the collars of γ_0 and γ_1 are disjoint and lie in the domain, equal to the Euclidean coordinates (t, Im) near the endpoints and to $(\text{Im} \mp \frac{1}{2}, \text{Re})$ along the horizontal segment, and replace the metric by $du^2 + ds^2$ near the rounded corners and along the horizontal segments, where [DFL21] do not fix the planar metric to be Euclidean, interpolating in a compact region. Near the endpoints this is the product structure of the input arms, so Z_{cut} acquires a product collar. $\square$

Remark 10.19. Reversing the direction of the interval factors, the same description gives the forward cobordism $\mathcal{E}: Z \rightarrow Z'$ as $T \times U \cup [0, 1] \times Z'_{\text{cut}}$. Since Z'_{cut} is the union of $(T \times I_-) \# (-M)$ and $(T \times I_+) \# M^+$, this is $T \times P$ with $P = U \cup ([0, 1] \times I_-) \cup ([0, 1] \times I_+)$, a pair of pants, and with the products of the punctured summands inserted along two arcs from the input circle to the two output circles; this is the model used in Lemma 11.1(ii).

Data at the fixed width. Let $\mathfrak{d}_J$ be the regular joined data at width λ_0 : the primary data of Section 10.1, the end perturbations $f_{\pm}$ on the two gauge inputs, a split invariant secondary perturbation $\eta_- + \eta_+$ with each term supported in the core of its own side, and the split structure $J_- \times J_+$ on $\mathbb{U}_+$. For $\ell \geq 0$ let $\mathfrak{d}_Z(\ell)$ be the *closed-cut data*: the metric of Lemma 10.18(v) with $[-\ell, \ell] \times Z$ inserted along Z_{cut} ; the end perturbation f_Z of the data of N_Z at width λ_0 , multiplied by a cutoff of the neck coordinate supported in $[-\ell, \ell]$ and equal to one on $[-\ell + 1, \ell - 1]$; regular invariant secondary data on (I), chosen directly on (I), and those of $\bar{E}$ on (II), supported away from the neck; the same primary data, end perturbations $f_{\pm}$ and split structure $J_- \times J_+$. Here N_Z is the mixed map of Remark 3.9 for (D_0, D_1) with the metrics induced on Z_{cut} (the *data at width* λ_0), and $\bar{E}$ is computed at the same end data $f_{\pm}$ and f_Z .

Lemma 10.20 (data change at fixed width). *For every $\ell \geq 0$ for which $\mathfrak{d}_Z(\ell)$ is regular in index at most one, the joined counts at $\mathfrak{d}_J$ and at $\mathfrak{d}_Z(\ell)$ are chain homotopic over $\mathbb{F}_2$.*

Proof. Let ψ_{ℓ} be a diffeomorphism of $W_{\lambda_0}^J$ onto the domain with the inserted neck, equal to the identity outside a compact neighbourhood of Z_{cut} and stretching a collar of Z_{cut} . Pull back $\mathfrak{d}_Z(\ell)$ by ψ_{ℓ} . The two sets of data on $W_{\lambda_0}^J$ then differ only in a compact set K of the gauge domain disjoint from all ends: in the metric (near Z_{cut}), in the term f_Z (supported in the neck) and in the secondary perturbations. On the gauge inputs, the strip output and the mixed ends they coincide, with the same critical data. Interpolate linearly in the metrics and in the coefficients of the holonomy terms. The family is invariant under Φ_J : the metrics and secondary terms are invariant by choice, and f_Z is invariant under ϕ_Z and under the determinant-one group, while Φ_J restricts on Z into ϕ_Z (Lemma 10.18(iv)). The parameter derivative is supported in K . Theorem 5.4 applies and gives the chain homotopy. $\square$

Proposition 10.21. *For every $\lambda_0 \geq \lambda_*$, $\mathcal{K}N_{\lambda_0}^J \simeq \mathcal{K}N_Z \bar{E}$ over $\mathbb{F}_2$, where $N_{\lambda_0}^J$ is the joined map at the data $\mathfrak{d}_J$ and N_Z is taken with the data at width λ_0 .*

Proof. Fix $\lambda_0 \geq \lambda_*$. By Lemma 10.18, $W_{\lambda_0}^J$ with the data $\mathfrak{d}_Z(\ell)$ is the union, along a neck $[-\ell, \ell] \times Z$, of the DFL domain (I) for the pair (A_0, A_1) and the gauge cobordism (II) $= \bar{W}$, and its lifting group is the fibre product of theirs over ϕ_Z . The cut Z is connected, its critical points for the perturbed functional with f_Z are nondegenerate and irreducible, since the odd torus kills the stabilizers, the connecting H^0 term vanishes, and the framed groups glue by the fibre product [KM11b, Proposition 5.4]. Hypotheses (H1)–(H5) hold, (H1) by Theorem 10.13(ii). By Theorem 5.1, for $\ell \geq \ell_0$ the data $\mathfrak{d}_Z(\ell)$ are regular in index at most zero, and a further invariant perturbation, at this fixed finite ℓ , makes the index-one spaces regular. Its output region includes the compact neck where the fixed term χf_Z has output;

it remains away from the matching lines. Choose it small enough that openness of regularity and compactness preserve the already regular index-zero count. Its support may meet that closed neck; the joined count is then the composite of the count of (I) and that of (II). By Lemma 9.7, whose mixed-end clause covers the passage from the slices D_{i,λ_0} of (I) to the ends of the quintuple of (D_0, D_1) with the metrics of Z_{cut} , the count of (I) is chain homotopic to N_Z . The count of (II) is chain homotopic to $\bar{E}$, cobordism maps being independent of the interior data [KM11b, Section 5.1]. With Lemma 10.20 this gives $N_{\lambda_0}^J \simeq N_Z \bar{E}$, and composing with $\mathcal{K}$ gives the claim; Lemma 10.20 is applied to $\mathfrak{d}_Z(\ell)$ with the further perturbation above, which is regular in index at most one, and its proof is unchanged. $\square$

10.10. Torus completion. The torus completion follows the steps of the proof of Theorem 5.1: a uniform inverse at the preglued configuration, the contraction in a chart, a scalar Neumann inverse for the gauge slice, and the entry of nearby solutions into the chart. What is new are the inputs: the flat-fibre chart at the face (Lemma 10.28); the uniform Poincaré inequality on the torus domains (Lemma 10.26); the uniform inverse across the growing torus collar (Proposition 10.27), which rests on the invertibility of the odd flat torus operator (Proposition 10.3) and on the gap of Proposition 10.7 holding uniformly in the length; and the entry of nearby solutions into the chart (Proposition 10.31), through the torus slice of Lemma 10.9.

Conventions. Throughout this subsection L stands for R on the separate collared domains $W_L = W_{\pm,R}$ and for λ on the joined domains $W_L = W_\lambda^J$, and W_∞ stands for $W_{\pm,\infty}$ or $W_{-,\infty} \sqcup W_{+,\infty}$. The index e runs over the mixed ends; the critical manifold of the end e is $\mathcal{C}_e$ (a critical manifold of one side on a separate domain, a product $P_i \times Q_i$ on a joined end) and its slice is $Y_{e,L}$ (Y_R , Y_R' or $D_{i,\lambda}$). The level $m \geq 4$ is the one fixed before Lemma 10.11. The weight τ of Definition 10.1 is chosen with $\tau = \tau_0(\theta)$ on N_X and on the torus region, $|\tau_0'| \leq 1$, $|\tau_0''| \leq 1$, $\tau_0(\theta) = |\theta|$ for $|\theta| \geq 2$ and $\tau \geq 0$; it has zero normal derivative on ∂W_L and bounded derivatives of every order used. Fix δ with

$$0 < \delta \leq \delta_T, \quad \delta < \gamma/2, \quad \delta < 1, \quad \delta < \mu_o \text{ for every gauge and strip end } o,$$

where δ_T is as in Proposition 10.10, γ is the gap of Proposition 10.7, and $\mu_o > 0$ is the spectral gap of the asymptotic operator at o : the extended Hessian, gauge-fixing component included, at the nondegenerate irreducible critical point of a gauge end, and $J\partial_t$ with the Lagrangian boundary conditions at the transverse intersection point of a strip end. We also take δ below the weights of the original mixed ends in [DFL21, Definition 3.9]. Thus δ lies below the spectral gaps at all ends: at the mixed ends by $\delta < \gamma/2$, and at the gauge and strip ends by $\delta < \mu_o$. In particular the completed solutions converge on the gauge and strip ends at a rate larger than δ (Lemma 10.11(iii)), and model (O) in the proof of Proposition 10.27 applies. We further decrease δ so that $\delta C_P \leq \frac{1}{2}$, where C_P is the constant of Lemma 10.26, which does not depend on δ . All these conditions are independent of L . Constants C , C_{inv} , C_{quad} and r_1 depend on the fixed data and on the finite sets of completed solutions, not on L .

Lemma 10.22 (regular completed pieces). *For generic secondary perturbations in the invariant class of Theorem 5.6, one can choose a common compact output and dependence region in the original core, away from the inserted torus region and the matching lines, such that the completed domains carry no solutions of negative index, and all solutions of index at most one are regular. The index-zero sets are finite.*

Proof. By Theorem 10.13 and the energy identity, the index of a solution with given labels is bounded below. Proceed upward from the least index d that occurs, the minimum being taken

over all of the finitely many labels, since breaking changes labels. A sequence of solutions of index d on the fixed completed domain has, by Proposition 10.14 for a fixed completed domain, a limit of total index $d - 4n$ with $n \geq 0$ and nonnegative trajectory contributions; since no lower index occurs, the limit has index d , there are no chains and $n = 0$, so the space is compact. Apply Theorem 5.6(ii) with O in the original pieces, including their original ends, and excluding the inserted torus collar; every fixed primary or secondary perturbation has support there. Each side keeps an original end on which the limiting connection is flat on an odd surface collar; in the joined problem use the two sets O of the two sides. Thus the hypotheses of Theorem 5.6(ii) hold without loops crossing the bridge. Part (iii) selects finitely many directions and a generic small choice making index d regular; a negative index then does not occur. At the subsequent steps remove neighbourhoods of already regular broken trajectories before taking a finite cover. Ordinary end gluing remains valid with the completed torus tail: the cutoff and error are on the original end, while the fixed tail is controlled by its positive torus gap and the compact-remainder Fredholm estimate; only these two facts about the tail enter, and no invertibility of an unregularized completed operator is used. Repeat through index one. The selected loop images and outputs have compact union. Enlarge the common core and original-end truncations to contain this union, and fix them before the finite collar or bridge lengths vary. Beyond those truncations the original asymptotic equations are unchanged. Finiteness at index zero follows from compactness and regularity, and the lemma is proved. $\square$

We fix such data on $W_{\pm, \infty}$ and use the same secondary perturbations on $W_{\pm, R}$ and W_{λ}^J (split in the joined case). The primary data, end perturbations and almost complex structures are those of Section 10.1.

Asymptotic values and their tangent lifts. Let P be a smooth configuration on W_L ($L \leq \infty$) converging on each mixed end e to $R_e = (\pi^* B_{q_e}, q_e)$ in all the weighted norms used below. In the separate case suppose that $P = \alpha$ on a collar of the flat face. Choose local stationary representatives equal to α on the torus collar, in product gauge near the genus face, and put

$$k_e(v) = \left(\pi^* \frac{d}{ds} \Big|_{s=0} B_{\exp_{q_e}(sv)}, v \right), \quad v \in T_{q_e} \mathcal{C}_e.$$

The adapted exponential is fixed on the whole far strip of each mixed end [DFL21, Lemma 4.4 and Section 4.1]. Choose a cutoff $\beta_e(\theta)$ with a fixed transition band, after a common truncation on which P is close to R_e .

A cutoff of $k_e(v)$ need not satisfy the linearized matching condition at P . We use the following corrected lift $K_{P,e}(v)$. Write $a_e = B_{q_e}|_{\Sigma'}$. On the far genus trace express

$$A_P|_{\Sigma'} = E_{a_e}(b_P), \quad u_P|_{\mathbb{U}_{\partial}} = e_{q_e}(v_P), \quad [b_P] = v_P,$$

using the flat surface chart of [DFL21, Lemmas 4.1 and 4.4]. Transport the surface trace of $k_e(v)$ by $DE_{a_e}(b_P)$ and its section vector by $De_{q_e}(v_P)$, with the corresponding adapted exponential along the strip. Differentiating flatness and the identity $[E_{a_e}(b)] = e_{q_e}([b])$ proves the linearized matching condition at P and tangency on the Lagrangian arcs. Multiply by β_e . Extend the difference between this genus trace and that of $\beta_e k_e^A(v)$ into the fixed genus collar, with zero normal component, and add it to $\beta_e k_e^A(v)$. A weighted right inverse of the trace of fixed normal support is used for this extension. The difference has zero stationary limit. Thus $K_{P,e}$ has zero normal component, satisfies the actual linearized boundary conditions, vanishes on the torus region, and

$$(10.5) \quad K_{P,e}(v) - \beta_e k_e(v) \in H_{\delta}^m, \quad \|K_{P,e}(v) - \beta_e k_e(v)\|_{H_{\delta}^m} \leq C|v|.$$

Section differences here are read in the fixed end trivializations. The constants are uniform for the preglues below: the genus collars are fixed and their reference errors obey uniform weighted bounds. The extension may have longitudinal tails; only its trace identity and fixed normal support are used. Choose the local stationary parameter frames consistently on the fixed cores. In the joined case all these choices are products of the choices on the two sides.

Definition 10.23 (gluing spaces). Let P be a configuration on W_L as in the preceding paragraphs, for instance the preglued configuration $P_L(V)$ of Lemma 10.25 below. Let $\mathcal{X}_L^m$ be the space of pairs $w = (\zeta, \nu)$ with $\zeta \in H_\delta^m(W_L; \Lambda^1 \otimes \text{ad})$ and $\nu \in H_\delta^m(\mathbb{U}_+; u_P^* TM)$, where $\|\zeta\|_{H_\delta^m} = \|e^{\delta\tau}\zeta\|_{H^m}$ with derivatives taken with respect to the Levi-Civita connection and the connection of P , and similarly for ν , subject to the linearized boundary conditions: on the genus matching faces $\mathbb{U}_\partial \times \Sigma'$ the normal component of ζ vanishes, its restriction ζ_θ to each surface is d_{A_θ} -closed and $[\zeta_\theta] = \nu(0, \theta)$ [DFL21, Definition 3.9(iii)]; on the Lagrangian arcs ν is tangent to the Lagrangians; and on the flat face (separate case) $\zeta(\partial_r) = 0$ and $d_\alpha(\zeta|_{T \times \{(L, \theta)\}}) = 0$. Put

$$X_L = \mathcal{X}_L^m \oplus \bigoplus_e T_{q_e} \mathcal{C}_e, \quad \|(w, p)\|_{X_L} = \|w\|_{\mathcal{X}_L^m} + |p|,$$

$$Y_L = H_\delta^{m-1}(W_L; (\Lambda^0 \oplus \Lambda^+) \otimes \text{ad}) \oplus H_\delta^{m-1}(\mathbb{U}_+; \Lambda^{0,1} \otimes u_P^* TM),$$

$$D_L(w, p) = \left(d_{A_P}^* \zeta, \mathfrak{L}_P(w + \sum_e K_{P,e}(p_e)) \right),$$

where $\mathfrak{L}_P$ is the linearization at P of the mixed operator $\mathfrak{M}$ (the perturbed anti-self-duality operator and the section operator).

Remark 10.24. The image of a profile is bounded in the weighted target. Indeed, $\mathfrak{L}_P(\beta_e k_e(v))$ is the sum of $\sigma(d\beta_e)k_e(v)$ and $\beta_e(\mathfrak{L}_P - \mathfrak{L}_{R_e})k_e(v)$; the second term decays with $P - R_e$, and (10.5) handles the added correction. Hence the part of D_L acting on asymptotic values has norm at most $C|p|$. To compare indices with Theorem 10.13, represent its extended domain by the same decaying space and the same finite stationary parameter space, using Coulomb stationary representatives corrected at the actual matching trace. The two operators agree on the decaying summand. Their bounded stationary columns, including their scalar rows, differ by a finite-rank map. Thus D_L is Fredholm with the index of that theorem. This argument compares indices, not kernels at a nonsolution. The gauge row of D_L acts on ζ alone; the scalar row of another realization must therefore also be retained in this comparison. The nonlinear slice below imposes Coulomb gauge on this decaying coordinate, rather than on the entire stationary increment as in [DFL21, Proposition 4.6].

Lemma 10.25 (preglued references). *For $V \in \mathcal{M}^0(W_\infty)$ let $V^{\text{ad}} = \alpha + a_V$ on $r \geq S_V$ be as in Lemma 10.11. For $L \geq 2S_V + 2$ define $P_L(V)$ on W_L to be V^{ad} outside $T \times [S_V, L] \times \mathbb{R}$ and $\alpha + \chi_L(r)a_V$ on $T \times [S_V, L] \times \mathbb{R}$, where $\chi_L = 1$ for $r \leq L/2$, $\chi_L = 0$ for $r \geq L/2 + 1$ and $|\chi'_L| \leq 2$. For a pair (V_-, V_+) define $P_\lambda(V_-, V_+)$ on W_λ^J by the same truncation of both tails at $r_\pm = \lambda/2$, with α on the middle of the bridge, joined by the bridge identification. Then, with $P = P_L$:*

- (i) P_L is a smooth configuration with the labels of V (of the pair), equals α near the face, and on each mixed end converges to $\pi^* B_{q_e}$ on $Y_{e,L}$ (to $B_p \cup \alpha \cup B_q$ on $D_{i,\lambda}$);
- (ii) $\mathfrak{M}(P_L)$ is supported in $T \times [L/2, L/2 + 1] \times \mathbb{R}$ (on both sides of the bridge) and $\|\mathfrak{M}(P_L)\|_{Y_L} \leq Ce^{-\gamma L/2}$;
- (iii) $\text{ind } D_L = 0$;

(iv) $\|P_L - \alpha\|_{C^m(T \times [S, L] \times \mathbb{R})} \leq C e^{-\gamma_T(S-S_0)}$ for $S \geq S_V$; on each end $\sup_L \|P_L - \pi^* B_{q_e}\|_{H_\delta^m(\{\theta \geq \Theta\})} \rightarrow 0$ as $\Theta \rightarrow \infty$; and $P_L = V^{\text{ad}}$ on $\{r \leq L/2\}$.

Proof. (i) is immediate from Lemma 10.11, whose representative V^{ad} is smooth, with a_V smooth and χ_L a smooth cutoff. (ii) Only the truncation band carries an error. There the equation is the unperturbed anti-self-duality equation, and since $F_{\alpha+a_V}^+ = 0$,

$$F_{\alpha+a_V}^+ = (d\chi \wedge a_V)^+ + (\chi^2 - \chi)(a_V \wedge a_V)^+,$$

whose H_δ^{m-1} norm is at most $C\|a_V\|_{H_\delta^m(r \geq L/2)}$, bounded by Proposition 10.10(ii); the section is unchanged. (iii) The path $\alpha + (1-s+s\chi_L)a_V$ shows, as in the proof of Proposition 10.14, that $\kappa(P_L) = \kappa(V)$ (the sum $\kappa(V_-) + \kappa(V_+)$ in the joined case). By Theorem 10.13(i)–(iii) the index is $8\kappa + \mathfrak{c}(x) - \mathfrak{c}(a)$ with constants independent of L , so $\text{ind } D_{P_L} = \text{ind } D_V = 0$ (the sum of the two indices, by Theorem 10.13(ii)). (iv) follows from Propositions 10.10(i) and 10.10(ii) and Lemma 10.11(i)–(ii), using (i) for the pointwise C^m bound. $\square$

Lemma 10.26 (uniform scalar inverse). *There are r_0 , L_0 , C_P and C such that for $L \in [L_0, \infty]$ and every connection A with $\|A - A_{P_L(V)}\|_{C^m} \leq r_0$:*

- (i) $\|\xi\|_{L^2} \leq C_P \|d_A \xi\|_{L^2}$ for every $\xi \in L_1^2(W_L; \text{ad})$, with no boundary condition on ξ ;
- (ii) the Neumann map $\xi \mapsto (d_A^* d_A \xi, \nabla_n \xi|_{\partial W_L})$ is an isomorphism from $H_\delta^{m+1}(W_L; \text{ad})$ to $H_\delta^{m-1}(W_L; \text{ad}) \oplus H_\delta^{m-1/2}(\partial W_L; \text{ad})$, with inverse norm at most C .

At infinity $P_\infty(V) = V^{\text{ad}}$. The boundary consists of the genus matching faces and, when present, the flat face.

Proof. (i) On the far torus region the fibre gap and Lemma 10.25(iv) give $\|\xi\|_{L^2} \leq 2\mu_T^{-1} \|d_A \xi\|_{L^2}$ by integration over the planar base. There is no boundary term or factor depending on its area. The stationary spatial compressions have a uniform scalar gap as well. Otherwise normalized sections with covariant derivatives tending to zero would have vanishing mass on their torus collars, and a parallel limit on each fixed compact core. That limit vanishes on the overlap with the torus collar and hence on the connected core, a contradiction. The same argument covers the two cores of a joined slice. Integrating over the mixed-end coordinate and absorbing the uniformly small reference error gives the mixed-end estimate. The original gauge ends use their fixed irreducible cross-sections.

On the whole domain a normalized counterexample now has mass only on the fixed compact cores. Rellich compactness gives a parallel limit there, zero on the controlled overlap bands, again a contradiction. Thus

$$\|\xi\|_{L^2} \leq C_P \|d_A \xi\|_{L^2}$$

uniformly at finite and infinite lengths. A sufficiently small C^0 change of A preserves this estimate by absorption; take C_P for that neighbourhood.

(ii) This is the Lax–Milgram argument of the proof of Theorem 5.1. For $U = e^{\delta\tau}\xi$ the weak Neumann form becomes $\langle d_A U - \delta d\tau U, d_A V + \delta d\tau V \rangle$, whose diagonal $\|d_A U\|^2 - \delta^2 \|d\tau U\|^2 \geq \frac{3}{4} \|d_A U\|^2$ is coercive by (i), $|d\tau| \leq 1$ and $\delta C_P \leq \frac{1}{2}$. Since $\partial_n \tau = 0$, the conjugation does not change the Neumann condition. Local Neumann regularity on the unit patches of Step 3 of the proof of Proposition 10.27, on which the coefficients of A are bounded in C^m by Lemma 10.25(iv) and the choice of r_0 , gives the estimate in H_δ^{m+1} and surjectivity, directly on the completed domain. $\square$

Proposition 10.27 (uniform inverse). *There are L_0 and C_{inv} such that for $L \geq L_0$ and every V (every pair) in the finite completed index-zero sets, $D_L: X_L \rightarrow Y_L$ at $P = P_L(V)$ is an isomorphism with $\|D_L^{-1}\| \leq C_{\text{inv}}$.*

Proof. By Lemma 10.25(iii) and the remark after Definition 10.23, D_L is Fredholm of index zero, so it suffices to prove $\|(w, p)\|_{X_L} \leq C_{\text{inv}} \|D_L(w, p)\|_{Y_L}$. Write $D_L^{\text{dec}} w = D_L(w, 0)$.

Step 1: models. Fix parameters S, Θ, T_0, ℓ , chosen below in this order.

(T) The torus region $\{r \geq S\}$ (the part of the bridge at distance at least S from both pieces in the joined case), including its corners inside the mixed ends. There $D_L^{\text{dec}} = D_\alpha + \mathfrak{a}$, where D_α is the operator of Proposition 10.3 and $\mathfrak{a}$ is of order zero with $|\mathfrak{a}| \leq C e^{-\gamma_T(S-S_0)}$ by Lemma 10.25(iv); no perturbation acts there, and the corners of the torus region inside the mixed ends carry only the gauge field. The base is a strip $[S, L] \times \mathbb{R}$ with the flat-fibre condition at $r = L$, or a strip in the bridge; on it Proposition 10.3 with the weight $e^{\delta\tau_0(\theta)}$ gives $\|e^{\delta\tau}\zeta\| \leq (\mu_T - \delta)^{-1} \|e^{\delta\tau} D_\alpha \zeta\|$ for ζ vanishing near $r = S$. For a straight boundary the Bochner identity for $d + d^*$ on the base with absolute boundary conditions has no boundary term, so the identity in the proof of Proposition 10.3 also bounds $\|\nabla \zeta\|$ by $\|D_\alpha \zeta\|$. Hence $\|\zeta\|_{H_\delta^1} \leq C_T \|D_\alpha \zeta\|_{L_\delta^2}$ with C_T depending only on μ_T and δ .

(E) The mixed end e , beyond Θ , restricted to $r \leq S + \ell$. First identify the actual boundary domain with the stationary one. The surface Hodge and harmonic-block maps of [DFL24, Proposition 5.27], applied at paired depths in the fixed gauge and section collars, give an isomorphism $Q(\theta)$ from the stationary domain to the actual domain. It is the identity off those collars and extends by the identity over the added torus region. It and its inverse are bounded on L^2 and H^1 , and their differences from the identity and their longitudinal derivatives tend uniformly to zero as $\Theta \rightarrow \infty$ [DFL24, Lemma 5.67]. These results are stated for a fixed compact slice, locally in θ ; they apply uniformly on the infinite end with a growing torus collar because $Q(\theta)$ is supported near the genus surface, where the geometry is fixed, and is the identity on the torus collar. Use the corresponding fixed trivializations of the target. The transformed operator is $\sigma(\partial_\theta + \mathcal{A}_{q_{e,L}}) + \mathfrak{b}$, where $\mathcal{A}_{q_{e,L}}$ is the stationary operator of Proposition 10.7 on the whole slice $Y_{e,L}$ [DFL21, Section 4.2.1, (4.16)] and $\|\mathfrak{b}\|_{L^2} \leq \eta(\Theta) \|\zeta\|_{H^1}$ with $\eta(\Theta) \rightarrow 0$ uniformly in L by Lemma 10.25(iv) and these transport bounds. For a cutoff field vanishing near $\theta = \Theta$, apply Q^{-1} before extending by zero. The resulting field belongs to the stationary domain on $\mathbb{R}_\theta \times Y_{e,L}$. Since $\mathcal{A} = \mathcal{A}_{q_{e,L}}$ is self-adjoint with spectrum meeting $(-\gamma, \gamma)$ only in 0, and $0 < \delta < \gamma$, Fourier transform in θ gives $\|(\partial_\theta + \mathcal{A} - \delta)^{-1}\| \leq \min(\delta, \gamma - \delta)^{-1}$; the identity $\|\partial_\theta \xi\|^2 + \|(\mathcal{A} - \delta)\xi\|^2 = \|(\partial_\theta + \mathcal{A} - \delta)\xi\|^2$ and the elliptic estimate $\|\xi\|_{H^1(Y_{e,L})} \leq C(\|\mathcal{A}\xi\| + \|\xi\|)$, which is uniform in L because it is local, $Y_{e,L}$ has bounded geometry and on the collar $\mathcal{A} = \mathcal{A}_\alpha$ (Corollary 10.4), give $\|e^{\delta\theta}\xi\|_{H^1} \leq C_E \|e^{\delta\theta}(\partial_\theta + \mathcal{A})\xi\|$ with C_E independent of L . Transporting back and absorbing the H^1 -to- L^2 error gives the estimate on the actual mixed end.

(O) The original gauge and ordinary ends beyond T_0 : translation-invariant operators with invertible asymptotic operators, as in Theorem 5.1, perturbed by terms that are small for T_0 large.

(K) A fixed compact region K' containing the rest: the local elliptic estimates of [DFL21, Theorem 4.17 and Remark 4.19] at the matching faces and interior estimates give $\|\chi w\|_{H^1} \leq C(\|D_L^{\text{dec}} w\|_{L^2(K'')} + \|w\|_{L^2(K'')})$ for a cutoff χ supported in K' and a slightly larger compact K'' .

Step 2: the compact-remainder estimate. Let χ be a smooth function on $\mathbb{R}$ with $\chi = 0$ on $(-\infty, 0]$ and $\chi = 1$ on $[1, \infty)$. Patch the models with the partition of unity $\psi_T + \sum_e \psi_{E,e} + \psi_G + \psi_O + \psi_K = 1$, where $\psi_T = \chi((r - S)/\ell)$ on the torus region (in the joined case, the product of the two such functions of r_- and r_+), $\psi_{E,e} = (1 - \psi_T)\chi(\theta - \Theta)$ on the end e , ψ_G, ψ_O are cutoffs of fixed slope on the original ends beyond T_0 , and ψ_K has compact support. The only noncompact transition region is the corner band $\{\theta \geq \Theta, S \leq r \leq S + \ell\}$, where the

torus tail enters the mixed ends; there both (T) and (E) apply, and the cutoffs vary in r with slope at most C/ℓ . The gradient of ψ_T is at most C/ℓ on all of $\{S \leq r \leq S + \ell\}$; the other gradients are supported in the compact set $K = \{\Theta \leq \theta \leq \Theta + 1, r \leq S + \ell\}$, in the fixed parts of the slices and in compact transition regions on the original ends. Choose S and Θ with $C_T \sup |\mathfrak{a}| \leq \frac{1}{2}$ and $C_E \eta(\Theta) \leq \frac{1}{2}$, which does not depend on ℓ since C_T and C_E do not, then T_0 similarly. Applying the model estimates to $\psi_i w$ and summing,

$$\|w\|_{H_\delta^1} \leq C(\|D_{\mathbf{L}}^{\text{dec}} w\|_{L_\delta^2} + \ell^{-1} \|w\|_{L_\delta^2} + \|w\|_{L^2(K_1)}),$$

where K_1 is a compact set containing K , K'' and the compact transition regions. The coefficient of $\ell^{-1} \|w\|_{L_\delta^2}$ comes from the cutoff gradients through the models (T) and (E), whose constants do not depend on ℓ . A choice of ℓ with $C\ell^{-1} \leq \frac{1}{2}$ therefore gives

$$(10.6) \quad \|w\|_{H_\delta^1} \leq C(\|D_{\mathbf{L}}^{\text{dec}} w\|_{L_\delta^2} + \|w\|_{L^2(K_1)}) \quad (\mathbf{L} \geq \mathbf{L}_0),$$

with $\mathbf{L}_0 = 2(S + \ell) + 2$, so that $K_1 \subset \{r \leq \mathbf{L}/2\}$ and $P_{\mathbf{L}} = V^{\text{ad}}$ on K_1 . In the joined case $K_1 = K_{1,-} \sqcup K_{1,+}$, one compact set in the core region of each side, at a fixed position relative to that side.

Step 3: higher order and asymptotic values. Cover $W_{\mathbf{L}}$ by unit patches of finitely many isometry types: flat patches $T \times Q$ in the torus region, flat half-patches at the face, translates of fixed patches along the mixed ends and the original ends, and fixed patches of the core. The coefficients of $D_{\mathbf{L}}$ are bounded in C^m uniformly in $\mathbf{L}$ (Lemma 10.25(iv)). On the flat patches the local estimate is that of $D_\alpha + \mathfrak{a}$ with the flat-fibre domain at the face: the tangential derivatives commute with D_α and preserve the domain, and the normal derivatives are recovered from the equation, as in the proof of Theorem 10.5(b); on the other patches it is [DFL21, Theorem 4.17 and Remark 4.19]. The weight varies by a bounded factor on each patch. Summing with bounded overlap and using (10.6),

$$\|w\|_{\mathcal{X}_{\mathbf{L}}^m} \leq C(\|D_{\mathbf{L}}^{\text{dec}} w\|_{H_\delta^{m-1}} + \|w\|_{L^2(K_1)}).$$

Since the part of $D_{\mathbf{L}}$ acting on asymptotic values is bounded by $C|p|$,

$$(10.7) \quad \|(w, p)\|_{X_{\mathbf{L}}} \leq C(\|D_{\mathbf{L}}(w, p)\|_{Y_{\mathbf{L}}} + \|w\|_{L^2(K_1)} + |p|).$$

Step 4: the limit operator is injective. Let D_V be the operator of Definition 10.23 at $P = V^{\text{ad}}$ on W_∞ , and suppose $D_V(w, p) = 0$. Write $K_{V,e} = (K_{V,e}^A, K_{V,e}^S)$ and let $h_e(v) = (h_e^A(v), v)$ be the Coulomb stationary kernel representative of Theorem 10.13. Since $k_e^A(v)$ and $h_e^A(v)$ represent the same primary deformation and have zero normal component, their difference is $d_{B_{q_e}} \psi_e(v)$, with ψ_e satisfying the spatial Neumann condition. Put

$$\zeta_1 = \zeta + \sum_e K_{V,e}^A(p_e) - d_V \left(\sum_e \beta_e \psi_e(p_e) \right), \quad \nu_1 = \nu + \sum_e K_{V,e}^S(p_e).$$

This variation satisfies the geometric linearized matching condition and $\mathfrak{L}_V(\zeta_1, \nu_1) = 0$: the original variation uses the corrected lifts, and the subtracted term is an infinitesimal gauge transformation at the solution V . Its asymptotic representatives are $h_e(p_e)$. The remaining field is decaying by (10.5); the other errors contain $d\beta_e$ or $V - R_e$ and its derivatives.

The divergence of ζ_1 and its normal trace are decaying. The latter need not vanish before the final gauge correction: the spatial Neumann condition on ψ_e uses B_{q_e} , whereas d_V uses the actual connection. Apply Lemma 10.26 with both these data to solve

$$d_V^* d_V \xi = -d_V^* \zeta_1, \quad \nabla_{n,V} \xi = -\iota_n \zeta_1.$$

Then $(\zeta_1 + d_V \xi, \nu_1)$ belongs to the kernel of the extended Coulomb realization in Theorem 10.13. At V that operator is onto and has index zero, so its kernel vanishes. Hence $p = 0$, $\nu = 0$ and $\zeta = -d_V \xi$. The original scalar and normal conditions give the homogeneous Neumann equation for ξ , and the scalar lemma implies $\xi = 0$ and $\zeta = 0$.

In the joined case the section, the Lagrangians and all the profile corrections were chosen as products. The limiting operator is therefore the direct sum of the two separate operators, with their independent stationary coordinates, and is injective by the same argument.

Step 5: contradiction. If the bound failed, there would be $L_j \rightarrow \infty$ and (w_j, p_j) of norm one with $D_{L_j}(w_j, p_j) \rightarrow 0$, for a fixed V (the sets are finite), and $\|w_j\|_{L^2(K_1)} + |p_j| \geq 1/C$ by (10.7). Every compact subset of W_∞ embeds isometrically in W_{L_j} for large j , with $P_{L_j} = V^{\text{ad}}$ on it, where D_{L_j} coincides with D_V . A diagonal subsequence therefore converges, weakly in H_{loc}^m and strongly in L_{loc}^2 , to some (w_∞, p_∞) of norm at most one that satisfies the linear boundary conditions and $D_V(w_\infty, p_\infty) = 0$. Step 4 gives $(w_\infty, p_\infty) = 0$, contradicting the lower bound on K_1 . In the joined case the exhaustions are taken separately on the two sides: the part of $W_{\lambda_j}^J$ within distance $\lambda_j/2$ of the $\pm$ core is identified with the corresponding part of $W_{\pm, \infty}$, the bridge between them escapes to infinity, and the limit is a pair $(w_{\pm, \infty}, p_{\pm, \infty})$ on $W_{-, \infty} \sqcup W_{+, \infty}$, which vanishes by Step 4, while $K_1 = K_{1, -} \sqcup K_{1, +}$ retains mass. Invertibility follows from index zero. $\square$

Lemma 10.28 (the flat-fibre chart). *Let $s = m - \frac{1}{2}$ and let H_δ^s denote the weighted Sobolev space of ad-valued torus one-forms on the face $T \times \mathbb{R}_\theta$, with weight $e^{\delta\tau_0(\theta)}$. Let Π_{cl} be the fibrewise projection onto d_α -closed forms and $Q = d_\alpha^*(d_\alpha d_\alpha^*)^{-1}$ on fibrewise two-forms.*

- (a) *There are r_3, C such that for every closed b with $\|b\|_{H_\delta^s} \leq r_3$ the equation $c = -\frac{1}{2}Q[(b+c) \wedge (b+c)]$ has a unique coexact solution $c(b)$ with $\|c\|_{H_\delta^s} \leq r_3$. Then $\alpha + b + c(b)$ is flat on every fibre, $c(0) = 0$, $Dc(0) = 0$, and $\|c(b) - c(b')\|_{H_\delta^s} \leq C(\|b\| + \|b'\|)\|b - b'\|$. Conversely, if $\alpha + \tilde{b}$ is flat on every fibre and $\|\tilde{b}\|_{H_\delta^s} \leq r_3/C$, then $\tilde{b} = b + c(b)$ with $b = \Pi_{\text{cl}}\tilde{b}$.*
- (b) *There is a linear map $\text{Ext}_\delta: H_\delta^s(T \times \mathbb{R}) \rightarrow H_\delta^m(W_L)$, with norm bounded independently of L , whose values are torus one-forms supported in the collar $T \times [L-1, L] \times \mathbb{R}$ with vanishing dr and $d\theta$ components, and whose restriction to the face is the identity.*
- (c) *The map $\Psi^{\text{face}}(\zeta) = \zeta + \text{Ext}_\delta(c(\zeta_T|_{r=L}))$, defined on the ball of radius r_3/C of fields satisfying the linear flat-fibre condition, takes values in fields x for which $\alpha + x$ satisfies the nonlinear flat-fibre condition and $x(\partial_r) = \zeta(\partial_r)$ at the face; in particular it preserves the zero-normal subspace. It is inverted by $x \mapsto x - \text{Ext}_\delta(c(\Pi_{\text{cl}}x_T|_{r=L}))$, and both maps have first and second derivatives bounded independently of L .*

Proof. (a) The operators Π_{cl} and Q act fibrewise and are Fourier multipliers in the eigenmodes of the flat bundles $\text{ad } \alpha$ on T ; they commute with ∂_θ and with multiplication by functions of θ , in particular with $e^{\delta\tau_0}$. Since $\Delta_\alpha \geq \mu_T^2$ on each fibre, Q is bounded on the isotropic space $H^s(T \times \mathbb{R})$ (its symbol is bounded by μ_T^{-1} in the torus frequencies and does not involve the θ -frequency). As $s > \frac{3}{2}$, $H^s(T \times \mathbb{R})$ is an algebra, and $\|e^{\delta\tau_0}fg\|_{H^s} \leq C\|e^{\delta\tau_0}f\|_{H^s}\|e^{\delta\tau_0}g\|_{H^s}$ because $\tau_0 \geq 0$ and $e^{-\delta\tau_0}$ has bounded derivatives. Hence the right side of the equation is a contraction of the ball of coexact forms of radius r_3 for r_3 small, and the contraction estimates give the stated bounds. Flatness: $F_{\alpha+b+c} = d_\alpha c + \frac{1}{2}[(b+c) \wedge (b+c)] = 0$, because $d_\alpha b = 0$ and $d_\alpha Q$ is the identity on two-forms, $H^2(T; \text{ad } \alpha)$ being zero. Conversely, write $\tilde{b} = b + \tilde{c}$ with $b = \Pi_{\text{cl}}\tilde{b}$ and $\tilde{c}$ coexact; flatness gives $d_\alpha \tilde{c} = -\frac{1}{2}[\tilde{b} \wedge \tilde{b}]$, and Qd_α is the identity on coexact forms, so $\tilde{c}$ solves the equation and equals $c(b)$ by uniqueness. (b) Let $\Lambda = (1 + \Delta_{T \times \mathbb{R}})^{1/2}$,

acting on torus one-forms on $T \times \mathbb{R}$ with the connection α , and $E_0(c)(\rho) = \chi(\rho)e^{-\rho\Lambda}c$, $\rho = \mathbb{L} - r$, with a fixed cutoff χ supported in $[0, 1)$ and equal to one near 0. Mode by mode $\int_0^\infty |\partial_\rho^j e^{-\rho\mu}|^2 d\rho = \mu^{2j-1}/2$, so E_0 is bounded from $H^{k-1/2}$ to H^k of the half-space for integers k , and by interpolation from H^s to H^m . Put $\text{Ext}_\delta(c) = e^{-\delta\tau_0}E_0(e^{\delta\tau_0}c)$. The collar is a translate of a fixed product collar, so the norm does not depend on $\mathbb{L}$. (c) By (b) and the trace theorem, $\zeta_T|_{r=\mathbb{L}}$ lies in H_δ^s with norm at most $C\|\zeta\|_{H_\delta^m}$; the torus restriction of $\zeta + \text{Ext}_\delta(c)$ at the face is $\zeta_T|_{r=\mathbb{L}} + c(\zeta_T|_{r=\mathbb{L}})$, flat by (a) since $\zeta_T|_{r=\mathbb{L}}$ is closed, and the normal component is unchanged. For the inverse, if $\alpha + x$ satisfies the nonlinear condition, (a) gives $x_T|_{r=\mathbb{L}} = b + c(b)$ with $b = \Pi_{\text{cl}}x_T|_{r=\mathbb{L}}$, so $x - \text{Ext}_\delta(c(b))$ has closed restriction b . The derivative bounds follow from those of c and the boundedness of Ext_δ , the trace and Π_{cl} . $\square$

The chart. We specify the genus trace extension, including its stationary part. For a genus trace f with stationary limits f_e , put

$$\mathcal{T}^{\text{ext}}f = \mathcal{T}_\delta\left(f - \sum_e \beta_e f_e\right) + \sum_e \beta_e \mathcal{T}_e f_e.$$

Here $\mathcal{T}_\delta$ is a weighted right inverse of trace on fixed genus collars, and $\mathcal{T}_e$ is its spatial stationary counterpart. Both lift surface-tangential forms with zero normal and longitudinal components. Their norms are uniform, as for the Poisson extension in Lemma 10.28(b). Thus no constant longitudinal mode is assigned a weighted Sobolev norm.

For $z = (w, p)$, $w = (\zeta, \nu)$, set

$$(\zeta', \nu') = w + \sum_e K_{P,e}(p_e), \quad Q_a(b) = E_a(b) - a - b,$$

where E_a is the flat surface chart compatible with the adapted section exponential. Apply Q fibrewise to $b = \zeta'|_{\Sigma'}$, using $a = A_P|_{\Sigma'}$. At a stationary reference the resulting connection is

$$\widehat{B}_{e,p} = B_{q_e} + k_e^A(p) + \mathcal{T}_e Q_{a_e}(k_e^A(p)|_{\Sigma'}).$$

Choose $B_{\exp_{q_e} p}$ to have the same genus trace as $\widehat{B}_{e,p}$, and take this re-chosen family as the fixed family B_q of stationary references used throughout, in particular in Proposition 10.31; it is in product gauge on the genus collar, equals α near the torus, and any θ -independent normalization is admissible in Proposition 10.8(ii). The two traces represent the same class, so a smooth near-identity surface gauge provides this choice; extend it through the fixed genus collar and as the identity near the torus. Their first variations agree, so the adjustment is quadratic in p . In product gauge the difference $C_e(p) = B_{\exp_{q_e} p} - \widehat{B}_{e,p}$ has zero genus tangential and normal trace, vanishes on the torus collar, and satisfies $C_e(0) = DC_e(0) = 0$.

For fixed end cutoffs φ_e , equal to one sufficiently far out, define $\Psi_{\mathbb{L}}(z)$ to have connection

$$A_P + \zeta' + \mathcal{T}^{\text{ext}}(Q_{A_P|_{\Sigma'}}(\zeta'|_{\Sigma'})) + \sum_e \varphi_e \pi^* C_e(p_e) + \text{Ext}_\delta(c(\zeta_T|_{r=\mathbb{L}}))$$

and section $e_{u_P}(\nu')$, with the adapted exponentials of [DFL21, Lemma 4.4]. The face term is absent in the joined and completed cases. The genus trace is flat and has precisely the class of the section. The C_e term preserves that trace and makes the limiting connection exactly $B_{\exp_{q_e} p_e}$. All profile and genus corrections vanish on the torus region. Write $P^{(p)} = \Psi_{\mathbb{L}}(0, p)$; it equals P there. The same formula gives an unsliced chart if the normal component of the decaying field is allowed to vary. We use that version before imposing the Neumann gauge condition.

Define

$$\mathcal{F}_{\mathbb{L}}(z) = (d_{A_P}^* \zeta, \Pi_P \mathfrak{M}(\Psi_{\mathbb{L}}(z))) \in Y_{\mathbb{L}}.$$

The anti-self-duality row is read in $\Lambda^+ \otimes \text{ad}$ for the given metric. For the section row use its tangent-vector representation $\partial_s u + J(u)\partial_\theta u$ from (3.1). Embed M isometrically in $\mathbb{R}^N$ and project this vector orthogonally to $T_{u_P}M$. When the target is written as a $(0,1)$ -form, reconstruct that form from its projected value using the complex structure at u_P . This defines Π_P on the section row; it is the identity on the other rows. The reference section has uniformly bounded derivatives, so this projection is bounded on the weighted target. For z small it is an isomorphism on the tangent spaces along $u_{\Psi_L(z)}$, with uniformly bounded inverse.

Lemma 10.29 (chart and quadratic estimate). *There are L_0 , r_1 and C_{quad} such that for $L \geq L_0$ and $P = P_L(V)$:*

- (a) $\Psi_L(z)$ satisfies the nonlinear matching, Lagrangian and flat-fibre conditions, and zeros of $\mathcal{F}_L$ in the ball of radius r_1 are solutions of the mixed equation. Conversely, if U satisfies these conditions and $\|U - P^{(p)}\|_{H_\delta^m} + |p| \leq r_1/C_{\text{quad}}$, then there are ξ with $\|\xi\|_{H_\delta^{m+1}} \leq C_{\text{quad}}\|U - P^{(p)}\|_{H_\delta^m}$ and w with

$$\exp(\xi)^*U = \Psi_L(w, p), \quad d_{A_P}^*\zeta = 0, \quad \|w\|_{\mathcal{X}_L^m} \leq C_{\text{quad}}(\|U - P^{(p)}\|_{H_\delta^m} + |p|).$$

- (b) $\mathcal{F}_L(0) = (0, \mathfrak{M}(P_L))$ and $D\mathcal{F}_L(0) = D_L$;
 (c) $\mathcal{F}_L$ is of class C^2 on the ball of radius r_1 in X_L , with $\|D^2\mathcal{F}_L\| \leq C_{\text{quad}}$. Consequently $\mathcal{F}_L(z) = \mathcal{F}_L(0) + D_L z + N_L(z)$ with

$$\|N_L(z) - N_L(z')\|_{Y_L} \leq C_{\text{quad}}(\|z\|_{X_L} + \|z'\|_{X_L})\|z - z'\|_{X_L}.$$

Proof. (a) The trace identities in the construction give the exact genus matching and Lagrangian conditions. Lemma 10.28 gives the flat-face condition; the collars are disjoint and their corrections have zero normal component. The injectivity of Π_P on the actual section target proves the assertion about zeros.

For the converse first use the unsliced chart. Extract p from the stationary limits and subtract $\sum_e \varphi_e \pi^* C_e(p_e)$. The inverse surface exponential recovers the closed genus trace, and Π_{cl} recovers the closed torus trace. Subtract their extended nonlinear remainders and the corrected profiles $K_{P,e}$. Together with the inverse section exponential, this gives the unique decaying coordinate $\tilde{w}$. The trace and extension bounds give $\|\tilde{w}\| \leq C\|U - P^{(p)}\|_{H_\delta^m}$ uniformly.

Let $Z(\exp(\xi)^*U)$ be the connection coordinate of $\exp(\xi)^*U$ in the unsliced chart, and solve $(d_{A_P}^* Z, \iota_n Z)(\exp(\xi)^*U) = 0$ for a small decaying ξ ; the unsliced chart is needed since a gauge transform need not have zero normal coordinate. As in the entry step of the proof of Theorem 5.1, the derivative in ξ at $\xi = 0$, $U = P$ is, up to sign, the Neumann map of Lemma 10.26(ii), the second derivatives are bounded uniformly for $m \geq 4$, and the quantitative implicit function theorem gives the fixed radius and the bounds; at $U = P^{(p)}$ the coordinate vanishes, so the bound for ξ involves only $\|U - P^{(p)}\|_{H_\delta^m}$.

- (b) Each nonlinear surface correction has zero first derivative at zero, as does $C_e(p)$. Hence

$$D\Psi_L(0)(w, p) = w + \sum_e K_{P,e}(p_e).$$

The reference section solves its equation, so differentiating its target projection adds no term. The gauge row acts on ζ alone. This gives exactly D_L .

(c) The genus chart and its inverse have uniform C^2 bounds by the argument of Lemma 5.3(b): they are assembled from the surface Hodge decomposition, the exponential of the exact potential and a finite-rank harmonic part along the fixed genus collars, the surface

Hodge inverses have a common gap, the trace space $H^{m-1/2}$ on each three-dimensional face is an algebra, and the trace extensions are uniformly bounded. The flat-face chart has the bounds of Lemma 10.28. On the torus the equation is quadratic curvature; bounded geometry and bounded overlap give multiplication constants independent of its length. Fixed holonomy families obey the corresponding local bounds by differentiating parallel transport; for the primary terms see [DFL21, Proposition 6.1]. Secondary dependence and output sets stay in the fixed compact cores.

Finally, on each far mixed end subtract the same chart expression at its stationary reference. Every remaining term contains a decaying reference error, a decaying coordinate, or a derivative of a cutoff on a fixed band. The separated extension $\mathcal{T}^{\text{ext}}$ respects this decomposition. The stationary pair with limit $\exp_{q_e} p_e$ solves the equation exactly, so it contributes no nondecaying term after any differentiation in p . The weighted product estimates therefore bound the second derivative of $\mathcal{F}_L$ uniformly. The displayed quadratic estimate follows by the mean value theorem. $\square$

Proposition 10.30 (existence and local uniqueness). *Let $r = \min\{r_1, (4C_{\text{inv}}C_{\text{quad}})^{-1}\}$. There is L_1 such that for $L \geq L_1$ and every V (every pair) in the completed index-zero sets there is a unique $z_L(V)$ in the ball of radius r in X_L with $\mathcal{F}_L(z_L(V)) = 0$. It satisfies $\|z_L(V)\| \leq 2C_{\text{inv}}\|\mathfrak{M}(P_L)\|_{Y_L} \leq Ce^{-\gamma T^{1/2}}$, and $\mathcal{G}_L(V) := \Psi_L(z_L(V))$ is a regular solution.*

Proof. The map $\mathcal{F}_L$ takes values in Y_L because its target is projected explicitly: the anti-self-duality component onto the self-dual forms of W_L , and the Cauchy–Riemann component pointwise onto the tangent spaces along the preglued section u_P . By Proposition 10.27 and Lemma 10.29, the map $z \mapsto -D_L^{-1}(\mathcal{F}_L(0) + N_L(z))$ has Lipschitz constant at most $2C_{\text{inv}}C_{\text{quad}}r \leq \frac{1}{2}$ on the ball of radius r , and maps it into itself once $C_{\text{inv}}\|\mathcal{F}_L(0)\| \leq r/2$, which holds for $L \geq L_1$ by Lemma 10.25(ii). As in the proof of Theorem 5.1, this gives $z_L(V)$, the only zero in that ball, with the stated bound; since $r \leq r_1$, $\mathcal{G}_L(V)$ is a solution (Lemma 10.29(a)). The derivative $D\mathcal{F}_L(z_L) = D_L + DN_L(z_L)$, with $\|DN_L(z_L)\| \leq 2C_{\text{quad}}r \leq (2C_{\text{inv}})^{-1}$, is invertible. It is the linearization of the mixed equation at $\mathcal{G}_L(V)$ in the chart, with the Coulomb condition on the decaying coordinate and the corrected stationary lifts $K_{P,e}$, followed by Π_P ; at the solution $\mathcal{G}_L(V)$ the Cauchy–Riemann part of the linearization takes values in $\Lambda^{0,1} \otimes u_{\mathcal{G}_L(V)}^* TM$, on which Π_P is an isomorphism onto $\Lambda^{0,1} \otimes u_P^* TM$, bounded with bounded inverse. Since this slice and the Coulomb slice of the solution itself are both transverse to the gauge orbits, surjectivity of the linearization does not depend on the slice, and $\mathcal{G}_L(V)$ is regular. $\square$

A gauge transformation is *end-even* if it is even on the torus region and on all ends.

Proposition 10.31 (convergence into the gluing chart). *Let $L_j \rightarrow \infty$ and let U_j be index-zero solutions on W_{L_j} which, in the sense of Proposition 10.14, converge without loss to V (to a pair (V_-, V_+)) in the completed index-zero sets. Then for large j there are end-even gauge transformations g_j with $g_j^* U_j = \Psi_{L_j}(z_j)$, $\mathcal{F}_{L_j}(z_j) = 0$ and $\|z_j\|_{X_{L_j}} \rightarrow 0$. In particular $[U_j] = [\mathcal{G}_{L_j}(V)]$ for large j .*

Proof. We treat the separate case; the joined case is identical, with two pieces, the bridge in place of the tail and the slices $D_{i,\lambda}$. Let V^{ad} , S_V , Θ_V be as in Lemma 10.11, and let $q_{j,e}$ be the limits of U_j at the mixed ends. Fix $\Theta \geq \Theta_V + 2$ and $S \geq S_V + 2$, to be taken large, in this order. Consider the regions

$$R_E^e = \{\theta \geq \Theta\} \times Y_{e,L_j} \text{ (with the section strip),} \quad R_T = T \times [S, L_j] \times [-\Theta - 1, \Theta + 1],$$

the compact core region R_K , consisting of the part of W_{L_j} with $r \leq S + 1$, $|\theta| \leq \Theta + 1$ and the original ends up to a fixed T_0 , and the original ends R_O beyond $T_0 - 1$. These regions cover W_{L_j} ; the overlaps $R_E^e \cap R_T = \{\Theta \leq |\theta| \leq \Theta + 1, r \geq S\}$ are bands of unit width and growing length, and all other overlaps are compact.

Energy exhaustion. By Proposition 10.14, for every $\varsigma > 0$ there are a compact set K_ς and j_ς with the energy of U_j outside K_ς at most ς for $j \geq j_\varsigma$. The energy density outside the fixed secondary supports and the regions where the cutoffs $\chi_\pm$ are nonconstant is nonnegative: the curvature density on the torus region and the kinetic energy on the ends. We write $E_j(\Omega)$ for the energy of U_j on a region Ω and $o_j(1)$ for quantities tending to zero as $j \rightarrow \infty$ with Θ and S fixed.

The mixed ends. Replacing U_j by an even gauge transform (Lemma 10.12), we may assume that U_j is smooth. The kinetic energy of U_j beyond $\Theta - 2$ converges to that of V , which is at most $Ce^{-2c(\Theta-2-\Theta_V)}$ by Proposition 10.8(i); for Θ large it is below the threshold e'_m of Proposition 10.8(ii), which is uniform in L . In a temporal gauge on R_E^e , normalized by a gauge transformation independent of θ so that the limit is exactly $\pi^*B_{q_{j,e}}$, the representative U_j^E satisfies, by (10.1) with $\theta_0 = \Theta - 2$, summed over unit intervals with the weight $e^{\delta\theta}$ and $\delta < c$,

$$\|U_j^E - \pi^*B_{q_{j,e}}\|_{H_\delta^m(R_E^e)} \leq Ce^{\delta\Theta} E_j(\{\theta \geq \Theta - 2\})^{1/2} \leq Ce^{-(c-\delta)\Theta} + o_j(1).$$

The slices Y_{e,L_j} include the whole torus collar, so this controls the corners $\{\theta \geq \Theta, r \geq S\}$ of the torus region. The same estimate for V , the comparison of the sections at $\theta = \Theta$ and the smooth convergence on compact sets give $d(q_{j,e}, q_e) \leq Ce^{-c\Theta} + o_j(1)$, so the coordinates $p_{j,e} = \exp_{q_e}^{-1} q_{j,e}$ satisfy $|p_j| \rightarrow 0$ along the diagonal choice below. On R_E^e , where $\beta_e = \varphi_e = 1$, the moved reference is $P_{L_j}^{(p_j)} = \pi^*B_{q_{j,e}} + (P_{L_j} - \pi^*B_{q_e}) + O(|p_j| |P_{L_j} - \pi^*B_{q_e}|)$, the last term coming from the E -corrections, so

$$\|U_j^E - P_{L_j}^{(p_j)}\|_{H_\delta^m(R_E^e)} \leq \|U_j^E - \pi^*B_{q_{j,e}}\|_{H_\delta^m(R_E^e)} + C \sup_L \|P_L - \pi^*B_{q_e}\|_{H_\delta^m(\{\theta \geq \Theta\})},$$

and both terms are small for Θ large, uniformly in j large, by the display and Lemma 10.25(iv).

The torus strip. Let $R'_T = T \times [S - 2, L_j] \times [-\Theta - 3, \Theta + 3]$. For $\varsigma > 0$ let K_ς, j_ς be as in the energy exhaustion; if $S - 2$ exceeds the largest value of r on K_ς , then R'_T is disjoint from K_ς for every Θ , so $E_j(R'_T) \leq \varsigma$ for $j \geq j_\varsigma$. We write $\varsigma(S)$ for the least such ς ; it tends to zero as $S \rightarrow \infty$, independently of Θ . By small-energy regularity at the face and in the interior (Theorem 10.5(b), Remark 10.6), for S and j large the curvature of U_j and its covariant derivatives up to order m are small on R_T and there are local gauges with $\|U_j - \alpha\|_{C^1} \leq e_T$. Lemma 10.9 on the simply connected base $[S - 1, L_j] \times [-\Theta - 2, \Theta + 2]$ gives the torus slice $U_j^T = \alpha + a_j$, and (10.3), integrated over unit patches with small-energy regularity, gives

$$\|a_j\|_{H^m(R_T)} \leq CE_j(R'_T)^{1/2} \leq C\varsigma(S)^{1/2} \quad (j \geq j_\varsigma(S)).$$

The weight on R_T is at most $e^{\delta(\Theta+1)}$. Since $P_{L_j} = \alpha + \chi_{L_j}a_V$ on R_T ,

$$\|U_j^T - P_{L_j}^{(p_j)}\|_{H_\delta^m(R_T)} \leq e^{\delta(\Theta+1)} (C\varsigma(S)^{1/2} + Ce^{-\gamma_T(S-S_0)}),$$

which is small once S is chosen large after Θ .

The core and the original ends. On R_K , which is compact and contained in $\{r \leq L_j/2\}$ for large j , there are end-even gauge transformations with $\|U_j^K - V^{\text{ad}}\|_{C^{m+1}(R_K)} = o_j(1)$, by the convergence of Proposition 10.14; and $P_{L_j}^{(p_j)} - V^{\text{ad}} = O(|p_j|)$ there. On R_O the standard exponential decay at nondegenerate ends, with constants uniform for small end

energy [Don02, Section 4.1], [RS01], gives $\|U_j^O - P_{L_j}\|_{H_\delta^m(R_O)} \leq \varsigma(T_0) + o_j(1)$ in temporal gauges normalized at the critical points.

Overlap logarithms. On $R_E^e \cap R_T$ both U_j^E and U_j^T are α plus a term that is small in C^1 , the first because $B_{q_{j,e}} = \alpha$ on the collar. The relative gauge transformation is even, so it is close to the even stabilizer of α , which is trivial, and it equals $\exp \varphi_j$ with φ_j small. On each fibre $d_\alpha \varphi_j$ equals the difference of the two torus components up to quadratic terms, and the base derivatives of φ_j equal the difference of the base components up to quadratic terms; as in the proof of Lemma 10.9,

$$\|\varphi_j\|_{H_\delta^{m+1}(R_E^e \cap R_T)} \leq C(\|U_j^E - \alpha\|_{H_\delta^m(R_E^e \cap R_T)} + \|U_j^T - \alpha\|_{H_\delta^m(R_E^e \cap R_T)}),$$

with C independent of the length of the band. On the compact overlaps with R_K and R_O the representatives are close to V^{ad} , respectively to the same stationary or critical configuration, and V^{ad} is irreducible on each overlap, which contains torus fibres; so the relative gauge transformations are close to the identity after a choice of central sign, and their logarithms are small in H^{m+1} .

Patching. Join U_j^T to the two representatives U_j^E across the two bands by $\exp(\chi(\pm\theta - \Theta)\varphi_j)$ with a fixed cutoff χ (the overlap graph is a tree), then to U_j^K and U_j^O across the compact connected overlaps, each time with one choice of central sign and a cutoff of fixed slope. The result is an end-even gauge transformation g_j , and since all cutoffs have bounded derivatives and all logarithms are small,

$$\|x_j\|_{H_\delta^m(W_{L_j})} \leq \eta(\Theta, S, T_0) + o_j(1), \quad x_j = g_j^* U_j - P_{L_j}^{(p_j)},$$

where η tends to zero when Θ , then S , then T_0 tend to infinity. A diagonal choice of Θ, S, T_0 along the sequence gives $\|x_j\|_{H_\delta^m} + |p_j| \rightarrow 0$.

Slice and chart. Gauge transformations preserve the flat-fibre, matching and Lagrangian conditions, so $g_j^* U_j$ satisfies them. By Lemma 10.29(a) there are ξ_j with $\|\xi_j\|_{H_\delta^{m+1}} \leq C\|x_j\|_{H_\delta^m}$ and w_j with $\exp(\xi_j)^* g_j^* U_j = \Psi_{L_j}(w_j, p_j)$, $d_{A_P}^* \zeta_j = 0$ and $\|z_j\| \leq C(\|x_j\| + |p_j|) \rightarrow 0$, where $z_j = (w_j, p_j)$; since U_j is a solution, $\mathcal{F}_{L_j}(z_j) = 0$. For large j , $\|z_j\| < r$, and Proposition 10.30 gives $z_j = z_{L_j}(V)$. $\square$

Theorem 10.32 (torus completion). *Choose the secondary data of the completed domains by Lemma 10.22 and use them on $W_{\pm, R}$ and W_λ^J . There are R_0 and $\lambda_1 \geq \lambda_*$ such that for $R \geq R_0$ and $\lambda \geq \lambda_1$:*

- (i) *there are no solutions of negative index on $W_{\pm, R}$ and W_λ^J ;*
- (ii) *gluing defines bijections*

$$\mathcal{G}_R: \mathcal{M}^0(W_\infty) \rightarrow \mathcal{M}^0(W_R), \quad \mathcal{G}_\lambda: \mathcal{M}^0(W_{-, \infty}) \times \mathcal{M}^0(W_{+, \infty}) \rightarrow \mathcal{M}^0(W_\lambda^J)$$

onto sets of regular points, preserving labels, relative classes and markings.

In particular, the index-zero counts satisfy $N_\pm^{(R)} = N_{\pm, \infty}$ and $\mathcal{K}N_\lambda^J = N_{-, \infty} \otimes N_{+, \infty}$ as matrices.

Proof. Negative indices. By Theorem 10.13 and the energy identity there are finitely many negative indices at which solutions can occur for given labels. If solutions U_j of a fixed negative index d existed on W_{L_j} with $L_j \rightarrow \infty$, Proposition 10.14 would give $d = \sum \text{ind } V_\beta + \sum \text{ind } \Gamma_\ell + 4n$ with $n \geq 0$, completed pieces of nonnegative index (Lemma 10.22) and chains of index at least one, which is impossible.

The map. For $L \geq \max\{L_0, L_1\}$, Proposition 10.30 defines $\mathcal{G}_L(V)$ for each representative V . It is well defined on classes and equivariant under the lifting groups. If $V' = h^*V$ for

an end-even gauge transformation h of W_∞ , trivial at the original ends in the limit, then V'^{ad} and V^{ad} differ by an end-even gauge transformation h' which on $\{r \geq S_V\}$ preserves the torus slice, hence is constant there with value in the stabilizer of α ; it extends over the truncated tail and the collar as the same constant, which fixes α , so $P_L(V') = h'^* P_L(V)$, and the chart, the equation and the slice condition are equivariant, so $\mathcal{G}_L(V') = h'^* \mathcal{G}_L(V)$. The labels are those of V , and the relative class and bundle marking are those of $P_L(V)$, which equal those of V by Lemma 10.25(iii) and the construction. In the joined case the lifting group is $\Phi_J \cong \Phi_- \oplus \Phi_+$, the even stabilizer of α on the bridge is trivial, and the joined reference of a pair is determined up to Φ_J by the classes of $V_\pm$, so no gluing parameter occurs.

Injectivity and surjectivity. These are the last two steps of the proof of Theorem 5.1. Since $\|z_L(V)\| \rightarrow 0$ and $P_L(V) = V^{\text{ad}}$ on $\{r \leq L/2\}$, $\mathcal{G}_L(V)$ converges to V on compact subsets; if $\mathcal{G}_{L_j}(V)$ and $\mathcal{G}_{L_j}(V')$ were gauge equivalent for $L_j \rightarrow \infty$, the gauge transformations would be bounded on compact sets and a limit would give $[V] = [V']$, and the completed sets are finite. If for $L_j \rightarrow \infty$ there were index-zero solutions U_j on W_{L_j} outside the image, Proposition 10.14, the index balance and Lemma 10.22 would give a subsequence converging without loss to some V (a pair) of index zero, and Proposition 10.31 would give $U_j = \mathcal{G}_{L_j}(V)$ for large j .

Counts. The coefficient of $N_\pm^{(R)}$ at labels (a, x) is the number of index-zero solutions with these labels, which by (ii) equals that of $N_{\pm, \infty}$. In the joined case the strip output is at $A_0 \cap A_1 = (\tilde{P}_0 \cap \tilde{P}_1) \times (R_0 \cap R_1)$ and the labels of a joined solution are the pairs of labels of its two components, so the coefficient of N_λ^J at $(a_- \otimes a_+, (x_-, x_+))$ is the product of the coefficients of $N_{-, \infty}$ and $N_{+, \infty}$; composing with the identification $\mathcal{K}$ of generators gives the last identity. $\square$

10.11. Proof of Theorem 10.2. *Order of choices.* (1) Fix the separate mixed maps $N_\pm$ on $X_{D, \pm}$. (2) Fix $\lambda_0 \geq \lambda_*$, the data of N_Z at width λ_0 and the data of $\bar{E}$ (Proposition 10.21), and regular joined data $\mathfrak{d}_J$ at width λ_0 . (3) Choose regular data on the completed domains (Lemma 10.22). (4) Choose $R \geq R_0$ and $\lambda \geq \max(\lambda_1, \lambda_0)$ (Theorem 10.32). (5) At these finite lengths, add an invariant secondary perturbation supported in the cores, regular in index one and small enough that the finite regular index-zero sets and the absence of negative indices persist; by compactness of the index-zero sets and the implicit function theorem the index-zero counts are unchanged.

By Proposition 10.21, which contains Lemma 10.20; by Proposition 10.17 with $a = \lambda_0$ and $b = \lambda$, applied to the data $\mathfrak{d}_J$ changed by a small generic invariant secondary perturbation that is regular at both widths (Theorem 5.6), followed, at width λ , by Theorem 5.4 for the compactly supported change from these data to those of steps (3) and (5), both regular; and by Theorem 10.32,

$$[\mathcal{K}N_Z][\bar{E}] = [\mathcal{K}N_{\lambda_0}^J] = [\mathcal{K}N_\lambda^J] = [N_{-, \infty}] \otimes [N_{+, \infty}] = [N_-^{(R)}] \otimes [N_+^{(R)}] = [N_-] \otimes [N_+],$$

the last step by Proposition 10.16, applied to the data changed by a small generic invariant secondary perturbation that is regular at both lengths 0 and R (Theorem 5.6), and Theorem 5.4 for the compactly supported changes of secondary data between $N_\pm$, these data and $N_\pm^{(R)}$. All structures on the section are split, so no change of the section structure occurs. Since $[\bar{E}] = [E]^{-1}$, the theorem follows. $\square$

11. ONE- AND THREE-HANDLES

We keep the notation of Section 10.1.

11.1. The relative invariant of the one-handle cobordism. Let $W_k: M \rightarrow M^+$ be the trace of k one-handles attached away from the tube, and let $B_k = (W_k \setminus \text{int } \nu(\gamma)) \natural (T^2 \times D^2)$, where $\nu(\gamma)$ is the tube; the two end balls of the tube turn the boundary of the first factor into $(-M) \# M^+$, and its boundary sum with the torus filling gives Z .

Lemma 11.1. (i) $\mathcal{W} \cup_{Z_0} F_0 \cup_{Z_1} F_1 \cong B_k$ by an orientation-preserving diffeomorphism covered by a bundle isomorphism, where $\mathcal{W}$ is the gauge triangle assembled from D_0, C, D_1 over the two surface sheets $\Delta \times \Sigma_h, \Delta \times \Sigma_g$ and $F_i = V_i \times [0, 1]$ fill its auxiliary boundaries $Z_i = V_i \cup (-V_i)$. The diffeomorphism restricts on the boundary to the identification $Z = D_0 \cup (-D_1) \cong (-M) \# M^+ \# T^3$ of Section 10.1, carries the complement of the torus parts onto the complement of the tube $\nu(\gamma)$ in W_k , and the lifting group of the union is zero, as is that of B_k .

(ii) $B_k \cup_Z \mathcal{E} \cong W_k^\#$, with the incoming end $-M$ read as outgoing, and the lifting group of the union is ϕ_{W_k} .

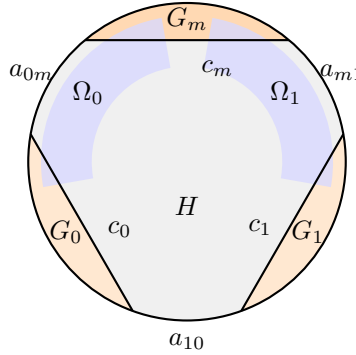


FIGURE 7. The planar base of the proof of Lemma 11.1(i). The fibre is Σ_h over the hexagon H , Σ_g over G_m and the torus T over G_0 and G_1 ; over the chords c_0, c_m, c_1 the pieces U_0, C, U_1 are inserted. Over the bands Ω_0 and Ω_1 the four-manifold is $F_0 = [0, 1] \times V_0$ and $F_1 = [0, 1] \times V_1$; the boundary over $\partial \mathbb{D}$ is Z .

Proof. (i) We use four-manifolds built from a planar base. Let $\mathbb{D}$ be a closed disk and $c_0, c_m, c_1 \subset \mathbb{D}$ three disjoint chords; they cut $\mathbb{D}$ into three outer half-disks G_0, G_m, G_1 (with $\partial G_e \setminus \partial \mathbb{D} = c_e$) and a hexagon H , whose boundary consists alternately of c_0 , an arc $a_{0m} \subset \partial \mathbb{D}$, c_m , an arc a_{m1} , c_1 and an arc a_{10} . Put

$$\mathfrak{X} = H \times \Sigma_h \cup G_m \times \Sigma_g \cup G_0 \times T \cup G_1 \times T \cup c_0 \times U_0 \cup c_m \times C \cup c_1 \times U_1,$$

where $c_e \times Y$ is glued to the two adjacent regions along $c_e \times \partial Y$, with $\partial U_i = \Sigma_h \sqcup T$ and $\partial C = \Sigma_h \sqcup \Sigma_g$, corners rounded in product collars, and the bundle given by the bundles of U_i, C and their boundary restrictions. The boundary $\partial \mathfrak{X}$ lies over $\partial \mathbb{D}$; reading it along $\partial \mathbb{D}$ from G_0 through a_{0m}, G_m, a_{m1}, G_1 and a_{10} gives $(T \times I) \cup U_0 \cup_{\Sigma_h} C \cup_{\Sigma_g} (-C) \cup_{\Sigma_h} (-U_1) \cup_T (T \times I) \cup U_1 \cup_{\Sigma_h} (-U_0)$, that is $V_0 \cup (-V_1)$ and $(-U_0) \cup U_1$ glued along two tori, which is $Z = D_0 \cup (-D_1)$.

Step 1: $\mathcal{W} \cup F_0 \cup F_1 \cong \mathfrak{X}$. Let $\Omega_0 \subset \text{int } \mathbb{D}$ be a closed rectangle $[0, 1]_y \times [0, 1]_x$ near a_{0m} that crosses, in the x -direction and in this order, G_0, c_0, H, c_m and G_m , with $c_0 \cap \Omega_0 = [0, 1] \times \{\frac{1}{4}\}$ and $c_m \cap \Omega_0 = [0, 1] \times \{\frac{3}{4}\}$; let Ω_1 be the analogous rectangle near a_{m1} crossing G_m, c_m, H, c_1, G_1 . Over Ω_0 the fibre across x is T -collar, U_0, Σ_h -collar, C, Σ_g -collar, so

$\mathfrak{X}|_{\Omega_0} \cong [0, 1] \times (U_0 \cup_{\Sigma_h} C) = [0, 1] \times V_0 = F_0$, and likewise $\mathfrak{X}|_{\Omega_1} \cong F_1$; the boundary $\partial\Omega_i$ carries Z_i . On $\mathbb{P} = \mathbb{D} \setminus \text{int}(\Omega_0 \cup \Omega_1)$, a disk with two holes, the chords are cut into the following arcs: two arcs of c_0 from $\partial\mathbb{D}$ to $\partial\Omega_0$, a short one near a_{0m} and a long one; three arcs of c_m , from $\partial\mathbb{D}$ to $\partial\Omega_0$, from $\partial\Omega_0$ to $\partial\Omega_1$, and from $\partial\Omega_1$ to $\partial\mathbb{D}$; and two arcs of c_1 from $\partial\Omega_1$ to $\partial\mathbb{D}$. Let $\mathcal{N}_0 \subset \mathbb{P}$ be a band from $\partial\mathbb{D}$ to $\partial\Omega_0$ containing the long and the short arc of c_0 , the region $G_0 \setminus \Omega_0$ between them, the thin region of H between the short arc of c_0 and the first arc of c_m , and that arc. Across $\mathcal{N}_0$ the fibre is a Σ_h -collar, U_0 traversed from Σ_h to T , a T -collar, U_0 traversed from T to Σ_h , a Σ_h -collar, C and a Σ_g -collar, that is $(-U_0) \cup_T U_0 \cup_{\Sigma_h} C = (-U_0) \cup_T V_0 = D_0$ with collars of its boundary; so $\mathfrak{X}|_{\mathcal{N}_0} \cong e_0 \times D_0$. Likewise a band $\mathcal{N}_1$ from $\partial\Omega_1$ to $\partial\mathbb{D}$ gives $e_1 \times D_1$, and a neighbourhood $\mathcal{N}_m$ of the middle arc of c_m gives $e_m \times C$. The complement of $\mathcal{N}_0 \cup \mathcal{N}_m \cup \mathcal{N}_1$ in $\mathbb{P}$ consists of two hexagons, the rest of H with fibre Σ_h and the rest of G_m with fibre Σ_g , each with its three edges on $\mathcal{N}_0$, $\mathcal{N}_m$, $\mathcal{N}_1$ and its three vertex arcs on $\partial\mathbb{D}$, $\partial\Omega_0$, $\partial\Omega_1$. This is the assembly of $\mathscr{W}$ over the two sheets, with the output vertex at $\partial\mathbb{D}$ and the inputs at $\partial\Omega_0$, $\partial\Omega_1$. The identifications are the identity on the pieces U_i , C and on the surfaces, so the bundles correspond.

Step 2: the Heegaard pieces. Write $M = H_0 \cup_{\Sigma} H_1$ for the Heegaard splitting of genus $h-1$ of the presentation, and $U_i = H_i \cup_D ([0, 1] \times T)$, where the sum disk $D \subset \partial H_i$ is identified with a disk $D \subset \{0\} \times T$; by Definition 3.3(ii), D is the equatorial disk of the ball $b(B^3)$, and we write o' for its centre. So $\Sigma_h = (\Sigma \setminus D) \cup_c T^\circ$, $T^\circ = T \setminus D$, $c = \partial D$. The compression body C is $C'' \cup_{c \times [0, 1]} (T^\circ \times [0, 1])$, where C'' is obtained from $(\Sigma \setminus D) \times [0, 1]$ by attaching k three-dimensional one-handles to $(\Sigma \setminus D) \times \{1\}$ away from ∂D , and $\Sigma_g = (\Sigma^+ \setminus D) \cup_c T^\circ$ with Σ^+ the surgered surface. Accordingly $\mathfrak{X} = \mathfrak{X}_M \cup \mathfrak{X}_T$, where $\mathfrak{X}_T$ collects the torus parts: T° over H and G_m , $T^\circ \times [0, 1]$ over c_m , the factor $[0, 1] \times T$ of U_i over c_i , and T over G_i ; and $\mathfrak{X}_M$ collects the rest: $\Sigma \setminus D$ over H , $\Sigma^+ \setminus D$ over G_m , C'' over c_m and H_i over c_i . They meet along $H' \times c \cup (c_0 \sqcup c_1) \times D$, where $H' = H \cup G_m \cup (\text{band of } c_m)$.

Step 3: $\mathfrak{X}_T$. Thicken each c_i into a band $b_i = c_i \times [0, 1]$, the factor $[0, 1]$ being the interval factor of $[0, 1] \times T \subset U_i$, and similarly c_m . Then $\mathbb{D}' = H' \cup b_0 \cup G_0 \cup b_1 \cup G_1$ is a disk, H' is a band across it separating $Q_i = b_i \cup G_i$, and the fibres are T° over H' and T over $Q_0 \cup Q_1$. Hence

$$\mathfrak{X}_T = (\mathbb{D}' \times T) \setminus \text{int}(H' \times D),$$

and $H' \times D$ is a tubular neighbourhood of the arc $\beta' = \beta_0 \times \{o'\}$, where β_0 crosses the band H' from a_{10} to the opposite boundary arc and $o' \in D$ is the centre; the interface with $\mathfrak{X}_M$ is its lateral boundary $H' \times c \cup (c_0 \sqcup c_1) \times D \cong S^2 \times [0, 1]$. The arc β' is a chord of the disk $\mathbb{D}' \times \{o'\}$, so it cobounds a half-disk δ with an arc of $\partial\mathbb{D}' \times \{o'\}$. A regular neighbourhood B of δ is a four-ball meeting $\partial(\mathbb{D}' \times T)$ in a three-ball, and we may take $H' \times D \subset B$ with $\nu(\beta') \cap \partial B = \nu(\beta') \cap \partial(\mathbb{D}' \times T)$. Then $(\mathbb{D}' \times T) \setminus \text{int} B \cong D^2 \times T^2$, and $B \setminus \text{int} \nu(\beta')$ is the complement of a trivial arc in a four-ball: in the model $B = D^3 \times [0, 1]$, $\beta' = \{0\} \times [0, 1]$, $\nu(\beta') = D_{1/2}^3 \times [0, 1]$, it is $S^2 \times [\frac{1}{2}, 1] \times [0, 1]$, with the lateral boundary $S^2 \times \{\frac{1}{2}\} \times [0, 1]$. Hence

$$\mathfrak{X}_T \cong (T^2 \times D^2) \natural (S^2 \times [\tfrac{1}{2}, 1] \times [0, 1]),$$

the boundary sum being taken along the three-ball $\partial B \cap \text{int}(\mathbb{D}' \times T)$, and the interface with $\mathfrak{X}_M$ is $S^2 \times \{\frac{1}{2}\} \times [0, 1]$.

Step 4: $\mathfrak{X}_M$. Refill the disk: $\mathfrak{X}'_M = \mathfrak{X}_M \cup (H' \times D) \cup (c_0 \times \mathcal{V}_0 \cup c_1 \times \mathcal{V}_1)$, where $\mathcal{V}_i \cong D \times [0, 1]$ is a collar of D in H_i removed from $\mathfrak{X}_M$ beforehand (which does not change its diffeomorphism type). Then $\mathfrak{X}'_M = H \times \Sigma \cup c_0 \times H_0 \cup c_1 \times H_1 \cup c_m \times C' \cup G_m \times \Sigma^+$, with $C' = C'' \cup D \times [0, 1]$ the compression body from Σ to Σ^+ , and $\nu = H' \times D \cup c_0 \times \mathcal{V}_0 \cup c_1 \times \mathcal{V}_1$ is a tubular neighbourhood of β' . Identify H with $[-1, 1]_u \times [0, 1]_t$ so that $c_0 = \{u = -1\}$,

$c_1 = \{u = 1\}$, $a_{10} = \{t = 0\}$ and $c_m = \{t = 1\} \times [u_1, u_2]$ with $0 < u_1 < u_2 < 1$. Then $H \times \Sigma \cup c_0 \times H_0 \cup c_1 \times H_1 = [0, 1]_t \times (H_0 \cup [-1, 1]_u \times \Sigma \cup H_1) = [0, 1] \times M$. The part over c_m and G_m is, after absorbing the collar $G_m \times \Sigma^+$ of the upper boundary, $[u_1, u_2] \times C'$ attached along $[u_1, u_2] \times \Sigma = \{1\} \times [u_1, u_2] \times \Sigma \subset \{1\} \times M$. As $C' = \Sigma \times [0, 1] \cup h_1 \cup \dots \cup h_k$, this is a collar on $\{1\} \times M$ together with the pieces $[u_1, u_2] \times h_j = [u_1, u_2] \times D^1 \times D^2 \cong D^1 \times D^3$ attached along $\partial D^1 \times (D^2 \times [u_1, u_2])$, two three-balls in the top boundary straddling Σ : these are four-dimensional one-handles. Hence $\mathfrak{X}'_M \cong W_k$, with incoming end $-M$ over a_{10} and outgoing end M^+ over the opposite arc. Choosing $\beta_0 = \{0\} \times [0, 1]$, which ends on a_{0m} , the arc β' becomes $\gamma = [0, 1] \times \{q\}$ with $q = (0, o') \in [-1, 1] \times \Sigma \subset M$, the centre of $b(B^3)$, which is the product path, and ν becomes the tube $\nu(\gamma)$ of the decorated cobordism; the one-handles are attached away from it. So $\mathfrak{X}_M \cong W_k \setminus \text{int } \nu(\gamma)$, and the interface with $\mathfrak{X}_T$ is the lateral boundary $S^2 \times [0, 1]$ of the tube.

Step 5: assembly. By Steps 3 and 4, $\mathfrak{X}$ is obtained from $W_k \setminus \text{int } \nu(\gamma)$ by attaching $S^2 \times [\frac{1}{2}, 1] \times [0, 1]$ along $S^2 \times \{\frac{1}{2}\} \times [0, 1]$, which is a collar and does not change the diffeomorphism type, boundary summed with $T^2 \times D^2$ at a ball in the lateral part of the boundary. Since $\partial(W_k \setminus \text{int } \nu(\gamma)) = (-M) \# M^+$ is connected, the boundary sum does not depend on the ball, and $\mathfrak{X} \cong B_k$. With Step 1 this proves the diffeomorphism. The bundle of $\mathfrak{X}$ is trivial on $\mathfrak{X}_M$ and the pullback of the odd bundle P_T under $\mathfrak{X}_T \subset \mathbb{D}' \times T \rightarrow T$ on $\mathfrak{X}_T$, glued along the interface by the fixed trivialization of P_T over D ; the summand $S^2 \times [\frac{1}{2}, 1] \times [0, 1]$ lies over D , where P_T is trivialized. So the diffeomorphism is covered by an isomorphism with the bundle of B_k , trivial on $W_k \setminus \nu(\gamma)$ and pulled back from T^2 on $T^2 \times D^2$, boundary summed along the trivialization over the ball. On the boundary, the pieces over a_{10} and over the opposite arc are $-M \setminus B$ and $M^+ \setminus B$, and the torus part over $\partial \mathbb{D}'$ is T^3 minus the two balls at the ends of $\nu(\beta')$, which are the connected-sum balls, in the identification of Section 10.1; the cut tori T_a, T_b of Lemma 10.18 are the fibres over points of the arcs $\partial G_0 \cap \partial \mathbb{D}$ and $\partial G_1 \cap \partial \mathbb{D}$.

For the lifting groups: restriction $H^1(\mathcal{W} \cup F_0 \cup F_1; \mathbb{F}_2) \rightarrow H^1(\mathcal{W}) \oplus H^1(F_0) \oplus H^1(F_1)$ is injective, since $H^0(\mathcal{W}) \oplus H^0(F_0) \oplus H^0(F_1) \rightarrow H^0(Z_0) \oplus H^0(Z_1)$, $(a, b, c) \mapsto (a + b, a + c)$, is onto. The caps carry the determinant-one group, so a lifting class of the union vanishes on Z_0 and Z_1 . A lifting class of $\mathcal{W}$ vanishing on the surface sheets restricts to zero on C and on each D_i : H^1 of C , of U_i and of V_i injects into H^1 of Σ_g , of Σ_h and of Σ_g respectively, and since T separates $D_i = (-U_i) \cup_T V_i$, $H^1(D_i)$ injects into $H^1(U_i) \oplus H^1(V_i)$. In addition, the incidence graph of $\mathcal{W}$ (two sheets, three edge pieces and six contact regions) has first Betti number two, and the two corresponding classes restrict to zero on every piece. These two classes restrict nontrivially and independently to Z_0 and Z_1 , each being the dual of the surface sheet in the vertex slice. A lifting class vanishing on the sheets is therefore determined by its restrictions to Z_0 and Z_1 , where the neck-torus duals and the incidence classes are nonzero, and so it vanishes. The lifting group of the union is therefore zero, which is the group of B_k . (ii) Model $\mathcal{E}$ as $T^2 \times P$ with the summand products inserted along two arcs. Attaching the disk base of $T^2 \times D^2$ in B_k caps the input circle c of the pair of pants and restores the product annulus, with the identity parametrizations at both exterior circles and no Dehn twist; across the cap the two sum collars join to the lateral collar of the original tube, with the prescribed normal frame. The complement of the tube and its bundle are unchanged. The cut Z is connected, so the lifting group of the union consists of the classes vanishing on B_k and restricting into $\Phi_{\mathcal{E}}$; compatibility on Z forces $s = 0$ and $s_- = s_+$, which is the torus lifting group ϕ_{W_k} . $\square$

Lemma 11.2. $[E](\Theta_{B_k}) = \text{ten}(I^\#(W_k))$ on homology over $\mathbb{F}_2$, where $\Theta_{B_k} = [\mu_{\mathcal{W}}](\theta_0, \theta_1)$ is the relative invariant of B_k , $\mu_{\mathcal{W}}$ is the gauge triangle map of $\mathcal{W}$ modulo the gauge group enlarged by Φ , $\theta_i \in C_G(Z_i)$ is the relative invariant of the filling F_i , defined as θ_G in Section 9,

and $\text{ten}(F) = \sum_i a_i \otimes b_i$ for $F(v) = \sum_i \langle a_i, v \rangle b_i$, the pairing being the gauge Kronecker pairing between $-M$ and M .

Proof. The relative invariant. By Lemma 11.1(i) and [KM11b, Proposition 5.4] with two caps, whose connecting images vanish because the auxiliary cuts are connected and meet the triangle, together with the projection argument of Proposition 9.5 for each input, $\Theta_{B_k} = [\mu_{\mathcal{W}}](\theta_0, \theta_1)$.

Gluing. For a long neck along Z with nondegenerate data, the index-zero moduli spaces of $B_k \cup_Z \mathcal{E}$ are, by Theorem 5.1, the fibre products over the critical points z_σ of Z (with their two even lifts σ) of those of B_k and of $\mathcal{E}$; the gluing parameter is a point since the determinant-one stabilizer $\{\pm 1\}$ extends over both pieces and the connecting image vanishes.

Counting. Let $e_{\sigma\tau}$ be the number of index-zero solutions on $\mathcal{E}$ from z_σ to the output lifts (a_0, b_τ) for fixed labels. Restriction of $\Phi_{\mathcal{E}}$ to the lifts of z and of the $-M$ output is an isomorphism, so the coefficient of $E([z])$ at $[a] \otimes [b]$ is $e_{00} + e_{01}$. The element of $\Phi_{\mathcal{E}}$ acting on the z and M^+ lifts gives a bijection between solutions from z_1 to b_τ and from z_0 to $b_{\tau+1}$, so $e_{1\tau} = e_{0,\tau+1}$. If β_σ are the coefficients of Θ_{B_k} at z_σ , the glued count at $(a_0, \cdot)$, summed over the output lifts, is $\sum_\tau \sum_{z,\sigma} \beta_\sigma e_{\sigma\tau} = \sum_z (\beta_0 + \beta_1)(e_{00} + e_{01})$, the coefficient of $E(\Theta_{B_k})$. Each orbit of lifts is counted once, because the lifting group acts freely and the counts are taken in the quotient.

Reversal. By Lemma 11.1(ii) the union is $W_k^\#$ with the incoming end read as outgoing; the same moduli spaces, with the incoming lift fixed and the output lifts summed, compute the framed coefficient $\langle I^\#(W_k)[a], [b] \rangle$ [Sca15, Section 7.2], the pairing between $C_G((-M)^\#)$ and $C_G(M^\#)$ being the reversal pairing [Sca15, Section 7.4]. Independence of data holds on homology by functoriality under continuation. $\square$

11.2. Reflection. For an object (M, b) with equatorial disk $D = \{x_3 = 0\}$ in the ball, put $\bar{b} = b \circ r_B$ with $r_B(x_1, x_2, x_3) = (x_1, x_2, -x_3)$; this is an orientation-preserving parametrization of a ball in $-M$ that fixes D and exchanges the two half-balls. On the torus $T^3 = T^2 \times S^1_C$ use $\rho_T(x, y, c) = (x, y, -c)$, which reverses orientation, preserves the odd two-torus with its bundle, the class α_T and the flat metric. Transporting the torus attachment by r_B and ρ_T defines the attachment of the *reflected object* $(-M, \bar{b})$, with $(-M)^\# = -(M^\#)$ as marked bundles; reflecting twice is the identity. For a presentation $p = (C_0, C_1)$ put $\bar{p} = (C_1, C_0)$, with all spatial data transported; ordinary strip data are reflected by $u(s, t) \mapsto u(-s, 1 - t)$, $J \mapsto J(-s, 1 - t)$, $H \mapsto -H(-s, 1 - t)$, with the full one-form transformation.

Reflection gives pairings, which are identifications of complexes for the reflected end perturbation and hold up to gauge continuation, hence on homology, for independently chosen end and secondary data; only the homology statement is used:

$$C_G((-M)^\#, \bar{p}) = C_G(M^\#, p)^\vee, \quad CF(\bar{p}) = CF(p)^\vee,$$

under which the gauge and ordinary continuation maps are transposed: $\text{cont}_G(\bar{P}, \bar{Q}) = \text{cont}_G(Q, P)^\vee$ and $\text{cont}_S(\bar{P}, \bar{Q}) = \text{cont}_S(Q, P)^\vee$. The mixed completion of a reflected presentation is *not* reflected: mixed solutions are not carried to mixed solutions. We choose independent regular invariant data (end perturbation and secondary perturbation) defining $N_{\bar{p}}$; by Theorem 3.8(ii) its homology class does not depend on the choice, and in particular $[N_{\bar{p}}] = [N_P]$.

11.3. Continuations and composition. The Lagrangians $\tilde{P}_i$ and R_i are simply connected with $\pi_2 = 0$. The correspondence L_C is an $\text{SU}(2)^k$ -bundle over M_h , simply connected with

$\pi_2(L_C) = \mathbb{Z}$; spheres in L_C have zero area and zero Chern number, so every pair and triangle used below is monotone with minimal Maslov number 4.

Continuation between primary data. For two admissible choices of primary data with the same splitting and metric data, the Lagrangians differ by Hamiltonian isotopies. Choose Hamiltonian isotopies on the two boundary components and extend them over disjoint boundary collars. Their interpolation in the strip interior gives a domain-dependent Hamiltonian change of variables which straightens both labels, with fixed data on the far ends. The transformed equation has a Hamiltonian one-form $K = K_s ds + K_t dt$; its tangential boundary restriction need not vanish. With $\iota_{X_H}\omega = -dH$, the equation is

$$v_s - X_{K_s} + J(v)(v_t - X_{K_t}) = 0.$$

On a truncated strip its full analytic energy is

$$E_K(v) = 2 \int v^* \omega - 2 \int_{\partial S} v^* K + 2 \int_S (\partial_s K_t - \partial_t K_s + \omega(X_{K_s}, X_{K_t}))(v) ds dt.$$

Combining the cut-end terms with fixed reference areas gives the prescribed chord-action contributions. The physical boundary work and curvature term have uniform bounds on compact families of these data. Monotonicity therefore bounds energy at fixed index. On an inserted autonomous neck the curvature vanishes and a stationary chord has zero covariant energy, so the bound is independent of neck length.

One may instead add a Hamiltonian boundary connection making the moving labels parallel before straightening. The resulting fixed-boundary perturbation has zero tangential boundary restriction. Interpolating the connections with fixed ends retains the preceding energy bound. Generic relative perturbations, the exclusion of bubbles (each costs index at least four), and ordinary strip gluing give a chain homotopy between the two constructions. This identifies $\text{cont}_S(P, P')$ with Wehrheim and Woodward's data-change map on fixed Lagrangians. It is a chain map, equals the identity for constant data, is compatible with composition and is invariant under homotopies of the transformed data with fixed ends [WW15b, Theorems 3.9, 3.13, 4.1, 4.2 and Example 2.6]. Both boundary Lagrangians may move. Paths of Lagrangians and their homotopies are taken among graphs of differentials of functions over the unperturbed Lagrangians, with the functions in a small convex set; so the continuations used here depend only on their endpoints. The functions of the Lagrangians in the neighbourhoods $\mathcal{N}(t)$ of Definition 11.12 form such a set, although the admissible sets of primary data defined there are not convex; the interpolating paths of Lagrangians may leave $\mathcal{N}(t)$, which does not affect the continuation, and the gauge continuations run in the convex space span F .

The auxiliary pairs. We use the actual product-target pairs (A_0, L_C) and (L_C, A_1) in $\mathbb{X}$. Their intersection points identify with those of $(R_0, \tilde{P}_0 \circ L_C)$ and $(\tilde{P}_1 \circ L_C, R_1)$, respectively, and hence, by continuation, with the critical sets of the height functions of Section 10.1. This is an identification of generators. For a general compression it is not an unfolding identification of strip moduli spaces; exact diagonal unfolding is used only when $k = 0$. For each ordered pair, let $\mathcal{A}_i$ be its critical action: the values at intersections of its local exact-graph primitive, with the sign determined by the order of the pair. Set

$$4\pi^2 \kappa_{\text{aux}}(u) = \int u^* \omega_{\mathbb{X}} - (\mathcal{A}_i(a) - \mathcal{A}_i(x)).$$

At the flat height data $\mathcal{A}_i$ is the corresponding signed height function th_i . Under small primary changes it is the actual graph primitive, including its higher-order terms. Continue the normalization from the flat reference. Along a critical branch the graph action and the

perturbed gauge action then have the same derivative, given by the varying perturbation evaluated on that branch. Their differences at the generators agree; a common additive constant in the primitive has no effect on these differences. On $CF_{\mathbb{X}}(A_0, L_C)$ and $CF_{\mathbb{X}}(L_C, A_1)$ define the relative grading m by $m(a) - m(x) = \text{ind}(u) - 8\kappa_{\text{aux}}(u)$ for any strip class u from a to x ; it is independent of u by monotonicity. For t small it coincides with the Morse index of the height function (for $i = 0$) or with the Morse index of its negative (for $i = 1$), and there is a unique generator q_i^* of maximal m .

Lemma 11.3 (Regular auxiliary strips and triangles). *Fix a lower almost complex path, regular for the ordinary strips, and an upper path near the two strip boundaries, leaving a nonempty free interval between them. There are split almost complex structures with zero Hamiltonian, extending these choices, that are regular for the strips of (A_0, L_C) , (L_C, A_1) and (A_0, A_1) and for the triangle with these boundary conditions. The relative parameter families used below are regular for generic such choices, with only the usual breaking and parameter-end strata in dimensions zero and one. For these choices the generators q_i^* are cycles.*

Proof. Put $M = M_h$, $N = M_g$, and write $J_{\mathbb{X}} = (-J_M) \oplus J_N$. The upper projection $L_C \rightarrow N$ is an embedding: restriction to the upper surface determines the representation of the compression group. In particular,

$$(11.1) \quad TL_C \cap (TM \oplus \{0\}) = 0.$$

For a strip $u = (u_M, u_N)$, constancy of u_N forces the lower trace along its L_C boundary to be constant. Extend u_M by that constant across a small interval of this boundary. Equal traces give a weak $W^{1,p}$ solution, $p > 2$, for smoothly extended coefficients. Interior regularity and unique continuation make u_M constant. Hence a nonconstant strip has nonconstant upper component, even if its lower component is constant.

For the upper component, the ordinary strip injective-point argument gives a dense open set of points where $\partial_s u_N \neq 0$ and the value $u_N(s, t)$ is neither an end value nor another value $u_N(s', t)$ with $s' \neq s$. The componentwise application is given in the proof of [WW11, Theorem 3.2, Step 1a]. It uses finite energy, the entire s -axis and convergence to the fixed end values; we do not invoke that theorem's later generic-Hamiltonian argument for arbitrary constant patches.

Let $\eta = (\eta_M, \eta_N)$ annihilate the linearization together with the allowed $J_N(t)$ variations. If η_N is nonzero, adjoint unique continuation gives a nonzero point in the free t -interval, and one may choose it in the preceding injective set. Compatible variations Y send the nonzero vector $u_{N,t}$ onto the full target tangent space. A variation supported near $(t, u_N(s, t))$ consequently has nonzero pairing with η_N , contrary to the annihilator condition. Thus $\eta_N = 0$. Green's formula identifies the remaining adjoint boundary trace with an element of TL_C whose upper component is zero. Equation (11.1) makes the lower trace zero. Extending it by zero across that boundary interval and using adjoint regularity and unique continuation gives $\eta_M = 0$. This proves full universal surjectivity without varying J_M . Reversing the strip boundary order gives the other auxiliary pair.

At a constant strip x , the operator is $\partial_s + B_x$. The self-adjoint strip operator B_x has kernel $T_x A_i \cap T_x L_C = 0$. Its spectral gap and Fourier transform in s give an inverse, also for small end weights. Constants are thus regular of index zero. Sard–Smale and a countable exhaustion now give regular upper paths for every required auxiliary pair. Intersect the residual sets with those for the ordinary upper strips, having fixed regular ordinary lower data first. The

output operator for (A_0, A_1) is their direct sum. Different triangle ends permit independent almost-complex paths; these paths extend to split structures on the triangle.

For the triangle the same constant-upper argument would force a constant full map and hence a triple intersection, which is empty. Domain-dependent upper variations in a free interior disc therefore kill the upper adjoint; no global injectivity condition is required for these domain-dependent variations. Equation (11.1) and boundary unique continuation kill the lower adjoint. This proves triangle regularity while retaining zero Hamiltonian, the fixed strip ends and the prescribed collar of the collapse interval. For $k = 0$ choose the collars along which the configuration is doubled at the outset; the variations in the free interior discs remain available.

For a parameter family multiply these variations by parameter bumps, zero near the prescribed regular endpoint collars. The parametrized operator includes the data parameter once; if several ends share that parameter, their independently supported variations enter the one parametrized operator. The Sard–Smale theorem, applied relative to the endpoint collars, makes the parametrized moduli spaces regular, which is all that is used.

The area-index relation above gives a fixed-index energy bound. Every positive sphere or disk bubble removes at least four from the core index. A regular nonconstant autonomous strip has its nonzero translation vector, so its full index is at least one. A constant strip has index zero. Thus in full strip indices one and two no bubble can occur, and the index-two translation quotient has exactly two index-one strips at its boundary. The same index count applies to triangles of dimensions zero and one and to the corresponding regular parametrized families, with the parameter counted once. When several strips share a parameter, the quotient by translation is taken after forming the parametrized operator. Ordinary nondegenerate strip gluing gives the boundary collars. Its converse follows from zero-loss compactness and exponential decay; a sequence outside the finitely many local gluing neighbourhoods would converge strongly into one of them.

Finally the energy and index of an auxiliary strip from the maximal generator are

$$E(u) = 8\pi^2 \kappa_{\text{aux}} + 2(\mathcal{A}_i(q_i^*) - \mathcal{A}_i(x)), \quad \text{ind}(u) = m(q_i^*) - m(x) + 8\kappa_{\text{aux}}.$$

The small action oscillation is less than $2\pi^2$ and $2\kappa_{\text{aux}}$ is integral, so energy excludes negative classes. The Morse difference is a nonnegative multiple of three, and $8\kappa_{\text{aux}}$ a nonnegative multiple of four. Their sum is not one. No differential trajectory leaves q_i^* . Changing J has altered neither these ordered actions nor their relative grading. This proves the cycle assertion directly on the product-target problem. $\square$

Definition 11.4. The *compression cup* of a k -fold one-handle, at height Lagrangians, is

$$U_C = \mathcal{BK}(\mu_{\mathbb{X}}(q_0^*, q_1^*)): HF(\tilde{P}_0, \tilde{P}_1) \rightarrow HF(R_0, R_1),$$

where $\mu_{\mathbb{X}}: CF_{\mathbb{X}}(A_0, L_C) \otimes CF_{\mathbb{X}}(L_C, A_1) \rightarrow CF_{\mathbb{X}}(A_0, A_1)$ is the triangle map in $\mathbb{X}$, $\mathcal{K}$ is the folding isomorphism and $\mathcal{B}$ turns a cycle $\sum x_i^\vee \otimes y_i$ into the map $v \mapsto \sum x_i^\vee(v) y_i$. At other admissible presentations it is conjugated by continuation maps. The *compression cap* $V_C: HF(R_0, R_1) \rightarrow HF(\tilde{P}_0, \tilde{P}_1)$ of a three-handle is the transpose of the cup of the reflected one-handle, $V_C = (U_{\tilde{C}})^\vee$ under the pairings of Section 11.2.

The cup is a count of holomorphic triangles, parallel to the two-handle map $\mu_S(\cdot \otimes \theta_S)$; its definition involves no gauge theory and no geometric composition.

Lemma 11.5. For $k = 0$, $L_C = \Delta \subset M_h^- \times M_h$ and U_C equals the continuation map $\text{cont}_S(\tilde{P}, R)$ on homology.

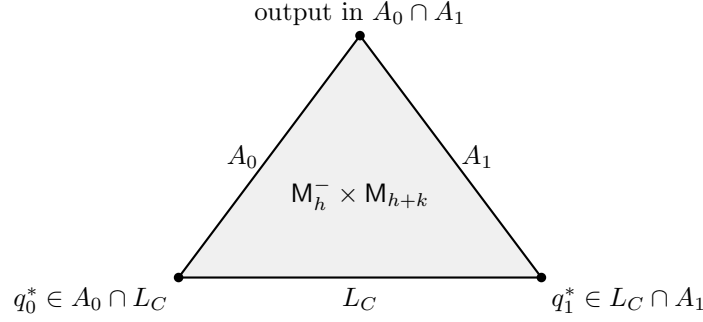


FIGURE 8. The folded triangle defining the compression cup (Definition 11.4). The inputs are the top generators q_0^* and q_1^* of the auxiliary pairs, and the output lies in $CF_{\mathbb{X}}(A_0, A_1)$, which the folding isomorphism $\mathcal{K}$ identifies with $CF(\tilde{P}_0, \tilde{P}_1)^\vee \otimes CF(R_0, R_1)$, that is, with maps from $CF(\tilde{P}_0, \tilde{P}_1)$ to $CF(R_0, R_1)$. Unfolded, the triangle is a quilted surface whose seam carries the correspondence L_C (Remark 11.6).

Proof. Let S be the triangle with boundary arcs labelled A_0 , Δ , A_1 and D the disk obtained by doubling S along its Δ -arc. For data (J_D, K_D) on D with K_D vanishing on the doubling arc, put $J_{\mathbb{X}}(z) = (-J_D(\bar{z})) \times J_D(z)$ on S . The map $v \mapsto (v \circ \bar{\cdot}, v)|_S$ is a bijection from solutions on D to solutions on S with boundary on (A_0, Δ, A_1) : a solution on S glues to a Lipschitz map on D that solves the equation weakly across the arc, hence is smooth; linearized operators correspond. The two ends of S adjacent to the Δ -arc double to strip ends of D labelled $(R_0, \tilde{P}_0)$ and $(\tilde{P}_1, R_1)$, respectively, and the output end doubles to one incoming and one outgoing end, which is the folding $\mathcal{K}$ followed by $\mathcal{B}$. So $U_C(x)$ is the count of disks with boundary arcs labelled $R_1, \tilde{P}_1, \tilde{P}_0, R_0$ with x at the incoming end and q_0^*, q_1^* at the two auxiliary ends. We construct the unit cap for each of these ordered pairs. In a Weinstein neighbourhood of $\tilde{P}_i$, write $R_i = \text{graph}(dg_i)$. For the first pair, fibre translation by $-dg_0$ takes $(\text{graph}(dg_0), 0)$ to $(0, \text{graph}(-dg_0))$. Realize this translation by a Hamiltonian isotopy with cutoff equal to one near the whole graph path. The second pair is already $(0, \text{graph}(dg_1))$. Thus the ordered primitives are $f_0 = -g_0 - \min(-g_0)$ and $f_1 = g_1 - \min g_1$. Put $M_i = \max f_i$. For the small primary data in use, f_i has its unique maximum at q_i^* . Extend f_i constantly in the cotangent fibres near the zero section and cut it off farther away by a function in $[0, 1]$. The resulting K_i satisfies $0 \leq K_i \leq M_i$ and $K_i(q_i^*) = M_i$. Choose the cutoff equal to one on the whole graph path. With $\iota_{X_{K_i}} \omega = -dK_i$, the inverse time-one flow carries the zero section to $\text{graph}(df_i)$.

Use a disk with one outgoing strip end, with boundary on that zero section and Hamiltonian one-form $\beta(s)\chi(\theta)(K_i - M_i)d\theta$. Here β increases from zero to one, while $\chi \geq 0$ is supported away from the strip sides and has integral one. On the end, conjugating by the inverse Hamiltonian flow gives the ordinary strip for $(0, \text{graph}(df_i))$ and fixes its critical chords. Transport the first cap back by the actual translation, including its almost-complex and Hamiltonian data. Its ordered labels are then $(R_0, \tilde{P}_0)$; those of the second cap are $(\tilde{P}_1, R_1)$. A cap with output x has index $m(q_i^*) - m(x) + 8\kappa_{\text{aux}}$: by additivity of the index, gluing the constant cap at q_i^* to a strip from q_i^* to x gives the grading difference of Section 11.3, and the constant cap has index zero: by spectral flow of $\beta \text{Hess}(f_i)$ along the end, the index of the constant cap at a critical point c is the number of positive eigenvalues of $\text{Hess}(f_i)(c)$, which

is zero exactly at the maximum q_i^* . The full energy identity for this cap is

$$E_{\text{cap}} = 8\pi^2 \kappa_{\text{aux}} - 2(K_i(x) - M_i) + 2 \int \beta' \chi(K_i(u) - M_i) ds d\theta.$$

The integral is nonpositive and the endpoint term is at most $2M_i$. Thus $2M_i < 4\pi^2$ excludes negative relative classes. The index formula then forces an index-zero cap to have $\kappa_{\text{aux}} = 0$ and output q_i^* . Its energy is zero, so it is the constant cap. At this solution the linear energy identity, with the nonpositive Hessian of $K_i - M_i$, forces a decaying kernel field to be horizontal with zero outgoing value, hence zero. Index zero gives regularity. Gluing the two unit caps into the auxiliary ends [WW15b, Theorem 3.13] produces a strip with moving boundary from $(\tilde{P}_0, \tilde{P}_1)$ to (R_0, R_1) , whose map is the continuation [WW15b, Theorem 4.1, Example 2.6]. $\square$

Remark 11.6. By the unit-insertion formula of Wehrheim and Woodward [WW15b, Corollary 5.5], U_C agrees with the morphism map of the quilted correspondence L_C [WW10] followed by the geometric composition isomorphisms $\tilde{P}_i \circ L_C$ and continuation to R_i . That formula rests on the strip-shrinking theorem [WW15b, Theorem 5.1]; for the analysis of strip shrinking see [WW12, BW18]. The composition isomorphism itself is also established by Lekili and Lipyanskiy without strip shrinking [LL13, Theorem 2 and Corrigendum, Remark 1(1)], whose hypotheses hold here: the targets are monotone with the same constant, the Lagrangians $\tilde{P}_i$ and L_C are compact, connected, orientable and simply connected with minimal Maslov number 4, and the compositions are embedded. Neither statement is used in this paper.

Lemma 11.7. *The statements of Lemma 9.1 hold at the height data for the fillings $F_i = V_i \times [0, 1]$ of $Z_i = V_i \cup (-V_i) = (\#^{g-1} S^1 \times S^2) \# T^3$ and the pairs (A_0, L_C) and (L_C, A_1) , with $r = g - 1$, the height functions defining R_i , and the top generators q_i^* . Fix the height t and a finite family $F \supseteq F_*$ as in Definition 3.3. The end data of an auxiliary pair are its lower primary datum, in $\text{span } F$, and the primary datum of its height Lagrangian, in $\text{span } F_*$; the height data are the end data $(0, th_i)$. For $k \geq 1$ the height datum, the family F_* and the function δ of Section 11.6 are those of the splitting (V_0, V_1) of M^+ , whose height functions are those of Section 10.1 on V_i , so that R_i is a height Lagrangian in the sense of that section with $k = 0$.*

- (i) *Let R be a height Lagrangian at height t with primary datum $th_i + g$, $g \in \text{span } F_*$, $|g| \leq \delta(t)$ (Section 11.6). There is a ball $\mathcal{N}_{C,R}$ about zero in $\text{span } F$, depending on C , on R and on F , such that, for end and secondary perturbations below the threshold of (ii) with $c = 2\pi^2$, under changes of the lower primary data within $\mathcal{N}_{C,R}$ the relative classes and integer gradings are transported from the height data, the endpoint height differences in the energy identities of the gauge trajectories, mixed solutions and strips are replaced by the differences of the actual critical actions $\mathcal{A}_i$, and the coefficient of q_i^* in $N_i \theta_i$, and in the filling invariant, remains 1; the representing solutions need not remain flat.*
- (ii) *Let (h_s, r_s) , $0 \leq s \leq 1$, be a piecewise smooth path of end data starting at the height data, such that at every s the generators of the auxiliary pairs are nondegenerate and continue those at the height data, with their gradings m and relative classes, and $2 \max_{a,x} |\mathcal{A}_i(a) - \mathcal{A}_i(x)| < c$ for a constant $c < 4\pi^2$ independent of s , where $\mathcal{A}_i$ are the critical actions of the auxiliary pair at (h_s, r_s) , lifted continuously along the branches of generators. Then at (h_1, r_1) , for end perturbations f along the path and secondary perturbations η at the endpoint with $4 \sup \|f\|_\infty + 2 \sup_A \|\eta(A)\|_{L^2}^2 < 4\pi^2 - c$, below*

the thresholds of Theorem 3.8: every counted gauge trajectory, mixed solution and strip has $\kappa_{\text{aux}} \geq 0$; the differential of the auxiliary gauge complex vanishes; the filling invariant is the generator θ_i of maximal m ; and $N_i\theta_i = q_i^*$, both on chains.

Proof. The proof of Lemma 9.1 uses only that the filling retracts onto a wedge of r circles and a torus, so that $H_C^1 = H^2 = 0$ with twisted coefficients, and that the critical manifold is $\text{SU}(2)^r$ with the height function; both hold for F_i . The mixed maps of the pairs (A_0, L_C) and (L_C, A_1) are those of Remark 3.9. We prove (ii) first.

Energy. The gradings and relative classes are transported along the path by hypothesis, and the critical actions are the perturbed Chern–Simons values at the generators, lifted continuously along the branches from the height data; their differences at one parameter are the differences of the graph primitives (Section 11.3). On the gauge pieces of the domains of the gauge trajectories, mixed solutions and strips the end data are autonomous, so they enter the analytic energy only through these critical values, as in the proof of Theorem 3.8: the energy identity of Lemma 9.1(ii) holds at every s with the endpoint height differences replaced by $2(\mathcal{A}_i(a) - \mathcal{A}_i(x))$, and the bounds on c , f and η , with nonnegative analytic energy and $\kappa_{\text{aux}} \in \frac{1}{2}\mathbb{Z}$, give $\kappa_{\text{aux}} \geq 0$ as in Lemma 9.1(iii). The fillings are not treated in this way: their energy identity contains the profile term of the lower datum, bounded only by its C^0 norm, which the hypotheses do not control.

Vanishing differentials. At every parameter at which the data are regular the auxiliary gauge complex has 2^r generators. It is the framed complex of $\#^r(S^1 \times S^2)$, with the lifting-group quotient of Section 3.4, so its homology is $I^\#(\#^r(S^1 \times S^2); \mathbb{F}_2)$, of dimension 2^r : the integral group is free of rank 2^r [Sca15, Sections 7.7–7.8], as recalled in the proof of Proposition 9.4, and universal coefficients give the dimension over $\mathbb{F}_2$. Hence its differential vanishes, and a chain map between two such complexes that is a homotopy equivalence is an isomorphism.

The mixed coefficient. At the endpoint N_i is a chain isomorphism by Theorem 3.8(i), and by the index formula of Lemma 9.1(ii) its matrix entry from a to x vanishes unless $m(x) = m(a) + 8\kappa_{\text{aux}}$ with $\kappa_{\text{aux}} \geq 0$. Lemma 9.2 gives $N_i\theta_i = q_i^*$ on chains for the generator θ_i of maximal m . This step uses only the data at the endpoint.

The filling invariant. At the height data, for end and secondary regularizations below the threshold of Proposition 9.4, the filling invariant is θ_i . The filling with the end data $(\mathfrak{h}_1, \mathfrak{r}_1)$ is joined, through a path of interior data with fixed ends, to the filling with the height data glued to the continuation cylinder $\mathbb{R} \times Z_i$ carrying the path $(\mathfrak{h}_s, \mathfrak{r}_s)$ and an interpolation of the end perturbations; by Theorems 5.4 and 5.1, the filling invariant at the endpoint is, on homology and hence on chains, cont_G of the filling invariant at the height data, where cont_G is the gauge continuation map along the path, the map of the product cobordism with varying end data. It remains to show $\text{cont}_G\theta_i = \theta_i$. Choose a finite partition s_j of $[0, 1]$, including the corners of the path, fine enough for the work and action bounds below. At its finitely many vertices choose regular end perturbations f_j within the stated bound by Proposition 3.4, retaining the prescribed regular data at the two endpoints. Then regularize the finitely many continuation cylinders. For a continuation trajectory from a at s_j to x at s_{j+1} the relative class κ_{aux} is taken with respect to the reference at s_j concatenated with the branch of x from s_j to s_{j+1} ; it is quantized, and the index formula holds for it, the spectral flow along a nondegenerate branch being zero. The continuation map along the path is the composite of the maps along the intervals $[s_j, s_{j+1}]$, by functoriality of the cobordism maps for the composite of the product cobordisms, up to chain homotopy, hence exactly, the differentials vanishing. A continuation trajectory of index zero from θ_i at s_j to x at s_{j+1} has $m(x) = m(\theta_i) + 8\kappa_{\text{aux}}$,

the gradings being continued, and its analytic energy is $8\pi^2\kappa_{\text{aux}}$ plus $2(\mathcal{A}_i(\theta_i) - \mathcal{A}_i(x))$, the actions taken at s_j and at s_{j+1} , plus the end terms, bounded by $2\|f_j\|_\infty + 2\|f_{j+1}\|_\infty$, plus the work term $2 \int \partial_\sigma(\mathfrak{h}, \mathfrak{r})_{s(\sigma)}(A_\sigma) d\sigma$ of the interpolation $\sigma \mapsto s(\sigma)$, plus the term of the small compactly supported holonomy perturbation that makes the continuation cylinder regular (Theorem 5.6(iii)), bounded by twice its squared norm, which is as small as we please. The work term is bounded by $2 \int_{s_j}^{s_{j+1}} \sup |\partial_s(\mathfrak{h}_s, \mathfrak{r}_s)| ds$, and the action term differs from the difference at the single parameter s_j by a quantity tending to zero with the length of the interval, the actions being continuous along the branches. The partition depends on this fixed path in the finite-dimensional span: the sup norm of its derivative is integrable, and the finitely many action branches are uniformly continuous. Choose the partition first and the cylinder regularizers afterward, so that these corrections are less than the margin $4\pi^2 - c - 4 \sup \|f\|_\infty$, which is positive by hypothesis, so $\kappa_{\text{aux}} \geq 0$; hence the continuation map along $[s_j, s_{j+1}]$ has matrix entries vanishing unless $m(x) = m(a) + 8\kappa_{\text{aux}}$ with $\kappa_{\text{aux}} \geq 0$. It is an isomorphism, being a homotopy equivalence between complexes with vanishing differentials, so Lemma 9.2 gives $\theta_i \mapsto \theta_i$. Composing, $\text{cont}_G \theta_i = \theta_i$, and the filling invariant at the endpoint is θ_i .

(i) follows from (ii) along the path that first moves the height datum from th_i to $th_i + g$ along $\sigma \mapsto th_i + \sigma g$, at lower datum zero, and then the lower datum from zero to $\mathfrak{h}$ along $\sigma \mapsto \sigma \mathfrak{h}$. On the first segment the hypotheses of (ii) hold with $c = 2\pi^2$ by the choice of $\delta(t)$ in Section 11.6, which is made so that at lower datum zero and for $|g| \leq \delta(t)$ the auxiliary pairs have 2^r transverse generators continuing those at $g = 0$ with their gradings and relative classes, and critical actions of oscillation less than π^2 . On the second segment they hold for $\mathfrak{h}$ in a sufficiently small ball of span F about zero, which is convex: nondegeneracy of the finitely many generators and the bound on the critical actions are open conditions in the lower datum, satisfied at $\mathfrak{h} = 0$. Under these changes the relative classes and gradings are transported by construction, and the representing solutions need not remain flat. $\square$

11.4. The folded triangle.

Theorem 11.8. *Let N_0, N_1 be the mixed maps of the product-target pairs (A_0, L_C) and (L_C, A_1) , with gauge inputs $Z_0 = D_0 \cup (-C)$ and $Z_1 = C \cup (-D_1)$, both diffeomorphic to $V_i \cup (-V_i) = (\#^{g-1} S^1 \times S^2) \# T^3$, and N_Z the mixed map of (A_0, A_1) with the data at width λ_0 of Theorem 10.2. Then*

$$N_Z \circ \mu_{\mathcal{W}} \simeq \mu_{\mathbb{X}} \circ (N_0 \otimes N_1) \quad \text{over } \mathbb{F}_2,$$

where $\mu_{\mathcal{W}}$ is the gauge triangle of the pieces D_0, C, D_1 . Moreover $N_i \theta_i = q_i^*$.

Proof. The proof is that of Theorem 9.10 with the following inputs. *Collapse.* The middle compression body $C = C_{h,g}$ is an odd compression body (Section 6.1), with $L(C) = L_C \subset M_h^- \times M_g$ and both boundary surfaces matched. Hypotheses (X1)–(X3) hold as in the proof of Theorem 9.13: the middle primary perturbation vanishes, the end perturbations are the fixed autonomous perturbations of the gauge input ends Z_0, Z_1 , with outputs and loops in those ends and cutoff bands placed beyond the compact set N_I , and all secondary perturbations have outputs and loop regions in a compact subset of the interiors of D_0 and D_1 , disjoint from N_I . The triangle piece in $\mathbb{X}$ is regular for a generic domain-dependent almost complex structure supported away from N_I , as in the proof of Theorem 9.10. For (X4), stretch the two ordinary necks cutting off the DFL disks of the pairs (A_0, L_C) and (L_C, A_1) from the triangle domain of (A_0, L_C, A_1) ; Theorem 5.1 gives regularity in index at most zero with count $\mu_{\mathbb{X}}(N_0 \otimes N_1)$. The linear inputs of Section 7 hold in this instance with the two-trace

slice space $H^1(\Sigma_h; \text{ad}) \oplus H^1(\Sigma_g; \text{ad})$: the gap is uniform in the scale because no harmonic field on C has vanishing harmonic trace ($H^1(C, \partial C; \text{ad}) = 0$), and the reversed lower sheet enters the Green identity with the sign of Convention 3.1. The period lattice of capped configurations is the image of $\mathbb{Z}^2/\mathbb{Z}(1, 1)$, of index step four, as in Section 3.6. *Outgoing cut.* Stretch along the closed cut $Z \cong (-M) \# M^+ \# T^3$, which is connected. Its critical points are nondegenerate and irreducible, the connecting H^0 term vanishes, and the framed groups glue by the fibre product; Theorem 5.1 gives $N_Z \circ \mu_{\mathcal{W}}$. *Family.* The path of data has the five stages of Lemma 9.12: (A) raise the scale of $C = C_{h,g}$ from ε to one; (B) interpolate to a metric that is a product $a^2 du^2 + g_Z$ on a collar of the cut Z , where the cut crosses the edge pieces D_0 and D_1 , on which the collar coordinate is the edge coordinate, and the two scaled sheets over B (Lemma 9.11), and g_Z restricts to the metrics of D_0 and D_1 ; (C) lengthen the collar; (D) switch on the end perturbation of Z on the collar; (E) interpolate the remaining data to the data with the collar of the cut inserted. Each stage lies in a convex set of data satisfying the hypotheses of Theorem 5.4, the parameter derivative is compactly supported away from the ends, and invariance holds because the classes of the lifting group vanish on $\Sigma_h \sqcup \Sigma_g$. Theorem 5.4 applies. *Classes.* The pair $(R_i, \tilde{P}_i \circ L_C)$ is the auxiliary pair of a height Lagrangian with unique top generator q_i^* , and Lemma 11.7 gives $N_i \theta_i = q_i^*$: part (i) for lower data in $\mathcal{N}_{C,R}$, and part (ii), for $k = 0$, for lower data in the set $\mathcal{U}_F(t)$ of Definition 11.12, along the paths provided there. $\square$

Proposition 11.9 (cup formula). *Put $N_P^\dagger = N_-^\vee: CF(\tilde{P}_0, \tilde{P}_1) \rightarrow C_G(M^\#)$, the transpose of the mixed map $N_- = N_{\tilde{P}}$ of the reflected presentation, so that $N_P^\dagger = N_{\tilde{P}}^\vee$, defined by $\langle a, N_P^\dagger x \rangle_G = (N_- a)(x)$. Then on homology*

$$U_C = N_+ \circ I^\#(W_k) \circ N_P^\dagger.$$

For $k = 0$ this reads $\text{cont}_S(\tilde{P}, R) = N_R \circ \text{cont}_G(\tilde{P}, R) \circ N_P^\dagger$, with cont_G the gauge continuation.

Here and below the mixed map and its transpose are indexed by the primary data when these vary: for a presentation p with primary data P we write $N_P = N_p$ and $N_P^\dagger = N_p^\dagger$.

Proof. Insert $\theta_0 \otimes \theta_1$ into Theorem 11.8. Here $N_i \theta_i = q_i^*$ at the data of Definition 11.4, and also at the generic data of Theorem 11.8: a continuation strip from q_i^* of index zero has $\kappa_{\text{aux}} = 0$ and ends at q_i^* , and a strip of zero energy is constant, so the continuation maps q_i^* to q_i^* on chains; and $\mu_{\mathbb{X}}(q_0^*, q_1^*)$ is natural under continuation with fixed ends. By Definition 11.4 and Theorem 10.2,

$$U_C = \mathcal{BK} \mu_{\mathbb{X}}(q_0^*, q_1^*) = \mathcal{BK} N_Z \mu_{\mathcal{W}}(\theta_0, \theta_1) = \mathcal{B}(N_- \otimes N_+) \mathbf{E}(\Theta_{B_k}),$$

and $\mathbf{E}(\Theta_{B_k}) = \text{ten}(I^\#(W_k)) = \sum a_i \otimes b_i$ by Lemma 11.2. Hence

$$U_C(x) = \sum_i (N_- a_i)(x) N_+ b_i = \sum_i \langle a_i, N_P^\dagger x \rangle N_+ b_i = N_+ I^\#(W_k) N_P^\dagger x. \quad \square$$

The factor $N_P^\dagger$ is not yet known to be $N_{\tilde{P}}^{-1}$; that is proved below, after naturality, in this order.

11.5. Order of choices. In each application of Proposition 11.9: first the upper height t and the upper Lagrangians R_i ; then the lower Lagrangians $\tilde{P}_i$ in a neighbourhood small relative to t , subject to the generic incidence conditions of Section 10.1 (expected dimension $-(3h + 3g - 6) < 0$); then the data at width λ_0 of Theorem 10.2. A finite change of torus collar length returns the torus factors to the fixed data by Proposition 10.16, and the

secondary data are then replaced by the prescribed ones by Theorem 3.8(ii). Hence the maps N appearing in one application are the mixed maps on homology. Squares proved with the data of one application are moved to other data at the vertices by Proposition 11.10, whose own quantifiers are ordered in the same way: height first, then the neighbourhood $\mathcal{N}(t)$ of Definition 11.12, which determines $\mathcal{U}_F(t)$, then the lower data, and last a generic height Lagrangian.

11.6. The one- and three-handle squares. A *height Lagrangian at height t* for $k = 0$ is an upper Lagrangian R of Section 10.1 for the product compression body: the primary datum of U perturbed by t times the height function h_i of Section 10.1, with its unique maximal generator q^* . A further perturbation of its primary datum by $g \in \text{span } F_*$ with $|g| \leq \delta(t)$ keeps it a height Lagrangian at height t , for a function $\delta > 0$ depending only on the splitting, the metric data and the height function, with $\delta(t) \rightarrow 0$ as $t \rightarrow 0$: $\delta(t)$ is so small that for $|g| \leq \delta(t)$ and at unperturbed lower data the generators of the auxiliary pairs, the intersection points of R with the unperturbed Lagrangian, are 2^r transverse points continuing those at $g = 0$ with their gradings m and relative classes, and that their critical actions have oscillation less than π^2 . Such $\delta(t)$ exists, for t below a threshold depending only on the splitting, the metric data and the height function: at $g = 0$ the generators are the nondegenerate critical points of th_i on the unperturbed Lagrangian, with critical actions of oscillation $t \text{osc } h_i$, and both conditions are open in g , by the implicit function theorem and compactness. Then q^* continues to the unique generator of maximal m ; the generators need not remain at the critical points of the height function. The generic perturbations used below are taken from this ball. They are not required to vanish near the generators: a perturbation vanishing near a generator cannot move it, and the incidence conditions may require that it move, for instance at a common flat point of the two compression Lagrangians that is critical for both height functions.

Proposition 11.10 (small-primary naturality). *Fix a splitting, metric data and a finite family $F \supseteq F_*$ of primary perturbations. For every height $t > 0$ below a threshold depending only on the splitting, the metric data and the height functions, and any two regular data P, Q in the neighbourhood $\mathcal{U}_F(t)$ of zero in $\text{span } F$ of Definition 11.12, $N_Q \circ \text{cont}_G(P, Q) = \text{cont}_S(P, Q) \circ N_P$ on homology.*

Proof. The quantifiers are chosen in this order: first the height t , then the neighbourhood $\mathcal{N}(t)$ of Definition 11.12, which determines $\mathcal{U}_F(t)$, then the lower data, and last a generic height Lagrangian. The conditions defining $\mathcal{U}_F(t)$ are symmetric under reflection, which exchanges the two Lagrangians of the pair, and the choices for the reflected splitting are the reflections of those for the splitting (Definition 11.12), so $\bar{P}$ and $\bar{Q}$ lie in the corresponding set for the reflected family. Given regular $P, Q \in \mathcal{U}_F(t)$, choose one height Lagrangian A at height t for the reflected presentation which, by a small generic perturbation $g \in \text{span } F_*$, $|g| \leq \delta(t)$, satisfies the incidence conditions of Section 10.1 with respect to both $\bar{P}$ and $\bar{Q}$; this is possible because the universal restriction map of $\text{span } F_*$ is a submersion at the perturbed flat connections of $th_i + g$, for t and $\delta(t)$ below a threshold fixed with the splitting, by openness as in the proof of Lemma C.14(ii), and the conditions are generic for each of the two lower pairs. By Lemma 11.7(ii), applied along the path that first moves the height datum to that of A and then the lower datum from zero to $\bar{P}$, respectively to $\bar{Q}$, along the paths provided by Definition 11.12, its conclusions hold for the auxiliary pairs of $(\bar{P}, A)$ and $(\bar{Q}, A)$, so the product case of Proposition 11.9 applies to $(\bar{P}, A)$ and $(\bar{Q}, A)$ at once and gives $\text{cont}_S(\bar{P}, A) = N_A \text{cont}_G(\bar{P}, A) N_{\bar{P}}^\vee$ and similarly for $\bar{Q}$; with $[N_{\bar{P}}] = [N_P]$ (Section 11.2)

this is $B_{\bar{P}} = N_A G_{\bar{P}} N_P^\vee$, $B_{\bar{Q}} = N_A G_{\bar{Q}} N_Q^\vee$, where G and B are the gauge and symplectic continuations to A . Eliminating the invertible N_A gives $G_{\bar{Q}}^{-1} G_{\bar{P}} N_P^\vee = N_Q^\vee B_{\bar{Q}}^{-1} B_{\bar{P}}$. The composites are the transposes of $\text{cont}_G(Q, P)$ and $\text{cont}_S(Q, P)$ under reflection; transposing gives the claim. $\square$

Proposition 11.11 (mixed duality). *Let t and F be as in Proposition 11.10. There is a height $t' \leq t$, depending only on t , the splitting, the metric data and the height functions, such that $N_{\bar{P}} = (N_P^{-1})^\vee$ on homology for every regular P in the path component $\mathcal{U}'_F(t)$ of zero in $\mathcal{U}_F(t) \cap \mathcal{U}_F(t')$ (Definition 11.12).*

Proof. Choose $t' \leq t$ so small that every height Lagrangian at height at most t' , with any $g \in \text{span } F_*$, $|g| \leq \delta(t')$, and its reflection lie in $\mathcal{N}(t)$; this is possible because these Lagrangians form a compact family converging to the unperturbed pair as the height tends to zero. For a height Lagrangian R at height t' with primary datum $t'h_i + g$, the segment from zero to $t'h_i + g$ in $\text{span } F_* \subseteq \text{span } F$ then lies in the preimage of $\mathcal{N}(t)$, so $R \in \mathcal{U}_F(t)$. Given regular $P \in \mathcal{U}'_F(t)$, let R be a height Lagrangian at height t' for the presentation of P , generic with respect to P as in the proof of Proposition 11.10. The product case of Proposition 11.9 at height t' , available at $P \in \mathcal{U}_F(t')$ by Lemma 11.7(ii), gives $\text{cont}_S(P, R) = N_R \text{cont}_G(P, R) N_P^\vee$, and Proposition 11.10 at height t , applied to the primary data $P, R \in \mathcal{U}_F(t)$, gives $N_R \text{cont}_G(P, R) = \text{cont}_S(P, R) N_P$. Hence $\text{cont}_S(P, R) = \text{cont}_S(P, R) N_P N_P^\vee$; cancelling the isomorphism $\text{cont}_S(P, R)$ gives $N_P N_P^\vee = 1$. $\square$

Definition 11.12 (admissible neighbourhoods). Fix the splitting and the metric data; the height functions, the family F_* of Section 3.2 and the function δ above are fixed with them, and so is a Weinstein neighbourhood of each $L(U_i)$ and of its reflection, in which the Lagrangians of small primary data are graphs of differentials of functions over $L(U_i)$ (Section 3.2). All choices for the reflected splitting are the reflections of those for the splitting. For a height t let $\mathcal{R}(t)$ be the family of height Lagrangians at height t for $k = 0$, of the presentation and of its reflection, with $g \in \text{span } F_*$ and $|g| \leq \delta(t)$; it is C^∞ -compact and independent of any further family. Let $\mathcal{N}(t)$ be an open C^1 -neighbourhood of the pair $(L(U_0), L(U_1))$, together with the corresponding neighbourhood of the reflected pair, each invariant under the symplectomorphisms τ_c for the lifting classes c that extend over both compression bodies (Proposition 12.6), the reflected part under the reflected τ_c , and so small that for every lower pair in it and every $R \in \mathcal{R}(t)$: the unfolding statements of Section 11.3 hold; the lower Lagrangians are graphs of differentials of functions whose C^2 -norms are so small relative to t that the generators of the auxiliary pairs are the 2^r nondegenerate critical points of a Morse function C^2 -close to t times the height function, with the same indices; and $2 \max_{a,x} |\mathcal{A}_i(a) - \mathcal{A}_i(x)| < 2\pi^2$ for the critical actions of the auxiliary pairs, the values at the generators of the difference of the primitives of the two graphs (Section 11.3). For a finite family $F \supseteq F_*$ let $\mathcal{U}_F(t)$ be the path component of zero in the set of $\mathfrak{h} \in \text{span } F$ whose Lagrangians, and those of their reflections, lie in $\mathcal{N}(t)$, and let $\mathcal{U}'_F(t)$ be the path component of zero in $\mathcal{U}_F(t) \cap \mathcal{U}_F(t')$, where $t' \leq t$ is the height chosen in the proof of Proposition 11.11. Fix once and for all a height t_* below the thresholds of Propositions 11.10 and 11.11, and for a finite family $F \supseteq F_*$ as in Definition 3.3 put $\mathcal{U}(F) = \mathcal{U}'_F(t_*)$.

The neighbourhood $\mathcal{N}(t)$ exists and depends only on the splitting, the metric data and the height functions. The conditions imposed on a lower pair are open, and they are uniform over the compact family $\mathcal{R}(t)$, which is independent of F because the generic perturbations g are taken from the fixed span of F_* ; and they concern the Lagrangians alone: the critical

actions of an auxiliary pair agree with the perturbed Chern–Simons values at its generators (Section 11.3), so that their differences are bounded by the oscillation of the difference of the primitives, which is controlled by the C^1 -size of the Lagrangians, and at the unperturbed lower pair they are less than $2\pi^2$ by the choice of $\delta(t)$. Intersecting with the finitely many images under the τ_c gives the invariance. Along any path of lower data within the preimage of $\mathcal{N}(t)$ the generators continue: at every point they are 2^r transverse intersection points, so they form a trivial covering of the path, and their relative classes, quantized and continuous, and their indices, invariant under homotopy, are constant. Hence the hypotheses of Lemma 11.7(ii) hold, with $c = 2\pi^2$, along the path that first moves the height datum to that of a given $R \in \mathcal{R}(t)$ and then follows a smooth path of lower data in the preimage, and the conclusions of that lemma hold at every point of $\mathcal{U}_F(t)$ for end and secondary perturbations below the thresholds of Theorem 3.8 with $4\sup \|f\|_\infty + 2\sup_A \|\eta(A)\|_{L^2}^2 < 2\pi^2$; by Theorem 3.8(ii) and Section 11.5 they then hold on homology for the prescribed data. Propositions 11.10 and 11.11 use Theorem 3.8 only for presentation data, so the definition is not circular.

The sets $\mathcal{U}_F(t)$, $\mathcal{U}'_F(t)$ and $\mathcal{U}(F)$ are open neighbourhoods of zero in $\text{span } F$: the map $\mathfrak{h} \mapsto (L_{\mathfrak{h}}, L_{\bar{\mathfrak{h}}})$ is defined and continuous on the set of primary perturbations in $\text{span } F$ in the sense of Section 3.2, which is open, transversality and embeddedness being open conditions by compactness; the preimage of the open set $\mathcal{N}(t)$ in it is open in the finite-dimensional space $\text{span } F$, hence locally path connected, and so are its path components; any two points of a path component are joined by a smooth path in it. They are monotone in the family: $\mathcal{U}(F) \subseteq \mathcal{U}(F')$ for $F \subseteq F'$, since a path in $\text{span } F$ is a path in $\text{span } F'$. In particular, if $P \in \mathcal{U}(F)$ and G is a finite family, then $P + g \in \mathcal{U}(F \cup G)$ for all sufficiently small $g \in \text{span } G$; and two admissible choices of primary data in $\mathcal{U}(F_1)$ and $\mathcal{U}(F_2)$ both lie in $\mathcal{U}(F_1 \cup F_2)$, where Proposition 11.10 applies to them. Admissibility is thus a condition on the primary data as functions, through the path joining them to zero within the family, and not on the position of their Lagrangians alone: a datum whose Lagrangians lie in $\mathcal{N}(t_*)$ but which is not joined to zero within the preimage is not admissible.

Theorem 11.13 (one-handle square). *Let $W_k: M \rightarrow M^+$ be a k -fold one-handle cobordism with the product bundle, p an admissible presentation adapted to its attaching balls, and $C \circ p$ the enlarged presentation with height Lagrangians, with primary data in the admissible neighbourhoods $\mathcal{U}(F)$ of Definition 11.12 at p and at $C \circ p$. Then $N_{C \circ p} \circ I^\#(W_k) = U_C \circ N_p$ on homology.*

Proof. Choose auxiliary lower data for p in $\mathcal{N}_{C,R} \cap \mathcal{U}(F)$, with $\mathcal{N}_{C,R}$ the neighbourhood of Lemma 11.7(i) for the given height Lagrangians R of $C \circ p$; they exist because both sets are neighbourhoods of zero in $\text{span } F$ and small generic data are regular. At these data Propositions 11.9 and 11.11 give $U_C = N_{C \circ p} I^\#(W_k) N_p^\dagger = N_{C \circ p} I^\#(W_k) N_p^{-1}$. At the given data, conjugate by the continuations and apply Proposition 11.10 at both ends. $\square$

Theorem 11.14 (three-handle square). *Let $W_3: M^+ \rightarrow M$ be a three-handle cobordism, read as the one-handle cobordism $\bar{W}_3: -M \rightarrow -M^+$ of the reflected presentation. Then $N_p \circ I^\#(W_3) = V_C \circ N_{C \circ p}$ on homology, with primary data in the admissible neighbourhoods of Definition 11.12.*

Proof. Reading the same oriented four-manifold from the opposite ends gives $I^\#(W_3) = I^\#(\bar{W}_3)^\vee$: the anti-self-duality equation on W_3 with ends read in reflected charts has the same index-zero moduli spaces with input and output exchanged. By Definition 11.4, $V_C = (U_{\bar{C}})^\vee$. Transposing Theorem 11.13 for the reflected presentation and applying Proposition 11.11 at

both ends,

$$N_p I^\#(W_3) = (N_{\bar{p}}^{-1})^\vee I^\#(\overline{W}_3)^\vee = (U_{\bar{C}})^\vee (N_{\bar{C}op}^{-1})^\vee = V_C N_{Cop}. \quad \square$$

Remark 11.15. A one-handle followed by a three-handle does not cancel: the composite is $(M \times [0, 1]) \# (S^1 \times S^3)$. Nor do two-handle data determine the one-handle map: if a and b are maps with $b \circ a = 0$ and $b \circ f = 1$, then $b \circ (f + a \circ g) = 1$ for every g . The one-handle square therefore needs its own degeneration.

12. PRESENTATION INDEPENDENCE AND FRAME CHANGES

12.1. The presentation graph. Let (M, b) be an object. Let $\mathcal{P}(M, b)$ be the graph whose vertices are the admissible presentations of Definition 3.3 and whose edges are:

- (P1) changes of the meridian disk systems in a fixed embedded splitting (Definition 3.3 records no disk systems, so these edges are loops at a vertex; they are listed to match the edges of [JTZ21]);
- (P2) ambient isotopies of the splitting fixed near $b(B^3)$;
- (P3) stabilization and destabilization away from $b(B^3)$;
- (P4) changes of the metric data, of the primary data and of the regularizing data (Definition 3.3(v)), including the almost complex structure on the strip end, of a fixed splitting.

Proposition 12.1. $\mathcal{P}(M, b)$ is connected.

Proof. Remove the interior of $b(B^3)$ and put an annular suture around the equator ∂D of its boundary sphere. The result is a balanced sutured manifold $M(b)$ whose positive and negative regions are disks. Removing D from an embedded Heegaard surface adapted to b gives a sutured Heegaard diagram with one boundary circle, and conversely. By [JTZ21, Definitions 2.34–2.35, Proposition 2.36], any two vertices of the graph of embedded sutured diagrams of $M(b)$ are joined by a path of edges given by compression-equivalent changes of attaching sets, stabilizations, and diffeomorphisms isotopic to the identity in $M(b)$ relative to the suture [JTZ21, Definition 2.34]. The first are (P1). The last are diffeomorphisms isotopic to the identity relative to the suture, which need not preserve the standard collar of the boundary sphere; since the image surface meets the boundary sphere transversely in the same circle, the collar neighbourhood theorem gives an ambient isotopy, fixed on a smaller collar, that makes it standard on the collar again, and the composite is an edge (P2). The stabilizations of [JTZ21, Definition 2.18] need not be local: the punctured torus T replacing the disk D is not required to lie in a small ball. Nevertheless every such edge $H_1 \rightarrow H_2$ factors in $\mathcal{P}(M, b)$ as a local stabilization (P3) in a ball Q with $Q \cap \Sigma_1 \subset \text{int } D$, followed by a change of disk system and an isotopy fixed near $b(B^3)$ (Lemma 12.2). An additional stabilization, in a small ball of the standard collar, makes the genus at least one throughout: fix a small ball Q_0 in the common standard collar, where all diagrams of the path are standard and all isotopies are the identity, and stabilize every vertex of the path in Q_0 . The stabilized vertices form a path of the same kind, and the two ends are joined to their unstabilized versions by edges (P3). Hence the torus genus is at least two throughout. For a fixed splitting, shrink the admissible collars so that the fixed height solid tori remain in the compression interiors. Cutting along the framing sphere S gives neat collars on the two sides, matched along the collar of ∂D . Uniqueness of neat collars up to relative isotopy, followed by isotopy extension fixing Σ_h and preserving S , joins the collar choices away from the fixed torus collar. Product metrics on a fixed collar form a convex set. The admissible neighbourhoods $\mathcal{U}(F)$ of Definition 11.12 are path connected, being path components. An edge (P4) joins

admissible endpoints within these neighbourhoods; its intermediate primary pairs need not stay transverse. Thus the choices are joined by edges (P4). $\square$

Lemma 12.2. *Let $H_1 = (\Sigma_1, \alpha_1, \beta_1)$ and $H_2 = (\Sigma_2, \alpha_2, \beta_2)$ be embedded diagrams of $M(b)$, standard on a common collar of the boundary, such that H_2 is obtained from H_1 by a stabilization with data $D \subset \Sigma_1$, $T \subset \Sigma_2$, $a = \alpha_2 \cap T$, $b = \beta_2 \cap T$. Then there are a ball Q disjoint from the curves of H_1 and from $b(B^3)$, with $Q \cap \Sigma_1 \subset \text{int } D$, and an isotopy ψ_t of $M(b)$ with compact support in the interior, such that $\psi_1(\Sigma_2)$ is the local stabilization of Σ_1 in Q , with the same sides.*

Proof. The curves a and b bound disks D_a, D_b on the two sides of Σ_2 , meeting in one point. Destabilizing Σ_2 along a regular neighbourhood N of $D_a \cup D_b$, chosen to meet Σ_2 exactly in T , gives a surface Σ'' that agrees with Σ_1 outside D . Then $\Sigma'' \setminus (\Sigma_1 \setminus D)$ and D are two disks with the same boundary. They lie in the union of the two sides of Σ_1 glued along D , a boundary sum of irreducible compression bodies, which is irreducible. After an innermost-disk argument removing the intersections of their interiors, the sphere they form bounds a ball, and the two disks are isotopic relative to their boundary, by an isotopy supported in the interior and away from the curves of H_1 and from $b(B^3)$. This isotopy carries N , which contains T , into a ball Q , which a final small isotopy near D places with $Q \cap \Sigma_1 \subset \text{int } D$, and carries Σ_2 to the local stabilization of Σ_1 in Q , with the same sides. The framing ball and the torus are never moved. $\square$

Lemma 12.3. *Let $Q \subset M \setminus b(B^3)$ be a ball, $W_1: M \rightarrow M_m$ a one-handle and $W_2: M_m \rightarrow M'$ a two-handle attached in $Q \times [0, 1]$ whose attaching circle meets the belt sphere of W_1 transversely once, with product bundles and vertical tube. For every diffeomorphism $\iota: M' \rightarrow M$ equal to the identity outside Q , $(W_2, \iota) \circ W_1 \cong 1_M$ in $\mathbf{B}$.*

Proof. Give $W_2 \circ W_1$ a Morse function with the two critical points and a gradient-like field equal to ∂_t on $(M \setminus Q) \times [0, 1]$. By the first cancellation theorem [Mil65, Theorem 5.4] it can be replaced by one without critical points, the change supported near the connecting trajectory between the two critical points, which misses $(M \setminus Q') \times [0, 1]$ for a slightly larger ball Q' . Integrating the new field gives a diffeomorphism to $M \times [0, 1]$ equal to the identity on the incoming end and on $(M \setminus Q') \times [0, 1]$. The induced self-diffeomorphism of M at the outgoing end, composed with ι^{-1} , is supported in Q' , hence isotopic to the identity relative to the boundary of Q' by Cerf's theorem [Cer68], and the mapping cylinder of the isotopy is an isomorphism of decorated cobordisms fixed on the tube. $\square$

12.2. The edge squares. For each edge $e: p \rightarrow q$ we define an independent symplectic map $\Gamma_e: HF(M, p) \rightarrow HF(M, q)$ and prove

$$(12.1) \quad H(N_q) \circ G(q, p) = \Gamma_e \circ H(N_p),$$

where $G(q, p)$ is the gauge continuation map, i.e. $I^\#$ of the product cobordism with vertical tube and product bundle. In the notation of Section 11.3, $G(q, p) = \text{cont}_G(p, q)$; here the arguments are written in the order of composition, so that $G(r, q)G(q, p) = G(r, p)$.

Proposition 12.4. *Every edge of $\mathcal{P}(M, b)$ has an independently defined map Γ_e satisfying (12.1).*

Proof. (P1): The Lagrangians $L(C_i)$ depend on the embedded compression bodies, their metrics and primary perturbations, and not on the disk systems, so $\Gamma_e = 1$ and (12.1) is trivial.

(P4), primary data: Γ_e is the symplectic continuation of Section 11.3, and (12.1) is Proposition 11.10 inside one finite family; two finite families are bridged through their union, which is again a finite family containing both.

(P4), regularizing data: for changes of f, χ, η the map Γ_e is the identity and the square is Theorem 3.8(ii). For a change of the almost complex structure J on the strip end, Γ_e is the symplectic continuation. Insert a strip carrying the continuation from J to J' at the ordinary end of the mixed domain; by Theorem 5.1 at the ordinary neck, for a long neck the count on the glued domain is $\Gamma_e \circ N_p$, and the glued domain is a compactly supported deformation of the domain for J' , so Theorem 5.4 identifies the count with $N_{p'}$ up to chain homotopy.

(P3): Let r be the local stabilization of p in a ball Q . Let $W_1: M \rightarrow M\#(S^1 \times S^2)$ be the one-handle and W_2 the cancelling two-handle, both attached in $Q \times [0, 1]$. By Lemma 12.3, $W_2 \circ W_1 \cong 1_M$ for any identification of the outgoing end that is the identity outside Q , so $G(r, p) = I^\#(W_2)I^\#(W_1)$. The one-handle square (Theorem 11.13) ends at the enlarged presentation $C \circ p$, with compression bodies (V_0, V_1) carrying height Lagrangians (R_0, R_1) at a height t , while the two-handle square (Theorem 9.13) starts at a vertex whose second body is unperturbed. The two are joined by primary-data changes, as follows. Give $C \circ p$ the metric data that agrees on V_0 with the metric of r on its first body; the output of the cancelling two-handle has bodies (V_0, C_2) , where C_2 is the second body of r with the metric of r . Take first the heights t_* of Definition 11.12 for the splittings and metric data of $C \circ p$ and of r , then t and the two-handle height t_2 so small that all data below lie in the corresponding neighbourhoods and in the admissible neighbourhoods, and then the lower Lagrangians of p as in Theorem 11.13. Let R'_0 be a height Lagrangian obtained from R_0 by a small generic perturbation from $\text{span } F_*$ as in Section 11.6, transverse to $L(V_1)$ and to $L_{t_2 h_Q}(C_2)$ and missing their intersection; this is a generic condition, and the perturbation may move the generators of R_0 , whose labels are continued. Put

- $(C \circ p)'$: the presentation $C \circ p$ with primary data $(R'_0, 0)$;
- r' : the presentation r with primary data $(R'_0, t_2 h_Q)$.

Both are admissible. The subordinate triple of the cancelling two-handle is $(C_0, C_1, C_2) = (V_0, V_1, C_2)$ with primary data $(R'_0, 0, t_2 h_Q)$, which satisfies the hypotheses of Section 9.1, so Theorem 9.13 holds from $(C \circ p)'$ to r' . Define

$$S_{r,p} = \text{cont}_S(r', r) \circ S_{W_2} \circ \text{cont}_S(C \circ p, (C \circ p)') \circ U_C.$$

The squares of Theorem 11.13, of Proposition 11.10 for the two changes of primary data, and of Theorem 9.13 compose to $H(N_r)G(r, p) = S_{r,p}H(N_p)$.

The reverse composite is a two-handle $W'_2: M \rightarrow M\#(S^1 \times S^2)$ followed by a three-handle W_3 , both in $Q \times [0, 1]$. Reversing the orientation of the cobordism exchanges them with a one-handle followed by a cancelling two-handle, so Lemma 12.3 applied to the reversed cobordism gives $W_3 \circ W'_2 \cong 1_M$ and $G(p, r) = I^\#(W_3)I^\#(W'_2)$. The triple of W'_2 is (V_0, C_2, V_1) with primary data $(R'_0, 0, t'_2 h'_Q)$, where h'_Q is the corresponding height function on V_1 , chosen in the same way, and

$$D_{p,r} = V_C \circ \text{cont}_S((C \circ p)'', C \circ p) \circ S_{W'_2} \circ \text{cont}_S(r, r''),$$

where r'' carries $(R'_0, 0)$ and $(C \circ p)''$ carries $(R'_0, t'_2 h'_Q)$. Theorems 9.13 and 11.14 and Proposition 11.10 give $H(N_p)G(p, r) = D_{p,r}H(N_r)$. Then $D_{p,r}S_{r,p}H(N_p) = H(N_p)G(p, r)G(r, p) = H(N_p)$, and since $H(N_p)$ is invertible, $D_{p,r}S_{r,p} = 1$; similarly $S_{r,p}D_{p,r} = 1$.

(P4), metric data: for two presentations p, q of one splitting that differ in their metric data, choose a common stabilization r and put $\Gamma_e = D_{q,r}S_{r,p}$. This is possible because the upper metric data in Theorems 9.13 and 11.13 is free within the class of Section 10.1: the

flat-connection Lagrangians do not depend on the metric, so one upper choice serves both p and q . The square follows from the stabilization squares and $G(q, r)G(r, p) = G(q, p)$; independence of r follows by invertibility of N .

(P2): An ambient isotopy ϕ_t fixed near $b(B^3)$ transports all data. Transporting data conjugates every equation, operator and perturbation, so the counts agree exactly, and the mapping cylinder of the isotopy is the decorated product. The isotopy transports the primary data as well as the metrics, so the transported vertex differs from the prescribed one in both. Hence Γ_e is the map induced by the transported data followed by the metric-data change and then the primary-data change (P4), and the square follows from the previous cases. $\square$

Corollary 12.5. *For paths γ in $\mathcal{P}(M, b)$ from p to q , the composite Γ_γ depends only on p and q ; the maps $\Gamma(q, p)$ satisfy $\Gamma(r, q)\Gamma(q, p) = \Gamma(r, p)$ and $\Gamma(p, p) = 1$.*

Proof. By (12.1) and functoriality of gauge continuation, $\Gamma_\gamma H(N_p) = H(N_q)G(q, p)$, and $H(N_p)$ is invertible. $\square$

12.3. Frame changes.

Proposition 12.6. *Let $u: M \rightarrow \mathrm{SO}(3)$ equal 1 near $b(B^3)$, and let p, q be admissible presentations. The restriction $u|_\Sigma$, extended by 1 over the torus summand, is an automorphism $\hat{u}$ of the odd surface bundle that extends over both compression bodies; its lifting class $c(\hat{u})$ therefore lies in $\Lambda_0 \cap \Lambda_1$, and the induced symplectomorphism $\tau_{c(\hat{u})}$ of M_h carries the Lagrangians of p to those of the transported presentation u_*p . Let u_S be the induced pushforward chain isomorphism and $S_{[u]} = \Gamma(q, u_*p) \circ u_S$. Then $H(N_q) \circ I^\#([u]) = S_{[u]} \circ H(N_p)$.*

Proof. On chains $N_{u_*p} \circ u_G = u_S \circ N_p$, where u_G is the gauge pushforward: applying the bundle automorphism u to all data transports the equations of the mixed domain, the compression bodies, the torus marking and the tube (on which $u = 1$), and preserves the lifting group, whose support is disjoint from that of u . The transported presentation u_*p is admissible for the family $F \cup u_*F$, with the transported mixed map: the path joining the primary data of p to zero in $\mathrm{span} F$ is transported to a path in $\mathrm{span} u_*F$ whose Lagrangians, the images under τ_c of the original ones, and their reflections lie in $\mathcal{N}(t_*) \cap \mathcal{N}(t'_*)$, these neighbourhoods being invariant under the symplectomorphisms τ_c for these classes c (Definition 11.12). The gauge map of the frame change is $I^\#([u]) = G(q, u_*p) \circ u_G$. Hence, by Corollary 12.5, $H(N_q)I^\#([u]) = H(N_q)G(q, u_*p)u_G = \Gamma(q, u_*p)H(N_{u_*p})u_G = \Gamma(q, u_*p)u_S H(N_p)$. $\square$

A lifting class in $\Lambda_0 + \Lambda_1$ would not suffice: if $a \in \Lambda_0 \setminus \Lambda_1$, then τ_a changes the sign of the lifted holonomy of a meridian of H_1 and moves $L(C_1)$ off itself. The condition $\zeta \in \Lambda_0 + \Lambda_1$ on the filling-frame class ζ is the triviality condition for the bundle on $H_0 \cup H_1$ and is used in the construction of the global frames of Section 4.4.

13. PROOF OF THEOREM A

13.1. A descent lemma. Let $V(M)$ denote the set of admissible presentations of an object M (Section 3). Consider the category $\mathbf{B}^V$ whose objects are pairs (M, p) with $p \in V(M)$ and whose morphisms $(M, p) \rightarrow (M', q)$ are the morphisms $M \rightarrow M'$ of $\mathbf{B}$. The projection $\mathbf{B}^V \rightarrow \mathbf{B}$ is an equivalence. For each (M, p) the gauge complex $C_G(M, p)$ computes $I^\#(M)$: the gauge continuation maps provide canonical isomorphisms $H(C_G(M, p)) \cong I^\#(M)$ compatible with the maps $I^\#(W, E, \nu)$ (Proposition 2.4). We therefore regard $I^\#$ as a functor on $\mathbf{B}^V$ with $I^\#(M, p) = H(C_G(M, p))$, and write $N_{(M, p)} = H(N_p): I^\#(M, p) \rightarrow HF(M, p)$.

An *elementary arrow* $e: (M, p) \rightarrow (M', q)$ is a triple consisting of an underlying morphism $[e] \in \mathbf{B}(M, M')$, source and target presentations, and a linear map $S_e: HF(M, p) \rightarrow HF(M', q)$ defined by holomorphic curve counts. The arrows used are listed in Section 13.2.

Lemma 13.1. *Let $\mathcal{A}$ be a set of elementary arrows such that:*

- (a) *each $N_{(M,p)}$ is an isomorphism;*
- (b) *every $e \in \mathcal{A}$ satisfies $N_{(M',q)} \circ I^\#([e]) = S_e \circ N_{(M,p)}$;*
- (c) *for all (M, p) , (M', q) and every $W \in \mathbf{B}(M, M')$ there is a word $w = (e_1, \dots, e_m)$ of elements of $\mathcal{A}$ from (M, p) to (M', q) with $[e_m] \circ \dots \circ [e_1] = W$.*

For a word w put $S_w = S_{e_m} \circ \dots \circ S_{e_1}$. Then

- (i) $N_{(M',q)} \circ I^\#(W) = S_w \circ N_{(M,p)}$ for every word w representing W ;
- (ii) S_w depends only on W , p and q ; write it $S(W)_{qp}$;
- (iii) $W \mapsto S(W)_{qp}$ is a functor $\mathbf{B}^V \rightarrow \text{Vect}_{\mathbb{F}_2}$, and N is a natural isomorphism $I^\# \Rightarrow S$;
- (iv) the maps $\Gamma(q, p) = S(1_M)_{qp}$ satisfy $\Gamma(r, q)\Gamma(q, p) = \Gamma(r, p)$ and $\Gamma(p, p) = 1$. With $HF^\#(M) = \text{colim}_p HF(M, p)$ along Γ , the functor S and the transformation N descend to a functor $HF^\#$ on $\mathbf{B}$ and a natural isomorphism $I^\# \Rightarrow HF^\#$.

Proof. (i) Induction on m . For $m = 0$ the word is empty, $W = 1_M$ and $p = q$. For $m \geq 1$ let $w' = (e_1, \dots, e_{m-1})$ end at (M'', r) . By functoriality of $I^\#$, (b) for e_m , and the induction hypothesis,

$$N_q I^\#(W) = N_q I^\#([e_m]) I^\#([w']) = S_{e_m} N_r I^\#([w']) = S_{e_m} S_{w'} N_p.$$

(ii) By (i) and (a), $S_w = N_q I^\#(W) N_p^{-1}$, which does not involve w . (iii) $S(W' \circ W) = N I^\#(W') I^\#(W) N^{-1} = S(W') S(W)$ and $S(1) = 1$; naturality is (i). (iv) is (iii) applied to identity morphisms; a system of isomorphisms satisfying the cocycle law has a colimit with isomorphisms $i_p: HF(M, p) \rightarrow HF^\#(M)$, and $N_M = i_p N_{(M,p)}$ is independent of p by (i). $\square$

Lemma 13.1 is formal. All of the content of Theorem A lies in (a), (b) and (c). In (b) the maps $N_{(M,p)}$ must be the ones fixed at the vertices: a square proved with an auxiliary mixed map at the source or target is usable only after that map has been identified, on homology, with the mixed map. This is the role of Theorem 3.8(ii) and of the data-change squares of Section 12.

13.2. Generation. The set $\mathcal{A}$ consists of the following arrows.

- (A1) *Presentation moves* $(M, p) \rightarrow (M, q)$ with $[e] = 1_M$: changes of disk system, stabilization and destabilization away from the ball, isotopies fixed near the ball, and changes of the metric and primary data (Section 12).
- (A2) *Frame changes* $[u]$ (Section 4.5), with $S_{[u]}$ the map induced by the symplectomorphism of M_h given by $u|_\Sigma$ (Proposition 12.6).
- (A3) *One-handles* $(M, p) \rightarrow (M^+, C \circ p)$, where $C \circ p$ carries height Lagrangians at some height t and p , $C \circ p$ satisfy the neighbourhood conditions of Theorem 11.13, with the compression cup.
- (A4) *Two-handle cobordisms* $(W_2, E; \phi_0^{\text{gl}}, \phi_1^{\text{gl}})$ with bundles trivial at both ends, between the presentations p_0^ϕ, p_1^ϕ of a subordinate triple with the global frames of Section 4.4, carrying the transported data of Section 9.1 with t below the threshold of Lemma 9.1(iii) and so small that the primary data are joined to zero within the admissible neighbourhoods; these vertices are admissible presentations (Definition 11.12). The map is $S_{W_2, E}$ (Theorem 9.13).

- (A5) *Three-handles* $(M^+, C \circ p) \rightarrow (M, p)$, with the same vertex conditions as in (A3), with the compression cap (Theorem 11.14).

Proposition 13.2. *A satisfies condition (c) of Lemma 13.1.*

Proof. Let $(W, E, \nu): M \rightarrow M'$ and presentations p, q be given. By Propositions 4.1 and 4.2, $(W, E, \nu) = W_3 \circ (W_2, E) \circ W_1$ with product bundles on W_1 and W_3 and connected levels Z_0, Z_1 . Write W_1 and W_3 as composites of single one- and three-handles. By Lemma 4.7, $(W_2, E) = [u_1] \circ (W_2, E; \phi^{\text{gl}}) \circ [u_0]^{-1}$, a two-handle arrow between frame changes, or a single frame change if there are no two-handles. Each handle can be attached in a presentation adapted to it (Section 4.3 and the paragraph preceding it), and the two-handle cobordism in the presentation given by a subordinate triple (Proposition 4.3). By Proposition 12.1, any two presentations of the same object, and in particular consecutive presentations at each level as well as p, q at the ends, are joined by presentation moves; in particular each handle arrow can be taken between vertices satisfying the conditions of (A3)–(A5), since such vertices exist for every splitting (small heights and generic small data) and are reached by presentation moves. Concatenation gives a word of arrows in $\mathcal{A}$ from (M, p) to (M', q) whose underlying composite is (W, E, ν) , since presentation moves have underlying morphism the identity. $\square$

13.3. Proof of Theorem A. Condition (a) of Lemma 13.1 is Theorem 3.8. Condition (b) is Proposition 12.4 for (A1), Proposition 12.6 for (A2), Theorem 11.13 for (A3), at presentations with primary data in the admissible neighbourhoods of Definition 11.12 (arrows may be taken at any convenient vertices, since presentation moves connect them), Theorem 9.13 for (A4) and Theorem 11.14 for (A5). Condition (c) is Proposition 13.2. Lemma 13.1 gives the equation of Theorem A for every word, the independence of S_d , the functor $HF^\#$ and the natural isomorphism. The statement in the homotopy category of finite complexes follows from Lemma B.1. $\square$

Remark 13.3. No relation among symplectic words is proved directly. Independence of the decomposition follows from the squares, the connectivity of the presentation graph and the invertibility of N . The argument does not give integral statements: over $\mathbb{Z}$ the squares would be needed with signs, and Lemma B.1 fails.

13.4. The isomorphism of Daemi, Fukaya and Lipyanskiy.

Corollary 13.4. *Every object has admissible presentations with DFL-regular primary data, at every Heegaard splitting and all metric data. Let p_0, p_1 be admissible presentations of M_0, M_1 with DFL-regular primary data, and let $N_{p_i}^{\text{DFL}}$ be the isomorphisms of [DFL21, Theorem 3.29] at the data of Proposition 3.11. Then for every morphism (W, E, ν) and every admissible decomposition d ,*

$$H(N_{p_1}^{\text{DFL}} \otimes \mathbb{F}_2) \circ I^\#(W, E, \nu) = S_d(W, E, \nu) \circ H(N_{p_0}^{\text{DFL}} \otimes \mathbb{F}_2).$$

In particular, for two such choices of data at presentations p, q of one object, $H(N_q^{\text{DFL}} \otimes \mathbb{F}_2) \circ G(q, p) = \Gamma(q, p) \circ H(N_p^{\text{DFL}} \otimes \mathbb{F}_2)$ as maps from the gauge homology at p , where $G(q, p)$ is the gauge continuation and $\Gamma(q, p)$ the symplectic transitive system of Theorem A, defined by holomorphic curve counts. So within this class of data, including changes of the Heegaard splitting, the reduction modulo two of the isomorphism of [DFL21] on homology does not depend on the choices.

Proof. The first sentence is Theorem 3.6, applied with any separated family as in Definition 3.3 and generic primary data near zero in $\mathcal{U}(F)$. By Proposition 3.11, $N_{p_i}^{\text{DFL}} \otimes \mathbb{F}_2 = N_{p_i}$

at these data, and Theorem A applies to N_{p_i} . The statement for two choices of data at one object is the case of the identity morphism: on the gauge homology at p and q the identity of $I^\#(M)$ is the gauge continuation $G(q, p)$, and $S(1_M)_{qp} = \Gamma(q, p)$ by Lemma 13.1. $\square$

Let p be an admissible presentation of M , with family F_p , and let q be an admissible presentation with the same splitting and metric data and DFL-regular primary data, admissible for a family F_q (Theorem 3.6). The case $W = 1_M$ of Theorem A, with Proposition 3.11 at q , gives

$$H(N_p) = \Gamma(p, q) \circ H(N_q^{\text{DFL}} \otimes \mathbb{F}_2) \circ G(q, p)$$

on homology, where $G(q, p)$ is the gauge continuation map. Both primary data lie in $\mathcal{U}(F_p \cup F_q)$, by the monotonicity of the admissible neighbourhoods (Definition 11.12 and the paragraph following it), so Proposition 11.10 applies to them in the family $F_p \cup F_q$; with the invertibility of $H(N_p)$ it identifies $\Gamma(q, p)$ with the symplectic continuation map. So the mixed map at any admissible presentation is, on homology, the reduction modulo two of the isomorphism of [DFL21] at an admissible presentation with the same splitting and metric data, composed with gauge and symplectic continuation maps.

APPENDIX A. RESULTS USED FROM THE LITERATURE

We list the published results used in the proof, where they enter, and the hypotheses that have to be checked.

- (1) [DFL21, Theorem 1, Propositions 2.21, 3.19, 3.20, Lemma 3.15, Theorem 3.29]: monotonicity with minimal Maslov number four, embeddedness of $L(C)$, the index of constant pairs, the index-energy law and triangularity of the mixed map. Used in Sections 3 and 6. Hypotheses: the bundles are odd on the matching surface; the Lagrangians are embedded, which holds for compression bodies.
- (2) [DFL21, Theorem 4.17, Remark 4.19, Section 4.2.1, (4.20)–(4.24), Theorem 5.2, Lemma 5.23, Proposition 5.29, Section 5.4, Section 6]: local elliptic estimates for the mixed operator, theory of mixed ends, small-energy regularity, exponential decay, compactness and perturbations. Used throughout. Hypotheses: product metrics and the complex structure $*_\Sigma$ near the seams, which Convention 3.1 and Hypothesis 6.5 arrange.
- (3) [DFL21, Section 2.3, Proposition 2.53, Section 3.1 and Figure 1, Definitions 3.8–3.9, (3.24), Proposition 3.18]: primary perturbations and the submersion property of the universal restriction map, the special quintuple and its planar region, the domains of DFL type with their weighted spaces, and the Fredholm property of the mixed operator. Used in Sections 3, 6 and 9. Hypotheses: those of (1).
- (4) [DFL21, Section 4.1 with (4.3), (4.5), (4.7), (4.8), Lemmas 4.1, 4.2 and 4.4, Proposition 4.6, Section 4.2.1 with (4.16) and (4.24), Propositions 2.10, 4.65, 4.69, 5.27, 5.28 and 5.29]: the chart P_p and the cutoff maps E_τ , the flat extension of [DFL21, Lemma 4.2], a metric in which the Lagrangian is totally geodesic, the Coulomb slice and its variant, the Coulomb gauge on the three-manifold, the slice operator on a cylinder quintuple, additivity of energy and index under cutoff gluing, and exponential decay on mixed ends. Used in Sections 5 and 10. Lemma 4.1 is used with its radius measured in L^q for some $q > 2$ (Remark 5.2). Hypotheses: nondegenerate or Morse–Bott ends as stated there.
- (5) [DFL24, Theorems 1 and 3, Theorem 4.2, Lemmas 4.25 and 4.50, Theorem 4.44]: seam regularity, compactness, seam estimates and removal of seam singularities for the mixed

equation. Hypotheses: fixed product collars after conformal rescaling; the disconnected odd matching surface is allowed.

- (6) [KM11b, Sections 5.1–5.2, Definition 5.1, Propositions 5.2 and 5.4, Theorem 5.6]: the enlarged gauge groups, the quotient by ϕ , the composition law with the connecting-map condition, and excision. Used in Sections 2, 9 and 10. Hypotheses: the ϕ -non-integral condition at every end, and the connecting image ψ contained in ϕ ; both are checked where used.
- (7) [Sca15, Sections 7.1–7.4, 7.6–7.8, Proposition 7.1]: the framed gauge transformations, the $\mathbb{Z}/4$ grading and degree formula, and $I^\#(\#^r(S^1 \times S^2))$ with its relative invariants.
- (8) [OS06, Definition 4.2, Proposition 4.3, Lemma 4.5]: subordinate Heegaard triples and the identification of the filled triangle with the two-handle trace; the tube is controlled by Proposition 4.3.
- (9) [JTZ21, Definitions 2.34–2.35, Proposition 2.36]: connectivity of the graph of embedded sutured diagrams; applied to the sutured manifold $M(b)$ in Proposition 12.1.
- (10) [WW15b, Theorems 3.9, 3.13, 4.1, 4.2, Example 2.6 and Corollary 5.5]: quilted Floer maps, gluing, data-change isomorphisms, the strip example (Example 2.6) and unit insertion (Corollary 5.5). Hypotheses: monotonicity with a common constant and minimal Maslov number at least three for all Lagrangians and seams, and Hamiltonian one-forms vanishing on the boundary; the continuation of Section 11.3 is arranged in this form.
- (11) [Oh15, Section 14.4]: continuation with moving Lagrangian boundary conditions.
- (12) [Don02, Propositions 3.8–3.10, 4.3, 4.4 and 4.16, Theorems 4.17 and 5.4, Sections 5.3 and 5.5]: additivity of the index under gluing along acyclic ends, weights and kernels, exponential decay of solutions of small energy on long tubes, also for the holonomy-perturbed equation, neck gluing with surjectivity, and gluing at index -1 in families; the addition property of [Don02, Section 3.3], on which the excision argument of Lemma 6.10 is modelled.
- (13) Gluing along ordinary strip necks with transverse intersections is proved in Theorem 5.1 by the argument of [Don02, Section 3] with the local estimates of [DFL24, Section 4] and the exponential decay of holomorphic strips of small energy [Sim14, Proposition 2.3.1], whose hypotheses (H) and (H1), closed Lagrangians with compact clean intersection, hold here; monotonicity enters only through compactness. For exact Lagrangians the corresponding statement is [Sei03, Proposition 2.22].
- (14) [MS12, Theorem B.4.1]: boundary regularity for J -holomorphic curves with Lagrangian boundary conditions, from $W^{1,p}$, $p > 2$.
- (15) [SW08, Proposition 8.6, Theorem C.1]: unique continuation for the trajectory equation.
- (16) [BB16], [Sch95, Corollary 2.1.6, Theorem 2.1.7], [RS95], [Aro57]: elliptic boundary value problems for first-order operators, the Gaffney–Friedrichs inequality, regularity for families of self-adjoint operators, and unique continuation.

The quilted description of the compression cup in Remark 11.6 uses in addition [WW15b, Corollary 5.5], [WW10], [WW12] and [LL13, Theorem 2 and Corrigendum, Remark 1(1)]; none of these enters the proofs of Theorems A and B.

APPENDIX B. HOMOLOGICAL ALGEBRA OVER A FIELD

Lemma B.1. *Let (C, d_C) and (D, d_D) be finite-dimensional chain complexes over a field k , and $f, g: C \rightarrow D$ chain maps with $H(f) = H(g)$. Then f and g are chain homotopic. Consequently, the homology functor from the homotopy category of finite-dimensional complexes*

over k to graded vector spaces is fully faithful, and an equation between composites of chain maps that holds on homology holds up to chain homotopy.

Proof. It suffices to treat $g = 0$. Over a field every finite complex splits: there are maps $i_C: H(C) \rightarrow C$, $p_C: C \rightarrow H(C)$ and $h_C: C \rightarrow C$ of degree one with $p_C i_C = 1$, $1 - i_C p_C = d_C h_C + h_C d_C$, and $d_C i_C = 0$, $p_C d_C = 0$; similarly for D . Put

$$H = h_D f + i_D p_D f h_C.$$

Using $d_D i_D = 0$ and $d_D f = f d_C$,

$$d_D H + H d_C = (d_D h_D + h_D d_D) f + i_D p_D f h_C d_C = (1 - i_D p_D) f + i_D p_D f (1 - i_C p_C - d_C h_C).$$

Since $p_D f d_C = p_D d_D f = 0$ and $p_D f i_C = H(f) = 0$, the last term equals $i_D p_D f$, and $d_D H + H d_C = f$. $\square$

The splitting maps depend on choices, so the homotopy H is not canonical, and no higher coherence is asserted. Over $\mathbb{Z}$ the lemma fails, which is one reason Theorem A is stated over $\mathbb{F}_2$.

APPENDIX C. EXISTENCE OF DFL-REGULAR PRIMARY DATA

We prove Lemma 3.5 and Theorem 3.6. The scheme of the induction is that of [DFL21, proof of Proposition 6.8] and [Don02, Section 5.5.1]; the new input is Lemma 3.5, whose proof occupies Sections C.1–C.8.

C.1. Setting. Fix a Heegaard splitting and metric data as in Definition 3.3(i)–(iii), and write $\Sigma = \Sigma_h \sqcup T^2$. Following [DFL21, Section 2.1, (2.1)], $M^\#$ contains a product region $(-2, 2) \times \Sigma$ with the metric $dt^2 + g_\Sigma$, and

$$M^\# = C_0^\circ \cup N \cup C_1^\circ, \quad N = [-1, 1] \times \Sigma, \quad C_0^\circ \cap N = \{1\} \times \Sigma, \quad C_1^\circ \cap N = \{-1\} \times \Sigma;$$

in the notation of [DFL21], $C_0^\circ = Y_0$ and $C_1^\circ = Y'_0$. All gauge groups are determinant-one [DFL21, Section 2.2], connections are of class L_l^2 with $l \geq 2$, and L^2 inner products use $\langle a, b \rangle = -\text{tr}(ab)$. Gauge transformations of C_i° carry no boundary condition. At an irreducible connection B on C_i° the operator $d_B: L_1^2(C_i^\circ) \rightarrow L^2(C_i^\circ)$ is injective and satisfies $\|\zeta\|_{L_1^2} \leq C(\|d_B \zeta\| + \|\zeta\|)$, so by the compactness of $L_1^2 \subset L^2$ it has closed range. The projection π_B^i of Section 3.4 is $\pi_B^i b = b - d_B \zeta$, where ζ solves the Neumann problem $d_B^* d_B \zeta = d_B^* b$, $*d_B \zeta = *b$ on ∂C_i° , and it depends continuously on B in operator norm near irreducible connections.

Let $\mathfrak{h}$ be primary data and g a split addition, and put $h = \mathfrak{h} + g = h_0 + h_1$, where $h_i = \mathfrak{h}_i + g_i$ depends only on the restriction to C_i° . Put

$$V(A) = *F_A + \nabla h(A), \quad K_A = DV_A = *d_A + \text{Hess}_A h.$$

For a cylinder function h , the gradient $\nabla h(B)$ lies in $\ker d_B^*$ and is supported in the images of the immersions of h ; the map $B \mapsto \nabla h(B)$ is smooth, with $\|\nabla h(B) - \nabla h(B')\|_{L^p} \leq C\|B - B'\|_{L^p}$, and $\text{Hess}_B h$ is bounded on L_k^2 for $0 \leq k \leq l$ and depends smoothly on B [DFL21, Proposition 6.1]; it is symmetric [DFL21, Lemma 2.13].

The images of the immersions γ_α of the separated family and of the immersions used by g are compact subsets of $\text{int } C_0^\circ \sqcup \text{int } C_1^\circ$. Fix $\varepsilon \in (0, 1)$ such that $U = (-1 - \varepsilon, 1 + \varepsilon) \times \Sigma$ is disjoint from all of them, and put

$$V_0 = (1, 1 + \varepsilon) \times \Sigma \subset \text{int } C_0^\circ, \quad V_1 = (-1 - \varepsilon, -1) \times \Sigma \subset \text{int } C_1^\circ.$$

On U we have $V(A) = *F_A$. As in Section 3.4, $X_i \subset \mathcal{B}(C_i^\circ)$ is the set of restrictions of the critical points of CS_h .

Lemma C.1 (locality). *For $x \in C_i^\circ$ the value $V(A)(x)$ depends only on $A|_{C_i^\circ}$, and $(K_A b)(x)$ only on $b|_{C_i^\circ}$; write $V^{(i)}$ and $K^{(i)}$ for the resulting maps on C_i° . For every $\zeta \in \Omega^0(C_i^\circ; \text{ad})$, with no boundary condition,*

$$K_A^{(i)}(d_A \zeta) = [V^{(i)}(A), \zeta].$$

The same identity holds on $M^\#$ with V and K .

Proof. The gradient of h_{1-i} vanishes on C_i° , and $*F_A$ is local. The function h_i is invariant under all determinant-one gauge transformations of C_i° , because its loops lie in the interior and traces of holonomy are invariant under conjugation; so $V^{(i)}(u \cdot A) = uV^{(i)}(A)u^{-1}$ for such u . Differentiating at $u = 1$ in the direction ζ , with $u \cdot A = A - (d_A u)u^{-1}$, gives the identity. The argument on $M^\#$ is the same. $\square$

C.2. Trajectories and cokernels. Trajectories of CS_h are written in temporal gauge as solutions $\gamma = (A_s)_{s \in \mathbb{R}}$ of $\partial_s A_s = -V(A_s)$ between determinant-one lifts of critical points, in any path class. Reversing s gives the opposite convention and changes the signs of all eigenvalues below; the argument uses only that two of them have opposite signs. The *extended Hessian*

$$L_A(b, c) = (K_A b - d_A c, -d_A^* b)$$

acts on $H = L^2(M^\#; (\Lambda^1 \oplus \Lambda^0) \otimes \text{ad})$ with domain L_1^2 (compare [Don02, (2.23)]) and is symmetric. Let $\tilde{a}$ be a lift of a nondegenerate critical point; it is irreducible (Lemma C.5). Then $L_{\tilde{a}}$ is self-adjoint with compact resolvent, and $\ker L_{\tilde{a}} = 0$: if $K_{\tilde{a}} b = d_{\tilde{a}} c$ and $d_{\tilde{a}}^* b = 0$, then $\|d_{\tilde{a}} c\|^2 = \langle b, K_{\tilde{a}} d_{\tilde{a}} c \rangle = \langle b, [V(\tilde{a}), c] \rangle = 0$ by Lemma C.1, so $c = 0$ by irreducibility and $b = 0$ by nondegeneracy. The linearization at a trajectory γ , with the Coulomb condition, is $D_\gamma = \partial_s + L_{A_s}$ (compare [Don02, Section 3.1]), and its cokernel consists of the L^2 solutions of $\partial_s y = L_{A_s} y$, which are smooth and decay exponentially at both ends.

Lemma C.2. *Let $\gamma = (A_s)$ be a trajectory of CS_h .*

- (i) $x(s) = (\partial_s A_s, 0)$ satisfies $\partial_s x = -L_{A_s} x$.
- (ii) If $y = (\phi, c)$ is in the cokernel of D_γ , then $c \equiv 0$; hence $d_{A_s}^* \phi_s = 0$ and $\partial_s \phi = K_{A_s} \phi$.
- (iii) Two solutions of the trajectory equation of CS_h on an interval that agree at one time agree on the whole interval; the proof is local in time. If γ is nonconstant, then $\partial_s A_s \neq 0$ for every s , and if $y \neq 0$ in (ii), then $\phi_s \neq 0$ for every s .

Proof. (i) Differentiating the flow gives $\partial_s \dot{A} = -K_A \dot{A}$, where $\dot{A} = \partial_s A$, and $d_A^* \dot{A} = -d_A^* V(A) = 0$ by the Bianchi identity and the gauge invariance of h .

(ii) The equation reads $\partial_s \phi = K_A \phi - d_A c$, $\partial_s c = -d_A^* \phi$. Define $\{a, b\}$ by $\langle \zeta, \{a, b\} \rangle = \langle [a, \zeta], b \rangle$ for all sections ζ . Then $\partial_s (d_{A_s}^* b) = \{\dot{A}_s, b\}$, and taking adjoints in Lemma C.1 on $M^\#$ gives $d_A^* K_A b = \{V(A), b\}$. Hence

$$\partial_s (d_A^* \phi) = \{\dot{A} + V(A), \phi\} - d_A^* d_A c = -d_A^* d_A c,$$

so $\partial_s^2 c = d_A^* d_A c$ and $\frac{d^2}{ds^2} \|c\|^2 = 2\|\partial_s c\|^2 + 2\|d_A c\|^2 \geq 0$. The function $\|c\|^2$ is convex and tends to zero at both ends, so $c \equiv 0$.

(iii) We use the theorem of Agmon and Nirenberg in the form [SW08, Theorem C.1], in $L^2(M^\#)$, as in the proof of [SW08, Proposition 8.6]; since $M^\#$ is closed there are no boundary terms. For two trajectories A, B put $x = A - B$. Then

$$\partial_s x + *d_B x = -\frac{1}{2} * [x \wedge x] - (\nabla h(A) - \nabla h(B)),$$

whose L^2 norm is at most $c_1(s)\|x\|$ with c_1 locally bounded, by [DFL21, Proposition 6.1(ii)]. The operators $*d_{B(s)}$ are symmetric, and $\frac{d}{ds}\langle x, *d_Bx \rangle - 2\langle \partial_s x, *d_Bx \rangle = \langle x, *[\partial_s B \wedge x] \rangle \leq c_3(s)\|x\|^2$. So [SW08, Theorem C.1(i)] gives $x \equiv 0$ forward in time from a time at which x vanishes, and backward after reversing time, which replaces the operators by their negatives. The linear equations of (i) and (ii) are treated in the same way, with the symmetric operators $\pm L_{A_s}$, for which $c_1 = 0$ and $c_3 = \|\partial_s L_{A_s}\|$ is locally bounded by [DFL21, Proposition 6.1(iii)]. $\square$

C.3. Asymptotics at $-\infty$.

Lemma C.3. *Let H be a real Hilbert space, L a self-adjoint operator on H with compact resolvent and $\ker L = 0$, and $B: (-\infty, 0] \rightarrow \mathcal{L}(H)$ continuous with $\beta(s) := \|B(s)\| \leq C_0 e^{\delta s}$ for some $\delta > 0$. Let $x \in C^1((-\infty, 0]; \text{dom } L)$, with $\text{dom } L$ carrying the graph norm, solve $x' = -(L + B)x$, with $x(s) \neq 0$ for all s , $x(s) \rightarrow 0$ as $s \rightarrow -\infty$, and*

$$(C.1) \quad \|Lx(s)\| \leq C_1 \sup_{|\sigma-s| \leq 1} \|x(\sigma)\| \quad (s \leq -1).$$

Then there are $\lambda > 0$ and $v \neq 0$ with $Lv = -\lambda v$ such that $e^{-\lambda s}x(s) \rightarrow v$ as $s \rightarrow -\infty$. If instead $x' = (L + B)x$, the same holds with $Lv = \lambda v$.

Proof. Put $\hat{x} = x/\|x\|$ and $N = \langle L\hat{x}, \hat{x} \rangle$.

(a) We have $(\log \|x\|)' = -N - \langle B\hat{x}, \hat{x} \rangle$ and $\hat{x}' = -(L - N)\hat{x} - B\hat{x} + \langle B\hat{x}, \hat{x} \rangle \hat{x}$. Since $\langle (L - N)\hat{x}, \hat{x} \rangle = 0$,

$$N' = 2\langle (L - N)\hat{x}, \hat{x}' \rangle = -2\|(L - N)\hat{x}\|^2 - 2\langle (L - N)\hat{x}, B\hat{x} \rangle \leq -\|(L - N)\hat{x}\|^2 + \beta^2.$$

(b) Hence $N - \int_{-\infty}^s \beta^2$ is nonincreasing, N is bounded below on $(-\infty, 0]$, and $N_{-\infty} = \lim_{s \rightarrow -\infty} N(s) \in (-\infty, \infty]$ exists. If $N_{-\infty} > 0$, then by (a) $(\log \|x\|)' \leq -c < 0$ near $-\infty$, and $\|x(s)\| \rightarrow \infty$ as $s \rightarrow -\infty$, which is excluded; so $N_{-\infty} \leq 0$. By (a), $\int_{-\infty}^0 \|(L - N)\hat{x}\|^2 < \infty$, so $\|(L - N_{-\infty})\hat{x}(s_k)\| \rightarrow 0$ along a sequence $s_k \rightarrow -\infty$. Hence $N_{-\infty} \in \text{spec } L$, and $N_{-\infty} = -\lambda$ with $\lambda > 0$ since $0 \notin \text{spec } L$.

(c) Let P be the orthogonal projection onto the finite-dimensional space $\ker(L + \lambda)$, $Q = 1 - P$, and $Q_{\pm}$ the spectral projections of L onto the parts of the spectrum above and below $-\lambda$; let $g > 0$ be the distance from $-\lambda$ to the rest of the spectrum. Choose s_1 with $|N + \lambda| \leq g/4$ on $(-\infty, s_1]$. For $y = Q_+\hat{x}$ and $z = Q_-\hat{x}$, which have norm at most one, (a) gives on $(-\infty, s_1]$

$$(\|y\|^2)' \leq -\frac{3}{2}g\|y\|^2 + 4\beta\|y\| \leq -g\|y\|^2 + \frac{8}{g}\beta^2, \quad (\|z\|^2)' \geq \frac{3}{2}g\|z\|^2 - 4\beta\|z\| \geq g\|z\|^2 - \frac{8}{g}\beta^2.$$

Integrating the first inequality from $-\infty$, and the second backward from s_1 , gives $\|Q\hat{x}(s)\| \leq Ce^{\delta's}$ for any fixed $\delta' < \min(\delta, g/2)$.

(d) The function N is bounded on $(-\infty, 0]$, so $(\log \|x\|)'$ is bounded, and (C.1) bounds $\|L\hat{x}\|$. Hence $|N + \lambda| = |\langle (L + \lambda)Q\hat{x}, Q\hat{x} \rangle| \leq Ce^{\delta's}$ is integrable, and by (a) $e^{-\lambda s}\|x(s)\|$ converges to some $c \in (0, \infty)$.

(e) $(e^{-\lambda s}Px)' = -e^{-\lambda s}PBx$ has norm at most $C\beta$, so $e^{-\lambda s}Px$ converges to some v , while $e^{-\lambda s}Qx = e^{-\lambda s}\|x\|Q\hat{x} \rightarrow 0$. So $e^{-\lambda s}x \rightarrow v$, with $\|v\| = c > 0$ and $Lv = -\lambda v$. The second statement follows by applying the first to $-L$ and $-B$. $\square$

Corollary C.4. *Let $\gamma = (A_s)$ be a nonconstant trajectory of CS_h , and $y = (\phi, 0) \neq 0$ in the cokernel of D_γ . Let $\tilde{a}$ be the limit of A_s as $s \rightarrow -\infty$. There are $\lambda, \mu > 0$ and nonzero $v, w \in \ker d_{\tilde{a}}^*$ with $K_{\tilde{a}}v = -\lambda v$ and $K_{\tilde{a}}w = \mu w$ such that, in $L^2(M^\#)$,*

$$(C.2) \quad \partial_s A_s = e^{\lambda s}(v + o(1)), \quad \phi_s = e^{\mu s}(w + o(1)) \quad (s \rightarrow -\infty).$$

Proof. In a temporal gauge, A_s converges to a lift $\tilde{a}$ of a critical point exponentially fast in every C^k norm [Don02, Theorem 4.2 and Proposition 4.3], whose proofs apply to the holonomy-perturbed equation with nondegenerate critical points [Don02, Section 5.5]. Apply Lemma C.3 with $H = L^2(M^\#; (\Lambda^1 \oplus \Lambda^0) \otimes \text{ad})$, $L = L_{\tilde{a}}$ and $B(s) = L_{A_s} - L_{\tilde{a}}$ to $x = (\partial_s A_s, 0)$ and to y (Lemma C.2(i),(ii)). The operator $B(s)$ consists of brackets with $A_s - \tilde{a}$ and of $\text{Hess}_{A_s} h - \text{Hess}_{\tilde{a}} h$, so $\|B(s)\| \leq C\|A_s - \tilde{a}\|_{L^2_i} \leq C_0 e^{\delta s}$ by [DFL21, Proposition 6.1(iii)]. The solutions vanish nowhere by Lemma C.2(iii), and (C.1) is the interior elliptic estimate for $\partial_s + L_{A_s}$ on the slabs $[s-1, s+1] \times M^\#$, whose coefficients are bounded uniformly in s , since $\text{Hess}_{A_s} h$ is bounded on L^2_k for $k \leq 2 \leq l$; the estimate at a fixed time uses the slab estimate one derivative higher, for which $s \mapsto \text{Hess}_{A_s} h$ is C^1 into the bounded operators on L^2_1 [DFL21, Proposition 6.1(iii)]. The limits have vanishing Λ^0 -components, because x and y have, so they are eigenvectors of $K_{\tilde{a}}$ in $\ker d_{\tilde{a}}^*$. $\square$

The two eigenvalues in (C.2), $-\lambda < 0 < \mu$, have opposite signs; this is what replaces the L^2 -orthogonality of $\partial_s \gamma$ and ψ used in [DFL21, p. 79].

C.4. Eigenvectors on one piece.

Lemma C.5. *The restriction to C_i° of a lift $\tilde{a}$ of a critical point of CS_h is irreducible. If $A_s \rightarrow \tilde{a}$ as $s \rightarrow -\infty$, then $A_s|_{C_i^\circ}$ is irreducible for all s near $-\infty$.*

Proof. On $V_i \cap ((-2, 2) \times T^2)$, an interval times T^2 , the connection $\tilde{a}$ satisfies $*F_{\tilde{a}} = V(\tilde{a}) = 0$. The bundle is odd on T^2 , so the holonomy of this flat connection lifts to the quaternion group in $\text{SU}(2)$, which fixes no nonzero vector of $\mathfrak{su}(2)$. A parallel section of ad over the connected C_i° therefore vanishes on this open set, hence everywhere. Irreducibility is an open condition, and $A_s \rightarrow \tilde{a}$ in C^∞ . $\square$

Lemma C.6. *Let a be a nondegenerate critical point of CS_h , $i \in \{0, 1\}$, and $v_1, v_2 \in \ker d_a^*$ eigenvectors of K_a with eigenvalues $\nu_1 \neq \nu_2$.*

- (i) *If $v_j|_{C_i^\circ}$ lies in the closure of $d_a L_1^2(C_i^\circ)$, then $v_j|_{C_i^\circ} = 0$.*
- (ii) *If $v_1|_{C_i^\circ} \neq 0 \neq v_2|_{C_i^\circ}$, then $\pi_a^i v_1$ and $\pi_a^i v_2$ are linearly independent.*

Proof. Since $V(a) = 0$, Lemma C.1 gives $K_a^{(i)}(d_a \zeta) = 0$ for smooth ζ , and hence, by continuity of $K_a^{(i)}$ from L^2 to distributions on $\text{int } C_i^\circ$, for every b in the closure of $d_a L_1^2(C_i^\circ)$. On C_i° the eigenvector equation reads $K_a^{(i)} v_j = \nu_j v_j$, and $\nu_j \neq 0$ by nondegeneracy; this gives (i). For (ii) suppose $\pi_a^i v_2 = c \pi_a^i v_1$, so that $v_2 = c v_1 + b$ on C_i° with b in the closure of $d_a L_1^2$. If $c = 0$ this contradicts (i). Otherwise applying $K_a^{(i)}$ gives $\nu_2(c v_1 + b) = c \nu_1 v_1$, so $c(\nu_2 - \nu_1)v_1 = -\nu_2 b$ on C_i° , and $v_1|_{C_i^\circ}$ lies in the closure of $d_a L_1^2$, which contradicts (i). $\square$

C.5. Non-trapping. The following Carleman estimate is proved by the method of [Boo00, Lemmas 5 and 6], after a change of variables that makes the inner product on the slices independent of the normal variable; we give the proof in full because the slices below are tori and the operator contains a nonlocal term acting slice by slice.

Lemma C.7. *Let N_1 be a compact region in a Riemannian three-manifold and $t \in C^\infty(N_1)$ with $dt \neq 0$ and $N_1 = t^{-1}([\tau_0, \tau_1])$, whose level sets S_t are closed surfaces. Let $E \rightarrow N_1$ be a Hermitian bundle with a Clifford multiplication $\mathbf{c}: T^*N_1 \rightarrow \text{End}(E)$, $\mathbf{c}(\xi)^* = -\mathbf{c}(\xi)$, $\mathbf{c}(\xi)^2 = -|\xi|^2$, let ∇ be a Hermitian connection on E and Θ a smooth endomorphism, and put $D = \sum_j \mathbf{c}(e^j) \nabla_{e_j} + \Theta$ for local orthonormal frames. Let K act on $L^2(N_1; E)$ slice by slice: there are operators K_t on $L^2(S_t; E)$, with the area measure, such that $\|K_t\| \leq b$ and*

$(Ku)|_{S_t} = K_t(u|_{S_t})$. Then there are $T_0, C, R_0 > 0$, depending on N_1, D and b , such that for every $[t_0, t_1] \subset [\tau_0, \tau_1]$ with $T := t_1 - t_0 \leq T_0$:

(i) for every $R \geq R_0$ and every $v \in L_1^2$ with compact support in $t^{-1}((t_0, t_1))$,

$$(C.3) \quad R \int e^{R(t_1-t)^2} |v|^2 \leq C \int e^{R(t_1-t)^2} |(D+K)v|^2;$$

(ii) if $u \in L_1^2(t^{-1}([t_0, t_1]); E)$ satisfies $(D+K)u = 0$ and vanishes on $\{t < t_0 + \eta\}$ for some $0 < \eta < T$, then $u = 0$ on $\{t \leq t_0 + T/2\}$.

Proof. It suffices to prove (C.3) for smooth v , since both sides are continuous in L_1^2 .

(a) *Coordinates.* Let $X = \nabla t / |\nabla t|^2$. Its flow gives a diffeomorphism $\Phi: [\tau_0, \tau_1] \times S \rightarrow N_1$, $S = S_{\tau_0}$, with $t(\Phi(t, y)) = t$. Since X is orthogonal to the level sets, $\Phi^*g = |\nabla t|^{-2}dt^2 + h_t$ with no cross terms. Write $\Phi^*d\text{vol} = \kappa dt d\text{vol}_S$ with $\kappa > 0$, for a fixed area form $d\text{vol}_S$ on S .

(b) *A unitary change of variables.* Let P_t be parallel transport for ∇ along the flow lines, and put $(\mathcal{U}u)(t, y) = \kappa^{1/2} P_t^{-1} u(\Phi(t, y))$. Then $\mathcal{U}$ is a unitary map from $L^2(N_1; E)$ to $L^2([\tau_0, \tau_1] \times S, dt d\text{vol}_S; E|_S)$, and the inner product on each slice is now $L^2(S, d\text{vol}_S)$, independent of t .

(c) *Normal form.* Let $e_0 = \nabla t / |\nabla t|$, and let e_1, e_2 be tangent to the level sets. Then $D = |\nabla t| \mathbf{c}(e^0) \nabla_X + D_S + \Theta$ with $D_S = \sum_{k=1,2} \mathbf{c}(e^k) \nabla_{e_k}$, and $\mathcal{U} \nabla_X \mathcal{U}^{-1} = \partial_t - \frac{1}{2} \partial_t \log \kappa$. Hence

$$\mathcal{U} D \mathcal{U}^{-1} = \sigma(\partial_t + \mathcal{B}_t), \quad \sigma = |\nabla t| P_t^{-1} \mathbf{c}(e^0) P_t, \quad \mathcal{B}_t = -\frac{1}{2} \partial_t \log \kappa + \sigma^{-1} \mathcal{U}(D_S + \Theta) \mathcal{U}^{-1},$$

where σ is invertible, $\sigma^{-1} = -|\nabla t|^{-1} P_t^{-1} \mathbf{c}(e^0) P_t$, and $\mathcal{B}_t$ is a first-order operator on S . For $\xi \in T_y^* S$ let η be the covector at $\Phi(t, y)$ with $\eta(X) = 0$ whose pullback to S is ξ ; since X is normal to the level set, η is orthogonal to e^0 . The principal symbol of $\mathcal{B}_t$ at ξ is $-i|\nabla t|^{-1} P_t^{-1} \mathbf{c}(e^0) \mathbf{c}(\eta) P_t$. It is self-adjoint, because $\mathbf{c}(\eta) \mathbf{c}(e^0) = -\mathbf{c}(e^0) \mathbf{c}(\eta) - 2\langle e^0, \eta \rangle = -\mathbf{c}(e^0) \mathbf{c}(\eta)$, and invertible for $\xi \neq 0$, because $(\mathbf{c}(e^0) \mathbf{c}(\eta))^2 = -|\eta|^2$. Put $B_t = \frac{1}{2}(\mathcal{B}_t + \mathcal{B}_t^*)$ and $C_t = \frac{1}{2}(\mathcal{B}_t - \mathcal{B}_t^*)$, adjoints in $L^2(S, d\text{vol}_S)$. Then B_t is symmetric and elliptic of first order, C_t is a skew-adjoint endomorphism, since the principal symbols cancel, and both depend smoothly on t . Fix constants c, c_1, s_σ , uniform over $[\tau_0, \tau_1]$, with $\|f\|_{H^1(S)} \leq c(\|f\| + \|B_t f\|)$, $|\langle f, (-\partial_t B_t + [B_t, C_t])f \rangle| \leq c_1 \|f\| \|f\|_{H^1(S)}$ (the operator is of first order) and $\|\sigma^{-1}\| \leq s_\sigma$.

(d) *The estimate for $\partial_t + B_t + C_t$.* Let w be smooth on $[t_0, t_1] \times S$ and vanish near t_0 and t_1 . Put $\mathcal{S}_t = B_t + R(t_1 - t)$, which is symmetric, and $J = \int_{t_0}^{t_1} \|\partial_t w + C_t w + \mathcal{S}_t w\|^2 dt$. Since the slice inner product does not depend on t ,

$$2 \operatorname{Re} \int \langle \partial_t w, \mathcal{S}_t w \rangle dt = - \int \langle w, (\partial_t \mathcal{S}_t) w \rangle dt = R \int \|w\|^2 dt - \int \langle w, (\partial_t B_t) w \rangle dt,$$

and since C_t is skew, $2 \operatorname{Re} \langle C_t w, \mathcal{S}_t w \rangle = \langle w, [B_t, C_t] w \rangle$. Expanding the square,

$$J \geq \int \|\mathcal{S}_t w\|^2 + R \int \|w\|^2 + J_3, \quad J_3 = \int \langle w, (-\partial_t B_t + [B_t, C_t]) w \rangle.$$

With $\|B_t w\| \leq \|\mathcal{S}_t w\| + RT\|w\|$,

$$|J_3| \leq cc_1 \int \|w\| (\|\mathcal{S}_t w\| + (1 + RT)\|w\|) \leq \frac{1}{2} \int \|\mathcal{S}_t w\|^2 + (cc_1(1 + RT) + \frac{1}{2}(cc_1)^2) \int \|w\|^2.$$

For $T \leq T_0 := 1/(4cc_1)$ and $R \geq 4cc_1 + 2(cc_1)^2$ this gives $J \geq \frac{R}{2} \int \|w\|^2$. For smooth v supported in $t^{-1}((t_0, t_1))$ put $w = e^{R(t_1-t)^2/2} \mathcal{U}v$. Then $e^{R(t_1-t)^2/2} (\partial_t + B_t + C_t) \mathcal{U}v =$

$(\partial_t + B_t + C_t + R(t_1 - t))w$ and $(\partial_t + B_t + C_t)Uv = \sigma^{-1}UDv$. Since U is unitary and the weight depends only on t ,

$$R \int e^{R(t_1-t)^2} |v|^2 \leq 2s_\sigma^2 \int e^{R(t_1-t)^2} |Dv|^2.$$

(e) *Absorption of K .* By the coarea formula $d\text{vol} = |\nabla t|^{-1} dA_t dt$, and the weight is constant on each slice, so $\int e^{R(t_1-t)^2} |Kv|^2 \leq b'^2 \int e^{R(t_1-t)^2} |v|^2$ with $b'^2 = b^2 \sup |\nabla t| / \inf |\nabla t|$. With $C_D = 2s_\sigma^2$,

$$R \int e^{R(t_1-t)^2} |v|^2 \leq 2C_D \int e^{R(t_1-t)^2} |(D+K)v|^2 + 2C_D b'^2 \int e^{R(t_1-t)^2} |v|^2,$$

and for $R \geq R_0 := \max(4cc_1 + 2(cc_1)^2, 4C_D b'^2)$ this is (C.3) with $C = 4C_D$.

(f) *Continuation.* Let $\varphi(t) = 1$ for $t \leq t_0 + 0.8T$ and $\varphi(t) = 0$ for $t \geq t_0 + 0.9T$, and put $v = \varphi u$. Then $v \in L_1^2$ has compact support in $t^{-1}((t_0, t_1))$, because $u = 0$ on $\{t < t_0 + \eta\}$. Since K commutes with multiplication by functions of t , $(D+K)v = \varphi'(t) \mathbf{c}(dt)u$, which is supported where $(t_1 - t)^2 \leq T^2/25$, while $(t_1 - t)^2 \geq T^2/4$ on $\{t \leq t_0 + T/2\}$, where $v = u$. By (C.3),

$$\int_{\{t \leq t_0 + T/2\}} |u|^2 \leq \frac{C}{R} e^{-\frac{21}{100}RT^2} \int |\varphi' \mathbf{c}(dt)u|^2,$$

which tends to zero as $R \rightarrow \infty$. $\square$

Proposition C.8. *Let $\mathfrak{h}$ be primary data in the span of a separated family with embeddings γ_α , and choose $r < 1$ such that every two-form μ in the averaged trace functions of $\mathfrak{h}$ is supported in the disk D_r of radius r . Put $\Omega' = M^\# \setminus \bigcup_\alpha \gamma_\alpha(S^1 \times \bar{D}_r)$. Let a be a critical point of $\text{CS}_\mathfrak{h}$ and $v \in \ker d_a^*$ an eigenvector of K_a . If v vanishes on a nonempty open subset of Ω' , then $v = 0$.*

Proof. A critical point of $\text{CS}_\mathfrak{h}$ is gauge equivalent to a smooth connection, by elliptic bootstrapping with $\nabla \mathfrak{h}: L_k^2 \rightarrow L_k^2$ for every k [DFL21, Proposition 6.1]; we work in such a gauge. Let ν be the eigenvalue. *Outside the solid tori.* On Ω' the Hessian of $\mathfrak{h}$ vanishes, so a is flat there and $*d_a v = \nu v$, $d_a^* v = 0$. Applying $*d_a$ gives $d_a^* d_a v = \nu^2 v$, so $\Delta_a v = \nu^2 v$ on Ω' , a second-order system whose principal part is the scalar Laplacian in every component. The set Ω' is connected: it is the complement in the connected $M^\#$ of finitely many disjoint closed solid tori, each of which has a connected collar $\gamma_\alpha(S^1 \times \{r < |z| < 1\})$ in Ω' , so a decomposition of Ω' into two open sets would give one of $M^\#$. By elliptic regularity v is smooth on Ω' . By the unique continuation theorem of [Aro57], which applies to such systems [Aro57, Remark 3] (see also [Boo00, Remark 8(a)]), applied on an exhaustion of Ω' by connected relatively compact open subsets, $v = 0$ on Ω' .

Inside a solid torus. Fix α and use the coordinates $(\theta, z) \in S^1 \times D^2$ of γ_α . Since the images are disjoint and $\mathfrak{h}$ is a linear combination of averaged trace functions, on $\gamma_\alpha(S^1 \times D^2)$ the operator $\text{Hess}_a \mathfrak{h}$ is a linear combination of Hessians of averaged trace functions of γ_α . The gradient of such a function at $\gamma_\alpha(\theta, z)$ is determined by $\mu(z)$ and by the traceless part of the holonomy around the loop $\gamma_\alpha(S^1 \times \{z\})$ based at that point [Don02, Lemma 5.12]; its derivative in a direction b is therefore an integral along that loop of b , transported by a , with bounded kernel. So the operator $\tilde{B}(c, b) = (0, \text{Hess}_a \mathfrak{h}(b))$ maps sections over a union of loops $\gamma_\alpha(S^1 \times \{z\})$ to sections over the same union, with a uniform bound on L^2 of such unions. With $\mathcal{D}_a(c, b) = (d_a^* b, d_a c + *d_a b)$ on $E = (\Lambda^0 \oplus \Lambda^1) \otimes \text{ad} \otimes \mathbb{C}$, the section $u = (0, v)$ satisfies

$$(\mathcal{D}_a - \nu)u + \tilde{B}u = 0.$$

The operator $\mathcal{D}_a - \nu$ has the form of Lemma C.7, with $\mathbf{c}(\xi)(c, b) = (-\iota_{\xi^\sharp} b, \xi c + *(\xi \wedge b))$, which is skew and satisfies $\mathbf{c}(\xi)^2 = -|\xi|^2$, with ∇ the Levi-Civita connection coupled to a , and $\Theta = -\nu$.

Let ω' be the set of $z \in D^\circ$ such that u vanishes on $\gamma_\alpha(S^1 \times W)$ for a neighbourhood W of z . It is open, and it contains $\{r < |z| < 1\}$ by the first step. Suppose $\omega' \neq D^\circ$. Its complement in D° is compact, being closed in D° and contained in $\bar{D}_r$, and D° is connected; so there is $z_1 \in \omega'$ whose distance ρ to $D^\circ \setminus \omega'$ is less than $(1-r)/2$, hence less than $\text{dist}(z_1, \partial D)$. Let $z_0 \in D^\circ \setminus \omega'$ attain this distance. Then $D(z_1, \rho) \subset \omega'$. Fix $0 < \eta_1 < \rho$ and $T_1 > 0$ with $\rho + T_1 < \text{dist}(z_1, \partial D)$, put $t = |z - z_1| - \rho$, and let $N_1 = \gamma_\alpha(S^1 \times \{-\eta_1 \leq t \leq T_1\})$. On N_1 we have $dt \neq 0$, and the level sets $S_t = \gamma_\alpha(S^1 \times \{|z - z_1| = \rho + t\})$ are closed tori, each a union of loops $\gamma_\alpha(S^1 \times \{z\})$, so $K = \tilde{B}$ acts slice by slice with a uniform bound. Let T_0 be given by Lemma C.7 for N_1 , and choose $0 < \eta < T'$ with $T' \leq T_1$, $\eta \leq \eta_1$ and $T' + \eta \leq T_0$. Lemma C.7(ii) with $[t_0, t_1] = [-\eta, T']$, of length $T' + \eta$, where $u = 0$ for $t < 0 = t_0 + \eta$, gives $u = 0$ on $\{t \leq (T' - \eta)/2\}$. Together with $u = 0$ over $D(z_1, \rho)$, this is a neighbourhood of $\gamma_\alpha(S^1 \times \{z_0\})$, so $z_0 \in \omega'$, a contradiction. Hence v vanishes on every solid torus, and $v = 0$. $\square$

Corollary C.9. *Let $\mathfrak{h}$ and g be as in Lemma 3.5, with ε as in Section C.1. For every critical point a of $\text{CS}_{\mathfrak{h}+g}$, every eigenvector $v \neq 0$ of K_a in $\ker d_a^*$ and each i , we have $v|_{V_i} \neq 0$, and hence $\pi_a^i v \neq 0$.*

Proof. The critical points of $\text{CS}_{\mathfrak{h}+g}$ are those of $\text{CS}_{\mathfrak{h}}$, and g vanishes on a neighbourhood of each of them, so K_a is the Hessian of $\text{CS}_{\mathfrak{h}}$ at a . The open set V_i lies in Ω' by the choice of ε , so Proposition C.8 gives $v|_{V_i} \neq 0$, and Lemma C.6(i) gives $\pi_a^i v \neq 0$. $\square$

C.6. Restricted paths. For a trajectory $\gamma = (A_s)$ of CS_h put $\Gamma_i(s) = [A_s|_{C_i^\circ}] \in \mathcal{B}(C_i^\circ)$, extended continuously to $\pm\infty$ by the restrictions of the limits.

Lemma C.10. *Suppose that CS_h and $\text{CS}_{\mathfrak{h}}$ have the same critical points, and let γ be a trajectory of CS_h and $i \in \{0, 1\}$. Either Γ_i is constant, or it is injective on $\mathbb{R}$ and $\Gamma_i(\mathbb{R}) \cap X_i = \emptyset$.*

Proof. Take $i = 0$; the case $i = 1$ is symmetric. Put $O_1 = \mathbb{R} \times \text{int } C_0^\circ$ and $O_2 = \mathbb{R} \times (U \cup C_1^\circ)$. Then $O_1 \cup O_2 = \mathbb{R} \times M^\#$ and $O_1 \cap O_2 = \mathbb{R} \times V_0$.

Returns. Suppose $\Gamma_0(s_1) = \Gamma_0(s_0)$ with $\tau = s_1 - s_0 \neq 0$. Choose a gauge transformation k of C_0° with $k \cdot A_{s_1} = A_{s_0}$ on C_0° ; it is smooth. It extends to a determinant-one gauge transformation of $M^\#$, since the obstructions lie in $H^j(M^\#, C_0^\circ; \pi_{j-1}\text{SU}(2)) = 0$ for $j \leq 3$. Put $\tilde{\gamma}(s) = k \cdot \gamma(s + \tau)$, a trajectory of CS_h in temporal gauge with $\tilde{\gamma}(s_0) = \gamma(s_0)$ on C_0° . On the open set V_0 both $\tilde{\gamma}$ and γ solve the unperturbed equation $\partial_s A + *F_A = 0$, so they agree on $\mathbb{R} \times V_0$ by [SW08, Lemma 8.7(i)]. Define Ξ to be $\tilde{\gamma}$ on O_1 and γ on O_2 . It is smooth, and it is a trajectory of CS_h : on O_1 the equation involves only the restriction to C_0° (Lemma C.1), where $\Xi = \tilde{\gamma}$; on O_2 it involves the unperturbed region U and the restriction to C_1° , where $\Xi = \gamma$. Since $\Xi(s_0) = \gamma(s_0)$ on $M^\#$, Lemma C.2(iii) gives $\Xi = \gamma$, so $\tilde{\gamma} = \gamma$ on $\mathbb{R} \times C_0^\circ$. Hence Γ_0 is periodic with period τ ; it converges as $s \rightarrow -\infty$, so it is constant.

Critical passages. Suppose $\Gamma_0(s_1)$ is the restriction of a critical point, with lift $\tilde{c}$. The same argument, with $\tilde{\gamma}$ the constant trajectory at $k \cdot \tilde{c}$ for a gauge transformation k with $k \cdot \tilde{c} = A_{s_1}$ on C_0° , applies: $\tilde{c}$ is flat on V_0 and $V(\tilde{c}) = 0$ on $M^\#$. It gives $\Gamma_0 \equiv \Gamma_0(s_1)$. $\square$

C.7. Holonomy detection and localization.

Lemma C.11. *Let B be a smooth irreducible connection on C_i° . All trace functions below are averaged trace functions of embeddings into $\text{int } C_i^\circ$.*

- (i) *If a smooth one-form b on C_i° satisfies $D\tau_B(b) = 0$ for every trace function τ , then $b \in d_B\Omega^0(C_i^\circ)$.*
- (ii) *If S is a finite-dimensional space of smooth one-forms on C_i° with $S \cap d_B\Omega^0(C_i^\circ) = 0$, there are finitely many trace functions whose derivatives at B are jointly injective on S .*
- (iii) *If $\mathcal{K} \subset \mathcal{B}(C_i^\circ)$ is compact and $[B] \notin \mathcal{K}$, finitely many trace functions separate $[B]$ from $\mathcal{K}$.*

Proof. (i) Letting μ concentrate at a point shows that the derivative in the direction b of the trace of the holonomy around every embedded loop in $\text{int } C_i^\circ$ vanishes, and by approximation [Don02, p. 141] the same holds for every piecewise smooth loop. Fix $p \in \text{int } C_i^\circ$. Since B is irreducible, there are loops $\delta_1, \dots, \delta_m$ based at p whose holonomies do not commute. Fix a unitary frame of the fibre at p , and write the holonomies of $\delta_1, \dots, \delta_m$ and of a further loop δ based at p as $\lambda_j g_j$ and λg , with $g_j, g \in \text{SU}(2)$ and $\lambda_j, \lambda \in U(1)$ independent of B , as in Section 3.2. Identify $\text{SU}(2)$ with the unit quaternions, let ρ be the homomorphism from the free group on $m+1$ letters to $\text{SU}(2)$ sending the letters to $g_1, \dots, g_m, g$, and extend it to an algebra map $\rho: \mathbb{R}[F_{m+1}] \rightarrow \mathbb{H}$. For a word W the holonomy around the corresponding loop is $\lambda_W \rho(W)$ with $\lambda_W \in U(1)$ independent of B , so the derivative of $\text{tr } \rho(W)$ in the direction b vanishes. Let $\dot{\rho}(W) \in \mathbb{H}$ be the derivative of $\rho(W)$ in the direction b , extended linearly. Then $\dot{\rho}(UV) = \dot{\rho}(U)\rho(V) + \rho(U)\dot{\rho}(V)$ and $\text{tr } \dot{\rho} = 0$ on $\mathbb{R}[F_{m+1}]$. The image of ρ is the real span of a noncommutative subgroup of the unit quaternions; the proper subalgebras of $\mathbb{H}$ are $\mathbb{R}$ and copies of $\mathbb{C}$, which are commutative, so the image is $\mathbb{H}$. If $\rho(x) = 0$, then $\text{tr}(\dot{\rho}(x)\rho(y)) = \text{tr } \dot{\rho}(xy) - \text{tr}(\rho(x)\dot{\rho}(y)) = 0$ for all y , and since the trace form of $\mathbb{H}$ is nondegenerate, $\dot{\rho}(x) = 0$. Hence $\dot{\rho} = D \circ \rho$ for a derivation D of $\mathbb{H}$. Every derivation of $\mathbb{H}$ is inner, so $D = [\eta, \cdot]$ with $\eta \in \text{Im } \mathbb{H} = \mathfrak{su}(2)$, and η is unique because $g_1, \dots, g_m$ do not commute. So the derivative of the holonomy in the direction b is the infinitesimal conjugation by η for every loop based at p , and one η serves for all of them. Let ζ_0 be a smooth section whose value at p induces this infinitesimal conjugation. Then the derivative of the holonomy in the direction $b' = b - d_B \zeta_0$ vanishes for every loop based at p , and $\zeta_1(q) = \int_\ell b'$, computed in a parallel frame along a path ℓ from p to q , is independent of ℓ and satisfies $d_B \zeta_1 = b'$ [DFL21, (6.4)]. So $b = d_B(\zeta_0 + \zeta_1)$.

(ii) By (i), no nonzero element of S is annihilated by the derivatives of all trace functions, so these derivatives span the dual of S , and $\dim S$ of them suffice.

(iii) A connection is determined up to gauge equivalence by its holonomy representation up to conjugacy, and this representation by the traces of all words [Don02, Lemma 5.13 and p. 141]. So for each $[B'] \in \mathcal{K}$ some loop, which may be taken embedded, has trace of holonomy at B' different from that at B , and so has an averaged trace function with μ concentrated near the centre. This is an open condition in $[B']$, and $\mathcal{K}$ is compact. $\square$

Lemma C.12. *Let $\nu: \mathbb{R} \rightarrow \mathbb{R}^n$ be C^2 and $W: \mathbb{R} \rightarrow \mathbb{R}^n$ be C^1 and integrable. Let $s_0 \in \mathbb{R}$ and $\delta > 0$ be such that ν is an injective immersion on $(s_0 - \delta, s_0 + \delta)$, $\nu(s_0)$ is not in the closure of $\nu(\mathbb{R} \setminus (s_0 - \delta, s_0 + \delta))$, and $W(s_0) \notin \mathbb{R} \nu'(s_0)$. Then every neighbourhood of $\nu(s_0)$ contains the support of a function $G \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}} \langle \nabla G(\nu(s)), W(s) \rangle ds \neq 0$.*

Proof. Write $\nu_0 = \nu(s_0)$. Near s_0 the distance $|\nu(s) - \nu_0|$ is strictly monotone on each side of s_0 , because $\nu'(s_0) \neq 0$, and the rest of $\nu(\mathbb{R})$ stays at positive distance from ν_0 , by injectivity on $(s_0 - \delta, s_0 + \delta)$ and the closure hypothesis. So for small r the preimage of the ball of radius

r about ν_0 is an interval $J \ni s_0$ of length $O(r)$. Choose $n \perp \nu'(s_0)$ with $\langle n, W(s_0) \rangle = 1$, a function $\chi \in C_c^\infty([0, 1])$ with $\chi \geq 0$ and $\chi = 1$ near 0, and put $G(x) = \chi(|x - \nu_0|/r) \langle n, x - \nu_0 \rangle$. Along ν we have $\langle n, \nu(s) - \nu_0 \rangle = O((s - s_0)^2)$, so the term of $\int \langle \nabla G(\nu), W \rangle$ containing the derivative of χ is $O(r^2)$, while the term $\int \chi(|\nu - \nu_0|/r) \langle n, W \rangle$ equals $c_\chi r / |\nu'(s_0)| + O(r^2)$ with $c_\chi = \int_{\mathbb{R}} \chi(|x|) dx > 0$. For small r the integral is nonzero. $\square$

Lemma C.13. *Let $\mathfrak{h}, g, \gamma = (A_s)$ and $\psi = (\phi, 0)$ be as in Lemma 3.5, $i \in \{0, 1\}$ and $s_0 \in \mathbb{R}$. Suppose that $B = A_{s_0}|_{C_i^\circ}$ is irreducible, that $\pi_B^i \partial_s A_{s_0}$ and $\pi_B^i \phi_{s_0}$ are linearly independent, and that $\Gamma_i(s_0) \notin \Gamma_i([-\infty, \infty] \setminus \{s_0\}) \cup X_i$. Then the conclusion (iii) of Lemma 3.5 holds.*

Proof. Let S be the span of $\pi_B^i \partial_s A_{s_0}$ and $\pi_B^i \phi_{s_0}$. These are smooth, by elliptic regularity for the Neumann problem that defines π_B^i , and orthogonal to $d_B L_1^2(C_i^\circ)$, so $S \cap d_B \Omega^0 = 0$. Trace functions are invariant under all gauge transformations of C_i° , so their derivatives at B vanish on $d_B L_1^2$. By Lemma C.11(ii) there are trace functions τ' with $D\tau'_B$ injective on S ; then $D\tau'_B(\partial_s A_{s_0}) = D\tau'_B(\pi_B^i \partial_s A_{s_0}) \neq 0$, so $\tau' \circ \Gamma_i$ is an injective immersion on some interval $(s_0 - \delta, s_0 + \delta)$. The set $\mathcal{K} = \Gamma_i([-\infty, s_0 - \delta] \cup [s_0 + \delta, \infty]) \cup X_i$ is compact and does not contain $\Gamma_i(s_0)$, so Lemma C.11(iii) gives trace functions τ'' separating $\Gamma_i(s_0)$ from $\mathcal{K}$. Put $\tau = (\tau', \tau'')$, $\nu = \tau \circ \Gamma_i$ and $W(s) = D\tau_{A_s}(\phi_s)$, which is integrable because ϕ decays exponentially and the gradients of trace functions are bounded. The hypotheses of Lemma C.12 hold: $\nu(\mathbb{R} \setminus (s_0 - \delta, s_0 + \delta)) \subset \tau(\mathcal{K})$, a compact set not containing $\nu(s_0)$, and $\nu'(s_0) = D\tau_B(\pi_B^i \partial_s A_{s_0})$ and $W(s_0) = D\tau_B(\pi_B^i \phi_{s_0})$ are linearly independent. It gives G supported in a ball about $\nu(s_0)$ whose closure misses $\tau(\mathcal{K}) \supset \tau(X_i)$. The cylinder function $g'_i = G \circ \tau$ has loops in $\text{int } C_i^\circ$ and vanishes on the neighbourhood $\tau^{-1}(\mathbb{R}^n \setminus \text{supp } G)$ of X_i , and by the chain rule $\int \langle \nabla g'_i(A_s), \phi_s \rangle ds = \int \langle \nabla G(\nu(s)), W(s) \rangle ds \neq 0$. $\square$

C.8. Proof of Lemma 3.5. By Lemma C.2(ii) the cokernel element is $\psi = (\phi, 0)$. Fix i and let $\tilde{a}$ be the limit of A_s as $s \rightarrow -\infty$. By Corollary C.4, (C.2) holds with eigenvalues $-\lambda < 0 < \mu$ of $K_{\tilde{a}}$. By Corollary C.9, $\pi_{\tilde{a}}^i v \neq 0 \neq \pi_{\tilde{a}}^i w$, and since $-\lambda \neq \mu$, Lemma C.6(ii) shows that $\pi_{\tilde{a}}^i v$ and $\pi_{\tilde{a}}^i w$ are linearly independent. The restriction of $\tilde{a}$ to C_i° is irreducible (Lemma C.5), so $\pi_{A_s}^i \rightarrow \pi_{\tilde{a}}^i$ in operator norm, and by (C.2)

$$e^{-\lambda s} \pi_{A_s}^i \partial_s A_s \rightarrow \pi_{\tilde{a}}^i v, \quad e^{-\mu s} \pi_{A_s}^i \phi_s \rightarrow \pi_{\tilde{a}}^i w.$$

Linear independence of a pair of vectors is an open condition, so there is T such that (ii) holds for all $s_0 < -T$. In particular $\pi^i \partial_s A_{s_0} \neq 0$, so Γ_i is not constant, and Lemma C.10 gives (i). The limits $\Gamma_i(\pm\infty)$ lie in X_i , so by (i) $\Gamma_i(s_0) \notin \Gamma_i([-\infty, \infty] \setminus \{s_0\}) \cup X_i$ for every s_0 , and Lemma C.13 at any $s_0 < -T$ gives (iii). $\square$

C.9. Proof of Theorem 3.6.

Lemma C.14. (i) *For every Heegaard splitting and metric data there are finite separated families of primary perturbations containing the height functions of Section 10 with the submersion property of Section 3.2: the universal restriction map of their span is a submersion at every flat connection of C_0 and of C_1 . The family F_* fixed there may be taken to be one of them.*

(ii) *For every finite family F containing such a family, in particular for every $F \supseteq F_*$, generic primary data in a neighbourhood of zero in $\text{span } F$ have transverse Lagrangians.*

Proof. (i) Fix i . For a flat connection B on C_i let $\mathcal{H}_B = \{b : d_B b = 0, d_B^* b = 0, *b|_{\partial C_i} = 0\}$. It maps isomorphically onto $H^1(C_i; \text{ad } B)$. By Definition 3.2(C2), B is irreducible and $H^2(C_i; \text{ad } B) = 0$, so $\dim \mathcal{H}_B = -3\chi(C_i)$ is independent of B , and $\mathcal{H}_B$ depends continuously

on B in a local slice of the flat connections. Let $0 \neq b \in \mathcal{H}_B$. If $b|_{C_i^\circ} \in d_B \Omega^0(C_i^\circ)$, then the class of b vanishes, because $C_i^\circ \hookrightarrow C_i$ is a homotopy equivalence; so $b = d_B \zeta$ on C_i and $\|b\|^2 = \langle \zeta, d_B^* b \rangle + \int_{\partial C_i} \langle \zeta, *b \rangle = 0$, a contradiction. Hence the restrictions to C_i° of the elements of $\mathcal{H}_B$ form a space S with $S \cap d_B \Omega^0(C_i^\circ) = 0$, and Lemma C.11(ii) gives finitely many averaged trace functions of embeddings into $\text{int } C_i^\circ$ whose derivatives at B are jointly injective on $\mathcal{H}_B$. At a flat connection B' and for $d_{B'} b = 0$, the derivative in the direction b of $\text{tr } g_\ell$ is unchanged by a homotopy of ℓ , with λ transported along it: the first variation of the holonomy between homotopic loops differs by the integral of $d_{B'} b$ over the swept surface. So on $\mathcal{H}_{B'}$ the derivative of an averaged trace function equals that of $\text{tr } g_\ell$ for its core ℓ , and up to sign it depends only on the free homotopy class of ℓ . Injectivity on $\mathcal{H}_{B'}$ is open in B' , and the space of gauge equivalence classes of flat connections on C_i is compact, so the cores $\ell_1, \dots, \ell_n$ of finitely many of the functions above serve every flat connection on C_i . Let W be the complement in $\text{int } C_i^\circ$ of the solid tori of the height functions. It is connected, and every loop in C_i° is freely homotopic into W : push it off the cores of those solid tori by general position, then radially out of each of them. By general position there are pairwise disjoint embedded loops $\ell'_k \subset W$ with ℓ'_k freely homotopic to ℓ_k . Take pairwise disjoint thin tubular neighbourhoods of them in W , and any μ . At every flat B' the derivatives on $\mathcal{H}_{B'}$ of the resulting averaged trace functions agree up to sign with those of the $\text{tr } g_{\ell_k}$, so they are jointly injective. Let F consist of the height functions and these averaged trace functions; it is separated.

By the proofs of [DFL21, Propositions 6.5 and 6.7], with the Banach space of perturbations replaced by $\text{span } F$, the universal restriction map is a submersion at a flat connection B on C_i provided two cokernels vanish. The first, $H^1(C_i, \partial C_i; \text{ad } B) \cong H^2(C_i; \text{ad } B)^*$, is zero by Definition 3.2(C2). The second consists of the $b \in \mathcal{H}_B$ annihilated by $D\tau_B$ for every $\tau \in F$, and it is zero by construction. So $\text{span } F$ has the submersion property of Section 3.2.

(ii) The submersion property holds for $\text{span } F$ at every flat connection, since it holds for a subspace, and is an open condition, so it holds over a neighbourhood of zero in $\text{span } F$, and transversality of the two Lagrangians for generic primary data there follows as in [DFL21, proof of Lemma 2.56, p. 78]. $\square$

Lemma C.15. *Let $\mathfrak{h}$ have transverse Lagrangians. For every split addition g sufficiently small in the norm $\sup_B \|\nabla g(B)\|_{L^\infty} + \sup_{B \neq B'} \|\nabla g(B) - \nabla g(B')\|_{L^2} / \|B - B'\|_{L^2}$, which is finite by [DFL21, Proposition 6.1], $\text{CS}_{\mathfrak{h}+g}$ and $\text{CS}_{\mathfrak{h}}$ have the same critical points, with the same Hessians.*

Proof. Critical points correspond to intersection points of the Lagrangians [DFL21, Lemmas 2.45 and 2.47]. For g small the perturbed flat connections on C_i are C^1 -close to those for $\mathfrak{h}$, by the implicit function theorem. Those whose restrictions to Σ_h are near a point of $L_{\mathfrak{h}_0}(C_0) \cap L_{\mathfrak{h}_1}(C_1)$ are near the corresponding perturbed flat connection, where g_i vanishes; so each intersection point remains an intersection point with unchanged tangent spaces, and transversality together with C^1 -closeness of the Lagrangians, uniform in that norm, gives uniqueness of the intersection near it. Away from a neighbourhood of the intersection the two Lagrangians for $\mathfrak{h}$ are at positive distance, and so are those for $\mathfrak{h} + g$ when g is small. The Hessians agree because g vanishes near the critical points. $\square$

Proof of Theorem 3.6. The first statement is Lemma C.14(i) and (ii). Let $\mathfrak{h} \in \mathcal{U}(F)$ have transverse Lagrangians; its critical points are nondegenerate [DFL21, Lemmas 2.45 and 2.47]. The functional $\text{CS}_{\mathfrak{h}+g}$ is invariant under α_M , which is dual to the T^2 component of Σ ; that torus misses $\text{int } C_i^\circ$, so every trace function with loops in $\text{int } C_i^\circ$ is α_M -invariant; the gauge

complex modulo ϕ_M is therefore defined. We follow [DFL21, proof of Proposition 6.8], by induction over the energy as in [Don02, p. 145]. Nonconstant trajectories have positive energy, and the positive energies of path classes of index at most seven in the determinant-one theory form a finite set $e_1 < \dots < e_N$, independent of g since g vanishes at the critical points: there are finitely many critical points, and within each class the index determines the energy. Suppose a split addition g_{k-1} has been found, as small as Lemma C.15 requires, such that all trajectories of $\text{CS}_{\mathfrak{h}+g_{k-1}}$ of index at most seven and energy less than e_k are regular. The non-regular trajectories of index at most seven and energy e_k form a compact set. The components of a broken limit have smaller energy. None has index at least eight, since the others would then have total index at most -1 and one of them negative index; so each has index at most seven, hence is regular by the induction hypothesis, and has index at least one. Bubbling is excluded by the energy and index bounds [DFL21, p. 79], and trajectories near a broken trajectory with regular components are themselves regular, by gluing [Don02, p. 144]; so a limit of non-regular trajectories is unbroken. So does the set of pairs (γ, ψ) with γ such a trajectory, up to translation, and ψ a unit cokernel element [Don02, p. 144]. For each such pair Lemma 3.5(iii), applied with $i = 0$ to $\mathfrak{h}$ and g_{k-1} , gives a cylinder function with loops in $\text{int } C_0^\circ$, vanishing near X_0 , whose gradient pairs nontrivially with ψ . The pairing is continuous in (γ, ψ) and invariant under translation, so finitely many such functions $g^{(1)}, \dots, g^{(m)}$ suffice for all pairs. The linearization of the family of equations parametrized by $P_\# = \text{span}\{g^{(j)}\}$ is then onto at every non-regular trajectory of energy e_k , and the Sard–Smale theorem gives arbitrarily small $\rho \in P_\#$ for which all trajectories of index at most seven and energy e_k of $\text{CS}_{\mathfrak{h}+g_{k-1}+\rho}$ are regular [DFL21, p. 80], [Don02, pp. 143–144]. For ρ small the trajectories of smaller energy stay regular, by compactness and openness, and $g_k = g_{k-1} + \rho$ is a split addition to which Lemma C.15 applies. After the stage $k = N$ the primary data $\mathfrak{h} + g$, $g = g_N$, are DFL-regular and have the critical points of $\mathfrak{h}$. Using Lemma 3.5(iii) with $i = 1$ as well gives additions in both pieces. The additions have loops in $\text{int } C_0^\circ \sqcup \text{int } C_1^\circ$, and the metric data are never changed.

Fix the required smallness of the total addition before the finite induction, choosing each increment small enough to meet this requirement. Let G be the family of all cylinder functions added; this family may depend on the prescribed smallness bound. Admissibility is preserved stage by stage: if $\mathfrak{h} + g_{k-1} \in \mathcal{U}(F \cup G_{k-1})$, where G_{k-1} is the family added so far, then $\mathfrak{h} + g_{k-1} \in \mathcal{U}(F \cup G_k)$, which is open in $\text{span}(F \cup G_k)$ (Definition 11.12), so the arbitrarily small increment ρ_k may be chosen with $\mathfrak{h} + g_k \in \mathcal{U}(F \cup G_k)$; after the last stage $\mathfrak{h} + g \in \mathcal{U}(F \cup G)$. $\square$

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