COBORDISM MAPS AND ATIYAH–FLOER ISOMORPHISMS FOR FRAMED INSTANTON HOMOLOGY III

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Abstract. We construct integral chain homotopy equivalences intertwining the usual framed instanton cobordism maps in the singular Hopf-link construction with symplectic cobordism maps oriented by transport from gauge theory. At common regular small data, the equivalences are represented by the Daemi–Fukaya–Lipyanskiy mixed count with a specified coherent orientation. For connected framed cobordisms, they are natural up to integral chain homotopy and one overall sign, with an explicit grading conversion, and extend by geometric changes of presentation. We also construct equivalences to the normalization in which attachment of a generating sphere in the surface moduli space preserves orientation, for every closed connected oriented three-manifold. Surgery generation gives a finite model built from one-generator complexes for the three-sphere. Counts through the three-sphere realize its grading-preserving matrix entries, so an algebraically chosen normalization becomes a finite integer linear combination of products of geometric counts. A polynomial in forward and reflected mixed counts transfers this construction to the symplectic complex. The proof uses the companion analytic and naturality results and the stated ordinary surgery and oriented gluing inputs. Identification with prescribed relative-spin maps and the nonsingular cap convention remains open.

1. Introduction

This is the third of three papers on the compatibility of Atiyah–Floer isomorphisms with cobordism maps. Paper I [Wue26a] establishes the comparison over $\mathbb{F}_2$ and the underlying mixed-equation analysis. Paper II [Wue26b] proves integral naturality, up to one overall sign for each cobordism, with symplectic orientations transported from the almost-complex gauge convention. The present paper uses those results to identify the comparison with the usual singular Hopf-link instanton maps and, at common regular data, with the Daemi–Fukaya–Lipyanskiy mixed count in a specified coherent orientation system, and to construct the further orientation normalization described below. The series thus separates the analytic comparison, its integral orientation theory, and the identification and normalization of the resulting maps. The additional surgery and oriented gluing inputs used here remain explicit.

1.1. **The comparison problem.** The integral form of an Atiyah–Floer comparison depends on the orientation systems used to define its two chain complexes. Even when the unsigned moduli spaces coincide, identifying their counts requires identifying the determinant lines, the action of the gauge group, and the conventions for quotienting by translation. A comparison of cobordism maps must in addition control these identifications under gluing. The allowed ambiguity is a single sign for the whole map.

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Paper II [Wue26b] constructs integral symplectic cobordism maps by transporting almost-complex gauge orientations through the mixed equation. Its mixed map intertwines those symplectic maps with the gauge maps in that convention. Two comparisons are needed to express this theorem using the usual framed instanton maps and the actual mixed count of Daemi, Fukaya and Lipyanskiy [DFL21]. The first compares the usual gauge reference lines with the almost-complex reference lines on the torus-summed manifold. The second compares Paper II's transported orientations with a specified choice in DFL's coherent orientation construction. We first establish these two comparisons and then construct the change to DFL's sphere-positive normalization for every object. The analytic constructions and integral naturality theorem of Papers I–II remain inputs; their scope is recalled in Section 2.

The gauge comparison uses the singular Hopf-link definition of framed instanton homology in [KM11b, Sections 4.3–4.4] and the excision cobordism of [KM11b, Proposition 5.7]. The incoming lifting group is trivial. The outgoing group is the single group specified by the framing torus in Paper II. Keeping the subgroup on the cobordism, rather than only its images at the ends, allows an integral comparison of coefficients and of the ordinary gluing families.

The orientation choice on the DFL side is specified before defining any mixed count. In [DFL21, Section 2.4], a coherent gauge system is determined by reference paths and an orientation of a primitive loop of index four in the framed quotient. We orient this loop and the reference paths by Paper II's gauge reference. We call this the *gauge-calibrated* DFL system. This is one of the choices in the original construction of [DFL21, Sections 2.4 and 4.7]. The further choice discussed in [DFL21, Remark 4.76], for which attachment of a sphere representing the positive generator of the second homotopy group of the surface moduli space preserves orientation, is a separate normalization. We call it the *sphere-positive* system. Theorem B compares the two systems for every closed connected oriented three-manifold by finite products of actual counts. The integer coefficients in this construction are chosen algebraically; the resulting comparison is noncanonical at the level of its outer maps.

Table 1 separates these conclusions from the underlying Atiyah–Floer isomorphism. The new assertions concern the identification and normalization of maps over $\mathbb{Z}$, with the orientation conventions specified here.

TABLE 1. The comparison results used in this paper. Each row is under the hypotheses of the cited construction.

Work	Result and orientation convention
DFL [DFL21]	Integral Atiyah–Floer isomorphism defined by a mixed count for the admissible gauge and Lagrangian data, including the framed specialization; coherent orientations (Theorem 3.29 and Sections 4.7, 7.1).
Paper I [Wue26a]	Geometric cobordism compatibility over $\mathbb{F}_2$, with the analytic constructions used by the integral theory (Theorem A).
Paper II [Wue26b]	Integral cobordism compatibility up to one overall sign, with symplectic orientations transported from the almost-complex gauge convention (Theorem A).
Paper III	Identification with the usual singular Hopf-link maps and the gauge-calibrated DFL count (Theorem A); normalization by finite count expressions for every closed connected oriented three-manifold (Theorem B).

1.2. **Objects, maps and statement.** An object is a closed connected oriented three-manifold M with an oriented ball and the framing data for the torus construction of Paper II.

A morphism $(W, E, \nu) : M_0 \to M_1$ consists of a connected oriented cobordism, a normally framed arc joining the specified balls, and an SO(3) bundle trivialized at the ends and along that arc. All bundle and framing data are part of the morphism. In particular, a change of an endpoint bundle frame is not identified with the identity merely because it preserves the underlying manifold.

Write $C_{\mathrm{H}}(M, p)$ for the usual singular Hopf-link instanton complex at regular data induced by a presentation p, with the parity relabeling of Section 3.4. Its differential is unchanged by that relabeling. The usual strict cobordism map is $F_{\mathrm{H}}(W, E, \nu)$. It uses a homology orientation and the I-orientation prescribed by the singular construction. For a general permitted relative bundle, any additionally required I-orientation is retained; its two choices change the whole map by one sign. We do not impose a liftability condition on E. If f is the parity of this map after the relabeling, put

$$F_{\mathrm{H}}^{\mathrm{r}} = F_{\mathrm{H}}\varepsilon^{f}, \qquad \varepsilon(x) = (-1)^{|x|}x.$$

This is the conversion from strict to graded maps used in Paper II. It is explained in Section 2.3 and is part of the statement, not an unspecified scalar sign.

Let $C_J(M, p)$, $C_S(M, p)$ and $N_p : C_J(M, p) \to C_S(M, p)$ be Paper II's complexes and mixed map. A *small presentation* has all the presentation data of Paper I, Definition 3.3, including original Heegaard genus at least one, compatible product collars and metrics, and the prescribed regularizer bounds. Its primary perturbations lie in Paper II's small admissible ball. These conditions are recalled in Section 2. For a small decomposition d of a morphism, $S_d(W, E, \nu)$ denotes the composite of Paper II's signed geometric symplectic maps. At objects with zero integral gauge homology we retain its zero-representative convention, with the contractions described below.

At common regular data for DFL and Papers I–II, write $C_G^{\mathrm{cal}}(p)$, $C_S^{\mathrm{cal}}(p)$ and N_p^{cal} for the complexes and the actual DFL mixed count with the gauge-calibrated orientation choice. These data meet both constructions' hypotheses, including zero additional gauge-end perturbation; the remaining conditions are specified in Section 2. The active Heegaard genus at such data will be at least two. The following theorem uses the parity-graded integral homotopy category $\mathcal{K}(\mathbb{Z})$; $F \simeq \pm G$ means that one sign makes the difference null-homotopic over $\mathbb{Z}$.

Theorem A. *For every small presentation p there is an even integral chain homotopy equivalence*

$$\widehat{\Psi}_p : C_{\mathrm{H}}(M, p) \longrightarrow C_S(M, p)$$

whose projective homotopy class is independent of the auxiliary choices and is compatible with geometric changes of presentation. At common DFL data it is represented by

$$\Psi_p = B_p N_p^{\mathrm{cal}} U_p, \qquad U_p = P_p A_p, \tag{1}$$

where $A_p : C_{\mathrm{H}}(M, p) \to C_J(M, p)$ is the Hopf-link excision equivalence and

$$P_p : C_J(M, p) \longrightarrow C_G^{\mathrm{cal}}(p), \qquad B_p : C_S^{\mathrm{cal}}(p) \longrightarrow C_S(M, p)$$

are orientation-line chain isomorphisms. These maps are constructed without using an inverse of a mixed map. Moreover,

$$B_p N_p^{\mathrm{cal}} P_p = N_p \tag{2}$$

as signed chain maps at common data.

For every permitted (W, E, ν), all small endpoint presentations, and every small decomposition d, one has

$$\widehat{\Psi}_{p_1} F^{\mathrm{r}}_{\mathrm{H}}(W, E, \nu) \simeq \pm S_d(W, E, \nu) \widehat{\Psi}_{p_0}. \tag{3}$$

The sign is one sign for the entire map. The assertion includes arbitrary permitted relative bundles, frame changes, and empty or acyclic complexes. At an arbitrary small presentation, $\widehat{\Psi}_p$ is obtained by geometric transport of an actual common-data DFL count; no original DFL mixed problem is asserted at data outside its analytic hypotheses.

On homology the theorem gives a projective natural isomorphism from the usual framed instanton theory, with the stated convention conversion, to Paper II's symplectic theory. The comparison maps at common data respect the relative mod-four gradings on the DFL and Paper II complexes. The full statement for geometric words is made in the parity-graded category, because Paper II permits inhomogeneous representatives of such words. It does not assert that every such representative is homogeneous modulo four.

The substantial steps are the naturality of the geometric excision comparison (Section 3) and equality of the signed mixed counts (Section 4). Section 5 assembles them using Paper II's geometric presentation maps and gives explicit integral homotopies. No numerical computation of a sphere orientation character is used in this proof.

1.3. **Normalization for every manifold.** Write $C^+_G(p)$, $C^+_S(p)$ and N^+_p for the DFL complexes and actual mixed count with the sphere-positive system. Reference paths and their orientations are matched with the calibrated system. The two systems can have different differentials even though the matrices of their index-zero mixed counts agree. The problem is therefore to construct chain equivalences on both sides of that count, with compatible inverse and mixed homotopies.

We specify the class of maps used in this construction. A *weighted count expression* is a finite integer linear combination of composable products of the following maps: ordinary gauge trajectory, continuation and cobordism counts; the zero-dimensional counts in the interval, pentagon and cylinder families of Section 7; the geometric symplectic words of Paper II; and the forward and reflected mixed counts of Section 10. Orientation-line identifications, formal degree labels and weights depending on endpoint relative grading and parity are allowed. A separately specified product may have any fixed integer multiplicity. Every physical intermediate critical set is summed over in full. Thus an expression counts products of existing zero-dimensional moduli spaces. This definition does not permit an arbitrary projection onto a Floer generator or an algebraically selected mixed inverse as a factor. Integral algebraic homotopies may be used to verify relations between the resulting maps.

Theorem B. *Retain the companion inputs of Section 2 and the ordinary surgery and oriented gluing inputs of Section 7.1. Let M be any closed connected oriented three-manifold with the specified ball and framing data. At every common regular small presentation p there are even integral chain homotopy equivalences of relative degree zero, defined by weighted count expressions,*

$$u_{X,p} : C^{\mathrm{cal}}_X(p) \longrightarrow C^+_X(p), \qquad v_{X,p} : C^+_X(p) \longrightarrow C^{\mathrm{cal}}_X(p), \quad X = G, S.$$

They are mutually inverse up to integral homotopy and satisfy

$$v_{S,p} N^+_p u_{G,p} \simeq N^{\mathrm{cal}}_p. \tag{4}$$

The assertion includes nonempty acyclic complexes.

A choice of these maps at each common presentation defines positive presentation maps by weighted count expressions. They satisfy projective composition, inverse and word-independence relations and mixed squares with the actual counts N_p^+. Different choices give equivalent systems in the projective homotopy category.

For every regular small presentation, geometric transport gives an even equivalence

$$\widehat{\Psi}_p^+ : C_{\mathrm{H}}(M,p) \longrightarrow C_S(M,p),$$

whose common-data representative is

$$\Psi_p^+ = B_p v_{S,p} N_p^+ u_{G,p} U_p. \tag{5}$$

Its projective homotopy class equals that of $\widehat{\Psi}_p$ in Theorem A. In particular it intertwines the usual gauge and Paper II symplectic cobordism maps for every permitted relative bundle and framed morphism, with the convention conversion of Theorem A.

The theorem constructs a system of positive presentation maps. It does not identify that system with independently prescribed positive stabilization or relative-spin counts. The individual maps $u_{X,p}$ and $v_{X,p}$ depend on choices. Their mixed composite (5), however, has the fixed projective homotopy class of Theorem A.

1.4. **Proof of the calibrated comparison.** Theorem A has two geometric ingredients. First, the Hopf-link excision equivalence intertwines the usual singular cobordism maps and the almost-complex gauge maps, after the stated grading conversion. Second, the calibrated orientation lines identify Paper II's mixed count with the actual DFL count coefficient by coefficient. Composing these identifications with Paper II's naturality homotopies proves the assertion at common data. Geometric presentation maps then extend it to every small presentation. This argument requires no normalization of the index-four loop and no inverse of a mixed count.

1.5. **Normalization by surgery generation.** Reversing the primitive index-four orientation changes the sign of just one component of the mod-four differential. Abstractly, any finite free integral complex is isomorphic to the complex with this sign change: split off its cycles and change signs on a complement. Such a splitting does not itself give a map defined by moduli spaces. The main step is to realize enough actual count expressions that the algebraic choice can be made among them.

Ni's surgery-generation theorem [Ni25, Theorem 1.4] gives a finite derivation of M from copies of S^3. The ordinary instanton surgery families turn this derivation into a finite iterated mapping cone P, together with actual count expressions $i : C_G^{\mathrm{cal}} \to P$, $p : P \to C_G^{\mathrm{cal}}$ and an actual homotopy $pi \simeq 1$. All leaves are labelled copies of the same fixed one-generator sphere datum. Consequently every row of i and column of p is a finite sum of count products with a sphere end. Joining one row and one column by the sphere identity cylinder realizes a matrix unit in the formal leaf coordinates. This realizes all grading-preserving matrices before one chooses a normalizing matrix. A finite integer search, whose termination follows from the cycle-splitting argument, then gives the required gauge equivalences.

To pass to the symplectic complex, we use the mixed count at the reflected presentation with literally paired ordinary ends. The ordinary index-energy identity shows that this transposed count and the forward mixed count preserve the same finite filtration. Their diagonal coefficients are units. A finite polynomial in the two counts therefore supplies a strict reverse map. This transfers the gauge normalization to the symplectic side and gives (4). Contractibility treats the acyclic endpoints without extending the reflection pairing beyond its stated domain.

Sections 3–5 prove the calibrated comparison. Section 6 describes the change of differential and its grading signs. Sections 7–9 construct and normalize the finite gauge model; Section 10 transfers it through the actual mixed count. The final argument in Section 11 proves presentation independence and cobordism compatibility. Only the ordinary gauge surgery families enter the model construction; no mixed surgery family is needed. Figure 1 records the proof dependencies. In particular, the augmented oriented gluing input at the reducible cut in Section 7.1 remains an input; the comparison of its two local unit faces is proved here.

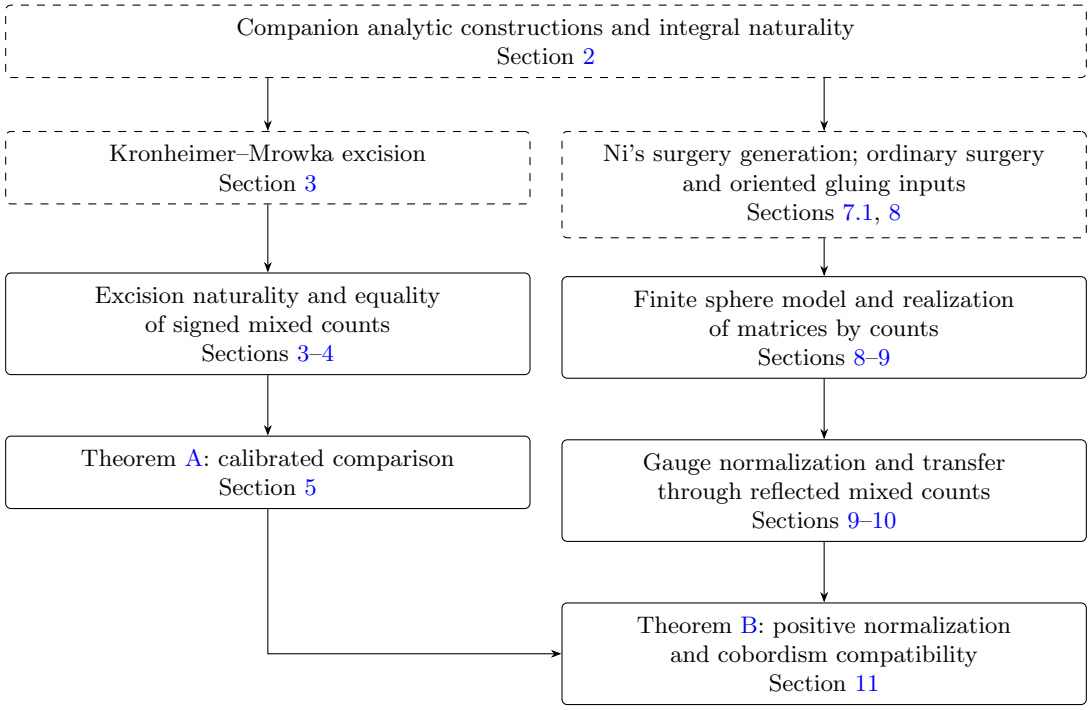

FIGURE 1. Dashed boxes denote imported results and retained inputs; solid boxes denote arguments of this paper. The companion inputs are used in both branches, as well as in the final presentation comparisons.

2. Conventions and imported constructions

2.1. **The gauge and symplectic objects.** Form $M^{\#} = M\#T^3$ along the specified ball. Fix an oriented torus T_0 in the punctured T^3 and an odd bundle on T^3, so that $\langle w_2, [T_0]\rangle = 1$. Join it to the trivial bundle on M. The framing quotient enlarges the determinant-one gauge group by

$$\phi_M = \langle \alpha_M \rangle, \qquad \alpha_M = \mathrm{PD}[T_0] \pmod 2.$$

A lifting class is the obstruction to lifting a bundle automorphism to the determinant-one gauge group. Only this cyclic subgroup is added; arbitrary classes from $H^1(M;\mathbb{Z}/2)$ are not included. For a morphism insert the punctured torus times an interval in the framed tube, obtaining $(W^{\#}, E^{\#})$ and its specified subgroup ϕ_W. On disconnected cuts the full subgroup on the cobordism must be retained.

Choose at each object an almost-complex reference J_0 on its cylinder with the fixed product germ near T_0. Paper II uses the Kronheimer–Mrowka almost-complex orientations, invariant

under bundle automorphisms, and converts the restrictions of cobordism structures to these object references. A reference conversion is the determinant-line isomorphism obtained by gluing the operator between the two references. Its degree can be odd. Paper II, Section 4, specifies its position relative to the generator line; we use its right-placed, graded form for $I_J(W)$.

For a Heegaard splitting of genus $g \geq 1$ adapted to the fixed ball, adjoining the torus through its equatorial disk produces two compression bodies with boundary $\Sigma_h \sqcup T^2$, where $h = g + 1 \geq 2$. Their boundary collars and product metrics agree on Σ_h and on the torus collar, whose flat metric is fixed for all presentations. The collar of Σ_h meets the gluing sphere in the product of an interval with the boundary of the equatorial disk: its lines stay in the punctured torus or in the original M piece according to their starting point. These are the collar compatibility conditions of Paper I, Definition 3.3. The symplectic space $\mathcal{M}_h$ is the moduli space of flat connections on the odd surface bundle, with the lift convention of [DFL21, Section 2]. Restriction from the compression bodies gives Lagrangians, moved to P_0, P_1 by primary holonomy perturbations. The primary functions are finite sums of averaged holonomy functions supported away from the fixed collars. Their trace convention uses a chosen continuous square root of the determinant holonomy to take the trace of the SU(2) part. At regular data $P_0 \cap P_1$ is finite and transverse and corresponds to the gauge critical orbits.

If $L_G(a_x)$ is the real orientation line at a gauge generator, write $\mathrm{or}_{\mathbb{Z}}(L_G(a_x))$ for the free abelian group on its two orientations, with opposite orientations negatives of one another. Paper II defines

$$C_S(M,p) = \bigoplus_{x \in P_0 \cap P_1} \mathrm{or}_{\mathbb{Z}}(L_G(a_x)).$$

Its differential counts strips oriented by transport through the mixed equation. A *mixed configuration* is a pair (A, u) consisting of a connection on the gauge region and a map from the symplectic region, satisfying the matching, Lagrangian boundary and asymptotic conditions. The matching condition requires that the boundary connection be flat and represent the boundary value of u. Denote this ambient space, modulo the specified gauge group, by $\mathfrak{B}_{\mathrm{mix}}$. A *mixed solution* is a configuration that also satisfies the perturbed anti-self-duality and Cauchy–Riemann equations. Determinant lines below are defined on $\mathfrak{B}_{\mathrm{mix}}$, and their transport paths need not consist of solutions. Secondary perturbations are supported away from the matching boundary and end collars. The regularizing data include the end perturbation, its mixed cutoff, secondary data, and the allowed interior almost-complex changes. Ordinary ends have fixed irreducible nondegenerate limits; mixed ends retain their varying limiting values as finite-dimensional stationary coordinates.

The domain defining the comparison is the *special quintuple* of [DFL21, Section 3.1, Figures 1–2]. A quintuple records the gauge four-manifold X, its bundle V, the symplectic surface S, the surface moduli space, and its Lagrangian boundary conditions. Here X is assembled from the products of the two compression-body cores with $\mathbb{R}$ and the negative half of a four-ended planar domain times the matching surface. The positive half of that domain is S; the common dividing line is the matching boundary. The two remaining boundary components of S carry the compression-body Lagrangians. The resulting domain has one ordinary gauge input, one ordinary symplectic output, and two mixed ends, one for each compression body. The matching surface is $\Sigma_h \sqcup T^2$; its moduli-space target is written as $\mathcal{M}_h$, since the odd torus factor is a point. This is the specific mixed domain used in Section 4.

2.2. **Smallness and the precise companion inputs.** For a primary function $\mathfrak{h}_i$ on a compression body use Paper II's norm, modulo constants,

$$\|\mathfrak{h}_i\|_i = \sup_B \|\nabla_B \mathfrak{h}_i\|_{L^\infty} + \sup_{B \neq B'} \frac{\|\nabla_B \mathfrak{h}_i - \nabla_{B'} \mathfrak{h}_i\|_{L^2}}{\|B - B'\|_{L^2}},$$
$$\|\mathfrak{h}\| = \max_i \|\mathfrak{h}_i\|_i.$$

The fixed finite trace family F_0 gives embedding coordinates on the compact unperturbed moduli spaces. In addition, Paper I fixes for each splitting a finite family F_* whose universal restriction map is a submersion at every flat connection of either compression body. It contains the fixed torus-compression height functions of Paper I, Section 10; the two-handle height functions are included in the enlarged families of the relevant comparisons. The allowed ambient families F contain both F_0 and F_*. The datum lies in Paper I's neighborhood $\mathcal{U}(F)$: a path of primary functions from zero to that datum keeps the two Lagrangians and their reflections C^1-close to the unperturbed ones, with closeness small relative to those fixed auxiliary heights. Thus admissibility includes this path condition, as well as transverse intersections at the chosen endpoint.

A finite family is *separated* if its functions can be written as affine combinations of averaged sign-sensitive traces on pairwise disjoint embedded solid tori in the interiors of the compression bodies. This is a condition on a chosen realization of the family, not an additional property of a critical point. For primary data $\mathfrak{h}$, let X_i be the finite set of restrictions of its gauge critical points to the interior of the ith compression body, modulo determinant-one gauge transformations. A *split addition* is a finite sum $g = g_0 + g_1$ of cylinder functions whose loops lie in the respective compression-body interiors and for which g_i vanishes on a neighborhood of X_i. It leaves the common collar unchanged; if sufficiently small, it also leaves the critical points and Hessians unchanged. Separatedness is used for the initial family in the regularization theorem below; an enlarged ambient family need not remain separated.

The end perturbation is invariant under the framing lift, vanishes near the critical set, and makes all nonconstant gauge trajectories of index at most seven in the determinant-one theory regular. Both it and the secondary data obey the smallness thresholds for the mixed isomorphism in Paper I, Theorem 3.8. We call primary data *DFL-regular* if these gauge trajectories are already regular with zero additional end perturbation. For fixed splitting and metric data, Paper II, Sections 10, 11 and 13, supplies a radius $r_\chi > 0$, independent of enlargements of the finite family, such that perturbations of norm less than r_χ retain its required admissibility and orientation-character properties. Small presentations include regularizing data making the relevant counts regular. Small cobordism words are the permitted elementary words with these data. No assertion below enlarges that smallness domain.

We use the following results of the fixed companion versions [Wue26a, Wue26b].

(i) Paper I, Theorem 3.6 and Appendix C, supplies arbitrarily small transverse primary data in a separated finite family, and arbitrarily small split additions making the gauge data DFL-regular without changing the metric or losing admissibility. Proposition 3.11 identifies its common mixed problem with DFL's: the extra gauge-end perturbation is zero, the primary data agree, the secondary perturbation is invariant under the framing lift, and the special quintuple, metrics and almost-complex data satisfy the DFL restrictions.

(ii) Paper II, Proposition 2.1 and its analytic appendices, supplies the Fredholm realizations on configuration families, with the primal matching domain, stationary coordinates and

the adjoints on dual weighted spaces. It includes ordinary gluing and invertible constant mixed pairs. Proposition 2.2(a) and its proof in Appendix B supply the following preparation and shifting statement on the comparison domain. A compact family in $\mathfrak{B}_{\mathrm{mix}}$ with fixed asymptotic labels can first be made stationary beyond a common end cutoff and then deformed to gauge paths followed by constant mixed pairs C^c, where c is the actual representative over the output label. Both deformations preserve the matching and Lagrangian conditions and retain the stationary coordinates. Preparation preserves members already stationary beyond the cutoff, and shifting keeps members already of terminal form within that subspace. They are continuous in configurations and data and coherent with ordinary input gluing. Its Proposition 5.4 defines the transported orientations by ordinary gauge orientations and canonically oriented constant pairs. These are statements on configuration spaces, not only on solutions.

(iii) Paper II gives an even chain isomorphism N_p (Theorem A and Proposition 6.3) and, for every morphism and small word, a sign t_W and an integral homotopy L_W satisfying

$$N_1 I_J(W) - t_W S_d(W) N_0 = \delta L_W. \tag{6}$$

Its Section 14 constructs the actual geometric continuation, stabilization, metric, isotopy and decorated-isomorphism maps. With fixed object references the identity-presentation maps are even, satisfy mixed squares, and have projective inverse, composition and word-independence homotopies. General reference conversions retain their degrees and the sign rules of Section 2.3.

These are the analytic existence results and geometric symplectic maps used in the comparisons below.

2.3. **Strict and graded maps.** All chain groups are finite free abelian groups with parity grading and an odd differential. For a homogeneous map T put

$$\delta T = d_D T - (-1)^{|T|} T d_C.$$

Extend this differential componentwise to inhomogeneous maps. The category $\mathcal{K}(\mathbb{Z})$ has closed maps modulo δ-boundaries; composition uses the usual graded Leibniz rule. We further identify a morphism with its negative when speaking of a projective class.

A strict homogeneous map satisfies $d_D F = F d_C$, regardless of its parity f. Its graded form is

$$\mathcal{R}(F) = F\varepsilon_C^f. \tag{7}$$

The identities $d\varepsilon = -\varepsilon d$ and $\varepsilon_D F = (-1)^f F \varepsilon_C$ give

$$\delta\mathcal{R}(F) = 0, \qquad \mathcal{R}(G)\mathcal{R}(F) = (-1)^{gf}\mathcal{R}(GF) \tag{8}$$

for strict maps of parities f, g. Thus the conversion preserves composition projectively, not necessarily strictly. If $F_1 - F_0 = d_D H + H d_C$ and the endpoint maps have parity f, then

$$\mathcal{R}(F_1) - \mathcal{R}(F_0) = \delta(H\varepsilon_C^f). \tag{9}$$

The exponent is the parity of the endpoint maps, whereas H has parity $f+1$. Throughout, the original differential is retained.

We will also use that an acyclic finite free parity-graded complex over $\mathbb{Z}$ is contractible. Indeed, write $C_i = Z_i \oplus L_i$, where $Z_i = \ker d_i = \operatorname{im} d_{i+1}$. The quotient by Z_i is the free subgroup $\operatorname{im} d_i$ of the next chain group, so this splitting exists, and $d_i : L_i \to Z_{i-1}$ is an isomorphism. The map

$$h(z+\ell) = (d|_{L_{i+1}})^{-1} z \tag{10}$$

satisfies $dh + hd = 1$. Consequently a quasi-isomorphism of finite free integral complexes is a chain homotopy equivalence, by applying this construction to its cone. This includes nonempty complexes with zero homology, without a torsion-freeness hypothesis on the homology of a general complex.

At a zero-homology object, maps incident to it are null-homotopic. Paper II uses zero representatives there; its geometric descriptions at nonzero-homology objects and this zero convention are both retained. Zero maps between contractible complexes are homotopy equivalences. The empty complex requires no critical basepoint or reference path.

3. The usual gauge maps and Hopf-link excision

3.1. **The singular realization.** Place the standard Hopf link H_M in the specified ball of M, with its arc ω_M joining the two components. Before the parity relabeling below, set

$$C_{\mathrm{H}}(M) = C_{\omega_M}(M, H_M; \mathbb{Z}).$$

The gauge group is determinant-one. This is the usual definition of $I^{\sharp}(M, \varnothing)$ in [KM11b, Section 4.3]; it is also $I^{\natural}(M, U)$ for the small unknot whose meridian is the other Hopf component. Its reference torsors are those of [KM11b, Section 4.4]. The incoming reference is the preferred reducible singular connection of that construction.

Transporting the Hopf link through the framed tube of W gives two oriented annuli S_H. For any permitted relative bundle E, retain it outside that tube and perform the singular replacement inside, using its specified trivialization. Denote the resulting bundle data by P_H. The map $F_{\mathrm{H}}(W)$ is the usual count for (W, S_H, P_H). For the untwisted bundle, the oriented annuli S_H make the I-orientation equivalent to a homology orientation, as described after Definition 4.3 and in equation (26) of [KM11b]. For a general fixed bundle, its I-orientations form a two-element torsor [KM11b, Definition 3.9]. The determinant factorization there compares all endpoint generators and relative classes; changing the I-orientation multiplies every coefficient by the same sign. We use the usual homology and lift data when they specify this element, and otherwise retain this single projective choice.

Proposition 3.1. *There are geometric even integral chain homotopy equivalences $A_p : C_{\mathrm{H}}(M, p) \to C_J(M, p)$, with the parity convention of Section 3.4, and, for every permitted morphism, a sign s_W and an integral homotopy G_W such that*

$$A_1 F^{\mathrm{r}}_{\mathrm{H}}(W) - s_W I_J(W) A_0 = \delta G_W. \tag{11}$$

Their construction uses singular excision and ordinary reference conversions and is independent of mixed maps.

3.2. **The excision subgroup and its coefficients.** Apply [KM11b, Proposition 5.7] to the small unknot U. Cutting the two tori around the Hopf components and interchanging meridians and longitudes gives a cobordism X_M with outgoing components (S^3, H) and $M \# T^3$. To identify the latter marking, recall that the complement of a coordinate circle in T^3 is $S^1 \times (T^2 \setminus \operatorname{int} D^2)$. Under the exchange, the disk bounded by the zero longitude of the unknot fills the boundary circle of the punctured torus. The remaining circle is the S^1 factor. The original M lies outside the ball. Its framing fixes the longitude and hence the required marked torus summand.

The incoming cutting tori are separating, so their dual classes vanish. The incoming lifting subgroup is zero. The outgoing sphere also has trivial subgroup, and the outgoing torus-summed component has precisely ϕ_M. On X_M we use the actual subgroup H_X of [KM11b,

Theorem 5.6], generated by the duals of the four edge hypersurfaces $T \times [0,1]$ in its local product $T \times U$, where U is the excision surface with corners.

Here $H_X \cong \mathbb{Z}/2$ and restriction $H_X \to \phi_M$ is an isomorphism. Indeed, the local surface U is a disk with four attaching edges. The exterior has three components; two edges attach to the middle Hopf exterior, and the remaining edges to the two solid-torus components. The incidence graph therefore has four vertices and four edges, with one cycle. Each prescribed dual class restricts to zero on the local piece and on each exterior piece. It belongs to the degree-zero Mayer–Vietoris contribution, which is the one-dimensional cycle space of this graph. The outgoing transverse circle detects its nonzero element. Thus the prescribed subgroup has no kernel on restriction to the outgoing end. Additional cohomology of M is not part of this subgroup.

The freeness needed for the quotient is the strengthened *ϕ-non-integral condition* of [KM11b, Definition 5.1]. It holds: T_0 is odd for the bundle and

$$\alpha_M|_{T_0} = \mathrm{PD}[T_0]|_{T_0} = 0$$

because its normal line is trivial. On the Hopf component the group is trivial and the standard torus is non-integral. Fixing a representative of an outgoing critical orbit therefore selects exactly one representative in every H_X-orbit of solutions. This remains true when the action permutes relative configuration classes: restriction is injective and surjective on the prescribed group. The coefficient dictionary uses this bijection, not division by the order of the group.

Use the almost-complex orientations of [KM11b, Sections 5.1 and 5.3] on X_M, with a product structure on $T \times U$ and complex cutting tori. The outgoing germ can be chosen to be Paper II's germ near T_0. The excision count is a strict chain map

$$e_M : C^{J_-}_{\mathrm{Hopf}}(M) \longrightarrow V \otimes C^{J_+}_G(M \# T^3), \tag{12}$$

where $V = C_\omega(S^3, H)$ has one nondegenerate generator and zero differential [KM11b, Proposition 4.1]. By the integral excision theorem [KM11b, Theorem 5.6], its homology map is an isomorphism. The rank-one free factor entails neither an eigenspace projection nor a Künneth assertion for two groups with torsion.

3.3. **Reference lines.** Compare the usual singular reference torsor at the incoming end with the J_- torsor by determinant gluing. There is no incoming lifting quotient. Choosing an orientation of the single reference comparison line determines its transport to every generator. The trajectory factor in the gluing is the same on both sides; ordinary instanton and monopole addition have the common conventions of [KM11b, Sections 3.8 and 4.4]. This gives a strict reference isomorphism

$$P^{\mathrm{H}}_M : C_{\mathrm{H}}(M) \longrightarrow C^{J_-}_{\mathrm{Hopf}}(M).$$

The trivial-subgroup case of [KM11b, Section 5.1, p. 154] identifies this construction with the usual reference comparison. In a cobordism the two reference conventions factor the same determinant line into endpoint factors and one I-orientation torsor. The remaining sign therefore belongs to the whole map, not separately to its generators or relative classes.

At the outgoing end use the strict form of Paper II's reference conversion

$$T_M : C^{J_+}_G(M \# T^3) \longrightarrow C_J(M).$$

Both structures have the fixed torus germ. Their difference class restricts trivially to T_0, so the conversion has trivial character on ϕ_M, by Paper II, Proposition 4.2. It consequently descends to the required quotient. This is the almost-complex conversion of that proposition.

Changing the chosen structures on the excision cobordism changes the map, after these reference conversions, by a single sign and an ordinary data homotopy. To see the sign assertion, compare the two complex orientations on a fixed-end configuration space. Their ratio is locally constant and is invariant under bundle automorphisms. The incoming subgroup is zero; the outgoing conversion has trivial character. Reference gluing and the common addition conventions compare all endpoint and relative classes, leaving the one I-orientation choice. This is the determinant factorization used for Paper II's gauge functor, applied to these specified singular and nonsingular ends.

3.4. **The unit factor and the grading.** Fix a complex reference on the Hopf factor and let v be its generator of parity τ. In the left tensor convention of (12),

$$d(v \otimes x) = (-1)^{\tau} v \otimes d_C x.$$

The strict contraction is

$$r_V(v \otimes x) = \varepsilon_C^{\tau} x. \tag{13}$$

It has parity τ, commutes strictly with the differential, and has inverse $x \mapsto v \otimes \varepsilon_C^{\tau} x$. For a strict homogeneous F of parity f, the same left tensor convention gives $(1_V \otimes F)(v \otimes x) = (-1)^{\tau f} v \otimes Fx$. Hence

$$r_V(1_V \otimes F) = F r_V. \tag{14}$$

The tensor sign cancels the sign in $\varepsilon_D^{\tau} F = (-1)^{\tau f} F \varepsilon_C^{\tau}$, although the contraction itself may have generator-dependent signs. Changing v to $-v$ changes the contraction by one sign.

If the excision induces a Hopf reference of parity τ', its strict conversion to the fixed one includes $\varepsilon_C^{\tau - \tau'}$ on the other tensor factor. Composition with (13) gives $\varepsilon_C^{\tau'}$, up to the chosen sign of the unit reference. This retains the grading operator when the free-end structure changes; it does not require prescribing an arbitrary Hopf plane field at both ends.

In strict conventions define

$$A_M = T_M r_V e_M P_M^{\mathrm{H}}. \tag{15}$$

Let $a_M \in \mathbb{Z}/2$ be its degree, computed by adding the Fredholm shifts of these four factors. The relative grading in [KM11b, Proposition 5.7] makes the excision shift independent of the generators; the reference shifts are fixed per object. The same definition uses reference operators if a critical set is empty and does not depend on a nonzero matrix coefficient. On the source relabel parity by

$$|x|_{\mathrm{H},new} = |x|_{\mathrm{H}} + a_M. \tag{16}$$

Keep its differential unchanged. The strict map (15) is now even and closed. A usual strict map of old parity f has new parity $f + a_{M_1} - a_{M_0}$ and is converted by (7) with that new parity. Index additivity for the cobordism diagram below identifies it with the parity of the corresponding torus map. Applying the same index identity to continuation cylinders shows that a_M is independent of regular end data. This conclusion does not require a nonzero continuation coefficient. It is a parity transfer, with the usual grading recoverable by (16); it is not an identification of every absolute grading normalization.

Excision and the rank-one factor make A_M a quasi-isomorphism. For completeness its integral inverse homotopies can be obtained from (10). With cone differential

$$D = \begin{pmatrix} d_J & A_M \\ 0 & -d_H \end{pmatrix}$$

and contraction $h = \left(\begin{smallmatrix} h_J & t \\ B & h_H \end{smallmatrix}\right)$, the equation $Dh + hD = 1$ gives

$$d_H B = B d_J, \qquad A_M B = 1 - d_J h_J - h_J d_J, \qquad B A_M = 1 + d_H h_H + h_H d_H.$$

This algebra verifies equivalence of the already defined geometric map. No splitting is used to define A_M or its naturality.

3.5. **The cobordism diagram and the chain homotopy.** The framed tube is $[0,1] \times B^3$ with its given end identifications. Move the local excision through this tube. Sliding the surface-with-corners factor in $T \times U$ gives an identification, relative to the outer ends, of singular cobordisms with bundle data:

$$X_{M_1} \circ (W, S_H, P_H) \cong \big(([0,1] \times (S^3, H)) \sqcup (W^{\#}, E^{\#})\big) \circ X_{M_0}. \tag{17}$$

It is the identity outside the tube and carries the four actual edge hypersurfaces to the corresponding hypersurfaces. In particular it preserves the prescribed subgroup, not just its endpoint images. The relative bundle outside the tube is unchanged. A frame-change morphism uses its own tube and endpoint identifications in this construction.

For composition, the left cut in (17) is connected. On the right it is $S^3 \sqcup (M_0 \# T^3)$. The degree-zero Mayer–Vietoris map is

$$H^0(X_{M_0}; \mathbb{Z}/2) \oplus H^0(([0,1] \times S^3) \sqcup W^{\#}; \mathbb{Z}/2) \longrightarrow H^0(S^3 \sqcup (M_0 \# T^3); \mathbb{Z}/2),$$

with coordinate expression $(a, b, c) \mapsto (a - b, a - c)$. It is surjective. Thus neither cut supplies an additional connecting class. The restrictions of the common subgroup are exactly H_X, ϕ_W, and zero on the Hopf product component. This verifies the subgroup hypothesis for the disconnected composition in [KM11b, Proposition 5.4] and excludes an extra multiplicity.

Choose a common almost-complex structure for (17), product near its cutting tori and with complex singular annuli. One may first choose it in the framed tube. In the local Hopf model a nonzero vector field is tangent to the cores and normal to the two oppositely oriented cutting tori; in the middle $T^2 \times [0,1]$ it is the interval direction. Extending across the solid tori interpolates to the chosen core directions. This gives the local complex structure and its product extension over the excision piece. Extend the tube germ over W, leaving the outer ends free. The primary obstruction is $W_3(W) = 0$ for an oriented four-manifold; the tube is contractible, and the remaining obstruction lies in $H^4(W, \text{tube}; \mathbb{Z}) = 0$, since W is connected with nonempty boundary. Adjust collars to be invariant at the outer ends. After excision the resulting germ on $W^{\#}$ is the permitted torus germ. Fixed object references are restored by the conversions already specified.

There are three ordinary metric families: stretch the left cut, join the two finite metrics on the common composite, and reverse the stretch at the right cut. Their endpoints give $e_{M_1} F_{\mathrm{H}}(W)$ and $(1_V \otimes F_J(W)) e_{M_0}$ in the induced complex conventions. The parameterized constructions are those of [KM11b, Section 3.9, following Proposition 3.10], with the enlarged subgroups from its Section 5.1. Two details of this application are useful. First, the lifting action is free. Choose perturbation variations in separated neighborhoods of an orbit and translate them equivariantly. On one representative the available variations are the original local variations, so they still give surjectivity. This is the invariant-family argument; averaging an arbitrary regular perturbation would not by itself prove regularity. Second, the disconnected cut has two neck parameters. At a rigid broken solution, regular gluing gives a product chart in the two large lengths. Restrict this chart to equal lengths. It supplies the one-parameter end used here. The invertible end operators give the direct sum of the two

end parametrices; the subgroup calculation above leaves no stabilizer multiplicity. Interior family perturbations may be chosen relative to these regular ends.

The low-dimensional compactness and gluing statements of [KM11b, Sections 3.8–3.9 and 5.1] apply. In dimensions zero and one the boundaries are metric endpoints and ordinary Floer breaking; instanton and monopole bubbling have higher codimension. Orient the one I-orientation torsor throughout the concatenated families. In the fiber-first parameter convention the signed parameter count H_W satisfies the strict relation $m_1 - m_0 = dH_W + H_W d$. Insert the reference conversions and unit contraction. Equation (14) introduces no sign; the comparison of I-orientations introduces only a constant sign. Convert to the fixed representatives in (15) and then use (9). With the new endpoint-map parity f, the graded homotopy is $G_W = H_W \varepsilon^f_{\mathrm{H},new}$, including these fixed conversions. This proves (11) and Proposition 3.1.

The same argument for ordinary continuation cylinders makes the construction at every regular end datum natural. Consequently A_p is available for arbitrary small presentations, independently of the additional conditions needed for a common DFL count.

4. Identification of the signed DFL count

4.1. **Choice of the coherent system.** Fix common regular data as in Section 2, of active genus $h \geq 2$. Suppose first that the critical set is nonempty. In the determinant-one cover, a primitive framed path joins a base critical point a_* to its other framed lift and has index four modulo eight. Its image in the framed quotient is the primitive index-four loop. DFL, Section 2.4, orients this path and chooses oriented reference paths from the other generators to a_*. Path gluing and ordinary instanton addition extend the system. We choose both the primitive orientation and reference orientations from J_0 before applying Sections 4.4 and 4.7 of that source. We also fix the translation direction specified in Section 4.3.

The resulting N^{cal} counts DFL's mixed solutions with DFL's determinant transport from gauge paths and constant mixed pairs. Its symplectic orientations are the ones obtained in that construction. This specifies an orientation system independently of the value or inverse of any mixed map.

Choose relative mod-four representatives $j(a) \in \{0,1,2,3\}$ with $j(a_*) = 0$, and reference paths of index $j(a)$, inserting primitive loops when necessary. The corresponding gauge and symplectic degrees agree by [DFL21, Proposition 2.54]. Setting this origin does not change determinant-line parity: the constant difference between the parity and j is retained on both sides. The reference-path bases and the symplectic bases induced by constant mixed pairs give even diagonal orientation-line maps

$$P_p : C_J(p) \longrightarrow C_G^{\mathrm{cal}}(p), \qquad B_p : C_S^{\mathrm{cal}}(p) \longrightarrow C_S(p). \tag{18}$$

They are initially identifications of chain groups; their compatibility with the differentials is proved below.

Proposition 4.1. *The maps (18) are chain isomorphisms and satisfy $B_p N_p^{\mathrm{cal}} P_p = N_p$. This is equality of all signed mixed coefficients, including off-diagonal coefficients. It uses no comparison with the sphere-positive normalization.*

4.2. **A common Fredholm realization.** We first identify operators on the configuration space, because the orientation constructions transport along nonsolutions. In Paper I's common-data coordinates $(t,r) = (\theta,s)$ and $J_D = -J$, the equations agree. Order the target as Coulomb, anti-self-dual, and strip rows, as in [DFL21, (3.16)]. There is a row convention to check. If $\mathcal{J}$ is the strip structure in DFL coordinates, write

$$F(u) = \partial_s u + \mathcal{J}(u)\partial_\theta u, \qquad Q(u) = \partial_\theta u - \mathcal{J}(u)\partial_s u = -\mathcal{J}(u)F(u).$$

Differentiation gives

$$D_Q\xi = -\mathcal{J}D_F\xi - (\nabla_\xi \mathcal{J})F(u). \tag{19}$$

The second term vanishes at solutions, but cannot be dropped along a shifting path. Use Q as the common row. After the permitted preparation of reference configurations to stationary ones outside a compact set, the correction is a compactly supported zeroth-order operator. Rellich compactness makes it compact from the first Sobolev domain to the target. Its action on stationary coordinates has finite rank. Equivalently, before truncation, compact-core Rellich and the vanishing tail give the same conclusion. Scalar interpolation of this correction is a Fredholm homotopy with unchanged asymptotic operators.

The coordinate pullback identifies the other rows and the primal matching condition. Ordinary ends are unweighted; at mixed ends use one common sufficiently small positive weight, with the stationary coordinates and the same cutoff extensions separated from the decaying part. These are the spaces in [DFL21, Definitions 3.8–3.9] and Paper I's common-data construction. Paper II's realized operators and formal adjoints use precisely these matching domains and dual weights. The bounded target isomorphism in (19) transports the adjoint boundary condition; the zeroth-order correction leaves the principal boundary pairing unchanged. No spectral wall is crossed.

The target isomorphism and Fredholm homotopy identify determinant lines on the common space. They also identify ordinary determinant gluing: the section pullbacks conjugate the pregluing maps, and the stabilized corrected kernel maps are homotopic through isomorphisms for sufficiently long necks. At an invertible operator the determinant line has its canonical orientation. The target isomorphism preserves it, and the correction in (19) is zero at a constant reference pair. Thus the comparison preserves the terminal constant calibration. The coordinate notation $J_D = -J$ alone introduces no further complex-conjugation orientation operation.

4.3. **Transport from a common terminal configuration.** The gauge orientation systems agree on each reference path, on the primitive framed path, and on ordinary instanton addition. The path classification and extension rule of [DFL21, Section 2.4, the classification preceding (2.48) and the gluing map (2.55)] therefore make them agree on every gauge path class. Here the quotient adds exactly the single framing class, as specified above; no further lifting class requires a separate comparison.

For any mixed configuration use one finite shifting path in the common special-quintuple configuration space. The relative preparation and shifting results of Paper II keep the matching condition, Lagrangian condition, and stationary coordinates, even when the limiting mixed value moves. They end at a gauge path followed by an invertible constant mixed pair C^c, for the representative c actually reached. A finite path suffices; the orientation comparison requires no limiting infinite translation. Choose the shift beyond the compact support and dependence of the secondary perturbation. Compare the terminal constant operator to its zero-secondary reference through the same small perturbation path. This path stays invertible by Paper II, Proposition 2.1(b), and preserves the canonical orientation.

At this terminal configuration both determinant constructions use the same ordered gauge factor and the same canonical constant factor. If c is the other framed lift, use the actual bundle action on these operators. Conjugation preserves the canonical orientation of the invertible constant factor, and the calibrated gauge system accounts for the trajectory factor. Replacing that action by an arbitrarily declared positive identification would not be sufficient.

The ratio of the two orientations along the common finite path is continuous and takes values in $\{1,-1\}$. It is $+1$ at the terminal configuration, hence at the original one. This proves the determinant comparison for every mixed component. The relative shifting statements in the two constructions ensure independence of the chosen preparation and shift. Alternatively, one may compare both on the single common path just described; their individually defined orientations already have that independence.

For strips glue a constant pair at the input and cancel its canonically oriented factor. This proves agreement of strip orientations. We use DFL's fiber-first convention for the quotient by translation [DFL21, p. 17]. In both constructions we choose the positive translation vector to be $\partial_s u$ in DFL coordinates, equivalently $\partial_r u$ in the original coordinates, in agreement with Paper II's convention. This choice of direction is part of our gauge-calibrated system. Gauge trajectories use the same increasing cylindrical coordinate. Therefore the rigid trajectory and strip counts give the same differentials under (18). The isolated mixed counts give exactly (2), proving Proposition 4.1.

Changing a reference-path orientation changes the corresponding gauge and symplectic bases together. With P_p and B_p changed accordingly, the identity remains the same orientation-line statement. It does not assert numerical basis independence of a matrix. If the critical set is empty, all the complexes and maps in this comparison are zero and no base critical point is chosen. Nonempty acyclic complexes require no alteration of the determinant argument.

5. Presentation changes and the calibrated comparison

5.1. **Common regular data in the small ball.** Every permitted splitting and metric admits common regular data inside Paper II's small ball. Indeed, choose the transverse separated primary datum of Paper I, Theorem 3.6, with norm less than $r_\chi/2$. Choose its DFL-regularizing split addition smaller than the remaining margin. The sum is admissible and has norm less than r_χ. Include F_0 in the ambient finite family after constructing this datum. This changes neither the function nor its norm; separatedness is needed at the stage where the existence theorem is applied, not as a new condition on an unused enlarged family. Use zero additional gauge-end perturbation, DFL-compatible almost-complex data and the small invariant secondary data of Paper I, Proposition 3.11. These are among the allowed data of Paper II. All presentations considered here already have $g \geq 1$ and $h \geq 2$. Every object admits such a presentation, including by stabilizing an initial genus-zero splitting.

For an arbitrary small presentation put

$$X_p = N_p A_p : C_{\mathrm{H}}(p) \longrightarrow C_S(p). \tag{20}$$

This is an even chain homotopy equivalence. At common data Proposition 4.1 gives

$$\Psi_p = B_p N_p^{\mathrm{cal}} P_p A_p = X_p. \tag{21}$$

Thus X_p is available for verifying relations at all small data, whereas (21) identifies its actual DFL representative when the data are common.

5.2. **The geometric transition maps.** Let d be a presentation word from q to p representing the decorated identity. Write

$$G_{pq} : C_J(q) \longrightarrow C_J(p), \qquad \Gamma_{pq} : C_S(q) \longrightarrow C_S(p)$$

for the gauge cylinder and Paper II's actual geometric word. Its elementary maps include changes of disk systems, primary perturbations, regularizing and strip-end data, the fixed-reference conversions, decorated isomorphisms, stabilization, metric and splitting changes.

Stabilization uses the canceling one/two-handle pair with its small primary continuations; the reverse uses the two/three-handle pair. Common stabilization treats metric changes. Paper II, Section 14, proves the mixed squares and descent for these maps. In particular Γ_{pq} is not defined using a mixed-map inverse.

With fixed object references these identity-presentation maps are even. For chosen representatives and signs,

$$N_p G_{pq} - t\Gamma_{pq} N_q = \delta L_{pq}, \tag{22}$$

$$G_{pq} G_{qp} - u1 = \delta R_p, \qquad t, u \in \{1, -1\}. \tag{23}$$

Here G_{qp} comes from a reverse identity word. The inverse, composition and word-independence homotopies for Γ are those of Paper II. Its cancellation of a mixed equivalence is valid over $\mathbb{Z}$: if $FN = \delta H$, $NV - 1 = \delta T$, and F is homogeneous of parity f, then

$$F = \delta\big(HV - (-1)^f FT\big). \tag{24}$$

This follows directly from the graded Leibniz rule. Apply it componentwise if required. An inverse V and its homotopy come from the finite free cone argument. Such an inverse verifies a relation between already constructed geometric maps; it does not define those maps. In particular equality of induced maps on homology is never used as a substitute for a chain homotopy.

For two common presentations q, r, the actual calibrated DFL complexes inherit geometric transitions by the diagonal line conversions

$$G^{\mathrm{cal}}_{rq} = P_r G_{rq} P_q^{-1}, \qquad \Gamma^{\mathrm{cal}}_{rq} = B_r^{-1} \Gamma_{rq} B_q.$$

The inverses here are only those of orientation-line isomorphisms. Substitution in (22) gives

$$N^{\mathrm{cal}}_r G^{\mathrm{cal}}_{rq} - t\Gamma^{\mathrm{cal}}_{rq} N^{\mathrm{cal}}_q = \delta(B_r^{-1} L_{rq} P_q^{-1}). \tag{25}$$

Converting the corresponding geometric homotopies gives composition, inverse and word independence. Intermediate data in the word need not be common DFL data, and the argument does not follow individual critical points through births.

5.3. **Transport to arbitrary small data.** Choose common data q, an identity word from q to p, and the usual gauge cylinder $H_{qp} : C_{\mathrm{H}}(p) \to C_{\mathrm{H}}(q)$. Define

$$\widehat{\Psi}_{p;q,d} = \Gamma_{pq} \Psi_q H_{qp}. \tag{26}$$

This uses an actual DFL count at q, the geometric excision map, ordinary gauge continuation and the geometric symplectic word. It is the asserted extension, with no claim about an original DFL moduli problem at arbitrary p.

Gauge naturality for this cylinder gives

$$A_q H_{qp} - sG_{qp} A_p = \delta K.$$

Using (22) and (23) in succession shows

$$\widehat{\Psi}_{p;q,d} - stuX_p = \delta E_{p;q,d}, \tag{27}$$

where the explicit integral homotopy is

$$E_{p;q,d} = \Gamma_{pq} N_q K - stL_{pq} G_{qp} A_p + stN_p R_p A_p. \tag{28}$$

All outer maps in this expression are even. Applying δ to the first term substitutes gauge continuation; the second substitutes the mixed square, and the third removes the reverse-word gauge composite. This verifies all signs in (27).

Since X_p is a homotopy equivalence, so is $\widehat{\Psi}_{p;q,d}$. For two choices write their signs and homotopies as e, E and e', E'. Subtraction gives

$$\widehat{\Psi} - ee'\widehat{\Psi}' = \delta(E - ee'E').$$

Thus independence is integral chain-homotopy independence up to one sign. It includes changes of splitting and metric. At a contractible object the same equations hold with Paper II's zero representatives and the contractions of Section 2.3; no geometric count is asserted for that zero convention.

5.4. **Cobordism naturality.** At arbitrary small endpoints, combine (11) and (6). Their even comparison factors give

$$X_1 F^{\mathrm{r}}_{\mathrm{H}}(W) - vS_d(W)X_0 = \delta J_W, \quad v = s_W t_W, \quad J_W = N_1 G_W + s_W L_W A_0. \tag{29}$$

This is already the desired formula at common data. For arbitrary chosen extensions use $\widehat{\Psi}_i - e_i X_i = \delta E_i$ from (27), and put $w = e_1 v e_0$. If $S = S_{\mathrm{even}} + S_{\mathrm{odd}}$ is a possibly inhomogeneous closed representative, define $S^{\varepsilon} = S_{\mathrm{even}} - S_{\mathrm{odd}}$. Then $\delta(S^{\varepsilon}E) = S\delta E$, so

$$\widehat{\Psi}_1 F^{\mathrm{r}}_{\mathrm{H}}(W) - wS_d(W)\widehat{\Psi}_0 = \delta\big(e_1 J_W + E_1 F^{\mathrm{r}}_{\mathrm{H}}(W) - wS_d(W)^{\varepsilon}E_0\big). \tag{30}$$

For a homogeneous S of parity f, $S^{\varepsilon} = (-1)^f S$. This formula retains the possible inhomogeneous long words of Paper II while the usual gauge map has its specified parity. Changing the small decomposition changes S_d by its projective homotopy, which composes with this formula. The subgroup, bundle and framing arguments in Proposition 3.1 apply to every morphism in the stated category. All zero-homology cases were included in the construction. This proves Theorem A.

6. Changing the primitive loop orientation

6.1. **The change in the differential.** Fix common data and relative grading representatives $j(a) \in \{0, 1, 2, 3\}$ as in Section 4. Match the reference-path orientations in the calibrated and positive systems. Let $\sigma \in \{1, -1\}$ be the ratio of the two primitive index-four orientations. A primitive path initially of index $8k+4$ is normalized to index four using the common index-eight addition convention. The value of σ is fixed before any mixed count; the argument works for either value.

By the path classification preceding (2.48) in [DFL21, Section 2.4], a path of index m from a to b has orientation ratio

$$\sigma^{(m-j(a)+j(b))/4}. \tag{31}$$

Removing the reference paths leaves a multiple of the primitive loop, and ordinary instanton addition has the same sign in the two systems. The shifting construction of [DFL21, Sections 4.4 and 4.7] gives the same formula on mixed configurations, since the terminal constant pair is canonically oriented in both systems. Gluing a constant pair and cancelling its determinant line gives the strip formula. An isolated mixed solution has $m = 0$ and $j(a) = j(b)$, so its coefficient is unchanged. A differential coefficient has $m = 1$; for $\sigma = -1$ its sign changes exactly from relative degree zero to degree three.

For a calibrated complex C, write TC for the same graded group with differential

$$d_{TC}|_{C_j} = s_j d_C|_{C_j}, \qquad s_0 = -1, \quad s_1 = s_2 = s_3 = 1. \tag{32}$$

This squares to zero and has the same kernels and images as d_C. In matched references the positive complex is TC if $\sigma = -1$ and is C if $\sigma = 1$. The degree-zero mixed matrix is unchanged, but in the first case its source and target differentials are twisted.

6.2. **Degrees, parity and homotopies.** The relative mod-four grading and determinant parity are kept as separate gradings. Write $C_{j,e}$ for their simultaneous pieces. The differential has relative degree -1 and parity one; the Hom differential δ always uses parity. This distinction is necessary when summands with different reference shifts occur in a mapping cone.

For an integer lift k of a relative degree, put

$$n(j,k) = \frac{j+k-[j+k]_4}{4}, \qquad T_k(F)|_{C_j} = (-1)^{n(j,k)}F|_{C_j}.$$

Here $[\,\cdot\,]_4$ denotes the representative in $\{0,1,2,3\}$, and $T_k(F)$ is read between the twisted complexes. The identity $n(j,k)+n([j+k]_4,\ell) = n(j,k+\ell)$ gives

$$\delta T_k(F) = T_{k-1}(\delta F), \qquad T_\ell(G)T_k(F) = T_{k+\ell}(GF). \tag{33}$$

Indeed, the two terms in the Hom differential have the same weight, while their parity sign is unchanged. In particular T_0 leaves a degree-zero matrix unchanged and T_1 changes the sign of a degree-one homotopy on degree-three inputs. Changing the lift k to $k+4$ multiplies the whole map by -1.

A formal shift $C[b;t]$ has labelling map s of relative degree b and parity t, and differential $(-1)^t s d_C s^{-1}$. Simultaneously shifting both ends of an even degree-zero map conjugates it by the labels. For an odd degree-one homotopy h, the corresponding homotopy is $(-1)^t shs^{-1}$. We write $\Sigma C = C[1;1]$. If $f : A \to B$ is closed, even and of relative degree zero, then

$$\mathrm{Cone}(f) = B \oplus \Sigma A, \qquad d_{\mathrm{Cone}(f)} = \begin{pmatrix} d_B & fs^{-1} \\ 0 & -sd_As^{-1} \end{pmatrix}. \tag{34}$$

In block formulas below we suppress these suspension labels; every upper-right entry is followed by the inverse source label.

Lemma 6.1. *Every finite free bigraded integral complex P admits a bigrading-preserving involution A of its underlying group such that $d_{TP}A = Ad_P$.*

Proof. For each (j,e), the cycles $Z_{j,e}$ form a direct summand of $P_{j,e}$: the quotient is isomorphic to the free subgroup $\mathrm{im}(d|_{P_{j,e}})$ of the next chain group. Choose $P_{j,e} = Z_{j,e} \oplus L_{j,e}$ and let A fix $Z_{j,e}$ and multiply $L_{j,e}$ by s_j. Since $d_P(P_{j,e})$ consists of cycles, $d_{TP}A = s_j^2 d_P = d_P = Ad_P$. Also $A^2 = 1$. □

This lemma explains why there is no objectwise algebraic obstruction. It supplies neither geometric factors for A nor compatibility with prescribed maps. We next construct actual count expressions in which a matrix of this kind can be used.

7. The ordinary surgery triangle with its unit homotopy

7.1. **Bundles and analytic inputs.** A surgery triad consists of three fillings Y_0, Y_1, Y_2 of a connected torus-boundary manifold Z along slopes with pairwise geometric intersection one. Exactly two filling cores vanish in $H_1(Y_i;\mathbb{Z}/2)$. To see this, the kernel of

$$H_1(\partial Z;\mathbb{Z}/2) \longrightarrow H_1(Z;\mathbb{Z}/2)$$

is a one-dimensional Lagrangian subspace. The three slopes are the three distinct nonzero boundary classes. For a slope outside the kernel, filling kills the image of the boundary group, so the core vanishes; for the slope in the kernel, a complementary slope survives and represents the core.

Use the bundle triad of [Sca15, Sections 3.1–3.7 and 7.5], putting its meridian decoration at one of the two nullhomologous cores and taking zero exterior decoration. All three endpoint bundles are then trivial on the original manifolds. Adjoin the same odd torus outside the surgery region and retain the framed arcs, endpoint trivializations and relative cobordism bundles. For a bundle E on the original cobordism W, before the odd torus is inserted, its endpoint trivializations give $w_2(E)|_{\partial W} = 0$. Define

$$P(E) = \langle \widetilde{w}_2(E) \smile \widetilde{w}_2(E), [W, \partial W] \rangle \in \mathbb{Z}/2,$$

where $\widetilde{w}_2(E) \in H^2(W, \partial W; \mathbb{Z}/2)$ is a relative lift. This value is independent of the lift: their difference maps to zero in absolute cohomology, so its square pairs trivially, and the cross terms cancel modulo two. At least one relative bundle in the triangle has odd $P(E)$; its degree contribution is retained. The bundle-gluing parameter of [Sca15, (3)] has cube one. Use that periodic identification, fixed on the common exterior, torus and ball, for the third return to the original marked object.

We use the ordinary compactness, regularity and gluing for the zero-, one- and two-parameter surgery families of [Sca15, Sections 4.4–5.3], with nondegenerate admissible ends and perturbations chosen relative to their broken faces. The boundary and gluing formulas for these parameterized families use [KM11b, Section 3.9]. At the reducible cut $T = S^1 \times S^2$, we use the Morse–Bott fiber-product and stabilizer description and the oriented count comparison of [Sca15, Lemma 5.4 and the proof of Lemma 5.2]. The determinant factorization includes both stabilizer and moving limit terms. Its compatibility with adjoining external trajectories and reference operators is the ordinary determinant-gluing associativity of [KM11b, Sections 3.8–3.9] and Paper II. This compatibility is retained as an oriented gluing input; it is not quoted as a separately stated theorem of [Sca15]. A gluing theorem restricted to non-integral cuts alone would not supply this reducible-cut input.

The author's errata [ScaE] are respected. For item 2, the original ends become nontrivial admissible bundles after adjoining the odd torus, and the families here have at most two parameters. For item 5, in the local piece E of Lemma 5.5, $H_2(E)$ has rank two; the image of relative cohomology carrying the intersection form has rank one. All counts below use the fixed almost-complex endpoint references. An I-orientation changes a whole cobordism count by one sign, rather than assigning independent signs to its coefficients.

7.2. **Interval and pentagon relations.** Let C_i be the three calibrated gauge complexes, with indices read modulo three. The actual graded surgery maps $F_i : C_i \to C_{i+1}$ have relative degrees a_i and parities p_i. The triple composite returns to the same marked endpoint and has $\chi = 3$, signature -1 and $P(E) = 1$ by [Sca15, Section 7.5]. The degree formula of [Sca15, Proposition 7.1, (29)] therefore gives

$$\delta F_i = 0, \qquad \sum_i a_i = -1 \pmod 4, \qquad \sum_i p_i = 1 \pmod 2. \tag{35}$$

The fixed endpoint grading constants telescope. Parity is computed with those constants, independently of the displayed relative degrees.

The interval of metrics on the double composite has an ordinary breaking end and an exceptional-sphere end. The latter has no relevant solutions by [Sca15, Section 5.2], using the nontrivial bundle on the punctured negative projective plane and the index bound. Orient its zero-dimensional parameter count H_i so that

$$\delta H_i = F_{i+1} F_i. \tag{36}$$

It has relative degree $a_i + a_{i+1} + 1$ and parity $p_i + p_{i+1} + 1$. For the pentagon on the triple composite, the two exceptional faces again vanish. The two ordinary breaking faces give

$$O_i = F_{i+2}H_i - (-1)^{p_{i+2}}H_{i+1}F_i. \tag{37}$$

The relative sign can be checked at their common two-break corner. Interchanging the two outward neck normals changes sign; moving the interval boundary past F_{i+2} contributes $(-1)^{p_{i+2}}$. This gives the coefficients in (37) before taking counts. Equivalently,

$$\delta(F_{i+2}H_i) = (-1)^{p_{i+2}}F_{i+2}F_{i+1}F_i, \qquad \delta(H_{i+1}F_i) = F_{i+2}F_{i+1}F_i.$$

Thus O_i is closed, even and of relative degree zero. Writing B_i for the remaining face with its induced sign, the pentagon count K_i is odd of relative degree one and satisfies

$$\delta K_i = O_i - B_i. \tag{38}$$

These identities use the graded determinant convention throughout. They do not import an unsigned strict matrix as a graded map.

7.3. **Comparison of the local unit faces.** The remaining cut has trivial bundle on $T = S^1 \times S^2$. Its even-gauge flat moduli space, after a spin lift, is the interval $I = [-1, 1]$, with the non-even gauge action given by reflection [Sca15, Lemma 5.3 and the preceding discussion]. Write I_0 for its noncentral interior. The triple composite is $V \cup_T E$, while the corresponding cut cylinder is $V \cup_T W$, where $W = S^1 \times D^3$. Keep the same exterior V, its perturbations, and the actual endpoint reference lines in both decompositions.

The index exclusions in [Sca15, Lemma 5.4 and proof of Lemma 5.2] keep all relevant limits in I_0 and all E metrics in the interior of their parameter interval. The external moduli space has dimension zero after allowing its limit to vary in I_0. The local E and W spaces both have dimension one. More explicitly, their fixed-limit indices, stabilizer dimensions, metric dimensions and moving-limit dimensions are respectively

	μ	stabilizer	metrics	limit	dimension
V	-1	0	0	1	0
E	-2	1	1	1	1
W	-1	1	0	1	1

At a noncentral limit, the stabilizers on E, W and T are circles and restriction identifies them. Thus both gluing quotients have one point; no residual circle or multiplicity two occurs.

The oriented gluing input gives the augmented determinant factorizations, in the same order,

$$\begin{aligned} o(\mathcal{M}_V \times_{I_0} \mathcal{M}_E) &\simeq o(\mathcal{M}_V) \otimes o(\mathcal{M}_E) \otimes o(I_0)^*, \\ o(\mathcal{M}_V \times_{I_0} \mathcal{M}_W) &\simeq o(\mathcal{M}_V) \otimes o(\mathcal{M}_W) \otimes o(I_0)^*. \end{aligned} \tag{39}$$

The local moduli factors are both odd and are followed by the same odd matching factor. Their comparison is an even local line. Orient it once. The interval has no monodromy, so the comparison extends over all regular local configurations. Gluing associativity retains these factorizations after adjoining reference operators, external trajectories or another relative class of V. The external factors cancel in the same order, and any Koszul reordering gives the same sign on the two sides. Hence their ratio is independent of both endpoint generators and the external relative class. No almost-complex structure common to E and W is required for this comparison of determinant factors.

The limit map $\mathcal{M}_E \to I$ has degree ± 1 by [Sca15, Lemmas 5.5–5.7]; the map $\mathcal{M}_W \to I$ is a homeomorphism. Summing the oriented preimages of each regular external limit in (39)

therefore gives $B_i = \eta_i C_{\mathrm{cut},i}$ for one sign η_i. The cylinder neck family from the product metric to this cut supplies an actual count J_i with $\delta J_i = C_{\mathrm{cut},i} - 1$. Consequently

$$O_i - \eta_i 1 = \delta L_i, \qquad L_i = K_i + \eta_i J_i. \tag{40}$$

This is a unit relation with the prescribed endpoint identity. In particular the unit homotopy is made from actual families.

7.4. **The framing quotient.** The added torus lies in the common external region V. Its single lifting generator extends over every family by the identity near the surgery handles, and restricts trivially to E, W and T. It acts only on the common external factor in (39). The local comparison is therefore equivariant. Perturbations and their relative extensions are chosen on the quotient configuration spaces; equivalently they are invariant upstairs.

The torus generator shifts a critical point by four modulo eight in the determinant-one theory [Sca15, Section 7.1], so it fixes no critical orbit there. A fixed finite-energy solution on a connected piece would have a fixed limiting critical orbit. Every piece with nontrivial action has such an original end; the action on the local pieces is the identity. It follows that the relevant solution-space actions are free and their gluing charts descend. One can therefore count the boundary identities directly on the quotient. No coinvariant exactness or division by two is used. In particular the local interval is not quotiented by reflection: the non-even gauge transformation on T used in the local degree calculation is not part of the framing subgroup.

7.5. **Actual maps to a cone.** Take $i = 0$, let s_1 label $C_1[a_1 + 1; p_1 + 1]$, and let s_0 label $D = C_0[-a_2; p_2]$. Set

$$\mathcal{C} = C_2 \oplus s_1 C_1, \qquad d_{\mathcal{C}} = \begin{pmatrix} d_2 & F_1 s_1^{-1} \\ 0 & (-1)^{p_1+1} s_1 d_1 s_1^{-1} \end{pmatrix}. \tag{41}$$

The count maps

$$P = \begin{pmatrix} -H_0 \\ s_1 F_0 \end{pmatrix} s_0^{-1}, \qquad Q = s_0 \big(F_2, (-1)^{p_2} H_1 s_1^{-1} \big) \tag{42}$$

are closed, even and of relative degree zero. The upper equation for P is $-\delta H_0 + F_1 F_0 = 0$; the lower uses $p_0 + p_1 + 1 = p_2$. The off-diagonal equation for Q is $\delta H_1 - F_2 F_1 = 0$ with the displayed shift signs. Their product is $QP = -s_0 O_0 s_0^{-1}$. Put

$$\widehat{Q} = -\eta_0 Q, \qquad h_D = \eta_0 (-1)^{p_2} s_0 L_0 s_0^{-1}. \tag{43}$$

Then $\delta h_D = \widehat{Q} P - 1_D$. The homotopy is an actual count expression, not a homotopy chosen from a cycle splitting.

Lemma 7.1. *The map $P : D \to \mathcal{C}$ is a quasi-isomorphism. Hence $\widehat{Q}$ is a homotopy inverse over $\mathbb{Z}$.*

Proof. All three cyclic unit relations (40) are needed. For the detection argument, temporarily suppress the relative grading and choose periodic parity shifts c_i with $c_{i+1} - c_i = 1 - p_i$ and $c_2 = 0$. They exist because the cyclic sum is even. If E_i are the labels, the converted maps

$$F_i' = E_{i+1} F_i E_i^{-1}, \qquad H_i' = (-1)^{c_{i+2}} E_{i+2} H_i E_i^{-1}$$

are odd, $\delta H_i' = F_{i+1}' F_i'$, and

$$F_{i+2}' H_i' + H_{i+1}' F_i' = (-1)^{c_{i+2}} E_i O_i E_i^{-1}.$$

Their diagonal differentials are periodic. The resulting cone and source are evenly chain-isomorphic to (41) and D, with maps (42).

Suppress primes and labels in this all-odd convention. The cone differential is $\left(\begin{smallmatrix} d_2 & F_1 \\ 0 & d_1 \end{smallmatrix}\right)$, the source differential is $-d_0$, and $P = (-H_0, F_0)^{\mathsf{t}}$, $Q = (F_2, -H_1)$. Define

$$W = \begin{pmatrix} O_2 & -H_0H_1 \\ 0 & O_1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 0 \\ H_2 & 0 \end{pmatrix}.$$

Direct multiplication using (36) gives $PQ - \delta Z = -W$. Thus W is closed. On the two quotients of the cone filtration it acts by the homotopy units O_2, O_1, so it is a homology isomorphism. Both PQ and QP are now homology isomorphisms; the first gives surjectivity of $H(P)$ and the second injectivity. Undoing the parity labels proves that the original P is a quasi-isomorphism. Its acyclic finite free cone is contractible by Section 2.3. The relation $\delta h_D = \widehat{Q}P - 1$ identifies the actual $\widehat{Q}$ with a homotopy inverse. A homotopy on the cone side may be chosen algebraically; it is not used as a defining factor below. □

8. Finite gauge models built from the sphere

The useful output of a surgery triangle is more than an abstract cone equivalence. It includes the actual left-unit homotopy in (43). Keeping this homotopy through successive triangles ensures that every component of the final model maps still has an expression in actual counts.

8.1. **Propagation of the actual unit.** Suppose an actual gauge complex C and a finite formal complex P are equipped with count expressions

$$i : C \longrightarrow P, \qquad p : P \longrightarrow C, \qquad \delta h = pi - 1_C, \tag{44}$$

where i, p are closed, even and of relative degree zero, h is odd of relative degree one, and i is a quasi-isomorphism. The reverse unit $ip \simeq 1_P$ follows by finite free algebra, but only the actual homotopy h is used in constructing further maps. Successive such data compose by

$$i = i_2 i_1, \qquad p = p_1 p_2, \qquad h = h_1 + p_1 h_2 i_1. \tag{45}$$

The boundary of the last expression is $p_1 p_2 i_2 i_1 - 1$. Ordinary continuation, its reverse and its concatenation and path-contraction families give data of this form. Marked pullbacks and their actual inverses are included when transporting between occurrences of the same manifold.

Let (i_A, p_A, h_A) and (i_B, p_B, h_B) be two such choices, and let $f : C_A \to C_B$ be an actual attaching map. Shift its source by its relative degree and parity so that f is even of relative degree zero. Put

$$m = i_B f p_A, \qquad C_f = \mathrm{Cone}(f), \qquad E_m = \mathrm{Cone}(m).$$

With the suspension convention of (34), set

$$I = \begin{pmatrix} i_B & -i_B f h_A \\ 0 & i_A \end{pmatrix} : C_f \to E_m, \qquad J = \begin{pmatrix} p_B & h_B f p_A \\ 0 & p_A \end{pmatrix} : E_m \to C_f,$$

$$H = \begin{pmatrix} h_B & -h_B f h_A \\ 0 & -h_A \end{pmatrix} : C_f \to C_f. \tag{46}$$

Here I, J are closed and even; H is odd of relative degree one. The off-diagonal chain equations are

$$\delta(-i_B f h_A) = i_B f - m i_A, \qquad \delta(h_B f p_A) = p_B m - f p_A.$$

More importantly,

$$\delta H = JI - 1_{C_f}. \tag{47}$$

For example, with $q_X = p_X i_X$, its upper-right component is

$$\delta(-h_B f h_A) - f h_A + h_B f = -q_B f h_A + h_B f q_A,$$

which is the corresponding component of $JI - 1$. The other components give the two original unit equations; suspension accounts for the negative lower homotopy. The two filtration quotients of I are i_B and the shifted i_A, so I is a quasi-isomorphism.

Apply this to the surgery cone for a third complex D, with $P_D, \widehat{Q}_D, h_D$ from (42)–(43). The new maps and actual homotopy are

$$i_D = IP_D, \qquad p_D = \widehat{Q}_D J, \qquad h_D^{\mathrm{new}} = h_D + \widehat{Q}_D H P_D. \tag{48}$$

They satisfy (44), and i_D is a quasi-isomorphism. Undo the simultaneous shifts on D and its model with the sign rule of Section 6. Every entry in (48) is a finite sum of products of actual factors from the children and the surgery triangle. In particular no algebraic reverse-unit homotopy enters this induction.

8.2. **A finite surgery derivation.** Ni proves that S^3 generates all closed connected oriented three-manifolds under surgery triads [Ni25, Theorem 1.4]. The statement was recorded without proof in [Cul14, Proposition 1]. We use Ni's theorem as a topological input. The relevant notion is closure of a set of oriented diffeomorphism classes under the two-of-three rule, with the two given classes allowed to coincide; it is not generation of a group by surgery relations. Ni's triad consists of a manifold and the 0- and 1-surgeries on a framed knot [Ni25, Definitions 1.1–1.3]. Its filling slopes are the meridian, a framing longitude and their sum. They have pairwise intersection number one in absolute value and form a surgery triangle of the kind used here.

Here is the finite consequence required by the construction. Let $\mathcal{S}_0 = \{S^3\}$ and let $\mathcal{S}_{n+1}$ add every third filling whose other two occurrences lie in $\mathcal{S}_n$. The union of the $\mathcal{S}_n$ is closed under the rule: two known occurrences belong to a common finite stage, so the third belongs to the next stage. Thus the union is the full set in Ni's theorem. Recursively decreasing the least possible stage gives a finite binary derivation tree for every M. Repeated diffeomorphism classes are allowed, with their filling occurrences kept distinct. No effective bound on the size of this tree is required.

At each vertex use the endpoint bundle choice of Section 7.1, keeping the odd torus and framed arc outside the surgery region. Choose marked diffeomorphisms and bundle identifications to the required child occurrences. Their pullbacks, inverses, and ordinary continuation units transport (44) to the local triangle data by (45). A nontrivial endpoint bundle automorphism is retained as its actual map, not replaced by an identity. This involves finitely many choices; it does not require one common perturbation for every possible surgery tree.

8.3. **The one-generator sphere datum.** Fix one actual minimal gauge datum D for S^3 with its reference line and relative grading origin. The odd-torus calculation of [Sca15, Section 7.6] gives one nondegenerate critical orbit in the framing quotient. We recall the elementary holonomy calculation: the two anticommuting lifted torus holonomies can be fixed to the unit quaternions i, j, and the remaining holonomy is ± 1; the framing action interchanges these choices. There is no deformation after quotienting. On the original sphere there is no additional holonomy. Thus the critical set consists of one point. The transverse small-data construction of Paper I preserves it, and the trajectory regularizer vanishes near it.

The complex has rank one and zero differential. Its marked identity cylinder is the canonical identity of its orientation line, with coefficient $+1$.

At every leaf use a labelled copy of this same fixed datum, with the actual marked pullbacks and continuation maps to the chosen occurrence. The iteration above proves the following statement.

Proposition 8.1. *Under the inputs of Section 7.1, every permitted actual calibrated gauge complex C has a finite formal iterated-cone model P and count expressions i, p, h satisfying (44), with i a quasi-isomorphism. As a bigraded group, P has supplied coordinates*

$$P = \bigoplus_{\alpha=1}^{r} D[b_\alpha; t_\alpha]. \tag{49}$$

Every row of i and every column of p, in these coordinates, is a finite sum of actual count products with one external end at the common sphere datum D.

Proof. The finite tree provides the induction. At its leaves, continuation and marked transport provide (44). At an internal vertex use (46)–(48). Quasi-isomorphisms and the actual left units are preserved at each step. Expanding a row or column gives a finite sum of products of child components, child units and actual attaching counts. The same is true of each new unit homotopy, so the induction also proves the asserted component expansions. The shifts are carried by their labels throughout. □

The direct sum in (49) is a decomposition of groups. Its differential contains the iterated-cone entries; it is not a direct sum of sphere complexes. The model need not be the critical complex of another Floer datum. Acyclic intermediate complexes and summands are retained, since the construction uses their actual maps and units.

9. Normalization by counts through the sphere

We now use the supplied coordinates of the complete model. There is no need to extend previously chosen normalizations across its individual attaching maps. This permits the whole model to be normalized even when those separate extension problems have no solution.

9.1. **Realizing the matrix units.** Let the generator line of D have relative grade j_* and parity e_*, and write U_α for its shift label in (49). The final grade and parity of this leaf are $[j_* + b_\alpha]_4$ and $e_* + t_\alpha$. If i_α and p_β denote the corresponding row and column, define

$$r_\alpha = U_\alpha^{-1} i_\alpha : C \longrightarrow D, \qquad c_\beta = p_\beta U_\beta : D \longrightarrow C. \tag{50}$$

Their relative degrees are $-b_\alpha, b_\beta$ and their parities are t_α, t_β. They need not be closed individually. Proposition 8.1 gives their actual count expansions. Selecting one formal component selects the corresponding expansion; all intermediate Floer generators in that expansion are still summed over. If a representative word is inhomogeneous, its required component is selected by a zero-one endpoint grading and parity weight.

For leaves with equal final grade and parity, let $E_{\beta\alpha}$ be the matrix unit given by the identity of the common sphere line in that position. Read it either from P to TP or in the reverse direction. Since T_0 has weight one, the underlying matrices of

$$T_0(p) E_{\beta\alpha} i, \qquad p E_{\beta\alpha} T_0(i) \tag{51}$$

are both

$$c_\beta \, 1_D \, r_\alpha. \tag{52}$$

The middle factor is the actual sphere identity cylinder. Expanding the two outer factors therefore expresses each map in (51) as a finite sum of permitted count products. The only new join is at the same fixed rank-one datum D, where no choice of a generator projection is involved.

The integer lifts in this identification must be kept. If the output column is instead read through TD, put $\epsilon_\beta = (-1)^{n(j_*,b_\beta)}$. The expression is

$$\epsilon_\beta T_{b_\beta}(c_\beta) 1_{D\to TD} r_\alpha. \tag{53}$$

The twist contributes ϵ_β on the sole sphere input, and the prefactor cancels it. Equal final grades imply $b_\beta - b_\alpha = 4k$, not equality of the integer lifts. The word (52) naturally has lift $4k$, whereas the complete maps in (51) have chosen lift zero. If one twists the word, the conversion is $T_{4k}(F) = (-1)^k T_0(F)$. No grading origin has been changed, and no index-four sphere trajectory is used.

Lemma 9.1. *For every bigrading-preserving integral matrix A in the supplied leaf coordinates of P, both $T_0(p)Ai$ and $pAT_0(i)$ are weighted count expressions, with their respective endpoint lines. This assertion does not require A to be closed.*

Proof. Expand $A = \sum_{\alpha,\beta} A_{\beta\alpha} E_{\beta\alpha}$ over matching grades and parities, and use (51)–(52) for each summand. There are finitely many leaves and finitely many words in every component. Each integer entry is a fixed signed multiplicity of an already specified word. □

For example, let A be supported on two leaves of matching bigrading, with entries $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$. Its realization is

$$\begin{aligned} T_0(p)Ai = {} & \lambda_{11} c_1 1_D r_1 + \lambda_{12} c_1 1_D r_2 \\ & + \lambda_{21} c_2 1_D r_1 + \lambda_{22} c_2 1_D r_2. \end{aligned}$$

Each summand passes through the same sphere line, although the corresponding leaves occupy different positions in the formal model. Individual summands need not be chain maps. Closedness will be imposed on their full sum by the choice of A.

9.2. **Selecting the coefficients.** Fix an order of the leaves. Enumerate bigrading-preserving integral matrices in increasing maximum absolute entry, and lexicographically within each finite set. Choose the first matrix satisfying

$$d_{TP} A = A d_P, \qquad A^2 = 1. \tag{54}$$

These are finite exact integer equations. The search terminates by Lemma 6.1, whose splitting proof supplies an integral solution. Read the same matrix as $B : TP \to P$. Multiplying the first equation by A on both sides shows that B is the inverse chain map.

This is an algebraic, noncanonical choice of coefficients. Lemma 9.1 is what makes it useful here: every candidate already has a count realization before its equations are tested. A cycle complement proves existence of a candidate, but is not inserted into a defining word. Neither a bound on these coefficients nor a canonical surgery tree is asserted.

9.3. **A two-generator example.** Consider the finite mod-four graded complex

$$P_0 = \mathbb{Z}x, \qquad P_3 = \mathbb{Z}y, \qquad P_1 = P_2 = 0, \qquad d_P x = 2y, \qquad d_P y = 0,$$

with x even and y odd. The twist changes precisely the degree-zero-to-degree-three component, so $d_{TP} x = -2y$. The bigrading-preserving involution

$$Ax = -x, \qquad Ay = y$$

satisfies $d_{TP}A = Ad_P$: both sides send x to $2y$ and y to zero. Thus $A : P \to TP$ is an integral chain isomorphism, with the same matrix for its inverse. In particular the example does not require torsion-free homology: $H_3(P) = \mathbb{Z}/2$.

This is a formal model calculation, not a computation of the Floer complex of a specified manifold. If P occurs with the supplied sphere coordinates and maps i, p of Proposition 8.1, its two diagonal entries give

$$T_0(p)Ai = -c_x\, 1_D\, r_x + c_y\, 1_D\, r_y. \tag{55}$$

The row and column labels absorb the different degrees and parities of x and y; each term joins the corresponding row and column through the same physical sphere datum. The reverse expression $pAT_0(i)$ has the same underlying matrix, with its opposite endpoint orientation lines. The equation $d_{TP}A = Ad_P$ makes the full expressions closed, even though their individual summands need not be closed. The following proposition verifies their inverse homotopies in general.

Proposition 9.2. *For every permitted actual calibrated gauge complex C, there are even relative-degree-zero weighted count expressions $u_C : C \to TC$ and $v_C : TC \to C$ which are mutually inverse up to integral chain homotopy.*

Proof. For the selected matrix set

$$u_C = T_0(p)Ai, \qquad v_C = pBT_0(i). \tag{56}$$

Lemma 9.1 gives their count expressions, and (54) gives their chain equations. For verification, choose an odd degree-one integral homotopy K with $\delta K = ip - 1_P$. It exists because the finite free cone of the quasi-isomorphism i is contractible, and $\delta h = pi - 1_C$ identifies p with a homotopy inverse. The splittings for this verification can be made separately in both gradings. Define

$$\begin{aligned} a_C &= h + pBT_1(K)Ai, \\ b_C &= T_1(h) + T_0(p)AKBT_0(i). \end{aligned} \tag{57}$$

Using $BA = AB = 1$ and (33) gives

$$\delta a_C = v_C u_C - 1_C, \qquad \delta b_C = u_C v_C - 1_{TC}.$$

The degree-one twists in (57) include their degree-three input signs. The homotopy K verifies the inverse identities and is absent from (56). The proof applies to nonempty acyclic complexes as well; for an empty complex the unique maps and zero homotopies suffice. □

No vanishing differential component or torsion-freeness assumption is used. The full formal model supplies the intermediate sphere lines even when every differential component of an actual endpoint complex is nonzero.

10. Transfer through forward and reflected mixed counts

10.1. **The reflected count.** Fix common regular small data p with nonzero integral gauge homology. Use literally reflected ordinary gauge and strip ends and paired reference structures at the reflected presentation $\bar{p}$. Its mixed interior data are chosen independently, sufficiently small and regular. This is the reflected-presentation construction of Paper I, Section 11.2, with the paired chain-level references of Paper II, Section 11.1; it is not a reflection of the original mixed solutions.

Suppress p and write G, S, N for the calibrated complexes and mixed map. Paper II's bent-cylinder gauge pairing Φ_G and its symplectic reflection pairing Φ_p identify reflected

generator lines with signed units. The latter pairing, including its orientation comparison on all strip configurations, is used only in its stated nonzero-homology domain. In the graded dual and evaluation conventions of Paper II, set

$$M = \mathrm{ev}_G^{-1}(N_{\bar p}\Phi_G^{-1})^\vee \widehat{\Phi}_p, \qquad \widehat{\Phi}_p = \Phi_p^\vee \,\mathrm{ev}_S\,. \tag{58}$$

This is the closed even map of Paper II's reflected pairing construction. The inverse symbols refer only to canonical line pairings and evaluation. Each coefficient is an actual reflected-presentation mixed count, read through those pairings with the graded evaluation sign. Inserting the orientation-line maps of Section 4 expresses it on G, S. These signed line identifications do not introduce a mixed inverse or a new quotient multiplicity.

10.2. **A common finite filtration.** Use the real index constants $c_p(a)$ of [DFL21, Proposition 2.53]. For any ordinary cylindrical gauge configuration A from a to b, including a nonsolution,

$$\operatorname{ind} D_A = 8\mathcal{E}_p(A) + c_p(b) - c_p(a). \tag{59}$$

Read the same cylinder from $\bar b$ to $\bar a$. Reversing both time and the spatial orientation preserves the four-dimensional orientation. Use the reflected assembled primary functional: its sign changes together with the spatial Hodge star. Thus the full modified curvature form and its topological energy are preserved by pullback, as is the full scalar/ASD operator. The paired end regularizer vanishes near the primary critical set; removing it by scalar homotopy keeps the Hessians and index fixed, and has zero endpoint transgression. It therefore does not change the constants in (59). Subtracting the two index identities gives

$$c_{\bar p}(\bar a) - c_{\bar p}(\bar b) = c_p(b) - c_p(a). \tag{60}$$

Smooth configurations exist between any two selected critical representatives of the fixed bundle, so this identifies all differences of the index constants.

The small-data triangularity theorem of Paper I, Theorem 3.8, and Paper II, Proposition 6.3, say that a nonzero off-diagonal coefficient of N from a to b requires $c_p(b) < c_p(a)$. At equal filtration the only coefficient is the unit of the constant pair, with sign $+1$ in the chosen bases. The corresponding DFL statements are [DFL21, Proposition 3.23(iii) and the proof of Theorem 3.29]. Apply the same theorem independently at $\bar p$. A coefficient of M from a to b is, up to its prescribed pairing signs, the coefficient of $N_{\bar p}$ from $\bar b$ to $\bar a$. By (60), its off-diagonal support has the same inequality $c_p(b) < c_p(a)$. Consequently N and M preserve the same finite filtration, and the diagonal entries of M are ± 1. The pairing signs may depend on the generator; the argument does not require them to be constant.

Proposition 10.1. *At these data, the mixed map N has a strict inverse expressible as a finite integer polynomial in forward and reflected mixed counts and identities.*

Proof. Let $n \geq 1$ be the number of generators. On S put

$$Z = NM, \qquad L = Z^2 - 1, \qquad Q = \sum_{k=0}^{n-1} (-L)^k, \qquad V = MZQ : S \longrightarrow G. \tag{61}$$

The triangular map Z has diagonal entries ± 1, so L strictly lowers the finite filtration and $L^n = 0$. Hence

$$NV = Z^2 Q = (1 + L)Q = 1_S.$$

The chain isomorphism N is injective, and applying it to $VN - 1_G$ gives zero; thus $VN = 1_G$ as well. The maps N, M are closed and even, so the same is true of V. Since N is invertible in each relative degree, $NV = 1$ also proves that V has relative degree zero. This conclusion

concerns the complete polynomial, without imposing an unproved degree on each reflected factor separately.

Every coefficient in (61) comes from a fixed finite polynomial in actual counts. The square Z^2 is essential because the diagonal signs of M may vary. Changing reflected data can change the expression for M; with N fixed, the two inverse identities show that the resulting V is unchanged. This verifies the formula rather than defining its coefficients by matrix inversion. □

10.3. **The actual positive mixed comparison.** Take the gauge maps u_G, v_G and homotopies a_G, b_G of Proposition 9.2. In the branch $\sigma = -1$, define

$$\begin{aligned} u_S &= T_0(N)u_G V, & v_S &= N v_G T_0(V), \\ a_S &= N a_G V, & b_S &= T_0(N) b_G T_0(V). \end{aligned} \tag{62}$$

Twists are applied to the complete degree-zero map V. The defining maps are weighted count expressions, now using the reflected mixed factors. The strict identities $VN = NV = 1$ give

$$\begin{aligned} u_S N &= T_0(N)u_G, & v_S T_0(N) &= N v_G, \\ v_S T_0(N) u_G - N &= \delta(N a_G). \end{aligned} \tag{63}$$

They also give $\delta a_S = v_S u_S - 1$ and $\delta b_S = u_S v_S - 1$. By (31), the twisted complexes and $T_0(N)$ are precisely the actual positive complexes and N^+ with the matched references. If $\sigma = 1$, use the matched line identities and zero homotopies instead. This proves (4) in the nonzero-homology case. The new maps on S include mixed factors; they are not asserted to be pure symplectic polygon counts.

For an acyclic endpoint, the reflected pairing is unnecessary. The actual mixed equivalence implies that G and S are both acyclic. Their positive versions are acyclic as well, since the twist changes neither kernels nor images. Choose integral contractions κ_X with $\delta\kappa_X = 1_X$, separately in the two gradings, and let $\kappa_X^+ = T_1(\kappa_X)$ in the negative branch or its matched line transport in the positive branch. For this endpoint choose

$$u_X = v_X = 0, \qquad a_X = -\kappa_X, \qquad b_X = -\kappa_X^+, \qquad t = -\kappa_S N. \tag{64}$$

The zero maps are empty weighted sums between the unchanged actual complexes, and $\delta t = -N = v_S N^+ u_G - N$. They are homotopy equivalences because those complexes are contractible. The contractions verify the relations; they do not define the maps. In particular the actual N^+ has not been replaced by zero, and no acyclic intermediate summand of a gauge model has been discarded. This proves the endpoint assertions of Theorem B in every case.

11. Positive comparisons under presentation changes

11.1. **The common-data system.** The preceding construction gives $u_{X,p}, v_{X,p}$ at every common presentation. Write their inverse and mixed homotopies as

$$v_{X,p}u_{X,p} - 1 = \delta r_{X,p}, \qquad u_{X,p}v_{X,p} - 1 = \delta s_{X,p}, \tag{65}$$

$$v_{S,p} N_p^+ u_{G,p} - N_p^{\mathrm{cal}} = \delta T_p. \tag{66}$$

The endpoint reference paths and relative grading origins remain fixed in these identities. Changes of such bases are recorded by their orientation-line maps; they are not silently replaced by a scalar sign.

For a calibrated presentation word F_{qp} on either side, define

$$F_{qp}^+ = u_q F_{qp} v_p. \tag{67}$$

These are weighted count expressions with the actual calibrated geometric word as middle factor. Its intermediate presentations need not be common DFL data. In particular this definition does not require an ordinary continuation equation joining opposite sphere characters. An even representative may be chosen by endpoint-parity projection; the other component is null-homotopic by its calibrated mixed square with the even gauge presentation map. If an inhomogeneous closed representative is retained, write $F^{\varepsilon} = F_{\mathrm{even}} - F_{\mathrm{odd}}$ and use the graded Hom differential componentwise.

The graded Leibniz rule and (65) give

$$F^{+}_{qp}u_p - u_q F_{qp} = \delta(u_q F^{\varepsilon}_{qp} r_p),$$
$$v_q F^{+}_{qp} - F_{qp} v_p = \delta(r_q F_{qp} v_p).$$

If $F_{rq}F_{qp} - eF_{rp} = \delta C$, with one sign e, then

$$F^{+}_{rq}F^{+}_{qp} - eF^{+}_{rp} = \delta\big(u_r(C + F^{\varepsilon}_{rq} r_q F_{qp})v_p\big). \tag{68}$$

The unit $u_p v_p$ is homotopic to 1 by s_p. Applying (68) to a reverse word and adding that unit homotopy supplies inverse homotopies. A word-independence homotopy $F - F' = \delta E$ becomes $u_q E v_p$. For arbitrary choices of the outer maps, the formulas below give equivalent presentation systems. The calibrated word homotopies retain their provenance from Section 5; some use algebraic cancellation to prove a relation between geometric maps. We do not assert that each is one parameterized count.

To check compatibility with the actual mixed count, put

$$D_p = u_{S,p}T_p - s_{S,p}N^{+}_{p}u_{G,p},$$
$$E'_p = v_{S,p}N^{+}_{p}s_{G,p} - T_p v_{G,p}.$$

Then

$$\delta D_p = N^{+}_{p}u_{G,p} - u_{S,p}N^{\mathrm{cal}}_{p}, \qquad \delta E'_p = N^{\mathrm{cal}}_{p}v_{G,p} - v_{S,p}N^{+}_{p}.$$

If the calibrated square is $N^{\mathrm{cal}}_{q}G_{qp} - e\Gamma_{qp}N^{\mathrm{cal}}_{p} = \delta R_{qp}$, the positive square has homotopy

$$N^{+}_{q}G^{+}_{qp} - e\Gamma^{+}_{qp}N^{+}_{p} = \delta\big(D_q G_{qp} v_{G,p} + u_{S,q}R_{qp}v_{G,p} + e u_{S,q}\Gamma^{\varepsilon}_{qp}E'_p\big). \tag{69}$$

This includes changes of genus. The construction specifies the positive words themselves; comparison with independently prescribed positive continuation or stabilization counts is a further question.

11.2. **Changing the outer normalization maps.** Suppose $u'_p, v'_p, r'_p, s'_p, T'_p$ are another choice on the same actual positive and calibrated complexes. Set $K_p = u'_p v_p$. Its inverse is $u_p v'_p$, with inverse homotopies $s_p + u_p r'_p v_p$ and $s'_p + u'_p r_p v'_p$. For the corresponding presentation maps,

$$K_q F^{+}_{qp} - F^{+\prime}_{qp}K_p = \delta\big(u'_q(r_q F_{qp} - F^{\varepsilon}_{qp}r'_p)v_p\big). \tag{70}$$

On the two sides, using E' from the unprimed data above and D' from the primed data, we also have

$$K_S N^{+} - N^{+}K_G = -\delta(u'_S E' + D' v_G).$$

Both identities follow by substituting the unit and mixed homotopies (65)–(66). For three choices, the composite of the adjacent K maps is homotopic to the direct comparison by the intermediate unit homotopy. Thus these are equivalent systems in the projective homotopy category. The assertion does not require K_p to be the identity or u'_p to be homotopic to u_p.

11.3. **Arbitrary regular small endpoints and cobordisms.** At a common presentation q define Ψ_q^+ by (5). Equation (66) gives

$$\Psi_q^+ - X_q = \delta J_q^+, \qquad J_q^+ = B_q T_q U_q. \tag{71}$$

For an arbitrary prescribed regular small presentation p, choose a common presentation q, a geometric identity word Γ_{pq} and the reverse usual gauge cylinder H_{qp} as in Section 5. Set

$$\widehat{\Psi}^+_{p;q,d} = \Gamma_{pq}\Psi_q^+ H_{qp}.$$

Combining (71) with (27) gives

$$\widehat{\Psi}^+_{p;q,d} - e_{p;q,d}X_p = \delta L^+_{p;q,d}, \qquad L^+_{p;q,d} = E_{p;q,d} + \Gamma^{\varepsilon}_{pq} J_q^+ H_{qp}. \tag{72}$$

It is therefore an equivalence with the projective class asserted in Theorem B. For two choices, subtraction gives the independence homotopy $L^+ - ee'L^{+\prime}$. If p is common, the constant word recovers its actual positive representative. Here $\Gamma^{\varepsilon} = \Gamma$ for the chosen even representative; the displayed form also retains an inhomogeneous word. The independence includes all choices inside u and v, because (71) holds separately for each choice. The prescribed endpoint p is not changed to a DFL-regular one; only q must meet the original mixed problem's hypotheses.

For any permitted cobordism, write $F = F^{\mathrm{r}}_{\mathrm{H}}(W)$ and $S = S_d(W)$. The calibrated square is $X_1F - vSX_0 = \delta J_W$. With $w = e_1 v e_0$, the homotopy for the positive comparison is exactly

$$\widehat{\Psi}_1^+ F - wS\widehat{\Psi}_0^+ = \delta\big(e_1 J_W + L_1^+ F - wS^{\varepsilon} L_0^+\big),$$

where $S^{\varepsilon} = S_{\mathrm{even}} - S_{\mathrm{odd}}$ as in (30). Its intermediate objects may be arbitrary: all middle cobordism maps are the calibrated ones of Theorem A, and normalization maps are needed only at the two endpoints. If an intermediate object has zero homology, retain Paper II's zero representatives and integral contractions, as in Section 5. This proves Theorem B.

12. Two further orientation comparisons

Theorem B changes the primitive-loop normalization by constructing new outer equivalences and positive presentation maps. Two comparisons with independently named constructions remain. They concern prescribed orientation actions, so their answers cannot be fixed by choosing a convenient sign in the maps constructed here.

12.1. **The nonsingular cap convention.** The usual gauge maps in this paper are identified through the singular Hopf-link realization and excision. The nonsingular framed construction of [Sca15, Sections 7.1–7.3] instead uses a particular torus cap to define its reference lines and homology-oriented maps.

Question 12.1. How does the actual boundary gauge action on this cap reference line compare with the singular Hopf-link and almost-complex reference actions, and how does that comparison behave under cobordism gluing?

One must retain the incoming reference operator, the outgoing bundle action and the quotient identification in a single determinant-gluing diagram. A closed index formula alone does not identify this boundary action. The excision comparison in Section 3 does not need its sign, so this is an additional comparison of the named nonsingular realization.

12.2. **Prescribed relative-spin orientations.** A relative-spin system fixes background classes and spin data for the Lagrangian correspondences before counts are defined. The symplectic orientations used here are transported from the gauge theory through the mixed equation. Agreement on strip complexes would not by itself identify the auxiliary operations used for cobordism maps or stabilization.

Question 12.2. For a specified compatible system of relative-spin backgrounds and structures, do the resulting strip, cup, cap, triangle and stabilization maps agree, up to the required integral homotopies, with the transported symplectic maps and their mixed comparison?

The target system must be specified together with its behavior under changes of genus. Determinant-line monodromy at all ends of an auxiliary operation is a compatibility condition for its existence, before one compares signs of individual solutions. The positive presentation maps of Section 11 provide a coherent system of weighted count expressions. Question 12.2 asks for its relation to prescribed geometric operations, rather than merely for the existence of some equivalent system.

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