ANTIDIAGONAL LOCAL CURVES AND INTEGRAL EXPONENTS IN UNIVERSAL CURVE COUNTING

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Abstract. Assuming Pardon's Gromov–Witten evaluation for analytic families of threefolds, we lift the elementary-cluster expansion of Ionel and Parker to his rational algebra of one-cycles. For every smooth projective Calabi–Yau threefold the universal series factors uniquely with the elementary-cluster integers as exponents, indexed by effective classes in integral homology. We construct the genus-g factor using the volume-preserving circle in the torus that scales the two line summands of a balanced local curve. Symmetric functions of the two fiber projections isolate the multiples of the zero section and give compactly supported classes compatible with addition of cycles and multiplication by positive integers. Among series with these compatibilities, the local Gromov–Witten series of an elementary cluster determines the universal factor. Among split local Calabi–Yau curves with the antidiagonal action, agreement with the elementary cluster in degree two already forces the splitting degrees to be balanced.

1. Introduction

Let C be a smooth projective curve of genus g and let $Y \to C$ be a rank-two vector bundle with determinant K_C. Two circle actions on the total space give different answers to a question about integrality. Scalar multiplication is intrinsic to the bundle. For a splitting into two lines the fiber-scaling torus acts on a holomorphic volume form by pullback with character $(s,t) \mapsto st$. Its kernel is the antidiagonal torus. Up to inversion its only faithful one-parameter parametrization has weights $(1,-1)$. For a splitting of degrees $(g-1, g-1)$, called *balanced*, Bryan and Pandharipande computed the antidiagonal Gromov–Witten series [BP08, Corollary 7.3]. The same opposite weights occur in the normal operator of an elementary cluster. In the spin model of Ionel and Parker the normal operator is the direct sum of two copies of a real Cauchy–Riemann operator. Real rotations of these copies commute with that operator. In the fiber coordinates $v_1 + iv_2$ and $v_1 - iv_2$ the rotations have characters t and t^{-1}. In the obstruction-bundle calculation, specializing to these weights gives the Chern class signs of the dual index bundle and hence the elementary-cluster contribution [IP18, Propositions 3.2–3.3, (3.2)–(3.3)].

Pardon associates a universal class to each compact open-and-closed locus in the space of one-cycles on a complex threefold. Gromov–Witten and stable-pair invariants are evaluations of this class. His scalar equivariant local classes freely generate the Chern-zero, virtual-dimension-zero part of the rational semi-Fano algebra [Par25, Theorem 1.2]. For the local series formed from those generators, Conjecture 1.4 of [Par25] predicts a product expansion with integer exponents. The compact example of [Wue26a] has a scalar exponent in $\mathbb{Z} + 2/3$.

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It leaves open whether the elementary local series themselves have geometric representatives in the universal algebra. We construct such representatives using the antidiagonal action.

Write H for the Chern-zero, virtual-dimension-zero part of Pardon's rational semi-Fano algebra. For a smooth projective Calabi–Yau threefold X over $\mathbb{C}$, let Γ_X be its monoid of effective integral curve classes. We use the convention $K_X \simeq \mathcal{O}_X$ and $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$. The coefficient of Q^β in the universal series $\mathcal{Z}_X$ is the class of the characteristic function of the compact cycle locus of class β; the constant term is 1. We complete the monoid ring in degree against an ample divisor. Write $c_{\beta,g}(X)$ for the coefficient of the genus-g elementary cluster at class β in the connected Gromov–Witten expansion [IP18, Theorem 8.1]. It multiplies the logarithm of the genus-g local hook series (4) evaluated at Q^β.

For the identification of the local factors and the compact product formula we assume Pardon's universal Gromov–Witten evaluation [Par25, §§3.4–3.5]. Its definition requires stable-map spaces over arbitrary separated submersions of reduced complex analytic spaces of relative dimension three. Pardon indicates how the construction for algebraic families should extend and notes that he knows no reference for this step [Par25, Remark 1.8].

Theorem 1.1. *Assume the universal Gromov–Witten evaluation for analytic families. Let C be a smooth connected projective curve of genus g and let L be a line bundle of degree $g-1$ on C. The antidiagonal action on $L \oplus (K_C \otimes L^{-1})$ defines classes $a_{g,m} \in H$ for $m \geq 1$ which depend only on g and m. Put $A_g(q) = 1 + \sum_{m\geq 1} a_{g,m} q^m$. For every smooth projective Calabi–Yau threefold X,*

$$\mathcal{Z}_X = \prod_{\beta\in\Gamma_X\setminus\{0\}} \prod_{g\geq 0} A_g(Q^\beta)^{c_{\beta,g}(X)}. \tag{1}$$

The exponents are unique, integral, and zero for all sufficiently large g at each fixed β.

The integer exponents and their genus finiteness are known for the *numerical* elementary expansion [IP18, Theorem 8.1], [DIW26, Theorem 3.50]. The assertion of (1) is the identity in $H[\![\Gamma_X]\!]$. The construction must therefore respect the operations on cycles before numerical evaluation. For a cycle of degree m over C, the elementary symmetric functions of its two fiber projections have opposite weights. Their common vanishing over sufficiently many points of C isolates $m[C]$. The relative Euler classes of these functions can be normalized for either sign of the weight. In the coproduct calculation the two summands of a cycle lie over independent Borel bases. The diagonal of those bases has just one Künneth summand that survives in virtual dimension zero. Reconstruction identifies the resulting factor as the unique series compatible with both operations whose evaluation is the elementary cluster series (Corollary 4.3). For compact X we keep the full class in $H_2(X;\mathbb{Z})$. When several effective classes have the same d-fold multiple, pullback under multiplication by d sums their contributions.

Sections 2 and 3 construct the classes and prove their compatibility with cycle operations independently of the family evaluations. The second integrality argument in Section 5 assumes the corresponding stable-pair evaluation.

2. The antidiagonal local classes

Set $L_+ = L$, $L_- = K_C \otimes L^{-1}$ and $Y = \mathrm{Tot}(L_+ \oplus L_-)$. The torus acts trivially on C and by $(v_+, v_-) \mapsto (tv_+, t^{-1}v_-)$ on the fibers. Denote by Z_m the space of compact effective one-cycles in Y of degree m over C. Every irreducible component maps finitely onto C. Indeed, an affine plane fiber contains no complete curve, and the projection of each compact component

is proper. The space Z_0 is the point representing the empty cycle. Since K_Y is trivial, all these cycles have Chern number zero and belong to the semi-Fano locus.

2.1. **Symmetric functions on the fibers.** For $p \in C$ the fiber intersection of $D \in Z_m$ is a multiset of m points in Y_p. Project it to each of $L_{+,p}$ and $L_{-,p}$. For $1 \le j \le m$ the jth elementary symmetric function of a projection gives an element

$$b_{\pm,j}(D)(p) \in L_{\pm,p}^{\otimes j}.$$

These evaluations are continuous functions of D by the fiber-intersection map of [Par25, (4.15)].

Lemma 2.1. *For $m > 0$, choose $1 + \max\{0, m(g-1)\}$ distinct points of C. The common zero set of the functions $b_{\pm,j}$ evaluated at these points consists of $m[C]$. This cycle is also the unique fixed point of the circle on Z_m.*

Proof. The coefficients $b_{\pm,j}(D)$ are holomorphic sections on C. To see this across a branch point, normalize each irreducible component. Its finite map to the smooth base curve is flat. In a local trivialization of the line, the fiber coordinate acts by multiplication on the finite locally free direct image of the structure sheaf. Its characteristic polynomial records the projected branches with multiplicities. Take the product of these polynomials over the components, raising each to its cycle multiplicity. The jth coefficient, up to the sign $(-1)^j$, transforms as a section of $L_{\pm}^{\otimes j}$.

Both line bundles have degree $g-1$. If all the chosen evaluations vanish, each coefficient vanishes identically by the degree bound. The two projected polynomials are thus T^m on every fiber. All points of the multiset have both coordinates zero, which forces $D = m[C]$. Conversely $m[C]$ is a common zero. For a circle-fixed cycle the circle preserves each finite fiber multiset. Connectedness forces it to fix each point individually. The origin is the only fixed point of the fiber action. □

For later use, Z_m is finite dimensional. Embed Y in the bundle of lines in $\mathcal{O}_C \oplus L_+ \oplus L_-$. On the open chart Y the relative $\mathcal{O}(1)$ is trivial. Tensoring it with a sufficiently ample line bundle pulled back from C gives a projective polarization whose degree on D is m times its degree on the zero section. Thus Z_m is the open subset of a fixed-degree projective Chow space parametrizing cycles disjoint from the divisor at infinity. This description gives the compact-cycle topology on that open set and the bounded dimension needed for equivariant localization.

2.2. **The finite Borel construction.** Write $B_N = \mathbb{P}^N$, $h = c_1(\mathcal{O}_{B_N}(1))$, and

$$Z_{m,N} = \big((\mathbb{C}^{N+1} \setminus \{0\}) \times Z_m\big)/\mathbb{C}^\times \longrightarrow B_N$$

for the diagonal quotient. It is the relative degree-m cycle space of the corresponding Borel family of local threefolds. Trivializing $L_\pm$ at each chosen point makes the tests of Lemma 2.1 into functions of weights $+j$ and $-j$. A function f of weight r gives a section of the pullback of $\mathcal{O}_{B_N}(r)$. Our sign convention follows from the quotient: the class of (v, s) of weight one defines the functional on $\mathbb{C}v$ taking v to s.

For any tuple $F = (f_1, \dots, f_n)$ of equivariant functions of nonzero weights r_i with compact common zero set, put

$$u_N(F) = \frac{1}{\prod_i r_i}\, F^* \operatorname{Th}\left(\bigoplus_{i=1}^{n} \mathcal{O}_{B_N}(r_i)\right) \quad \text{in } H_c^{2n}(Z_{m,N}; \mathbb{Q}). \tag{2}$$

Here the Thom class is taken relative to the complement of the zero section. Its pullback has support in the common zero set over B_N, which is compact. In particular the definition allows noncompact individual zero sets and needs no transversality. On forgetting supports the normalized Euler class of a single section is $c_1(\mathcal{O}(r))/r = h$ for either sign of r.

Lemma 2.2. *The universal image of $u_N(F)$ depends only on $n-N$ for the given local model and degree.*

Proof. For the inclusion $i : B_N \hookrightarrow B_{N+1}$ of a hyperplane use the same symbol for the inclusion of Borel families. The class $u_{N+1}(F)$ restricts to $u_N(F)$, so the hyperplane Gysin map gives

$$i_! u_N(F) = h\, u_{N+1}(F) = u_{N+1}(F, f)$$

for every additional function f of nonzero weight. The second equality uses the compact support already supplied by F, which allows the last relative Euler class to be replaced by its ordinary class. The first equality is the projection formula for the complex normal line pulled back from the base. The product description locally over the base suffices even for a singular cycle space.

These Gysin maps induce the stabilization of relative classes in [Par25, (4.5)–(4.7)]. Given two tuples of lengths n, n' at stages N, N' with $n - N = n' - N'$, append the second tuple to the first and the first to the second. Both products now live at stage $N + n' = N' + n$, and the even Thom factors commute. Hence the images in the universal group coincide. □

Take $n = N$ in (2) and let $a_m(C, L)$ denote the image in Pardon's algebra. Such a tuple exists by Lemma 2.1. The oriented base has real dimension $2N$, so a compact class of degree $2n$ has relative degree $2n-2N$ [Par25, Lemma 3.5]. The class $a_m(C, L)$ has virtual dimension and Chern number zero and therefore lies in H. In particular its expression in the scalar generators is a polynomial. The counit evaluates on the empty cycle, hence vanishes on $a_m(C, L)$ for $m > 0$. We set $a_0(C, L) = 1$, as given by the empty tuple at $N = 0$.

3. Addition and multiplication of cycles

The coproduct Δ in H is defined by pullback under cycle addition; the operation ρ_d is pullback under $D \mapsto dD$, for $d \geq 1$ [Par25, Definitions 3.9 and 3.12]. The two arguments of the coproduct are cycles on independent families of threefolds. In the Borel construction this distinction introduces the class of the diagonal in a product of two projective spaces.

Proposition 3.1. *For a fixed balanced pair (C, L), write $a_m = a_m(C, L)$. Then*

$$\Delta a_m = \sum_{s+t=m} a_s \otimes a_t, \qquad \rho_d a_m = \begin{cases} a_{m/d} & d \mid m, \\ 0 & d \nmid m. \end{cases} \tag{3}$$

Proof. The preimage of Z_m under multiplication by d is empty unless $d \mid m$. Otherwise pull the tuple of Lemma 2.1 back to $Z_{m/d}$. The pulled-back functions have the same weights, with common zero $(m/d)[C]$. Naturality of the relative Thom class and Lemma 2.2 give the formula for ρ_d.

For the coproduct fix $s + t = m$. Over the diagonal in B_N^2, addition gives a map from the common-base family of $Z_s \times Z_t$ to $Z_{m,N}$. In the two-base description of Δ [Par25, (4.17)–(4.21)], the (s, t) component is represented by pulling back $u_N(F)$ along addition and then applying the oriented Gysin map $\delta_{N!}$ over this diagonal. Its target is the independent product $Z_{s,N} \times Z_{t,N}$. We can take $|F| = N$ with common zero $m[C]$. The tuple pulled back along addition has common zero $(s[C], t[C])$, since both summands are effective cycles.

Choose compactly supported cut-downs given by tuples F_s, F_t on the two factors, of lengths n_s, n_t, and set $M = n_s+n_t$. On the common-base family their concatenation also has compact common zero set. The proof of Lemma 2.2, applied to the diagonal action on $Z_s \times Z_t$, identifies its class at stage M with the pulled-back class at stage N after stabilization. We may make this substitution before applying the diagonal Gysin map. Indeed, for the simultaneous hyperplane inclusion i the identity

$$(i \times i)_!\, \delta_{N!} = \delta_{N+1,!}\, i_!$$

follows by composition of oriented embeddings on the bases. Hyperplane pushforward in each of the two independent bases preserves the image of the corresponding relative class.

At stage M the concatenated class on the diagonal is the restriction of $u_M(F_s) \boxtimes u_M(F_t)$. By the projection formula its diagonal pushforward is this external product times

$$[\delta_M(B_M)] = \sum_{i+j=M} h_1^i h_2^j.$$

The two factors of the (i, j) summand have relative cohomological degrees $2(n_s + i - M)$ and $2(n_t + j - M)$, whose sum is zero. A positive degree here is a negative virtual dimension. Its image in the semi-Fano algebra vanishes by [Par25, Theorem 1.2], as in the proof of [Par25, Theorem 7.6]. Consequently only $i = n_t$, $j = n_s$ survives. Multiplication by these powers of the hyperplane classes completes each tuple to length M at stage M, giving $a_s \otimes a_t$ with coefficient one. An empty factor admits the empty tuple and the same argument, with any extra copies of h represented by the Euler class of $\mathcal{O}(1)$. Summation over $s + t = m$ proves (3). □

4. Local Gromov–Witten evaluation and reconstruction

Let u be the Gromov–Witten variable and q the local curve-degree variable. We use disconnected maps with no wholly contracted connected components. Contracted components attached to positive-degree maps are included. For each partition λ let $h(b)$ denote the hook length of a box b of its Young diagram. The empty partition contributes 1 to the following series:

$$E_g(q, u) = \sum_{\lambda} q^{|\lambda|} \prod_{b \in \lambda} \big(2 \sin(h(b)u/2)\big)^{2g-2}. \tag{4}$$

Proposition 4.1. *The disconnected Gromov–Witten evaluation of* $\sum_{m \geq 0} a_m(C, L) q^m$ *is* $E_g(q, u)$.

Proof. Fix $m > 0$ and the Euler characteristic of the source. The stable-map cycle map is proper at fixed Euler characteristic, and its equivariant virtual pushforward defines the Gromov–Witten class on Z_m [Par25, §3.5, (3.21)–(3.23)]. By Lemma 2.1 its fixed-locus localization is a class v at $m[C]$. More precisely, equivariant homology after inverting h identifies with that of the fixed locus by [Par25, Proposition 4.5], and v is the inverse image under fixed-locus pushforward. Indeed, off $m[C]$ the tuple of Lemma 2.1 is a nowhere-zero section. Its Euler class $(\prod_i r_i)h^n$ vanishes there, so h^n annihilates the equivariant homology of the complement. The fixed-locus isomorphism follows from the localization sequence. The class v is a scalar because the virtual dimension is zero.

We check that the compact class (2) makes precisely this scalar an ordinary universal evaluation. Fix a tuple of length n as in Lemma 2.1. At each Borel stage its compact class restricts on the fixed section to h^n. Thus capping with the equivariant virtual class and pushing forward to the point gives $h^n v$ after inverting h, by the projection formula. The

compact support of the cut-down already defines this pushforward over $\mathbb{Q}[h]$, which injects into $\mathbb{Q}[h,h^{-1}]$. At stage $N=n$ the ordinary integral is v since $\int_{B_n} h^n = 1$. This is the evaluation of $a_m(C,L)$, by the finite Borel description in [Par25, Proposition 4.8].

It remains to identify v. A fixed map has fixed image cycle, so all its positive-degree components map into C. Attached contracted components map into C as well. Conversely every stable map into C is fixed. Write $\mathcal{F}$ for this proper fixed moduli stack and p for the cycle map. Along C the tangent bundle splits as $T_C \oplus L_+ \oplus L_-$. The fixed part of the obstruction theory is the stable-map theory of C; the moving virtual normal complex is $R\pi_* f^* L_+ \oplus R\pi_* f^* L_-$ with the respective characters $+1, -1$, where (π, f) is the universal curve and map. Both characters stay nonzero on the antidiagonal. Equivariant virtual localization and proper pushforward give

$$v = (p|_{\mathcal{F}})_* \left(\frac{[\mathcal{F}]^{\mathrm{vir}}}{e_{S^1}(N^{\mathrm{vir}})} \right).$$

This is the residue integral of [BP08, §2.2]. Its degree-m normalization shifts the power of u by $m(2-2g+\deg L_+ + \deg L_-) = 0$. Formula (4) now follows from [BP08, Corollary 7.3]. □

The degree-two term distinguishes the balanced degrees among all splittings with determinant K_C. Indeed, take line degrees $g-1+\ell, g-1-\ell$ and retain the antidiagonal action. Write $E_{g,\ell}$ for its disconnected residue series, so $E_{g,0} = E_g$. In the formula of [BP08, Corollary 7.2] the change of degrees in degree two multiplies the two partition contributions by $e^{i\ell u}$ and $e^{-i\ell u}$. Indeed, the partitions (2) and $(1,1)$ have opposite total contents ± 1 and the same hook lengths, and the sign in degree two is $(-1)^{2\ell} = 1$. Thus

$$[q^2]E_{g,\ell} = \cos(\ell u)\,[q^2]E_g. \tag{5}$$

The balanced coefficient is a nonzero Laurent series. Equality with the elementary-cluster coefficient therefore forces $\ell = 0$.

To pass from this evaluation to a class in H, we extend the reconstruction argument of [Wue26b, §2]. Write $x_{h,j} = x_{h,j,0}$ for Pardon's scalar generators, $x_{h,0} = 1$, and set

$$f_h(q) = \sum_{j \geq 0} x_{h,j} q^j, \qquad p_{h,j} = [q^j] \log f_h(q) \quad (j > 0).$$

Over $\mathbb{Q}$ the $p_{h,j}$ are polynomial coordinates on H. Pardon's scalar formulas [Par25, Lemmas 4.12–4.13 and Theorem 7.6] give (3) for $x_{h,m}$. Thus $p_{h,j}$ is primitive for Δ, and $\rho_d p_{h,j}$ is $p_{h,j/d}$ if $d \mid j$ and zero otherwise.

Let Γ be a submonoid of a finitely generated abelian group with an additive degree on Γ, positive and integral off zero, and with finite sublevel sets. Complete the monoid ring in this degree. Call a series $U = \sum_\beta U_\beta Q^\beta$ *compatible* if $U_0 = 1$, its other coefficients lie in the augmentation ideal of H, and

$$\Delta U_\beta = \sum_{\gamma+\eta=\beta} U_\gamma \otimes U_\eta, \qquad \rho_d U_\beta = \sum_{\substack{\gamma \in \Gamma \\ d\gamma = \beta}} U_\gamma. \tag{6}$$

The second identity says that $\rho_d U = \psi_d U$, where ρ_d acts on coefficients and ψ_d substitutes $Q^\gamma \mapsto Q^{d\gamma}$. The latter is a continuous ring endomorphism even when multiplication by d identifies distinct classes. The coefficient at β sums all their contributions.

Lemma 4.2. *A compatible series is determined by its disconnected Gromov–Witten evaluation. It admits a unique product expansion in the $f_h(Q^\beta)$ with rational exponents and finite genus support in each class.*

Proof. The primitive polynomials in the $p_{h,j}$ are precisely their finite linear combinations, since an additive polynomial over $\mathbb{Q}$ is linear. If a primitive R is killed by every ρ_d for $d > 1$, it is a linear combination of $p_{h,1} = x_{h,1}$. Indeed applying ρ_j for the largest multiplicity $j > 1$ in such a polynomial detects all its coefficients at j. Set $w = (2\sin(u/2))^2$. The scalar degree-one evaluations are $\mathrm{GW}(x_{h,1}) = w^{h-1}$ by the degree-one local formula [BP08, §8]. These functions are linearly independent by their distinct lowest powers of u. If two compatible series differ, choose a class β of least degree at which they differ. Every decomposition of β into two nonzero summands uses lower degrees. The first identity in (6) therefore makes the difference at β primitive. For $d > 1$ each root $d\gamma = \beta$ has degree $\deg(\beta)/d$, so the second identity kills the difference under ρ_d. Equal evaluations force it to vanish.

For the expansion, every coefficient of $\log U$ is primitive. Indeed the coproduct identity gives $\Delta U = (U \otimes 1)(1 \otimes U)$ with one common monoid variable, so taking the logarithm gives the primitive identity coefficientwise. Write $[Q^\beta] \log U = \sum_{h,j} b_{\beta;h,j} p_{h,j}$. Since ρ_j and ψ_j commute with logarithms, comparison of the $p_{h,1}$ coefficients gives

$$b_{\beta;h,j} = \sum_{j\gamma=\beta} b_{\gamma;h,1}.$$

This is exactly the $p_{h,j}Q^\beta$ coefficient of $\sum_{\gamma\neq 0,h} b_{\gamma;h,1} \log f_h(Q^\gamma)$. The sums are finite in each degree, because positive-class degrees are at least one and each coefficient in H is a polynomial. Exponentiating gives the expansion. Its exponents are unique since $p_{h,1}Q^\gamma$ in the logarithm recovers $b_{\gamma;h,1}$. □

Corollary 4.3. *Assume the universal Gromov–Witten evaluation for analytic families. For each g there is exactly one compatible series in $H[\![q]\!]$ whose disconnected evaluation is the contribution of a genus-g elementary cluster. Every balanced antidiagonal model of genus g represents this series.*

Proof. Propositions 3.1 and 4.1 give existence, since the elementary-cluster contribution is E_g [IP18, Proposition 3.3]. Lemma 4.2 with $\Gamma = \mathbb{N}$ gives uniqueness. In particular the coefficients are independent of the curve, the line bundle, and the order of the two summands. We denote them by $a_{g,m}$ and the series by A_g. □

5. The product expansion on a compact threefold

Let X be a smooth projective Calabi–Yau threefold. Choose a projective embedding of X and use its degree on Γ_X. Every nonempty effective cycle has positive degree. The integral homology class is locally constant on the cycle space [Par25, §2.1]. The Chow variety of cycles of each fixed degree is compact, so its image in the discrete group $H_2(X;\mathbb{Z})$ is finite. Thus Γ_X has finite degree sublevel sets, and the locus $\mathscr{C}_\beta(X)$ of cycles of class β is compact and open and closed, including when different classes have the same image modulo torsion.

Put $U_{X,\beta} = (X, 1_{\mathscr{C}_\beta(X)}) \in H$. Addition pulls the characteristic function back to the sum over decompositions of β. Under multiplication $D \mapsto dD$, the inverse image of $\mathscr{C}_\beta(X)$ is the disjoint union of the $\mathscr{C}_\gamma(X)$ over all effective γ with $d\gamma = \beta$. Pullback of its characteristic function counts each locus once. This proves (6) for $\mathcal{Z}_X = \sum_\beta U_{X,\beta} Q^\beta$.

Proof of Theorem 1.1. Use the symplectic form of the polarization. Proposition 3.3 and Theorem 8.1 of Ionel and Parker [IP18] and the projective comparison theorem of Siebert [Sie99, Theorem 0.1] give

$$\log \mathrm{GW}(\mathcal{Z}_X) = \sum_{\beta\neq 0} \sum_{g\geq 0} c_{\beta,g}(X) \log E_g(Q^\beta, u). \tag{7}$$

Here $\log \mathrm{GW}(\mathcal{Z}_X)$ is the connected positive-class series. The integers in (7) have finite genus support at each β by [DIW26, Definition 3.45 and Theorem 3.50], which gives finiteness for this elementary expansion itself. Both the cluster expansion and the comparison of virtual classes are indexed by $H_2(X;\mathbb{Z})$. In particular [IP18, (8.9)] sums over every pair (d, B) with $dB = A$ in integral homology. Siebert's comparison fixes the integral class and the connected genus. The semipositive description of the symplectic invariants is recalled in [IP18, Remark 1.1].

Only classes of positive symplectic area can occur in the sum. Suppose β has positive ample degree and is ineffective for the projective structure on X. Every γ with $d\gamma = \beta$, $d > 1$, is then ineffective, since a positive multiple of an effective cycle is effective. There are finitely many such pairs because d divides $\deg(\beta)$ and the roots for a fixed d, if any, form a coset of the finite kernel of d on $H_2(X;\mathbb{Z})$. Compare coefficients in (7), inductively on the positive integral degree. The left side vanishes at β, and all terms from proper divisors already vanish. Thus $\sum_g c_{\beta,g}(X) w^{g-1} = 0$, using $[q] \log E_g(q,u) = w^{g-1}$. Linear independence gives $c_{\beta,g}(X) = 0$ for every g.

The right side of (1) is now well defined in $H[\![\Gamma_X]\!]$. There are finitely many effective classes of bounded degree, finitely many contributing genera per class, and polynomial coefficients in each factor. Products and inverses preserve the coproduct identity. By (3), $\rho_d A_g(Q^\beta) = A_g(Q^{d\beta}) = \psi_d A_g(Q^\beta)$. Both ρ_d and ψ_d are ring endomorphisms, so the full product also satisfies the root sum in (6). Proposition 4.1 and (7) identify the disconnected GW evaluation of the product with $\mathrm{GW}(\mathcal{Z}_X)$. Lemma 4.2 identifies the universal series. Uniqueness follows upon evaluation from the unique elementary expansion, or by comparing the multiplicity-one term at a first class of difference. □

The elementary-cluster evaluation in Corollary 4.3 is essential to the characterization. For example, fix $g \neq h$ and replace $A_g(q)$ by $A_g(q)A_h(q^2)$, keeping every other factor. The new series is compatible and has the same multiplicity-one coefficient as A_g. In the compact product its exponent stays $c_{\beta,g}$, and the exponent of $A_h(Q^\beta)$ becomes $c_{\beta,h} - \sum_{2\gamma=\beta} c_{\gamma,g}$. Thus this change preserves integer exponents in every class, although it changes the degree-two evaluation of the genus-g factor.

5.1. **Integrality from stable pairs.** There is a second way to see why the choice of action in (1) yields integer exponents. It uses the Gromov–Witten/stable-pair correspondence for H and the form of the local hook series. We first obtain rational exponents with finite genus support without using the numerical cluster decomposition.

By Lemma 4.2 the compatible $A_g(q)$ is a product of rational powers of $f_h(q^d)$. Its degree-one evaluation is w^{g-1}. Thus the entire factor at $d = 1$ is $f_g(q)$, by the independence of the powers w^{h-1}. All other factors have $d \geq 2$. At each multiplicity m this gives

$$a_{g,m} = x_{g,m} + P_{g,m}\big(\{x_{h,j} : 1 \leq j < m\}\big) \tag{8}$$

with $P_{g,m}$ a finite rational polynomial. Induction on m inverts the substitution $x_{g,m} \mapsto a_{g,m}$. It is therefore an automorphism of the polynomial algebra H. The identities (3) show that it intertwines both the coproduct and each ρ_d. Apply Lemma 4.2 to the inverse image of $\mathcal{Z}_X$, then apply the automorphism. This gives a product in the $A_g(Q^\beta)$ with rational exponents $n_{\beta,g}$ and finite genus support in every class.

Write z for the stable-pair variable. Pardon's Theorem 1.7 [Par25], extended by linearity over $\mathbb{Q}$, identifies $\mathrm{PT}(a_{g,m})$ with the expansion at $z = 0$ of the rational function $[q^m]E_g(q,u)$ under $e^{iu} = -z$. The identification is unambiguous since expansion at $e^{iu} = 1$ determines a

rational function. For a box of hook length k one has

$$(2\sin(ku/2))^2 = 2-(-z)^k-(-z)^{-k} = -(-z)^{-k}(1-(-z)^k)^2.$$

This is a unit in $\mathbb{Z}((z))$. Hence every coefficient of $\mathrm{PT}(A_g(q))$ lies in $\mathbb{Z}((z))$, including when $g=0$ and the hook factors are inverted. The compact stable-pair invariants $\mathrm{PT}(U_{X,\beta})$ also lie in $\mathbb{Z}((z))$ [Par25, §3.5].

Suppose all exponents in classes of smaller ample degree than β are known to be integral. Divide the stable-pair series by the corresponding powers of the local factors. Their constant terms are one, so the residual series still has coefficients in $\mathbb{Z}((z))$. Its coefficient at β is

$$\sum_g n_{\beta,g}\, z^{1-g}(1+z)^{2g-2}, \tag{9}$$

by the degree-one hook formula. In this finite sum the highest genus with nonzero exponent has the unique lowest Laurent power, and its local factor has leading coefficient one. The coefficient of that power is therefore the exponent itself. It is integral. Subtract its summand in (9) and continue in descending genus. This proves integrality for every β. Equation (7) identifies these integers as $c_{\beta,g}(X)$.

6. Comparison with scalar local curves

The first discrepancy between the two actions can be read in H itself. Equation (8), or reconstruction in degree one, gives $a_{g,1}=x_{g,1}$. Thus the difference $a_{2,2}-x_{2,2}$ is primitive: the mixed coproduct term in each case is $x_{2,1}\otimes x_{2,1}$. It is killed by all ρ_d with $d>1$ and hence is a finite linear combination of $x_{h,1}$. Each partition of two has hook lengths $1,2$. Its genus-two contribution to (4) is $w^2(4-w)$. At scalar weights the corresponding total is $8w^2$ [BP08, §8]. The difference of the evaluations is $-2w^3$, which is the evaluation of $-2x_{4,1}$. It follows that

$$a_{2,2} = x_{2,2} - 2x_{4,1}. \tag{10}$$

In degree three the change of local factors affects the integrality of compact product exponents. We give the comparison to fix the direction of the change. Let $e_{\beta,g}$ be the exponent of $f_g(Q^\beta)$ in the scalar factorization of $\mathcal{Z}_X$, and put

$$F_g = \mathrm{GW}(f_g), \qquad K_g = \log F_g, \qquad J_g = \log E_g.$$

Let α be an effective class whose image in $H_2(X;\mathbb{Z})/\mathrm{tors}$ is primitive. If $d\gamma = 3\alpha$ with $d>1$, passage to the free quotient gives $d=3$. The effective roots γ all have the same primitive image as α, so they have no proper integral roots and $e_{\gamma,g}=c_{\gamma,g}$. Comparison of the two connected series at 3α gives

$$\sum_h (c_{3\alpha,h}-e_{3\alpha,h})w^{h-1} = \sum_{3\gamma=3\alpha}\sum_g c_{\gamma,g}[q^3](K_g-J_g),$$

where the sum is over effective roots. The scalar local calculation of [Wue26a, §3] gives

$$[q^3](K_2-J_2) = 120w^4-20w^5+2w^6,$$
$$[q^3](K_3-J_3) = 14448w^7-1136w^8+\tfrac{4720}{3}w^9-332w^{10}+40w^{11}-2w^{12}.$$

For the second row the scalar disconnected coefficient is $2673w^6+9612w^7+2534w^8+(292/3)w^9+w^{10}$. The three antidiagonal partitions contribute $2U^2+V^2$, with $U=w^3(4-w)(3-w)^2$ and $V=w^3(3-w)^2$. Passing to the logarithmic difference subtracts the quadratic cross term $w^2(24w^5-2w^6)$. For $g=0,1$ the local series agree, and for $g\geq 4$ their cubic

difference is $O(u^{6(g-1)+2})$ [Wue26a, §2]. Thus only the genus-three row contributes to w^9. Taking this coefficient gives

$$e_{3\alpha,10} = c_{3\alpha,10} - \tfrac{4720}{3} \sum_{3\gamma=3\alpha} c_{\gamma,3}. \tag{11}$$

On the smooth projective toric hypersurface of [Wue26a] there is a first Hirzebruch surface with negative section s and ruling fiber f. The group $H_2(X;\mathbb{Z})$ is torsion-free, and the class $\alpha = 3s + 4f$ is primitive, so the sum has only one term. The minimal stable-pair moduli space in this class is $\mathbb{P}^{13}$, and the primitive genus-three coefficient is $c_{\alpha,3} = -14$. For this threefold (11) becomes

$$e_{3\alpha,10} = c_{3\alpha,10} + \frac{66080}{3}.$$

The antidiagonal exponent on the right is an integer by Theorem 1.1, while the scalar exponent on the left has fractional part $2/3$. Both products represent the same universal cycle series.

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