BRANCHED TRIPLE COVERS AND WALL CROSSING FOR LOCAL CURVE COUNTS

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Abstract. A branched triple cover of a genus-two curve can change the local curve count while preserving all determinant signs on unramified covers of degrees at most seven. At the crossing the degree-three elementary multiplicity in genus five changes by two in absolute value. At a generic transverse crossing on a prescribed simply branched triple cover with symmetric monodromy over a curve of genus at least two, we determine the local Gromov–Witten change in degree seven and every source genus. Each nonempty subset of branch values gives a nodal map joining the sextic Galois closure to the original curve. Its contribution is the degree-one series of a smooth curve, with sign determined by the number of nodes and the sum of their branch weights, up to a common orientation sign. On a fixed symplectic six-torus the other curves compensate each change in the same integral homology class and elementary genus.

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1. Introduction

A connected triple cover of a genus-two curve, simply branched at two values, has genus five. Suppose the curve is embedded in a symplectic six-manifold and we vary its normal Cauchy–Riemann operator while keeping the curve fixed. A deformation of the cover away from the curve becomes possible when its pullback normal operator acquires a kernel. We show that passage through this wall can change the local curve count both in the class of the triple cover and in degree seven. Every connected unramified cover through degree seven can have the same determinant sign on both sides.

To specify the local count, let C have genus $g \geq 2$ and let $N \to C$ be a complex rank-two bundle of degree $2g-2$, as for the normal bundle of a curve in a threefold of vanishing first Chern class. A real Cauchy–Riemann operator D on N defines an almost complex structure on its total space. Over a holomorphic map u, a section ξ of u^*N gives a holomorphic graph precisely when $u^*D\xi = 0$. Call D rigid through degree m if every connected smooth finite cover of degree at most m has zero pullback kernel, and super-rigid if this holds in all degrees. On an unramified cover the normal operator has index zero. At a rigid endpoint its determinant has a sign, taken with the orientation continued from complex-linear operators.

The contribution of an isolated curve in class $[C]$ has an integral decomposition into the elementary clusters of Ionel and Parker [IP18, DIW26, Wue26b]. We write $n_{d,h}$ for the signed multiplicity of the elementary factor of genus h in class $d[C]$ in the isolating neighborhood. Its normalization is the local series of a genus-h curve with opposite normal weights. Bai and Swaminathan coorient the wall of smooth double covers and describe the bifurcation of a smooth cubic cover [BS24, Theorem 4.12(ii)]. We realize the cubic crossing on a prescribed cover and compute the higher-degree change due to its nodal joins. Doan and Walpuski study chambered invariants of real Cauchy–Riemann operators [DW25]. Along paths of the realized unramified crossings, the local series on intermediate covers determine the change on the original curve [Wue26j, Theorems 4.1 and 7.1]. We ask what the unramified determinant signs can miss at a branched crossing.

At an ordinary crossing the pullback kernel is a real line with embedded graph. Differentiating the pullback equation on a generator in the branch values and the path parameter identifies those parameters with the cokernel. We write Δ for the value after a crossing minus the value before it.

Theorem 1.1. *On any genus-two curve there is an ordinary crossing on a prescribed connected S_3 triple cover with two distinct simple branch values, with super-rigid endpoints arbitrarily close on opposite sides. All determinant signs on the connected unramified covers of degree at most seven agree at these endpoints. The local elementary changes satisfy*

$$|\Delta n_{3,5}| = 2, \qquad |\Delta n_{7,13}| = 1.$$

The cover of degree three has trivial deck group. A displacement produces one embedded graph on each side of the crossing. Their fixed-parameter normal signs are opposite. Subtracting the incoming count therefore gives two contributions of the same sign. In degree seven a normal height on the sextic Galois closure vanishes above each branch value, where its graph can join the zero section. The proof of Theorem 1.1 is completed in §6.6.

1.1. The wall in degree seven.

For a rigid operator, matching across nodes also gives zero kernel on connected stable maps in the same degree. Its oriented cokernels form a real bundle V_D on the compact moduli stack $\overline{\mathcal{M}}_k(C,d)$ of unmarked connected degree-d stable maps from curves of arithmetic genus k. Its real rank is twice the virtual complex dimension

$2k-2-2d(g-1)$. Set

$$L_{d,k}(D)=\int_{[\overline{\mathcal{M}}_k(C,d)]^{\mathrm{vir}}} e(V_D), \qquad L_d(D;v)=\sum_{k\geq 0} L_{d,k}(D)v^{2k-2}, \qquad x=(2\sin(v/2))^2.$$

The normal obstruction comparison identifies this Euler integral with the local threefold invariant [Zin11, Theorem 1.2]. An elementary factor of genus h contributes x^{h-1} in its first degree [Pan99, Theorem 2]. This distinction between the domain genus k and the elementary genus h will matter whenever a stable map has contracted components. Write $H_d=1+d(g-1)$ for the genus of an unramified degree-d cover.

Fix a connected simply branched cubic $f:Y\to C$ with monodromy S_3. Write E for its set of branch values, $b=|E|=2\ell>0$, and $h_j=1+7(g-1)+3\ell+j$. The sextic closure has genus $1+6(g-1)+3\ell$. Attaching it to C at j distinct points gives precisely the arithmetic genus h_j.

Theorem 1.2. *At a generic transverse cubic crossing there is one punctured interval on which every pullback to a connected stable map of degree seven is injective, in every source genus. Choose super-rigid endpoints on opposite sides in this interval and write Δ for the value on the positive side minus the value on the negative side. There are real weights w_i indexed by E, with all nonempty subsums nonzero, and one orientation sign κ such that, with $a_I=\kappa(-1)^{|I|}\operatorname{sgn}(\sum_{i\in I} w_i)$,*

$$\sum_{k\geq 0}\Delta L_{7,k}\,v^{2k-2}=\sum_{\varnothing\neq I\subseteq E} a_I x^{h_{|I|}-1}, \tag{1}$$

$$\Delta n_{7,h_j}=\sum_{|I|=j} a_I. \tag{2}$$

All other degree-seven elementary multiplicities are constant across the crossing. For fixed f and N, every chamber of the real hyperplanes where a nonempty subsum vanishes occurs in this way. The common sign can equivalently be absorbed by reversing all the weights.

The interval assertion concerns degree exactly seven. For the endpoint factorizations we also need rigidity in other degrees, which is a residual condition on operators [Wen23, DW23]. On a generic path through the prescribed crossing one may ensure super-rigidity at a countable dense set of noncentral times on both sides: evaluation at each of these times is a submersion on paths with fixed first jet at zero, by variations supported away from zero. Intersect the corresponding countably many residual conditions. Rigidity in all degrees on the whole punctured interval is unnecessary.

1.2. **The sign at the nodes.** The three sheet values of a cubic kernel section have sum zero. On the sextic Galois closure their conjugates span the real standard representation of S_3. Fix one transposition τ. Over each branch value the three ramification points of the sextic have the three different inertia transpositions. Exactly one is fixed by τ. At each of these chosen points a height in the negative eigenspace of τ vanishes: its value equals its own negative. A nonzero vector on this line has trivial stabilizer, so its graph requires all six sheets. In degree seven only one sheet remains to carry the zero height. Selecting a nonempty set of joins connects the two components.

At a chosen node there are two complex variables: one moves the attaching point along the graph and the other smooths the node. Modulo the tangent of the graph, the leading matching term is antilinear in the smoothing variable. It contributes one minus sign per node. The other line of the standard representation, fixed by the transposition, supplies an

even height on the entire sextic component. Every matching equation sees this same real parameter. After the complex variables are eliminated there is one real equation left. Pairing the next term at each node with the cubic adjoint gives a real weight w_i. The coefficient of the common height is, up to common sign, $\sum_{i\in I} w_i$. In particular one cannot recover the sign of a two-node join from the individual node signs. For two branch values on a genus-two target the four possible pairs $(\Delta n_{7,12}, \Delta n_{7,13})$ are

$$(-2,1), \qquad (0,1), \qquad (0,-1), \qquad (2,-1).$$

A displacement replaces each isolated join by regular graphs. Degree-one maps to these graphs account for contracted tails and bridges and give the full factor x^{h_j-1}.

1.3. **Compensation on a compact target.** When the normal model is realized on a closed threefold, deformation invariance constrains the curves outside its isolating tube. At each generic endpoint perturbation the bounded-energy cycle space splits into parts supported in the tube and in its complement. Call the elementary multiplicities of the latter part complementary.

Theorem 1.3. *On the standard symplectic six-torus, fix a genus-g symplectic curve of primitive class A as in §8. For each positive even b the cubics in Theorem 1.2 with b branch values can be realized on this curve. The changes in the complementary elementary multiplicities in class $7A$ are the negatives of (2). The sharp maximum of the sum of their absolute values over all genera is 2^b-1.*

The exterior change is determined in each integral class and genus even when the curves realizing it differ at the two endpoints. With all branch weights of the same sign, all subsets of a given size contribute with the same sign, which gives the sharp bound.

The smooth-cover calculation and the first nodal comparison, where a double cover joins the original curve, occupy §§2–3. The application to two spin endpoints in §3.3 uses the absolute series of [Wue26f, Theorem 1.1]. For the degree-seven theorem we classify the central kernels in §4, compute the local degree at the joins in §5, and count contracted components in §6. Sections 7 and 8 give the smooth degree-seven covers near a change of sign and the compact compensation. Appendix D compares two normalizations at a double wall, using the compact example of [Wue26a]. Appendix E treats the first possible genera in higher degrees. Simultaneous crossings of several branched covers remain open.

2. Single walls in degrees two and three

Set $c=g-1$, and let $\mathcal{U}$ be the open set of operators invertible on C. The sign $\epsilon(D)$ is the sign of the determinant at the base. Along a generic path in $\mathcal{U}$, the numbers of smooth covers crossing with a kernel are signed using the derivative on the moving branch values and path parameter. Consider a generic oriented path from D_- to D_+ in $\mathcal{U}$, with both endpoints rigid through degree three. For $d=2,3$ let $W_{d,l}$ count pairs (t,f), up to isomorphism of f over C, for which the pullback of D_t to the smooth connected cover f has a kernel and f has $2l$ simple branch values. The term $l=0$ counts unramified covers. The source genus is $1+dc+l$. Each pair is counted once with the determinant sign of the Hurwitz and path derivative on its kernel line (4). Put $W_d(x)=\sum_l W_{d,l}x^l$ and $\Delta L_d = L_d(D_+;v)-L_d(D_-;v)$.

Theorem 2.1. *For a generic path as above the polynomials W_2, W_3 have finite support. With $\epsilon = \epsilon(D_-) = \epsilon(D_+)$ one has*

$$\begin{aligned} x^{-2c}\Delta L_2 &= W_2(x), \\ x^{-3c}\Delta L_3 &= 2W_3(x) + \epsilon\big(W_2(x(1-x)^2) - W_2(x)\big). \end{aligned} \tag{3}$$

In particular all the integers $W_{3,l}$ depend only on the endpoints.

A smooth cubic crossing of source genus h contributes twice its sign times x^{h-1}. At a double crossing with $2l$ branch values, a connected cubic map may join the cover to the identity at any nonempty subset of these values. The change for j joins has sign $(-1)^j\epsilon\eta_f$, where η_f is the double crossing sign. The factor $(1-x)^{2l}$ in (3) results from the sum over these subsets. The sign at a join is the real degree of an antilinear smoothing equation (Theorem 3.1).

For a smooth simply branched or unramified covering f of genus h with a single transverse kernel and an embedded graph isolated in the first affected degree, the two real amplitudes give $2\eta_f/|\operatorname{Deck} f|$ after quotienting by the covering symmetries acting on the line. The genus-h sine monomial already incorporates every higher-genus contracted decoration of the displaced embedded curve. For a double the deck involution exchanges the amplitudes. For the ramified S_3 cubic the deck group is trivial. In the next arithmetic genus the coefficient is $-(2\eta_f/|\operatorname{Deck} f|)(h-1)/12$. For the double with two branch values on a genus-two target it splits into $-\eta_f/6$ on the moving-tail locus and $-\eta_f/24$ at each branch value. The double-cover displacement and the separate elliptic-tail terms are proved in Appendix A. We first establish the corresponding smooth cubic calculation. The joined maps enter the chamber formula in §3.

2.1. **Smooth covers at a crossing.** Let $f : Y \to C$ be a smooth connected cover, and write $D_f = f^*D$. We regard covers up to isomorphism over C. Let $\mathcal{H}$ denote the fixed-target Hurwitz space of the ramification type of f. At a point (f,t) with kernel K and cokernel P the differential of the family of operators induces a map

$$W : T_f\mathcal{H} \oplus \mathbb{R}_t \longrightarrow \operatorname{Hom}_{\mathbb{R}}(K,P). \tag{4}$$

It is obtained by differentiating on K and then dividing by the image of D_f, so it is independent of the trivializations. When K is a line and W is an isomorphism, write η_f for its sign. We orient the source by the complex orientation of the Hurwitz space and increasing time. On the target we use the full Cauchy–Riemann determinant orientation, continued from complex-linear operators. Indeed $\dim_{\mathbb{R}} P = 2\dim_{\mathbb{C}}\mathcal{H} + 1$ in this case, and $\det\operatorname{Hom}_{\mathbb{R}}(K,P)$ differs from $(\det D_f)^*$ by an even tensor power of K^*. This convention also applies to a double cover.

Proposition 2.2. *Fix endpoints rigid through degree three. For a residual set of smooth paths between them with fixed symbol and invertible base operator, every smooth cubic cover with kernel has monodromy S_3, trivial deck group, and only simple ramification over distinct branch values. Its kernel is a real line, the graph of a generator is embedded, and (4) is an isomorphism. At that parameter every other smooth connected cubic cover and every smooth connected double cover have injective pullback, in every source genus.*

Proof. We first vary the operator and a nonzero kernel vector u on one cubic cover. On an unramified disk write u_1, u_2, u_3 for its three values. Their sum is single-valued on the punctured target and solves D there. It is bounded at the branch points and hence extends across them by removal of singularities. Base invertibility gives $u_1 + u_2 + u_3 = 0$. On the connected Galois closure these three sections span the real sum-zero representation of the monodromy group. In fact the map from the permutation representation sending the

coordinate vectors to the three conjugate sections kills the invariant vector, and the remaining real plane is irreducible for either transitive group C_3 or S_3. In particular each pair of sheet values is real-linearly independent on an open dense subset. A relation $u_i = au_j$ on an open set with $u_j \neq 0$ and a real gives $(\bar{\partial}a)u_j = 0$, so a is constant. Unique continuation on the Galois closure contradicts the representation just computed.

Consider the universal equation $D_f u = 0$. A covector p annihilating its differential in the section and operator directions solves the homogeneous adjoint equation. On an unramified disk use the base coordinate to write p_i for its sheet values, with the density in the real duality pairing expressed in that coordinate. An arbitrary real zeroth-order variation A of D supported there gives the variations (Au_1, Au_2, Au_3). Where the first two vectors are real-linearly independent, these range over all triples of form values with sum zero. Thus $p_1 = p_2 = p_3$ on such a disk. Testing the section variation f^*w gives $\langle p, f^*(Dw)\rangle = 0$ for all smooth w. Every smooth base form equals Dw by base invertibility, so $\sum_i p_i = 0$ on every unramified disk. Hence p vanishes there and identically by unique continuation. The differential is surjective since its range contains the closed finite-codimensional image of D_f.

Suppose the cover has b simple and q totally ramified values. These exhaust the cubic profiles. In a chart with ordered branch values, the $b + q$ distinct values are local complex coordinates. Indeed the finite covering of their complement is locally fixed by monodromy and has a unique completion by the prescribed power maps. Riemann–Roch and Riemann–Hurwitz give $\operatorname{ind}_{\mathbb{R}} D_f = -2(b + 2q)$. Quotienting the vector by nonzero real scaling removes one kernel direction. The projectivized incidence thus projects to the space of operators with index

$$2(b + q) - 2(b + 2q) - 1 = -1 - 2q.$$

Relative path transversality excludes $q > 0$ and leaves a regular zero-dimensional incidence for $q = 0$. A kernel of dimension greater than one would give a positive-dimensional projective fiber. A cyclic cubic has no invariant kernel vector, by descent to the invertible base, and every nontrivial irreducible real representation of C_3 has dimension two. Thus the monodromy is S_3. A point stabilizer in S_3 is self-normalizing, so the deck group is trivial. At the remaining covers, projection of the regular linearization to P gives the surjectivity of W. Both sides have dimension $2b + 1$.

We give the marked transversality argument because separation of two cubic covers will require a condition on their complex sheet spans. At distinct points of a fiber evaluate the difference of their values. At a ramification point take the complex-linear first derivative of the section in a frame pulled back from C. Its antiholomorphic derivative already vanishes at a solution since $df = 0$ there. Finally, over a fixed regular base point, evaluate the pair

$$(u_1 - u_3, u_2 - u_3) \in N_x^2.$$

Each of these linear maps J on sections kills all base pullbacks. To check surjectivity for the equation augmented by J, hold the marks fixed and let (p, γ) annihilate the section and operator variations. Off the marks p solves the homogeneous adjoint equation. The same test with base pullbacks kills its trace, and the localized variations A force p to vanish on an unmarked disk. By unique continuation p is supported on the marks. A distribution supported at these finitely many points has finite order. At each mark we may prescribe the value and the holomorphic first derivative of a test section w while arranging that $D_f w$ vanish to any fixed jet order. In a local expression $D_f = \bar{\partial} + B$ the Taylor coefficients with positive antiholomorphic degree are determined successively, leaving the holomorphic coefficients free. Multiplying the resulting polynomials by cutoffs gives the required global sections. Now

$\langle p, D_f w\rangle + \gamma(Jw) = 0$ forces $\gamma = 0$. Elliptic regularity for the remaining adjoint equation gives $p = 0$.

The coincidence of two values has real codimension four. The base point in an unramified fiber varies in two real dimensions, and a mark over a branch value, including the pair consisting of the ramified and the unramified preimage, is discrete over its cover. Vanishing of the derivative at a ramification point likewise imposes four real conditions at a discrete mark. A generic path avoids all three defects, so the graph is an injective immersion of a compact curve. For the last evaluation the pair has complex rank one in real codimension two and rank zero in codimension eight. Taking a countable dense set of fixed base points shows that at every crossing there is an unramified disk on which the two differences, and hence u_1, u_2, are complex-linearly independent. The rank condition is unchanged by relabeling the sheets.

It remains to exclude a second cover at the same time. Use separate Hurwitz and section variables and a common variation A for the two equations. An annihilating pair of adjoints solves the homogeneous adjoint equation on each cover and has zero trace on each by the base-pullback test. For a nonzero cubic adjoint the three values on an unramified disk span a real plane on an open dense set. Indeed, view the adjoint there as a dual-bundle-valued differential in the base coordinate. On the connected regular cover of the complement of the branch set the trace-free conjugates form the same irreducible plane as above. Real proportionality on a disk forces a constant ratio by their common adjoint equation. The argument needs no extension of the adjoint divided by the base differential across a branch point.

The tensor pairing with A on that disk is $\sum_i u_i \otimes p_i$. Substituting the trace identities gives coefficient matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ in the first two values of each triple. Thus, when both pairs are real-linearly independent, the tensor has real rank two and image $\operatorname{span}_{\mathbb{R}}(u_1, u_2)$. For a double cover the corresponding tensor is $2u_1 \otimes p_1$, of rank one on an open dense set for nonzero adjoint. Tensors of these different ranks cannot cancel. This proves surjectivity for a cubic and a double cover with marked kernel lines.

For two cubics restrict to the open locus where the first kernel has a complex-independent pair on some disk. If both adjoints are nonzero, cancellation for all A forces the two real sheet planes to agree on a smaller disk on which all the rank conditions hold. Express any kernel sheet of the second cover as $au_1 + bu_2$ there, with a, b real. Its equation gives $(\bar\partial a)u_1 + (\bar\partial b)u_2 = 0$. Complex independence makes a, b constant. On the connected regular cover of the common punctured target defined by the intersection of the two monodromy kernels, unique continuation extends all these relations. The two planes of conjugate sections coincide as representations of the punctured target's fundamental group. The sum-zero representation of a transitive cubic monodromy group is faithful, so the monodromy kernels agree. In their common quotient, all index-three subgroups are conjugate, both for S_3 and for C_3. The two covers are isomorphic on the punctured curve and then on its unique normal completion. This excludes cancellation for distinct covers. If only one adjoint were nonzero, its tensor would already have positive rank. The paired universal equation is therefore transverse off the diagonal. Its projection index is the sum of the two single-cover indices, at most -2. Here the double-cover index is -1 by the same branch count and Riemann–Hurwitz. A generic path avoids these pairs.

All uses of transversality may be made on sheet-labelled Hurwitz charts, with Sobolev order large enough for moving first jets and a higher differentiability class for coefficient variations. Scaling is a free kernel direction before taking projective lines. The equations augmented by evaluations have closed finite-codimensional range in the section variables. Consequently smooth coefficient variations suffice for all the surjections above. More explicitly, exhaust

each marked chart by compact subsets away from the relevant diagonals and normalize kernel vectors in L^2. On each compact subset elliptic estimates give precompactness of the vectors for smoothly converging paths. Transversality there is open and can be achieved by finite-dimensional smooth perturbations supported away from the endpoints, by the parametric transversality theorem. For the open locus in the paired calculation one also exhausts the locus where a specified complex minor is nonzero. Countably many such charts, profiles and genera suffice. This yields the asserted residual set in the smooth topology. For the analogous double-cover marked incidence see [BS24, Lemmas 5.11 and 5.15]. □

Proposition 2.3. *Suppose a path as in Proposition 2.2 meets the wall at a smooth cubic cover $f : Y \to C$ of genus h, at time zero. There is a punctured interval of operators rigid through degree three. For any sufficiently small $t_- < 0 < t_+$ the series on the original curve satisfy*

$$L_3(D_{t_+}; v) - L_3(D_{t_-}; v) = 2\eta_f x^{h-1}, \qquad L_d(D_{t_+}; v) = L_d(D_{t_-}; v) \quad (d = 1, 2).$$

At a simple double crossing of genus h, the corresponding degree-two change is $\eta_f x^{h-1}$.

Proof. Fix a generator ξ of K at the cubic cover. For a simply branched cover the quotient of $TY \to f^*TC$ is torsion of length b, the number of simple branch values. Its H^0 is the Hurwitz tangent and its H^1 vanishes. Thus the tangential deformation problem is regular and contributes the complex orientation.

We realize the path in a neighborhood of the zero section by prescribing the graph equation over a holomorphic projection $\varphi : Y' \to C$ to be

$$\varphi^* D_t u + \varphi^* B(u) = 0. \tag{5}$$

Here B is a smooth fiberwise real quadratic with values in $\Omega_C^{0,1} \otimes N$ and φ^* pulls back its form argument. In a connection splitting, the horizontal-to-vertical block of the almost complex structure prescribes precisely the zeroth-order term in this equation, with standard complex structure on the fibers [BS24, Lemma 5.9]. Cut off this block outside a disk to extend the structures to $\mathbb{P}(N \oplus \mathbb{C})$. A fiber Fubini–Study form extended using a unitary connection, plus a sufficiently large multiple of a positive base form, tames the compact family. The extension can be made for all fiber dilations used below on a fixed disk of radius greater than one, by applying the cutoff outside that disk.

Choose a nonzero adjoint p annihilating $W(T_f\mathcal{H})(\xi) \subset P$. On an unramified disk where one of its values p_1 is nonzero and the three ξ_i lie on distinct real lines, choose smooth real covectors α_2, α_3 with $\alpha_i(\xi_i) = 0$ and $\alpha_i(\xi_1) = 1$. Multiplying $\alpha_2(u)\alpha_3(u)$ by a compactly supported vector-valued form pairing positively with p_1 gives B with $\langle p, f^*B(\xi)\rangle \neq 0$. Only the first sheet contributes. This choice is possible for a cubic because its deck group is trivial.

Use local Hurwitz coordinates y and a real kernel amplitude z. Write $u = z(\xi + w)$ with w in a fixed complement of K. After division by z the image equation determines $w = w(y, t, z)$ by the implicit function theorem. The remaining equation is $F(y, t, z) = 0$ in P. Its (y, t) derivative at zero is $W(\cdot)(\xi)$, and F_z pairs nontrivially with p by the choice of B. The terms from differentiation of w lie in the image of D_f. Thus the solution is $(y(z), t(z))$, where $t(0) = 0$ and $t'(0) \neq 0$. The undivided reduced equation is $zF = 0$. There is one nonzero graph on each nearby slice, and it is embedded since dilation by $1/z$ makes it converge smoothly to (f, ξ).

For the fixed-time reduced section zF, order its columns by the complex-oriented y variables followed by z. Along the nonzero branch the first $2b$ columns are zF_y, and modulo these the last is $-zt'(z)F_t$. The identifications of the invertible image complement and the tangential

determinant are the same as those used in (4). The normal determinant sign of the embedded graph is consequently

$$-\operatorname{sgn}\bigl(z^{2b+1}t'(z)\bigr)\eta_f = -\operatorname{sgn}(t)\eta_f.$$

Changing ξ changes the orientations induced on z and on P together. The signed number of nonzero curves therefore changes by $-2\eta_f$. This is the smooth-domain bifurcation of [BS24, Theorem 4.12(ii)], realized here for the prescribed path of first-order operators.

We next isolate these supports in all normalization genera. Suppose there are additional simple curves of degree at most three contained in tubes shrinking to the zero section at parameters tending to zero. Divide their heights by their fiber maxima $\rho > 0$. Their equations on the unit disk have quadratic term ρB and converge smoothly to the linear graph equation. Their image cycles have bounded mass for a common taming form, since its class on the disk bundle comes from the base and the projection degrees are bounded. Cycle compactness gives a limiting holomorphic cycle with convergence of currents and Hausdorff convergence of supports [DIW26, Theorem 2.11]. Integration of a pulled-back area form preserves projection degree, and the limit contains a point of height one. Every irreducible component has positive projection degree, since a compact complex curve in an affine fiber is constant. The normalization of a component reaching positive height therefore carries a nonzero pullback kernel. By Proposition 2.2 this component is the graph of a nonzero multiple of ξ. Its degree exhausts the available three. The limit is thus embedded of multiplicity one.

Near such a cycle the simple parametrizations are embeddings and converge up to reparametrization [DIW26, Theorem 2.20 and Proposition 2.21]. In particular their genus stabilizes, and elliptic regularity gives smooth convergence. After scaling back, the covers and sections belong to the Hurwitz chart just analyzed. This proves the existence of one closed tube and one interval with no additional simple supports. Shrink the interval so the nonzero branches lie strictly inside. For the linear endpoint equation use the same argument with unit-height kernel graphs for a sequence of nonzero times tending to zero, retaining the mapping multiplicity in each pushforward cycle. The embedded multiplicity-one limit again gives convergence of the covers and normalized sections. The reduced equation for this linear problem is $F(y, t, 0) = 0$, whose unique nearby solution is $(0, 0)$. Thus on the punctured interval all smooth pullbacks through degree three are injective. On a nodal domain every positive normalization then has zero section and the matched constants on contracted components are zero as well.

Fix an arithmetic genus k and a positive degree $d \leq 3$. The stable maps whose images lie in the closed tube form an open and closed part of the parametrized moduli in the compact completion. Indeed all their simple supports lie in the interior, and compactness preserves containment in the closed tube. Their endpoint virtual degrees agree by deformation invariance [DIW26, §3.1, (3.4)–(3.5)]. Maps supported on C contribute the fixed-target Euler number by [Zin11, Theorem 1.2, (1.7) and (1.10)]. Using even one nonzero graph exhausts degree three, so a map with that support has degree one onto the embedded graph, with arbitrary contracted components.

For clarity, let D' be the invertible normal operator of one such embedded curve. A degree-one stable map to the curve has one effective normalization mapping isomorphically, and each connected contracted subcurve attaches at just one point. For any deformation of D', restriction to the effective normalization identifies the kernels of the base and pullback operators and is a left inverse on their cokernels. Thus the quotient of the two Fredholm complexes has zero kernel throughout, and the stabilized quotient cokernels form a bundle, including over nodal maps [BS24, Appendix B]. At an invertible endpoint its orientation

differs from the full normal orientation by the determinant sign of D'. The complex normal degree is $2h-2$, since the ambient first Chern class vanishes on the disk bundle. Deform to the Dolbeault operator of two acyclic lines of degree $h-1$. The quotient bundle there has complex orientation, and its degree-one Euler series is x^{h-1} by [Pan99, Theorem 2]. Consequently the change contributed by the nonzero graphs is $-2\eta_f x^{h-1}$, including every contracted configuration at its own fixed arithmetic genus. Conservation gives the asserted change on C and constancy in lower degrees.

For a double cover the involution negates its kernel generator and identifies the two nonzero amplitudes in an odd cubic perturbation. The fixed-target comparison then gives the change $\eta_f x^{h-1}$ in degree two (Theorem A.1). □

3. A double cover joined to the zero section

For a double wall the first reducible support occurs already in degree three. A branch point supplies a joining flag, and different flags may occur together. This leads to a relation between the signed counts of smooth double and cubic covers.

3.1. **Smoothing a double cover joined to the identity.** Let D_t be a smooth path of real Cauchy–Riemann operators with fixed symbol on a complex rank-two bundle N of degree $2g-2$ over C. Suppose $D = D_0$ is invertible, of determinant sign ϵ. Let $f : Y \to C$ be a connected smooth double cover with $b > 0$ simple ramification points over distinct values. Suppose that $K = \ker f^*D$ is a real line, that its generator ξ is odd under the deck involution, and that $(f, \xi) : Y \to N$ is an embedding. Put $P_f = \operatorname{coker} f^*D$. Write s for the b complex branch parameters and the increasing real path parameter together. We assume the double crossing is transverse, so that the derivative of the pullback operator on its kernel induces an isomorphism

$$W_f : T_{(f,0)}(\mathcal{H}_2 \times \mathbb{R}) \longrightarrow \operatorname{Hom}_{\mathbb{R}}(K, P_f).$$

Here $\mathcal{H}_2$ denotes the fixed-target double Hurwitz chart. Write η_f for the sign of W_f with the absolute Fredholm orientation, obtained by deformation to a complex operator.

Choose a nonempty subset I of the ramification points. Join each $r_i \in I$ to $x_i = f(r_i)$ on one copy of C, and denote the resulting finite cubic map by h_I. Its arithmetic genus is $H = g(Y) + g + j - 1$, where $j = |I|$. Near r_i take coordinates $w = z^2$ centered at x_i, r_i, and a local frame holomorphic for the complex-linear part of D. Write $D = \bar\partial + B(w)\, d\bar w$ with $B(w)$ antilinear, and put $v_i = \partial_z \xi(r_i)$. Since $df(r_i) = 0$, $d\xi(r_i)$ is complex linear, and embeddedness gives $v_i \neq 0$. The condition

$$B(x_i)v_i \neq 0 \quad \text{in } N_{x_i}/\mathbb{C}v_i \tag{6}$$

is independent of the coordinates, complex frame and real generator. Indeed the change of frame adds only a complex-linear term to the zeroth-order coefficient, and $\xi(r_i) = 0$.

Theorem 3.1. *With the double crossing and joining set I as above, assume (6) at every point of I. The pullback kernel at h_I is isolated in the full fixed-target chart of genus H times the path. Its local contribution to the change of the absolutely oriented normal Euler number is*

$$(-1)^j \epsilon \eta_f.$$

If (6) holds at every ramification point, summing over all choices of j joins gives $\binom{b}{j}(-1)^j \epsilon \eta_f$. Condition (6) holds at all such crossings on a generic smooth path of base-invertible fixed-symbol operators with rigid endpoints.

The attaching parameter moves the common value along the tangent of the double graph. The smoothing parameter contributes in the quotient by that tangent. Its leading term is conjugate-linear after a radial change of variable, which accounts for the sign at a node.

The full local calculation is given in Appendix B.1. Its normalized equation has one complex tangent column from the moving branch and one complex antilinear column from the smoothing at each selected point. The projected coefficient in (6) makes both columns regular after the radial change of variable. The base determinant and the absolute crossing coorientation supply the two remaining signs. Unlike the sextic calculation below, the double has only an odd kernel line; imposing its normalization leaves no even height common to the nodes.

To pass from this local degree to the Gromov–Witten change, one must also allow contracted curves at the joins and at smooth points of the two branches. The displacement comparison of Theorem B.3 replaces the joined map by regular embedded graphs of genus H. In source genus k their degree-one contribution is $[v^{2k-2}]x^{H-1}$ times the signed degree above. Proposition B.2 gives the finiteness needed to sum these contributions along a generic path.

3.2. **The chamber formula and the balanced product.**

Proof of Theorem 2.1. By Proposition B.2 and its double-cover counterpart there are finitely many crossings on the path. At a smooth double wall with $2l$ branch values the degree-two change is $\eta_f x^{2c+l}$ (Theorem A.1). At a smooth cubic wall with the same branch count the degree-three change is $2\eta_f x^{3c+l}$ by Proposition 2.3. The separation in Proposition 2.2 applies to covers of every genus.

Consider the degree-three virtual cylinder at a double crossing. A nonzero section must restrict nontrivially to a nonconstant normalization, since matching determines its values on every connected contracted block. There is no cubic normalization with kernel at this parameter. There must therefore be a degree-two normalization carrying the odd kernel and a copy of C at zero height. Both graphs are embedded. Their common image points are the $2l$ ramification points, each with one preimage on each graph. There is no self-node and a contracted block has at most two nonconstant attaching branches. Thus at least one of these $2l$ points is used to connect the graphs, either directly or by a contracted bridge. The set I of such points determines the compact locus of Theorem B.3; every other contracted component belongs to its allowed decorations. When $l = 0$ there is no connected support of this type.

Put $j = |I|$. The arithmetic genus of the bare join is $(1+2c+l)+(1+c)+j-1 = 1+3c+l+j$. Theorem B.3 therefore gives the total change at this double crossing as

$$\sum_{j=1}^{2l} \binom{2l}{j} (-1)^j \epsilon \eta_f x^{3c+l+j} = \epsilon \eta_f x^{3c+l} \big((1-x)^{2l} - 1 \big).$$

For each fixed source genus the supported Euler change is additive on the finite collection of disjoint compact loci just classified. Outside them the ordinary oriented cokernel extends over the parameter. Summing coefficientwise over all arithmetic genera proves (3). Substitution $x \mapsto (2\sin(v/2))^2$ is injective on polynomials, since the substituted series begins with v^2. The first equation determines W_2 from the endpoints and the second then determines W_3. □

Let H be Pardon's rational algebra of complex cycle families [Par25], and F_h the balanced antidiagonal local curve series of genus h in this algebra [Wue26b]. In the comparison algebra S of almost complex cycle families, let $P_d(D)$ denote the class of the isolated cycle $d[C]$ for the homogeneous normal equation D. The variable q records the degree over C. A *balanced*

lift through degree three is a product of $F_h(q^d)$ with integer exponents, finite at each $d \le 3$, whose image under the comparison $H \to S$ agrees with $1 + \sum_{d=1}^{3} P_d(D) q^d$ modulo q^4.

Theorem 3.2. *Each endpoint in Theorem 2.1 has a unique balanced lift $P^H_{\le 3}(D)$ through degree three. Set $V(x) = 2W_3(x) + \epsilon(W_2(x(1-x)^2) - W_2(x))$. Then*

$$\frac{P^H_{\le 3}(D_+)}{P^H_{\le 3}(D_-)} = \prod_{l \ge 0} F_{1+2c+l}(q^2)^{W_{2,l}} \prod_{l \ge 0} F_{1+3c+l}(q^3)^{[x^l]V(x)} \pmod{q^4}.$$

Proof. We apply the finite-degree factorization of [Wue26j, Corollary 3.2] using only the stated endpoint rigidity. Smooth the intersections of a horizontal torus with $g-1$ parallel vertical tori in a product of two symplectic two-tori, and include the product in a six-torus. The resulting genus-g surface represents a primitive class A, as pairing in the first two factors with a vertical torus gives one. Its symplectic normal bundle has degree $2g-2$. Identify the normal germ with the given one along the surface and extend its graph almost complex structure to a global tame structure, by interpolation to a compatible extension in a sufficiently small tube [Wue26j, proof of Proposition 3.1].

Through energy $3\omega(A)$ the only cycles contained in the closed tube are multiples of the zero section. Indeed a component in that tube projects holomorphically in positive degree; a component of nonzero height would give a nonzero kernel on its smooth normalization. The degree agrees with the positive energy label since the tube retracts to C. The cycles in the interior give a compact open and closed insertion through this cutoff. Compactness of cycles and convergence of supports preserve separation from the tube boundary under a small perturbation of the ambient structure [DIW26]. Transport to a generic perturbation and finite-energy cluster decomposition express this insertion as a product of the images of the balanced factors with integral exponents [Wue26j, Corollary 3.2]. Their coefficients generate the integral cycle algebra by [Wue26e, Theorem 1.1]. The construction at that energy involves only labels through three. Lift each factor to H. If two products first differed at label d, the difference at that label would be a finite linear combination of $[q]F_h(q)$ whose image vanishes in S. Independence of these images under fixed-genus detection [Wue26b, Theorem 5.1] gives uniqueness.

It remains to identify the exponents of these geometric products. At each endpoint, fixed-target comparison in the compact realization [Zin11, Theorem 1.2] identifies the connected local series in degrees $1, 2, 3$ with the integrals defining L_d. The series of an elementary factor in its first label is x^{h-1} [Pan99, Theorem 2]. In degree one the endpoint factor is $F_g(q)^\epsilon$ at both ends. At labels two and three the only proper divisor is one. Its contribution to the logarithm of the ratio cancels, and an exponent difference in genus h at either new label therefore contributes exactly its multiple of x^{h-1}. The finite Laurent monomials in x are linearly independent after sine substitution. Equation (3) determines all the differences asserted above. □

Write $n_{d,h}(D)$ for the exponents in this lift. The original fixed-target series in degree d vanishes below genus $1 + dc$ by Riemann–Hurwitz. For $d = 2, 3$ the only proper-divisor contribution comes from the genus-g balanced reference in label one, whose degree-d term satisfies the same bound. The remaining monomials x^{h-1} have distinct leading orders, so $n_{d,h}(D) = 0$ for $h < 1 + dc$ [Wue26j, Proposition 3.1 and Corollary 3.2]. Set $N_d(D; x) = \sum_{l \ge 0} n_{d,1+dc+l}(D) x^l$.

Corollary 3.3. *The polynomial $N_3(D; x) - N_2(D; x + x^3) + N_2(D; x)$ modulo two is constant between rigid endpoints joined in $\mathcal{U}$.*

Proof. The two exponent changes are $\Delta N_2 = W_2$ and $\Delta N_3 = V$. Modulo two one has $x(1-x)^2 = x + x^3$, $2W_3 = 0$ and $\epsilon = 1$. □

3.3. **Two spin endpoints.** Let A, B be distinct theta characteristics on C with $H^0(C, A) = H^0(C, B) = 0$. For a theta characteristic S, take the positive spin operator D_S associated with a lift of the uniformizing representation to $\mathrm{SL}(2, \mathbb{R})$. In a spin frame on the upper half-plane it is given by

$$D_S u = \bar\partial u + \frac{i\bar u}{2\,\mathrm{Im}\, z}\, d\bar z.$$

The choice of lift specifies the signs in the transition functions of the frame. On the same underlying smooth complex rank-two bundle consider the ordered endpoints

$$D_- = D_A \oplus D_A, \qquad D_+ = D_A \oplus D_B.$$

They are rigid in every degree. Indeed, on a finite cover f, the equation for one spin summand makes $d(u^2 f^* dz)$ a nonnegative area form whose density is positive wherever $u \neq 0$ and $df \neq 0$. Integration forces $u = 0$ on every positive-degree component, and nodal matching gives vanishing on each attached contracted subcurve. This also proves base injectivity when the antilinear term is multiplied by any positive scalar. At scalar zero each base Dolbeault operator is invertible by non-effectivity and Riemann–Roch. Hence the two endpoint determinants have sign $\epsilon = +1$. To connect the operators in the base-invertible locus, turn off the antilinear term on the second summand, join the two Dolbeault operators in the complement of the theta divisor in $\mathrm{Pic}^{g-1}(C)$, and restore the term on B. The complement is path connected since the divisor has real codimension two. Trivialize the smooth bundles along the interval and perturb the path relative to its endpoints.

We evaluate the two endpoints using the absolute partition formula of [Wue26f, Theorem 1.1]. Write $c = g - 1$ and $X_k = (2\sin(kv/2))^2$. At the repeated spin the disconnected degree-d coefficient is the sum over partitions $\lambda \vdash d$ with hook weight $\prod_{a\in\lambda} X_{h(a)}^c$. At the distinct pair with positive determinant the same weights are summed over just the self-conjugate partitions. These two absolute evaluations are the input from spin theory. The signed number of crossings between them follows from Theorem 2.1.

Let Z_d^- and Z_d^+ be the respective disconnected coefficients, requiring positive degree on every connected component. For $d \le 3$ the partition sums read

d	Z_d^-	Z_d^+
1	x^c	x^c
2	$2(xX_2)^c$	0
3	$2(xX_2X_3)^c + (x^2X_3)^c$	$(x^2X_3)^c$.

Indeed the row and column of size three have hooks $1, 2, 3$, while $(2, 1)$ has hooks $1, 1, 3$. The logarithm of the disconnected series has degree-two coefficient $Z_2 - Z_1^2/2$ and degree-three coefficient $Z_3 - Z_1Z_2 + Z_1^3/3$. Using $X_2 = x(4-x)$ and $X_3 = x(3-x)^2$ gives the connected differences in the order D_+ minus D_-,

$$x^{-2c}\Delta L_2 = -2(4-x)^c,$$
$$x^{-3c}\Delta L_3 = 2(4-x)^c\big(1-(3-x)^{2c}\big).$$

Corollary 3.4. *For every generic path between these ordered endpoints in the base-invertible locus, the signed polynomial of smooth cubic crossings is*

$$W_3(x) = \big(4 - x(1-x)^2\big)^c - \big((4-x)(3-x)^2\big)^c. \tag{7}$$

The largest normalization genus with nonzero signed count is $6c$*. Every such path contains at least* $8c$ *smooth cubic cover–parameter pairs in that genus.*

Proof. Insert $W_2(x) = -2(4-x)^c$ and the cubic difference into (3). The terms in $(4-x)^c$ cancel, giving (7). The two cubics on the right, before taking cth powers, have common leading coefficient -1 and difference $-8(x-2)^2$. Their difference of cth powers thus has degree $3c-1$ and leading coefficient $-8c(-1)^{c-1}$. By Riemann–Hurwitz, a cubic cover with $2l$ simple branch values has genus $1+3c+l$. Taking $l = 3c-1$ proves both assertions. □

On a genus-two target one can take any two distinct even theta characteristics. Each has degree one, hence at most one independent section, and even parity forces non-effectivity. Here $W_2(x) = -8+2x$ and $W_3(x) = -32+32x-8x^2$. The signed counts of smooth cubics are -32, $+32$, and -8 in normalization genera four, five, and six, respectively. In particular the half-coefficient of x^2 in $x^{-3}\Delta L_3$ is -10, and subtracting the double-cover term in (3) raises the signed four-branch count to -8. Every such path therefore contains at least 72 smooth cubic cover–parameter pairs across these three genera. In higher genera the signed count vanishes.

4. Normal kernels on the sextic Galois closure

In degree seven the normal kernel at a ramified cubic has two real directions on its Galois closure. Projectivizing this plane does not separate the equations at the different nodes. We identify the one remaining real equation after solving the complex matching equations.

4.1. **The cubic and its Galois closure.** Fix a complex rank-two bundle N of degree $2g-2$ over a closed Riemann surface C of genus $g \geq 2$. A real Cauchy–Riemann operator D on N has the Dolbeault symbol and a real-linear term of order zero. All kernels, cokernels and indices below are real. Let $f : Y \to C$ be a connected cubic with monodromy S_3 and $b > 0$ distinct simple branch values. Write $Z \to C$ for the smooth Galois closure and V for the real plane $\{(a_1,a_2,a_3) : \sum a_i = 0\}$ with its permutation action. Locally the involution at a ramification point acts by $z \mapsto -z$, with $w = z^2$ on the base. The fixed-target Hurwitz space of f has complex dimension b.

Let $D = D_0$ be invertible on C. At an *ordinary transverse cubic crossing* of a path D_t, the kernel of $D_f = f^*D$ is a line $\mathbb{R}\xi$, the graph of ξ in N is embedded, and differentiation at ξ in the branch positions and increasing time gives an isomorphism

$$W_f : T_f\mathcal{H}_f \oplus \mathbb{R} \longrightarrow \operatorname{coker} D_f. \tag{8}$$

Indeed $\operatorname{ind} D_f = -2b$. The trace of ξ is zero by removability on C and base invertibility. Its three conjugates on Z therefore span one copy of V. Over a regular disk they give three solutions with sum zero. Evaluation of this plane will have real rank two at regular points and complex span of dimension two on at least one regular disk. We explain both the generic assertion and a construction on any prescribed f.

Lemma 4.1. *Suppose two of the sheet values form a complex frame on a regular disk. For a residual set of arbitrarily small smooth variations on this disk preserving the three solutions, every connected smooth finite cover* $u : X \to C$ *has*

$$\ker u^*D = V^{K_u}.$$

Here $K_u \leq S_3$ *is the image in the lower monodromy of a sheet stabilizer for* X *in the joint monodromy of* X *and* Z*. The identification uses pullback to a common normal cover.*

Proof. Use real variations annihilating the two frame vectors s_1, s_2. On any smaller disk a dense set of these variations attains the infinite-order Petri condition at a specified interior point. This condition says that a tensor of formal solution and adjoint-solution jets with zero pointwise product jet is zero. For unrestricted variations use [Wen23, Corollary 5.10] and [DW23, Theorems 1.6.17 and 2.5.3]. For a nearby unrestricted operator D' with such a jet, solve $D's'_j = 0$ locally with s'_j close to s_j: a Cauchy right inverse on a disk of radius r has norm $O(r)$ on bounded functions, and interior elliptic estimates give closeness in any prescribed finite smooth norm. Conjugate by the complex gauge taking s_j to s'_j, then cut off the difference from D away from the point. This preserves the Petri condition there and annihilates the original columns on the entire disk.

Consider an irreducible real finite monodromy system W inequivalent to the prescribed standard system, with its ineffective punctures removed. A nonzero twisted solution provides columns w_i which, together with s_1, s_2, are real linearly independent as germs on a common normal cover. This follows from inequivalence, irreducibility and unique continuation. An annihilator of the section and operator variations has adjoint columns v_i with

$$\sum_i w_i \otimes v_i = s_1 \otimes \alpha_1 + s_2 \otimes \alpha_2.$$

This identity is the pointwise annihilator condition for maps killing the two prescribed columns. Test it against $[D, \Phi]$, with Φ a compactly supported complex-linear endomorphism of N. The integral on the left vanishes. The sections $\Phi s_1, \Phi s_2$ are independently arbitrary, so both α_j solve the adjoint equation on the disk. The Petri condition and independence of the input jets force all v_i to vanish to infinite order. Adjoint unique continuation gives surjectivity.

If the a effective punctures move, the projective incidence has projection index

$$2a - 1 - 2\sum_x \dim(W/W^{I_x}) \le -1,$$

by the orbifold index formula [DW23, Proposition 2.8.6]; I_x denotes inertia at x. Exhaust each configuration chart by compact blocks whose punctures avoid a common smaller test disk. At a good Petri jet over a block, the unit-normalized solution set is compact by elliptic estimates on the compactified normal covers. Finitely many allowed variations give surjectivity near that set. Finite-dimensional Sard gives arbitrarily small perturbations excluding all such solutions over the block. Absence is open there. There are countably many blocks and discrete monodromy types. The remaining type V is faithful, so each of its copies on a common normal cover descends to Z. Each has a one-dimensional invariant line on Y; openness of the given simple kernel on the fixed Y permits just one copy. Taking stabilizer invariants proves the formula, by bounded descent and removability over the branch points. □

At an ordinary crossing on a generic path the required regular-fiber evaluation is injective everywhere. Indeed the two independent differences of the three heights at a marked regular value can be varied independently. For completeness, the universal section equation at a cubic with complex frame disk admits arbitrary trace-free forcing there by variation of D. In the adjoint test the trace vanishes by base invertibility, hence the annihilator vanishes off the marked points by unique continuation. Any remaining adjoint distribution has finite order and is supported at the marks. Prescribe holomorphic value and derivative jets there and solve recursively for mixed jets so that the image under D_f vanishes to higher order than that distribution. Taylor polynomials with cutoffs realize these jets. Independence of the resulting two value differences kills their multipliers, and elliptic regularity removes the supported adjoint. The same argument with a holomorphic linear jet at the ramifying point

gives independent variations of the height difference at the two points in a branch fiber and the ramification derivative. One may perform these tests without first assuming a complex frame disk: two real-independent sheet values suffice for the forcing test. Such a disk of real independence exists, since collinear nonzero solutions on an open disk have constant real ratio by their equation, contradicting the standard monodromy. Real rank one for a pair of real four-vectors has codimension three, and rank zero has codimension eight. The regular mark contributes only two parameters along a transverse crossing. The complex dependence locus at a fixed regular mark has codimension two. These tests give the claimed ranks, including complex independence on some disk. On continued transverse crossing charts the preceding residual perturbations are available by multiplication by a time cutoff.

If a nonzero section on a positive component is described by $a \in V^{K_u}$, the component factors through $Z/\operatorname{Stab}(a)$. Each nonzero stabilizer orbit has size three or six. At a branch value the heights have values $u, u, -2u$ with $u \neq 0$ by embeddedness. The kernel of evaluation is the negative eigenline of inertia, generated, in a suitable ordering, by $(1, -1, 0)$. A nonzero vector in this line has trivial stabilizer. Thus a positive component with a nonzero section vanishing somewhere factors through Z.

In total degree seven some positive component must have zero height. Connectedness forces a nonzero component to meet height zero since contracted components carry constant sections. The degree count leaves exactly one copy of Z and one copy of C. Up to the deck group their nonzero height on Z is a scalar multiple of a fixed root s, negated by a transposition τ. Over each branch value x_i there is a unique ramification point r_i of Z fixed by τ. Its height is zero. The other two heights over x_i are nonzero and opposite. At a regular value the six heights are distinct by real injectivity. At all three ramification points the vertical derivatives are nonzero, being proportional to the difference of two distinct coordinates of the root times the ramification derivative of the lower embedded graph. The sextic graph is therefore embedded. Its intersections with the zero section are precisely the b points just described, with vertical and horizontal tangent lines respectively. The direct nodes or connected contracted subcurves joining the two components determine a unique nonempty subset I of these points. No point is used twice since the positive normalizations are embedded. Riemann–Hurwitz gives $g(Z) = 1 + 6(g-1) + 3b/2$. Joining at j points before attaching any contracted components gives genus $1 + 7(g-1) + 3b/2 + j$.

4.1.1. *Construction on a fixed covering curve.* Here and in Section 5, prescribing the crossing permits variation of the operator and its solution, with C, N, f fixed.

Proposition 4.2. *Every connected simply branched cubic f as above admits an ordinary transverse crossing with the stated evaluation ranks and central kernels. Its branch weights can be chosen in any prescribed chamber of the nonempty subset-sum hyperplanes.*

Proof. Over the regular locus map the flat V-bundle real-linearly to $N_{\mathbb{R}}$. At a branch send an even and an odd basis vector to e_0 and zv_0, where e_0, v_0 form a complex basis. This is equivariant under $z \mapsto -z$. Fix also a complex frame on a regular disk. Extend relatively over the complement by a smooth bundle map; transversality avoids the rank-one and rank-zero strata of codimensions three and eight over the surface. The maps of the three sheet vectors define an embedded cubic graph. Start with a normal operator equal to $\bar{\partial}$ in the branch frames. Off the branches prescribe a real zeroth-order correction on the image of the flat plane by sending each parallel column to minus its image under this operator. Extend smoothly on a complement, using the zero correction near the branch centers. This makes the prescribed heights solutions.

We verify regularity on the fixed f. On the frame disk variations killing the two heights can first make D invertible. A nonzero base kernel vector cannot belong to their real span on an open disk: its two real coefficients would have zero $\bar\partial$ and be constant, contrary to the absence of an invariant in V. Testing such a vector against a nonzero adjoint therefore gives a nonzero kernel-to-cokernel matrix entry, by variation of D and unique continuation. In a finite reduction the rank increases under a small multiple of this variation. Iterate until the index-zero base problem is invertible.

Augment D_f by b complex pole columns at the ramifying points. Each column is the image off the pole of $\chi a/z$, where a is in the complex derivative line of the height there and χ is an even cutoff. The pulled-back antilinear coefficient contains a factor $\bar z$, so the forcing lies in L^p, $p>2$. The derivative of the inverse sheet $(w-\alpha)^{1/2}$ is $-\alpha/(2z)$, giving exactly the branch-motion column modulo the regular domain. The augmented index is zero and the known kernel line persists. Suppose an additional augmented solution exists. Its trace extends by cancellation of the two odd poles at each ramifying pair and is zero by invertibility. The adjoint of the augmented problem also has trace zero by testing base pullbacks. Two of its three components are real-independent on an open dense subset: proportionality on an open set would give a constant real relation by the adjoint equation and hence on the normal cover, whereas the nonzero trace-free conjugates span the irreducible standard representation. Writing the three conjugates of the extra solution as h_i, annihilation of the constrained variations would put both h_1-h_3, h_2-h_3 in the real span of s_1, s_2 there. Their coefficients would again be constant. By trace zero and continuation all h_i would lie in the original standard copy and have no poles, so its sheet-invariant line is the only possibility. Thus a rank-increasing variation exists until the augmented kernel is just the prescribed line. One unrestricted time variation on the disk then pairs nontrivially with the remaining adjoint: the available forcing on the three sheet heights is any triple of sum zero. This proves (8). Lemma 4.1 applies in its open regular neighborhood.

Section 5 below proves the assertion concerning strict branch weights by changing the coefficients at the branch points and restoring the equation on this same Y. This step also supplies the additional regularity needed for the signed local degrees. □

5. One real obstruction at several joining nodes

Fix the root s on Z and its transposition τ, and choose once and for all an even generator s^+ of the other eigenspace in V. Over x_i write $v_i=\partial_z s(r_i)$ and $e_i=s^+(r_i)$. Use the quotient cubic $Y=Z/\langle\tau\rangle$. The image y_i of r_i is an unramified point of $Y\to C$. Trivialize N in a holomorphic frame for the complex-linear part of D, writing $D=\bar\partial+B(w)\,d\bar w$, with B antilinear. In that frame put $B_i=B(x_i)$ and $L_i=N_{x_i}/\mathbb{C}v_i$. We shall have $[B_iv_i]\neq 0$ and $[e_i]\neq 0$ in L_i. Let ψ generate the adjoint line on Y annihilating its branch columns in (8). In coordinate w it pairs $a\,d\bar w$ with $\operatorname{Re}\psi(w)(a)\,dA(w)$. Define

$$2\overline{\beta_i}[B_iv_i]=[e_i],\qquad H_i(q)=\tfrac{\pi}{2}\operatorname{Re}\psi(y_i)(qv_i),\qquad w_i=H_i(\beta_i). \tag{9}$$

Here the w_i on the right are real weights. The local domain coordinate z and the target coordinate w are those used in the smoothing chart below. The same s, s^+, ψ are used at all the joining flags.

Proposition 5.1. *Join the sextic to the identity over a nonempty set I of branch values. Under the regularity conditions below, if $\sum_{i\in I} w_i\neq 0$, the joined map without contracted components (the bare map) is an isolated zero in the full finite-cover chart times the parameter*

interval. Its local Euler change is

$$a_I = \kappa(-1)^{|I|}\operatorname{sgn}\Big(\sum_{i\in I} w_i\Big), \tag{10}$$

where $\kappa \in \{1,-1\}$ is common to all subsets at the crossing. All nonempty subsums are nonzero at a generic crossing. Every strict sign pattern of these sums can be realized on the fixed cubic.

5.1. **The derivative along the nodal stratum.** There are $3b$ independently movable simple ramification points on Z, even when their images coincide. At each join there are two more complex parameters p, q. In coordinates on the two branches the family is $zy = q$, $w = (z+p)^2 + y$ after translating the branch center. Elimination gives the algebra with basis $1, z, z^2$ and relation

$$z(z+p)^2 - wz + q = 0. \tag{11}$$

At the node the relative deformation sheaf has length three: at the central fiber its local algebra is the cokernel of the Jacobian of (zy, z^2+y), or equivalently $\mathbb{C}[z]/(z^3)$ after eliminating y. The other simple ramification points have length one. The relative deformation complex is supported at these finitely many source points and the finite-map chart is smooth of complex dimension $3b+2j$, $j = |I|$. Its central stabilizer is $\langle\tau\rangle$. Write $P_Z = \operatorname{coker} D_Z$. Normalization and evaluation give the joined kernel $\mathbb{R}s$ and the joined cokernel P_I with

$$0 \longrightarrow (\bigoplus_{i\in I} N_{x_i})/\mathbb{R}(e_i)_{i\in I} \xrightarrow{\iota} P_I \longrightarrow P_Z \longrightarrow 0. \tag{12}$$

Use the convention that the ordinary forcing of a cutoff height with value a on the sextic and zero on the identity represents $\iota(a)$.

To identify the image in P_Z, split under S_3. The trivial part is zero. The sign part is the odd cokernel on the discriminant double $T = Z/A_3$ and has dimension $2b$. The standard part has multiplicity $2b+1$, its multiplicity space being $\operatorname{coker} D_f$. These identifications follow by stabilizing with forms away from the branches on each quotient. A bounded invariant solution descends and extends across the punctures, and the Fredholm indices agree by deformation to a complex-linear operator. The odd operator on T has no smooth kernel. Augment it by a complex plane of odd poles $\chi a v_i/z$ at each ramification point above x_i, as in the proof of Proposition 4.2. Its resulting index is zero. We require it to be invertible. This is open dense with the lower graph fixed. A nonzero odd polar solution cannot be a real combination of the two lower heights on an entire open disk: the coefficients there would be constant and the sign representation on the punctured normal cover has zero intersection with the standard one. An allowed zeroth-order perturbation can prescribe arbitrary forcing on this solution on a smaller disk. The two odd adjoint contributions add. Unique continuation and rank increase in a finite reduction give the assertion.

Write B_0 for the real space of sextic branch velocities and increasing time. Their derivative on s embeds B_0 as a hyperplane in P_Z. To see this, first take the common shift of the three ramification points over each x_i. These shifts and time preserve S_3 symmetry. On the even vector their standard component is (8); on the odd vector the same columns span the odd standard summand since $\operatorname{End}_{S_3} V = \mathbb{R}$. The additional separate shifts of the fixed flags give on the sign part the discriminant pole columns computed above, with nonzero factor $1/3$ from averaging over A_3. Indeed a branch shift α produces the principal part $-\alpha v_i/(2z)$, by differentiation at fixed regular target point. Subtract the principal part and use removability for the bounded remainder; changing the cutoff changes the forcing by the image of a regular section. Finally the opposite shifts at the two τ-exchanged flags, applied to $(1,-1,0)$, give

one third of the common shift applied to $(1,1,-2)$. At inertia (13) and (23) this follows by taking the ratios $1/3$ and $-1/3$ of the coordinate differences. The even vector has zero derivative at the remaining flag. These columns give exactly the cubic branch columns on the invariant quotient. The remaining functional Λ is thus the half-trace to the cubic paired with ψ. Extend it to P_I by (12). For the full $q_i = 0$ derivative, each attaching column contributes $v_i p_i$ in matching. They are independent modulo the one diagonal real line since $[e_i] \neq 0$. Denote this injective nodal derivative, including time, by J, and its variables by h.

5.2. **The smoothing terms.** Let F be the normalized obstruction equation, q_i the complex smoothing parameters, and $Q_i = q_i \log(1/|q_i|)$, continued by zero. In the matching fibers the leading displacement is $2\iota((\overline{Q}_i B_i v_i)_{i\in I})$. It vanishes under the remaining real functional Λ. On the cone where all $|q_i|$ are comparable and the nodal parameters h are $O(R_0 \log(1/R_0))$, with $R_0 = \max_i |q_i|$, the next term is $\Lambda F = \sum_i H_i(q_i) + o(R_0)$. It is the residue of the variation of the two sextic sheets on the quotient cubic. Appendix C proves the two estimates, including uniformity at the faces where some q_i vanish.

5.3. **Degree and orientation.** Choose $n \in P_I$ with $\Lambda(n) = 1$. The boundary degree is that of the real-linear map

$$(h,q) \longmapsto Jh + 2\iota\big((\bar{q}_i B_i v_i)_{i\in I}\big) + n\sum_i H_i(q_i). \tag{13}$$

Indeed first homotope by linear interpolation to this model with a common factor $\log(1/R_0)$ on the matching term. Replacing the separate logarithms there by the common one costs $O(R_0)$, uniformly up to the coordinate faces. At a sequence of noncentral zeros (39) and injectivity of J would force $|h| = O(R_0 \log(1/R_0))$, $R_0 > 0$. Projecting away the nodal columns, then dividing by $R_0 \log(1/R_0)$, puts each limiting tuple (q_i/R_0) on the real line spanned by (β_i). Indeed the remaining matching space is $(\bigoplus L_i)/\mathbb{R}([e_i])$. Every entry of a nonzero tuple on that line is nonzero. Hence the comparable cone applies, and division of (40) by R_0 contradicts $\sum_i w_i \neq 0$. On the model the common positive logarithmic factor can then be changed to one. This establishes both isolation and the asserted boundary degree.

Restore the even height as a real variable, projecting to its evaluation vector $E_I = (e_i)$ before taking the quotient in (12). Eliminate B_0 and order the remaining rows as the matching fibers followed by the complementary normalized line. The remaining matrix is

$$\begin{pmatrix} M_I & E_I \\ H & 0 \end{pmatrix}, \qquad M_i(p_i,q_i) = v_i p_i + 2\bar{q}_i B_i v_i, \qquad H(p,q) = \sum_i H_i(q_i).$$

Each M_i has real sign -1 by complex linearity in p_i and antilinearity in q_i modulo $\mathbb{C}v_i$. The Schur complement is $-HM_I^{-1}E_I = -\sum_i w_i$. The sign from cancellation of the branch-and-time block, the identity operator, and the even height is common to all I: the normalization determinant is the determinant on the two smooth components times the dual determinant of the complex matching fibers. Every added pair (p_i, q_i) and matching fiber has real dimension four. Moving these past a fixed real line or past each other preserves orientation. The projective Euler link has Euler number twice the oriented degree, since its cokernel quotient is the pullback of the tangent bundle of an even-dimensional sphere under the normalized column. Dividing by the order-two stabilizer leaves the signed unit (10).

5.4. **Regularity and prescription of the weights.** The marked cubic height difference and the derivative test in Section 4.1 give generically $[e_i] \neq 0$. Adding the value of B_i to the marked test is submersive: after forcing off the marks the adjoint vanishes there, the

local section values and holomorphic jets kill the marked multipliers, and the zeroth-order coefficient can then vary freely at the center. The failure of $[B_i v_i] \neq 0$ has real codimension two.

With these jets fixed, adjoin to the cubic branch-augmented operator one real column whose residues at the unramified y_i, $i \in I$, are $\beta_i v_i$. A cutoff representative with residue R there is $R/w - B_i R \log|w|^2$. The off-pole forcing is locally $O(1 + |\log|w||)$ and its pairing with the remaining cubic adjoint is $-2\sum_i H_i(\beta_i)$. Quotient out the known smooth kernel. The resulting index-zero operator is invertible exactly when the sum is nonzero. At a defect a polar solution has nonzero coefficient on the new column. Its three conjugates at any chosen x_i, pulled back to Z, have an order-two pole at three different flags, while the other conjugates there have at most order one. Together with the bounded s_1, s_2 these give five independent solution germs. The constrained Petri argument of Lemma 4.1 therefore gives a rank-increasing perturbation at a defect. More precisely, first insert a good Petri jet on the test disk, then apply the annihilator argument there to the three polar columns and three local adjoint components. This gives open density with fixed graph; intersect over the finitely many subsets. Open conditions on a continued cubic crossing transfer to generic paths by a time cutoff. For the countably many kernel exclusions of Lemma 4.1 the same argument is applied to the individual compact configuration blocks before taking the intersection.

Finally we explain how to prescribe the chamber on the fixed cover in Proposition 4.2. The branch model used there starts with $B_i = 0$. Insert any bounded antilinear values with $[B_i v_i] \neq 0$ by cutoffs on disks of radius $\rho \to 0$. The perturbation tends to zero as an L^p multiplier on the fixed covers for each finite $p > 2$. To restore the equation on the same Y add $2b + 1$ smooth real operator variations on the regular frame disk. Their values on ξ may be chosen to span the cokernel of D_f, by trace-free forcing and adjoint continuation. The implicit function theorem with a thick real normalization now gives coefficients tending to zero and a new normalized solution converging to ξ in $W^{1,p}$. At ramification the pulled-back cutoff coefficient has the form $2\bar{z}\chi(z^2/\rho)B(z^2)$. It is uniformly Lipschitz and $O(\sqrt{\rho})$, hence small in C^α, $0 < \alpha < 1$. Interior estimates give C^1 convergence of the heights at the flags. In particular the even value stays off the complex derivative line, the odd height is $v_i z + O(|z|^{1+\alpha})$ uniformly after choosing $\alpha < 1 - 2/p$, and regular-sheet separation persists up to each ramification point. The branch pole columns converge in L^p since the factor $\bar{z}$ cancels the conjugate polar denominator. Consequently (8) remains invertible, and its branch-annihilating adjoint, normalized by time, converges uniformly on the cubic by the adjoint estimates. Apply the preceding regular-disk tests to ensure the singleton $H_i(\beta_i)$ are nonzero and the discriminant augmentation is invertible.

Given real c_i with all nonempty subsums nonzero, now choose γ_i with $H_i(\gamma_i) = c_i$. Insert bounded coefficients $\widetilde{B}_i$ satisfying $2\overline{\gamma_i}[\widetilde{B}_i v_i] = [e_i]$, and restore the fixed-cover equation in exactly the same way. The normalized cubic and discriminant pole columns and the time column converge on their fixed sources. Thus the remaining orientation choices transport continuously and $\beta_i \to \gamma_i$, $w_i \to c_i$. For sufficiently small supports all desired strict inequalities hold. Restoring the residual kernel exclusions on the regular disk preserves them. This proves both propositions, including realizability on a prescribed f. For every even $b > 0$ such cubics exist over the same C: send the b peripheral loops to (12) and one handle generator to (123), all other handle generators to the identity. The surface relation holds and compactifying gives the required connected holomorphic cover.

6. Contracted components and the degree-seven wall formula

Let $Z \to C$ be the sextic Galois closure. Put $b = 2l$ for the number of branch values and $h_j = 1 + 7(g-1) + 3l + j$. Fix a nonempty set I of these values. Choose the negative root line of one inertia transposition τ in the standard representation. Over each value in I join the τ-fixed point of Z to the point of the zero section. Denote the resulting finite map by a_I^0 and its signed local degree from Proposition 5.1 by a_I. Its arithmetic genus is $h_{|I|}$. We write $G_{I,k}$ for the locus obtained from a_I^0 by attaching one-pointed contracted curves at smooth points or inserting two-pointed contracted curves at its nodes, with total arithmetic genus k.

Proposition 6.1. *At a cubic crossing satisfying the kernel and smoothing hypotheses of Sections 4.1 and 5, the contribution of $G_{I,k}$ to the change of the degree-seven local invariant is*

$$a_I[v^{2k-2}]\big(2\sin(v/2)\big)^{2h_{|I|}-2}.$$

A single interval about the crossing suffices to compute the changes for all k on the same two sides.

6.1. **The central locus.** By Section 4.1, a central degree-seven map carrying a projective height has positive normalizations Z and C, each with multiplicity one. Its matching height on Z spans the negative line of one transposition. Its embedded graph meets the zero section once over each branch value, with distinct complex tangent lines.

Each connected contracted subcurve at a node or at a smooth point has one image. Maximal such subcurves have at most two positive branches, one on each normalization when there are two. The joining values are therefore recovered from the map with its projective kernel. For fixed k, $G_{I,k}$ is compact. Take stable degree-one maps to the two normalizations with marks evaluated at the indicated flags. Glue their marks directly or insert stable two-pointed curves of positive genus. Summing these proper gluing maps over the finitely many genus distributions gives the stated locus, including a tail specializing to a bridge. Distinct joining sets have disjoint images in the projective incidence. The unique matching height line is spanned by the chosen root on Z extended by zero on C.

6.2. **Displacing the line of sections.** For a stable map a to C write $L_{a,t} = a^*D_t$. Near its one-dimensional kernel, take a finite-dimensional stabilization and projectivize its domain over the reals. The zero on the tautological line represents the local degeneracy class: the derivative in the projective directions cancels the image directions of the linear map. In the transverse matrix space, with a generator of the kernel chosen, the cokernel on the linking sphere is its tangent bundle. The rank off the central locus is $r = 4(k-1-7(g-1))$, the real virtual dimension of the moduli of maps to C. The Euler number of this even-dimensional link is two. Thus the relative projective class capped with the fixed-target virtual class times the oriented parameter interval, and multiplied by two, computes the terminal minus initial Euler number [BS24, §6.2, Lemma 6.10 and Appendix B].

Choose a closed disk strictly inside a larger disk of regular values. On the central map the seven sheets over the larger disk are distinct as pieces of the normalization, including possible contracted attachments. For nearby maps the preimage of the smaller disk stays in seven disjoint neighborhoods. Each has degree one by conservation of degree at an ordinary value avoiding all attaching points. Order these pieces using an identification of the sextic with the fixed model over C taking a chosen joining flag to its fixed representative. There are two such identifications, interchanged by τ. In the finite covering of all sheet orderings these choices and their complement are disjoint closed sets over the compact central locus.

Shrinking the neighborhood extends the selection to a stack cover of degree two, retaining the automorphisms of contracted components.

On one distinguished nonzero sheet choose a real covector density on the disk with integral $\ell_a(s) \neq 0$ on the chosen root s. Integration uses pulled-back area, so constant components give zero. Normalize by $\ell_a(u) = 1$. For independent smooth vector-valued $(0,1)$-forms on the seven sheets, compactly supported in that disk, write j_a for their sheetwise pullback. Replace $L_{a,t}u$ by $L_{a,t}u - \ell_a(u)j_a$. On the normalized slice the equation is

$$L_{a,t}u = j_a, \qquad \ell_a(u) = 1. \tag{14}$$

This defines a continuous perturbation in nodal stabilizations. Indeed introduce $z = \ell_a(u)$, solve $D_t p = j$ locally on each disk containing a nodal value on each sheet, and replace u by $u' + za^*p$, using cutoffs equal to one near the nodes. The new forcing vanishes there, and the scalar row becomes $z = \ell_a(u') + z\ell_a(a^*p)$. These substitutions are invertible row and variable changes. On thick parts the remaining forcing converges smoothly, and on a shrinking disk the area test on bounded sections is bounded by seven times its area times the supremum of the integrand. These are precisely the continuous perturbations of the stabilized inverse in [BS24, Appendix B] and [Par16, Propositions B.7.9 and B.10.4]. To obtain a single ambient almost complex structure, split the tangent bundle of the total space by a complex connection and write $D_t = \nabla^{0,1} + A_t$. Over the support disk take disjoint cutoffs χ_m equal to one near the seven normalized sheet graphs. Write $F_{t,j}(x,n) = A_t(x)n - \sum_m \chi_m(x,n)j_m(x)$, a fiber-valued antilinear one-form in the base direction. The block formula

$$(X,V) \longmapsto \big(j_C X,\ iV + 2iF_{t,j}(x,n)X\big)$$

squares to minus the identity. With $\bar\partial u = (\nabla u + i\nabla u\, j)/2$, its graph equation near these sheets is (14) before imposing the scalar row. The construction is independent of the source, including its contracted components. It extends the homogeneous graph correspondence of [BS24, Lemma 5.9]. For a compact completion first cut off $A_t(x)n$ radially, identically one on a tube containing the normalized central graphs and their sufficiently small displacements. Use the connection structure near the divisor at infinity in the projective bundle. A closed coupling form positive on the projective fibers, plus a sufficiently large multiple of the base area form, tames this compact family. All comparisons take place in the smaller tube. In the bundle tube the area of a closed graph is bounded by its projection degree times a constant.

On a finite-map chart the sheetwise forms surject onto the cokernel: a dual solution annihilating them vanishes on an open set of each positive normalization, and hence everywhere off its nodal flags by unique continuation. The equation-space dual in the L^p formulation has no point masses. Sections with prescribed independent values at the flags then eliminate the dual node-matching rows. Finitely many forms suffice near the finitely many bare joins. On each smooth stratum of the finite-map chart the displaced problem is transverse for a regular choice. A persistent node or a collision of simple branch values has positive complex codimension, whereas the full normalized problem with time has index zero. Hence the displaced zeros are finite, smooth, simply branched covers. Their graphs may also be taken embedded. A double point over a common value imposes four real conditions with at most two real parameters for that value. At a simple ramification point a zero vertical derivative imposes four conditions. For transversality of these marked problems the same adjoint test vanishes off the marks. Prescribe arbitrary independent values and holomorphic first jets there, solving the symbol equation recursively for the antiholomorphic derivatives of a formal jet. Testing against these jets eliminates the distributions at the marks. Immersion excludes the diagonal in the marked-pair test, and finitely many further forms then suffice by compactness.

By conservation of the local degree their signs in the labelled chart sum to a_I. Here and below all displacements at the finitely many joins are made using the same ordered disk and orientations.

6.3. **Isolation and contracted components.** Consider a convergent sequence of normalized graph solutions approaching $G_{I,k}$, with displacements tending to zero. Take stable limits of the positive normalizations separately, first retaining the attaching marks. Their nonconstant limits together are the root graph and the zero section, each once. Every positive component has positive projection degree, so there are at most two such normalizations. On forgetting marks, their individual stable limits have no contracted component. Indeed the theorem of Ekholm and Shende [ES25, Theorem 1.1] would require a vanishing derivative, equal complex tangent lines at two preimages, or three attaching preimages of one point. The two central embeddings exclude each possibility. The theorem applies to these bare maps because the displaced equation is an ordinary graph equation. For finitely many smooth perturbation parameters, put the varying structures and the standard complex structure of an even-dimensional parameter ball (adjoining one dummy coordinate if needed) in block diagonal form on the product. Maps in constant slices are holomorphic for this one structure, with the same attaching tangents.

If there are two positive normalizations, each converges smoothly to one of the embeddings. Each nearby pair can meet at most once near each central intersection: projecting the difference of the two parametrizations to the real four-plane of the limiting tangents has invertible derivative in the two local source parameters. Away from these neighborhoods the limiting images are disjoint. Deleting the one-ended contracted blocks and replacing the two-ended ones by nodes now gives a finite map satisfying the same equation. Its joining flags remain distinct. Properness of the marked factors shows that their limiting subset is I. At zero displacement the local isolation at each bare join forces precisely the central finite map. A regular displacement has no finite nodal zero there.

With one positive normalization its unmarked limit has direct joins at a nonempty set J. The bare isolation for J again excludes a sequence of smooth finite zeros at zero displacement. At a regular displacement these zeros belong to the smooth embedded graphs just constructed. Each remaining contracted block on such a graph attaches once. On restoring marks to its limit one can insert rational trees at smooth points or at existing nodes, and the one-pointed tails can specialize to these trees. Distinct smooth points over an unjoined intersection have disjoint fibers under marked stabilization. Hence no additional joining value results: $J = I$. Conversely, every degree-one expansion of a displaced graph solves (14), and any of its limits at zero displacement lies in $G_{I,k}$ by the marked description. This proves fixed-genus isolation and identifies its entire displaced zero set. In particular a bridge in a limit is counted with its own joining subset.

6.4. **The virtual class at a displaced graph.** Write Γ for one of the embedded displaced graphs and t_0 for its parameter. On its smooth projection the undisplaced homogeneous operator is injective by isolation in the finite-map chart. This remains true on each degree-one expansion by restriction and matching. In a chart for maps to C choose stabilizing marks, incidence slices, horizontal perturbations and obstruction section s_C. Use the same data for maps to the total space, lifting the slices by inverse image and the horizontal forms by a complex-linear splitting of projection. The splitting may be chosen affine in height near each sheet by taking the complex-linear part of a connection lift. Choose an equivariant supplement BW to the image of $L_{a,t}$ at the chart center, with representatives on the smooth parts of all

components including contracted ones. On $s_C = 0$ the stabilized vertical equation is

$$L_{a,t}u - Bw = j_a.$$

Its derivative in (u, w) is invertible. Projection identifies the regular simultaneous thickening with the horizontal one over the common family of stabilized domains [Swa21, Theorem 3.5 and Lemma 5.10]. The residual ambient obstruction rows are s_C and w. For the homogeneous projective equation introduce $z = \ell_a(v)$ as before. Its stabilized solution is $(v, w') = z(u, w)$ and injectivity forces $z \neq 0$ at a projective zero. In the chart $z = 1$ the joint rows are therefore

$$s_C = 0, \qquad w = 0, \qquad \ell_a(u) - 1 = 0.$$

Their Thom product computes both the compactly supported projective pairing and the constrained ambient class. To compare charts, take the union of stabilizing marks and direct sums of perturbations. Adding a vertical supplement pairs its new variable with the same new obstruction coordinate. Changing horizontal lifts adds rows vanishing on $s_C = 0$, and eliminating a different vertical supplement adds a multiple of w to the scalar row. These changes give the same localized Thom product on common refinements [Par16, §9.2]. Both r and the horizontal virtual dimension are even. The scalar orientation fixed on the smooth finite-map chart thus applies at every arithmetic genus.

In unparametrized normal coordinates at Γ the augmented linearization with time and the area test is $(\xi, \dot{t}) \mapsto (A\xi + B_0\dot{t}, \lambda\xi)$. It is invertible by transversality of the normalized finite-map zero: the horizontal deformation complex of a smooth cover has torsion cokernel sheaf and no obstruction. The fixed-time operator A has index zero. If it has a kernel, the augmented isomorphism makes both kernel and cokernel lines. In an unramified disk where their generators are nonzero, add a small zeroth-order term linear in the height minus the height of Γ. The vertical tangent there is the normal quotient, so the variation can have nonzero pairing on these lines. This fixes the graph and makes A invertible, preserving the augmented isomorphism. In the affine presentation this changes the homogeneous operator and the forcing by the corresponding linear and constant terms. Compactness of the degree-one expansions permits this homotopy inside the isolating chart. Any extra constrained graphs approaching the retained one would have one nonconstant normalization: in its retracting tube each positive component projects in a positive multiple of seven. Ghost censorship at the embedded multiplicity-one limit gives smooth convergence of the bare normalization, contrary to uniqueness for the augmented equation.

The nearby ordinary graphs form a regular family Γ_t. The same normalization and compactness argument shows that its fixed-genus locus in a small tube consists of degree-one expansions. On these maps the area test is the function $\gamma(t) = \ell(\Gamma_t) - 1$, with $\gamma'(t_0) \neq 0$ by the augmented isomorphism. Replacing the scalar row by γ preserves its Thom product with the two other rows since the functions agree on their zero set. The time slice contributes $\operatorname{sgn}\gamma'(t_0)$ times the ordinary fixed-time degree-one class. The normal bundle of Γ has degree $2h_{|I|} - 2$, since $c_1(TC) + c_1(N) = 0$. Its regular normal operator pulls back injectively to every degree-one map. Zinger's comparison identifies the local invariant with the normal Euler class on the moduli of degree-one maps [Zin11, Theorem 1.2]. To transport the sign to a complex-linear reference, quotient the pullback Fredholm complex by the complex on the base curve Γ. Restriction to the unique positive normalization splits the pullback on cokernels, and matching identifies the two kernels throughout any deformation. The quotient thus has vector-bundle cokernel with its quotient determinant orientation [BS24, Appendix B]. At an invertible end the remaining orientation is the base determinant sign. At a Dolbeault reference on two non-effective degree $h_{|I|} - 1$ lines the Hodge integral gives the series $(2\sin(v/2))^{2h_{|I|}-2}$

[Pan99, Theorem 2]. The product of the time and normal signs is the sign of the original displaced zero, by evaluating in its own smooth genus. Summing these signs and cancelling the factor two from the Euler link against the degree two of the labelling proves the asserted coefficient at fixed k. Isotropy on contracted curves is included in the degree-one virtual class.

6.5. **One interval for exact degree seven.** We record the genus estimate needed to pass to a common interval. For a compact smooth family of homogeneous normal operators on C there is a constant $K \geq 0$ such that a simple graph projecting in degree d has normalization genus at most $1 + d(g-1) + Kd(d-1)$. Indeed, take the normalization of the off-diagonal part of the ordered fiber square of its projection. The difference of the two heights is generically nonzero on each component. For a fixed unitary connection and M a uniform norm bound on the zeroth-order term, it satisfies $|\bar{\partial}_\nabla s| \leq M|s|\,|df|$, up to a fixed metric constant, for the induced projection f. Its isolated zeros have positive integral orders by the similarity principle. Off these zeros let $e = s/|s|$ and $\alpha = \operatorname{Im}\langle e, \nabla e\rangle$. The curvature identity expresses $d\alpha$ as the curvature evaluated on e plus twice the difference of the squared norms of the orthogonal $(1,0)$ and $(0,1)$ derivatives of e. The latter $(0,1)$ norm is bounded by $M|df|$. Thus $d\alpha \geq -f^*\kappa$ for one positive area form κ on C uniform over the family. Stokes on the complements of the zeros bounds their total order by $d(d-1)\int_C \kappa/(2\pi)$. A ramification point of index a gives $a-1$ distinct off-diagonal branches over the repeated point, all with zero difference. The genus bound follows by Riemann–Hurwitz.

Suppose a connected stable map of exact degree d has a nonzero height. For its positive normalizations let d_i be the degree of the simple graph and m_i the covering multiplicity, including graphs of height zero. Thus $\sum_i m_i d_i = d$. Replace each covering by a connected unramified cover of the simple normalization with the same multiplicity. Such a cover exists since its base has genus at least two. The new genus is at most $1 + m_i d_i(g-1) + K m_i d_i(d_i-1)$. Reconnect along a tree of equal graph values, taken from the original matching after collapsing each connected contracted block to its image. Lift each selected flag to its replacement. If flags coincide there, place them on a constant rational component attached at that point with one node for each of the coincident flags. This keeps the dual graph a tree and the resulting map stable. Its degree is d, its height is nonzero, and its genus is at most $dg + Kd(d-1)$. The replacement is used only to detect existence of a kernel.

Apply this test at $d = 7$ on a compact subarc. In each of the finitely many genera below the bound the central compact loci just described exhaust all projective zeros. Fixed-genus isolation and compactness give a kernel-free punctured interval. Intersecting the finitely many intervals and applying the test proves the assertion in every genus at exact degree seven. The Euler integrals are then constant on each punctured side coefficient by coefficient. This finishes the proof of Proposition 6.1. Since the only proper divisor of seven is one, and the base operator stays invertible, the degree-one factor has constant exponent. Subtraction of its unchanged contribution in the connected series gives $\Delta n_{7,h_j} = \sum_{|I|=j} a_I$ at super-rigid endpoints, with all other degree-seven exponent changes zero. This proves Theorem 1.2.

6.6. **The possible changes in elementary multiplicity.** Write $c_j = \Delta n_{7,h_j}$. With a common choice of sign as in Theorem 1.2, the set of attainable vectors on the given cubic is the image of the complement of the real hyperplanes

$$H_I = \{w \in \mathbb{R}^b : w_I = 0\}, \qquad w_I = \sum_{i \in I} w_i, \qquad \varnothing \neq I \subseteq E,$$

under $w \mapsto ((-1)^j \sum_{|I|=j} \operatorname{sgn} w_I)_{1 \leq j \leq b}$. These hyperplanes form the resonance arrangement [Küh23]. Several chambers may determine the same vector. The sharp binomial bound used

in Theorem 1.3 follows at once from this description:

$$|c_j| \le \binom{b}{j}, \qquad \max_w \sum_{j=1}^{b} |c_j| = 2^b - 1.$$

Indeed choosing all weights positive gives equality for every j simultaneously. Such a choice is realizable by Proposition 4.2 and the branch variations of §5.

Proposition 6.2. *The subgroup generated by the changes on a prescribed cubic is*

$$2\mathbb{Z}^b + \mathbb{Z}\left(\binom{b}{1}, \dots, \binom{b}{b}\right).$$

Its index in $\mathbb{Z}^b$ *is* 2^{b-1}.

Proof. Every summand has absolute value one, so the reduction of any attainable vector modulo two is the displayed binomial vector. For each nonempty I choose a point of H_I outside the finitely many other hyperplanes. Crossing there changes only $\operatorname{sgn} w_I$. The difference of the two attainable vectors is a signed double coordinate vector in coordinate $|I|$. These differences give all of $2\mathbb{Z}^b$; the last binomial coefficient is one, so the reduction of the generated subgroup has order two. □

There is a more restrictive relation on chambers meeting the full-sum hyperplane H_E at a point on no proper-subset hyperplane. Complementation pairs each proper subset I with a subset of opposite sum on H_E. Since $b = 2\ell$, the two cardinalities have the same parity. On either side of this point it follows that

$$c_{b-j} = -c_j \quad (1 \le j < b), \qquad c_\ell = 0.$$

Only c_b changes when crossing H_E there. Consequently neither of these two chambers attains the unrestricted binomial bound. For example, at $b = 2$ they give $(0, 1)$ and $(0, -1)$. The same-sign chambers give $(-2, 1)$ and $(2, -1)$. Expanding $x = v^2 - v^4/12 + O(v^6)$ gives, for $g = 2$, the connected genus-thirteen coefficient $c_2 - 11c_1/12$. Thus the two chambers with nonzero c_1 give $17/6$ and $-17/6$.

The subgroup from chambers adjacent to H_E also has a simple description. In the free coordinates $(c_1, \dots, c_{\ell-1}, c_b)$ it is

$$2\mathbb{Z}^\ell + \mathbb{Z}\left(\binom{b}{1}, \dots, \binom{b}{\ell - 1}, 1\right).$$

Indeed two proper-subset hyperplanes have the same restriction to H_E only when their indexing sets agree or are complements: their indicator vectors must be proportional modulo the constant vector. Cross one generic restricted hyperplane in H_E, and move both endpoints slightly to the same side of H_E. This gives twice any coordinate with $j < \ell$, up to sign. Crossing H_E itself gives twice the last coordinate. Reduction modulo two finishes the argument also for $\ell = 1$. Here the subgroup describes sums of changes over several crossings; an individual crossing must satisfy the subset-sign formula.

Proof of Theorem 1.1. Such covers exist by the Riemann existence theorem: send the two peripheral loops to the same transposition and one handle generator to a three-cycle, with its partner trivial. The product relation and transitivity hold. The construction of §4.1 realizes the crossing on any prescribed such cover. The standard two-plane of S_3 is the central kernel on its normal closure. The fundamental group of an unramified cover, restricted to the punctured base, contains the normal closure of the peripheral loops. Its image therefore

contains the normal closure of a transposition, which is S_3. The two-plane has no invariants. Lemma 4.1 gives an invertible pullback on each unramified cover at the center. There are finitely many covers of the bounded degree, so their determinants remain invertible on a common interval. The smooth cubic calculation of §2.1 gives the first change. Theorem 1.2 gives the second from the single subset containing both branch values. □

7. Smooth degree-seven covers near a change of sign

Changing the sum of the weights at a fixed set of flags through zero reverses the degree of the local smoothing equation. Smooth maps near the join account for this difference.

7.1. **Smooth covers approaching a joined map.** Fix a simply branched cubic $Y \to C$ with monodromy S_3 and $b = 2\ell > 0$ branch values. Let I be a nonempty subset of these values. Write $j = |I|$, $h_j = 1 + 7(g-1) + 3\ell + j$, and f_I for the finite map obtained by joining its Galois closure Z to C at the selected flags fixed by one transposition. We use increasing t as the positive direction across a cubic wall. A crossing on a smooth cover is counted once for each isomorphism class over C and each parameter, with the sign of the derivative of the normalized kernel equation in the complex-oriented Hurwitz directions and the oriented path direction.

Theorem 7.1. *On the prescribed cubic and normal bundle there is a two-parameter family of operators $D_{a,t}$ with the following properties. Along $t = 0$ the prescribed cubic has a normal kernel, transverse in the t direction. There is a sequence of parameters approaching a point of this segment for which smooth connected degree-seven covers with a nonzero normal section converge to f_I and its root section. Their source genus is h_j and their monodromy is S_7. If I consists of all branch values, the graphs in the convergent sequence are embedded after discarding finitely many terms.*

There are also arbitrarily small Hurwitz neighborhoods U of f_I and rectangles $[a_-, a_+] \times [-\delta, \delta]$ as above with the following signed count. Let d_L, d_R denote the local degrees in the uniformizing chart of the joined map along the left and right edges. On the lower and upper edges, oriented by increasing a, take sufficiently small generic perturbations relative to their ends. Their smooth crossings in U are regular, have simple branching over distinct values, trivial deck group and embedded kernel graph. Their signed counts N_-, N_+ satisfy

$$2(N_+ - N_-) = d_R - d_L \in \{-2, 2\}. \tag{15}$$

Proof. Changing the degree. Take a short segment $c(a) \in \mathbb{R}^b$ crossing the hyperplane $\sum_{i \in I} c_i = 0$ at a point of no other subset hyperplane. In the branch coordinates of Section 5, H_i is a nonzero real functional on a complex smoothing plane. Choose nonzero $\gamma_i(a)$ smoothly with $H_i(\gamma_i(a)) = c_i(a)$. A right inverse of H_i plus a fixed nonzero vector in its kernel gives such a choice even when $c_i = 0$. Assign the antilinear coefficient at the ith center so that $2\overline{\gamma_i(a)}[B_i v_i] = [e_i]$. The local replacement and restoration on a disk of regular values in Section 5 apply with one cutoff radius for this compact segment. Indeed the prescribed coefficients and the right inverses in the restoration are uniformly bounded. Thus the weights on the restored cubic are uniformly close to $c_i(a)$. Take the radius small enough to retain opposite signs of the I-sum at the two ends and to keep every other subsum away from zero. The ordinary and polar operators on the fixed covers retain the ranks used in the nodal derivative. The cubic remains transverse to one common time direction. The kernel at f_I remains the odd real line: evaluation at any chosen flag removes the even section on Z.

Write q_i for the complex smoothing parameters, and h for the remaining chart coordinates and time, centered at f_I, $t = 0$. Trivialize the reduced obstruction equation over the parameter

segment by stabilizing on disks of regular values and normalizing the root section by a real linear functional. The center kernel and cokernel have constant rank, so supplements and normalizations can be continued along the segment. By Proposition 5.1 the end zeros are isolated and have opposite degrees in this uniformizing chart. On the boundary of every sufficiently small ball in (h, q) a zero occurs at some intermediate a, by homotopy invariance of degree. Passing to a sequence of shrinking balls and a convergent subsequence of parameters, put $r = \max_{i\in I} |q_i|$. The nodal derivative is uniformly injective, hence $r > 0$ at every noncentral zero sufficiently close to the center and $|h| = O(r \log(1/r))$. After division by $r \log(1/r)$ the matching equation of Section 5 places every subsequential limit of (q_i/r) on the real line spanned by $(\beta_i)_{i\in I}$ at the limiting parameter. Here $2\overline{\beta_i}[B_i v_i] = [e_i]$. Every coordinate of the generator is nonzero. Thus all $|q_i|$ are comparable to r. The remaining adjoint row, divided by r, now gives $\sum_{i\in I} H_i(\beta_i) = 0$ at the limit. These estimates are uniform on the segment, since they use the bounds for the stabilized inverse and the leading nodal and matching maps, all of which retain their ranks at resonance. Every selected node has smoothed. The domain is connected, of arithmetic genus h_j, and the normalized sections converge to the odd section. This proves existence of the asserted sequence.

Ramification and separation. On the three local sheets involved at a selected flag, the projection in the neck coordinate z is

$$w - c_i' = (z + p_i)^2 + q_i/z.$$

Here c_i' is the displacement of the branch center. Both c_i' and p_i are $O(r \log(1/r))$. Put $d_i^3 = q_i/2$. The critical equation is $2z^2(z + p_i) = q_i$, with three distinct simple roots $z/d_i = \zeta + o(1)$, $\zeta^3 = 1$. At such a root,

$$w - c_i' = (z + p_i)(3z + p_i) = 3d_i^2(\zeta^2 + o(1)).$$

These critical values have mutual distances comparable to $r^{2/3}$. The two other local quadratic factors have critical values at distance $O(r \log(1/r))$ from the original branch value. Consequently each of the three displayed values supports exactly one simple ramification point. At each unselected branch disk the three quadratic factors persist. In total the ramification degree is $B = 3b + 2j$. A solitary simple branch gives a transposition on the seven sheets. The graph of the conjugates of that transposition has components permuted transitively by the monodromy, so their common size divides seven. There is an edge, hence this graph is connected and its edge transpositions generate S_7.

For the full join the two limiting images coincide only at the selected points. On a selected disk the normalized height on the cubic factor has the expression $U_0(w) + zU_1(w) + z^2U_2(w)$. In the fixed-base polynomial presentation its coefficients converge in $W^{1,p}$ for some $2 < p < 4$, hence uniformly, by the stabilized inverse in Section 5. Centrally $U_1(0) = v_i \neq 0$ and $U_2 = 0$. The difference of the heights at distinct roots z, z' over the same w, divided by $z - z'$, is

$$U_1(w) + (z + z')U_2(w).$$

Its norm is bounded below on a smaller disk. At a simple critical point with local coordinate $w = w_0 + \eta^2$, the smooth height has zero antiholomorphic derivative at $\eta = 0$, by the pullback equation. Taking the difference at η and $-\eta$ gives vertical derivative in the z coordinate equal to $U_1(w_0) + 2z_0U_2(w_0) \neq 0$. Thus this factor also immerses at ramification. The other two quadratic factors stay separated by their distinct nonzero limiting heights. Their normalized heights converge in C^1 on smaller source disks by ordinary elliptic regularity and have nonzero limiting vertical derivative at the ramification points. Away from these disks the projections

are unramified and the central heights are pairwise separated on a compact set. The resulting injective immersion of a compact source is an embedding.

Isolating a neighborhood for paths of operators. To obtain signed counts, the neighborhood must lie in Hurwitz space alone, independent of a and t. Arrange the infinite-order Petri condition at two distinct points of a regular disk where two cubic heights form a complex frame, by the local gauge construction of Section 4.1 and [Wen23, Corollary 5.10]. Restore the cubic, and choose its transverse time direction, on another subdisk. Both Petri jets are then fixed along the family. Vary the central segment relative to its ends on the frame disk by small a-dependent tensors flat at the test points and annihilating the two current cubic heights. This preserves the prescribed cubic.

For connected smooth covers sufficiently close to f_I, monodromy outside the branch disks contains the regular S_3 on six sheets fixing the seventh. Connectedness now implies double transitivity. The reduced real sheet representation W is absolutely irreducible of dimension six: the complex commuting algebra of the permutation representation has dimension two by the two orbits on ordered pairs. The constant sector is invertible. The six components of a nonzero solution in W and the two independent cubic heights are linearly independent as real solution germs on a regular disk. Indeed on the joint normal cover they span inequivalent irreducible systems, and unique continuation identifies their linear relations on the disk.

Here is the constrained transversality argument. An adjoint annihilating the allowed variations gives on the disk a tensor identity

$$\sum_{k=1}^{6} u_k \otimes v_k = s_1 \otimes \alpha_1 + s_2 \otimes \alpha_2,$$

where u_k, v_k are the test solution and adjoint components and the s_i are the prescribed cubic heights. The smooth frame gives the α_i smoothly across the test points. Pair with $[D, \Phi]$ for compactly supported complex-linear endomorphisms Φ. The left integral vanishes by the solution and adjoint equations. Since $\Phi s_1, \Phi s_2$ can be chosen freely, both α_i solve the adjoint equation. The Petri condition on infinite jets and independence of the eight solution germs force the jets of all v_k to vanish. Unique continuation gives $v_k = 0$. This is the local-system Petri argument of [DW23, §§1.6, 2.7–2.9] with the two prescribed solutions retained.

A smooth branching profile with s distinct values has complex dimension s. The pullback operator has real index $-2B$ by Riemann–Roch. After allowing a and dividing by real kernel scalings the expected dimension is $2s - 2B$. The preceding test gives transversality for a generic constrained central segment. To apply it on the noncompact smooth strata, exhaust their projective incidences by countably many compact subsets. Finitely many supported variations suffice near the zeros on each such set, and their parameter cutoffs preserve the end collars. The regularity conditions on compact subsets are open, so the countable intersection gives regular segments. Only simple branching at distinct values remains, in dimension zero. The uniform matching estimate above excludes partially smoothed zeros sufficiently near the center throughout the segment. Thus all off-center zeros at $t = 0$ sufficiently near f_I form a countable set.

Choose arbitrarily small invariant Hurwitz balls U with radii outside the set of radii of these zeros. Compactness gives a $\delta > 0$ for which ∂U is free of kernels throughout the rectangle. The uniform estimates exclude all nonsmooth zeros on the horizontal faces after shrinking the ball and δ, since a zero with $q = 0$ can only be the center at $t = 0$. The sign condition at each end leaves only the joined zero on each vertical face. Small perturbations of the horizontal paths preserve these boundary properties. Applying the preceding transversality test there

with unrestricted operator variations gives regular zeros, with distinct simple branch values and one-dimensional kernel. There is a transposition, hence full monodromy by the prime degree argument. The deck group, the centralizer in S_7, is trivial.

For completeness, the marked tests for embeddedness are also transverse. At two distinct preimages over a common value, equality of heights has real codimension four, while a free base value has two real parameters and a branch value is already determined. Vanishing of the vertical complex derivative at a ramification point gives another four real conditions. For either test a putative annihilator solves its adjoint equation away from the marked fiber. Its invariant part vanishes distributionally by base invertibility: each test vanishes on sections pulled back from the base. One of the two Petri points lies away from the marked value. The unrestricted adjoint test there and unique continuation kill the reduced part away from the marks. To test the remaining point-supported distributions, vary a local section with prescribed value or holomorphic first derivative. Its equation can be made zero to any given finite order, since the coefficients involving a $\bar\partial$ derivative are determined recursively by its pure holomorphic jet. These variations kill the marked multipliers, and the remaining homogeneous adjoint vanishes by elliptic regularity. At a zero of a marked test it vanishes also on the infinitesimal rescaling. The resulting negative expected dimensions exclude both coincidences and nonimmersions. The signed zero sets on the compact horizontal faces are therefore finite and have embedded graphs.

Boundary orientations. Orient the parameter rectangle by $da \wedge dt$. On the product of its boundary with the uniformizing Hurwitz ball, the total signed degree is zero because the section extends over the product and has no zeros on the Hurwitz boundary. The right and left faces contribute d_R and $-d_L$. On each horizontal face a smooth zero has two lifts, interchanged by the central involution. Their signs agree. Indeed this involution acts complex linearly on the Hurwitz chart. On the normalized obstruction target it preserves orientation, as follows by equivariance from the nonzero local degree at either vertical end. This orientation character persists across the rectangle. The lower and upper faces thus contribute $2N_-$ and $-2N_+$, respectively. This proves (15). □

When I contains every branch value, complementary nonempty proper subsets have opposite sums at the limiting weights. In particular the size-$b/2$ terms cancel in either adjacent chamber. Weights of one sign give the largest sum of absolute changes over all genera and belong to different chambers. Equation (15) determines the difference of the two counts in U. Extending that identity to counts in the whole Hurwitz space would require control of its complement.

8. Compensation in a fixed integral homology class

Let (X, ω) be a closed symplectic six-manifold with $c_1(TX) = 0$. At an endpoint J of a smooth path of ω-tame structures, mark finitely many disjoint embedded curves C_i in nonzero integral classes A_i. Suppose each admits arbitrarily small tubes in which the only cycles below a prescribed energy bound are its multiples. Suppose also that each marked cycle series has a geometric factorization into elementary clusters. These hypotheses hold for the super-rigid homogeneous germs used in Theorem 1.2: projection expresses each irreducible compact curve in the bundle as a finite map with a height section, and a vertical fiber contains no compact curve. The cluster construction below then applies to the insertion consisting of the multiple zero sections at each energy bound.

Let H be the virtual-dimension-zero, Chern-number-zero part of Pardon's rational algebra of complex curves [Par25]. Write S for its almost-complex cycle-space counterpart and $\Phi : H \to S$ for the comparison of [Wue26b, §2]. In S a compact open and closed part of the

relative cycle space defines an insertion. Excision on disjoint supports gives multiplication, and the two restrictions of such an insertion over a closed interval agree. Denote by F_h the normalized balanced factor of a genus-h elementary curve: it is the equivariant series on the sum of two degree-$h-1$ line bundles with product the canonical bundle and circle weights $+1, -1$. Its linear term is the multiplicity-one class $x_{h,1}$ and its image under Φ is the elementary cluster series [Wue26b, §§3–4]. We use the full integral label $\beta \in H_2(X;\mathbb{Z})$ in the monomial Q^β. Products at a fixed energy cutoff are taken modulo terms above that cutoff. For each marked curve send the local variable q^d to Q^{dA_i}, and let P_J be the product of its marked factors over all the marked curves.

Proposition 8.1. *At each such endpoint there is a product R_J of elementary factors with integral exponents, obtained from the connected supports outside the marked sectors after a generic perturbation. It is independent of the perturbation, the cluster decomposition, and sufficiently small isolating tubes. For endpoints of a smooth tame path,*

$$P_{J_-} R_{J_-} = P_{J_+} R_{J_+}. \tag{16}$$

The equality holds in the class-labelled completion with coefficients in H. It holds also when the marked curves differ at the two ends.

Proof. First take an integral taming form and fix an energy cutoff L. Choose smaller disjoint tubes V_i with closure in the respective isolation neighborhoods. At the endpoint no nonempty cycle contained in $\overline{V}_i$ meets its boundary. This persists for nearby structures: a contrary sequence would have a bounded-mass subsequence converging as cycles with Hausdorff convergence of supports [DIW26, Theorem 2.11]. A nonempty limit has positive energy, and integrality prevents a sequence of positive cycles from limiting to the empty cycle. Thus containment in each tube is open and closed in the bounded-energy cycle space over a short path to a generic endpoint perturbation.

At the perturbed endpoint distinct irreducible supports are disjoint. The bounded-energy space of connected supports, counted with possible multiplicities in the compactification, splits into compact open and closed parts according to containment in the marked tubes and their complement. To see closedness, a limit of connected supports is connected, and disjointness of distinct irreducible curves forces it to have a single irreducible support. The respective unions of supports are disjoint compact subsets of X. Restriction to disjoint open neighborhoods splits each effective current into currents labelled by the parts. This factors the entire insertion in S, with mixed configurations included by multiplying the part series. In particular R_J is constructed from configurations all of whose connected supports are complementary.

Subdivide each part by a finite cluster partition. Cluster isotopy, excision and the fold relation decompose each cluster into a signed elementary factor and corrections in classes $d\gamma$, $d \geq 2$, if the original class is γ [DIW26, Propositions 3.18, 3.21 and Theorem 3.25] [Wue26b, §4]. The latter reference gives the identities of cycle insertions needed here. Each step decreases $\lfloor L/\omega(\gamma) \rfloor$. This yields finite products for the parts with integral exponents. The image of their product under Φ is precisely the insertion of the entire target at this cutoff. For each integral class the entire class locus is compact over the original path by cycle compactness; the class is locally constant. The interval relation identifies its two endpoint insertions. The argument requires isolation of the marked sectors only at the endpoints.

To prove uniqueness, cancel factors in order of increasing positive energy. At a first differing label β, the difference of the comparison images is a finite sum $\sum_h d_h \Phi(x_{h,1})$. All nonlinear terms at this label are composed of smaller-energy contributions. Other labels at the same energy have distinct monomials, also when they differ by torsion. Independence of the

multiplicity-one classes [Wue26b, Theorem 5.1] gives $d_h = 0$ for every h. This proves equality of the two total products, uniqueness at an endpoint and consistency on increasing the cutoff. Applying the same argument in each isolating region fixes the marked factors. Equation (16) and independence of the exterior follow.

For an arbitrary taming form, approximate its cohomology class on the compact path by a rational class represented by a nearby closed taming form and rescale the latter to integral periods. The energies are uniformly comparable on the complex lines along this path. At a fixed endpoint let M_J be the monoid of classes of effective cycles. Any nonzero exponent of the full product has label in M_J: induct on positive integral energy. At an ineffective class the entire insertion is zero and smaller effective labels have sums in M_J. Multiplicity-one detection forces all the new exponents there to vanish. This also applies to the marked products in their tubes. Cycle compactness and local constancy of class imply finiteness of effective labels at bounded energy. Two choices of taming form therefore give products in a common locally finite completion at this endpoint, since their energies are comparable on M_J. Their images coincide at each label and the same induction identifies their exponents. Subtracting the marked multiplicities identifies the complementary ones as well. □

For example, in class $7A$ over the genus-two core with two branch values the change in the marked Euler integral of source genus thirteen is

$$a_{\{1,2\}} - \tfrac{11}{12}(a_{\{1\}} + a_{\{2\}}).$$

Indeed $x^{11} = v^{22} - (11/12)v^{24} + O(v^{26})$, whereas $x^{12} = v^{24} + O(v^{26})$. In the elementary product the two genera give separate integral changes. The exterior multiplicities in genera twelve and thirteen have the opposite changes, respectively, by Proposition 8.1.

Proof of Theorem 1.3. On $X = \mathbb{R}^6/\mathbb{Z}^6$ with its standard symplectic form there exists a constant tame linear complex structure admitting no nonempty compact effective curve. To see this, identify the integral classes with integral bivectors. For every nonzero integral bivector B the equation $(\Lambda^2 J)B = B$ defines a proper analytic subset of the space of positive linear complex structures. Indeed this space is the connected homogeneous space $\mathrm{GL}^+(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C})$. If the equation held on an open subset, the standard J would fix the span of the entire GL^+-orbit of B in the exterior square. A diagonal matrix with distinct pairwise products of its diagonal entries extracts the nonzero coordinate bivectors by polynomial interpolation; oriented signed permutations then give every basis bivector in the span. The standard J does not fix a wedge of real coordinate vectors from two different complex lines. Baire's theorem applied to the open tame subset gives a constant J avoiding these equations for all $B \neq 0$. The structure descends under translations. Every effective current would give an invariant integral bivector by integration of its complex tangent planes, with positive symplectic area. Hence the total series is one there. The space of tame structures is path connected, so Proposition 8.1 gives $R_J = P_J^{-1}$ at every marked endpoint on this torus.

In a product of two symplectic two-tori smooth the positive transverse intersections of one horizontal torus with $g-1$ vertical tori, using $zw = \delta$ in disjoint coordinate balls. The intersection graph is a tree. The result is a genus-g symplectic surface of class $[T_1]+(g-1)[T_2]$. Take its product with a point of the last two-torus. The resulting class A is primitive by pairing with the area class of the first factor. Its symplectic normal bundle has Chern number $2g-2$; unitary bundles of rank two over a surface are classified by this number. A tubular identification thus realizes the given homogeneous structure with its prescribed normal derivative. It is tame near the zero section. Cutting off in a chart of tame structures extends it over X while preserving the germ. At every positive even b a connected cubic as required

exists on the same base: send all puncture loops to (12), one handle generator to (123), and the rest to the identity, and fill the finite cover over the punctures. Applying Theorem 1.2 changes the exterior multiplicities by $-c_j$ at h_j, and by zero in every other genus at label $7A$. The binomial bound of §6.6 proves sharpness. □

In particular at least one of the two endpoints attaining this maximum has $\sum_h |n^{\mathrm{ext}}_{7A,h}| \geq 2^{b-1}$, by the triangle inequality and integrality. The area and the class are fixed as b increases. These endpoints may approach a smooth reference structure whose marked section is itself super-rigid. Use, for example, the homogeneous split spin model of [Wue26j, Lemma 14.1] on the same bundle. All the homogeneous structures agree with it along C. On a tube of radius R, cutting off their difference in a smooth chart gives sup norm at most $K_b R$ and first derivative bounded by K_b; the constant may depend on the crossing. For each b take R_b so small that $K_b R_b^{1/b} \leq 1/b$. Interpolation yields convergence in every $C^{0,\alpha}$, $0 < \alpha < 1$. Real dilations of the full-join graphs in §7.1 likewise place their embedded representatives in arbitrarily small tubes, using the operators of that construction. Their genus is $h_b = 1 + 7(g-1) + 5b/2$. It is unbounded along this sequence of smooth tame structures at fixed symplectic area.

The compensation determines the total signed change outside the isolating tube at the two endpoints. Allocating that change to the individual intermediate curves of an arbitrary compact deformation remains open. For simultaneous ramified crossings there is a further local question: several independent heights can match on a common component, and the resulting strata need not have a common dimension or stabilizer. The single adjoint equation of §5 treats the prescribed cubic crossing.

Appendix A. The double-cover displacement and its elliptic tails

We prove the double-cover displacement used in §2.1 and separate its first contracted elliptic contribution into the moving and ramification-point terms.

A.1. **The local contribution at a double-cover wall.** Let C be a smooth curve of genus g, and let $N \to C$ be a complex bundle of rank two and degree $2g-2$. Consider a smooth path D_t of real Cauchy–Riemann operators on N, with D_0 invertible. We orient the parameter interval by increasing t. Write $\mathcal{H}$ for the space of smooth connected double covers of the fixed target, locally taken in a uniformizing chart. At a cover $f : Y \to C$ with b branch values, these values are local complex coordinates up to ordering. The deck involution acts trivially on these coordinates.

Suppose $K = \ker f^*D_0$ is a real line, and put $P = \operatorname{coker} f^*D_0$. Differentiating the family of pullback operators on a section in K and projecting to P gives the intrinsic derivative

$$W_f : T_f\mathcal{H} \oplus \mathbb{R}_t \longrightarrow \operatorname{Hom}_{\mathbb{R}}(K, P). \tag{17}$$

We use the complex orientation of $\mathcal{H}$ and the absolute Cauchy–Riemann determinant orientation of f^*D_0. Indeed, Riemann–Hurwitz gives $\dim_{\mathbb{R}} P = 2b+1$, and the determinant of the space on the right is $(K^*)^{\otimes 2b} \otimes (\det f^*D_0)^*$. Write η_f for the sign of (17) when it is an isomorphism. This sign uses the full pullback operator, including its invariant summand [BS24, (4.59) and (5.59)–(5.60)].

Theorem A.1. *Fix an arithmetic genus bound k_*. Suppose the smooth connected double covers of genus at most k_* with nonzero pullback kernel at $t = 0$ form a finite set up to isomorphism. For each such f, assume that the kernel is an anti-invariant real line, the map $(f, \xi) : Y \to N$ is an embedding for a generator ξ, and (17) is an isomorphism. For sufficiently small $\delta > 0$ the pullback operators at the two endpoints are injective on every*

degree-two stable map of genus at most k_*. In each genus $k \le k_*$ the change of the absolute normal Euler number is

$$L_{2,k}(D_\delta) - L_{2,k}(D_{-\delta}) = \sum_{f:Y\to C} \eta_f \,[v^{2k-2}]\big(2\sin(v/2)\big)^{2g(Y)-2}. \tag{18}$$

Each summand is the change supported on stable maps that factor through a degree-one stable map to the source of the indicated cover. In particular a single wall with smooth source of genus H contributes the series $\eta_f(2\sin(v/2))^{2H-2}$ coefficientwise.

Proof. We first treat a single cover f of genus H. Choose a complex connection on N. On a disk bundle about the zero section we realize D_t as the normal operator of an almost complex structure with holomorphic projection and standard complex structure on each fiber. For a holomorphic map h to C, prescribe the equation of a lift (h, u) by

$$h^*D_t u + h^*B(u,u,u) = 0, \tag{19}$$

where B is a smooth real symmetric trilinear form on N with values in $\Omega_C^{0,1}\otimes N$. Here pullback of the form includes composition with dh. In a connection splitting, the difference of D_t from the connection Dolbeault operator and the cubic both specify vertical-valued $(0,1)$-forms in the horizontal variable. Such a form specifies the off-diagonal block of the almost complex structure, giving (19) for every h [BS24, Lemma 5.9]. We multiply the change of horizontal structure by a radial cutoff equal to one in the disk under consideration and extend across $\mathbb{P}(N\oplus\mathbb{C})$. A unitary connection gives a closed form restricting to the Fubini–Study form on the fibers. Adding a sufficiently large multiple of a positive area form from C tames the extended family, since the horizontal terms are bounded on the compact bundle.

The smooth branch and its sign. The invariant part of f^*D_0 has index zero. To see this, deform equivariantly to a pullback Dolbeault operator and use $(f_*\mathcal{O}_Y)^+ = \mathcal{O}_C$, together with exactness of invariants under the deck group in characteristic zero. The hypothesis on K thus implies that P is entirely anti-invariant. Take equivariant complements to the kernel and image in a Sobolev completion of (19) over the Hurwitz chart. The implicit function theorem solves the image equation for the kernel complement. In local coordinates y on the chart and with kernel coordinate $a\xi$, the remaining equation is

$$aF(y,t,a) = 0, \qquad F(y,t,-a) = F(y,t,a).$$

Indeed the zero section solves the equation at all (y,t) and the deck change negates both a and the obstruction in P. The differential of F in (y,t) at the origin equals W_f evaluated on ξ. Its coefficient of a^2 at $(y,t)=0$ is the image of $f^*B(\xi,\xi,\xi)$ in P. The solution in the kernel complement starts in degree three there, and its linear image projects to zero.

Let p be a nonzero adjoint section annihilating the image of the Hurwitz directions in P. On an unramified disk where p and ξ are nonzero on one sheet, prescribe an arbitrary compactly supported form on that sheet as $f^*B(\xi,\xi,\xi)$. This uses a real covector taking value one on ξ and its third tensor power. Such a disk exists by unique continuation. On the other sheet both the prescribed form and the adjoint change sign, so their two integrals agree. Choose the form to give a nonzero pairing. The implicit function theorem for $F=0$ now gives even functions $y(a), t(a)$ with

$$t(a) = ca^2 + O(a^4), \qquad c \neq 0.$$

The two signs of a give the same curve by the deck action. Dividing its vertical coordinate by a gives smooth convergence to (f,ξ), so the curve for small $a>0$ is embedded and has trivial stabilizer.

At a nonzero solution the derivative of aF in (y, a) has first columns aF_y. Its last column modulo these is $-at'(a)F_t$ by differentiation along the branch. It is invertible, and comparison with W_f gives the sign $-\operatorname{sgn}(t)\eta_f$. In this comparison the tangential deformation of the cover has the complex Hurwitz orientation. Indeed its holomorphic deformation complex reduces to the torsion cokernel of df, supported at the simple ramification points. At the zero section the vertical and tangential determinants split, and continuation gives the determinant of the embedded graph. This is the absolute sign convention of [BS24, Theorem 4.12 and (4.30), (4.59)]. Thus the change in the signed number of embedded curves with nonzero vertical coordinate is $-\eta_f$.

Isolation through genus k_*. Suppose there were other simple curves, of degree at most two over C and normalization genus at most k_*, in arbitrarily small closed tubes at parameters tending to zero. Pass to a sequence with fixed normalization genus. There is no nonconstant compact curve in an affine fiber. In degree one the projection of the normalization is an isomorphism, and invertibility of D_0 excludes a small nonzero graph. Equivalently, a sequence of such graphs can be divided by its maximum norm, and elliptic compactness over the fixed base would give a nonzero element of $\ker D_0$.

In degree two write the maps from the smooth normalizations as (h_n, u_n) and put $\rho_n = \max|u_n| > 0$. The graphs $(h_n, u_n/\rho_n)$ solve (19) with cubic coefficient $\rho_n^2 B$. On a disk bundle of radius greater than one these structures extend to a smooth two-parameter family in (t, ρ) including $(0, 0)$. A common taming form as above has bounded area on the graphs, since its cohomology class on the disk bundle is pulled back from C and the projection degree is fixed. Gromov compactness, with a point marked where the norm is one, gives a stable limit in the linear structure of D_0 whose vertical coordinate is nonzero. Forget the extra mark and stabilize. If two normalized components had degree one, invertibility of D_0 would force both sections to vanish and all constant components would have the same zero vertical coordinate at their attachments. Thus the unique effective normalization is a smooth double cover with a nonzero kernel section. By hypothesis its graph is an embedded curve $\Gamma = (f, \lambda\xi)(Y)$, with $\lambda \neq 0$.

The stable limit factors in degree one through Γ. Its effective normalization maps isomorphically to Γ, so no self-node identifies two points on that normalization. Each maximal connected contracted subcurve attaches at a single point, since its image has a single preimage on the effective normalization. For the resulting map w to Γ and every real normal Cauchy–Riemann operator D_Γ, restriction and pullback give inverse isomorphisms

$$\ker w^* D_\Gamma \simeq \ker D_\Gamma. \tag{20}$$

A section on a contracted component is holomorphic with values in a fixed vector space, hence constant, and the node conditions prescribe its value on the whole contracted subcurve. The compactness theorem for sequences of simple maps to a finite-dimensional family of almost complex structures [BS24, Theorem 3.1 and Remark 3.2(b)] says that a limit over an embedded curve, unless the limiting map is an isomorphism to that curve, has a normal kernel section outside the pullbacks from the embedded curve. By (20) the limit is an isomorphism. We thus have smooth convergence modulo parametrization. The original graphs eventually belong to the Hurwitz chart and its implicit-function neighborhood, contradicting their choice.

We also need injectivity at the endpoints on the entire fixed-target space in genus $k \leq k_*$. Apply the same compactness argument to graphs of normalized kernel sections of the linear operators at $t_n \to 0$. The invariant part of any smooth double-cover kernel descends to a base section. It descends on the unramified locus by invariance and extends across the branch values by bounded removability for the base equation. Since D_t remains invertible, the kernel

sections are odd. Their nonzero graphs are simple because the two sheets have different values wherever the section is nonzero. After smooth convergence in the rescaling argument, the linear problem in the Hurwitz chart reduces to $F(y,t,0)=0$, whose only solution near the origin is $(0,0)$ by (17). Thus all smooth covers through the cutoff have injective pullback for small $0<|t|$. For a nodal source, restriction to its effective normalizations and the constant matching on its contracted components give injectivity as well. At $t=0$ the same reasoning shows that every stable map with kernel lifts in degree one to Y via its graph in Γ. Conversely every such lift carries the pullback of ξ.

Conservation of the local degree. The isolation just proved gives a closed tube of some radius $r>0$ and an interval about zero in which the only simple curves in that tube through the cutoff are C and the embedded curves found above. Shrink the interval so the latter lie in the tube of radius $r/2$. Every effective component of a degree-two stable map contained in the closed tube factors through a simple normalization of degree at most two and genus at most k_*. Connectedness therefore prevents any such map from meeting the boundary. In the compactified target these maps form a compact open and closed part of the parametrized stable-map space. Compactness gives closedness, and support in the interior gives openness. Its virtual degrees at the two ends agree by fixed-genus deformation invariance and the splitting of the virtual class over open and closed substacks [DIW26, §3.1, (3.4)–(3.5)].

At an endpoint the maps supported on C contribute $L_{2,k}(D_t)$ by the normal Euler comparison [Zin11, Theorem 1.2, (1.7)]. The maps on the additional embedded curve, when present, have degree one over it. We compute their degree including its orientation. Let D' be the invertible normal operator on that embedded curve of genus H. For any deformation of D' on the curve, the pullback and base operators admit compatible finite-dimensional stabilizations. The quotient of the two resulting complexes has zero kernel on every degree-one stable map w. Indeed (20) identifies the kernels, and restriction to the effective normalization is a left inverse on cokernels to the pullback of base forms. The quotient cokernel is therefore a vector bundle throughout the deformation, by the stabilized construction in [BS24, Appendix B, (B.8)–(B.10)]. Its determinant is the quotient of the pullback and base determinants. At an invertible endpoint the Euler class in the absolute orientation is the Euler class of this bundle multiplied by $\epsilon(D')$.

The normal bundle has degree $2H-2$, since the ambient first Chern class on the tube vanishes. Deform D' to the Dolbeault operator of two line bundles of degree $H-1$ off the theta divisor, using $\mathcal{O}(-1)^{\oplus 2}$ if $H=0$. There the cokernel has its complex orientation and its virtual Euler degree is the ordinary degree-one local invariant. Pandharipande's degree-one formula [Pan99, Theorem 2] gives, in our genus variable, the series $(2\sin(v/2))^{2H-2}$. This also gives the degree of the reduced cokernel at D' by deformation. The change from the embedded curves off C is consequently $-\eta_f$ times this series through the cutoff. Conservation gives (18) with the opposite sign on C.

Simultaneous covers. For finitely many covers f_i a single cubic B can be chosen outside the finite union of hyperplanes in the adjoint tests above. The compactness argument then gives one branch in each indicated Hurwitz chart and the sum in (18). To identify the individual supported changes, we separate their times. With generators ξ_i and adjoints p_i as above, the conormal to the ith wall on real zeroth-order variations of the base operator is

$$\ell_i(A)=\langle p_i, f_i^* A\,\xi_i\rangle.$$

For distinct covers these functionals are nonproportional. Work on a common unramified disk where the sections and adjoints in question are nonzero. The two sheet integrals for each

ℓ_i agree. Testing with arbitrary real-linear A on this disk, proportionality would therefore imply proportionality of the rank-one tensor densities, and in particular real proportionality $\xi_i = c\xi_j$ on chosen sheets. The two sections solve the same equation there, so $\bar\partial c = 0$ and c is a nonzero constant. Off the union of the branch values the two covers have distinct nonzero mod-two characters, since a cover of the complement has a unique finite normal completion. Their characters are independent and their fiber product is connected. Unique continuation extends the equality over this fiber product. A deck transformation flipping only the first sheet then negates exactly one side, a contradiction.

Choose $\chi(t)$ equal to one near zero and supported away from the endpoints. Write $t_i(s)$ for the crossing time of f_i in the family $D_t + s\chi(t)A$. Then

$$t_i'(0) = -\ell_i(A)/\ell_i(\dot D_0).$$

The denominators are nonzero by (17). Pairwise nonproportionality lets us choose A so all derivatives are distinct. The compactness argument for the linear graphs, now including s, confines all kernels through the cutoff to the same charts. Thus the perturbation separates the crossings and preserves their signs.

For completeness, let $M = \overline{\mathcal{M}}_k(C, 2)$, let I be the time interval, and let Z be the closed noninjectivity locus in $M \times I$. On its complement the absolutely oriented pullback cokernel has an Euler class e. The image of e under the connecting morphism of the pair $(M \times I, (M \times I) \setminus Z)$ pairs with $[M]^{\mathrm{vir}} \times [I, \partial I]$ to give the final Euler degree minus the initial one. At $s = 0$ the kernel supports of the distinct covers are disjoint compact images of $\overline{\mathcal{M}}_k(Y_i, 1)$, since the effective normalization recovers f_i. Choose disjoint neighborhoods. For sufficiently small s all kernels remain in them and avoid their boundaries, by properness and the local stabilized operators [BS24, Appendix B, (B.8)]. The relative class for the whole (t, s)-family splits over these neighborhoods by excision. Pairing over the product virtual family shows that each summand has the same degree at $s = 0$ as at small positive s. There each is the single-cover contribution already computed. All compactness choices were made at the prescribed finite cutoff. □

On a fixed-target path with finitely many degree-at-most-two wall times in all genera, the coefficients in Theorem A.1 may be evaluated at common regular endpoints on either side of a single time. On a regular parameter interval the pullback cokernels form an oriented bundle on each compact fixed-genus map space, so their Euler degrees are constant there. For generic fixed-target paths the finiteness assertion follows from [DW25, Definition 2.6, Proposition 2.7 and Theorem 3.40].

A.2. **A contracted elliptic tail at a ramification point.** Let C be a smooth curve of genus $g \geq 2$, let $N \to C$ be a complex bundle of rank two and degree $2g - 2$, and let D_t be a smooth path of real Cauchy–Riemann operators on N. Fix a smooth connected cover $f : Y \to C$ of degree m and genus H, with b simple ramification points over distinct values. Write $G = \mathrm{Aut}_C(f)$ and $a = |G|$. The fixed-target Hurwitz stack $\mathcal{H}$ is smooth of complex dimension b at f. Indeed, the quotient of $T_Y \to f^*T_C$ is a skyscraper sheaf of length b, with one summand at each ramification point.

Suppose $K = \ker f^*D_0$ is a real line and the intrinsic derivative

$$T_f\mathcal{H} \oplus \mathbb{R}_t \longrightarrow \mathrm{Hom}_{\mathbb{R}}(K, \operatorname{coker} f^*D_0) \tag{21}$$

is an isomorphism. Its sign η uses the complex orientation of $\mathcal{H}$, increasing time and the absolute determinant orientation transported from complex-linear operators. For $0 \neq \xi \in K$ assume $d\xi_r \neq 0$ at each ramification point r. In a frame pulled back from the target,

$d\xi_r : T_rY \to N_{f(r)}$ is independent of the connection because $df_r = 0$, and is complex-linear by the pullback equation.

Theorem A.2. *In source genus $k = H+1$, the stable maps obtained by attaching a contracted one-pointed curve of arithmetic genus one to f form an isolated compact part of the locus with nonzero normal kernel at the crossing. The local degree of the change of the absolute normal Euler class supported on this family is*

$$-\frac{2\eta}{a}\left(\frac{m(g-1)}{12}+\frac{b}{24}\right) = -\frac{2\eta}{a}\frac{H-1}{12}. \tag{22}$$

The first term comes from the component on which the tail persists. The second is the sum over ramification points on the component on which the attaching node smooths. The corresponding local degree at the smooth cover in genus H is $2\eta/a$. When the endpoint pullbacks are injective in the genus under consideration, each local degree is a summand of the difference of the endpoint Euler numbers.

Proof. For fixed k let $X = \overline{\mathcal{M}}_k(C, m)$ be the stack of connected stable maps, let I be the parameter interval, and let $U \subset X \times I$ be the open set where the pullback normal operator is injective. Its oriented real cokernel bundle Q_U has rank $d = 4(k-1-m(g-1))$, the real virtual dimension of X. The relative class

$$u = \tfrac{1}{2}\delta e(Q_U) \in H^{d+1}(X \times I, U; \mathbb{Q}) \tag{23}$$

defines the local degrees in the statement. Here δ is the connecting homomorphism. With injective endpoints, $2u$ paired with the image of $[X]^{\mathrm{vir}} \times [I, \partial I]$ is the difference of the endpoint Euler numbers by the boundary formula. Excision gives the summand of this pairing supported near an isolated compact part of the complement of U. This local pairing also makes sense when other covers have kernel at the same time.

Near a map with kernel line K and cokernel Q, stabilization gives a finite-dimensional presentation of the normal operator [BS24, §6.1 and Appendix B, (B.8)]. In its matrix space the stratum with kernel line has normal space $\mathrm{Hom}_{\mathbb{R}}(K, Q)$. Its determinant is $(K^*)^{\otimes d} \otimes (\det D)^*$, where $\det D$ denotes the Fredholm determinant at that map. Thus the absolute orientation coorients this stratum. On the link of a normal disc, choosing a generator of K identifies the universal cokernel with the tangent bundle of S^d. Its Euler degree is two with the boundary orientation. For $d = 0$ the two invertible matrices have opposite determinant signs. Hence (23) pulls back the positive Thom class of this stratum on the corank-at-most-one neighborhood. Two stabilizations admit a common sum, which adds an identity block after a change of splitting. The normal space and its orientation are preserved. In particular (21) gives local degree η/a for u at the smooth cover.

Write (E, e) for the pointed tail, p for the attaching point on Y, and $\mathbb{E}$ for the Hodge line on $\overline{\mathcal{M}}_{1,1}$. The compact family over $(f, 0)$ is

$$T = [(Y \times \overline{\mathcal{M}}_{1,1})/G].$$

We describe the fixed-target virtual class near T. The cover deforms unobstructedly, and evaluation of the constant value on E eliminates the matching condition at the node. The remaining obstruction space is $T_{f(p)}C \otimes H^1(E, \mathcal{O}_E)$. Parameters consist of the cover, the point p, the pointed tail, and the attaching parameter $q \in T_pY \otimes T_eE$. The node-preserving maps exist for every value of the first three parameters, so their scalar holomorphic obstruction equation has the form $qF = 0$. On $q = 0$ its derivative in the smoothing direction is $df_p \otimes \delta_e$, where $\delta_e : T_eE \to H^1(E, \mathcal{O}_E)$ is the boundary map for $\mathcal{O}_E(e)$. Indeed, substitution of the plumbing relation into the linear term of the core map gives the simple principal part on E.

The dual of δ_e is evaluation of a dualizing differential at the mark, an isomorphism also on the irreducible nodal elliptic curve. Thus F is a unit away from ramification. At a simple ramification point it vanishes simply as p varies. Locally there are two smooth branches $B = \{q = 0\}$ and $M = \{F = 0\}$ near such a point. The fixed-target virtual class is the divisor of the scalar obstruction section and hence the sum of their fundamental cycles, each with multiplicity one. At a nodal tail the modulus of the pointed elliptic curve includes the independent smoothing parameter of its self-node.

On B, normalizing at the attaching node identifies the normal kernel with the core kernel. Constant tail sections evaluate onto the matching fiber, so the cokernel over T is the sum of the core cokernel and the complex two-plane $V = f^*N \otimes \mathbb{E}^\vee$. Taking the Thom class along (21) leaves the Euler class of $\mathrm{Hom}_{\mathbb{R}}(K, V)$. On the cover $Y \times \overline{\mathcal{M}}_{1,1}$, K is constant and either generator induces the complex orientation on this excess bundle. The degree on B is therefore

$$\frac{\eta}{a} \int_{Y \times \overline{\mathcal{M}}_{1,1}} c_2(f^*N \otimes \mathbb{E}^\vee) = -\frac{\eta}{a} \frac{2m(g-1)}{24}. \tag{24}$$

Indeed $c_2 = -c_1(f^*N)c_1(\mathbb{E})$ and $\int_{\overline{\mathcal{M}}_{1,1}} c_1(\mathbb{E}) = 1/24$. The latter degree follows from the Weierstrass presentation as the weighted projective stack of weights $4, 6$, with Hodge line $\mathcal{O}(1)$. Transfer from the cover accounts for the action of G, including its action on K.

It remains to restrict u to M at a ramification point r. Work on a uniformizing chart before division by G. Choose core and target coordinates so that $w = w_0 + z^2$. On the pointed elliptic family choose a meromorphic function P with its sole pole of order two at e. This choice extends locally across the nodal fiber by base change for $\mathcal{O}_E(2e)$, which has two sections, with only the constants in $H^0(E, \mathcal{O}_E(e))$. Take a coordinate y with $P = y^{-2}$ near e. The plumbing

$$zy = q, \qquad w = w_0 + z^2 \quad \text{on the core}, \qquad w = w_0 + q^2 P \quad \text{on the tail} \tag{25}$$

gives maps agreeing on the overlap. Varying the marked ramification cover and the elliptic modulus gives every tangent to $M \cap B$, and q smooths the attaching node to first order. The inverse function theorem on the smooth branch M proves that these are local parameters there.

Use cokernel representatives supported on the core outside the attaching disc and on the tail away from both nodes. The former span by unique continuation for the adjoint equation, the latter by duality with the Hodge line. Stabilization by these spaces is surjective with kernel a real line near the central map. Normalize its section s by a real evaluation on the thick part of the core, giving ξ at the center. The stabilized kernels are continuous across the nodal fibers [BS24, Appendix B]. In the plumbing norms, a continuous local section is uniformly bounded and converges on compact subsets away from the nodes [Sie98, §§4.3–4.4, Proposition 4.15]. For the pulled-back zeroth-order term the multiplication norm is bounded by the sup norm of the section times the weighted L^p norm of the map derivative. A map holomorphic on the total space has bounded derivative on each half-collar in its larger coordinate, so this latter norm tends to zero on shrinking collars. On the remaining compact sets the coefficients vary smoothly. The elliptic obstruction coefficient c_E vanishes identically along $q = 0$, where the stabilized core section extends by its constant value to the tail.

We compute c_E to first order. In a holomorphic frame for a reference Dolbeault operator on the target disc, the equation on the truncated elliptic side $|y| \geq |q|/\rho$ reads

$$\bar\partial s + A_t(w)(s)\, \overline{dw} = \theta_E(c_E).$$

Here $A_t(w)$ is real-linear on the fibers. Normalize the forcing θ_E by pairing with a regular differential $\omega = (a_0 + O(y))\, dy$ near e. With the trace convention for δ_e, the boundary integral

in Stokes' formula becomes

$$qa_0\,\frac{1}{2\pi i}\oint_{|z|=\rho} s(z)\frac{dz}{z^2}+O_\rho(|q|^2).$$

For fixed ρ the contour coefficient tends to that of ξ as all parameters approach the chosen point of T. Since $\bar\partial\xi(r)=0$, it is $\partial_z\xi(0)+O(\rho)$ in the limit. For fixed small $c>0$, the zeroth-order term on $|q|/\rho\le|y|\le c$ is bounded by

$$C|q|^2\int_{|q|/\rho}^{c} u^{-2}\,du\le C\rho|q|.$$

Outside this disc $dw=q^2dP$ and the integral of $|dP|\,|\omega|$ is uniformly bounded, also on a smoothing $uv=\tau$ of the self-node. Indeed ω is a bounded multiple of du/u, and the derivative of the regular function $P(u,\tau/u)$ is bounded on $|u|\ge\sqrt{|\tau|}$. The other half is covered by v. On the nodal fiber the two residues in Stokes' formula cancel by matching of s on the normalization.

Dividing by q, then sending ρ to zero, gives the elliptic coefficient to order $o(|q|)$ as all parameters tend to the wall. Its linear term in the attaching direction is the intrinsic homomorphism

$$\begin{aligned} T_rY\otimes T_eE &\longrightarrow \mathrm{Hom}_{\mathbb{R}}\big(K,N_{f(r)}\otimes H^1(E,\mathcal{O}_E)\big),\\ v\otimes v' &\longmapsto \big[\xi\longmapsto d\xi_r(v)\otimes\delta_e(v')\big]. \end{aligned}\tag{26}$$

This limit holds along arbitrary simultaneous variations of the cover, time and elliptic modulus. Compactness of $\overline{\mathcal{M}}_{1,1}$ gives a uniform estimate in finitely many charts. The map (26) embeds the complex smoothing line since $d\xi_r\neq0$. In a tubular neighborhood extend the splitting of the obstruction space from the divisor. The straight-line homotopy from the elliptic coefficient to the extended linear term stays nonzero for $q\neq0$ after shrinking the neighborhood. With this coefficient fixed, retract the core coefficient to its restriction at $q=0$, where its derivative is (21). These homotopies require only continuity of the core coefficient in q. They apply over the whole ramification-marked family: the leading map transforms as the tensor in (26), and splittings can be chosen equivariantly using a metric and common stabilizations.

After these two Thom isomorphisms the excess bundle on the deck cover is the complex line

$$\mathbb{E}^\vee\otimes\big(N_{f(r)}/d\xi_r(T_rY)\big).$$

Its degree is $-1/24$. The smoothing line has its complex orientation. Interchanging it with the oriented real normal space to the wall of smooth covers preserves the sign. Thus the degree of u on M for the orbit of r is $-\eta/(24|\operatorname{Aut}_C(f,r)|)$. Summation over the orbits gives $-\eta b/(24a)$. The transversality on B and the nonvanishing for $q\neq0$ on M isolate T in the kernel locus. Adding the two virtual-cycle contributions, doubling by (23), and using $2H-2=2m(g-1)+b$ proves (22). □

For a double cover of a genus-two target with two branch points, $H=4$ and $a=2$. The smooth-cover jump is η. In genus five the moving attachment contributes $-\eta/6$, and smoothing at either ramification point contributes $-\eta/24$, giving $-\eta/4$ in all. In general the ratio $-(H-1)/12$ of the two successive jumps is the quadratic coefficient of $(\sin(v/2)/(v/2))^{2H-2}$, as in the degree-one formula [Pan99, Theorem 2].

Appendix B. Estimates and virtual classes at a joined double cover

B.1. **The local equation for Theorem 3.1.** Fix the joining set and the branch frames of §3.1.

B.1.1. *The deformation space and the matching equations.* At one selected point the nodal map and its deformation are given by

$$zy = q, \qquad w = w_0 + (z+p)^2 + y. \tag{27}$$

When $q = 0$, the coordinate on the normalized double is $z + p$ and its attaching mark has coordinate p. Translating w_0 to zero and eliminating y gives the free algebra on $1, z, z^2$ with relation

$$z(z+p)^2 - wz + q = 0. \tag{28}$$

To see versality, use the relative cotangent complex of this hypersurface over the target [Stacks, Tag 08SL]. Its dual on the central fiber is multiplication by $3z^2 - w$ between two copies of the local ring. This element is a nonzerodivisor on the two branches. Its cokernel is $\mathbb{C}\{z\}/(z^3)$, with basis represented by the variations of q, w_0, p, respectively. An unselected branch contributes one complex parameter. The first deformation sheaf has finite support and the complex has no other cohomology, so the fixed-target chart is smooth of complex dimension $b + 2j$ and carries its ordinary fundamental class. Distinct branch disks supply independent pairs (p_i, q_i). The stabilizer is the single involution of Y over C. In (27) it acts by $(z, p, q) \mapsto (-z, -p, -q)$ at every join and fixes s.

Put $P_I = \operatorname{coker} h_I^* D$. A kernel section restricts to zero on the identity and to a multiple of ξ on Y. Every such pair matches because ξ vanishes at the marked points. The normalization sequence for the operator thus identifies $\ker h_I^* D = K$ and gives

$$0 \longrightarrow \bigoplus_{i \in I} N_{x_i} \longrightarrow P_I \longrightarrow P_f \longrightarrow 0. \tag{29}$$

The smoothing calculation takes place in the individual matching fibers of this sequence. We give it at one point, suppressing i.

B.1.2. *The pullback operator on the fixed target.* For a map h in the finite chart, let $F = h_*\mathcal{O}$ on C. It is a rank-three bundle by (28), the ordinary quadratic algebra at the remaining branch points, and the sheet decomposition off the branch disks. Translate the moving branch centers on smaller disks and extend the translations by diffeomorphisms supported in larger disks. Identify the family of bundles smoothly, using the polynomial frames on the smaller disks. On the intervening collars the sheets are unramified and the identifications are smooth in the parameters. Fix $2 < a < 4$. In these trivializations consider the operator

$$\mathcal{A}_h : W^{1,a}(C, N \otimes F) \longrightarrow L^a(C, \Omega_C^{0,1} \otimes N \otimes F)$$

which acts by D_t after evaluation on each regular sheet. The complex-linear part is the tensor product Dolbeault operator. In the polynomial frame the antilinear part applies B to the coefficients and replaces $1, \bar{z}, \bar{z}^2$ by their quadratic interpolants at the three roots. This determines a measurable matrix $M_{p,q}$ almost everywhere. At an ordinary simple branch the sole quadratic-sheet factor in the interpolation of $\bar{z}$ is $\bar{z}/z$, which is bounded.

Let d be the diameter of the three roots and m their maximum modulus. In a second divided difference order the two most distant roots first and last. The first divided differences of conjugation have modulus one and those of $\bar{z}^2$ have modulus at most $2m$. Hence the respective second differences are bounded by $2/d$ and $4m/d$. Newton interpolation bounds every entry of $M_{p,q}$ by $C(1 + d^{-1})$. Subtracting the mean of the roots depresses (28) to a monic cubic with linear coefficient $-(w + p^2/3)$, so $|w + p^2/3| \le Cd^2$. It follows that the matrices have uniformly bounded $L^{a'}$ norm for $a < a' < 4$. Away from the limiting branch

values they converge pointwise, hence they converge in L^a. Since $W^{1,a} \hookrightarrow C^0$ is compact, the antilinear part is a continuous family of compact perturbations of the differential part.

We identify its finite-dimensional reduction with that of the actual pullback operator. A polynomial section in the kernel evaluates to a bounded solution on the normalization off the branch points and node preimages. Removability extends it there, and the common root gives equal values at the two sides of a node. Evaluation is injective by interpolation on the open unramified locus. This argument also applies to solutions with smooth forcing supported in that locus. At the central map the generator on the two local components is represented by $zu_1(w)$, where $u_1(w) = \xi(z)/z$ on the double. Oddness and $\bar\partial\xi(0) = 0$ give $\xi(z) = vz + O(|z|^3)$ with derivative error $O(|z|^2)$. Thus $u_1(0) = v$ and its first weak derivatives in $w = z^2$ are bounded. The same division on each ordinary branch disk shows that this is a global coefficient solution. The two central kernels agree.

The real indices agree by deformation to the complex-linear operators and the finite projection formula, including on the nodal fiber. Consequently the central cokernels have equal dimension. Smooth forms supported on unramified disks span both cokernels: a dual annihilator vanishes on these disks and hence off the nodes by unique continuation. In the normalized matching presentation, testing arbitrary endpoint values then kills the matching multipliers. Inside a finite common span surjecting onto both, a generic tuple of minimal length projects to a basis of each. Use its span E as a stabilization. Both stabilized operators are surjective near the central map and their kernels have rank one, using the continuity of the pullback reduction on nodal curves [BS24, Appendix B, (B.6)–(B.9)]. Evaluation identifies the two lines and commutes with projection to E. Normalize a generator by a fixed real functional supported on an unramified disk and taking value one on ξ. The two obstruction maps to E then coincide. Enlarging E by further such forms gives the same identification of their cokernel classes, since evaluation again identifies the stabilized kernels of equal dimension.

B.1.3. *The logarithmic term.* Set $\ell = \max(|p|, |q|^{1/3})$ and write $Q = q\log(1/|q|)$ for $q \neq 0$, with $Q = 0$ at zero. The only entry of $M_{p,q}$ whose q-variation has nonintegrable first derivative is the coefficient of z^2 in the interpolant of $\bar z$. We compute its variation against a Hölder continuous test using Euclidean area $dA(w)$.

In the disk $|w| \le A\ell^2$, for fixed large A, the roots have size $O(\ell)$ by rescaling (28). The bound by a constant times $1 + |w + p^2/3|^{-1/2}$ gives an integral $O(\ell^3)$ there for the varying and central matrices. On the complementary annulus choose $r^2 = w$ and set $z = rZ$. The roots in Z are analytic near $0, 1, -1$ in p/r and q/r^3, which are uniformly small. Their interpolation matrix is real analytic in these parameters, with bounded second derivatives. The coefficient in question is multiplied by $\bar r/r^2$ on returning to z. All other nonconstant entries scale by factors with nonnegative power of $|r|$.

At $p = q = 0$ the interpolant of $\bar z$ is $(\bar r/r)z$. To first order in q, the root increments at $0, r, -r$ are $q/w, -q/(2w), -q/(2w)$. Subtract the old interpolant applied to those increments from their conjugates. The three differences are $a_0, -a_0/2, -a_0/2$ with $a_0 = \bar q/\bar w - (\bar r/r)q/w$. Their quadratic coefficient is

$$-\frac{3a_0}{2w} = -\frac{3}{2}\left(\frac{\bar q}{|w|^2} - \frac{\bar r}{r}\frac{q}{w^2}\right). \tag{30}$$

The second term has angular mode -3 in w and hence zero mean. In either term subtracting the value of the test at zero costs $O(|q|)$ in the integral by Hölder continuity. Against the constant test one, (30) integrates to $-6\pi\bar q\log(1/\ell) + O(|q|)$.

For completeness the Taylor remainders can be estimated before angular integration. Writing $R = |r|$, the largest is bounded by $CR^{-1}(|p|/R + |q|/R^3)^2$. Integration against $O(R^3\,dR)$ gives

$$O\big(|p|^2 + |pq|/\ell + |q|^2/\ell^3\big) = O(|p|^2 + |q|).$$

The q-linear terms of all other entries have integral $O(|q|)$ in absolute value. The p-linear terms are bounded by $C|p|/|w|$ on the disk, and deleting its inner part costs $O(|p|\ell^2)$. For $\ell = |q|^{1/3}$ the logarithmic integral above is $-2\pi\bar{Q} + O(|q|)$. For $|q| < |p|^3$ both $|q|\log(1/\ell)$ and $|Q|$ are uniformly $o(|p|)$. Since $\ell^3 \le |p|^3 + |q|$, this proves, also on the axes, that $M_{p,q} - M_{0,0}$ has L^1 norm $O(|p| + |Q|)$ and that its pairing with any fixed Hölder test is its integral of its first-order p-variation plus the point mass

$$-2\pi\bar{Q}\,\delta_x \quad \text{in the } (\bar{z}, z^2) \text{ coefficient,} \tag{31}$$

up to $o(|p| + |Q|)$. Translating the branch center before this calculation leaves the matrix on the smaller disk unchanged in form. Variation of its smooth factors and of the data on the collars adds their ordinary linear terms in s, p and an error $o(|s| + |p| + |Q|)$ in these pairings.

To pass to the obstruction map, pair with a central adjoint solution. At the central parameter the conjugation matrix is bounded, sending $\bar{z}$ to $(\bar{r}/r)z$ and $\bar{z}^2$ to $(\bar{w}/w)z^2$. The adjoint solution, initially in $L^{a/(a-1)}$, thus lies in $W^{1,k}$ for every finite k by elliptic estimates and Sobolev iteration. In particular it is Hölder continuous. The normalized stabilized generator converges in $W^{1,a}$ to the central one. In pairing the operator variation with the difference of these generators the multiplier has L^1 norm $O(|s| + |p| + |Q|)$ and the generator difference is $o(1)$ in C^0. The smooth variation of the principal part on the collars gives the same little-oh estimate from $W^{1,a}$ convergence. The central operator applied to the difference pairs to zero. A varying stabilizing form contributes only its smooth variation times its coefficient, which tends to zero. We may therefore apply (31) to the central generator in the actual reduced equation. Its coefficient at the node is $B(x)v$.

B.1.4. *Identification of the residue and the local degree.* The distribution $a_1 z^2 \delta_x$ represents in (29) a common nonzero real multiple of the matching class of $a_1 \in N_x$. To verify this, choose a local real-linear frame $S(w)$ of D-homogeneous sections, with $S(0) = 1$. Such a frame follows by solving the local equation for the constant real basis vectors with the Cauchy integral right inverse on a sufficiently small disk, and renormalizing their values at zero. On the punctured disk the section $z^2 S(w) a_1/w$ takes the value $S(w)a_1$ on the double and zero on the identity. Multiply by a cutoff equal to one near zero. Its distributional image is the ordinary cutoff forcing plus $\pi a_1 z^2 \delta_x$, since $\partial_{\bar{w}}(1/w) = \pi\delta_x$. The ordinary forcing represents the matching class up to the convention for the evaluation difference and projects to zero on P_f. The total image pairs to zero with every adjoint by integration by parts, using $1/w \in L^k$ for $k < 2$ and the adjoint regularity just proved. This identifies the residue in its own matching fiber, with the same real scalar for every a_1.

On the nodal stratum $q_i = 0$ the normalized component maps are independent of the p_i. Their obstruction in P_f has derivative W_f in s. The ith evaluation difference has derivative $\pm v_i p_i$ in its moving mark, and may have a linear term in s. These derivatives follow by stabilizing the normalized component operators off the marks and varying the evaluation row in (29). The comparison with the fixed-target stabilization preserves the derivatives at the zero, since a continuous change of trivialization only multiplies their linear part by its central value.

Proof of Theorem 3.1. Choose the stabilizing forms away from all selected disks. Each disk contributes (31) with its own cutoff, so none of its residue projects to another matching fiber.

With $Q_i = q_i \log(1/|q_i|)$ extended by zero, the combined error is

$$o\left(|s| + \sum_{i\in I}(|p_i| + |Q_i|)\right)$$

because there are finitely many disjoint disks. This holds on the partially smoothed strata as well. After choosing a generator of K and filtering each N_{x_i} by $\mathbb{C}v_i$, the linear part in these coordinates is block triangular. Its diagonal blocks are W_f evaluated on the generator, the complex-linear maps $p_i \mapsto \pm v_i p_i$, and the maps $Q_i \mapsto c_i \bar{Q}_i[B(x_i)v_i]$ to $N_{x_i}/\mathbb{C}v_i$ for nonzero real c_i. By (6) the last maps are antilinear isomorphisms of complex lines. The radial coordinate change $q_i \mapsto Q_i$ preserves orientation near zero. The little-oh estimate gives isolation and a homotopy on every sufficiently small link to the invertible linear part.

Normalization gives the Fredholm determinant isomorphism

$$\det h_I^* D \simeq \det f^* D \otimes \det D \otimes \bigotimes_{i\in I} (\det_{\mathbb{R}} N_{x_i})^*.$$

It preserves the orientations obtained by deformation to the complex normalization sequence. The evaluation fibers have complex orientation and the one invertible identity component contributes ϵ. Replacing the generator of K negates an even number $4j$ of real matching coordinates. Each attaching block has positive sign and each smoothing quotient negative sign. Thus the degree with the absolute orientation upstairs is $(-1)^j \epsilon \eta_f$. On its link the cokernel is the quotient by the image of the normalized obstruction vector, hence pulls back the tangent bundle of the sphere of dimension $2b + 4j$. Its Euler number gives twice the local degree. Dividing by the order two of the single deck group gives the asserted change. Distinct subsets give distinct maps since the branch values in the fixed target are distinct.

It remains to verify genericity. On a fixed double cover mark a simple ramification point r over x and consider, on odd sections, the map

$$(D, \xi) \longmapsto (f^* D\xi, \partial_z \xi(r), B_D(x)).$$

It is a submersion at a nonzero solution. Indeed an annihilator of its differential has an adjoint distribution solving the homogeneous adjoint equation off r. On an unramified disk disjoint from x where ξ is nonzero on one sheet, real zeroth-order variations prescribe arbitrary forcing on that sheet and its negative on the other. Oddness and unique continuation therefore force the distribution to be supported at r. For any prescribed first holomorphic jet, choose a section whose equation vanishes to higher order than the distribution there. This is obtained by prescribing the pure holomorphic Taylor coefficients and recursively solving the equation for coefficients involving $\bar{z}$, then cutting off and antisymmetrizing. These variations kill the first-jet multiplier. The adjoint distribution now solves the homogeneous equation across r, so is smooth and zero. Free variation of $B_D(x)$ kills the remaining multiplier. In Sobolev spaces of sufficiently high order the linearized range is closed, since the section operator is Fredholm and the two evaluations are finite dimensional.

At $v \neq 0$, variation of B surjects onto $N_x/\mathbb{C}v$. Thus failure of (6) has real codimension two. The marked projectivized odd incidence has index -1 over the operator space. Indeed, the odd operator has real index $-2b$, since the invariant part has index zero by Dolbeault descent. Varying the Hurwitz parameters adds $2b$ real dimensions and projectivizing removes one. The branch mark has finite choices. Adding a path parameter leaves the bad incidence of negative expected dimension. Parametric transversality, followed by countable intersection over covering genera and charts, gives the assertion relative to rigid endpoints. □

B.2. **A ramification bound and compactness.** Let N for the moment be a Hermitian bundle of any rank over C. Fix a unitary connection ∇ and a metric on C in its conformal class. The graph equation for a lift u of a finite holomorphic map $f : Y \to C$ is

$$\bar\partial_{f^*\nabla} u + f^* P(\cdot, u) = 0.$$

Here P is a smooth fiber-dependent $(0,1)$-form on a bounded fiberwise convex tube. For a positive orthonormal pair X, JX on C, let M bound the fiber Lipschitz constant of $P(\cdot,\cdot)(X)$, uniformly in X. Put

$$\kappa = \big(\lambda_{\max}(iF_\nabla(X, JX)) + 2M^2\big)\, dA.$$

We use the Hermitian product conjugate-linear in the first variable. For a compact family of linear normal equations one may use a common M, independent of the height of the graph.

Lemma B.1. *Suppose (f, u) is generically one-to-one on its smooth normalization, with distinct component images. Let $m = \deg f$, let R be its total ramification, and suppose n disjoint pairs of distinct normalization points are identified as nodes at which the graph matches. Then*

$$R + 2n \le \frac{m(m-1)}{2\pi} \int_C \kappa.$$

In particular a connected graph with these nodes and no contracted components has arithmetic genus at most $1 + m(g-1) + \frac{m(m-1)}{4\pi} \int_C \kappa$.

Proof. Normalize the closure of the off-diagonal pairs in $Y \times_C Y$ over regular values of f. The resulting smooth, possibly disconnected curve Z has projections p_1, p_2 and a map $b = fp_i$ of degree $m(m-1)$ to C. The section $s = p_1^* u - p_2^* u$ is nonzero somewhere on every component. Subtracting the two equations gives $|(\bar\partial_{b^*\nabla} s)(T)| \le M|s|\,|db(T)|$ for each unit tangent T. Integrating the vertical derivative of P along the segment between the two heights writes the difference as a smooth real-linear zeroth-order operator applied to s. Its zeros are isolated and have finite order by the similarity principle. In a smooth unitary frame the leading Taylor term of order a is $z^a v$ with $v \neq 0$, since the principal equation on that term is $\bar\partial = 0$.

Away from the zeros, set $e = s/|s|$ and $\alpha = \operatorname{Im}\langle e, b^*\nabla e\rangle$. Project the derivative perpendicular to $\mathbb{C}e$ and denote the result by $\nabla^\perp e$. Differentiation in a positive unit frame T, jT gives

$$\begin{aligned} d\alpha(T, jT) = {} & \operatorname{Im}\langle e, (b^* F_\nabla)(T, jT) e\rangle \\ & + 2\big(|(\nabla^\perp e)^{1,0}(T)|^2 - |(\nabla^\perp e)^{0,1}(T)|^2\big). \end{aligned}$$

The components parallel to e have purely imaginary coefficients and drop out of the exterior product. The bound on $\bar\partial s$ gives $|(\nabla^\perp e)^{0,1}(T)| \le M|db(T)|$, whence $d\alpha \ge -b^*\kappa$. On a positively oriented small circle about a zero of order a, the integral of α tends to $2\pi a$. Stokes' theorem on the complement of the disks bounds the sum of the orders by $(2\pi)^{-1}\int_Z b^*\kappa$. At a point of index k there are $k-1$ off-diagonal branches over the repeated point, one for each nontrivial kth root of unity. Each forces a zero. Each identified pair forces a zero in each order of the pair, at points distinct from the ramification pairs. This proves the inequality. Riemann–Hurwitz on the components and the arithmetic genus formula for a nodal curve give the last claim. □

Proposition B.2. *Over a generic compact path as in Theorem 2.1, the projectivized incidence of smooth connected cubic covers with kernel is compact and finite, with the genera of all covers included.*

Proof. Proposition 2.2 gives discreteness. For a sequence of kernel lines choose generators of fixed small maximum norm and mark points attaining that norm. Their embedded graphs lie in a compact subtube. The family of graph almost complex structures agrees there with a smooth family on a compact completion, tamed by a common form as in the proof of Proposition 2.3. Projection determines the homology class and hence gives a uniform bound on area. Lemma B.1 bounds the genus, so a subsequence has fixed genus. Stable-map compactness gives a limit whose marked point has nonzero height. Forget that mark and stabilize.

A nonconstant component projects in positive degree, since the portion of a single fiber in the tube lies in a complex vector space. Invertibility on C excludes a degree-one component of nonzero height. If a component has degree three and nonzero height, its parametrization is simple, since the alternative factors through a nonzero graph of degree one. This cubic normalization is embedded by Proposition 2.2. Otherwise the positive components are a nonzero double graph and one copy of the zero section, both with multiplicity one. The double generator is odd. It can vanish only at ramification points, since a zero over a regular value would identify the two sheets of its embedded graph. Its derivative at a ramification point gives a vertical tangent, distinct from the tangent to the zero section.

For a maximal stable contracted subcurve in a limit of smooth holomorphic curves, the attaching branches include a critical branch, two branches with coincident tangent lines, or at least three branches [ES25, Theorem 1.1]. This theorem applies to the varying structure by extending the parameter interval to a complex disk, smoothly extending the family in its real coordinate, and using the fixed block structure $J_t \oplus i$ on the product. The curves are constant in the parameter direction. Each candidate limit above has at most two branches through any image point, and they are immersed with distinct tangent lines when both occur. Consequently there is no contracted subcurve. Neither embedded normalization admits a self-node. A single cubic limit therefore converges in the smooth-cover chart and retains its kernel line. In the two-component case connectedness requires direct joins at a nonempty subset of the ramification points. Theorem 3.1 isolates the projective kernel there even in the full smoothing chart. Such a point cannot be approached by smooth cubic solutions. Rigidity excludes endpoint limits. This proves compactness. □

For degree two a normalized nonzero graph has only the embedded degree-two possibility in the preceding proof. The same argument gives finiteness, also proved in [DW25, Proposition 3.73]. The contracted components excluded in Proposition B.2 do enter the Euler class on the full stable-map space.

B.3. **Contracted components at a reducible cubic cover.** Let $f : Y \to C$ be the double cover at a crossing as in Theorem 3.1, and take the crossing parameter to be 0. Write $r_1, \dots, r_b$ for its ramification points, $x_i = f(r_i)$, and ξ for a nonzero odd kernel section. For a nonempty set $I \subset \{1, \dots, b\}$, let $h_I : X_I \to C$ be the finite map obtained from $Y \amalg C$ by identifying r_i with x_i for $i \in I$. Put

$$H = g(Y) + g(C) + |I| - 1, \qquad d_I = (-1)^{|I|} \epsilon \eta_f.$$

In arithmetic genus $k \geq H$, let $G_{I,k}$ consist of the stable maps obtained from h_I by attaching connected contracted curves at smooth points and by inserting connected contracted curves between the branches of its nodes. We include all stable configurations in these operations. In particular an inserted curve may have rational components. We regard $G_{I,k}$ as a locus at $t = 0$ in the product of the ordinary fixed-target moduli stack with the parameter interval.

Theorem B.3. *Suppose the hypotheses of Theorem 3.1 hold with its projected jet condition at every ramification point. Then $G_{I,k}$ is compact and isolated in the locus of stable maps with nonzero pullback kernel. Its contribution to the change of the absolute normal Euler number in arithmetic genus k is*

$$d_I\,[v^{2k-2}]\big(2\sin(v/2)\big)^{2H-2}. \tag{32}$$

Thus the contribution of the specified set of joining nodes to the connected series is $d_I x^{H-1}$, where $x = (2\sin(v/2))^2$.

We displace the nodal kernel to a signed collection of embedded graphs on smooth domains. Near each such graph the Euler integral is a constrained virtual count in the total space. Cutting its parameter interval at one regular value leaves the ordinary degree-one series.

B.3.1. *The compact locus and the choice of sheets.* A presentation by proper gluing maps makes explicit the allowable specializations at an attachment. Choose nonnegative integers e_Y, e_C, e_i with $e_Y + e_C + \sum_{i\in I} e_i = k - H$. Take the two evaluation fibers

$$\overline{\mathcal{M}}_{g(Y)+e_Y,|I|}(Y,1)_{(r_i)_{i\in I}}, \qquad \overline{\mathcal{M}}_{g(C)+e_C,|I|}(C,1)_{(x_i)_{i\in I}}.$$

At i, join the corresponding marks directly if $e_i = 0$ and insert a member of $\overline{\mathcal{M}}_{e_i,2}$ if $e_i > 0$. Compose the maps to Y with f. The union of the images over these finitely many choices is $G_{I,k}$. Indeed a degree-one stable map to a smooth curve has one nonconstant normalization, mapping isomorphically, and every connected contracted subcurve maximal for inclusion meets this normalization just once. Its markings may lie on contracted components. This accounts also for tails whose attaching points come together or specialize to a joining point. Conversely, cutting out all maximal contracted subcurves that join the two positive components, and marking the ends of each direct joining node, recovers such gluing data.

In the lift to N, the positive components are the graph of ξ over Y and the zero section over C. They are embedded and meet precisely over the x_i, with distinct complex tangent lines. Every contracted connected subcurve has a single image point. Thus its external positive branches number at most two, and two can occur only on opposite components. The direct nodes and the contracted subcurves with two ends recover I intrinsically. The proper gluing images for distinct I are therefore disjoint. Restriction to the positive components identifies the kernel throughout $G_{I,k}$ with $\mathbb{R}\xi$, since the base operator is invertible and on contracted subcurves the sections are constants with prescribed attaching values.

Choose concentric closed disks $U \subset \operatorname{int} U'$ of regular values disjoint from the x_i. Over U' the three heights $0, \xi, -\xi$ are separated. Even if a contracted curve lies over a point of U, a nearby stable map has three pieces over U, each of total degree one. In fact at a central map the three preimages of U' with their contracted components have disjoint neighborhoods. The collapse maps of converging domains keep the preimage of U in these neighborhoods, and evaluation at a regular value away from the limiting nodal images gives degree one in each. Order the pieces with the identity sheet first. These two orderings over the central locus extend to a finite étale cover of a neighborhood, by taking all sheet orderings locally and continuing the two selected ones. Its stack degree is two. The deck involution of f exchanges the orderings and fixes each r_i. This removes the double-cover symmetry at the finite map and retains the isotropy of the contracted curves. We work on this cover until the last step of the proof.

We recall the local class computing the Euler change. In a finite stabilization $S : E \to F$ of the normal family, projectivize E over $\mathbb{R}$ and restrict S to its tautological line. The pushforward of the resulting relative Euler class is supported on the kernel locus. Near $G_{I,k}$

we can exclude kernels of dimension two by upper semicontinuity. In the space of matrices the normal space at a matrix of kernel dimension one is $\mathrm{Hom}(K,P)$ for its kernel K and cokernel P. The pushforward gives the Thom generator there, since the projective vertical differential cancels the image of the matrix. On the normal sphere the oriented cokernel is its tangent bundle after choosing a generator of K. The sphere has even dimension, and its Euler integral is two [BS24, Lemma 6.10]. Thus the connecting class of the cokernel Euler class off the kernel locus equals twice the projective class by the Thom isomorphism in the matrix space and pullback. Its pairing with the fixed-target virtual class times the oriented interval is the terminal Euler number minus the initial one. Both this identity and its restriction by excision are preserved by common stabilization.

B.3.2. *Displacement on the finite-map charts.* On one labelled double sheet let $\ell_a(u)$ be the integral of a smooth real covector applied to u, against the pullback by a of a cutoff area form supported in $\operatorname{int} U$. Choose the test so that $\ell_a(\xi) \neq 0$. It gives the same normalization of the kernel through every attachment on that sheet, since pullback area vanishes on contracted components. Let $j_a(e)$ denote a linear combination of triples of smooth N-valued $(0,1)$-forms supported in $\operatorname{int} U$, pulled back separately on the corresponding labelled pieces. For the variable stable map a put $L_{a,t} = a^* D_t$ and consider the finite-rank perturbation

$$T^e_{a,t} u = L_{a,t} u - \ell_a(u) j_a(e). \tag{33}$$

For continuity of the reduction at contracted curves we explain how to use the nodal stabilization of [BS24, Appendix B]. Add a scalar z with $L_{a,t} u = z j_a(e)$ and $z = \ell_a(u)$. Fix a central stable map. Enclose the distinct nodal values meeting the supports of the forcings in small disjoint disks inside U'. On each disk and each labelled piece choose a smooth primitive b_i solving $D_t b_i = j_i(e)$ on the base. These primitives can be chosen smoothly in t and linearly in e: on a small disk the Cauchy right inverse for $\bar\partial$ followed by the zeroth-order term has norm less than one after rescaling. With cutoffs equal to one on smaller disks about the nodal values, pull the primitives back on their respective labelled pieces and extend by zero. Write b_a for their sum. The substitution $u = u' + z b_a$ changes the forcing to $j_a(e) - L_{a,t} b_a$ and the scalar equation to $z = \ell_a(u') + z \ell_a(b_a)$. The new forcing vanishes on neighborhoods of all the nodes. This uses the pullback identity on maps to C, including the constant components.

On shrinking the chart, the remaining forms vary smoothly on compact thick subsets away from the necks in the sense of convergence of stable maps. In a domain trivialization they converge there in every smooth norm; pullback by a involves only its first derivative. One may freeze these forms at the center in the nodal stabilization: the error tends to zero in operator norm and is absorbed by the uniformly bounded stabilized inverse [Par16, Propositions B.7.9 and B.10.4]. At the center a minimal cokernel supplement B gives the invertible augmented operator $(v,w) \mapsto (L_{a,t} v - Bw, \ell_a(v))$ since ℓ_a is nonzero on the kernel line. The area test on shrinking base disks has size at most a constant times their area for bounded sections, by the degree bound. This proves continuity also of the scalar row. On overlaps the substitutions are translations by the difference of the two sums of primitives times z, with the induced scalar row, and hence give the same projective class. Near a central normalized line, its stabilized sections converge on thick regions and are uniformly bounded on the domains. Thus the area test is continuous, and the normalization $\ell_a(u) = 1$ is available near the central projective kernel. On this slice the equation becomes

$$L_{a,t} u = j_a(e), \qquad \ell_a(u) = 1. \tag{34}$$

The first equality is a local graph equation. In the three disjoint fiber neighborhoods over the support of the forms, multiply the corresponding forcing by a smooth cutoff equal to one

near that sheet. The resulting affine term in the horizontal-to-vertical block defines an almost complex structure on N with holomorphic projection and standard fibers [BS24, Lemma 5.9]. On the normalized graphs it gives the indicated equation even when a contracted curve lies over the support. Cutting off outside a larger disk bundle extends the structure over $\mathbb{P}(N \oplus \mathbb{C})$. The fiberwise Fubini–Study form extended by a connection, plus a sufficiently large pullback of an area form on C, tames the family.

Fix k. Use the same sheet data on the finite-map charts for all h_J of genus at most k, $J \neq \varnothing$. By Theorem 3.1 each has an isolating closed neighborhood in its own genus. Independent forms on the labelled sheets span the cokernel at every one of their centers. To see this, present the operator on the normalization with one evaluation-difference row at each join. A dual annihilator of the sheet forms vanishes on a disk of each positive component. The adjoint equation and unique continuation give vanishing off the joins. The equation-space multipliers lie in the dual of L^p for $p > 2$, so none is supported at an isolated point. Sections with independently prescribed endpoint values then kill the remaining evaluation multipliers. Hence one finite-dimensional space of triples suffices for all these centers.

In a minimal stabilization the normalized obstruction section takes values in this space of representatives, and (34) translates the section by the constant parameter e. If more forms are used, the derivative in the forcing parameters is still surjective. On each stratum of fixed nodal and branching type the section is smooth by normalization with the matching equations. Persistence of a node has positive complex codimension by the local smoothing parameters of Theorem 3.1. On smooth sources collision of branch values also has positive complex codimension by the Hurwitz dimension formula. The unrestricted normalized section has expected real dimension zero at fixed e, with the path parameter included. Regular values therefore avoid these strata and give isolated zeros on smooth simply branched coverings. Compactness of the isolating charts makes their number finite.

We may arrange also that their graphs are embedded. On a smooth source the vertical derivative at a simple ramification point takes values in a complex two-plane. Equality of heights at two distinct points over a regular value imposes four real conditions with two real parameters for the value. At a branch value the marks are discrete. The corresponding marked evaluation and first-derivative conditions are transverse under the sheet forcings. Indeed an annihilating distribution vanishes on the open test sheets away from the marks by the independent variations. The adjoint equation there eliminates any multiple of the normalization density by testing where that density is nonzero. Unique continuation leaves a distribution supported at the marks. In a local trivialization one can prescribe independent values and complex first derivatives at the marks with all jets of the linearized equation vanishing up to any specified finite order. This follows by solving recursively for the derivatives involving $\bar{z}$ from the Cauchy–Riemann symbol. Testing on these sections removes the point multipliers. One first obtains immersion, which excludes coincidences in a neighborhood of the diagonal. On the compact remaining marked pairs, finitely many additional forms give the required transversality near the finitely many regular solutions. Arbitrarily small regular choices in the smooth space of triples result, each in a finite-dimensional perturbation. Write σ_Γ for the oriented projective degree of a resulting smooth solution in the chart of h_I. Conservation of the degree of the normalized obstruction section in that chart gives

$$\sum_{\Gamma} \sigma_\Gamma = d_I \tag{35}$$

by the oriented local calculation of Theorem 3.1.

B.3.3. *Removing the contracted components.* Consider a sequence of projective zeros tending to $G_{I,k}$ with $e \to 0$, allowing also $e = 0$. Use the normalized graph equation and take stable limits of the nonconstant normalizations separately, first retaining their nodal flags. Their limiting nonconstant images together are the graph of ξ and the zero section, each with multiplicity one. This follows by gluing the marked limits and forgetting the flags. Each nonconstant normalization retains positive projection degree, since every compact curve in an affine fiber is constant. Thus there are either one or two such normalizations.

In the unmarked stable limit of each normalization there are no contracted components. In fact ghost bubble censorship requires a vanishing derivative at a positive attaching branch, equal complex tangent lines at a pair of attaching branches, or at least three positive attaching branches at one point [ES25, Theorem 1.1]. The two central embeddings exclude all three possibilities. To use the fixed-structure theorem in a smooth finite-dimensional family, extend the parameters locally to an open subset of $\mathbb{C}^m$ and put the block-diagonal almost complex structure on the product. The graphs on constant parameter slices are holomorphic for this single structure, and the attaching directions are unchanged. If a sequence uses successively larger spaces of smooth forcings, choose a subsequence whose first n seminorms differ from the limit by at most 2^{-n^2} at step n. Disjoint smooth bumps of width 2^{-n-3} centered at 2^{-n} interpolate it to a smooth arc flat at zero. The same product argument then applies.

If there are two positive normalizations, their individual unmarked limits are smooth embedded copies of Y and C. An identification on either limit would give a self-intersection of its embedding. The normalizations thus converge smoothly and remain embeddings. Near a point of intersection of the two limiting images, project the difference of their two local parametrizations onto the real four-plane spanned by the two tangent lines. Its derivative in the two source variables is invertible. On smaller neighborhoods the inverse function theorem gives at most one intersection for each nearby pair, and there are none away from these neighborhoods. Consequently every maximal contracted subcurve has one end on a positive normalization or one on each. Delete the one-ended subcurves, replace the two-ended ones by nodes, and retain the direct nodes. Matching and the thick integral show that the finite nodal map so obtained satisfies the same normalized equation. After subselection its joining sites approach a fixed subset J. The flags for different connections have distinct limits on either component. Marked stabilization and properness of the one- and two-pointed contracted factors recover a limit in $G_{J,k}$, even when other flags coalesce with a joining flag. Hence $J = I$ and the finite zeros converge in the chart of h_I. At $e = 0$ they are h_I itself, by Theorem 3.1. The regular displacements just chosen avoid all finite nodal zeros there.

With one positive normalization, its unmarked limit is a connected finite union of the two central embeddings with direct nodes at some nonempty subset J. Its normalized equation is unchanged on deleting the other components, which are constant. For $e = 0$, smooth finite zeros approaching h_J contradict Theorem 3.1. For the chosen regular displacements they are smooth embedded graphs from the chart of h_J. Embeddedness excludes self-nodes and forces each contracted connected subcurve maximal for inclusion to attach once. Mark these attachments before passing to the limit. Forgetting the marks on the positive limit recovers X_J. Restoring them can insert a rational tree over a smooth point or between the two branches of an existing node. Proper limits of the one-pointed tails attach to these trees. In particular flags approaching two distinct smooth points above an unjoined graph intersection stay on distinct fibers of the stabilization. The limit belongs to $G_{J,k}$, so again $J = I$.

The zero-displacement argument proves isolation. Properness in fixed genus and closedness of the kernel locus give a relatively compact neighborhood with kernel-free boundary, which

persists through sufficiently small perturbations. The two cases show that within it the displaced zeros are precisely the stable degree-one expansions of the smooth graphs from the chart of h_I. For the reverse inclusion, every such expansion solves (34) by constant extension. If expansions escaped the neighborhood as $e \to 0$, their marked stable limits would lie in $G_{I,k}$ by the same gluing argument. This also separates the compact families over the individual displaced graphs.

B.3.4. *The virtual class of the normalized graph equation.* Fix one displaced graph Γ, at t_0, and write a_0 for its projection. The homogeneous operator L_{a_0,t_0} is injective. Indeed a_0 has smooth source in the isolating finite-map chart of the original nodal zero. Its pullback to every degree-one expansion is still injective, by restriction to the positive normalization and matching constants on the other components. On this compact family the cokernel has real rank

$$r = 4\big(k - 1 - 3(g(C) - 1)\big),$$

the real virtual dimension of $\overline{\mathcal{M}}_k(C, 3)$. We compare the projective Euler pairing for (33) with the virtual degree of ordinary maps to the almost complex total space subject to the scalar equation in (34).

Take an equivariant chart for maps to C, with stabilizing markings, incidence slices, horizontal perturbation forms, and obstruction section s_C. Use the same markings for maps to the total space, taking inverse images of the target slices. Lift the horizontal forms using a complex-linear splitting of projection for the graph structure. On a horizontally thickened map the vertical equation is still affine in u. Indeed in connection coordinates the off-diagonal term of the almost complex structure is affine in height where a sheet forcing acts, and linear elsewhere near the graph. The horizontal lift can be chosen affine in height there by projecting a connection lift to its complex-linear part. Taking the $(0,1)$-part on the source then gives an affine vertical equation with the usual Dolbeault symbol, restricting to $L_{a,t}u - j_a(e)$ on $s_C = 0$.

Choose vertical forms Bw, $w \in W$, inducing an isomorphism from W to the cokernel of $L_{a,t}$ at the chart center. Forms on the smooth interiors of every component, including contracted ones, are available. They span by density off the nodes in the equation spaces for sections with matching values. Choose equivariant representatives by averaging and extend them over the stabilized universal domain. A contracted component can require a vertical stabilizer even though the forcing $j_a(e)$ vanishes there. In these conventions the vertically stabilized equation on $s_C = 0$ is

$$L_{a,t}u - Bw = j_a(e). \tag{36}$$

The derivative $(L_{a,t}, -B)$ in (u, w) is an isomorphism at the center. In the simultaneous thickening the first row is the regular stabilized horizontal equation, independent of the height. The second has this isomorphism as its vertical derivative. Projection therefore identifies the ambient thickening locally with the horizontal one, with (u, w) uniquely determined on it. This holds over a nodal fiber by the regular perturbed-map chart and inverse function theorem relative to the common base of stabilized domains [Swa21, Theorem 3.5 and Lemma 5.10]. The ambient obstruction section on the common chart is (s_C, w).

On the horizontal zero set introduce $z = \ell_a(v)$ for a homogeneous section v. Eliminating the variables in $L_{a,t}v - Bw' = zj_a(e)$ by (36) gives $v = zu$, $w' = zw$. At a projective zero $w' = 0$ implies $z \neq 0$ by injectivity of $L_{a,t}$. Thus in the chart $z = 1$ the residual equations are exactly

$$w = 0, \qquad \ell_a(u) - 1 = 0. \tag{37}$$

Capping the horizontal virtual class with the localized projective Euler class amounts on this common thickening to multiplying the Thom classes of s_C and of (37). Their product is the class of the constrained ambient equation.

For compatibility, on overlaps take the union of stabilizing marks and the sum of the horizontal and vertical perturbation spaces, transporting earlier forms by the forgetful maps of marked domains. The lifted slices and projection commute with these forgetful maps. An enlarged vertical space adds free variables in (36) paired with identical additional obstruction coordinates. Switching the minimal space W changes cokernel coordinates and adds a multiple of w to the scalar row. A change of horizontal lift adds terms vanishing on $s_C = 0$. The resulting stabilizations and triangular row operations preserve the localized Thom product, since throughout the homotopies the joint zeros stay in the compact interior. The same sums refine triple intersections, equivariantly throughout, as in the ordinary Gromov–Witten implicit atlas [Par16, §9.2, Theorem 9.2.10]. The determinant orientation under projection is the product of the horizontal and vertical orientations, by the triangular linearization and homotopy to its complex symbol. Moving the scalar row past the horizontal virtual dimension or the vertical rank r has even parity. Its orientation convention can thus be fixed at the genus-H crossing and used in every arithmetic genus.

B.3.5. *Restriction to one parameter value.* At Γ let A be the fixed-time normal linearization of the ordinary ambient equation. Its real index is zero since the ambient first Chern number on the graph is zero. The derivative with time and the integral constraint, in unparametrized normal coordinates, is

$$(\zeta, \tau) \longmapsto (A\zeta + b_0\tau, \lambda\zeta). \tag{38}$$

Here b_0 is the parameter derivative and λ the differential of the time-independent integral test. This operator is invertible. Indeed the horizontal equation at a smooth finite cover is regular with variable domain, as its deformation complex reduces to the torsion cokernel of the derivative of the cover. Eliminating that equation and the vertical complement leaves the transverse normalized equation that defined σ_Γ. Hence $\dim_{\mathbb{R}} \ker A \leq 1$.

If A is singular, choose an unramified subdisk of a labelled sheet on which generators of its kernel and adjoint cokernel are nonzero. Such a subdisk exists by unique continuation. If $u_\Gamma(w)$ is its height, add to the affine equation a small term

$$\delta B_1(w)\big(u - u_\Gamma(w)\big)$$

with support in this subdisk and a fiber cutoff equal to one near the sheet. On an unramified graph the vertical tangent projects isomorphically onto the normal quotient. We can therefore prescribe the zeroth-order normal variation so that its pairing between the two generators is nonzero. The one-dimensional Fredholm reduction makes A invertible for sufficiently small $\delta \neq 0$, leaves Γ a solution, and preserves invertibility of (38). Near the graphs this is the homogeneous deformation (33) with L and j replaced by $L + \delta B_1$ and $j + \delta B_1 u_\Gamma$ on the chosen sheet. By compactness and continuous stabilization its size can be chosen to preserve vertical injectivity near the expansions and projective nonvanishing on the boundary of their isolated neighborhood, both in genus k and in genus H. Thus it preserves the localized degrees.

To see that no part of this degree lies away from the retained graph after the perturbation, consider a sequence of other constrained solutions as $\delta \to 0$ in the original isolated neighborhood. Its compact limit is an expansion of Γ. In a tube retracting to that multiplicity-one curve only one nonconstant normalization can occur, since a positive component projects with degree a positive multiple of three. The unmarked normalization then converges smoothly by ghost bubble censorship at the embedded limit. The implicit function theorem for the

augmented operator (38), with δ as parameter, gives the unique nearby constrained curve, namely the retained Γ at t_0. All remaining components attach to it at one point each.

At the regular endpoint the nearby ordinary embedded graphs form a smooth family Γ_t. In a small tube retracting to Γ, the projection of every positive class represented by a holomorphic component has degree a positive multiple of three. A map of total projection degree three has one nonconstant component, of multiplicity one. At fixed genus, a sequence of its unmarked normalizations approaching Γ has smooth limit. Indeed the limiting positive map is the embedding with multiplicity one, and ghost bubble censorship excludes any contracted component of the unmarked limit. Smooth convergence places the normalization in the regular chart of Γ_t. Compare also the embedded-cycle compactness and openness results of [DIW26, Propositions 2.21 and 2.29]. Embeddedness then implies that every other component belongs to a one-ended contracted subcurve. Consequently, after shrinking the tube and parameter interval, the entire fixed-genus ordinary locus near the compact family consists of degree-one stable maps to Γ_t.

On this locus the integral constraint equals the function of time $\gamma(t) = \ell(\Gamma_t) - 1$, since every attaching subcurve has zero pullback area. Invertibility of (38) gives $\gamma'(t_0) \neq 0$ by applying its scalar row to the tangent of the implicit family. Interpolating between $\ell_a(u) - 1$ and $\gamma(t)$ on the common thickening preserves the localized product, as the two functions agree wherever $s_C = w = 0$. The real Thom class of γ cuts the parametrized ambient virtual class down to the ordinary fixed-time class with sign $\operatorname{sgn}\gamma'(t_0)$.

Write D_Γ for the regular normal operator on Γ. Its normal bundle has degree $2H - 2$, since $c_1(TC) + c_1(N) = 0$. Every degree-one pullback of D_Γ is injective by restriction and matching. Zinger's comparison therefore expresses the local degree as the normal Euler pairing on the ordinary degree-one stable-map class of Γ [Zin11, Theorem 1.2]. To record its orientation, deform D_Γ on the fixed bundle. For each degree-one map the pullback and base operators have the same kernel throughout the deformation. On cokernels restriction to the positive normalization is a left inverse to pullback of forms. Hence the quotient of their two stabilized Fredholm complexes is injective with vector-bundle cokernel throughout [BS24, Appendix B]. Its determinant is the quotient determinant. At the regular endpoint its Euler class gives the absolute pairing multiplied by the base determinant sign $\epsilon(D_\Gamma)$. At the Dolbeault operator of a sum of two non-effective line bundles of degree $H-1$ the orientation is complex. Pandharipande's degree-one Hodge integral [Pan99, Theorem 2] gives its generating series $\big(2\sin(v/2)\big)^{2H-2}$. The oriented fixed-time series at the regular endpoint is therefore $\epsilon(D_\Gamma)x^{H-1}$.

Completion of the proof of Theorem B.3. The product of the time-slice sign and the fixed-time determinant sign is σ_Γ, as follows by evaluating their product in genus H at the transverse normalized graph. The scalar orientation and quotient determinant comparison above give the same product in genus k. Thus this graph contributes $\sigma_\Gamma[v^{2k-2}]x^{H-1}$ to the projective pairing upstairs. Summing by (35), multiplying by two for the full Euler change, and dividing by the stack degree two of the sheet labelling gives (32). All isolating neighborhoods and perturbation sizes were chosen at fixed genus. In particular the formula includes tails and bridges of every arithmetic genus allowed by k. □

Appendix C. The two estimates in the smoothing at a cubic branch

We use the parameters $h, (q_i)$, the nodal derivative J, and its remaining real adjoint Λ from §5. The different orders of the matching and adjoint equations determine the degree at a join. Here we prove (39) and (40) for the stabilized equation.

Choose thick stabilizing forms representing P_I away from the moving disks and fix a real normalization of the height, equal to one on s. Solving the stabilized equation and taking its forcing coordinate gives a continuous column $F(h,(q_i))$ with values in P_I. The model (11) permits computation in a fixed vector bundle over C. Values at the three roots identify its sections with sections of the pullback away from the branches. The conjugates $\bar z, \bar z^2$ are replaced by their polynomial interpolants of degree at most two. At the central fiber this imposes the ordinary matching of the two normalizations at the node. For $2 < p_0 < 4$ these matrices are continuous as L^{p_0} multipliers in the parameters, by the divided-difference bound $O(1+|w+p^2/3|^{-1/2})$. Bounded sections on the source extend over the branches. Interpolation identifies the central stabilized kernel by even and odd decomposition on the double sheet and matching to the third sheet. At the smooth branch factors the same argument applies, and the indices agree by the complex-linear finite projection formula. Forms on thick unramified disks span both cokernels by adjoint continuation and the matching sequence. Choose a minimal common supplement. Both stabilized problems have a line of solutions nearby by continuity of the pullback stabilization [BS24, Appendix B]. Evaluation identifies the lines and their forcing coordinates. The details of this comparison for the algebra (11) are in §3.1.

Set $Q_i = q_i \log(1/|q_i|)$, with $Q_i = 0$ when $q_i = 0$. First, in the entire supplement one has

$$F = Jh + 2\iota\big((\overline{Q}_i B_i v_i)_{i\in I}\big) + o(|h| + \max_i |Q_i|). \tag{39}$$

Here is the coefficient of the logarithm. On a single disk put $\ell = \max(|p|, |q|^{1/3})$. Outside $|w| \le A\ell^2$, take $r^2 = w$ and expand the roots in $p/r, q/r^3$. The quadratic coefficient of the interpolant of $\bar z$ has linear q-term

$$-\frac{3}{2}\left(\frac{\bar q}{|w|^2} - \frac{\bar r}{r}\frac{q}{w^2}\right).$$

Only its first summand has nonzero angular mean. For a fixed Hölder test the integrated logarithmic term is $-2\pi\bar Q\delta_0$ in the z^2 component. Indeed the inner integral is $O(\ell^3)$, the outer quadratic error is $O(|p|^2 + |pq|/\ell + |q|^2/\ell^3)$, and if $|q| < |p|^3$ the logarithmic terms are already $o(|p|)$. All other entries have integrable q-derivatives. The full matrix difference has L^1 norm $O(|p| + |Q|)$. Continuous convergence of the normalized stabilized solution in $W^{1,p_0} \subset C^0$ therefore permits using its central value in this pairing to first order. The section $\chi z^2 S(w) a/w$, with $S(0) = 1$ a local real frame of solutions, has ordinary forcing of class $\iota(a)$ and point forcing $\pi a z^2 \delta_0$. Their sum pairs to zero with a central adjoint. Apply this to the generator with derivative v_i. The disks are disjoint, which proves (39).

The missing row requires a smaller remainder. Put $R_0 = \max_i |q_i|$. On each cone $|h| \le CR_0 \log(1/R_0)$ on which all $|q_i|$ are comparable and positive, one has

$$\Lambda F = \sum_{i\in I} H_i(q_i) + o(R_0). \tag{40}$$

We give the estimate on one disk. The half-trace of the forcing $a_0 + a_1 z + a_2 z^2$ is $a_0 + w a_2$. Write $\varepsilon = |q|^{1/3}$, $R = |w|^{1/2}$. On $|w| \le A\varepsilon^2$ the roots are $O(\varepsilon)$ and their diameter is bounded below by $c\varepsilon$: their mean is $O(p) = o(\varepsilon)$ and their product is $-q$. The constant and quadratic coefficients of the interpolant of $\bar z$ are $O(\varepsilon)$ and $O(1/\varepsilon)$. The weighted inner integral is therefore $O(\varepsilon^5)$. The central odd height is $z u_1(w)$ with $u_1(0) = v_i$ and $u_1(w) - v_i = O(|w|)$, by oddness and the pulled-back equation. On the exterior the weighted quadratic error from the three roots is $O(R(|p|/R + |q|/R^3)^2)$. Integration against $O(R^3\, dR)$ gives $O(|p|^2 + |pq| + |q|^2/\varepsilon) = o(|q|)$. The linear terms are bounded by $C(|p| + |q|/|w|)$; restoring their inner integrals costs $o(|q|)$. Smooth center and time changes of the coefficients and collars have Taylor remainder $O(|h|^2)$.

Their products with the varying interpolation matrix have integral $O(|h|(|h| + R_0 \log(1/R_0)))$ by the full matrix bound. Both are $o(R_0)$ on the comparable cone. Other simple branch centers may be fixed by translation on their disjoint source disks. Without translation the angular multiplier on such a disk has first and second derivative $O(|w|^{-1})$ and $O(|w|^{-2})$, giving integrated Taylor error $O(|h|^2(1 + |\log|h||))$.

For the varying generator, the worst unweighted matrix change in L^{p_0} on a joining disk is bounded by

$$C\big(\varepsilon^{4/p_0-1} + |p|\varepsilon^{4/p_0-2}\big).$$

Use the root diameter on the inner disk and the outer bound $CR^{-1}(|p|/R + |q|/R^3)$ elsewhere. Thus the bounded normalized stabilized inverse gives $O(R_0^{\delta})$ convergence of the generator in W^{1,p_0} on the comparable cone, for some $\delta > 0$. The product with the L^1 multiplier variation $O(|h| + R_0 \log(1/R_0))$ is $o(R_0)$ in the adjoint pairing; the principal-part variation on the thick collars satisfies the same bound. This justifies differentiation of the frozen height in the half-trace pairing. At fixed regular target point the two sextic roots move by $-q/(2w)$ to first order. Their averaged variation has pole $-qv_i/(2w)$ at y_i on the quotient cubic. Away from that point the differentiated forcing is the cubic operator applied to this sheet variation, including its collar cutoff. Green's formula against the adjoint gives minus π times the real pairing with the residue, since $\partial_{\bar{w}}(1/w) = \pi\delta_0$. The bounded remainder has zero limiting boundary pairing. The h-derivative pairs to zero by $\Lambda J = 0$. Adding the integrals proves (40).

Appendix D. Comparison with the double-cover count on a closed threefold

We turn to a second compact consequence. The chamber weight for smooth double covers is normalized at a complex-linear operator. In the elementary cluster expansion the reference is the balanced local curve. Computing the two references gives a correction in twice the primitive class. Let v be the genus parameter. Write $s = 2\sin(v/2)$ and $x = s^2$. For a partition λ and a box u of its diagram, let $\ell(u)$ denote its hook length. The connected elementary series of a curve of genus g is

$$G_g(q, v) = \log\left(\sum_{\lambda} q^{|\lambda|} \prod_{u\in\lambda} (2\sin(\ell(u)v/2))^{2g-2}\right).$$

The empty diagram contributes one before taking the logarithm. Ionel–Parker's $e^{\mathrm{IP}}_{B,h}$ is the coefficient of $G_h(Q^B, v)$ in the connected Gromov–Witten series [IP18, Proposition 3.3 and (8.1)]. In particular $[q]G_g = x^{g-1}$. The normal weights in the elementary series are opposite. The class variable Q records integral homology classes.

For a generic compatible almost complex structure, denote the determinant sign of the normal operator D_C of an embedded curve C by ϵ_C. The Bai–Swaminathan count in genus h is

$$\mathrm{Gr}^{\mathrm{BS}}_{2A,h} = \sum_{\substack{[C]=2A\\ g(C)=h}} \epsilon_C + \sum_{[C]=A} \epsilon_C w_{2,h}(D_C).$$

Here $w_{2,h}$ is the oriented intersection number with the wall of smooth genus-h double covers carrying an anti-invariant kernel [BS24, (1.5), Definitions 5.8 and 5.17]. Its value at a double-cover-rigid complex-linear reference is zero. We use the coorientation and path convention of that definition throughout. We extend $w_{2,h}$ by zero when $h < g(C)$.

Theorem D.1. *Let X be a closed symplectic six-manifold with $c_1(TX) = 0$ and torsion-free $H_2(X;\mathbb{Z})$. For every primitive class A of positive area and every $h \geq 0$,*

$$e^{\mathrm{IP}}_{2A,h}(X) - \mathrm{Gr}^{\mathrm{BS}}_{2A,h}(X) = \sum_{g\geq 0} e^{\mathrm{IP}}_{A,g}(X)\, a_{g,h}. \tag{41}$$

The coefficient $a_{g,h}$ vanishes for $g = 0, 1$. For $c = g - 1 \geq 1$ and $j = h - (2g-1)$ it is

$$a_{g,h} = \begin{cases} 2\,4^{c-j}\left(\binom{c}{2j} - (-1)^j\binom{c}{j}\right), & 1 \leq j \leq c, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

A binomial coefficient outside its usual range is zero.

The first two nonzero rows of the correction, written in the variable x with genus h contributing x^{h-1}, are $2x^3$ and $24x^5 - 2x^6$. On the compact threefold of Section D.3, a primitive class A has $e^{\mathrm{IP}}_{A,3} = -14$ and no higher-genus primitive coefficient. Only the term $(-14)(-2)$ contributes in genus seven. Corollary D.3 gives a family on the same threefold for which the absolute differences grow quadratically.

D.1. **The fixed-target comparison.** Let C be a smooth closed connected Riemann surface of genus g and let $N \to C$ be a complex bundle of rank two and degree $2g-2$. Let D be a real Cauchy–Riemann operator on N. Its index on C is zero. When D is invertible, $\epsilon(D)$ is its sign for the determinant orientation obtained by transport from complex-linear operators. Suppose also that f^*D is injective for every smooth connected double cover f of C. We call such an operator rigid through degree two. On a nodal stable map of degree two one can normalize the positive components, each of degree one or two. Kernel sections vanish there, and the constants on each connected contracted subcurve vanish by matching. Hence all stable pullbacks are injective.

Write $M_k = \overline{\mathcal{M}}_k(C, 2)$ for the stack of connected degree-two stable maps with unmarked source of arithmetic genus k. The oriented real cokernels of f^*D give a bundle on M_k of real rank $2r$, where $r = 2k - 2 - 2(2g-2)$ is the complex virtual dimension of M_k. Define

$$L_{2,k}(D) = \int_{[M_k]^{\mathrm{vir}}} e(\operatorname{coker} f^*D), \qquad B^{\mathrm{bal}}_{2,g}(v) = [q^2]G_g(q, v).$$

The superscript bal denotes the balanced antidiagonal series. The integral is zero for empty moduli. We use the orientation of the full pullback determinant for $L_{2,k}$, including its sign in rank zero. In the orientation relative to the base operator, this number is $\epsilon(D)L_{2,k}(D)$ [BS24, Remark 6.4].

Theorem D.2. *For every operator rigid through degree two one has*

$$\epsilon(D)\sum_{k\geq 0} L_{2,k}(D)v^{2k-2} = B^{\mathrm{bal}}_{2,g}(v) + \sum_{h\geq 0}\big(w_{2,h}(D) + a_{g,h}\big)x^{h-1}. \tag{43}$$

The equality is coefficientwise in v. In genus zero the series may have a pole of order two.

D.2. **Comparison with the chamber weight.**

D.2.1. *Transport from a complex-linear operator.* Fix C and N as in Theorem D.2. Use an invertible complex-linear operator β rigid through degree two as a reference. Such operators exist by the complex-operator transversality in [BS24, Lemmas 5.3, 5.6 and 5.15(ii), Definition 5.17]. Base acyclicity is open and is achieved by taking a direct sum of degree-$g-1$ line bundles off the theta divisor, or $\mathcal{O}(-1)^{\oplus 2}$ when $g = 0$. The reference need not split holomorphically.

Choose a generic path from β to D in the affine space of real Cauchy–Riemann operators with fixed symbol. At an ordinary crossing of the base wall the kernel is a real line of nowhere-zero sections. At an ordinary crossing of a double-cover wall its anti-invariant kernel line gives an embedded graph and its intrinsic derivative (17) is invertible. Here one uses the one-parameter transversality and embeddedness of the kernel graphs [BS24, Lemmas 5.6, 5.11, 5.15–5.16]. The variations in these tests are variations of the zeroth-order real tensor on an unramified disk. The fixed-symbol space also has the proper degree-at-most-two wall of [DW25, Definition 2.6, Proposition 2.7 and Theorem 3.40]. In particular, up to any prescribed genus there are only finitely many crossings.

We give the transversality detail that separates a base crossing from an odd crossing. Mark real kernel lines on the base and in the anti-invariant part of a smooth cover with b branch points. For a fixed cover the Fredholm indices of the two operators are 0 and $-2b$. Projectivizing both kernels and allowing the branch values to vary gives index -2. For surjectivity of the universal section, an adjoint pair annihilating all variations of the operators would, on an unramified disk, annihilate all pairs of values $(A\xi_0, A\xi_1)$, where A is a localized real tensor and the two generators are evaluated on one sheet. The odd kernel and its adjoint both change sign on the second sheet, so the contributions of the sheets add. The generators are real independent on a disk. Otherwise on a disk of nonvanishing a real multiple relating two solutions has zero $\bar\partial$, hence is constant. Unique continuation on the connected cover then contradicts their opposite deck characters. Arbitrary variations of A on the disk force both adjoints to vanish there and hence globally. Thus a generic path misses this incidence. Taking the countable intersection over source genera preserves genericity.

Put $E_{2,k}(D) = \epsilon(D)L_{2,k}(D)$ on regular slices. At a base crossing with no odd kernel this reduced Euler number extends across the crossing. To see this, for a smooth positive component of degree two the invariant sections in the pullback kernel descend away from the branch values. They are bounded there in a target trivialization. Removable singularities for the real Cauchy–Riemann equation extend them across the branch values. Since there is no odd kernel, all kernel sections are therefore pulled back from the base. If there are two degree-one components, their base sections agree at a point, either at a common node or along the chain of contracted components joining them. Their difference vanishes everywhere because a nonzero base kernel section is nowhere zero. Constants on each contracted subcurve are determined by their nodal values. The same description on a single positive component is compatible with any further node. Consequently the kernel on every stable map is precisely the base pullback.

Represent the base and pullback families by compatible stabilized two-term complexes of finite-dimensional bundles as in [BS24, Appendix B, (B.7)–(B.10)]. The reduced complex is the quotient in this finite-dimensional presentation. To see that its degree-zero cohomology vanishes, it remains, by the cohomology exact sequence and the kernel computation above, to check that pullback is injective on cokernels. Represent a base cokernel class by a smooth form α. If $f^*Du = f^*\alpha$ on a single effective double normalization, averaging u over the two sheets gives a base solution of $D\bar u = \alpha$ off the branch values. It extends by bounded removability for the inhomogeneous equation. On two degree-one components either restriction supplies such a solution. Thus the base class vanishes. The finite-dimensional reduced differential is injective over the whole compact stable-map space at the crossing. Its cokernels form an oriented bundle across the base crossing. The determinant identity is

$$\det(f^*D) \simeq \det D \otimes \det((f^*D)_{\mathrm{red}}).$$

In particular the reduced integral on either side is $E_{2,k}$.

At a double-cover crossing the base sign is constant. In the convention of [BS24, (5.59)–(5.60)], the forward change of $w_{2,H}$ is $\epsilon(D_0)\eta_f$ for a cover of genus H. Applying Theorem A.1 and summing the jumps within each source genus gives

$$\sum_{k\geq 0}\big(E_{2,k}(D)-E_{2,k}(\beta)\big)v^{2k-2}=\sum_{h\geq 0}w_{2,h}(D)x^{h-1}. \tag{44}$$

At the initial endpoint $w_{2,h}(\beta)=0$ by the constant path in its definition. The equality follows for every formal coefficient using compactness only through that coefficient's genus.

D.2.2. *The Euler integral at the reference.* Let π and f now denote the universal curve and evaluation map over M_k. At β the class $-R\pi_*f^*N$ is represented by its actual complex cokernel. Its rank is $r=\dim^{\mathrm{vir}}_{\mathbb{C}}M_k$ and the base determinant has positive sign. Thus its ordinary top Chern number is $E_{2,k}(\beta)$. To compute it give N scalar torus weight t. If V is the virtual index class of rank r, its equivariant Euler class is $t^r c_{1/t}(V)$, with c_z the total Chern polynomial. Integration on the virtual cycle selects $c_r(V)$. This description also permits deformation of the holomorphic structure to a sum of two lines of degree $g-1$, since the family index class extends along the deformation, including at split operators with kernel.

Equal weights compute the reference number, whereas opposite weights compute the elementary contribution already assigned to the base curve. The scalar and antidiagonal specializations have the same degree-one term x^{g-1}. In degree two taking the connected logarithm subtracts the same half-square from both. The scalar formula of [BP08, §8] and the two antidiagonal hook products in [BP08, Corollary 7.2] therefore give

$$\sum_{k\geq 0}E_{2,k}(\beta)v^{2k-2}-B^{\mathrm{bal}}_{2,g}(v)=x^{2g-2}\big((4-2s)^{g-1}+(4+2s)^{g-1}-2(4-x)^{g-1}\big).$$

The bracket vanishes for $g=1$. For $g=0$ the sum of the first two reciprocals is $2/(4-x)$. For $c=g-1>0$ the odd powers of s cancel. The coefficient of x^{2c+j} is (42) for $h=2g-1+j$, and the constant term in the bracket vanishes. Together with (44) this proves Theorem D.2.

D.2.3. *Passage to the compact invariants.* Choose a generic compatible almost complex structure for which all simple curves of index zero are embedded, regular, super-rigid and pairwise disjoint [BS24, §1]. The image of a connected stable map in class $2A$ lies on a single such support. Indeed, two distinct effective images cannot meet, and contracted components can only join them at a common image point. If the map has degree m on its support of class B, then $mB=2A$. Primitivity and the torsion-free lattice give $m=1$, $B=2A$, or $m=2$, $B=A$. The same argument in class A gives only degree one.

Through any fixed source genus there are finitely many contributing simple supports, by compactness at fixed genus and super-rigidity [BS24, §§3,5]. The degree-one series on a support of genus g is $\epsilon_C x^{g-1}$, by the degree-one calculation in Theorem A.1. Hence $e^{\mathrm{IP}}_{A,g}$ is the sum of signs of genus-g supports of class A. In class $2A$ first subtract their degree-two elementary terms $\epsilon_C B^{\mathrm{bal}}_{2,g}$. By (43), the remaining coefficient of x^{h-1} on each such support is $\epsilon_C(w_{2,h}(D_C)+a_{g,h})$. A support in $2A$ contributes its degree-one determinant sign in its own genus. The latter signs and the terms $\epsilon_C w_{2,h}$ give exactly [BS24, (1.5)]. The remaining terms sum to (41), proving Theorem D.1.

D.3. **A compact threefold with unbounded discrepancy.** Let X_9 be a general nondegenerate anticanonical hypersurface in the smooth projective toric fourfold with rays and maximal cones [Wue26a, (23)–(24)]. We equip it with a Kähler form. It has trivial canonical bundle and torsion-free $H_2(X_9;\mathbb{Z})$ [Wue26a, §4.1]. The divisor corresponding to ray 8 cuts

out a surface $T \simeq \mathbb{F}_1$. Write s for its negative section, f for a ruling fiber and $i : T \hookrightarrow X_9$ for the inclusion. For $d \geq 3$ let $D_d = ds + (d+1)f$ and $A_d = i_* D_d$. Set

$$G = \frac{d(d-1)}{2}, \qquad N = \frac{(d+1)(d+4)}{2}. \tag{45}$$

Corollary D.3. *The class A_d is primitive, and*

$$e^{\mathrm{IP}}_{2A_d,3G-2}(X_9) - \mathrm{Gr}^{\mathrm{BS}}_{2A_d,3G-2}(X_9) = (-1)^{d+1}(d+1)(d+4).$$

In particular, the two counts differ by 28 in class $2i_*(3s+4f)$ and genus seven. Their differences are unbounded on X_9.

D.3.1. *The surface classes.* The semiample divisor H_4 of [Wue26a, §4.2] contracts curves only in T, in a second surface $E \simeq \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$, or in two rational curves C_1, C_2. On E the effective cone is generated by its exceptional curves e_1, e_2 and the strict transform of the line through the two points, which equals $s = E \cap T$. The six integral nef divisors of [Wue26a, Lemma 4.1 and (30)] have degrees

	H_1	H_2	H_3	H_4	H_5	H_6
s	0	1	0	0	0	0
f	0	0	1	0	0	0
e_1	1	0	0	0	1	1
e_2	0	0	0	0	0	1
C_1	0	0	0	0	1	0
C_2	1	0	0	0	0	0

.

In particular $H_3 - H_2$ pairs to one with A_d, and H_2, H_3 distinguish all classes on T. Each effective summand of A_d has zero H_1, H_4, H_5, H_6 degrees. These conditions exclude C_1, C_2 and force any component on E to have class proportional to its negative curve s. Such a component equals s. Thus every summand is represented entirely on T. This is the argument of [Wue26a, Proposition 4.2] for arbitrary d.

D.3.2. *The primitive coefficient.* Adjunction gives $N_{T/X_9} = K_T$. To compute the connected invariants on these classes, we deform $T \subset X_9$ to $Z = \mathrm{Tot}(K_T)$. Blowing up $T \times 0$ in $X_9 \times \mathbb{A}^1$ and removing the strict transform of the central fiber gives a smooth family V with fibers X_9 away from zero and Z over zero. It contains the Cartier divisor $\mathcal{T} = T \times \mathbb{A}^1$ and has a natural morphism to X_9. Use this morphism to label curve classes. In Z every complete irreducible curve lies in T. Its projection must be nonconstant since the fibers are affine lines, and the tautological fiber coordinate on its normalization is a section of a line bundle of negative degree.

For every positive class on this face, stable maps from connected domains to the fibers of V factor through $\mathcal{T}$ in families. Indeed on a geometric fiber the image lies in T by the preceding support calculation. The pullback of $\mathcal{O}_V(\mathcal{T})$ then has negative degree on each nonconstant component and is trivial on each contracted component. Matching at the nodes makes its space of sections zero. For a family of nodal domains the derived direct image of this line bundle is locally a two-term complex of vector bundles in degrees zero and one. Fiberwise injectivity of its differential implies that its degree-zero cohomology sheaf vanishes after every base change. The defining section of $\mathcal{T}$ therefore pulls back to zero, giving the asserted factorization also over nonreduced bases. The relative stable-map stack in the specified class and genus is consequently the product of a proper stable-map stack of T with $\mathbb{A}^1$. Deformation invariance of its relative virtual class identifies the connected invariants of X_9 and Z in every genus.

Write $D = D_d$. On $T = \mathrm{Bl}_p \mathbb{P}^2$ it is the class of a plane curve of degree $d+1$ through p with one imposed simple base point. Thus $p_a(D) = G$ and $h^0(T, \mathcal{O}_T(D)) = N$. We claim that the least power of z in the stable-pair partition function of Z in class D is z^{1-G}, with coefficient $(-1)^{N-1}N$, and that this term survives in the logarithm. The case $d = 3$ of the following moduli calculation appears in [Wue26a, Propositions 5.2–5.3].

For any splitting $D = \sum_j D_j$ into at least two nonzero effective classes on T, write $D_j = a_j s + b_j f$ with $a_j, b_j \geq 0$. From $s^2 = -1$, $sf = 1$ and $f^2 = 0$ we get

$$\sum_j \chi(\mathcal{O}_{D_j}) - \chi(\mathcal{O}_D) = \sum_{j<k} D_j D_k = \sum_j b_j(d - a_j) - \sum_{j<k} a_j a_k > 0.$$

Indeed, let $m = \max a_j$ and $r = d - m$. If $r > 0$, the positive sum is at least $(d+1)r$, and the sum being subtracted is at most $mr + r(r-1)/2$. The difference is at least $r(r+3)/2$. If $r = 0$, the remaining terms are positive multiples of f, giving a difference of at least d.

The image of a stable pair is the structure sheaf of a pure curve B, and its cokernel has finite length. Filter $\mathcal{O}_B$ by powers of the ideal of T. Its jth quotient is $K_T^{-j} \otimes (\mathcal{O}_T/J_j)$ for some ideal J_j, since the conormal bundle is K_T^{-1}. Write $J_j^{\vee\vee} = \mathcal{O}_T(-D_j)$ and $\ell_j = \mathrm{length}(J_j^{\vee\vee}/J_j)$. Then

$$\chi(\mathcal{O}_B) = \sum_j \big(\chi(\mathcal{O}_{D_j}) - jK_T D_j + \ell_j\big), \qquad \sum_j [D_j] = [B].$$

The correction terms are nonnegative because $-K_T$ is ample. If there are several divisorial layers, the strict inequality above gives $\chi(\mathcal{O}_B) > 1 - G$ in class D. If there is only one, it is the zeroth layer by Nakayama's lemma at the generic points. The ideal of T then generates a subsheaf of $\mathcal{O}_B$ of finite support, which vanishes by purity. Hence equality occurs precisely for quotient pairs $\mathcal{O}_Z \to \mathcal{O}_B$ with $B \in |D|$. For a product of pairs in two or more nonzero classes summing to D, their combined filtrations have at least two nonzero divisorial layers. The same strict inequality shows that all product terms in the logarithm occur above z^{1-G}.

The quotient pairs give a closed $\mathbb{P}^{N-1} = |D|$ in the minimum pair space. They exhaust its closed points, and surjectivity of their sections persists in every infinitesimal deformation by Nakayama's lemma. For every $B \in |D|$, including nonreduced B, the normal sequence is

$$0 \longrightarrow \mathcal{O}_B(D) \longrightarrow N_{B/Z} \longrightarrow K_T|_B \longrightarrow 0.$$

Now $H^0(K_T) = 0$ and $H^1(K_T - D) = H^1(\mathcal{O}_T(D))^* = 0$ by Kodaira vanishing, since $D - K_T = (d+2)s + (d+4)f$ is ample. Thus $H^0(K_T|_B) = 0$, while $h^0(\mathcal{O}_B(D)) = N - 1$ by rationality of T. The Hilbert tangent dimension is $N - 1$ at every closed point. It equals the local dimension, so the pair space is regular and the closed immersion is an isomorphism. Its symmetric obstruction theory has virtual degree $(-1)^{N-1}\chi_{\mathrm{top}}(\mathbb{P}^{N-1}) = (-1)^{N-1}N$ [PT10, §1].

Finally, apply the toric Gromov–Witten/stable-pairs correspondence [PP14, Theorem 1] to Z, with empty insertions and $-z = e^{iu}$. The Calabi–Yau condition gives prefactor one. The stable-map and stable-pair spaces in these classes are proper. For pairs, compactify Z to the projective line bundle over T. In a flat family over a valuation ring the support is the closure of the generic support, so a pair supported set-theoretically on the zero section specializes there as well. Thus the open space of pairs supported in Z is proper, and the equivariant integrals give ordinary virtual degrees. The disconnected Gromov–Witten series in this correspondence omits degree-zero connected components, so taking its logarithm gives the connected series. The equality applies to every effective summand on the face.

On the projective fiber, the algebraic and symplectic virtual degrees agree [Sie99]. By primitivity, the coefficient of class A_d in the connected elementary expansion is the finite sum of degree-one terms [IP18, Proposition 3.3 and Theorem 8.1], [DIW26]. It follows that

$$[Q^D]\log Z^Z_{\mathrm{PT}}(z,Q)=\sum_{g\geq 0} e^{\mathrm{IP}}_{A_d,g}(X_9)\left(\frac{(1+z)^2}{z}\right)^{g-1}.$$

Both sides are rational in z. The correspondence identifies their expansions at $-z=e^{iu}$, since $(2\sin(u/2))^2=(1+z)^2/z$, and therefore also their Laurent expansions at zero. Each summand of genus g has lowest power z^{1-g} with leading factor one. The minimum computed above gives

$$e^{\mathrm{IP}}_{A_d,G}(X_9)=(-1)^{N-1}N,\qquad e^{\mathrm{IP}}_{A_d,g}(X_9)=0\quad (g>G). \tag{46}$$

Proof of Corollary D.3. In (41) a row with base genus $g<G$ vanishes in genus $3G-2$ by (42). All rows with $g>G$ vanish by (46). The remaining coefficient in (42) is $a_{G,3G-2}=-2(-1)^{G-1}$. Multiplication by $(-1)^{N-1}N$ gives the formula, since $G+N-2=d(d+2)$. For $d=3$ the minimum pair space is $\mathbb{P}^{13}$ and its virtual degree is -14, giving $(-2)(-14)=28$. For $d=15$ the difference is 304 in genus 313. As d ranges through $12\ell+3$ these positive differences tend to infinity. □

Appendix E. The first genera above a ramified double wall

Let $f:Y\to C$ be an ordinary double crossing with a single odd kernel line of embedded graph, $b=2\ell>0$ simple branch values, and sign η_f. Put $h=H_2+\ell$. Use the generic nonvanishing of the projected antilinear jet at each branch, the classification of smooth central kernels as pullbacks of the fixed graph line, and invertibility of the unramified upper normal twists used below. For clarity about genericity, on a regular disk one varies the real order-zero tensor while annihilating the graph vector. Perturbing a good Petri jet, locally solving for a nearby section, and using a complex gauge sending it to the prescribed section preserves that vector. The adjoint test on irreducible monodromy systems then leaves only its sign-character summand; the moving branch positions are included only at genuinely ramified punctures of each system. The fixed graph, polar and quartic transversality proofs for the complete residual intersection are given in Calculations G1–G3 of [Wue26g]. For any regular point of C put

$$B_m=\sum_{[u:T\to C]}\frac{m}{|\operatorname{Aut}_C(u)|}\epsilon(u^*D_0)$$

over connected unramified covers of degree m. Denote by A the normal operator of the graph in its homogeneous target, $T=\frac{1}{2}\sum_{\lambda\neq 0}\epsilon(A^\lambda)$ over the sign twists of Y, and $N_-=\#\{\lambda\neq 0:\epsilon(A^\lambda)=-1\}$.

Proposition E.1. *For $d\geq 3$ there is no change below $k_d=H_d+\ell+1$. At this genus the connected and elementary jumps, respectively, are*

$$-b\eta_f B_{d-2}+\mathbf{1}_{(d,\ell)=(4,1)}\eta_f T,\qquad -b\eta_f B_{d-2}-\mathbf{1}_{(d,\ell)=(4,1)}\eta_f N_-.$$

The comparison is coefficientwise. For elementary exponents the endpoints admit the balanced lift.

Proof. On the positive normalizations denote the total degree of the nonzero ones over Y by a, their number together with the zero-height positive components by n, and their total ramification over their respective targets Y,C by R. If G sums the contracted genera and β is

the graph Betti number, the genus identity is $k - H_d = n - 1 + a\ell + R/2 + G + \beta$. When $n \geq 2$, equality at the minimum forces $n = 2, a = 1, R = G = \beta = 0$. Thus a single branch of Y joins a pointed étale cover at zero height. The smoothing degree is $-\eta_f \epsilon(u^* D_0)$ per marked orbit, by the one-node logarithmic calculation of §3.1 on that sheet. The deck action is free on the marked fiber. Summing yields the first term. When $n = 1$ equality forces $(d, \ell) = (4, 1)$ and an unramified upper double. Its odd determinant is that of the graph normal twist and its covering involution divides the count by two. This adds $\eta_f T$. Proper-divisor contributions from a genus excess e are supported above excess je on a j-fold multiple. Only the exceptional pair requires subtraction at equality, namely $\eta_f(2^{2h} - 1)/2$, the elementary double of the lower curve. Since $T - (2^{2h} - 1)/2 = -N_-$, the claim follows. □

Modulo two the pointed sum counts index-m subgroups of $\pi_1(C)$. Its parity is odd precisely for m a square or twice a square, by the surface-group parity theorem of Müller and Puchta [MP03, Theorem 1]. Outside the exception the jump in these degrees is an odd multiple of b. For a cubic crossing the central classification of §4 gives no changes in degrees one, two, four or five. At $d = 6 + m,\ 3 \nmid d$, the first possible genus is $H_d + 3\ell + 1$, with a sextic closure joined to a pointed unramified m-cover. A normalized section that reaches the zero section is a negative eigenvector of a peripheral transposition; its stabilizer as a vector in the permutation plane is trivial. Its component must therefore cover the sextic closure. Applying the genus identity with this component gives the stated bound and forces just one join at equality. The adjoint and polar equations yield odd local degree for each pointed cover, with the upper determinant included. More precisely, writing a_i for the signed degree with one identity sheet at the i-th branch, each pointed upper cover u contributes $a_i \epsilon(u^* D_0)\epsilon(D_0)$. Hence the complete first-genus jump is $\epsilon(D_0) B_m \sum_i a_i$ for both connected and elementary counts. Calculations H2–H3 of [Wue26g] derive the discriminant adjoint line and its sign. The complete series at $d = 7$ is Theorem 1.2.

E.1. **The next quartic genus.** Set $k_0 = h + 2g - 1,\ k_1 = k_0 + 1$, and put $S_C = B_2$. At each lower ramification point r, let E_r be the integral of the Euler class of the pole-augmented upper obstruction line along the ramification divisor of the marked double-cover surface. The upper section is allowed the simple pole fixed by the projected derivative at r. Write $F_2(D)$ for the corresponding ordinary upper finite-map Euler integral over the two-branch surface. For $\ell \geq 2$ the complete elementary coefficient at k_1 is

$$\Delta n_{4,k_1} = \eta_f \left[(2S_C + 1)\binom{b}{2} - 2bF_2(D_0) - \sum_r E_r - \mathbf{1}_{\ell=2} N_- \right].$$

Two distinct flags give the binomial term. Each marked surface contributes $-\eta_f(2F_2 + E_r)$. A tower reaches this genus only for four lower branch values. For two lower branch values the two copies of the lower graph vary in their relative real height. Displace on one common path. Denote by D_θ the base operator displaced in this direction and by A_δ the displaced upper graph operator. Define

$$R = \eta_f \left[\epsilon(A_\delta) F_2(A_\delta) - 4F_2(D_\theta) - \sum_r E(D_\theta; r) \right].$$

Then $\Delta n_{4,k_1} = R + \eta_f[2S_C + 1 + (h-1)2^{2h-3}]$. In both cases the full connected coefficient is recovered by adding the lower degree-two factor and the next Taylor coefficient of the preceding degree-four factor. The detailed two-height calculation in the companion proves independence of the sign of displacement. Its interior chart has transverse degree -3. The

common-parameter perturbation excludes smoothings at unequal and zero heights; the equal-height endpoint uses the graph operator and includes all contracted expansions of its regular towers. The same comparison gives an upper ordinary odd Euler jump $-3b\eta_A$ with at least three smooth crossings near each lower branch after a generic perturbation. For $2k-1$ prescribed plus-axis poles the signed degree in genus $2g-1+k$ is $(-1)^{k+1}(2k-1)2^{2g}/4$. The pole residue and its stack normalization are worked out in the accompanying calculations on marked covers and polar crossings.

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