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The topological string on balanced local curves from two circle actions on a Calabi–Yau fivefold

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Abstract: In the membrane proposal of Nekrasov and Okounkov, the variable that counts the Euler characteristic of a stable pair on a threefold is a rotation fugacity of an ambient Calabi–Yau fivefold. We prove the predicted exchange of these roles for two circle actions over a smooth curve of positive genus. One fixes a local Calabi–Yau threefold with two normal lines of equal degree. The other fixes the product of the curve with a plane. With the orientations determined by the ordered normal lines, their stable-pair indices agree in every positive curve degree as integral classes on the common space of cycles, throughout the Picard variety of splittings. On the product threefold a deformation of the matrix equations makes the virtual contribution vanish above the minimum Euler characteristic. At the minimum, the rotation weights of plane Hilbert schemes reproduce the known local-curve partition function. For a doubled genus-two curve the connected index is supported where the two transverse points coincide. A first variation of the rotation weights detects the same nonzero multiple of the Picard theta class in both descriptions, although the corresponding coefficient vanishes on every individual target.

Keywords: Topological Strings, M-Theory, Differential and Algebraic Geometry

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1 Introduction

Fix the total wrapping charge of membranes on a curve. What accounts for the dependence of the topological-string partition function on its coupling at that charge? In the stable-pair calculation on a local Calabi–Yau threefold, this dependence is encoded by a variable q that grades the Euler characteristic of the sheaf: a pair of Euler characteristic n has weight $(-q)^n$. In the fivefold description proposed by Nekrasov and Okounkov [1], the same q is the fugacity for a circle rotating the two complex directions normal to the threefold. Selecting a different circle in the fivefold can change the threefold on which one counts. Can the rotation weights of the new geometry recover the Euler-characteristic series of the old one?

We answer this question for two explicit circles over a smooth curve of positive genus. One fixes a local Calabi–Yau threefold in which the curve has two normal line bundles of equal degree. The other fixes the product of the curve with a plane. On the product threefold, q rotates the plane, and a different variable counts the stable pairs. We show why this second counting variable disappears and how the weights of the plane rotation reproduce the entire q-dependence in each curve degree. The two calculations agree already on the space of curves in the fivefold, as the normal line bundle varies. This is a case of the independence of the external circle formulated by Holmes and Schuler [2, Conjecture 1.7 and Examples 1.13–1.14].

The distinction between the two variables of the topological string will be important. We write Q^m for curve degree m and $q = e^{iu}$ for the exponential of the string coupling u. The wrapping variable Q is common to both descriptions. In the Gopakumar–Vafa formula, the dependence on u at fixed total degree comes from spin traces and repeated windings of wrapped particles [3, 4]. Here we derive that dependence by comparing two geometric indices. The doubled curve in section 4 will let us ask how much this calculation can tell us about a particle of charge two.

1.1 Two threefolds in one fivefold

Fix a smooth projective complex curve C of genus $g \geq 1$, put $b = g - 1$, and let $B = \mathrm{Pic}^b(C)$. Over $C \times B$ choose the Poincaré line $\mathcal{L}$, trivialized along a chosen point of C, and set $\mathcal{M} = K_C \otimes \mathcal{L}^{-1}$. For a line bundle U and a torus character w, write Uw for the bundle with fiber weight w. Our family of fivefolds and its two fixed threefolds are

$$\begin{aligned} Z &= \mathrm{Tot}_{C\times B}(\mathcal{L}t \oplus \mathcal{M}t^{-1} \oplus \mathcal{O}q \oplus \mathcal{O}q^{-1}), \\ X = Z^{\mathbb{C}_q^\times} &= \mathrm{Tot}_{C\times B}(\mathcal{L} \oplus \mathcal{M}), \qquad\qquad X' = Z^{\mathbb{C}_t^\times} = C \times B \times S, \qquad S = \mathbb{A}^2. \end{aligned} \tag{1.1}$$

The dimensions are relative to B. Here a circle is a fiber rotation, written algebraically as its complexification $\mathbb{C}^\times$. On X the internal circle rotates the two line bundles with weights t, t^{-1}. On X' it rotates the plane with weights q, q^{-1}. Both external circles preserve the fivefold volume form. The canonical line of X' is the pullback of K_C; its two normal lines in Z supply the compensating determinant.

The first fixed threefold is the balanced antidiagonal local curve used to construct universal curve-counting classes supported on the multiples of its zero section [5, §2]. As the multiplicity and genus vary, including genus zero, these classes freely generate the part of Pardon's integral curve-counting algebra in first Chern number and virtual dimension zero [6, Theorem 1.1]. Embedding this model in Z supplies a second fixed threefold and a common space of cycles on which to compare the two stable-pair indices.

A stable pair consists of a pure one-dimensional sheaf $\mathcal{F}$ with proper support and a section $\mathcal{O} \to \mathcal{F}$ whose cokernel is zero-dimensional [7]. Its curve degree m is the degree of its fundamental cycle over

C, and $n = \chi(\mathcal{F})$. We use the modified virtual structure sheaf of [1]. The virtual structure sheaf counts with the deformation and obstruction spaces of the pair. Its modification uses a square root of the virtual canonical line and the determinant of cohomology from the two ordered external lines. Exchanging the circles changes those lines as well as the fixed threefold. We specify both roots in sections 2 and 3. Throughout the comparison we work relative to the fixed curve, without integrating over B.

1.2 The common index

Curve degree one illustrates the exchange. On the product threefold a copy of C can sit at any point of the plane. At the minimum Euler characteristic $1-g$, these translates are all the pairs. Their deformations are the two plane translations, and their obstructions are g copies of the same two directions, one for each element of a basis of $H^1(C, \mathcal{O}_C)$. The only fixed translate sits at the origin. Cancelling the deformation factor against one obstruction factor leaves the rotation character

$$Z_1(q) = \big[(1-q)(1-q^{-1})\big]^{g-1}.$$

For a genus-two curve this is $2-q-q^{-1}$. In the original threefold the three powers of q belong to three Euler characteristics. In the plane they all come from rotations at a single fixed pair. The all-degree theorem extends this observation to nonreduced curves and keeps track of their support.

Let W_m be the relative Chow space of compact effective one-cycles of degree m in Z/B. Stable pairs on either fixed threefold determine points of this space by taking their fundamental cycles in Z. Write $s_m : B \to W_m$ for the section $m[C]$. The group $G_0^{q,t}(W_m)$ records equivariant coherent sheaves modulo exact sequences, with their integral multiplicities and torus characters. Let $\mathcal{I}_m^{(q)}$ and $\mathcal{I}_m^{(t)}$ be the series of the proper direct images of the modified sheaves in the two presentations. Our counting convention gives weight $(-q)^n$ in the first. In the second the weight is $(-t)^{n+mb}$, including the shift for the ordered external line of degree b. The theorem in particular makes both sums finite in this group.

Theorem 1.1. *For the family (1.1), with the ordered determinant orientations above, the two stable-pair indices have the same class in $G_0^{q,t}(W_m)$ for each $m \geq 1$. More precisely,*

$$\mathcal{I}_m^{(q)} = \mathcal{I}_m^{(t)} = (s_m)_*\big(Z_m(q)[\mathcal{O}_B]\big), \qquad Z_m(q) = \sum_{\lambda \vdash m} \prod_{\square \in \lambda} \big[(1-q^{h(\square)})(1-q^{-h(\square)})\big]^{g-1}. \tag{1.2}$$

Here $h(\square)$ is one plus the number of boxes to the right and below $\square$ in its row and column. The equality holds over the full Picard variety, including the locus where the number of sections of the normal bundle jumps.

On a fixed target the character $Z_m(q)$ is known. Bryan and Pandharipande obtained the local Gromov–Witten partition function [8], and Monavari's stable-pair formulas give its q-character [9, 10]. For the related comparison with fivefold Gromov–Witten theory, Giacchetto, Pandharipande and Schuler derive a degree-one local-curve correspondence from their membrane conjectures [11]. The content of Theorem 1.1 is that two moduli problems give the same integral class supported on $m[C]$. Equality of the Laurent polynomials on individual targets would leave this undetermined: a nonzero class on the Picard variety can have rank zero at every point.

The two sides account for the equality in different ways. On X the infinitesimal rotation of the normal lines gives a cosection of the obstruction sheaf. It concentrates the virtual sheaf on invariant pairs. Their fixed deformation theory, with its determinant orientation, is constant over B, even where the dimension of the space of normal sections jumps (section 2).

On X' the two plane coordinates act as endomorphisms of a rank-m bundle E on C. Its degree $k = n + m(g-1)$ is precisely the exponent of the exchanged counting variable. Write x, y for the endomorphisms. The framed matrix equation admits the deformation

$$[x, y] + ij = a\, 1_E,$$

where i is the section of the pair, j a covector, and a a scalar. At $a \neq 0$, taking traces on invariant quotients and subspaces forces the words in x, y applied to i and to j to generate E and $E^\vee$, respectively. Both bundles are therefore generated by global sections. Their degrees must both be nonnegative, so $k = 0$. For $k > 0$, specialization back to the original equation makes the virtual structure sheaf vanish on the stable-pair space itself. At $k = 0$ the matrix data are constant along C and specify a length-m subscheme of the plane. The tangent weights of its Hilbert scheme supply the hook lengths in (1.2) (section 3). This accounts for the loss of the t-grading and the survival of the q-dependence in the same calculation.

1.3 What the doubled curve detects

In genus two, degree two is the first place where two independent degree-one objects contribute at the same total charge as a connected double. Over two distinct points of the plane the index sheaf factors. This prescribes the graded symmetric square to subtract for the independent pair. The remaining class is supported where the two points coincide. There the rotation has two fixed length-two schemes, one for each coordinate direction in which a point can thicken. Each contributes $+1$ at q^{-3} and -2 at q^{-2}. The graded symmetric square contributes zero at both powers, giving the connected coefficients 2 and -4 (section 4). To interpret the latter as the spin trace of a charge-two particle, one would also have to establish a normalizable state in the wrapped theory with the continuum of two separated branes open and derive its spin.

There is a further test for which even exact numerical agreement on every target would give no information. Vary the product κ of the two normal weights on X, compensating on the plane so the fivefold volume remains invariant. In curve degree two we compare the term linear in $1 - \kappa$ at the lowest original grading q^{-3}. On X it comes from two ribbons, one in each normal line. On X' it comes from divisors on the curve. At the exchanged grading t^2 the two kinds of divisors give 4θ and 8θ in cohomology, where θ is the principal polarization of $\mathrm{Pic}^1(C)$. The ribbons give the same 12θ. The integral class giving this coefficient has rank zero on every individual target. Section 5 establishes the integral identity at q^{-3} for every positive power of t in this first variation.

2 The index on the split threefold

Take q as the external circle. Its fixed threefold over $L \in B = \mathrm{Pic}^b(C)$ is the total space of $L \oplus K_C L^{-1}$, where $b = g - 1$. The internal circle $T = \mathbb{C}_t^\times$ rotates these two lines with opposite weights. We shall compute the class of the index on the space of cycles. This requires keeping the dependence on L. For example, a section of L translates the curve inside its threefold, and the number of independent translations can jump as L varies.

Write $\mathcal{L}$ for the normalized Poincaré line on $C \times B$, $\mathcal{M} = K_C \mathcal{L}^{-1}$, and $X = \mathrm{Tot}_{C\times B}(\mathcal{L}t \oplus \mathcal{M}t^{-1})$. Let $\mathcal{P} = P_n(X/B, m)$ parametrize stable pairs $\mathbb{I} = [\mathcal{O}_X \to \mathcal{F}]$ in the fibers of $X \to B$. The sheaf has proper pure one-dimensional support, generic degree $m > 0$ over C, Euler characteristic n, and a section with zero-dimensional cokernel [7]. Let $h : \mathcal{P} \to W_m$ send a pair to its support cycle in the fivefold, and let $s_m : B \to W_m$ denote the cycle $m[C]$. The support morphism is proper. Indeed,

compactifying X to the projective bundle with infinity divisor D_∞ gives the projective stable-pair support morphism. Its restriction over cycles disjoint from D_∞ parametrizes pairs on X, since purity forces the whole sheaf support to lie on its cycle. The inclusion of cycles on the fixed threefold into the fivefold cycle space is closed.

2.1 The orientation and the support of the index

Push the universal sheaf down to $C \times \mathcal{P}$ and call the result E. The projection of the support is proper and affine, hence finite. Purity makes every fiber of E over $\mathcal{P}$ torsion-free on C. Flatness over $\mathcal{P}$ then makes E a vector bundle of rank m. For the projection $\rho : C \times \mathcal{P} \to \mathcal{P}$, set

$$P = R\rho_* E - R\rho_*\big(\mathrm{End}(E)(1 - \mathcal{L}t)\big). \tag{2.1}$$

This is a polarization of the relative virtual tangent class:

$$T^{\mathrm{vir}}_{\mathcal{P}/B} = P - P^\vee, \qquad \mathrm{rk}\, P = n + m^2 b. \tag{2.2}$$

Indeed the two normal coordinates give a Koszul resolution for the self-Ext term, whose class before pushforward is $\mathrm{End}(E)(1 - \mathcal{L}t)(1 - \mathcal{M}t^{-1})$. Duality on C pairs its two halves and pairs the two terms linear in E in the framed deformation complex. Riemann–Roch gives the rank in (2.2). We use $(\det P)^{-1}$ as the square root of the relative virtual canonical bundle. The two ordered external lines are trivial as bundles in this presentation, so their determinant ratio introduces no further line. Write $\widehat{\mathcal{O}}^{\mathrm{vir}} = (\det P)^{-1}\mathcal{O}^{\mathrm{vir}}$ with the signed counting variable $-q$ [1].

Let $F = \mathcal{P}^T$ and $i : F \to \mathcal{P}$ denote its inclusion. A fixed pair has its entire reduced support on the zero section. Indeed, every fiber of its support over C is finite, so the connected torus fixes each point of that fiber. The origin is the only fixed point in each affine-plane fiber. In particular F is proper over B, by properness of the support morphism and invariance of properness under nilpotent thickenings. Fixed supports can themselves have nontrivial scheme structure.

To obtain an *integral* lift of the virtual sheaf to F, consider the vector field induced by the normal rotation. Relative Calabi–Yau duality identifies $\mathrm{Ob}_{\mathcal{P}/B}$ with $\Omega_{\mathcal{P}/B}$, and contraction with the vector field gives a cosection. We check that it descends to the absolute obstruction sheaf. Write ξ for the Euler field on X and $D = \iota_\xi \mathrm{At}_{X/B}(\mathbb{I})$ for the corresponding degree-one endomorphism. A Picard tangent represented by $\alpha \in H^1(C, \mathcal{O}_C)$ varies the transition functions of $\mathcal{L}$ and $\mathcal{M}$ inversely. Its Kodaira–Spencer class is therefore $\alpha\xi$, and its obstruction is the trace-free part of αD, up to the common infinitesimal-action sign [12]. Its pairing with D vanishes. In the Serre trace, commuting D past the even block αD preserves the sign, whereas commuting it past the odd scalar α changes the sign. This calculation also applies over nonreduced parameter schemes. For the proper Serre trace one can use the projective bundle compactification. The trace-free endomorphism complex vanishes near infinity, where $\mathbb{I}$ is the identity line. The relative canonical bundle on the compactification is $\mathcal{O}(-3D_\infty)$, and the section of $\mathcal{O}(3D_\infty)$ identifies the twisted trace-free complex with the untwisted one in the derived category. Multiplication by this section has degree zero and leaves the cyclic sign unchanged.

Thus the cosection kills the image of T_B in the relative obstruction sheaf and descends through the absolute obstruction sequence. Its zero scheme is F: on a homogeneous function of weight j the orbit derivation is multiplication by j, and the nonzero weights generate the fixed-scheme ideal in characteristic zero. Equivariant cosection localization now gives

$$\gamma \in G_0^T(F), \qquad i_*\gamma = [\mathcal{O}^{\mathrm{vir}}_{\mathcal{P}}]. \tag{2.3}$$

Here G_0^T denotes the group of equivariant coherent sheaves with integer coefficients. We use the absolute virtual sheaf induced by the relative theory over the smooth base and the relative orientation fixed above [13, 14].

2.2 The fixed class over the Picard variety

On F, decompose $P = P_0 + P_{\mathrm{mov}}$ into its zero and nonzero weights. The moving virtual normal class is $P_{\mathrm{mov}} - P_{\mathrm{mov}}^\vee$. To compute γ without losing torsion in the Picard family, choose a two-term bundle resolution $[N^{-1} \to N^0]$ of the moving cotangent complex on F. One can choose it weight by weight since F is quasi-projective and T acts trivially on it. For $e(V) = \lambda_{-1}(V)$, the self-intersection and composition identities in virtual pullback give [15, proof of Theorem 3.3 and Lemma 3.6]

$$e(N^0)\gamma = e(N^{-1})[\mathcal{O}_F^{\mathrm{vir}}]. \tag{2.4}$$

Indeed virtual self-intersection applied to $i_*\gamma$ gives the left side. Applied to the virtual sheaf on $\mathcal{P}$, the same pullback adjoins $N^{-1}[1]$ to the fixed obstruction theory and gives the right side. Both operations precede division by torus characters: the factors in (2.4) are the Koszul classes for zero sections.

On each component a moving bundle has Euler polynomial in t with invertible extremal coefficients, up to sign and a line bundle. Hence multiplication by it on $G_0(F)[t^{\pm 1}]$ is injective even in the presence of additive torsion. The identity $1 - a = -a(1 - a^{-1})$ applied to the lines of a splitting of the moving polarization gives

$$\frac{e(N^{-1})}{e(N^0)} = (-1)^{\mathrm{rk}\, P_{\mathrm{mov}}} \det P_{\mathrm{mov}}.$$

Clearing the Euler factors and using injectivity in (2.4) identifies the oriented lift integrally as

$$(\det P)^{-1}\gamma = (-1)^{\mathrm{rk}\, P_{\mathrm{mov}}} (\det P_0)^{-1} [\mathcal{O}_F^{\mathrm{vir}}]. \tag{2.5}$$

In particular its internal rotation weight is zero.

It remains to determine how this class varies with L. Fix $L_0 \in B$ and put $A = \mathcal{L} \otimes L_0^{-1}$. An equivariant stable pair on the fiber at L_0 gives a pair on the varying threefold by tensoring each weight-j summand E_j with A^j. Both normal actions have precisely the required changes of weight and the section has weight zero. The inverse construction uses A^{-1}. The degree of A on each curve is zero, so this preserves n. Stable pairs have trivial automorphisms and a fixed family has a unique compatible linearization. The construction commutes with arbitrary base change and gives $F \simeq F(L_0) \times B$.

This identification includes the fixed obstruction theory. In local frames of A, the weight-zero relative derived endomorphisms, the trace and the maps induced by the section glue with total power A^0. The same identification respects their obstruction morphism: the relative Atiyah class in the pair-parameter directions is taken with A held fixed on $C \times B$. Thus the relative fixed virtual sheaf is pulled back from $F(L_0)$. The terms of P_0 also have total power A^0, and the moving rank in (2.5) is constant on each product component. Proper pushforward of the oriented fixed class therefore gives a numerical coefficient $c_{m,n}$ times $[\mathcal{O}_B]$. On F_{red} the support map factors through s_m. Filtering coherent sheaves by the nilradical permits the same pushforward calculation even if the map on F has infinitesimal components normal to $s_m(B)$. Consequently

$$h_*[\widehat{\mathcal{O}}_{\mathcal{P}}^{\mathrm{vir}}] = (s_m)_*\big(c_{m,n}[\mathcal{O}_B]\big) \quad \text{in } G_0^T(W_m). \tag{2.6}$$

It suffices now to evaluate $c_{m,n}$ on one fiber. The antidiagonal local-curve stable-pair formula at levels (b, b) gives [9, 10]

$$\sum_n c_{m,n}(-q)^n = \sum_{\lambda \vdash m} \prod_{u \in \lambda} \big((1 - q^{h(u)})(1 - q^{-h(u)})\big)^b, \tag{2.7}$$

where $h(u)$ is one plus the number of boxes to the right and below u in its row and column. The sign follows from $(1-q^h)(1-q^{-h}) = -(q^{h/2}-q^{-h/2})^2$. In degree one this gives $((1-q)(1-q^{-1}))^b$. For $g \geq 1$ every summand is an integral Laurent polynomial. Equation (2.6) identifies the family class to which this numerical formula belongs.

3 The rotated threefold

Choose the circle with character t as the external circle. Its fixed threefold is $X' = C \times S$, where $S = \mathbb{A}^2$ carries the two weights q, q^{-1}. A curve can now move off the origin of S by a constant translation. We compute on this noncompact moduli space before pushing its index to cycles.

3.1 Deforming the framed matrix equation

Let $\mathcal{O}_{X'} \to F$ be a stable pair of curve degree $m > 0$ and Euler characteristic n. Its proper support projects finitely to C. Indeed, the projection is affine and its restriction to the support is proper. Purity then makes $E = p_*F$ a vector bundle of rank m, where $p : X' \to C$. Riemann–Roch gives

$$k = \deg E = n + m(g-1). \tag{3.1}$$

The two coordinates of S act on E, and the section of the pair gives a distinguished vector $i : \mathcal{O}_C \to E$. Over the function field of C, the vector generates E under the coordinate actions.

The framed ADHM description of the stable-pair theory adds a covector $j : E \to \mathcal{O}_C$ and writes the relation as a matrix equation [16, Theorem 1.9 and §7]. Write x, y for the two endomorphisms. We deform the equation, with its scalar part included, to

$$[x, y] + ij = a\, 1_E, \qquad a \in \mathbb{A}^1. \tag{3.2}$$

Framing fixes the copy of $\mathcal{O}_C$, and we divide by changes of basis of E. The stable open is specified by generic generation by the words in x, y applied to i. At $a = 0$ this is Diaconescu's presentation of the stable-pair moduli. Under the opposite weights on x, y, the parameter a is invariant.

At $a \neq 0$, fix any point of C. The span of the words applied to i is invariant under x, y. On its quotient the commutator would be a times the identity. Taking the trace forces this quotient to vanish. Dually, the common kernel of the covectors $j\, w(x, y)$, for all words w, is invariant and contained in $\ker j$. The same trace argument on this subspace makes it zero. Words of length at most $m-1$ suffice in each argument, since an increasing span which stops growing is already invariant. These words are global sections on C. Consequently E and $E^\vee$ are both globally generated and $\deg E = 0$. At $a = 0$ there are still m words generically independent on i. Their determinant is a nonzero section of $\det E$, so $k \geq 0$.

For $k > 0$ this calculation implies the integral identity

$$[\mathcal{O}^{\mathrm{vir}}_{P_n(X',m)}] = 0 \qquad \text{in } G_0^{\mathbb{C}_q^\times}(P_n(X', m)). \tag{3.3}$$

We give the specialization argument, since the equivariant weight of a matters here. Use the derived zero locus of (3.2) modulo changes of frame and take its stable space of sections over the fixed curve. Relative to the a-line its tangent complex is the derived push from C of the complex in degrees $-1, 0, 1$

$$\operatorname{End} E \longrightarrow (\operatorname{End} E)q \oplus (\operatorname{End} E)q^{-1} \oplus E \oplus E^\vee \longrightarrow \operatorname{End} E. \tag{3.4}$$

The first arrow is the infinitesimal change of basis and the second is the differential of the moment map. Generic cyclicity makes the first injective over the function field. The trace symplectic pairing on

the framed matrix space identifies its transpose, up to sign, with the second arrow, which is therefore generically surjective. On C the first kernel sheaf vanishes and the last cokernel has finite support. Pushing to the parameter space gives tangent amplitude $[0,1]$. This is the fixed-domain quasimap obstruction theory [17, §4.3]. Stabilizers are trivial by generic cyclicity. Boundedness at fixed k follows by dualizing m generically independent word sections. The resulting subbundle away from finitely many points, $E^\vee \subset \mathcal{O}_C^m$, has quotient of length k, hence belongs to a finite-type Quot scheme.

For clarity, the comparison on the central fiber can be made at the level of virtual structure sheaves. If π forgets the C-factor of the universal family, the stable-pair virtual tangent has class

$$R\pi_*\big(E + E^\vee - (\operatorname{End} E)(1-q)(1-q^{-1})\big)\,. \tag{3.5}$$

The Koszul resolution for the two plane coordinates gives the quadratic term in E in the trace-free pair complex. The two linear terms come from the section and its dual, with the product of the two plane weights equal to one. This agrees with the alternating class of (3.4) on the same moduli scheme. The K-theoretic Fulton class formula expresses the virtual structure sheaf of a quasi-projective scheme in terms of its intrinsic Fulton class and its virtual cotangent class [18, Theorem 4.2 and Corollary 4.5]. Its cone filtration and exterior powers preserve the q-linearizations, with a separate auxiliary grading for the filtration. Thus the comparison holds in integral equivariant coherent K-theory. See also the local-curve application in [9, §9].

The virtual structure sheaf on the central fiber is the refined pullback of the relative virtual family along $a = 0$, by derived base change or virtual specialization [15]. For $k > 0$ the underlying space of the entire family lies over zero. Filter a coherent sheaf there by powers of a. Every successive quotient is a sheaf on the central fiber, and its refined pullback has two equal Tor terms in adjacent degrees, since a has weight zero. The pullback thus kills every class of the total family. This proves (3.3), even when the parameter acts nilpotently on its ordinary structure sheaf.

3.2 The plane Hilbert scheme and its orientation

At $k = 0$ the determinant of the m independent word sections has degree zero and no zero. These sections trivialize E on C. The coordinate operators and the cyclic vector of the stable pair are then constant, and give a cyclic length-m quotient of the coordinate ring of S. This identification works in families. Locally on the parameter scheme choose words that form a basis on one fiber over a point of C. At the chosen parameter they form a basis on the entire curve. Properness of C allows one to shrink the parameter scheme until they form a basis everywhere. Since C is projective and connected, the matrix entries then descend to the parameter scheme, also over nonreduced bases. It follows that

$$P_{-mb}(X', m) \simeq H_m = \operatorname{Hilb}^m(S), \qquad \xi \longmapsto [\mathcal{O}_{X'} \longrightarrow \mathcal{O}_{C\times\xi}]. \tag{3.6}$$

For these constant cyclic data the framed complex on C is pulled back from the tangent bundle TH_m. The virtual tangent is $R\Gamma(C, \mathcal{O}_C) \otimes TH_m$. Its degree-zero cohomology is the ordinary tangent bundle of the smooth Hilbert scheme. Its obstruction bundle is $H^1(C, \mathcal{O}_C) \otimes TH_m$. The symplectic form on H_m is invariant under the antidiagonal circle and trivializes the tangent determinant. We use the resulting relative orientation, suppressing fixed one-dimensional vector spaces from C, which have trivial circle character.

There is also a determinant contribution from the two external lines [1]. Let V be the rank-m tautological bundle on H_m, let $\pi_B : C \times B \to B$, and put $A = R\pi_{B*}\mathcal{L}$. Relative duality gives $R\pi_{B*}\mathcal{M} = A^\vee[-1]$. The unweighted ratio of the two determinants on $H_m \times B$ is

$$D = \frac{\det(V \otimes A)}{\det(V \otimes A^\vee[-1])}. \tag{3.7}$$

Here both complexes on B have virtual rank zero and the same determinant. Thus D is trivial and we take its trivial square root. This argument uses perfect complexes throughout, including where the individual spaces of sections of $\mathcal{L}$ jump. The external prefactor is $(-t)^{n+mb}$, hence is one on H_m. The modified class on the minimum component is therefore the pullback from H_m of

$$\mathcal{F}_m = \lambda_{-1}(T^\vee H_m)^g, \qquad \lambda_{-1}(U) = \sum_i (-1)^i [\wedge^i U]. \tag{3.8}$$

The effect of balance can be tested on an actual family of coincident points. With $B = \mathrm{Pic}^d C$ set $\delta = d - (g-1)$. The two ranks in (3.7) become $\delta, -\delta$, whereas their determinants still agree. Thus $D = (\det V)^{2\delta}$ with oriented root $(\det V)^\delta$. The punctual Hilbert square is a $\mathbb{P}^1$ of one-dimensional quotients of the cotangent plane at the origin. On this line the length-two algebra has the constants and the universal quotient $\mathcal{O}(1)$ as its two graded pieces. The degree of the root on this collision family is δ. The choice $d = g-1$ is therefore detected by the disappearance of the twist on the Hilbert scheme itself.

3.3 Comparison on the space of cycles

The q-fixed ideals of H_m are the monomial ideals ξ_λ for partitions λ of m. To see that the one-parameter subgroup has no further fixed components, first observe that every fixed zero-cycle of S is concentrated at the origin. The punctual Hilbert scheme is projective, so every connected component of the fixed locus contains a point fixed by the full coordinate torus. Those points are monomial ideals. At ξ_λ the tangent weights restrict to

$$\{q^{h(s)}, q^{-h(s)} : s \in \lambda\}, \tag{3.9}$$

by the arm and leg formula [19, Lemma 5.4.5]. They are all nontrivial, so each such point is isolated and reduced in the fixed locus.

The infinitesimal q-rotation is an equivariant section of TH_m. Its linearization at a fixed ideal is invertible by (3.9). Its zeros are therefore simple and its Koszul complex resolves their structure sheaf. Since $g \geq 1$, one exterior Euler factor in (3.8) resolves the fixed points. The other $g-1$ factors restrict to them. This proves the identity

$$\mathcal{F}_m = \sum_{|\lambda|=m} [\mathcal{O}_{\xi_\lambda}] \prod_{s\in\lambda} \big((1-q^{h(s)})(1-q^{-h(s)})\big)^{g-1} \quad \text{in } G_0^{\mathbb{C}_q^\times}(H_m). \tag{3.10}$$

In particular no character denominator is needed for the transverse plane, even for an elliptic curve, when each point contributes one.

Let $s_m : B \to W_m$ be the section of the common fivefold cycle space which assigns the zero-section cycle $m[C]$. Hilbert–Chow sends each ξ_λ to the origin of $\mathrm{Sym}^m S$ with multiplicity m, so its associated curve has cycle $m[C]$. These pushforwards are proper on the displayed representatives. Equivalently one can embed both threefold cycle spaces in W_m. Compactification by compatible projective bundles makes containment in each closed subbundle a closed condition on cycles, and one then restricts to the open locus disjoint from infinity.

Writing $Z_m(q)$ for (1.2), the rotated pushforward is $(s_m)_*(Z_m(q)\mathcal{O}_B)$ in $G_0^{q,t}(W_m)$. In the original presentation section 2 gives the same class after summing the Euler-characteristic grading with its external weight. This proves the circle exchange of Theorem 1.1. One grading has become an ordinary rotation of the plane, and its hook weights now come from the tangent spaces to punctual ideals.

4 The connected index at a collision

Set $g = 2$. The plane Hilbert square parametrizes both pairs of distinct points and length-two schemes supported at a single point. To extract the connected contribution we compare its index with the symmetric square of the degree-one index, on the space of cycles itself. All constructions below can be pulled back over B; we suppress its trivial structure-sheaf factor.

Let $h : H = \mathrm{Hilb}^2(S) \to W = \mathrm{Sym}^2(S)$ be the Hilbert–Chow morphism, $p : S^2 \to W$ the finite quotient and $i : S \to W$ the map $x \mapsto 2[x]$. Put

$$K_1 = \lambda_{-1}(T_S^\vee)^2, \qquad K_2 = \lambda_{-1}(T_H^\vee)^2. \tag{4.1}$$

Here $\lambda_{-1}(U) = \sum_j (-1)^j [\bigwedge^j U]$. Its even and odd summands give the signs in a permutation of two such factors. The connected class is

$$D_+ = h_* K_2 - \big(p_*(K_1 \boxtimes K_1)\big)^{\mathfrak{S}_2} \quad \in G_0^q(W). \tag{4.2}$$

The invariant summand is exact over $\mathbb{C}$. Over two distinct points, h is an isomorphism, and on the ordered cover the tangent bundle is the direct sum of the tangent bundles at the two points. Exchanging them acts on the exterior algebra with the graded permutation sign. Thus the two terms of (4.2) restrict to the same class there. The localization sequence for coherent sheaves makes D_+ the pushforward of an integral class on $i(S)$.

Write $y = q + q^{-1}$, $e = (1-q)(1-q^{-1}) = 2 - y$, and $e_2 = (1-q^2)(1-q^{-2}) = 4 - y^2$. The degree-one index is e. At the two ideals (z_1^2, z_2) and (z_1, z_2^2), in plane coordinates z_1, z_2, the four tangent weights on H are q, q^{-1}, q^2, q^{-2}. Equivariant localization divides out one cotangent Euler factor and leaves ee_2 at each ideal. Their sum is $2e^2(4 - e)$, the disconnected genus-two double already obtained from the antidiagonal Gromov–Witten evaluation [5, §6]. For the independent pair the identity permutation has trace e^2; the transposition has trace $\psi^2(e) = e_2$, where the Adams operation ψ^2 replaces q by q^2. Taking invariants averages these traces. The connected index is therefore

$$\begin{aligned} F_2(q) = \chi_q(D_+) &= 2ee_2 - \tfrac{1}{2}(e^2 + e_2) = 2(y-2)(y^2-3) \\ &= 2q^{-3} - 4q^{-2} + 4 - 4q^2 + 2q^3. \end{aligned} \tag{4.3}$$

The lift to the collision diagonal is determined as well. Equivariant homotopy invariance identifies $G_0^q(S)$ with the Laurent character ring times $[\mathcal{O}_S]$. The localized Euler characteristic of $A(q)[\mathcal{O}_S]$ is $A(q)/e$. Since this map is injective, the class constructed in (4.2) is

$$D_+ = i_*\big(eF_2(q)[\mathcal{O}_S]\big). \tag{4.4}$$

Each punctual ideal contributes $+1$ at q^{-3} and -2 at q^{-2}. The graded symmetric square is $(e^2+e_2)/2 = 4 - 2y$. At q^{-2} the identity and the transposition have coefficients $+1$ and -1, respectively, so their average vanishes. Thus the two punctual ideals account for both leading coefficients of the connected index. The subtraction changes the constant and the $q^{\pm 1}$ terms.

For completeness one can make the connected subtraction in all degrees at the level of characters. Introduce a curve variable Q and put $Z = 1 + \sum_{m \geq 1} Q^m Z_m(q)$. Define integer Laurent polynomials $F_m = \sum_j a_{m,j} q^j$ successively by

$$Z = \prod_{m \geq 1} \prod_{j \in \mathbb{Z}} (1 - Q^m q^j)^{-a_{m,j}}. \tag{4.5}$$

After dividing by the previously determined factors, the coefficient of Q^m fixes F_m integrally. Each Z_m is invariant under inversion of q, so uniqueness gives $F_m \in \mathbb{Z}[2-q-q^{-1}]$. For $g>1$ each Z_m is divisible by e^{g-1}, since every hook Euler factor is divisible by e. The same divisibility holds for F_m by induction. Indeed the nonconstant contributions from lower degrees in the symmetric-algebra expansion contain an Adams image of a previous F_d, and $2-q^r-q^{-r}$ is divisible by e. This proves divisibility over $\mathbb{Q}$; division by the monic polynomial e^{g-1} in $\mathbb{Z}[e]$ proves it integrally. For $g=1$ the hook sum gives the ordinary partition numbers and the Euler product has $F_m=1$.

Writing $F_m=\sum_{h\ge1} n_{m,h}e^{h-1}$ and putting $q=\exp(iu)$ gives the local Gopakumar–Vafa form [3, 8],

$$\log Z=\sum_{m,h,r\ge1}\frac{n_{m,h}}{r}\bigl(2\sin(ru/2)\bigr)^{2h-2}Q^{mr}. \tag{4.6}$$

In the example (4.3), $F_2=-2e+8e^2-2e^3$. The construction (4.2) realizes this symmetric-algebra subtraction by an actual support calculation in coherent K-theory.

5 Varying the threefold volume character

Moving away from the volume-preserving torus of the threefold gives a test of circle exchange that the numerical index of a single curve cannot detect. Let C have genus two, $B=\mathrm{Pic}^1(C)$, and $\pi: C\times B\to B$. Choose a Poincaré line $\mathcal{L}$ trivialized over one point of C, and put $\mathcal{M}=K_C\mathcal{L}^{-1}$, $A=R\pi_*\mathcal{L}$. Give the four lines $\mathcal{L},\mathcal{M},\mathcal{O},\mathcal{O}$ weights $t,\kappa/t,q,\kappa^{-1}/q$, in that order. Their product remains one on the fivefold. Write $\delta=1-\kappa$ and retain the orientation of the ordered external lines in the modified pair index.

Let $\mathcal{I}_2^{\mathrm{orig}}$ and $\mathcal{I}_2^{\mathrm{rot}}$ denote the degree-two family characters with these weights in the original and rotated presentations. For $k\ge1$ their first variations satisfy

$$[\delta q^{-3}t^k]\,\mathcal{I}_2^{\mathrm{orig}}=[\delta q^{-3}t^k]\,\mathcal{I}_2^{\mathrm{rot}}=-2\bigl(1+2\,\mathbf{1}_{2|k}\bigr)\psi^k(A)\quad\text{in } K^0(B), \tag{5.1}$$

where ψ^k is the Adams operation. In the original presentation one takes the q^{-3} term, varies κ, and expands the result at $t=0$. After exchange, t counts pairs. One fixes k, takes the variation, and expands at $q=0$. In the latter expansion there are no powers below q^{-3} at this order. We give the two calculations because the fixed loci that produce the same class have different dimensions.

5.1 The two minimal ribbons

At the lowest Euler characteristic, $n=-3$, a torus-fixed degree-two pair is the structure sheaf of a ribbon in either normal direction. Indeed its generic cyclic quotient of length two has monomial basis $1,x$ or $1,y$. The corresponding ribbon in a line F has pushforward $\mathcal{O}_C\oplus F^{-1}$, of Euler characteristic -3. A zero divisor of the section or of the multiplication between the two weight lines raises the Euler characteristic. Thus each choice gives one fixed copy of B.

These two ribbons also occur as limits of smooth covers. For a genus-two splitting as in [20, Proposition 8.1], the degree-two cycles form two one-parameter families, one in each normal line. Their nonzero members are smooth genus-four curves. In a fiber coordinate z on either line, the family has equation $z^2=as$, where s is a section of the square of that line with two simple zeros. At $a=0$ the central ribbon has equation $z^2=0$. The index calculation below keeps its scheme structure and its equivariant deformation weights.

Write $F=\mathcal{L},\ p=t$ on the first ribbon and $F=\mathcal{M},\ p=\kappa/t$ on the second. The equations for the ribbon are the other normal coordinate and the square of the first. Tensoring the corresponding

normal lines with $\mathcal{O}_C \oplus (pF)^{-1}$ and using relative Serre duality gives the virtual tangent $H - \kappa H^\vee$, with polarization

$$H = pR\pi_* F + p^2 R\pi_* F^2, \qquad \operatorname{rk} H = 1.$$

For this polarization the symmetrized inverse Euler class, the external determinant and the sign of $(-q)^{-3}$ give

$$\kappa^{-1} \frac{\lambda_{-1}(\kappa^{-1} H)}{\lambda_{-1}(H)}, \qquad \lambda_{-1}(V) = \sum_j (-1)^j \bigwedge^j V.$$

The ratio at $\kappa = 1$ is identically one, so varying $p = \kappa/t$ in both its arguments contributes zero. A root w of H contributes $-w/(1-w)$ to its first variation. The expansions for $w = tx$ and $w = t^{-1}x$ have opposite signs in positive degree and involve x^k and x^{-k}, respectively. Using $R\pi_* \mathcal{M} = -A^\vee$, the resulting positive coefficients are

$$-2\psi^k(A) + \mathbf{1}_{2|k}\, \psi^{k/2}(D_M^\vee - D_L), \qquad D_L = R\pi_* \mathcal{L}^2, \quad D_M = R\pi_* \mathcal{M}^2. \tag{5.2}$$

Here and below the second term is included only for even k.

5.2 The fixed pairs after circle exchange

The rotated threefold is $C \times \mathbb{A}^2$. Write $u = q,\ v = \kappa^{-1}/q,\ c = uv$, so $1 - c^{-1} = \delta$. A fixed row is specified by effective divisors P, Q on C. The section vanishes on P in the inner weight line, and the map to the outer line vanishes on Q. The pushforward to C and the counting degree are

$$E = \mathcal{O}(P) \oplus u^{-1}\mathcal{O}(P+Q), \qquad k = 2 \deg P + \deg Q.$$

The transposed row exchanges u and v. Thus the fixed loci are products of symmetric powers of C. The virtual tangent of a row is the pushforward along C of the following character, obtained by resolving the pair in the two plane directions:

$$E + cE^\vee - (1-u)(1-v) \operatorname{End} E.$$

For a universal divisor D of length d, let p_D be its projection to $S_d = \operatorname{Sym}^d C$. Set $U_D = p_{D*}\mathcal{O}_D$ and $T_D = p_{D*}\mathcal{O}_D(D) = T_{S_d}$. The divisor sequences give the weight-one and weight-c terms $T_P + T_Q - c(U_P + U_Q)$. The unit in U_D splits by the normalized trace for $d > 0$. Each nonempty divisor therefore supplies a factor $1 - c^{-1}$. All remaining moving weights are nontrivial at $c = 1$ for generic q. When both divisors are nonempty the first variation vanishes, including along their incidence locus.

It remains to divide by the one factor $1 - c^{-1}$ on a pure component and set $c = 1$. Write $U = U_D,\ T = T_D,\ J = \det U$, and write $N(F)$ for the norm to S_d of a line F restricted to D. Finite duality gives $\det T = J^{-1} N(K_C)^{-1}$. The residual fixed obstruction contributes $\lambda_{-1}((U/\mathbf{1})^\vee)$. The moving tangent and modified root on the two components are

	q^{-2}	q^{-1}	q	q^2	modified root
$P=0,\ Q=D$	$T-1$	$-T-1$	$U-1$	$-U-1$	$q^d J N(\mathcal{L})$
$P=D,\ Q=0$	-1	$T-1$	$-U-1$	-1	$q^d J^2 N(\mathcal{L}^2)$.

(5.3)

For example, subtract the minimal tangent $-(u + v + u^2 + v/u)$ from the virtual tangent above. The divisor sequences give $T(1 + v/u - v) + Uu(1 - u - v)$ for the first row and $T(1 + u^{-1}) - cU(1+u)$ for the second.

For clarity, the orientation in the last column can be computed before restricting to a fixed component. Write $\mathcal{D}(F) = \det R\pi_* F$, with the projection now taken over the pair family, and let $\langle\,,\,\rangle$ be the Deligne pairing on the curve. The modified root, up to constant vector-space lines, is

$$c^{k+1}\mathcal{D}(E)^{-1}\langle \det E, \mathcal{L}K_C^{-1}\rangle.$$

Indeed the endomorphism term multiplied by $(1-u)(1-v)$ has trivial determinant. For rank-two E and a line F one has $\mathcal{D}(EF)/\mathcal{D}(E) = \langle \det E, F\rangle(\mathcal{D}(F)/\mathcal{D}(\mathcal{O}))^2$. This identity and Serre duality $\mathcal{D}(E^\vee) = \mathcal{D}(EK_C)$, $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{M})$ give the stated root on squaring, with external character exponent $(\chi(E\mathcal{M}) - \chi(E^\vee))/2 = k+1$. Restriction to the universal divisors gives the norm lines in (5.3). The counting sign $(-1)^k$ is still to be included.

A summand $q^a V$ of the virtual tangent with $a > 0$ and virtual rank r has inverse Euler class with leading term $(-1)^r q^{ar} \det V$ at $q = 0$. A negative weight has leading term one. The two rows of (5.3) start at q^{-3}, since the exponent from the root and positive weights is $d + (d-1) + 2(-d-1) = -3$ in the first and $d + (-d-1) + 2(-1) = -3$ in the second. At $q = \infty$ the extreme exponent is 3 for each row, which gives the same lower bound for their transposes. To extract the coefficient put

$$\Phi_d(F) = R\pi_{S_d *}\big(N(F)\lambda_{-1}(U/\mathbf{1})\big), \qquad \pi_{S_d} : S_d \times B \longrightarrow B.$$

Combining the leading determinants with $J\lambda_{-1}((U/\mathbf{1})^\vee) = (-1)^{d-1}\lambda_{-1}(U/\mathbf{1})$ and the counting sign gives $-\Phi_d(\mathcal{L})$ from each outer divisor, at $k = d$. The common divisor and its transpose, at $k = 2d$, give respectively $-\Phi_d(\mathcal{L}^2)$ and $-\Phi_d(\mathcal{L}^2 K_C^{-1})$. The factor $N(K_C)^{-1}$ in the transpose comes from $\det T$ at the opposite end of the q-expansion.

5.3 Adams operations and the Picard polarization

These symmetric-product indices admit an integral evaluation,

$$\Phi_d(F) = \psi^d(R\pi_* F). \tag{5.4}$$

To see it, let $h : C^d \to S_d$ be the ordered cover, let $V = \mathbb{C}^d$ be the permutation representation, and put $V_0 = V/\mathbf{1}$. Marking one point of a divisor gives $U = (h_*\mathcal{O} \otimes V)^{\mathfrak{S}_d}$. Exterior powers of these covariants identify with the covariants of the exterior powers. Locally at a divisor of multiplicities d_i this is Solomon's exterior algebra theorem for the product $\prod_i \mathfrak{S}_{d_i}$ acting on the local coordinates of the ordered points [21]. Splitting off the constant vector gives the same statement for $U/\mathbf{1}$ and V_0. The norm pulls back to $F^{\boxtimes d}$. Relative Künneth now expresses the pushforward in (5.4) as the alternating sum of the Schur operations corresponding to $\bigwedge^j V_0$, with the usual Koszul signs on the tensor powers of $R\pi_* F$. Their partitions are the hooks $(d-j, 1^j)$. The integral symmetric-function identity $p_d = \sum_{j=0}^{d-1}(-1)^j s_{(d-j,1^j)}$ identifies the sum with the Adams operation. Together with $D_M^\vee = -R\pi_*(\mathcal{L}^2 K_C^{-1})$, this gives (5.2) directly in the rotated presentation.

There is one more simplification in even counting degree. On S_2, J is the trace-free line of the ordered double cover, so $J^2 = \mathcal{O}(-i(C))$ for the doubled-point divisor $i : C \to S_2$. Its restriction $i^* J = K_C$ is the conormal of the diagonal on the ordered cover. Multiplication by a line minus one lowers the dimension of the support filtration on the smooth surface S_2. Hence $(J-1)^3 = 0$ in $K^0(S_2)$, and the divisor sequence gives

$$4(J-1) = (J-1)(2 + J + J^{-1}) = (J^2 - 1)(1 + J^{-1}) = -i_*(1 + K_C^{-1}).$$

Tensor with $N(\mathcal{L})$ and push to B. Since $U/\mathbf{1} = J$ and $i^* N(\mathcal{L}) = \mathcal{L}^2$, equation (5.4) gives $4\psi^2(A) = R\pi_*(\mathcal{L}^2(1 + K_C^{-1}))$. Applying $\psi^{k/2}$ proves (5.1) integrally.

Finally let θ be the principal polarization, with sign fixed by $\pi_*(\gamma^2)/2 = -\theta$ for the mixed Künneth component γ of $c_1(\mathcal{L})$. The degree-one line has Euler characteristic zero on C, and the chosen normalization removes the base component of its first Chern class. Riemann–Roch therefore gives $\mathrm{ch}(A) = -\theta$ in $H^{\mathrm{even}}(B, \mathbb{Q})$. The cohomological Chern character of (5.1) is $2k(1 + 2\,\mathbf{1}_{2|k})\theta$. At $k = 2$ the two outer components give 4θ and the common-divisor components give 8θ. Their sum 12θ measures the response of the family to varying the threefold volume character. Restriction to any one target retains only its rank, which is zero.

6 Discussion

In the family (1.1), changing the external circle exchanges an Euler-characteristic grading with a rotation of a plane. The commutator deformation accounts for the disappearance of positive excess in the plane description. The geometry that remains is the Hilbert scheme of points, and its tangent representation supplies the hook lengths in the local topological-string partition function. The determinant on the family of doubling directions distinguishes the balanced splitting: its degree is zero precisely when the two original normal lines have equal degree. The first variation in section 5 illustrates what a family retains even when a numerical trace vanishes.

An index on a space of cycles and the Hilbert space of a charged five-dimensional particle answer different questions. In degree two the index calculation identifies the locus supporting the connected term and fixes all its coefficients. A microscopic membrane calculation would have to determine the spectrum at the threshold where two branes on C can separate. To identify the coefficient at q^{-2} with a particle spin trace, one would have to establish a normalizable state in the full unframed theory with the two-particle continuum open, and then derive its spin from the wrapped-brane dynamics. The value -4 computed here provides an independent comparison for that calculation.

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