

MIRROR SYMMETRY FOR CONIFOLD RELATIONS IN CALABI–YAU THREEFOLDS

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ABSTRACT. For a class of Batyrev mirror families, we establish Morrison’s predicted codimension for nodal degenerations and strengthen it to an equality of integral relation lattices. The result holds on every connected component of the nodal coefficient locus meeting our construction near the maximally degenerate limit. After marking and orienting the nodes, the relations among exceptional curves of the original small resolution coincide with those among mirror vanishing three-spheres, which generate a torsion-free subgroup. Each such component is smooth of codimension equal to the relative Picard number. We realise every relation near the limit by an integral four-chain lifted from a tropical two-cycle relative to the discriminant on Gross’s degeneration base. The comparison characterises smoothability of the nodal partial resolution and, when it is smoothable, yields a symplectic conifold transition on the mirror. We treat specified projective toric partial resolutions of hypersurfaces with nodes and degree-six or degree-seven del Pezzo cone points. In an explicit four-node family, every member of the unique codimension-three nodal component admits a projective small resolution. For general members, these resolutions are birational to Batyrev mirrors of smoothings of the original nodal threefold, giving the reversed transition predicted by Morrison in this family.

Date: September 29, 2026.

2020 Mathematics Subject Classification. Primary 14J32; Secondary 14J33, 14T20, 14M25, 14D06, 32S30, 53D12, 53D37.

Key words and phrases. Calabi–Yau threefolds, conifold transitions, mirror symmetry, vanishing cycles, tropical 2-cycles, integral affine manifolds with singularities, toric degenerations, Lagrangian spheres.

CONTENTS

1. Introduction	5
1.1. Conifold transitions and Morrison's prediction	5
1.2. The main theorem and its consequences	6
1.3. Geometric realisation of the relations	7
1.4. Proof of the integral comparison	8
1.5. Hypotheses	9
1.6. Relation to other work	10
1.7. Organisation	11
Acknowledgements	11
2. Setting and statements	11
2.1. Conventions	11
2.2. Admissible polytopes and profiles	12
2.3. Gross's base and relative tropical 2-cycles	13
2.4. The mirror family and its vanishing spheres	15
2.5. Results	15
2.6. Scope	18
Part I. The base and its tropical cycles	19
3. The partial resolution and Gross's base	19
3.1. Admissible two-faces and profiles	20
3.2. The partial resolution, the small resolution and K	21
3.3. Gross's base	26
3.4. The discriminant	28
3.5. The mirror subdivision	35
3.6. When (Proj) fails	37
4. The complex of dual cell pairs	38
4.1. Faces, decompositions and charts	39
4.2. The complex of dual cell pairs	41
4.3. The discriminants and the flat pairing	44
5. Relative tropical 2-cycles	47
5.1. The coefficient system	48
5.2. Tropical 2-cycles in the sense of Castaño-Bernard and Matessi	50
5.3. Node coefficients of balanced networks	53
5.4. The neighbourhood of the discriminant	55
5.5. Filling	58
Part II. The mirror lift and the relation lattice	64
6. The transfer to the mirror base	64
6.1. Change of fan-side subdivision and of height	64
6.2. The cells near a node and near an arc of the discriminant	69
6.3. Transfer of relative tropical 2-cycles	71
7. The localized mirror and Morse charts at the nodes	74
7.1. The mirror member	76
7.2. The localized mirror and the smoothness of the interpolating family	82
7.3. The Morse chart at a node	84
7.4. Where the localized mirror is given by exact models	86

7.5. The node piece and the orientation of the vanishing sphere	87
8. The lift of relative tropical 2-cycles	93
8.1. Moving the discriminant to the local models; a normal form for the transferred cycle	94
8.2. Local lifts along the discriminant and at its three-valent vertices	102
8.3. Node pieces with multiplicity	105
8.4. Crossings with an arbitrary invariant coefficient	106
8.5. Joining the local lifts to the positive real section	106
8.6. Extending the lift over the regular part	111
8.7. The positive real locus and the fibre class	117
8.8. Assembly of the local lifts	118
8.9. What the construction uses, and the strict case	120
9. Periods and the relation lattice	121
9.1. The neighbourhood U and the periods	122
9.2. Relations: proof of Theorem A(1) and (2)	125
9.3. The nodal locus: proof of Theorem A(3)	126
9.4. Lagrangian representatives	128
9.5. Smoothings: proof of Corollary B	129
9.6. The factoring locus and the nodal locus	132
9.7. Continuation along the nodal locus	133
9.8. Two components of the four-node locus for X_9	139
9.9. The codimension-three component for X_9	143
9.10. A reverse mirror transition for X_9	148
Part III. The resolution side	151
10. Geometric realisation on the small resolution	151
10.1. The geometric statement	152
10.2. Mechanisms of the construction	152
10.3. The example X_9	153
Part IV. Examples	154
11. Examples and finite computations	154
11.1. The example X_9	154
11.2. The examples X_{19} and X_{20}	157
11.3. Hexagon examples	158
11.4. The 612-polytope collection	159
11.5. Evidence for Conjecture G	161
Appendix A. Unimodular heights: proof of Theorem F	163
A.1. Reformulation	164
A.2. The facets of Δ°	165
A.3. Certificates and classification completeness	166
A.4. What the proof rests on	168
Appendix B. Proofs for Section 3	169
Appendix C. The complex of dual cell pairs	172
C.1. Faces, cells and charts	172
C.2. The cells of the complex of dual cell pairs are balls	173
C.3. Piecewise linear details	176
Appendix D. Estimates for the localized mirror	177

D.1.	Deficits, cut-offs and a regularity criterion	177
D.2.	Smoothness of the interpolating family: proof of Lemma 7.11	179
D.3.	Near a node	180
D.4.	Where the localized mirror is given by exact models	183
D.5.	Change of diagonal: proof of Lemma 7.26	185
Appendix E.	Deferred proofs for Section 8	187
Appendix F.	Construction of the resolution-side chains	194
F.1.	The resolved base	195
F.2.	Standard position near the nodes	199
F.3.	The localized hypersurface	202
F.4.	The domain	205
F.5.	The local pieces	206
F.6.	The regular part	214
F.7.	Transport	214
F.8.	Proof of Theorem E	215
F.9.	Relative tropical 2-cycles	218
F.10.	Regularity of the family: proof of Lemma F.7(3)	218
F.11.	The regions: proof of Lemma F.12	219
F.12.	The positive section: proof of Lemma F.10	221
F.13.	Local fundamental groups: proof of Lemma F.18	222
F.14.	The compatible cover: proof of Lemma F.20	225
Appendix G.	Data of the examples	227
G.1.	The example X_9	229
G.2.	The example X_{19}	229
G.3.	The example X_{20}	230
G.4.	The hexagon examples X_{88} and X_{154}	231
References		232

1. INTRODUCTION

Morrison proposed that mirror symmetry reverses conifold transitions of Calabi–Yau threefolds, exchanging exceptional curves with vanishing three-spheres [Mor99]. For the class of Batyrev hypersurfaces considered here, we establish his numerical prediction on the connected components of the nodal coefficient locus meeting our near-limit construction, and strengthen it to an equality of integral relation lattices. Every relation near the limit is realised by an integral bounding chain on the mirror hypersurface, constructed from a relative tropical two-cycle. The comparison also characterises smoothability in terms of mirror vanishing cycles and yields a symplectic conifold transition whenever the original nodal threefold is smoothable.

1.1. Conifold transitions and Morrison’s prediction. Let X' be a projective threefold with trivial canonical bundle whose singular points $q \in Q$ are ordinary double points, and let $\widehat{X} \rightarrow X'$ be a projective small resolution, with exceptional curves $C_q \cong \mathbb{P}^1$. A smoothing X_s of X' , when it exists, contains a vanishing three-sphere for each node, and passing from $\widehat{X}$ to X_s through X' is a conifold transition.

In Morrison’s picture, a mirror family of $\widehat{X}$ acquires nodes with vanishing spheres δ_q corresponding to the curves C_q , and the nodal members admit small resolutions mirror to X_s . His numerical prediction is that the $|Q|$ mirror nodes impose only ρ_{rel} conditions on the complex moduli, where ρ_{rel} is the relative Picard number of $\widehat{X} \rightarrow X'$ [Mor99, §3.1]; his setting is a conifold transition, so X' is assumed smoothable. For the families below we prove this count in the space of coefficient vectors of the Batyrev mirror family, whether or not X' is smoothable, together with the following integral comparison. Let $K \subset \mathbb{Z}^Q$ be the lattice of relations among the classes $[C_q]$ in $H_2(\widehat{X}; \mathbb{Z})$, of rank $|Q| - \rho_{\text{rel}}$. On a suitable smooth member Y of the mirror family,

$$(1.1) \quad \left\{ k \in \mathbb{Z}^Q : \sum_q k_q [\delta_q] = 0 \text{ in } H_3(Y; \mathbb{Z}) \right\} = K,$$

and the classes $[\delta_q]$ generate a torsion-free subgroup. We also show that, near that member, the members on which all these spheres collapse form a nonempty smooth locus of codimension $|Q| - \text{rank } K = \rho_{\text{rel}}$ in the space of coefficient vectors. The *factoring locus*, where the face polynomials of the node parallelograms introduced below factor into binomials, has the same codimension there; the result concerns the nodal locus itself, on which all $|Q|$ nodes are present. The integral equality specifies exactly which combinations of the mirror spheres are null-homologous. The locus of hypersurfaces with exactly $|Q|$ nodes and no other singularities is smooth at every point. On each connected component meeting the constructed neighbourhood, its codimension is ρ_{rel} and the integral comparison continues with the nodes marked and oriented (Proposition 9.13). The component restriction is necessary: for X_9 , Proposition 9.19 exhibits another four-node component of codimension four, whose vanishing spheres have no nonzero relation. Theorem 9.21 identifies the entire codimension-three part of this family as one connected component defined by a rank-one coefficient matrix.

This equality compares curves on $\widehat{X}$ with spheres on its mirror. Within a single conifold transition, the relation spaces among exceptional curves and vanishing spheres are instead mutual annihilators [LLW18, Thm. 1.14], reflecting the topology studied by Clemens [Cle83].

Castaño-Bernard and Matessi gave a geometric construction of relations from tropical 2-cycles with boundary on the discriminant of an integral affine three-manifold [CBM14, Thm. 7.3, §9.5]. In their topological conifold models, such a cycle produces chains bounded

by the corresponding exceptional curves and vanishing spheres. Here we construct integral bounding chains on the algebraic Batyrev mirror hypersurfaces and prove that they realise every relation. We use *relative tropical 2-cycles*: integral two-chains on the affine base, with monodromy-invariant integral tangent-vector coefficients, whose boundary follows the discriminant in its vanishing directions (Definition 2.4). These extend the tropical cycles of [CBM14]. Their exhaustion of the relation lattice and their lift to the mirror hypersurface are the geometric steps in the proof.

1.2. The main theorem and its consequences. Let Δ be an admissible reflexive four-polytope (Definition 2.1) and $X \subset \mathbb{P}_\Delta$ a Δ° -regular anticanonical hypersurface. Its singularities are nodes and isolated cones over del Pezzo surfaces of degrees six and seven. A *profile* σ is a choice of subdivisions of its singular two-faces into unit parallelograms and unimodular triangles (Definition 2.3). It determines a nodal crepant partial resolution $X'_\sigma \rightarrow X$, with one node for each parallelogram (Section 3.2; for the hexagons compare [Wue26b, Prop. 3.7(2),(3)]). We assume that this partial resolution is induced by a projective toric morphism, Hypothesis (Proj). The precise combinatorial restrictions are recalled below and in Section 2.

Write Q for the nodes. There is a projective small resolution $\widehat{X} \rightarrow X'_\sigma$, and the relation lattice of its exceptional curves is computed by the parallelograms:

$$K = \left\{ \lambda \in \mathbb{Z}^Q : \sum_q \lambda_q \text{circ}_q = 0 \right\}$$

(Proposition 3.5). Here $\text{circ}_q = e_b + e_d - e_a - e_c$ is the *circuit* of the corresponding parallelogram $abcd$, with the sign fixed by the chosen diagonal, and e_v is the formal basis vector indexed by the lattice point v . On the mirror side, Y_t is a member of the Batyrev family near its maximally degenerate limit $t \rightarrow 0$. The construction starts with real coefficients of a fixed sign pattern, close to the factoring locus, where the face polynomial of each parallelogram factors into two binomials. For coefficients satisfying the bounds of Theorem A, there is one nondegenerate critical point with small nonzero critical value near each parallelogram; its vanishing sphere is $\delta_q \subset Y_t$.

Theorem (Theorem A). *Let Δ be admissible and satisfy Hypothesis (U), and let σ satisfy Hypothesis (Proj). For t sufficiently small and real coefficients satisfying the bounds of Theorem A:*

- (1) $\sum_q k_q [\delta_q]$ is a torsion class of $H_3(Y_t; \mathbb{Z})$ only if $k \in K$, and it vanishes for $k \in K$. So the relation lattice of the vanishing spheres is exactly K , and the $[\delta_q]$ span a free group $\mathbb{Z}^Q/K$ that meets the torsion trivially.
- (2) Every relation $k \in K$ is the boundary of an integral 4-chain on Y_t lifted from a relative tropical 2-cycle on the base of Gross's toric degeneration of X'_σ .
- (3) Near the given member, the coefficient vectors whose members have a node at every one of these critical points form a nonempty complex submanifold of the space of coefficient vectors, of codimension $|Q| - \text{rank } K$, the relative Picard number of a projective small resolution $\widehat{X} \rightarrow X'_\sigma$.

Continuation along the nodal locus. Let $S_{|Q|}$ be the locus in the full coefficient space whose members have exactly $|Q|$ nodes and no other singularities. This locus is smooth at every point, with local ideal generated by independent vanishing-cycle periods (Lemma 9.17). On every connected component meeting the nodal members of Theorem A, its codimension is $|Q| - \text{rank } K$. After marking and orienting the nodes, both the integral relation lattice and the

preimage of ambient torsion remain equal to K throughout that component. The bounding chains continue by transport (Proposition 9.13); their tropical construction is made near the limit.

Smoothings and symplectic transitions. Friedman’s relation criterion [Fri86, Rem. (4.5), Cor. (4.7)], in Namikawa’s necessary and sufficient form [Nam02, Thm. 2.5], characterises smoothability of X'_σ by the existence of a relation among the exceptional curves with every coefficient nonzero; compare [RT09, Thms. 1.2 and 1.3]. The integral comparison transfers this criterion to the mirror: X'_σ is smoothable if and only if the $[\delta_q]$ satisfy an integral relation with every coefficient nonzero. The spheres are smoothly isotopic to pairwise disjoint Lagrangian spheres for the restriction of a toric Kähler form, whether or not such a relation exists. When it exists, by Smith–Thomas–Yau [STY02, Thm. 2.9], one of their $2^{|Q|}$ conifold transitions carries a symplectic structure in which the exceptional curves are symplectic (Corollary B). Thus algebraic smoothability on the original side yields a symplectic conifold transition on the mirror. For hexagon-free Δ satisfying (Proj), the same relation criterion decides the smoothability of X itself (Corollary 9.11).

For X_9 we also establish the geometric reverse transition in the sense of Batyrev families, up to birational equivalence (Theorem 9.25). The projective small resolution of a general member of the codimension-three four-node component is birational to the Batyrev mirror of a smoothing of the exact partial resolution X'_9 used here. The corresponding geometric assertion for the whole admissible class remains open.

A four-node example. The polytope Δ_9 of [Wue26b] has two unit squares and one dP_7 pentagon, so X'_σ has four nodes, and $K = \mathbb{Z}(1, -1, -1, 1)$. On the base B of Gross’s degeneration there is a disc whose boundary runs along the discriminant through the four nodes along opposite legs, and whose node coefficients are $(1, -1, -1, 1)$ (Section 11.1). Its lift is a 4-chain on Y_t bounded by $\delta_1 - \delta_2 - \delta_3 + \delta_4$, and by the theorem this is the only relation: the four spheres span a free group of rank three, and, near the given member, the members of the mirror family with all four nodes form a submanifold of codimension three in the space of coefficient vectors. Theorem 9.21 shows that the whole codimension-three four-node locus is one determinantal component, including its zero-coefficient members. Every member of this component has a projective small resolution, obtained by blowing up a compact surface through the four nodes. Other four-node components have independent vanishing spheres and codimension four. To identify the reverse transition, we construct a simultaneous modification of the degree-seven cone smoothing whose central fibre is the two-node partial model of X'_9 . Its smooth fibres are those of the toric smoothing in [Wue26b]. An explicit Laurent birational substitution then identifies the resolved mirror family with their Batyrev partner (Section 9.10).

1.3. Geometric realisation of the relations. Let $B = B'_\sigma$ be the dual intersection complex of Gross’s toric degeneration of X'_σ [Gro05]: an integral affine three-sphere whose singularities lie along a graph Γ , the discriminant. Each node q gives a four-valent vertex p_q of Γ , which, after an isotopy supported in the two-cell over its parallelogram, is a negative node in the sense of [CBM14] (Proposition 3.18); the other vertices of Γ are trivalent of the two standard types. Each arc of Γ carries a vanishing vector, the direction of the image of the monodromy minus the identity.

In the definition of relative tropical 2-cycles, the coefficient sheaf is $\iota_*\Lambda$, where Λ is the local system of integral tangent vectors on $B \setminus \Gamma$ and ι is the inclusion. Its sections across Γ are monodromy-invariant. The boundary of a relative cycle is a *balanced network* on Γ , and at each node it has a *node coefficient* $\rho_q \in \mathbb{Z}$, the jump of the boundary coefficient across q

measured against the circuit circ_q (Definition 2.6). The tropical 2-cycles of [CBM14], which carry a primitive parallel vector field v as coefficient, are examples; relative tropical 2-cycles may in addition carry non-primitive coefficients, branch arbitrarily, cross the discriminant with any invariant coefficient, and pass a node in both directions. Their node coefficients always lie in K (Section 5.3). The converse is the integral filling theorem:

Theorem (Theorem C). *Every balanced network on Γ bounds a relative tropical 2-cycle, and the node coefficients of balanced networks are exactly K . Hence the node coefficients of relative tropical 2-cycles generate K over $\mathbb{Z}$.*

The mirror lift takes every relative tropical 2-cycle z to an integral 4-chain $\check{\Xi}(z)$ on Y_t with

$$\partial \check{\Xi}(z) = \sum_q \rho_q(z) [\delta_q]$$

(Theorem D). Together these two theorems construct a bounding chain for every $k \in K$ on the mirror hypersurface. The period argument then proves that there are no further relations and no nonzero torsion combinations of the spheres.

We call the tropical 2-cycles (S, j, v) of [CBM14, Def. 7.2], with j piecewise linear and two points of that definition made precise (the group in which a monodromy conjugacy is taken, and the meaning of a negative vertex; Definition 2.5), *strict*, to distinguish them from relative cycles. Away from the discriminant, their mirror lifts are, up to homotopy in each region, the bundles of two-tori annihilated by v in the dual torus fibres, as in the topological construction of [CBM14]. Remark 2.7 specifies the comparison near the discriminant and the corrections needed to complete the chains.

Theorem E is a supplementary geometric realisation on the resolution side. The integral relations among $[C_q]$ are already known from Proposition 3.5; the additional result constructs, from a strict cycle, a completed circle domain and a continuous map to $\hat{X}$ whose boundary spheres have degrees $-\rho_q(S)$ on the exceptional curves. Its negative fundamental chain, adjusted within those curves, has boundary $\sum_q \rho_q(S)[C_q]$. Thus the same tropical cycle gives geometric representatives of the corresponding relations on the original small resolution and the mirror hypersurface. This construction is independent of the proof of the central comparison, Theorem A, and of its smoothing consequences. Section 10 states its geometric content and explains the mechanisms through X_9 ; Appendix F contains the complete proof.

Whether strict cycles already generate K is Conjecture G; the evidence is discussed in Section 11.5. The integral exhaustion proved here uses relative cycles and addresses the relation-generating aspect of [CBM14, Conj. 8.3] for that enlarged class. It does not settle their full conjecture.

1.4. Proof of the integral comparison. The proof combines an integral filling theorem on the affine base, a geometric lift to the mirror, and a period calculation. The first two steps realise all elements of K as boundaries; the third establishes completeness and determines the nodal codimension.

The base. A circuit relation $\sum_q \lambda_q \text{circ}_q = 0$ is converted into a boundary problem on $\partial\Delta \cong S^3$. Using $H_1(S^3) = 0$, we construct a balanced network with node coefficients λ (Section 5.3). To fill it, we push the network off Γ along the dual cells. The resulting cycle in $B \setminus \Gamma$, with coefficients in Λ , must bound modulo meridians. A Mayer–Vietoris argument shows that ordinary $H_1(B \setminus \Gamma; \mathbb{Z})$ is generated by meridians; a computation on the boundary spheres of the facets, using $H^1(S^2; \mathbb{Z}) = 0$, controls the local coefficients. This integral complement argument is the main step in the filling theorem (Section 5).

The lift. The bases B'_σ and B_Y of the original and mirror degenerations have different cell decompositions. A complex of pairs of dual cells gives a piecewise linear homeomorphism between them, matching the discriminants and dualising the tangent local systems (Sections 4 and 6). We then use a localized hypersurface W^* , isotopic to Y_t , modelled on the localization of Mikhalkin and Yamamoto [Mik04, Yam16]; since the mirror subdivision is not unimodular, their results do not apply here, and the isotopy is proved directly. Near each node it has the conifold smoothing model $\{xy + z_1z_2 = \kappa\}$ with κ small and nonzero (Section 7). The chain is assembled from thimbles at the nodes, pieces of toric divisors along its boundary, and two-torus bundles over its interior. The local extensions leave a possible boundary in the class of the fibre three-torus. This obstruction vanishes because the positive real locus meets each fibre torus once and misses every vanishing sphere (Section 8). The isotopy carries the resulting integral chain to Y_t .

Periods. The chains give $K \subseteq \text{Rel}(\delta)$, where $\text{Rel}(\delta)$ is the relation lattice of the spheres, the left side of (1.1). The periods $\wp_q = \int_{\delta_q} \Omega$ extend holomorphically to the nodal members and equal their critical values ε_P times functions that do not vanish near the nodal locus, where P is the parallelogram of q . Up to nonzero scalars, the differentials of the critical values are close to the linear forms defined by the circuits. The period differentials have rank $|Q| - \text{rank } K$, which bounds the rank of $\text{Rel}(\delta)$ from above by $\text{rank } K$. Since K is saturated, the inclusion is an equality; the same argument excludes nonzero torsion combinations of the spheres. Critical-value equations for a basis of the circuit span meet transversally, and the remaining critical values vanish on their intersection. This gives the smooth nodal locus and its codimension (Section 9). The period argument is classical [GMS95, LLW18]; here it is made uniform near the maximally degenerate limit and combined with the integral chains.

Continuation. Admissibility makes the correction term in Batyrev’s formula for polynomial deformations vanish. Toric cohomology then shows that coefficient deformations induce all first-order deformations of a nodal member. The Friedman–Rollenske–Thomas node-smoothing sequence, applied to the blow-up of the nodes, identifies the rank of the local smoothing parameters with the rank of the vanishing cycles. Exact linear identities among their periods show that independent periods generate the whole nodal ideal, proving smoothness at every point of $S_{|Q|}$. Along this locus the complements of the nodes form a differentiably locally trivial family. Integral Mayer–Vietoris identifies their link maps with the vanishing-cycle maps, including the preimages of torsion. These data transport the near-limit comparison along every component meeting the construction (Section 9.7).

1.5. Hypotheses. Admissibility means that the edges of Δ are primitive and its two-faces are unimodular triangles, unit squares, or lattice pentagons and hexagons with one interior lattice point and no boundary lattice points besides their vertices. The squares give nodes, and the pentagons and hexagons give cones over del Pezzo surfaces of degrees seven and six. The edge of the polar polytope Δ° dual to each non-triangular two-face is required to have lattice length one, so that each such face contributes exactly one singular point. A profile specifies the subdivisions of these faces into unit parallelograms and unimodular triangles; the choices occur at the hexagons.

For the polytope and profile, two hypotheses are made beyond admissibility. Hypothesis (Proj) is needed for the projective partial resolution and Gross degeneration used here, and is a genuine restriction. On the list of 612 admissible polytopes of Section 11.4, it fails for 11 of the 375 hexagon-free polytopes with $K \neq 0$ and for every profile on 59 of the 142 polytopes with a hexagon (Section 2.6).

The initial mirror construction uses real coefficients satisfying the bounds of Theorem A; the resulting relations extend to neighbouring smooth complex members as explained in Section 9.2, and to small smoothings along every reached nodal component by Proposition 9.13.

Hypothesis (U) asks for the input of Gross’s construction of the base: a regular unimodular triangulation of $\partial\Delta^\circ$ with a height satisfying his convexity conditions. Theorem F verifies it, by computer, for every member of the fixed 51,827-class list $\mathcal{L}_{\text{adm}}$ (Definition A.1), with each triangulation and its height checked in exact arithmetic (Appendix A). The separate Proposition 2.8 asserts that these are all admissible polytopes; it relies on the exhaustive filter, whose exclusions have been independently checked only in part (Appendix A.4). The geometric theorems retain (U) explicitly and so apply to any admissible polytope for which it holds. This larger computation is distinct from the list of 612 polytopes used for the other statistics. One further condition holds automatically: the mirror side requires a subdivision of $\partial\Delta$ in which the cells next to each node parallelogram have their other vertices at lattice height one, and the pulling refinement of the subdivision given by (Proj) has this property (Proposition 3.26).

1.6. Relation to other work. Batyrev and Kreuzer studied conifold transitions for Batyrev hypersurfaces whose singular two-faces are unit squares [BK10]. They identified the curve classes with the circuits of the squares and specialised the mirror family to the locus where each square’s face polynomial factors into two binomials. They expected the specialised members to have a node in each of the local toric charts they single out, and observed that a projective, hence symplectic, small resolution of such a nodal mirror requires a relation among the vanishing spheres with every coefficient nonzero, but did not determine whether such a relation exists [BK10, §3]. For dual edges of lattice length one, the case considered here, their local equation at the expected node omits the monomials of the cells next to the square, which are linear in the normal coordinates. The nodal members constructed here approach the factoring locus as $t \rightarrow 0$; our estimates do not place them on it, and in a local model at a square node the critical value on the factoring locus is nonzero for general coefficients (Section 9.6). Corollary B answers the analogous question for these nodal members: the relation exists exactly when X'_σ is smoothable.

Prince constructs a locally rigid toric log Calabi–Yau space from a reflexive four-polytope and a choice of Minkowski decompositions of its two-faces (scaled by the lattice lengths of the dual edges) into standard simplices. For the dP_6 hexagon the decompositions are into two triangles or into three segments. When a regularity condition holds, the space admits a polarisation and is smoothed by the Gross–Siebert algorithm [Pri21, §1, Thm. 1.1, Rem. 1.2]. He expects the resulting threefolds to degenerate to Calabi–Yau toric hypersurfaces with certain Gorenstein singularities [Pri21, Abstract].

For his example $P_{6,6}$, whose anticanonical hypersurfaces have 36 nodes and 12 cones over dP_6 , he asks for a description of his regularity condition in terms similar to the homological conditions of Friedman and Tian [Pri21, Rem. 5.4]. His two decompositions of a hexagon correspond to the two smoothing components of the cone over dP_6 [Pri21, §5], [Alt97, (8.1)], while the profiles used here subdivide the same faces for a partial resolution. Corollary B is a homological criterion of this kind for the nodal threefolds X'_σ , and Corollary 9.11 one for X itself when Δ is hexagon-free and satisfies (Proj); neither concerns the smoothing of the dP_6 points, and we do not compare them with his regularity condition.

Castaño-Bernard and Matessi constructed Lagrangian torus fibrations over integral affine three-manifolds with singularities [CBM09] and the topological conifolds and cycles described above [CBM14]. Their Conjecture 8.3 relates tropical cycles, homological relations, and

tropical resolutions and smoothings; their result producing tropical cycles from vanishing-cycle relations in a family of compact models is conditional on a basis conjecture [CBM14, Prop. 9.2]. The present paper proves integral exhaustion by relative cycles on Gross’s bases and constructs their bounding chains on the Batyrev mirror hypersurfaces.

Gross and Siebert developed the correspondence between toric degenerations and their affine data, with the discrete Legendre transform that exchanges the two sides of the mirror [GS06]; Haase and Zharkov constructed the integral affine spheres of toric hypersurfaces and their torus fibrations [HZ02]. Cycles in Calabi–Yau hypersurfaces near a maximally degenerate limit have been built from tropical data in other settings: Lagrangian rational homology spheres from tropical curves with ends on the discriminant by Mak and Ruddat [MR20], and lifts of a class of closed tropical (p, q) -cycles in hypersurfaces with unimodular tropicalisation by Yamamoto [Yam26], who notes that the tropical 2-cycles of [CBM14] can be regarded as tropical $(1, 2)$ -cycles [Yam26, §1, footnote]. Ruddat and Siebert compute period integrals over lifts of tropical one-cycles [RS20], Ruddat develops a homology theory of tropical cycles on integral affine manifolds with a perfect pairing [Rud21], and Abouzaid, Ganatra, Iritani and Sheridan compute periods tropically in the setting of the Gamma conjecture [AGIS20]. The chains constructed here differ from these in two respects: they are relative, with boundary at the nodes, and the subdivision that defines the ambient toric variety of X'_σ is not unimodular, since it contains the node parallelograms. The period argument of Section 9 goes back to Greene, Morrison and Strominger [GMS95]; compare [LLW18].

1.7. Organisation. Section 2 states the precise hypotheses and results. Part I constructs the base and its cycles: Section 3 describes the partial resolution and Gross’s base, Section 4 constructs the complex of pairs of dual cells, and Section 5 proves Theorem C. In Part II, Section 6 transfers cycles to the mirror base, Section 7 constructs the localized mirror and Morse charts, Section 8 lifts the cycles (Theorem D), and Section 9 proves Theorem A and Corollaries B and 9.11. Part III, Section 10, explains the resolution-side realisation (Theorem E); the detailed construction is in Appendix F. It uses results of Parts I and II, whose proofs are independent of it. Part IV, Section 11, treats examples and the evidence for Conjecture G. Appendix A proves Theorem F and the separate classification-completeness assertion; the remaining appendices contain deferred proofs and data.

Acknowledgements. I am grateful to Mark Gross for first drawing my attention, many years ago, to the problem of smoothing canonical singularities on Calabi–Yau threefolds. The author used LLMs during the development of this work for literature exploration, editorial and coding assistance and as an adversarial reader.

2. SETTING AND STATEMENTS

This section fixes the objects and states the results. Sections 3–11 use the definitions made here without repeating them.

2.1. Conventions. We write $N \cong \mathbb{Z}^4$ and $M = \text{Hom}(N, \mathbb{Z})$, with pairing $\langle \cdot, \cdot \rangle$. The polytope $\Delta \subset N_{\mathbb{R}}$ is reflexive, with polar $\Delta^\circ = \{m \in M_{\mathbb{R}} : \langle m, n \rangle \geq -1 \text{ for all } n \in \Delta\}$. For a face G of Δ the dual face is $G^\circ = \{m \in \Delta^\circ : \langle m, n \rangle = -1 \text{ for all } n \in G\}$; for a two-face G it is an edge, of lattice length $\ell(G^\circ)$. We write $\mathbb{P}_\Delta$ for the toric variety of the fan over the faces of Δ . Its anticanonical hypersurfaces have Newton polytope Δ° , and such a hypersurface is Δ° -regular if it meets every torus orbit transversally [Bat94, §3]. The Batyrev mirror family consists of the anticanonical hypersurfaces with Newton polytope Δ in a crepant resolution $X_{\mathcal{T}}$ of $\mathbb{P}_{\Delta^\circ}$ (Section 2.4). In the notation of Gross [Gro05] our pair (Δ°, Δ) is (Δ, ∇) . We

call the polytope whose lattice points index the monomials of a hypersurface its *Newton side*, and the polytope whose subdivision gives the fan of the ambient toric variety its *fan side*; for X these are Δ° and Δ , for the mirror family Δ and Δ° . We write dP_k for a del Pezzo surface of degree k .

For a finite set of lattice points n with heights $H(n)$ the tropical polynomial is $m \mapsto \min_n (H(n) + \langle m, n \rangle)$, $m \in M_{\mathbb{R}}$, and its tropical hypersurface is the locus where the minimum is attained at least twice. This convention is compatible with the signs of [Gro05], and it is used in Sections 3–6 and 10; Sections 7–9 use $p = -m$ and maxima of valuations, as stated there.

2.2. Admissible polytopes and profiles. A dP_7 *pentagon* is a lattice pentagon with exactly one interior lattice point, its *centre* o , and no lattice points on its boundary besides its vertices; a dP_6 *hexagon* is a lattice hexagon with the same two properties. Up to $GL_2(\mathbb{Z})$ and translation the hexagon has vertices $w_0, \dots, w_5 = (1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1), (0, -1)$ and centre $o = (0, 0)$, and $w_j + w_{j+2} = o + w_{j+1}$ for all j (indices modulo 6).

Definition 2.1 (admissible polytopes). The reflexive polytope Δ is *admissible* if every edge of Δ is primitive, every two-face is a unimodular triangle, a unit square, a dP_7 pentagon or a dP_6 hexagon, and $\ell(G^\circ) = 1$ for every two-face G that is not a triangle. This is [Wue26c, Def. 8.1].

Remark 2.2 (two-faces with one interior lattice point). Let G be a lattice polygon with primitive edges and exactly one interior lattice point. Up to $GL_2(\mathbb{Z})$ and translation, G is the polygon spanned by the primitive ray generators of the fan of one of the five smooth toric del Pezzo surfaces S_G : $\mathbb{P}^2$, $\mathbb{F}_1$, $\mathbb{P}^1 \times \mathbb{P}^1$, dP_7 or dP_6 . As in Section 3.2, if G is a two-face of Δ with $\ell(G^\circ) = 1$, the point of X on the orbit of the cone over G is the anticanonical cone over S_G . For $\mathbb{P}^2$ this cone is $\mathbb{C}^3/\mu_3$, with μ_3 acting by scalars, which is rigid [Sch71]. For $\mathbb{F}_1$ the cone is not smoothable: G has no nontrivial Minkowski decomposition into lattice polygons, and the base space of the versal deformation is $\text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)$ [Alt97, (9.1)(iv), (9.2)]. For $\mathbb{P}^1 \times \mathbb{P}^1$, dP_7 and dP_6 the cones are smoothable [Wue26b, Table 1, Lemma 3.5]; the components of their reduced versal base spaces correspond to the maximal Minkowski decompositions of G into lattice polygons [Alt97, (2.5), (8.1)]. Admissibility therefore keeps every such type with a smoothable cone point except $\mathbb{P}^1 \times \mathbb{P}^1$. Its polygon $\text{conv}\{(\pm 1, 0), (0, \pm 1)\}$ is a lattice parallelogram of twice the area of a unit parallelogram, and its vertices, placed at height one, span a sublattice of index two; so its cone is the quotient of the conifold by an involution acting freely off the vertex, not a node. Its only lattice point besides the vertices is the centre, so it has no subdivision into unit parallelograms and unimodular triangles that contains a parallelogram, and the nodal partial resolutions used here do not exist for it. The smoothing criterion of [Wue26b, Thm. 4.7] for X itself allows cones over $\mathbb{P}^1 \times \mathbb{P}^1$; for hexagon-free Δ satisfying (Proj), the relation criterion of Corollary B decides the smoothability of X itself (Corollary 9.11).

The lattice points of an admissible Δ that lie on faces of dimension at most two are the vertices and the centres of the pentagons and hexagons; we denote this set by R . A singular two-face G of Δ (a square, a pentagon or a hexagon) gives one singular point of X , on the torus orbit of the cone over G : a node, a cone over dP_7 , or a cone over dP_6 respectively (Section 3.2).

Definition 2.3 (profiles and node parallelograms). A *profile* σ assigns to every singular two-face of Δ a subdivision into unit parallelograms and unimodular triangles, as follows.

A unit square is not subdivided. A pentagon receives its unique subdivision into two unit parallelograms and one triangle, all containing the centre (Section 3.1). A hexagon receives either

- an A_1 *subdivision*: two parallelograms $\{o, w_j, w_{j+1}, w_{j+2}\}$, $\{o, w_{j+3}, w_{j+4}, w_{j+5}\}$ and the two triangles $\{o, w_{j+2}, w_{j+3}\}$, $\{o, w_{j+5}, w_j\}$, for one $j \in \{0, 1, 2\}$; or
- an A_2 *subdivision*: the three parallelograms $\{o, w_j, w_{j+1}, w_{j+2}\}$ with $j \equiv j_0 \pmod{2}$, for one $j_0 \in \{0, 1\}$.

The parallelograms of these subdivisions are the *node parallelograms*; Q denotes their set. We say that a hexagon is on the A_1 *branch* or the A_2 *branch* of σ according to its subdivision. The names refer to the root sublattices generated by differences of exceptional curve classes on the dP_6 surface over its centre (Remark 3.10).

The vertices of a node parallelogram are labelled a, b, c, d in cyclic order with $a + c = b + d$, and one of its diagonals, $\{a, c\}$, is chosen; in a pentagon or hexagon we choose the diagonal through the centre. The *circuit* of $q \in Q$ and the *relation lattice* are

$$\text{circ}_q := e_b + e_d - e_a - e_c \in \mathbb{Z}^R, \quad K := \left\{ \lambda \in \mathbb{Z}^Q : \sum_q \lambda_q \text{circ}_q = 0 \right\}.$$

K is saturated in $\mathbb{Z}^Q$. Changing the chosen diagonal at q changes the signs of circ_q , of the coordinate λ_q and of every node coefficient ρ_q below. The results hold for every choice of diagonals, with K , the node coefficients and the orientations of the vanishing spheres (Definition 7.23) all formed from that choice; changing the diagonal at q reverses the orientation of δ_q , so $\rho_q \delta_q$ does not depend on it (Lemma 7.26).

Hypothesis (Proj) (projectivity). There is a projective toric variety $P'_\sigma \rightarrow \mathbb{P}_\Delta$ whose fan refines the fan over the faces of Δ , has its rays generated by lattice points of $\partial\Delta$, and induces the profile σ on every two-face of Δ .

Under (Proj) the cones of the fan of P'_σ cut $\partial\Delta$ into lattice polytopes; this subdivision is denoted Σ' . Let $X \subset \mathbb{P}_\Delta$ be a Δ° -regular anticanonical hypersurface and X'_σ its proper transform in P'_σ . Then $X'_\sigma \rightarrow X$ is crepant, X'_σ has exactly one singular point for each node parallelogram, an ordinary double point, and we write $q \in Q$ also for this node. A choice of diagonal at every node parallelogram defines a small resolution $\widehat{X} \rightarrow X'_\sigma$, with exceptional curves $C_q \cong \mathbb{P}^1$; for suitable choices, including the diagonals through the centres at the pentagons and hexagons, $\widehat{X}$ is projective, induced by a regular refinement of Σ' (Proposition 3.4). For every small resolution the integral relations among the classes $[C_q]$ in $H_2(\widehat{X}; \mathbb{Z})$ are exactly K , with K formed from the diagonals of $\widehat{X}$; moreover $H_2(\widehat{X}; \mathbb{Z})$ is torsion-free (Propositions 3.4 and 3.5). In particular, for a projective $\widehat{X}$ the number $|Q| - \text{rank } K$ is the relative Picard number of $\widehat{X} \rightarrow X'_\sigma$; it does not depend on the choice.

Hypothesis (Proj) is a linear feasibility problem in the heights of the lattice points of $\partial\Delta$. When every singular two-face is a square it holds with $P'_\sigma = \mathbb{P}_\Delta$. It is a genuine restriction in general; see Section 2.6.

2.3. Gross's base and relative tropical 2-cycles. Let $\check{\varphi}$ equal 1 on $\partial\Delta^\circ$ and be linear on the cones over the faces of Δ° .

Hypothesis (U) (unimodular heights). The boundary of Δ° has a unimodular triangulation $\mathcal{T}$ refining its faces, with an integral height $\check{h}$ on the fan over $\mathcal{T}$ such that $\check{h}$ and $\check{h} - \check{\varphi}$ are strictly convex on that fan.

Hypothesis (U) is a condition on the polar polytope alone. Theorem F below verifies it, by computer, for every member of the fixed list of Definition A.1; under (U) we fix one pair $(\mathcal{T}, \check{h})$, normalised as in Section 3.3. The *base* $B = B'_\sigma$ is the dual intersection complex, in the fan picture, of Gross's toric degeneration of X'_σ [Gro05]: the boundary of the polytope $\nabla^{\check{h}}$ with the polyhedral decomposition $\mathcal{P}$ determined by Σ' (Section 3.3). It is an integral affine three-sphere with singularities along a trivalent-and-four-valent graph $\Gamma \subset B$, the *discriminant*, which has edges called *arcs*. Each arc is a union of *legs*, the segments joining the barycentres of an edge and a two-cell of $\mathcal{P}$. Each leg E carries a primitive *vanishing vector* $d_E = n' - n \in N$, where $\{n, n'\}$ is the lower support of the edge, namely the cell of Σ' underlying it in Gross's construction (Section 3.3); the monodromy around E is the transvection $x \mapsto x + \langle \alpha_E, x \rangle d_E$ for a covector α_E . The legs of an arc A have the same lower support, so their vanishing vectors agree up to sign; we write d_A for one of them (Section 5.1). The four-valent vertices of Γ are in bijection with Q : the vertex p_q lies on the two-cell of $\mathcal{P}$ over the node parallelogram of q , and after an isotopy supported in that two-cell it is a negative node in the sense of Castaño-Bernard and Matessi [CBM14, Def. 6.3] (Proposition 3.18). We write Λ for the local system of integral tangent vectors on $B \setminus \Gamma$ and $\iota: B \setminus \Gamma \rightarrow B$ for the inclusion.

Definition 2.4 (relative tropical 2-cycles). Let $\mathfrak{K}$ be a triangulation of B in which Γ is a full subcomplex; the barycentric subdivision of $\mathcal{P}$ is one (Lemma 3.13). For a simplex s of $\mathfrak{K}$ let $F(s)$ be the group of flat sections of Λ over $V \setminus \Gamma$ for a small neighbourhood V of the closed simplex $\bar{s}$; faces restrict. Let $C_*(\mathfrak{K}; F)$ be the simplicial chain complex with these coefficients, and $C_*(\Gamma; \mathcal{D}) \subset C_*(\mathfrak{K}; F)$ the subcomplex of chains on Γ whose coefficient on every edge E of $\mathfrak{K}$ in Γ lies in $\mathbb{Z}d_E$, where $d_E := \pm d_A$ for the arc A whose closure contains E (the sign does not affect $\mathbb{Z}d_E$). A *relative tropical 2-cycle* is a chain $z \in C_2(\mathfrak{K}; F)$ with $\partial z \in C_1(\Gamma; \mathcal{D})$. We write $Z(\mathfrak{K})$ for the group of relative tropical 2-cycles, and call $\partial z = \sum_E c_E d_E [E]$ the *boundary network* of z .

In words: a relative tropical 2-cycle is an integral 2-chain on B with coefficients in the sheaf $\iota_*\Lambda$ whose boundary lies on the discriminant, and runs along each leg in the direction of its vanishing vector. The coefficient of a triangle is an integral tangent vector of B that is invariant under the local monodromy wherever the triangle meets Γ .

We name the clauses by their content; Section 5.2 gives the dictionary to both versions of [CBM14].

Definition 2.5 (strict tropical 2-cycles). A *strict tropical 2-cycle* on B'_σ is a tropical 2-cycle (S, j, v) in the sense of Castaño-Bernard and Matessi [CBM14, Def. 7.2], applied to B'_σ with the following three conventions, which are explained in Section 5.2:

- (1) the embedding j is piecewise linear;
- (2) at each point where S crosses Γ transversally, the monodromy on $\ker v$ is conjugate in $GL_2(\mathbb{Z})$ to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (the crossing clause of [CBM14, Def. 7.2], which does not name the group);
- (3) a point p of S is a boundary vertex if and only if $j(p)$ is a three-valent vertex of Γ of negative type (the boundary-vertex clause); negative type is a condition on the local monodromies only (Definition 3.16).

Castaño-Bernard and Matessi allow a topological embedding j ; the triangulations used below require it to be piecewise linear. Triangulated, a strict tropical 2-cycle is a relative tropical 2-cycle (Section 5.2). Reading the crossing clause as conjugacy in $SL_2(\mathbb{Z})$ gives a smaller class of cycles: Theorem E below covers it as well, while Conjecture G asks for

more under that reading. The relative ones are allowed non-primitive coefficients, arbitrary branching along edges, crossings of Γ with any invariant coefficient, and boundary networks that pass a node in both directions or with multiplicity.

The boundary network ∂z is a 1-cycle of $C_*(\Gamma; \mathcal{D})$; such cycles are the *balanced networks*, and $\text{Bal}(\Gamma)$ denotes their group. For a balanced network $\beta = \sum_E c_E d_E [E] \in \text{Bal}(\Gamma)$ let $\tilde{\beta} = \sum_E c_E (e_{n'} - e_n) [E] \in C_1(\Gamma; \mathbb{Z}^R)$, where $d_E = n' - n$, be its *canonical lift*. The boundary of $\tilde{\beta}$ vanishes away from the nodes, and at p_q it is a multiple of the circuit (Lemma 5.5).

Definition 2.6 (node coefficients). For $\beta \in \text{Bal}(\Gamma)$ and $q \in Q$ the *node coefficient* $\rho_q(\beta) \in \mathbb{Z}$ is defined by $\partial \tilde{\beta}(p_q) = -\rho_q(\beta) \text{circ}_q$. For $z \in Z(\mathfrak{K})$ put $\rho(z) := \rho(\partial z) \in \mathbb{Z}^Q$.

Equivalently, the four legs at p_q have lower supports $\{a, b\}$, $\{b, c\}$, $\{c, d\}$ and $\{d, a\}$; if the coefficients of a balanced network on the legs over $\{a, b\}$ and $\{b, c\}$ are oriented away from p_q and written $c_{ab}(b - a)$ and $c_{bc}(c - b)$, then $\rho_q = c_{ab} - c_{bc}$ (Section 5.3). For a tropical 2-cycle S in the sense of [CBM14], $\rho(S) \in \{0, \pm 1\}^Q$ records the jump of the boundary field of S across each node that ∂S meets.

2.4. The mirror family and its vanishing spheres. Let $X_{\mathcal{T}}$ be the smooth projective toric fourfold of the fan over $\mathcal{T}$, and put $\mathcal{A} = \Delta \cap N$. The members of the mirror family are

$$Y_{t,c} = \left\{ \sum_{n \in \mathcal{A}} c_n t^{H(n)} x^n = 0 \right\} \subset X_{\mathcal{T}}, \quad 0 < t < 1,$$

where x^n is the Cox monomial of n , $c = (c_n)$ is a coefficient vector, and the height is $H = k_0 \varphi + h_C$: here φ equals 1 on $\partial \Delta$ and is linear on the cones over the faces of Δ , h_C is a strictly convex height with integral slopes inducing the *pulling refinement* $\mathcal{F}_C$ of Σ' (Section 3.5), and k_0 is an integer bounded below in terms of the lattice data and h_C (Lemma 7.7). As $t \rightarrow 0$ the members approach the maximally degenerate limit.

For a node parallelogram $P = abcd$ the *node binomial ratio* of c is $\mu_P = c_a c_c / (c_b c_d)$; the face polynomial $c_a x^a + c_b x^b + c_c x^c + c_d x^d$ is a product of two binomials exactly when $\mu_P = 1$. Let $\mathcal{Z}$ be the set of real coefficient vectors with $c_0 > 0 > c_n$ for $n \neq 0$ and $\mu_P = 1$ for every P . Every real c with this sign pattern is uniquely $c = c^{(0)} e^{\xi}$ with $c^{(0)} \in \mathcal{Z}$ and $\xi \in \text{span}_{\mathbb{R}}\{r_P\}$, where $r_P = e_a + e_c - e_b - e_d$, so that $r_P = -\text{circ}_q$ under $\mathbb{Z}^R \subseteq \mathbb{Z}^{\Delta \cap N}$; then $\log \mu_P(c) = r_P \cdot \xi$. For t small and c in the window of Theorem A below, in which every $|\log \mu_P|$ lies between two fixed small positive bounds, the defining section of $Y_{t,c}$, trivialised by dividing by its term $c_a t^{H(a)} x^a$, has, for each P , a unique nondegenerate critical point q_P in a polydisc $\mathbb{D}_P$ about a point of the torus orbit associated with the edge dual to P (Section 7.3). Its critical value for this trivialisation is small and nonzero, and $\delta_q \subset Y_{t,c}$ denotes its vanishing 3-sphere, oriented by the convention of Definition 7.23. In chain-boundary identities, $[\delta_q]$ denotes a chosen fundamental singular cycle of the oriented sphere; elsewhere the same notation denotes its homology class.

2.5. Results.

Theorem A (relations among the vanishing spheres). *Let Δ be admissible and satisfy (U), with a profile σ satisfying (Proj). There are $\eta_1 > 0$ and $C \geq 1$, depending only on Δ and σ through the vectors r_P , $P \in Q$ (Lemma 7.2 fixes such a choice), with the following property. For every $\eta_0 \in (0, \eta_1]$ and every compact set $\mathcal{C}_0 \subset \mathcal{Z}$ there is $t_0 > 0$ such that for $0 < t < t_0$ and every real coefficient vector c with $c_0 > 0 > c_n$ ($n \neq 0$), component $c^{(0)} \in \mathcal{C}_0$, and $\eta_0/C \leq |\log \mu_P(c)| \leq \eta_0$ for every P , the member $Y_t = Y_{t,c}$ is smooth and:*

- (1) (Relations.) For $k \in \mathbb{Z}^Q$, the class $\sum_q k_q [\delta_q]$ is a torsion element of $H_3(Y_t; \mathbb{Z})$ if and only if $k \in K$, and it is then zero. So the relation lattice $\text{Rel}(\delta) = \{k : \sum_q k_q [\delta_q] = 0\}$ equals K , and the classes $[\delta_q]$ span a free subgroup of $H_3(Y_t; \mathbb{Z})$, isomorphic to $\mathbb{Z}^Q/K$, of rank $|Q| - \text{rank } K$, which meets the torsion subgroup only in 0.
- (2) (Realisation.) Every $k \in K$ is the vector of node coefficients $\rho(z)$ of a relative tropical 2-cycle z on B'_σ , and z lifts to an integral 4-chain $\tilde{\Xi}(z)$ on Y_t with $\partial \tilde{\Xi}(z) = \sum_q k_q [\delta_q]$.
- (3) (The nodal locus.) In an open neighbourhood U of c in the space of complex coefficient vectors (Section 9.1), the coefficient vectors c' for which $Y_{t,c'}$ is singular at the critical point $q_P(c') \in \mathbb{D}_P$ for every $P \in Q$ (the point q_P above, which moves with c') form a nonempty complex submanifold of codimension $|Q| - \text{rank } K$, the relative Picard number of a projective small resolution $\hat{X} \rightarrow X'_\sigma$.

Morrison proposed that mirror symmetry exchanges the two sides of a conifold transition, with the number of conditions imposed by the nodes on one side equal to the relative Picard number on the other [Mor99, §3.1]. His setting is a conifold transition, so X'_σ is assumed smoothable there. Item (3) is this count in the space of coefficient vectors of the Batyrev family, for smoothable and non-smoothable X'_σ alike. The factoring locus of Section 9.6, cut out in U by the equations $\mu_P = 1$, that is $\eta_P = r_P \cdot \xi = 0$ with $\xi \in \mathbb{C}^A$ as in Definition 7.3, has the same codimension, since the r_P span a space of dimension $|Q| - \text{rank } K$; item (3) concerns the nodal locus itself, on which all $|Q|$ nodes are present. Item (1), with Proposition 3.5, identifies the relations on the two sides integrally: the exceptional curves of $\hat{X}$ and the vanishing spheres of Y_t have the same relation lattice K . For a projective conifold transition the relations among the vanishing spheres and among the exceptional curves are mutual annihilators [LLW18, Thm. 1.14]; item (1) is the statement that crosses the mirror.

Corollary B (smoothings and mirror relations). *Under the hypotheses of Theorem A the following are equivalent:*

- (a) X'_σ admits a smoothing;
- (b) the exceptional curves satisfy a relation $\sum_q \lambda_q [C_q] = 0$ in $H_2(\hat{X}; \mathbb{Q})$ with every $\lambda_q \neq 0$;
- (c) the vanishing spheres satisfy a relation $\sum_q k_q [\delta_q] = 0$ in $H_3(Y_t; \mathbb{Z})$ with every $k_q \neq 0$.

The spheres δ_q are smoothly isotopic to pairwise disjoint Lagrangian spheres L_q for the restriction of a toric Kähler form. If (a)–(c) hold, one of the $2^{|Q|}$ conifold transitions of Y_t in the L_q carries a symplectic structure in which the exceptional curves are symplectic.

Here a *smoothing* is a flat deformation, over a germ of a complex space, whose general fibre is smooth. For the nodal members of Theorem A(3), Corollary B answers the relation question of [BK10, §3]: a relation with all coefficients nonzero exists exactly when X'_σ is smoothable. The distinction between these nodal members and the factoring locus is explained in Section 9.6. For hexagon-free Δ satisfying (Proj), the same relation criterion decides the smoothability of X itself (Corollary 9.11). The equivalence of (a) and (b) is Namikawa's form [Nam02, Thm. 2.5] of the criterion of Friedman [Fri86, Rem. (4.5), Cor. (4.7)] (compare [RT09, Thms. 1.2 and 1.3]); the symplectic statement is [STY02, Thm. 2.9].

Theorem A is assembled from two results, which are the first two steps of the proof; the third step is a period computation.

Theorem C (node coefficients on the base). *Let Δ be admissible and satisfy (U), with a profile satisfying (Proj), and let $\mathfrak{K}_0$ be the barycentric subdivision of $\mathcal{P}$. Every balanced network on Γ is the boundary network of a relative tropical 2-cycle on $\mathfrak{K}_0$, and the node*

coefficients of balanced networks form exactly K :

$$\partial Z(\mathfrak{K}_0) = \text{Bal}(\Gamma), \quad \rho(\text{Bal}(\Gamma)) = K, \quad \text{hence} \quad \rho(Z(\mathfrak{K}_0)) = K.$$

For every triangulation $\mathfrak{K}$ of B in which Γ is full, $\rho(Z(\mathfrak{K})) \subseteq K$.

Theorem D (the mirror lift). *With the members Y_t of Theorem A, for every triangulation $\mathfrak{K}$ of B'_σ in which Γ is full and every $z \in Z(\mathfrak{K})$ there is an integral 4-chain $\tilde{\Xi}(z)$ on Y_t , constructed from z and finitely many auxiliary choices, with*

$$\partial \tilde{\Xi}(z) = \sum_q \rho_q(z) [\delta_q].$$

Remark 2.7 (comparison with strict tropical cycles). For a strict tropical 2-cycle (S, j, v) (Definition 2.5), the chain $\tilde{\Xi}(S)$ realises on Y_t the four-dimensional object $\tilde{S}$ that Castaño-Bernard and Matessi attach to S in a topological smoothing [CBM14, §7] in the following sense. Over the part of S away from the discriminant it is, up to homotopy in each region, the bundle whose fibre over a point is the two-torus annihilated by v in the dual torus $\Lambda^\vee \otimes U(1)$. Completing this local construction to a chain with the stated boundary uses the fibre-torus corrections of Proposition 8.26. Near the discriminant the local pieces agree with those of [CBM14] under two conditions on the position of S near the discriminant, stated in Remark 8.29, which gives the precise comparison.

The period computation (Section 9) turns the inclusion $K \subseteq \text{Rel}(\delta)$ given by Theorems C and D into equality and gives the nodal locus.

On the resolution side the same tropical objects produce the relations among the exceptional curves, for tropical 2-cycles in the sense of [CBM14].

Theorem E (resolution-side realisation). *Let Δ be admissible and satisfy (U), with a profile satisfying (Proj), and let $\hat{X}$ be a projective small resolution of X'_σ induced by a regular refinement of Σ' . Label the node parallelograms by its diagonals and give the exceptional curves C_q their complex orientations. For every strict tropical 2-cycle (S, j, v) on B'_σ there are a three-dimensional CW complex $\mathfrak{D}(S)$, an integral fundamental chain $[\mathfrak{D}]$, and a continuous map $f_S: \mathfrak{D}(S) \rightarrow \hat{X}$, constructed with finitely many auxiliary choices, with the following properties.*

Over the smooth sheets away from a neighbourhood of the discriminant, the domain is the circle bundle $\mathbb{R}v/\mathbb{Z}v$, completed at interior edges and vertices by balancing triangles and tetrahedra and near the discriminant by the local domains of Definition F.13. Its distinguished boundary components are two-spheres $\partial_q \mathfrak{D}$, one for each node on $j(\partial S)$, with

$$\partial[\mathfrak{D}] = \sum_q [\partial_q \mathfrak{D}], \quad f_S(\partial_q \mathfrak{D}) \subset C_q, \quad \deg(f_S|_{\partial_q \mathfrak{D}}) = -\rho_q(S).$$

For chosen fundamental singular cycles $[C_q]$, there is an integral singular 3-chain A supported in the exceptional curves such that

$$\Xi(S) = -(f_S)_*[\mathfrak{D}] + A, \quad \partial \Xi(S) = \sum_q \rho_q(S) [C_q].$$

For any other small resolution an integral bounding chain exists after negating $\rho_q(S)$ at each flopped curve; the two chains may be chosen to determine the same relative class on the common exterior of the node neighbourhoods. For the members Y_t of Theorem A, Theorem D gives a bounding 4-chain for the vanishing spheres with the same coefficients, using the corresponding diagonal labels and sphere orientations.

The construction of the base uses the pair $(\mathcal{T}, \check{h})$ of Hypothesis (U), whose existence is a combinatorial question about the polar polytope.

Let $\mathcal{L}_{\text{adm}}$ be the finite set of lattice-equivalence classes of the polytopes specified by the vertex arrays in the published certificate files (Definition A.1). These files give 51,827 pairwise inequivalent admissible reflexive four-polytopes, 40,403 without hexagonal two-faces.

Theorem F (unimodular heights on the finite list; computer-assisted). *For every $[\Delta] \in \mathcal{L}_{\text{adm}}$, Hypothesis (U) holds. Equivalently, the polar polytope Δ° has a fine, regular, unimodular triangulation that is star-shaped from the origin.*

The proof verifies explicit integer heights for every member of this fixed list in exact arithmetic (Appendix A). The assertion that the list contains every admissible polytope is separate.

Proposition 2.8 (classification completeness; computer-assisted). *The set $\mathcal{L}_{\text{adm}}$ is the set of all admissible reflexive four-polytopes up to $GL_4(\mathbb{Z})$.*

This proposition uses the Kreuzer–Skarke classification and the original exhaustive admissibility filter (Appendix A.3, Steps 1–2). Every accepted polytope was independently checked admissible, but the independent replay of the exclusions covers only the subsets specified in Appendix A.4. The positive height certificates proving Theorem F do not prove classification completeness.

Theorem F removes Hypothesis (U) for every member of $\mathcal{L}_{\text{adm}}$; together with Proposition 2.8, it removes that hypothesis for all admissible reflexive four-polytopes. Theorems A–E are stated independently of the classification assertion: they apply to any admissible Δ satisfying (U), with a profile satisfying (Proj) and the indicated geometric hypotheses. Thus if an admissible class were omitted from the list, the same theorems would apply once (U) were verified for it.

Finally, the strict tropical 2-cycles of [CBM14] are expected to suffice on their own.

Conjecture G (strict realisation). *Under the hypotheses of Theorem C, the node coefficients of strict tropical 2-cycles (S, j, v) on B'_σ (Definition 2.5) generate K .*

By Theorems D and E, Conjecture G would realise every relation of K on both sides by the same objects. Computations find strict tropical 2-cycles whose node coefficients generate K on the five examples of Section 11 and on 340 of the 364 hexagon-free polytopes with $K \neq 0$ of the list of Section 11.4 that satisfy (Proj).

For X_9 this is proved in the text (Lemma 11.2), given the triangulation $\mathcal{T}$ and the triangles of the cycle given in the accompanying data (Appendix G). For the other examples the boundary networks and node coefficients of the cycles are verified from the accompanying data. These data also verify the shape of the cycles that are discs with one coefficient meeting Γ only along their boundary (the belt cycles of Definition 11.1). For their other cycles, verification of the interior conditions of [CBM14, Def. 7.2] rests on the computations described in Section 11.5, as do the results on the 340 polytopes.

On the other 24 polytopes the cycles found generate a sublattice of K of corank between 1 and 4, while balanced networks of the shape that such cycles have along their boundary already generate K (Section 11.5).

2.6. Scope. Apart from Hypothesis (U), which Theorem F verifies on $\mathcal{L}_{\text{adm}}$, Hypothesis (Proj) is the only restriction on admissible polytopes in Theorems A–E. It is needed by the method: without it X'_σ is not projective over X , and the base B'_σ is not defined. It is

a genuine restriction. On the list of Section 11.4 (612 admissible polytopes up to $GL_4(\mathbb{Z})$) it holds for 364 of the 375 hexagon-free polytopes with $K \neq 0$; of the 142 polytopes with a hexagonal two-face it holds for some profile on 83 (55 with every hexagon on the A_2 branch, 20 with every hexagon on the A_1 branch, and 18 with a mixed profile, with overlaps), and for no profile on 59. Where it fails there is an explicit obstruction: a nonzero effective combination of the proper transforms $\tilde{f}_\epsilon \subset \hat{X}$ of curves in the fibres of $X'_\sigma \rightarrow X$ over the del Pezzo points that is homologous in $H_2(\hat{X}; \mathbb{Q})$ to a combination of the exceptional curves C_q (Proposition 3.29(3)).

Theorem A is stated for members with real coefficients of the given sign pattern near $\mathcal{Z}$, for t small. The classes $[\delta_q]$ are flat sections of the local system $H_3(Y_{t,c}; \mathbb{Z})$ over the members of the neighbourhood U of item (3) that lie off the nodal hypersurfaces, and this set is connected; so item (1) holds for all of these members, complex coefficients included, with δ_q at such a member oriented by transport from c (Section 9.2). Proposition 9.13 extends the integral comparison, including the torsion assertion, to small smoothings along every connected component of the nodal coefficient locus containing a nodal member of Theorem A(3), on the component of the node-marking and orientation cover through the initial marking. The nodal locus is smooth everywhere; on each reached component its codimension is $|Q| - \text{rank } K$. The comparison can fail on other components: Proposition 9.19 gives an X_9 component with four independent vanishing spheres, whereas K has rank one. For X_9 , Theorem 9.21 classifies the entire codimension-three part as one reached component; the number of codimension-four components and the component classification for other polytopes remain open. Theorem 9.25 identifies the geometric reverse transition for X_9 up to birational equivalence with the Batyrev partner family. The tropical bounding chains are constructed near the limit and continued by transport.

Part I. The base and its tropical cycles

3. THE PARTIAL RESOLUTION AND GROSS'S BASE

This section constructs the two objects on which the rest of the paper works: the crepant partial resolution X'_σ , with the lattice K of relations among its exceptional curves, and Gross's integral affine base $B = B'_\sigma$, with its discriminant Γ . It proves the properties of both that later sections use.

Proposition 3.4 shows that X'_σ has one ordinary double point for each node parallelogram, and that in every small resolution the class of the exceptional curve C_q is determined by its intersection numbers with the divisors D_r , $r \in R$, which are the entries of the circuit circ_q ; so the rational relations among the $[C_q]$ are exactly K . The proof compares an arbitrary small resolution with a maximal projective crepant partial resolution of X and does not use (Proj). Proposition 3.5 shows that $H_2(\hat{X}; \mathbb{Z})$ is torsion-free and detected by the D_r , so the integral relations are exactly K as well; its proof combines a Lefschetz theorem for the affine part of X (Lemma 3.7) with a lattice property of admissible polytopes (Lemma 3.6) and duality. Proposition 3.9 describes the curves C_q inside the exceptional del Pezzo surface over a pentagon, an A_1 hexagon and an A_2 hexagon, and Remark 3.10 records the resulting linear conditions (*ties*) on the coordinates of K . On the base, Proposition 3.18 shows that Γ is a graph whose four-valent vertices are negative nodes in the sense of Castaño-Bernard and Matessi, one for each node parallelogram; Lemma 3.13 shows that Γ is a full subcomplex of the barycentric subdivision of Gross's decomposition, and Lemma 3.21 describes Γ inside the two-face of a pentagon or hexagon. Proposition 3.26 constructs the subdivision $\mathcal{F}_C$ of $\partial\Delta$ used on

the mirror side and proves that it satisfies a height condition at every node parallelogram. The proof rests on a count of lattice points in a pyramid over a unit parallelogram (Lemma 3.24). Section 3.6 identifies (Proj) with the projectivity of $X'_\sigma \rightarrow X$ and describes an explicit obstruction to it.

Throughout, Δ is admissible (Definition 2.1) and σ is a profile (Definition 2.3). Hypothesis (Proj) is assumed in Sections 3.3–3.5, and in Section 3.2 where stated; Section 3.6 does not assume it. From Section 3.3 on, Δ satisfies Hypothesis (U), and the pair $(\mathcal{T}, \check{h})$ is the one fixed in Section 2.3.

3.1. Admissible two-faces and profiles. Let G be a dP_7 pentagon or a dP_6 hexagon, with $k = 5$ or 6 vertices $w_0, \dots, w_{k-1}$ in cyclic order and centre o ; indices of the w_j are taken modulo k . For each j put

$$T_j = \text{conv}\{o, w_j, w_{j+1}\}, \quad P_j = \text{conv}\{o, w_j, w_{j+1}, w_{j+2}\}.$$

Up to $GL_2(\mathbb{Z})$ and translation there is exactly one lattice pentagon and exactly one lattice hexagon with one interior lattice point and no boundary lattice points other than their vertices, since the lattice polygons with exactly one interior lattice point form sixteen classes [PRV00]. The hexagon is the one of Section 2.2. The pentagon has, in suitable coordinates, the vertices

$$(3.1) \quad w_0, \dots, w_4 = (1, 0), (1, 1), (0, 1), (-1, 0), (0, -1), \quad o = (0, 0).$$

- Lemma 3.1** (subdivisions of the del Pezzo faces). (1) *Every triangle T_j is unimodular, every segment $[o, w_j]$ is primitive, and P_j is a unit parallelogram if and only if $w_j + w_{j+2} = o + w_{j+1}$. For the hexagon this holds for every j ; for the pentagon (3.1) it holds exactly for $j \in \{4, 0, 1\}$.*
- (2) *The subdivisions of G into unit parallelograms and unimodular triangles all of which contain o are the subdivisions into cells T_j and P_j whose edges at o are consecutive segments $[o, w_j]$.*
- (3) *A pentagon has exactly one such subdivision with two parallelograms; in the coordinates (3.1) it consists of P_4 , P_1 and T_3 . A hexagon has exactly three such subdivisions in which two parallelograms meet only in o , namely $\{P_j, T_{j+2}, P_{j+3}, T_{j+5}\}$ for $j \in \{0, 1, 2\}$ (the A_1 subdivisions of Definition 2.3), and exactly two with three parallelograms, namely $\{P_j : j \equiv j_0 \pmod{2}\}$ for $j_0 \in \{0, 1\}$ (the A_2 subdivisions).*
- (4) *Every cell of these subdivisions, and every face of such a cell, has no lattice points other than its vertices.*

Proof. (1) By Pick's formula G has normalised area $2 \cdot 1 + k - 2 = k$. It is the union of the k triangles T_j , whose normalised areas are positive integers; so each is 1. An interior lattice point of $[o, w_j]$ would be a second interior lattice point of G . If $w_j + w_{j+2} = o + w_{j+1}$, then o, w_j, w_{j+1}, w_{j+2} are the vertices of a parallelogram in this cyclic order, the union of the unimodular triangles T_j and T_{j+1} ; conversely, in a parallelogram with vertices in this cyclic order the sums of opposite vertices agree. For the hexagon the relation is stated in Section 2.2. For (3.1), $w_4 + w_1 = w_0$, $w_0 + w_2 = w_1$ and $w_1 + w_3 = w_2$, while $w_2 + w_4 = 0 \neq w_3$ and $w_3 + w_0 = 0 \neq w_4$.

(2) Unit parallelograms and unimodular triangles contain no lattice points besides their vertices, so o is a vertex of every cell, and the other vertices are among the w_j . The edges of the subdivision at o are segments $[o, w_j]$; they cut G into sectors, and a cell containing o is the sector between two consecutive such edges $[o, w_j]$ and $[o, w_{j+m}]$, because it contains the part of ∂G between w_j and w_{j+m} . That sector is $\text{conv}\{o, w_j, \dots, w_{j+m}\}$, a polygon with

$m + 2$ vertices. So $m = 1$ and the cell is T_j , or $m = 2$ and the cell is P_j , which must then be a unit parallelogram.

(3) By (2) the subdivisions correspond to the ways of cutting the cyclic sequence of the k boundary edges of G into runs of length 1 (a triangle) and 2 (a parallelogram P_j , allowed only where the relation of (1) holds). For the pentagon with two parallelograms the runs are 2, 2, 1, starting at j , $j + 2$ and $j + 4$; by (1) both j and $j + 2$ must lie in $\{4, 0, 1\}$, which forces $j = 4$. For the hexagon every run of length 2 is allowed. Two parallelograms meeting only in o are separated on both sides by a triangle, so the runs are 2, 1, 2, 1 starting at $j, j + 2, j + 3, j + 5$, and j is determined modulo 3. Three parallelograms fill the hexagon with runs 2, 2, 2, determined by the parity of j .

(4) This holds for unimodular triangles, unit parallelograms, the edges of G , which are primitive by admissibility, and the segments $[o, w_j]$, which are primitive by (1). $\square$

The remaining subdivisions of the hexagon in (2), with no parallelogram, with one, or with two parallelograms sharing an edge, do not occur in profiles. Up to rotation, the A_1 and A_2 subdivisions are the local models $W_{6,1}$ and $W_{6,2}$ of [Wue26b, Prop. 3.7(2),(3)]. The pentagon subdivision gives the two-node model described in Section 3.2.

By Lemma 3.1(4) and admissibility, the lattice points of Δ on faces of dimension at most two are the vertices of Δ and the centres of the pentagons and hexagons, which is the set R of Section 2.2; each of them is a vertex of the subdivision that σ induces on every two-face containing it. For a two-face G that is not a triangle we write $G^\circ = [u, u']$; since $\ell(G^\circ) = 1$, u and u' are lattice points and vertices of Δ° , and the facets $u^\vee = \{n \in \Delta : \langle u, n \rangle = -1\}$ and $u'^\vee$ of Δ satisfy $u^\vee \cap u'^\vee = G$. We write $N_G := \{u, u'\}^\perp \cap N$ for the lattice of the plane of G . Put

$$e_{u'}(n) := \langle u', n \rangle + 1, \quad e_u(n) := \langle u, n \rangle + 1 \quad (n \in N).$$

Both are nonnegative on Δ ; $e_{u'}$ vanishes on $u^\vee$ exactly along G .

Lemma 3.2 (the dual edge). *Let G be a two-face of Δ that is not a triangle, with $G^\circ = [u, u']$. Then $\mathbb{Z}u + \mathbb{Z}u'$ is a saturated sublattice of M . Hence there is $w_{u'} \in N$ with $\langle u, w_{u'} \rangle = 0$ and $\langle u', w_{u'} \rangle = 1$, and $u^\perp \cap N = N_G \oplus \mathbb{Z}w_{u'}$. In particular $e_{u'}$ is a primitive affine function on the lattice of $\text{aff}(u^\vee)$.*

Proof. Let $L = (\mathbb{R}u + \mathbb{R}u') \cap M$, a saturated sublattice of rank two. Since $\ell(G^\circ) = 1$, $u' - u$ is primitive in M , hence in L ; complete it to a basis $(u' - u, f)$ of L and write $u = x(u' - u) + yf$ with $x, y \in \mathbb{Z}$. For a vertex a of G , $\langle u, a \rangle = \langle u', a \rangle = -1$, so $\langle u' - u, a \rangle = 0$ and $-1 = y\langle f, a \rangle$; hence $y = \pm 1$, and $(u' - u, u)$ is a basis of L . So $L = \mathbb{Z}u + \mathbb{Z}u'$. A saturated sublattice spanned by u, u' extends to a basis of M , and the dual basis contains a vector $w_{u'}$ as stated. For $n \in u^\perp \cap N$ the vector $n - \langle u', n \rangle w_{u'}$ lies in N_G . $\square$

The vectors $w_{u'}$ (and w_u , with u and u' exchanged) are the ones used at the node points in Section 7.1. The lemma uses only reflexivity and $\ell(G^\circ) = 1$. The lattice point a on the level -1 of both u and u' is needed: $u = (1, 0)$ and $u' = (2, 2)$ have primitive difference and span a sublattice of index two.

3.2. The partial resolution, the small resolution and K . *The singular points of X .* For a two-face G of Δ let $\tau_G = \text{cone}(G)$. Since G lies in the hyperplane $\{\langle u, \cdot \rangle = -1\}$, the affine toric variety of τ_G is $U_G \times \mathbb{C}^*$, where U_G is the three-dimensional Gorenstein affine toric variety of τ_G in the lattice $N \cap \text{span}(G)$: the cone over the toric surface of the polygon G , embedded by its anticanonical class. The orbit $O(\tau_G) \cong \mathbb{C}^*$ is the curve $\{0\} \times \mathbb{C}^*$. As X is Δ° -regular, it meets $O(\tau_G)$ transversally in $\ell(G^\circ)$ points [Bat94, §3]: the equation of X restricted

to the orbit is a Laurent polynomial in one variable with simple zeros. If G is not a triangle, then $\ell(G^\circ) = 1$ by Definition 2.1, and we write x_G for this point. (For a triangle, $\ell(G^\circ)$ can exceed one: in the cross-polytope $\text{conv}\{\pm e_1, \dots, \pm e_4\}$ the triangle $\text{conv}(e_1, e_2, e_3)$ has dual edge $\{(-1, -1, -1, s) : |s| \leq 1\}$ of length two.) By the implicit function theorem, near x_G the hypersurface X is the graph of a holomorphic function $U_G \rightarrow \mathbb{C}^*$, so the projection to U_G identifies a neighbourhood of x_G in X with a neighbourhood of the vertex of U_G . Thus x_G is a node when G is a unit square, a cone over dP_7 when G is a pentagon, and a cone over dP_6 when G is a hexagon. The cones over the edges and the unimodular triangles of Δ are generated by parts of lattice bases, so $\mathbb{P}_\Delta$ is smooth along their orbits and so is X ; and X misses the torus-fixed points of $\mathbb{P}_\Delta$, since its equation restricted to such a point is the coefficient of a vertex of Δ° , which is nonzero. So the singular points of X are the points x_G , one for each square, pentagon and hexagon.

The partial resolution. The profile σ subdivides τ_G into the cones over its cells; let $U'_{G,\sigma} \rightarrow U_G$ be the corresponding toric morphism. The *partial resolution* $X'_\sigma \rightarrow X$ replaces a neighbourhood of each x_G by the corresponding neighbourhood of the fibre over the vertex in $U'_{G,\sigma}$, and is an isomorphism elsewhere; since the x_G are isolated, this defines a compact complex space. Under (Proj) it is the proper transform of X in P'_σ of Section 2.2: the cones of the fan of P'_σ in the cone over a face of dimension at most two are the cones over the cells of σ , and X meets no orbit of a cone that is not contained in such a cone. Similarly, a *small resolution* $\hat{X} \rightarrow X'_\sigma$ is locally, at each node, one of the two small resolutions of the cone $\{xy = zw\}$. At the node of the parallelogram $abcd$ they are the toric resolutions given by the triangles abc, acd (the diagonal $\{a, c\}$) or abd, bcd (the diagonal $\{b, d\}$). So near the preimage of x_G , a small resolution is the toric resolution of U_G given by the triangulation of G obtained from σ by one diagonal in each parallelogram. Every small resolution is a compact complex manifold with trivial canonical class, since $X'_\sigma \rightarrow X$ and $\hat{X} \rightarrow X'_\sigma$ are crepant and $K_X = 0$ [Bat94, §4].

The divisors D_r . For a vertex r of Δ let $D_r \subset \hat{X}$ be the proper transform of the surface $X \cap D_r^\Delta$, where D_r^Δ is the toric divisor of $\mathbb{P}_\Delta$ of the ray through r ; for the centre o of a pentagon or hexagon G let $D_o \subset \hat{X}$ be the exceptional surface over x_G . Near the preimage of x_G , D_r is the toric divisor of the ray through r in the local model, for every $r \in R \cap G$. As $\hat{X}$ is smooth, each D_r is a Cartier divisor, with a class $D_r \in H^2(\hat{X}; \mathbb{Z})$. When $\hat{X}$ is the proper transform of X in a toric variety refining the fan of $\mathbb{P}_\Delta$, D_r is the restriction of the toric divisor of the ray through r ; toric divisors of rays not through points of R restrict to zero, because their orbits lie over torus-fixed points of $\mathbb{P}_\Delta$.

Lemma 3.3. *For every small resolution $\hat{X}$ of X'_σ the classes D_r , $r \in R$, span $H^2(\hat{X}; \mathbb{Q})$. Hence*

$$(3.2) \quad H_2(\hat{X}; \mathbb{Q}) \longrightarrow \mathbb{Q}^R, \quad C \longmapsto (D_r \cdot C)_{r \in R},$$

is injective. This does not require (Proj).

The proof is in Appendix B.

Proposition 3.4. *Let Δ be admissible and σ a profile.*

- (1) X'_σ has exactly one singular point for each node parallelogram, an ordinary double point, and is smooth elsewhere. Under (Proj), $X'_\sigma \rightarrow X$ is induced by the projective toric morphism $P'_\sigma \rightarrow \mathbb{P}_\Delta$.
- (2) Let $\hat{X}$ be a small resolution of X'_σ . If $\hat{X}$ uses the diagonal $\{a, c\}$ at the node q of the parallelogram $abcd$, the image of $[C_q]$ under (3.2) is $\text{circ}_q = e_b + e_d - e_a - e_c$; if it uses

$\{b, d\}$, it is $-\text{circ}_q$. Consequently, for the small resolution with the chosen diagonals,

$$\left\{ \lambda \in \mathbb{Z}^Q : \sum_q \lambda_q [C_q] = 0 \text{ in } H_2(\widehat{X}; \mathbb{Q}) \right\} = K,$$

and the classes $[C_q]$ span a subspace of $H_2(\widehat{X}; \mathbb{Q})$ of dimension $|Q| - \text{rank } K$. When $\widehat{X}$ is projective, this is the relative Picard number of $\widehat{X} \rightarrow X'_\sigma$.

- (3) Assume (Proj). The small resolution with the chosen diagonals is projective if and only if no nonzero $\lambda \in K$ has all $\lambda_q \geq 0$; it is then the proper transform of X in the toric variety of a regular refinement of Σ' . The diagonals at the squares can always be chosen so that this holds.

Proof. (1) Let $P = abcd$ be a node parallelogram in the two-face G . The cone over P is generated by four vectors with $a + c = b + d$ that span a saturated sublattice of rank three, so its affine toric variety is $\{xy = zw\} \times \mathbb{C}^*$. In the local model $U'_{G,\sigma}$ over x_G it contributes the torus-fixed point of the cone over P , an ordinary double point; the other cells of $\sigma|_G$ are unimodular triangles and primitive segments (Lemma 3.1(4)), whose cones are smooth. Over the squares, $X'_\sigma = X$ near x_G , which is a node. The other points of X'_σ lie over smooth points of X . The second sentence was shown above.

(2) Near C_q , $\widehat{X}$ is the small resolution of the cone $\{xy = zw\}$ given by the three-dimensional cones $\text{cone}(a, b, c)$ and $\text{cone}(a, c, d)$, and C_q is the curve of their common wall $\text{cone}(a, c)$. The divisors $D_a, \dots, D_d$ are there the toric divisors of the local model, and the wall relation $b + d - a - c = 0$ gives $D_b \cdot C_q = D_d \cdot C_q = 1$ and $D_a \cdot C_q = D_c \cdot C_q = -1$. A divisor D_r with $r \notin \{a, b, c, d\}$ is disjoint from C_q , because in the local model the orbit closure of the ray through r contains the orbit of $\text{cone}(a, c)$ only if $r \in \{a, c\}$, and the other orbits of $\widehat{X}$ over x_G that meet C_q are those of $\text{cone}(a, b, c)$ and $\text{cone}(a, c, d)$. Hence $[C_q] \mapsto \text{circ}_q$, and the other diagonal exchanges the roles of $\{a, c\}$ and $\{b, d\}$. Since (3.2) is injective (Lemma 3.3), $\sum_q \lambda_q [C_q] = 0$ if and only if $\sum_q \lambda_q \text{circ}_q = 0$, and the span of the $[C_q]$ is isomorphic to that of the circ_q , of dimension $|Q| - \text{rank } K$. When $\widehat{X}$ is projective, the divisors D_r span $H^2(\widehat{X}; \mathbb{Q})$ (Lemma 3.3), so a rational combination of curves is numerically trivial if and only if its class in $H_2(\widehat{X}; \mathbb{Q})$ vanishes; hence the span of the contracted curves in $N_1(\widehat{X}/X'_\sigma) \otimes \mathbb{Q}$ has the same dimension.

(3) If an ample class L exists on $\widehat{X}$, then $L \cdot C_q > 0$ for all q , and a relation $\sum_q \lambda_q [C_q] = 0$ with $\lambda \geq 0$ forces $\lambda = 0$. Conversely, suppose no nonzero $\lambda \geq 0$ lies in K . By Gordan's alternative there is $\psi \in \mathbb{Q}^R$ with $\langle \text{circ}_q, \psi \rangle > 0$ for every q ; extend ψ by 0 to the other lattice points of $\partial\Delta$. Let h be a strictly convex function inducing Σ' . For $\epsilon > 0$ small, the strict inequalities by which h exceeds its linear piece on the cone over a cell of Σ' at the lattice points outside that cell persist for $h + \epsilon\psi$, so the subdivision it induces refines Σ' and subdivides each cell ω of Σ' by the regular subdivision of $\omega \cap N$ induced by $\psi|_\omega$. This subdivision may use lattice points of ω that are not vertices of ω ; these lie in the relative interiors of facets of Δ , and the proper transform of X meets no orbit of a cone containing one of them, since such an orbit lies over a torus-fixed point of $\mathbb{P}_\Delta$. A unimodular triangle or a primitive segment is not subdivided, and ψ subdivides a parallelogram $abcd$ by the diagonal $\{a, c\}$ if and only if $\psi(a) + \psi(c) < \psi(b) + \psi(d)$, that is, $\langle \text{circ}_q, \psi \rangle > 0$. The toric variety of the refined fan is projective, it is smooth along the orbits that X meets, and the proper transform of X in it is the small resolution with the chosen diagonals. For the last sentence take ψ equal to -1 at the centres and to small generic values at the vertices of Δ , and choose at each square

the diagonal $\{a, c\}$ with $\langle \text{circ}_q, \psi \rangle > 0$. At a parallelogram of a pentagon or hexagon, whose chosen diagonal passes through the centre, $\langle \text{circ}_q, \psi \rangle$ is 1 plus a small number. $\square$

When every singular two-face is a unit square, the identification of the classes $[C_q]$ with the circuits is stated in [BK10, §2]. The relations hold integrally (Proposition 3.5 below).

Integral relations among the exceptional curves. Proposition 3.4(2) determines the relations among the classes $[C_q]$ in rational homology. They hold integrally, because $H_2(\hat{X}; \mathbb{Z})$ is torsion-free and is detected by the divisors D_r . Let $T = N \otimes \mathbb{C}^*$ be the dense torus of $\mathbb{P}_\Delta$ and $W = X \cap T$. The singular points x_G of X lie on orbits of cones over two-faces, so W lies in the locus over which $\hat{X} \rightarrow X$ is an isomorphism, and we regard W as an open subset of $\hat{X}$. It is the affine hypersurface in T of a Laurent polynomial with Newton polytope Δ° .

Proposition 3.5 (integral relations among the exceptional curves). *Let Δ be admissible, σ a profile and $\hat{X}$ any small resolution of X'_σ , with the node parallelograms labelled by the diagonals of $\hat{X}$. Then:*

(1) *the homomorphism*

$$(3.3) \quad \text{ev}_D: H_2(\hat{X}; \mathbb{Z}) \longrightarrow \mathbb{Z}^R, \quad C \longmapsto (D_r \cdot C)_{r \in R},$$

is injective; in particular $H_2(\hat{X}; \mathbb{Z})$ is torsion-free;

(2) $\{\lambda \in \mathbb{Z}^Q : \sum_q \lambda_q [C_q] = 0 \text{ in } H_2(\hat{X}; \mathbb{Z})\} = K$.

Hypothesis (Proj) is not required, and $\hat{X}$ need not be projective.

Part (2) follows from part (1): by the proof of Proposition 3.4(2), $\text{ev}_D[C_q] = \text{circ}_q$, a local toric computation that is integral as it stands, so $\text{ev}_D(\sum_q \lambda_q [C_q]) = \sum_q \lambda_q \text{circ}_q$, which vanishes if and only if $\lambda \in K$. Part (1) also recovers the injectivity of (3.2). With Theorem A(1), the exceptional curves of $\hat{X}$ and the vanishing spheres of Y_t have the same relation lattice K , integrally.

The proof of part (1) rests on two lemmas. The first is a lattice property of admissible polytopes. For a two-face G of Δ let $L_G = \text{span}_{\mathbb{R}}(G) \cap N$, a saturated sublattice of rank three, and call three vertices of G a *unimodular triple* if they span a triangle of normalised area one in the affine lattice $\text{aff}(G) \cap N$.

Lemma 3.6 (unimodular triples). *Every two-face G of an admissible Δ has a unimodular triple y_1, y_2, y_3 . Every unimodular triple of G is a basis of L_G and extends to a basis of N . Consequently the classes $y_i \wedge n$, $i = 1, 2, 3$, $n \in N$, span $\Lambda^2 N$.*

The second lemma is a Lefschetz theorem for the hypersurface W and for its intersections with the torus orbits of the vertices. It follows from Oka's theorem on the complement of a nondegenerate hypersurface. A Laurent polynomial g is *nondegenerate* if for every face Θ of its Newton polytope, including the polytope itself, the truncation g_Θ and its logarithmic derivatives have no common zero in the torus; this is equivalent to Batyrev's regularity condition [Bat94, §3] and to Oka's global non-degeneracy [Oka80, Def. (5.1)].

Lemma 3.7. *Let T' be an algebraic torus of dimension $n \geq 2$, let g be a nondegenerate Laurent polynomial on T' whose Newton polytope has dimension n , and let $W' = \{g = 0\}$. Then the inclusion $j: W' \rightarrow T'$ induces isomorphisms $j_*: H_k(W'; \mathbb{Z}) \rightarrow H_k(T'; \mathbb{Z})$ for $k \leq n - 2$. In particular W' is connected.*

The proofs of the two lemmas and of part (1) are in Appendix B. Part (1) is proved in three steps. Step A uses the two lemmas to show that $H_2(W; \mathbb{Z}) \rightarrow H_2(\hat{X}; \mathbb{Z})$ is zero; Step B deduces from the exact sequence of the pair $(\hat{X}, W)$ that $H_2(\hat{X}; \mathbb{Z})$ injects into $H_2(\hat{X}, W; \mathbb{Z})$; and Step C identifies $H_2(\hat{X}, W; \mathbb{Z})$ with $H^4(D; \mathbb{Z}) = \mathbb{Z}^R$, D the union of the divisors D_r , by duality for the compact subset D , so that the composite is $\pm \text{ev}_D$.

Remark 3.8. The class of a relation $\sum_q \lambda_q [C_q]$, $\lambda \in K$, is torsion by Proposition 3.4(2), so Proposition 3.5(2) is equivalent to the vanishing of the torsion of $H_2(\hat{X}; \mathbb{Z})$ on these classes. For projective $\hat{X}$ it can also be derived from the computation of Batyrev and Kreuzer. By universal coefficients $\text{Tors } H_2(\hat{X}; \mathbb{Z})$ is dual to $\text{Tors } H^3(\hat{X}; \mathbb{Z})$, and for a smooth projective crepant resolution the latter is the Brauer group, isomorphic to $\text{Hom}(\Lambda^2 N / (N \wedge N''), \mathbb{Q}/\mathbb{Z})$ with N'' the sublattice generated by the lattice points of Δ on faces of dimension at most one [BK06, Cor. 3.9]. For admissible Δ , N'' is generated by the vertices, and Lemma 3.6 gives $N \wedge N'' = \Lambda^2 N$. Step A is the topological counterpart of this computation, and it does not require $\hat{X}$ to be projective.

The exceptional surfaces. Let G be a pentagon or a hexagon, and let $\hat{X}$ use the diagonal through the centre o at each parallelogram of G . The triangulation of G in the local model is then the star subdivision into the triangles T_j . Put $E = E_G := D_o$, the fibre of $\hat{X} \rightarrow X$ over x_G , and let $E' = E'_G \subset X'_\sigma$ be the fibre of $X'_\sigma \rightarrow X$ over x_G . For each j let $f_j := D_{w_j} \cap E$. We use the marking of dP_6 of [Wue26b],

$$(3.4) \quad f_0 = e_1, \quad f_1 = H_E - e_1 - e_2, \quad f_2 = e_2, \quad f_3 = H_E - e_2 - e_3, \quad f_4 = e_3, \quad f_5 = H_E - e_1 - e_3,$$

with H_E the pullback of the class of a line and e_1, e_2, e_3 the exceptional classes of $\text{Bl}_3 \mathbb{P}^2$, and for the pentagon in the coordinates (3.1)

$$(3.5) \quad f_0 = e_1, \quad f_1 = H_E - e_1 - e_2, \quad f_2 = e_2, \quad f_3 = H_E - e_2, \quad f_4 = H_E - e_1$$

on $\text{Bl}_2 \mathbb{P}^2$.

Proposition 3.9 (the classical description). *With these conventions, E is the smooth toric surface of the complete fan in N_G with rays $w_j - o$, and its boundary curves are the f_j .*

- (1) Pentagon. $E \cong \text{Bl}_2 \mathbb{P}^2$ with the marking (3.5). The node parallelograms are P_4 and P_1 , and their exceptional curves are $C_{q_1} = f_0 = e_1$ and $C_{q_2} = f_2 = e_2$, two disjoint (-1) -curves. $E' \cong \mathbb{P}^2$, and the two nodes lie at two of its three torus-fixed points.
- (2) A_1 hexagon. $E \cong \text{Bl}_3 \mathbb{P}^2$ with the marking (3.4). For the A_1 subdivision with parallelograms P_j and P_{j+3} , the exceptional curves are f_{j+1} and f_{j+4} , two opposite, hence disjoint, (-1) -curves. We label the nodes so that the middle vertex of q_1 has even index; then $C_{q_1} = e_i$ and $C_{q_2} = H_E - e_k - e_l$ with $\{i, k, l\} = \{1, 2, 3\}$. $E' \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the two nodes lie at two opposite torus-fixed points.
- (3) A_2 hexagon. $E \cong \text{Bl}_3 \mathbb{P}^2$ with the marking (3.4). For the A_2 subdivision $\{P_j : j \equiv j_0 \pmod{2}\}$ the exceptional curves are the three pairwise disjoint (-1) -curves f_{j+1} : for $j_0 = 1$ they are $C = e_2, e_3, e_1$ (for $j = 1, 3, 5$), and for $j_0 = 0$ the lines $C = H_E - e_1 - e_2, H_E - e_2 - e_3, H_E - e_1 - e_3$ (for $j = 0, 2, 4$). $E' \cong \mathbb{P}^2$, and the three nodes are its three torus-fixed points.

The proof is in Appendix B.

The two hexagon cases are [Wue26b, Prop. 3.7(2),(3)]. The subdivision $W_{6,1}$ there is the A_1 subdivision with $j = 2$, written with the representative $j = 5$: its parallelogram $P_5 \ni w_0$ is q_1 , with $C_{q_1} = f_0 = e_1$, and $C_{q_2} = f_3 = H_E - e_2 - e_3$. The subdivision $W_{6,2}$ is the A_2

subdivision with $j_0 = 1$. The labelling of q_1, q_2 in (2) matters for signs only: the rotation of the hexagon by three positions is the Cremona involution of dP_6 , which changes the sign of the class ℓ below.

Remark 3.10 (the ties). The centre o of a pentagon or hexagon G lies on no other face of dimension at most two, so it occurs only in the circuits of the node parallelograms in G , each time with coefficient -1 (the chosen diagonal passes through o). Hence every $\lambda \in K$ satisfies

$$\sum_{q \subset G} \lambda_q = 0,$$

a condition on two coordinates at a pentagon or an A_1 hexagon and on three at an A_2 hexagon. The same condition follows from $D_o|_E = K_E$, which holds by adjunction because $K_{\widehat{X}} = 0$: the curves C_q with $q \not\subset G$ are disjoint from E , each C_q with $q \subset G$ is a (-1) -curve of E , so $K_E \cdot C_q = -1$, and pairing a relation with D_o gives $-\sum_{q \subset G} \lambda_q = 0$. By Proposition 3.9, the part of a relation at G is a class in a root lattice of E :

- (1) at a pentagon, $\lambda_{q_1}[C_{q_1}] + \lambda_{q_2}[C_{q_2}] = \lambda_{q_1}(e_1 - e_2) \in \mathbb{Z}(e_1 - e_2)$;
- (2) at an A_1 hexagon, $\lambda_{q_1}[C_{q_1}] + \lambda_{q_2}[C_{q_2}] = \lambda_{q_1} \ell$ with $\ell = e_i - (H_E - e_k - e_l) = -H_E + e_1 + e_2 + e_3$, the A_1 root line of dP_6 ; the three A_1 subdivisions give the same line;
- (3) at an A_2 hexagon, $\sum_{q \subset G} \lambda_q[C_q] = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3$ with $\mu_1 + \mu_2 + \mu_3 = 0$, where $(\mu_1, \mu_2, \mu_3) = (\lambda_{q_5}, \lambda_{q_1}, \lambda_{q_3})$ for $j_0 = 1$ and $(\lambda_{q_2}, \lambda_{q_4}, \lambda_{q_0})$ for $j_0 = 0$, indexing the node q_j of P_j by j ; for $j_0 = 0$ the multiples of H_E cancel by the tie. The local tie plane $\{\lambda \in \mathbb{Z}^3 : \sum \lambda = 0\}$ maps isomorphically onto the A_2 root lattice $\{\sum \mu_i e_i : \sum \mu_i = 0\}$.

In each case every λ_q with $q \subset G$ is nonzero if and only if the local class is nonzero on every root coordinate: $\lambda_{q_1} \neq 0$ at a pentagon or an A_1 hexagon, and $\mu_1 \mu_2 \mu_3 \neq 0$ at an A_2 hexagon. When every singular point of X is a node or a dP_7 point, that is, when Δ has no hexagonal two-face, the criterion [Wue26b, Thm. 4.7] therefore says that X has a convergent one-parameter smoothing if and only if some $\lambda \in K$ has all $\lambda_q \neq 0$; the proof of Corollary 9.11 verifies the hypotheses of [Wue26b, Thm. 4.7] and the identification of its criterion with this condition, and that part of the proof does not use (Proj). For the nodal threefold X'_σ , with any profile, the corresponding criterion is Corollary B.

3.3. Gross's base. From now on until the end of Section 3.5 we assume (Proj), and Σ' denotes the subdivision of $\partial\Delta$ it provides. Let $h: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be a function that is piecewise linear and strictly convex on the fan over Σ' (convex, with the cones over the cells of Σ' as its maximal domains of linearity) and vanishes at 0; it exists by (Proj).

Heights. Let $\mathcal{T}$ and $\check{h}$ be as in Hypothesis (U): $\mathcal{T}$ is a unimodular triangulation of $\partial\Delta^\circ$ refining its faces, so that its vertex set is $\partial\Delta^\circ \cap M$ and the cone over each tetrahedron is generated by a basis of M , and $\check{h}$ is an integral piecewise linear function on the fan over $\mathcal{T}$ with $\check{h}(0) = 0$. Let $\check{\varphi}$ equal 1 on $\partial\Delta^\circ$ and be linear on the cones over the faces of Δ° . The two conditions of Hypothesis (U) are

- (G1) $\check{h}$ is strictly convex on the fan over $\mathcal{T}$;
- (G2) $\check{h}' := \check{h} - \check{\varphi}$ is strictly convex on the fan over $\mathcal{T}$.

Since $\mathcal{T}$ is unimodular, the linear function ℓ_κ that agrees with $\check{h}$ on the cone over a tetrahedron κ is integral, and strict convexity gives $\check{h}(m) \geq \ell_\kappa(m) + 1$ for every lattice point m outside that cone. The standing hypotheses of Gross's construction [Gro05, §3] are that $\check{h}$ is integral, piecewise linear and strictly convex and that $\check{h}'$ is convex, together with the corresponding conditions on the fan side (h integral and strictly convex, $h - \varphi$ convex, with

φ as in Section 2.4); the fan-side function enters the construction of the base and of its decomposition only through the subdivision $\tilde{\Sigma}'$ below [Gro05, Prop. 3.12, Def. 3.13, Prop. 3.14]. Some h that satisfies these fan-side hypotheses induces Σ' and the same $\tilde{\Sigma}'$: the functions h as above that induce Σ' and $\tilde{\Sigma}'$ form a nonempty set cut out by finitely many linear equations and strict linear inequalities with rational coefficients, so it contains one with rational slopes. Let m be a positive integer with mh integral. Replacing h by $mh + \varphi$ changes the heights $h(a)$ of the points $(a, 0)$ in the construction of $\tilde{\Sigma}'$ below to $mh(a) + 1$ and leaves the heights 0 of the points $(w, 1)$ unchanged; this is a scaling by m followed by the addition of the affine function $(y, l) \mapsto 1 - l$, which changes no regular subdivision. And $mh + \varphi$ is integral and strictly convex on the fan over Σ' , with $(mh + \varphi) - \varphi = mh$ convex. Strictness in (G2) is an additional requirement. Replacing $\check{h}$ by $\nu\check{\varphi} + 2\check{h}$ with $\nu \geq -1$ an odd integer preserves (G1) and (G2), because $\check{\varphi}$ is convex and linear on the cones over the simplices of $\mathcal{T}$, $2\check{h} - \check{\varphi} = \check{h} + \check{h}'$, and $(\nu\check{\varphi} + 2\check{h}) - \check{\varphi} = 2\check{h}' + (\nu + 1)\check{\varphi}$; for ν large it makes $\check{h}(m)$ odd and positive for every $m \in \partial\Delta^\circ \cap M$. We assume this normalisation, which Section 10 uses, and build the base from the normalised height. It is not a restriction: two heights on the same $\mathcal{T}$ satisfying (G1) and (G2) give bases related by a piecewise linear homeomorphism that preserves the cells, their lower supports and face labels, the discriminant and the local system Λ (Lemmas 4.9 and 6.5(2)).

The polytope. Put

$$\nabla^{\check{h}} := \{y \in N_{\mathbb{R}} : \langle m, y \rangle \geq -\check{h}(m) \text{ for all } m \in \partial\Delta^\circ \cap M\}.$$

By (G1) the normal fan of $\nabla^{\check{h}}$ is the fan over $\mathcal{T}$. Its facets F_u correspond to the lattice points $u \in \partial\Delta^\circ \cap M$, its two-faces $F_{uu'} = F_u \cap F_{u'}$ to the edges $[u, u']$ of $\mathcal{T}$, its edges to the triangles and its vertices to the tetrahedra; we write F_τ for the face of the simplex τ . Gross's base $B_{\check{\nabla}}^{\check{h}}$ is contained in $\partial\nabla^{\check{h}}$ [Gro05, Cor. 3.3]. Conversely every face of $\nabla^{\check{h}}$ is $F_\tau = \tau^\vee + \nabla_\tau^{\check{h}'}$, where $\tau^\vee$ is the face of Δ on which every vertex of τ takes the value -1 and $\nabla_\tau^{\check{h}'}$ is the face of $\nabla^{\check{h}'}$ with inner normal cone $\text{cone}(\tau)$ (Lemma 4.2), and these sets are the ones whose union defines $B_{\check{\nabla}}^{\check{h}}$ [Gro05, §3]. So the base is the three-sphere $B = \partial\nabla^{\check{h}}$; for a hypersurface this also follows from [Yam21, (5.36) and Prop. 5.4], cf. [Yam21, Ex. 5.16].

The decomposition. Let $\tilde{\Sigma}$ be the normal fan of the epigraph $\text{Epi} = \{(m, l) : m \in \Delta^\circ, l \geq \check{h}'(m)\}$ [Gro05, Prop. 3.5, 3.7], a fan in $N_{\mathbb{R}} \oplus \mathbb{R}$ whose support is $\{l \geq 0\}$, the dual of the recession cone $\{0\} \times \mathbb{R}_{\geq 0}$ of Epi . By (G1) and (G2) its maximal cones are the normal cones $\tilde{\Sigma}_u$ at the vertices $(u, \check{h}'(u))$, $u \in \partial\Delta^\circ \cap M$, and the cone $\tilde{\Sigma}_{\text{up}} = \text{cone}(\nabla^{\check{h}'} \times \{1\})$ at $(0, 0)$ [Gro05, Prop. 3.7]. We fix the following subdivision $\tilde{\Sigma}'$ of $\tilde{\Sigma}$: on each $\tilde{\Sigma}_u$ it is the cone over the regular subdivision of the point configuration $(u^\vee \cap N) \times \{0\} \cup \text{vert}(\nabla_u^{\check{h}'} \times \{1\})$ with the heights $h(a)$ at $(a, 0)$ and 0 at $(w, 1)$, and $\tilde{\Sigma}_{\text{up}}$ is not subdivided. These subdivisions agree on common faces, since the heights are global. The trace of $\tilde{\Sigma}'$ on $N_{\mathbb{R}} \times \{0\}$ is the fan over Σ' , and its other rays are generated by the vectors $(w, 1)$, w a vertex of $\nabla^{\check{h}'}$. Such a w is a lattice point: it is $\nabla_\kappa^{\check{h}'}$ for a tetrahedron $\kappa = \{u_1, \dots, u_4\}$ of $\mathcal{T}$, so $\langle u_i, w \rangle = -\check{h}'(u_i) \in \mathbb{Z}$, and $u_1, \dots, u_4$ is a basis of M . So $(w, 1)$ is primitive and $\tilde{\Sigma}'$ is good for Σ' in the sense of [Gro05, Def. 3.8]. It is the subdivision of [Gro05, Rem. 3.18] determined by the values h on $N_{\mathbb{R}} \times \{0\}$ and 0 on $\tilde{\Sigma}_{\text{up}}$ (Lemma 4.3), and it is the subdivision used in the computations of Section 11. A cone of $\tilde{\Sigma}'$ is *mixed* if it has rays of both kinds. A mixed cone of dimension $d + 2$ with lower generators $\omega \times \{0\}$ (ω a cell of Σ') and upper generators $\tau_1 \times \{1\}$ gives the d -cell $\omega + \text{conv } \tau_1$ of the polyhedral decomposition $\mathcal{P}$ of B [Gro05, Def. 3.13, Obs. 3.9]; ω is

its *lower support*. For a cell c the smallest face of $\nabla^{\check{h}}$ containing it is $F_{\tau(c)}$, and $\tau(c)$ is its *face label*. A vertex of $\mathcal{P}$ is $n + w$ with lower support one lattice point n , its *lower ray*. The same construction applies to any regular subdivision of $\partial\Delta$ refining the faces of Δ in place of Σ' ; Section 6 uses it for $\mathcal{F}_C$ (Section 3.5).

Lemma 3.11 (facet index). *Every maximal cell C of $\mathcal{P}$ lies in a facet F_{u_C} of $\nabla^{\check{h}}$, and $\langle u_C, n \rangle = -1$ for every lower ray n of C . We call u_C the facet index of C .*

Proof. The five-dimensional cone of C lies in a unique maximal cone of $\tilde{\Sigma}$. A lower ray $(n, 0)$ lies in $\tilde{\Sigma}_p$ if and only if $\langle \cdot, n \rangle$ is minimal on Δ° at p , that is, $\langle p, n \rangle = -1$; it never lies in $\tilde{\Sigma}_{\text{up}}$. So the cone lies in some $\tilde{\Sigma}_p$ with $\langle p, n \rangle = -1$ on the lower rays. An upper ray $(w, 1) \in \tilde{\Sigma}_p$ is an inner normal at $(p, \check{h}'(p))$ of a lower face containing $(0, 0)$, so $\langle p, w \rangle = -\check{h}'(p)$. Hence $\langle p, n + w \rangle = -1 - \check{h}'(p) = -\check{h}(p)$ on the vertices of C , that is, $C \subset F_p$. $\square$

The affine structure [Gro05, Prop. 3.14]. On the interior of a maximal cell C the chart is the inclusion into $\text{aff}(C) = \text{aff}(F_{u_C})$, with tangent lattice $u_C^\perp \cap N$. At a vertex $n + w$ the chart is the projection $\pi_n: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R}n$, defined on the open star of the vertex in the barycentric subdivision, with lattice $N/\mathbb{Z}n$. On the part of the open star of $n + w$ inside a maximal cell C , the two charts differ by the projection $\text{aff}(F_{u_C}) \rightarrow N_{\mathbb{R}}/\mathbb{R}n$, which is an integral affine isomorphism because $\langle u_C, n \rangle = -1$; its inverse on tangent lattices is $x \mapsto x + \langle u_C, x \rangle n$.

Lemma 3.12 (monodromy of a flag). *Let $e \subset f$ be an edge and a two-cell of $\mathcal{P}$, let $C_\pm$ be the two maximal cells containing f , with facet indices $u_\pm$, and let the lower rays of the end points of e be n, n' . Let γ be a loop based in C_+ that passes into C_- near the end point with lower ray n and returns near the end point with lower ray n' . Then parallel transport along γ is*

$$T_\gamma(x) = x + \langle u_+ - u_-, x \rangle (n' - n), \quad x \in u_+^\perp \cap N.$$

It is nontrivial if and only if $n \neq n'$ and $u_+ \neq u_-$.

Proof. Through the chart at the end point with lower ray n , $x \in u_+^\perp$ is carried to the representative $x + \langle u_-, x \rangle n$ of its class in $N/\mathbb{Z}n$ that lies in $u_-^\perp$. Returning through the chart at the other end point gives $x + \langle u_-, x \rangle (n - n')$, since $\langle u_\pm, n \rangle = \langle u_\pm, n' \rangle = -1$ and $\langle u_+, x \rangle = 0$. As $\langle u_-, x \rangle = \langle u_- - u_+, x \rangle$, this is the formula. This is [Gro05, Prop. 3.15] for hypersurfaces. $\square$

3.4. The discriminant. *Legs.* For an edge e and a two-cell $f \supset e$ as in Lemma 3.12 with nontrivial monodromy, the segment $E = [b_e, b_f]$ joining their barycentres is a *leg*. Its *vanishing vector* is $d_E = n' - n$ and its *covector* is $\alpha_E = u_+ - u_-$, both defined up to a common sign; the monodromy of a loop around E is the transvection $x \mapsto x + \langle \alpha_E, x \rangle d_E$. The *discriminant* Γ is the union of the legs. The lower support $\{n, n'\}$ of e is an edge of Σ' on which both u_+ and u_- take the value -1 , so it lies in the face $\{u_+, u_-\}^\vee$ of Δ , of dimension at most two; by admissibility and Lemma 3.1(4), d_E is primitive. A leg lies in a two-cell $f \subset F_{u_+} \cap F_{u_-} = F_{u_+ u_-}$, so Γ is contained in the two-skeleton $\bigcup F_{uu'}$ of $\nabla^{\check{h}}$ and meets no open facet.

Γ is contained in Gross's discriminant, and the affine structure extends over the rest of it. For a lattice point n let $\tilde{R}_n$, the *vertex region* of n , be the union of the open simplices of the barycentric subdivision $\text{Bar } \mathcal{P}$ whose smallest cell has lower support $\{n\}$. Every facet F_u through a point of $\tilde{R}_n$ has $\langle u, n \rangle = -1$, so B is locally a graph over $N_{\mathbb{R}}/\mathbb{R}n$ there, and the projection along n is a chart on $\tilde{R}_n$ compatible with the facet charts (Lemma 4.4). The open facets $\text{int } F_u$ and the vertex regions $\tilde{R}_n$ cover $B \setminus \Gamma$. We write Λ for the local system of integral tangent vectors on $B \setminus \Gamma$ and $\iota: B \setminus \Gamma \rightarrow B$ for the inclusion.

Lemma 3.13. Γ is a full subcomplex of $\text{Bar } \mathcal{P}$: every simplex of $\text{Bar } \mathcal{P}$ whose vertices lie on Γ lies in Γ .

Proof. A leg $[b_e, b_f]$ is the simplex of the chain $e \subsetneq f$, so Γ is a subcomplex. The relative interior of a leg lies in the relative interior of f , and its end points are b_e and b_f . So a barycentre b_c lies on Γ only if c is the edge or the two-cell of a leg: $b_e \in \Gamma$ forces e to have a two-point lower support $\{n, n'\}$ with $n \neq n'$, and $b_f \in \Gamma$ forces $u_+(f) \neq u_-(f)$. A simplex of $\text{Bar } \mathcal{P}$ with all its vertices on Γ is therefore the simplex of a chain consisting of at most one edge and at most one two-cell. If it is a chain $e \subsetneq f$, then the flag (e, f) has $n \neq n'$ and $u_+ \neq u_-$, so it has nontrivial monodromy by Lemma 3.12, and $[b_e, b_f]$ is a leg. $\square$

For a two-dimensional cell L of Σ' , let $G(L)$ be the smallest face of Δ containing L . We use the convention $\ell(G(L)^\circ) = 0$ when $G(L)$ is a facet, so its dual is a point.

Lemma 3.14 (stack lemma). *The two-cells of $\mathcal{P}$ with lower support L , together with the maximal cells containing them, form a linear sequence $C_0 | f_1 | C_1 | \cdots | f_k | C_k$ with $k \geq 1$, in which C_0 and C_k have as lower supports the two three-cells of Σ' containing L . Along the sequence the facet indices run monotonically through all lattice points of $G(L)^\circ$, and exactly $\ell(G(L)^\circ)$ of the f_i separate maximal cells with different facet indices.*

Proof. By goodness, $\kappa = \text{cone}(L) \times \{0\}$ is a cone of $\tilde{\Sigma}'$. Fix x in its relative interior. The cones of $\tilde{\Sigma}'$ containing x project to a complete fan of the closed half-plane that is the image of $\{l \geq 0\}$ in $(N_{\mathbb{R}} \oplus \mathbb{R})/\text{span } \kappa$. Its two boundary rays are the images of the cones over the two three-cells of Σ' through L ; its interior rays are the images of the four-dimensional mixed cones with lower part κ , that is, of the two-cells with lower support L ; its sectors are the maximal cells through them. A complete fan of a half-plane is a linear sequence of sectors, and there is at least one interior ray since every sector is pointed. The coarse fan $\tilde{\Sigma}$ at x consists of the $\tilde{\Sigma}_p$ with $\langle p, \cdot \rangle = -1$ on L , that is, $p \in G(L)^\circ \cap M = \{p_0, \dots, p_\lambda\}$. By (G1) and (G2) each such $\tilde{\Sigma}_p$ is five-dimensional, and $\tilde{\Sigma}_{p_j} \cap \tilde{\Sigma}_{p_{j+1}}$ is a wall because $[p_j, p_{j+1}] \in \mathcal{T}$. So the coarse sectors are $\lambda + 1$ sectors in the order of the p_j . Each refined sector lies in the coarse sector of its facet index (Lemma 3.11), so the facet indices are constant on coarse sectors and change exactly at the λ coarse walls. $\square$

Lemma 3.15 (edge lemma). *Let e be an edge of $\mathcal{P}$ with lower support $\{n, n'\}$, $n \neq n'$. The maximal cells through e carry $j \leq 3$ distinct facet indices, and the number of two-cells $f \supset e$ for which the flag (e, f) has nontrivial monodromy is j if $j \geq 2$ and 0 if $j = 1$. If e has a one-point lower support, no flag at e has nontrivial monodromy.*

Proof. The last sentence is Lemma 3.12. Fix x in the relative interior of the three-dimensional mixed cone of e ; the cones through x project to a complete fan of a plane, with rays the two-cells through e and sectors the maximal cells. The coarse cones through x are the $\tilde{\Sigma}_p$ with p a vertex of the bounded face ϕ of the epigraph whose normal cone contains x in its relative interior. By (G1) and (G2), ϕ is the lift of a simplex s of $\mathcal{T}$, and every vertex of s takes the value -1 on n and n' , so s lies in the face of Δ° dual to the smallest face of Δ containing $[n, n']$. That face of Δ has dimension at least one, so its dual has dimension at most two and s has at most three vertices; the facet indices around e are the vertices of s . So $j \leq 3$, and the j coarse sectors are separated by exactly j walls if $j \geq 2$ and by none if $j = 1$. The refined rays on the walls are the two-cells across which the facet index changes, and by Lemma 3.12 these are the flags with nontrivial monodromy. $\square$

Definition 3.16. Let p be a point of Γ at which three legs meet, and let T_1, T_2, T_3 be the monodromies of loops around the three legs, based near p and oriented compatibly with an orientation of B . The point p is of *negative type* if $T_i(x) = x + \langle \alpha, x \rangle d_i$ with a common primitive covector α and $d_1 + d_2 + d_3 = 0$ spanning a saturated sublattice of rank two of $\alpha^\perp$; it is of *positive type* if $T_i(x) = x + \langle \alpha_i, x \rangle d$ with a common primitive d and $\alpha_1 + \alpha_2 + \alpha_3 = 0$ spanning a saturated sublattice of rank two of $d^\perp$. An *edge point* of Γ , the term used in [CBM14, Def. 7.2(b)], is a point of Γ that is not a vertex of the graph Γ : an interior point of a leg, or a point at which exactly two legs meet.

These are the local monodromies at the two kinds of trivalent vertex of the discriminant in Gross's topological torus fibrations [Gro01, Thm. 1.11(2)], the negative vertex [CBM14, Ex. 2.3] and the positive vertex [CBM14, Ex. 2.4]. At a point where two legs meet, the two legs have the same vanishing vector and the same covector up to sign (Lemma 3.17), so near every edge point the monodromy is that of an edge point of the discriminant in [CBM14, Ex. 2.2], the product of the focus-focus singularity of [CBM14, Ex. 2.1] with an interval; [CBM14, §3.2, (4)] gives the corresponding monodromy on the first homology of the fibres of the model torus fibration.

Lemma 3.17 (vertex types). *The points of Γ at which the number of legs is not zero are the following, and no others:*

<i>point</i>	<i>position</i>	<i>lower support</i>	<i>type</i>
b_f	interior of a two-face $F_{uu'}$	two points	edge point, two legs
b_f	interior of a two-face $F_{uu'}$	unimodular triangle	negative, three legs
b_f	interior of a two-face $F_{uu'}$	node parallelogram	four legs
b_e	interior of a two-face $F_{uu'}$	two points	edge point, two legs
b_e	interior of an edge of ∇^h	two points	positive, three legs

Proof. Legs exist only on two-cells f contained in a two-face $F_{uu'}$, and at their edges e with a two-point lower support. The lower support of f is a cell of Σ' in the face $[u, u']^\vee$ of Δ , which has dimension at most two; by admissibility and Lemma 3.1(4) it is a point, a primitive segment, a unimodular triangle or a unit parallelogram. The number of legs at b_f is the number of edges of f with a two-point lower support (Lemma 3.12); a two-cell with a one-point lower support has none, a two-cell with lower support a segment $\{n, n'\}$ has exactly two such edges (the faces of f selected by the two normals of the segment in the plane of f), and a two-cell with lower support a polygon has one for each edge of the polygon. At b_e the count is Lemma 3.15: $j = 2$ if e lies in the interior of a two-face, and $j = 3$ if e lies on an edge of ∇^h , dual to a triangle $\{u_1, u_2, u_3\}$ of $\mathcal{T}$.

Negative type. At b_f with lower support a triangle $\{n_a, n_b, n_c\}$ in $F_{uu'}$, the three legs share f , hence the covector $\alpha = u' - u$, which is primitive since $[u, u']$ is an edge of the unimodular triangulation $\mathcal{T}$. The vanishing vectors are the oriented edge vectors of the triangle, which sum to zero; the triangle is unimodular in the plane $\{\langle u, \cdot \rangle = \langle u', \cdot \rangle = -1\}$, so they span the saturated lattice $\{u, u'\}^\perp \cap N \subset \alpha^\perp$.

Positive type. At b_e with e on the edge of ∇^h dual to $\{u_1, u_2, u_3\}$, the three legs share e , hence $d = n' - n$, which is primitive. The covectors are the differences $u_j - u_i$, which sum to zero with suitable orientations. Since $\mathcal{T}$ is unimodular, the tetrahedron $\{0, u_1, u_2, u_3\}$ is unimodular, so $u_2 - u_1$ and $u_3 - u_1$ span a saturated lattice of rank two, and it lies in $d^\perp$ since $\langle u_i, n \rangle = \langle u_i, n' \rangle = -1$.

Points on two legs. At b_e in the interior of $F_{uu'}$ both legs have $d = n' - n$ from e and $\alpha = u' - u$. At b_f with a two-point lower support $\{n, n'\}$ the two legs are the two edges of f with that lower support, and they share d and, through f , α . $\square$

The positive case uses the unimodularity of $\mathcal{T}$: for a triangulation containing an empty tetrahedron of normalised volume two, the three covectors at a positive vertex span a sublattice of index two. Gross's simplicity theorem [Gro05, Thm. 3.16] assumes elementary simplices on both sides and does not apply to Σ' , whose parallelograms produce the four-valent vertices.

Carrying cells. Let $q \in Q$, with node parallelogram P in the two-face G and $G^\circ = [u_-, u_+]$. By the stack lemma and $\ell(G(P)^\circ) = \ell(G^\circ) = 1$, exactly one of the two-cells of $\mathcal{P}$ with lower support P separates maximal cells with different facet indices. We call it the *carrying cell* $\mathbf{c}_q$ of q ; it lies in the two-face $F_{u_-u_+}$, and we put $p_q := b_{\mathbf{c}_q}$. Likewise every unimodular triangle L of σ in a two-face G is the lower support of exactly one two-cell $\mathbf{c}_L \subset F_{u_-u_+}$ with different facet indices on its two sides.

Proposition 3.18. (1) *The discriminant Γ of B'_σ is a graph and a full subcomplex of $\text{Bar } \mathcal{P}$. Its four-valent vertices are the points p_q , one for each $q \in Q$; its three-valent vertices are of negative or positive type; every other point of Γ is an edge point, near which the monodromy is that of [CBM14, Ex. 2.2]. In particular B'_σ has no positive node.*

(2) *After the vertex of Γ inside each carrying cell is moved as in Lemma 3.19, the germ of B'_σ at each four-valent vertex is integral affine isomorphic, with integral translations, to the germ of [CBM14, Ex. 6.1] at its vertex. So these vertices are negative nodes in the sense of [CBM14, Def. 6.3]. The moved and the barycentric discriminants are related by a homeomorphism of B that preserves every cell of $\mathcal{P}$ and the local system (Lemma 3.20).*

Proof of (1). Fullness is Lemma 3.13. By Lemma 3.17, Γ is a graph whose three-valent vertices are of negative or positive type and whose other points are edge points, except at the barycentres of two-cells with a node parallelogram as lower support. By the stack lemma these barycentres, on cells carrying legs, are exactly the p_q ; the other two-cells over a node parallelogram, and all two-cells over parallelograms interior to facets of Δ , carry no leg. Four-valent points b_e do not occur (Lemma 3.15), so every four-valent vertex of Γ is a point p_q . The legs at p_q are the flags $(e, \mathbf{c}_q)$ with e an edge of $\mathbf{c}_q$ over an edge of the node parallelogram, and $\mathbf{c}_q$ separates maximal cells with facet indices $u_\pm$; by Lemma 3.12 their monodromies are transvections with the common covector $\pm(u_+ - u_-)$ and with vanishing vectors the edge vectors of the parallelogram, two of which are linearly independent. The flat sections of Λ near p_q form the rank-two lattice $\{u_-, u_+\}^\perp \cap N$ (Lemma 5.1(2)). At a positive node the monodromies of the two loops computed in [CBM14, Ex. 6.2] are transvections with a common vanishing vector e_1 , and the vectors invariant under both form the line $\mathbb{R}e_1$. An integral affine isomorphism of neighbourhoods carries flat sections to flat sections, so no p_q is a positive node, and B'_σ has none. $\square$

The node germ. Let $q \in Q$ with node parallelogram $P = abcd$ in the two-face G , $G^\circ = [u_-, u_+]$, carrying cell $\mathbf{c} = \mathbf{c}_q$ and adjacent maximal cells $C_\pm$ with facet indices $u_\pm$. By Lemma 4.3(2), applied to the face $F_{u_-u_+}$ that contains $\mathbf{c}$, $\mathbf{c} = P + \text{conv } \tau$, where $\text{conv } \tau = \nabla_{\tau'}^{\check{h}'}$ for a simplex $\tau' \supseteq [u_-, u_+]$ of $\mathcal{T}$; it lies in a translate of $\{u_-, u_+\}^\perp$, so τ is a point, a segment or a polygon parallel to the plane of P . Put $\alpha = u_+ - u_-$ and $\check{h}_\pm = \check{h}(u_\pm)$, so that $C_\pm \subset \{\langle u_\pm, \cdot \rangle = -\check{h}_\pm\}$ and $\alpha \equiv \check{h}_- - \check{h}_+$ on $\mathbf{c}$. The four edges of $\mathbf{c}$ with a two-point lower

support have lower supports $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$, one for each edge of P , and the four legs at p_q join p_q to their barycentres. The legs over $\{a, b\}$ and $\{c, d\}$ have vanishing vector $\pm(b - a) = \pm(c - d)$, those over $\{b, c\}$ and $\{d, a\}$ have $\pm(c - b) = \pm(d - a)$, and all four have covector $\pm\alpha$. We call the pair of opposite legs over $\{a, b\}$ and $\{c, d\}$ the $b - a$ passage at q , and the pair over $\{b, c\}$ and $\{d, a\}$ the $c - b$ passage (Figure 1). The vector $b - a$ is fixed by the monodromies of all four legs, since $\langle \alpha, b - a \rangle = 0$, and so is $c - b$.

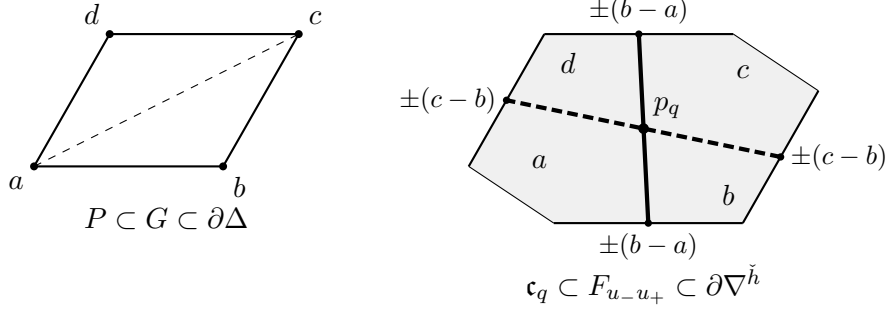


FIGURE 1. Left: a node parallelogram $P = abcd$ in a two-face G of Δ , with the diagonal $\{a, c\}$ dashed. Right: its carrying cell $\mathfrak{c}_q = P + \text{conv } \tau$ in the two-face $F_{u_-u_+}$ of ∇^h , drawn for τ a segment; the maximal cells $C_\pm$ lie on the two sides of the page. The thick edges have the two-point lower supports $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$, and the four legs of Γ join p_q to their barycentres, labelled with their vanishing vectors. The four complementary regions carry the charts of the lower rays a, b, c, d . The $b - a$ passage (solid) and the $c - b$ passage (dashed) are the two pairs of opposite legs.

Lemma 3.19 (the node germ). (1) For $n \in P$ let π_n be the chart $N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R}n$, identified with $H_0 = u_+^\perp$ by $x \mapsto x + \langle u_+, x \rangle n$. Then for $n, n' \in P$,

$$\pi_{n'} - \pi_n = \begin{cases} -\check{h}_+(n' - n) & \text{on } C_+, \\ -(\check{h}_+ + y_2(x))(n' - n) & \text{on } C_-, \end{cases} \quad y_2(x) := \alpha(\mathfrak{c}) - \alpha(x),$$

and $y_2 \geq 0$ on C_+ , $y_2 \leq 0$ on C_- , $y_2 = 0$ on $\mathfrak{c}$.

- (2) Let Γ' be any four-legged star in $\mathfrak{c}$ whose legs end at the barycentres of the four edges of $\mathfrak{c}$ with a two-point lower support. Giving the region between consecutive legs the chart of the lower ray of the boundary arc of $\mathfrak{c}$ that it contains defines an integral affine structure on a neighbourhood of $\mathfrak{c}$ minus Γ' , which agrees with that of B off $\mathfrak{c}$. Its local system of integral tangent vectors is given on each region by the chart of that region, by the same rule for every such star.
- (3) Let b' be the centre of a translate $P + z \subset \mathfrak{c}$, z a vertex of $\text{conv } \tau$, and let $\Gamma_{b'}$ be a star with vertex b' whose legs leave b' in the directions $\pm(b - a)$, $\pm(d - a)$ and continue inside $\mathfrak{c}$ to the barycentres of the edges with lower supports $\{a, d\}$, $\{b, c\}$, $\{a, b\}$, $\{c, d\}$, coinciding with the legs of Γ near these barycentres. Then a neighbourhood of b' is integral affine isomorphic, with integral translations, to a neighbourhood of the vertex of the discriminant in [CBM14, Ex. 6.1].

Proof. (1) Since $\langle u_+, n \rangle = -1$, $N = (H_0 \cap N) \oplus \mathbb{Z}n$, and $x \mapsto x + \langle u_+, x \rangle n$ is the lattice projection along n onto H_0 . Hence $\pi_{n'} - \pi_n = \langle u_+, x \rangle (n' - n)$. On C_+ , $\langle u_+, x \rangle = -\check{h}_+$. On

C_- , $\langle u_+, x \rangle = \alpha(x) + \langle u_-, x \rangle = \alpha(x) - \check{h}_- = -\check{h}_+ - y_2(x)$. Since $\nabla^{\check{h}}$ is convex, $\langle u_-, x \rangle \geq -\check{h}_-$ on C_+ and $\langle u_+, x \rangle \geq -\check{h}_+$ on C_- , which give the signs of y_2 .

(2) By (1) the charts π_n , $n \in P$, differ on C_+ by integral translations and on C_- by integral affine shears $x \mapsto x - (\check{h}_+ + y_2(x))(n' - n)$, with y_2 an integral affine function there; so on each of $C_{\pm}$ separately every π_n is a chart of the integral affine structure of that cell. Across $\mathfrak{c}$ the change of chart is piecewise linear and not affine, so near each point of $\mathfrak{c}$ off the star a single chart π_n must be used on both sides of $\mathfrak{c}$; the affine structure near such a point depends on which chart it receives. So inside $\mathfrak{c}$ one needs only a closed set separating four regions, each containing the vertices of $\mathfrak{c}$ with a given lower ray. Edges of $\mathfrak{c}$ with a one-point lower support join vertices with the same lower ray, and the four edges with a two-point lower support separate the boundary of $\mathfrak{c}$ into four arcs with lower rays a, b, c, d in this cyclic order.

(3) Choose the origin at the lattice point $a + z$ and the basis $v_1 = b - a$, w_{u_-} , $v_3 = d - a$ of $H_0 \cap N$, where w_{u_-} is the vector of Lemma 3.2 for $(u, u') = (u_+, u_-)$: by that lemma $H_0 \cap N = N_G \oplus \mathbb{Z}w_{u_-}$, and v_1, v_3 is a basis of N_G because P is a unit parallelogram. Then $\alpha(v_1) = \alpha(v_3) = 0$ and $\alpha(w_{u_-}) = -1$, so the second coordinate of a point of H_0 is y_2 . In the coordinates $y = (y_1, y_2, y_3)$ of this basis, the chart of the corner $a + iv_1 + jv_3$ ($i, j \in \{0, 1\}$) is, up to an integral translation, y on $\{y_2 \geq 0\}$ and $(I - (ie_1 + je_3)e_2^T)y$ on $\{y_2 \leq 0\}$, by (1). Near $b' = (\frac{1}{2}, 0, \frac{1}{2})$ the region of the corner (i, j) is the quadrant $\{\text{sign}(y_1 - \frac{1}{2}) = 2i - 1, \text{sign}(y_3 - \frac{1}{2}) = 2j - 1\}$. This is the atlas of [CBM14, Ex. 6.1]: two triangular prisms glued along a unit square, with perpendicular prism directions e_1, e_3 , the monodromy-invariant plane $\langle e_1, e_3 \rangle$, and the vertex at the barycentre of the square; the translations are integral. $\square$

The sign of the shear in (3), forced by the convexity of $\nabla^{\check{h}}$ through (1), is the one of [CBM14, Ex. 6.1], where it is forced by the two prisms. With the barycentric placement, the vertex of Γ sits at the barycentre of $\mathfrak{c}$, whose position modulo the lattice is in general not that of b' ; this is why the vertex is moved in (3).

Lemma 3.20 (placement). *Let Γ' be Γ with the star inside every carrying cell replaced by $\Gamma_{b'}$. There is a piecewise linear homeomorphism ϕ_{pl} of B which preserves every closed face of $\nabla^{\check{h}}$ and every cell of $\mathcal{P}$, carries Γ' onto Γ and the local system of $B \setminus \Gamma'$ to that of $B \setminus \Gamma$, and is supported in a compact subset of the union of the open stars of the points p_q in $\text{Bar } \mathcal{P}$. It is isotopic to the identity through homeomorphisms with the same support that preserve Γ outside the carrying cells.*

Proof. Both stars lie in the disc $\mathfrak{c}$, have the same four end points on $\partial\mathfrak{c}$, coincide near them, and have their legs in the same cyclic order. So there is a piecewise linear homeomorphism of $\mathfrak{c}$, equal to the identity near $\partial\mathfrak{c}$, carrying one to the other; by the Alexander trick [Ale23], [FM12, Lemma 2.1] it is isotopic to the identity relative to a neighbourhood of $\partial\mathfrak{c}$. Extend it and the isotopy across the two sides of $\mathfrak{c}$ by a product in a thin collar, tapering to the identity. The support is then a compact subset of the open star of p_q , which is an open neighbourhood of $\text{int } \mathfrak{c}$. By Lemma 3.19(2) both placements assign the charts to the four complementary regions by the same rule, and ϕ_{pl} preserves the regions, so it carries Λ to Λ . $\square$

The open stars of distinct points p_q in $\text{Bar } \mathcal{P}$ are disjoint, because no chain of cells contains two different two-cells; so the modifications at different nodes do not interact. This holds even when the closed carrying cells of two nodes meet, which happens: at an A_1 hexagon they may share a vertex with lower ray o , at a pentagon or an A_2 hexagon they may share

an edge with lower support $\{o, w\}$, and across an edge of Δ they may share an edge whose barycentre is a positive vertex of Γ .

Proof of Proposition 3.18(2). The germ is described by Lemma 3.19(3), and the comparison of the two placements is Lemma 3.20. $\square$

We work with the barycentric placement from now on; the node point is $p_q = b_{\mathfrak{c}_q}$. Relative tropical 2-cycles (Definition 2.4) are defined on triangulations in which Γ is full, and Bar $\mathcal{P}$ is one by Lemma 3.13.

The stalks of $\iota_*\Lambda$ at the points of Γ are computed in Lemma 5.1: on a leg in $F_{uu'}$, at a negative vertex and at a node point of $F_{uu'}$ the stalk is $\{u, u'\}^\perp \cap N$, and at a positive vertex on the edge of ∇^h dual to the triangle $\{u_1, u_2, u_3\}$ it is $\{u_1, u_2, u_3\}^\perp \cap N = \mathbb{Z}d$, with d the common vanishing vector of the three legs.

The discriminant in a two-face. Let G be a pentagon or a hexagon, $G^\circ = [u_-, u_+]$. Each node parallelogram P of $\sigma|_G$ has two legs whose lower supports are interior edges $[o, w]$ of $\sigma|_G$, which share the lower ray o and are therefore adjacent in the cyclic order at p_q , and two legs whose lower supports are edges of Δ .

Lemma 3.21 (arcs across interior edges). *Let $\epsilon = [o, w]$ be an interior edge of $\sigma|_G$, and let L and L' be the two cells of $\sigma|_G$ containing it, with two-cells $\mathfrak{c}_L, \mathfrak{c}_{L'} \subset F_{u_- u_+}$ as above. Then Γ contains an arc from $b_{\mathfrak{c}_L}$ to $b_{\mathfrak{c}_{L'}}$, that is a union of legs with lower support $\{o, w\}$, vanishing vector $\pm(w - o)$ and covector $\pm(u_+ - u_-)$, and whose interior points are edge points of Γ .*

Proof. Let Γ_ϵ be the union of the legs with lower support $\{o, w\}$, a subgraph of Γ . The facet indices of the maximal cells at such a leg take the value -1 at o , so they are u_- and u_+ , the only lattice points of Δ° with this property (o is interior to G , and $G^\circ \cap M = \{u_-, u_+\}$). So every such leg lies in a two-cell $f \subset F_{u_- u_+}$ whose lower support contains ϵ : the segment ϵ itself, or a two-cell of Σ' containing ϵ on which $u_\pm$ take the value -1 , that is, L or L' . We compute the number of legs of Γ_ϵ at each point.

- At b_e , e an edge with lower support ϵ : the facet indices around e lie in $\{u_-, u_+\}$. If e lies in one facet, so that $j = 1$ in Lemma 3.15, no leg ends at b_e . If $b_e \in \Gamma_\epsilon$, then $j = 2$, and by Lemma 3.15 exactly two legs end at b_e , both with the lower support of e .
- At b_f , f a two-cell with lower support ϵ and $u_+(f) \neq u_-(f)$: f has exactly two edges with lower support ϵ and no other edge with a two-point lower support, so two legs of Γ_ϵ end at b_f .
- At $b_{\mathfrak{c}_L}$: the unique edge of $\mathfrak{c}_L$ with lower support ϵ , the face selected by the outer normal of ϵ in L , gives exactly one leg of Γ_ϵ ; likewise at $b_{\mathfrak{c}_{L'}}$. By the stack lemma no other two-cell over L or L' carries a leg.

So Γ_ϵ is a finite graph in which $b_{\mathfrak{c}_L}$ and $b_{\mathfrak{c}_{L'}}$ have valency one and every other vertex has valency two. The component of $b_{\mathfrak{c}_L}$ is therefore a path, whose other end has valency one and is not $b_{\mathfrak{c}_L}$; it is $b_{\mathfrak{c}_{L'}}$. Its interior points are points b_e in the interior of $F_{u_- u_+}$ and points b_f with a two-point lower support, which are edge points by Lemma 3.17. All its legs have the vanishing vector $\pm(w - o)$ and the covector $\pm(u_+ - u_-)$. $\square$

By Proposition 4.10(2) this arc is the only arc of Γ with lower support $\{o, w\}$. Applied to the interior edges of Lemma 3.1(3), the lemma gives the shape of Γ in the two-face $F_{u_- u_+}$ of a del Pezzo face: at a pentagon, a cycle through the two nodes and the negative vertex of the triangle, with one arc from node to node (across $[o, w_1]$ in the coordinates (3.1)); at an A_1 hexagon, a cycle through the two nodes and the two negative vertices, with no arc from

node to node; at an A_2 hexagon, three arcs from node to node, across the edges $[o, w_j]$ with $j \equiv j_0 \pmod{2}$, forming a cycle of Γ through the three nodes with no trivalent vertex. These configurations are used in Section 5; the lift of Section 8 treats several nodes in one two-face in Section 8.8, and the resolution side in Lemma F.3.

3.5. The mirror subdivision. The mirror family of Section 2.4 uses a height $H = k_0\varphi + h_C$ on $\partial\Delta \cap N$, where h_C induces a refinement $\mathcal{F}_C$ of Σ' . The analysis near the node points (Section 7.1) needs two properties of $\mathcal{F}_C$: it agrees with Σ' on the faces of dimension at most two, so that every node parallelogram is a cell, and the three-cells of $\mathcal{F}_C$ containing a node parallelogram P have all their lattice points off P at lattice height one over the plane of P . We construct $\mathcal{F}_C$ by pulling the lattice points of $\partial\Delta$ that are not vertices of Σ' , and show that the second property then holds automatically. The subdivision Σ' itself need not have it (Remark 3.27).

Fans and subdivisions of $\mathcal{A}$. Let $\mathcal{A} = (\partial\Delta \cap N) \cup \{0\}$. We use the language of point configurations [DLRS10]. A subdivision $\mathcal{S}$ of $\mathcal{A}$ is *regular* if it is induced by a height $g: \mathcal{A} \rightarrow \mathbb{R}$: its cells are the subsets $\mathfrak{d} = \{a \in \mathcal{A} : g(a) = \ell_{\mathfrak{d}}(a)\}$ for the affine functions $\ell_{\mathfrak{d}} \leq g$ on $\mathcal{A}$ whose equality set spans a face of the lower hull. Let $\hat{g}: \text{conv } \mathcal{A} \rightarrow \mathbb{R}$ be the lower envelope, the largest convex function with $\hat{g} \leq g$ on $\mathcal{A}$; then $\hat{g} = \ell_{\mathfrak{d}}$ on $\text{conv } \mathfrak{d}$, and $\{\hat{g} = \ell_{\mathfrak{d}}\} = \text{conv } \mathfrak{d}$. A complete fan in $N_{\mathbb{R}}$ that refines the fan over the faces of Δ and has its rays generated by points of $\mathcal{A}$ is the fan of cones from 0 over the cells of a subdivision of $\mathcal{A}$ all of whose maximal cells contain 0, the cells on $\partial\Delta$ being those of the fan; the fan carries a strictly convex piecewise linear function if and only if the subdivision is regular, the function, normalised to vanish at 0, being a height that induces the subdivision. By (Proj), Σ' is regular in this sense. A point $j \in \mathcal{A}$ is a *vertex* of $\mathcal{S}$ if $\{j\}$ is a cell.

Definition 3.22 (pulling). Let $\mathcal{S}$ be a subdivision of $\mathcal{A}$ and $j \in \mathcal{A}$. The *pulling* $\text{pull}_j \mathcal{S}$ keeps every cell $\mathfrak{d}$ with $j \notin \text{conv } \mathfrak{d}$, and replaces every cell $\mathfrak{d}$ with $j \in \text{conv } \mathfrak{d}$ by the cells $\{j\} \cup (\mathfrak{d} \cap F)$, for the faces F of $\text{conv } \mathfrak{d}$ with $j \notin F$ [DLRS10, §4.3].

The definition applies to every cell whose convex hull contains j , also when j is not a point of the cell, as happens after an earlier pulling has placed j strictly above the lower hull.

Lemma 3.23 (pulling preserves regularity). *Let g induce $\mathcal{S}$ and let $j \in \mathcal{A}$. Put $g_{\eta}(j) = \hat{g}(j) - \eta$ and $g_{\eta} = g$ on $\mathcal{A} \setminus \{j\}$. For $\eta > 0$ small, g_{η} induces $\text{pull}_j \mathcal{S}$. In particular $\text{pull}_j \mathcal{S}$ is regular, j is one of its vertices, and every vertex of $\mathcal{S}$ other than j remains a vertex. If $j \in \partial\Delta$ and every maximal cell of $\mathcal{S}$ contains 0, then every maximal cell of $\text{pull}_j \mathcal{S}$ contains 0.*

Proof. Let $\mathfrak{d}$ be a cell with $j \in \text{conv } \mathfrak{d}$, and F a facet of $\text{conv } \mathfrak{d}$ with $j \notin F$. Let λ_F be the affine function on $\text{aff } \mathfrak{d}$, extended to the ambient space, with $\lambda_F = 0$ on F and $\lambda_F(j) = 1$; it is positive on $\text{conv } \mathfrak{d} \setminus F$. The affine function $\ell_{\mathfrak{d}} - \eta\lambda_F$ agrees with g_{η} at j , because $\ell_{\mathfrak{d}}(j) = \hat{g}(j)$, and on $\mathfrak{d} \cap F$; it lies strictly below $g_{\eta} = g = \ell_{\mathfrak{d}}$ on $\mathfrak{d} \setminus (F \cup \{j\})$. At a point $a \in \mathcal{A} \setminus (\mathfrak{d} \cup \{j\})$ we have $g(a) > \ell_{\mathfrak{d}}(a)$, and since $\mathcal{A}$ is finite, $\ell_{\mathfrak{d}}(a) - \eta\lambda_F(a) < g(a)$ for η small. So $\{j\} \cup (\mathfrak{d} \cap F)$ is a cell of the subdivision induced by g_{η} , with convex hull $\text{conv}(\{j\} \cup F)$; these convex hulls, over the facets F not containing j , cover $\text{conv } \mathfrak{d}$. If $j \notin \text{conv } \mathfrak{d}$, then $\hat{g}(j) > \ell_{\mathfrak{d}}(j)$, because $\{\hat{g} = \ell_{\mathfrak{d}}\} = \text{conv } \mathfrak{d}$; so for η small $\ell_{\mathfrak{d}}(j) < g_{\eta}(j)$, and $\mathfrak{d}$ remains a cell. Applying this to the maximal cells, the cells found cover $\text{conv } \mathcal{A}$, so they are all the maximal cells of the subdivision induced by g_{η} , which is therefore $\text{pull}_j \mathcal{S}$. The point j is a vertex of every cell $\text{conv}(\{j\} \cup F)$, and a vertex $v \neq j$ of $\mathcal{S}$ is kept because $j \notin \text{conv } \{v\}$.

For the last statement, a maximal cell of $\mathcal{S}$ with j in its convex hull is $\text{conv}(\{0\} \cup C)$ with C a cell on $\partial\Delta$, and $j \in \text{conv } C$ because $j \in \partial\Delta$ and C lies in a supporting hyperplane of

Δ . The facets of $\text{conv}(\{0\} \cup C)$ are C and the cones $\text{conv}(\{0\} \cup F')$ over the facets F' of C ; those not containing j are of the second kind and contain 0. $\square$

Let $\mathcal{F}_C$ be the subdivision of $\partial\Delta$ obtained from Σ' by pulling, one at a time and in any order, the lattice points of $\partial\Delta$ that are not vertices of Σ' . By Lemma 3.23 it is induced by a height on $\mathcal{A}$, which we may take rational; the function on $N_{\mathbb{R}}$ that is linear on the cone over each cell and equal to this height on its vertices is strictly convex on the fan over $\mathcal{F}_C$. Multiplying it by a positive integer makes its linear functions integral. We let h_C be such a function: strictly convex on the fan over $\mathcal{F}_C$, inducing $\mathcal{F}_C$, with integral slopes and $h_C(0) = 0$.

Lemma 3.24 (the pyramid over a unit parallelogram). *Let $P = abcd \subset G$ be a node parallelogram, $G^\circ = [u, u']$, and let $x \in u^\vee \cap N$ with $h_x := e_{u'}(x) \geq 1$. With $w_{u'}$ as in Lemma 3.2, write*

$$x - a = \alpha(b - a) + \beta(d - a) + h_x w_{u'}, \quad \alpha, \beta \in \mathbb{Z}.$$

Then $y := a + \lceil \alpha/h_x \rceil(b - a) + \lceil \beta/h_x \rceil(d - a) + w_{u'}$ is the only lattice point of $\text{conv}(P \cup \{x\})$ with $e_{u'} = 1$. If $h_x \geq 2$, then y is not a vertex of $\text{conv}(P \cup \{x\})$.

Proof. By Lemma 3.2, $(b - a, d - a, w_{u'})$ is a basis of $u^\perp \cap N$, since $b - a, d - a$ is a basis of N_G (P is a unit parallelogram). In the lattice coordinates (s, t, e) of this basis with origin a on $\text{aff}(u^\vee)$, $e = e_{u'}$, $P = [0, 1]^2 \times \{0\}$ and $x = (\alpha, \beta, h_x)$. The slice of $\text{conv}(P \cup \{x\})$ at $e = 1$ is $(1 - 1/h_x)P + x/h_x$, the product of the intervals $[\alpha/h_x, \alpha/h_x + 1 - 1/h_x]$ and $[\beta/h_x, \beta/h_x + 1 - 1/h_x]$. Each has length $1 - 1/h_x$ and end points in $\frac{1}{h_x}\mathbb{Z}$; writing $\alpha = mh_x + i$ with $0 \leq i < h_x$, the first interval is $[m + i/h_x, m + (i + h_x - 1)/h_x]$, which contains exactly one integer, $\lceil \alpha/h_x \rceil$; likewise for β . The vertices of $\text{conv}(P \cup \{x\})$ lie at the levels 0 and h_x , so for $h_x \geq 2$ the point y at level 1 is not a vertex. $\square$

Lemma 3.25 (a high apex forces a lattice point). *Let $\mathcal{S}$ be a subdivision of $\partial\Delta$ into lattice polytopes that induces σ on the two-faces of Δ . Let $P \subset G$ be a node parallelogram, $G^\circ = [u, u']$, and ω the three-cell of $\mathcal{S}$ in $u^\vee$ containing P . If some vertex x of ω has $e_{u'}(x) \geq 2$, then the point y of Lemma 3.24 is a lattice point of ω that is not a vertex of ω , and it lies in the relative interior of the facet $u^\vee$.*

Proof. $y \in \text{conv}(P \cup \{x\}) \subseteq \omega$. A vertex of ω lying in the subpolytope $\text{conv}(P \cup \{x\})$ is a vertex of it, and by Lemma 3.24 y is not. Suppose y lay in a face G'' of Δ of dimension at most two; we may take $G'' \subseteq u^\vee$. Then $\omega \cap G''$ is a face of ω , hence a cell of $\mathcal{S}$ contained in G'' , that is, a cell of σ or a face of one. Such a cell has no lattice points other than its vertices (Lemma 3.1(4)), so y would be a vertex of $\omega \cap G''$, hence of ω , a contradiction. $\square$

Proposition 3.26. *Let Δ be admissible and σ a profile satisfying (Proj), with any number of pentagons and of hexagons on either branch. Then:*

- (1) $\mathcal{F}_C$ is regular: there is a function h_C with integral slopes, strictly convex on the fan over $\mathcal{F}_C$, that induces $\mathcal{F}_C$; every maximal cell of the corresponding subdivision of $\mathcal{A}$ contains 0;
- (2) $\mathcal{F}_C$ refines Σ' and agrees with it on the faces of Δ of dimension at most two; so the two-cells of $\mathcal{F}_C$ in two-faces of Δ are unimodular triangles and node parallelograms;
- (3) every lattice point of $\partial\Delta$ is a vertex of $\mathcal{F}_C$;
- (4) (height condition) for every node parallelogram $P \subset G$, $G^\circ = [u, u']$, let $\omega_1 \subseteq u^\vee$ and $\omega_2 \subseteq u'^\vee$ be the three-cells of $\mathcal{F}_C$ containing P . Every lattice point of ω_1 not in P has $e_{u'} = 1$, and every lattice point of ω_2 not in P has $e_{u'} = 1$.

Proof. (1) By Lemma 3.23, applied at each step, $\mathcal{F}_C$ is induced by a height on $\mathcal{A}$ and every maximal cell of the corresponding subdivision of $\mathcal{A}$ contains 0; the integral slopes of h_C come from the choice of h_C after the definition of $\mathcal{F}_C$. (2) Pulling j changes only the cells whose convex hull contains j , and refines them. Every lattice point of a face of dimension at most two lies in R and is a vertex of Σ' (Section 3.1), so every pulled point lies in the relative interior of a facet of Δ , and no cell in a face of dimension at most two contains it. By induction these cells are never changed. (3) Each pulled point becomes a vertex, and vertices are never lost (Lemma 3.23).

(4) Let $\omega = \omega_1$. The set $\omega \cap u^\vee = \omega \cap G$ is a face of ω and, by (2), a cell of σ in G containing P ; so it is P . Hence every vertex of ω off P has $e_{u'} \geq 1$. If a vertex had $e_{u'} \geq 2$, Lemma 3.25 would give a lattice point of ω that is not a vertex of ω ; by (3) it is a vertex of $\mathcal{F}_C$, and a vertex of a polyhedral complex that lies in a cell is a vertex of that cell. So every vertex of ω off P has $e_{u'} = 1$. As $e_{u'}$ is affine and takes values in $[0, 1]$ on ω , every lattice point of ω has $e_{u'} \in \{0, 1\}$, and those with $e_{u'} = 0$ lie in $\omega \cap G = P$. The statement for ω_2 follows by exchanging u and u' .

The face type enters only through three facts, which hold for every profile: $\ell(G^\circ) = 1$ at the face of P (used in Lemma 3.2); every cell of σ , and every face of one, has no lattice points besides its vertices (used in Lemma 3.25; Lemma 3.1(4) for pentagons and for hexagons on both branches); and every point of R , the hexagon centres included, is a vertex of Σ' (used in (2)). A node parallelogram is a two-cell of $\sigma|_G$, so no other cell of $\sigma|_G$ contains it; at an A_2 hexagon this applies to each of the three rhombi, although they share edges pairwise. $\square$

The lattice points of $\omega_1 \cup \omega_2$ not in P are the *apexes* of P in Section 7.1. The proof of (4) shows more.

Remark 3.27. (1) Let $\mathcal{S}$ be a regular subdivision of $\partial\Delta$ into lattice polytopes that induces σ on the two-faces and in which the three-cells containing node parallelograms have no lattice points other than their vertices. Then the height condition holds for $\mathcal{S}$, by the argument of (4) with Lemma 3.25. In particular it holds for Σ' itself whenever every lattice point of $\partial\Delta$ is a vertex of Σ' ; this is the case for the subdivisions Σ' of the examples X_9 , X_{19} and X_{20} in Section 11, and there $\mathcal{F}_C = \Sigma'$.

(2) In general a subdivision Σ' satisfying (Proj) does not satisfy the height condition. The polytope Δ_9 of the example X_9 has a subdivision satisfying (Proj) whose vertices are all the lattice points of $\partial\Delta_9$ except $(-1, 0, 0, 0)$; in it the three-cell over one of the pentagon parallelograms P , in the facet $u = (1, -1, -1, -1)$ with $u' = (0, 1, -1, -1)$, is the pyramid $\text{conv}(P \cup \{(0, 1, 0, 0)\})$ with $e_{u'}((0, 1, 0, 0)) = 2$, whose only lattice point at height one is $(-1, 0, 0, 0)$. Pulling this point gives the Σ' of Section 11. On the list of Section 11.4, the subdivisions Σ' obtained by solving the linear feasibility problem of (Proj) violate the height condition on 7 of the 364 hexagon-free polytopes satisfying (Proj), with apexes at height 2 or 3, while $\mathcal{F}_C$ satisfies it on all of them (exact computation; the results for each polytope are in the accompanying data, <https://github.com/bwuebben/calabi-yau-smoothability>, directory paper6).

3.6. When (Proj) fails. This section does not assume (Proj). We show that (Proj) is equivalent to the projectivity of $X'_\sigma \rightarrow X$, and describe what a failure means for the exceptional surfaces. For an interior edge $\epsilon = [o, w_j]$ of $\sigma|_G$, G a pentagon or a hexagon, put

$$I_\epsilon := e_{w_{j-1}} + e_{w_{j+1}} - a_j e_{w_j} + (a_j - 2)e_o \in \mathbb{Z}^R, \quad (w_{j-1} - o) + (w_{j+1} - o) = a_j(w_j - o).$$

For a function ψ on $G \cap N$ that is affine on the cells $L \supset T_{j-1}$ and $L' \supset T_j$ of $\sigma|_G$ at ϵ , $\langle I_\epsilon, \psi \rangle = \psi(w_{j+1}) - \ell_L(w_{j+1})$, where ℓ_L is the affine function equal to ψ on L : the *bend* of ψ across ϵ , measured at a point at lattice distance one from ϵ .

Lemma 3.28. (*Proj*) holds if and only if there is $\psi \in \mathbb{Q}^R$ with $\langle \text{circ}_q, \psi \rangle = 0$ for every $q \in Q$ and $\langle I_\epsilon, \psi \rangle > 0$ for every interior edge ϵ of σ at a pentagon or a hexagon.

Proof. A function ψ on $G \cap N$ induces $\sigma|_G$ if and only if it is affine on each cell, which for a parallelogram is $\langle \text{circ}_q, \psi \rangle = 0$, and bends positively across each interior edge; local convexity across the interior edges implies convexity on G . If (*Proj*) holds, the restriction to R of a strictly convex function inducing Σ' has these properties. Conversely, given ψ , let $\mathcal{S}_\Phi$ be the regular subdivision of $\Phi \cap R$ induced by ψ on each facet Φ of Δ . On a face of Φ it restricts to the regular subdivision induced by the restriction of ψ , which on two-faces is σ and on edges is trivial; so the $\mathcal{S}_\Phi$ agree on common faces and form a subdivision $\mathcal{S}$ of $\partial\Delta$ with vertices in R . Let g be the function on $N_\mathbb{R}$ that is linear on the cone over each cell of $\mathcal{S}$ and equals ψ on R . It is convex across the walls inside the cones over facets. The function φ that equals 1 on $\partial\Delta$ and is linear on the cones over the facets is convex, with positive bend across each wall between two facet cones, and linear inside them; so for γ large, $g + \gamma\varphi$ is strictly convex on the fan over $\mathcal{S}$. The toric variety of this fan is projective, its rays pass through points of $R \subset \partial\Delta$, and it induces σ . $\square$

Let $\widehat{X}$ be the small resolution with the chosen diagonals, which exists as a compact complex manifold without (*Proj*) (Section 3.2). For an interior edge $\epsilon = [o, w_j]$ of $\sigma|_G$, let $f_\epsilon \subset E'_G$ be the torus-invariant curve of the cone over ϵ in the local model $U'_{G,\sigma}$, and $\tilde{f}_\epsilon \subset E_G$ its proper transform in $\widehat{X}$, which is the curve f_j of Proposition 3.9. In the local model of $\widehat{X}$ the curve $\tilde{f}_\epsilon$ is the curve of the wall cone(o, w_j) between the cones over T_{j-1} and T_j , and the wall relation $w_{j-1} + w_{j+1} - a_j w_j + (a_j - 2)o = 0$ gives $(D_r \cdot \tilde{f}_\epsilon)_{r \in R} = I_\epsilon$.

Proposition 3.29. *The following are equivalent:*

- (1) (*Proj*) holds;
- (2) $X'_\sigma \rightarrow X$ is projective;
- (3) there are no rational numbers $y_\epsilon \geq 0$, not all zero, and ν_q with $\sum_\epsilon y_\epsilon [\tilde{f}_\epsilon] = \sum_q \nu_q [C_q]$ in $H_2(\widehat{X}; \mathbb{Q})$.

Proof. (1) $\Rightarrow$ (2): $X'_\sigma \rightarrow X$ is induced by the projective morphism $P'_\sigma \rightarrow \mathbb{P}_\Delta$. (2) $\Rightarrow$ (3): let L be relatively ample for $X'_\sigma \rightarrow X$ and $\pi: \widehat{X} \rightarrow X'_\sigma$. Then $\pi^* L \cdot C_q = 0$, since C_q is contracted by π , and $\pi^* L \cdot \tilde{f}_\epsilon = L \cdot f_\epsilon > 0$, since f_ϵ is a curve in a fibre of $X'_\sigma \rightarrow X$. A relation as in (3) would pair with $\pi^* L$ to a positive number on the left and to 0 on the right. (3) $\Rightarrow$ (1): if (*Proj*) fails, the linear system of Lemma 3.28 is infeasible, and by the Farkas–Motzkin alternative there are $y_\epsilon \geq 0$, not all zero, and ν_q with $\sum_\epsilon y_\epsilon I_\epsilon = \sum_q \nu_q \text{circ}_q$ in $\mathbb{Q}^R$. By Proposition 3.4(2) and the computation above, the class $\sum_\epsilon y_\epsilon [\tilde{f}_\epsilon] - \sum_q \nu_q [C_q]$ has zero image under (3.2), which is injective without (*Proj*) (Lemma 3.3); so it vanishes. $\square$

A relation as in (3) is a *Farkas certificate*. Where (*Proj*) fails there is no toric variety P'_σ , no subdivision Σ' and no base B'_σ , and the methods of this paper do not apply.

4. THE COMPLEX OF DUAL CELL PAIRS

The base B'_σ of X'_σ and the base B_Y of the mirror family are built from the same two subdivisions, of $\partial\Delta^\circ$ by $\mathcal{T}$ and of $\partial\Delta$ by a fan-side subdivision, with the roles of Newton side

and fan side exchanged. Gross proved that the base built from Σ' is Legendre dual, after a rescaling of the polarization, to a base on $\partial\nabla^{m_0h+m_1\varphi}$, for a suitable h inducing Σ' and suitable positive integers m_0, m_1 , built from the same Σ' with a decomposition that he constructs from the polarization [Gro05, Thms. 3.21, 3.25, 3.26] (Remark 4.8); the discrete Legendre transform matches the cells of two polyhedral decompositions. The decompositions attached to the degenerations used here need not be Legendre dual to one another (compare [Gro05, §3, after Rem. 3.18]), and the mirror base is built from the pulling refinement $\mathcal{F}_C$ of Σ' rather than from Σ' itself. This section therefore introduces a coarser structure, used in both parts of the paper: a regular cell decomposition of each base whose cells are indexed by pairs of dual cells of $\mathcal{T}$ and of the subdivision of $\partial\Delta$ (Proposition 4.7). Its face poset is the same on both bases, it needs no polarization, and it carries the discriminants (Proposition 4.10) and a flat perfect pairing between the two local systems of integral tangent vectors (Proposition 4.11). Section 5 uses it to compute with relative tropical 2-cycles on B'_σ , and Section 6 to transfer them to B_Y . The proof that the cells are balls is in Appendix C.

4.1. Faces, decompositions and charts. Throughout this section $\mathcal{T}$ and $\check{h}$ are as in Section 2.3, with the normalisation of Section 3.3: $\mathcal{T}$ is a unimodular triangulation of $\partial\Delta^\circ$ refining its faces, and the integral height $\check{h}$ and $\check{h}' := \check{h} - \check{\varphi}$ are strictly convex on the fan over $\mathcal{T}$, that is, convex, with the cones over the tetrahedra of $\mathcal{T}$ as their maximal domains of linearity. In the language of Section 2.1, $(\Delta^\circ, \mathcal{T}, \check{h})$ is the Newton side of the degeneration of X'_σ .

Definition 4.1 (fan-side subdivisions). A *fan-side subdivision* is a subdivision $\mathcal{F}$ of $\partial\Delta$ into lattice polytopes, refining the faces of Δ , that is induced by a function $h: N_\mathbb{R} \rightarrow \mathbb{R}$ which is piecewise linear and strictly convex on the fan over $\mathcal{F}$: convex, with the cones over the cells of $\mathcal{F}$ as its maximal domains of linearity.

A fan-side subdivision is a regular polyhedral subdivision of $\partial\Delta$ in the sense of [DLRS10, §2.3], and the fan over it is the fan of a projective toric variety refining $\mathbb{P}_\Delta$. The fan-side subdivisions of this paper are Σ' (Hypothesis (Proj)), the pulling refinement $\mathcal{F}_C$ (Section 3.5) and the regular refinements of Section 10.

Cells of $\mathcal{T}$ are denoted τ , cells of $\mathcal{F}$ are denoted ω , vertices of $\mathcal{T}$ are denoted u, u' and vertices of $\mathcal{F}$ are denoted n, n' . For $\tau \in \mathcal{T}$ and $\omega \in \mathcal{F}$ put

$$\begin{aligned}\tau^\vee &= \{n \in \Delta : \langle u, n \rangle = -1 \text{ for all vertices } u \text{ of } \tau\}, \\ \omega^\vee &= \{m \in \Delta^\circ : \langle m, n \rangle = -1 \text{ for all vertices } n \text{ of } \omega\}.\end{aligned}$$

If G is the smallest face of Δ° containing τ , then τ meets the relative interior of G , and an affine function that is ≥ -1 on G and equal to -1 at a relative interior point is -1 on G ; so $\tau^\vee$ is the face of Δ dual to G , of dimension $3 - \dim G \leq 3 - \dim \tau$. The same holds for $\omega^\vee$. We write $\text{ri } Z$ for the relative interior of a convex set Z , $\text{cone}(\tau) = \mathbb{R}_{\geq 0}\tau$, and $L_\tau \subset M_\mathbb{R}$ for the linear span of τ ; since the vertices of τ lie in a hyperplane $\{\langle \cdot, n \rangle = -1\}$, $\dim L_\tau = \dim \tau + 1$.

Lemma 4.2 (faces). For $\tau \in \mathcal{T}$ let $\nabla_\tau^{\check{h}'} = \{w \in \nabla^{\check{h}'} : \langle u, w \rangle = -\check{h}'(u) \text{ for all vertices } u \text{ of } \tau\}$, where $\nabla^{\check{h}'} = \{w \in N_\mathbb{R} : \langle m, w \rangle \geq -\check{h}'(m) \text{ for all } m \in M_\mathbb{R}\}$.

- (1) $\nabla_\tau^{\check{h}'}$ is the face of the polytope $\nabla^{\check{h}'}$ with inner normal cone $\text{cone}(\tau)$; $\dim \nabla_\tau^{\check{h}'} = 3 - \dim \tau$, and $\nabla_\tau^{\check{h}'} \subseteq \nabla_{\tau'}^{\check{h}'}$ if and only if $\tau \supseteq \tau'$.
- (2) $\nabla^{\check{h}} = \Delta + \nabla^{\check{h}'}$, its normal fan is the fan over $\mathcal{T}$, and the face with inner normal cone $\text{cone}(\tau)$ is $F_\tau = \tau^\vee + \nabla_\tau^{\check{h}'}$. In particular $\tau^\vee \neq \emptyset$, $\dim F_\tau = 3 - \dim \tau$, and $F_\tau \subseteq F_{\tau'}$ if and only if $\tau \supseteq \tau'$.

We write $F_u = F_{\{u\}}$ for the facets and $F_{uu'} = F_{[u,u']}$ for the two-faces. The proof is in Appendix C.1.

Gross's decomposition. Let $\tilde{\Sigma}$ be the normal fan of the epigraph $\text{Epi} = \{(m, l) : m \in \Delta^\circ, l \geq \check{h}'(m)\}$ [Gro05, Prop. 3.7], and for a vertex u of $\mathcal{T}$ let $\tilde{\Sigma}_u$ be its normal cone at the vertex $(u, \check{h}'(u))$. The *Cayley polytope* of u is

$$\text{Cay}_u = \text{conv}(u^\vee \times \{0\} \cup \nabla_u^{\check{h}'} \times \{1\}) \subset N_{\mathbb{R}} \oplus \mathbb{R}.$$

It lies in the affine hyperplane $\{\langle u, y \rangle + (\check{h}(u) - 2)l = -1\}$, since $\langle u, a \rangle = -1$ for $a \in u^\vee$ and $\langle u, w \rangle = -\check{h}'(u) = 1 - \check{h}(u)$ for $w \in \nabla_u^{\check{h}'}$. The maximal cones of $\tilde{\Sigma}$ are the $\tilde{\Sigma}_u$ and the normal cone $\tilde{\Sigma}_{\text{up}} = \text{cone}(\nabla^{\check{h}'} \times \{1\})$ at the vertex $(0, 0)$ of Epi [Gro05, Prop. 3.7].

Given a fan-side subdivision $\mathcal{F}$ with function h , we fix the following subdivision $\tilde{\Sigma}'$ of $\tilde{\Sigma}$: on each $\tilde{\Sigma}_u$ its cones are the cones over the cells of the regular subdivision of the point configuration $(u^\vee \cap N) \times \{0\} \cup \text{vert } \nabla_u^{\check{h}'} \times \{1\}$ with the heights $h(a)$ at $(a, 0)$ and 0 at $(w, 1)$, and $\tilde{\Sigma}_{\text{up}}$ is not subdivided. These subdivisions agree on common faces, since the heights are global and vanish on the upper points. The trace of $\tilde{\Sigma}'$ on $N_{\mathbb{R}} \times \{0\}$ is the fan over $\mathcal{F}$, and its other rays are generated by the vectors $(w, 1)$, w a vertex of $\nabla^{\check{h}'}$. Such a w is a lattice point: it is $\nabla_\tau^{\check{h}'}$ for a tetrahedron $\tau = \{u_1, \dots, u_4\}$ of $\mathcal{T}$, so $\langle u_i, w \rangle = -\check{h}'(u_i) \in \mathbb{Z}$, and $u_1, \dots, u_4$ is a basis of M because $\mathcal{T}$ is unimodular. So $(w, 1)$ is primitive, and $\tilde{\Sigma}'$ is good for $\mathcal{F}$ in the sense of [Gro05, Def. 3.8]. It is the subdivision of [Gro05, Rem. 3.18] determined by the values h on $N_{\mathbb{R}} \times \{0\}$ and 0 on $\tilde{\Sigma}_{\text{up}}$, because on $\tilde{\Sigma}_u$ the lower envelope of these values is the cone over the lower envelope of the heights on the slice Cay_u (Lemma 4.3(1)). We let $B_X = B_X(\mathcal{F})$ be $\partial \nabla^{\check{h}}$ with the decomposition $\mathcal{P}_X = \mathcal{P}_X^{\tilde{\Sigma}'}$ of [Gro05, Def. 3.13]: a mixed cone with lower generators $\omega \times \{0\}$ and upper generators $\tau_1 \times \{1\}$ gives the cell $\omega + \text{conv } \tau_1$. For $\mathcal{F} = \Sigma'$ this is the base B'_σ with its decomposition $\mathcal{P}$ (Section 3.3).

Lemma 4.3 (the cells as a mixed subdivision). (1) $\tilde{\Sigma}_u$ is the cone over Cay_u .

(2) For $\tau \in \mathcal{T}$, the cells of $\mathcal{P}_X$ contained in F_τ are the cells of the regular mixed subdivision of $\tau^\vee + \nabla_\tau^{\check{h}'}$ with the heights h on $\tau^\vee \cap N$ and 0 on $\nabla_\tau^{\check{h}'}$: the projections to $N_{\mathbb{R}}$ of the bounded faces of $\hat{A}_\tau + \hat{\nabla}_\tau$, where

$$\hat{A}_\tau = \text{conv}\{(a, h(a)) : a \in \tau^\vee \cap N\} + \mathbb{R}_{\geq 0}(0, 1), \quad \hat{\nabla}_\tau = \nabla_\tau^{\check{h}'} \times \{0\} + \mathbb{R}_{\geq 0}(0, 1).$$

Each such cell is $c = \omega + \nabla_{\tau'}^{\check{h}'}$ with $\omega \in \mathcal{F}$, $\omega \subseteq \tau^\vee$, and $\tau' \in \mathcal{T}$, $\tau' \supseteq \tau$, and ω and τ' are determined by c .

The proof, by the Cayley trick, is in Appendix C.1. For a cell c of $\mathcal{P}_X$ we call $\omega(c)$ its *lower support*, as in Section 2.3, and the $\tau(c) \in \mathcal{T}$ with $\text{ri } c \subseteq \text{ri } F_{\tau(c)}$ its *face label*; $F_{\tau(c)}$ is the smallest face of $\nabla^{\check{h}}$ containing c . If $c \subseteq c'$, then $\tau(c) \supseteq \tau(c')$ and $\omega(c) \subseteq \omega(c')$, the latter because the lifted face of c is a face of the lifted face of c' . The barycentric subdivision $\text{Bar } \mathcal{P}_X$ has one simplex for each chain $c_0 \subsetneq \dots \subsetneq c_k$ of cells.

Regions, discriminant and charts. For a vertex n of $\mathcal{F}$ let $\tilde{R}_n$ be the union of the open simplices of $\text{Bar } \mathcal{P}_X$ whose chain has $\omega(c_0) = \{n\}$, and let Γ_X be the union of the closed simplices whose chain has $\dim \omega(c_0) \geq 1$ and $\dim \tau(c_k) \geq 1$. A maximal such chain is $e \subsetneq f$ with f a two-cell in a two-face $F_{uu'}$ and e an edge of f with a two-point lower support; we call the segment $[b_e, b_f]$ between their barycentres a *leg* of Γ_X . For $\mathcal{F} = \Sigma'$ these are the legs of Section 2.3 (Lemma 3.12), and Γ_X is the discriminant Γ of B'_σ .

- Lemma 4.4** (charts). (1) $\tilde{R}_n$ is open. If $b \in \tilde{R}_n$, then $\langle u, n \rangle = -1$ for every facet F_u containing b , and the projection $\pi_n: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R}n$ restricts to a homeomorphism of a neighbourhood of b in B_X onto an open set, which on each facet F_u is the restriction of an affine isomorphism $\text{aff } F_u \rightarrow N_{\mathbb{R}}/\mathbb{R}n$ with linear part the lattice isomorphism $u^\perp \cap N \rightarrow N/\mathbb{Z}n$.
- (2) $B_X \setminus \Gamma_X = \bigcup_u \text{int } F_u \cup \bigcup_n \tilde{R}_n$, and Γ_X is the union of the open simplices whose chain has $\dim \omega(c_0) \geq 1$ and $\dim \tau(c_k) \geq 1$.
- (3) The facet charts on the $\text{int } F_u$ and the projections π_n on the $\tilde{R}_n$ form an integral affine atlas on $B_X \setminus \Gamma_X$, which restricts to Gross's structure [Gro05, Prop. 3.14] on the complement of Gross's discriminant.

We write Λ_X for the local system of integral tangent vectors of this structure. For $\mathcal{F} = \Sigma'$ it is the local system Λ of Section 2.3. The proof is in Appendix C.1.

The mirror base. Let $H: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be piecewise linear on the fan over $\mathcal{F}$, with integral slopes (on each cone of that fan H is the restriction of an element of M), and such that $H' := H - \varphi$ is strictly convex on that fan. The height $H = k_0\varphi + h_C$ of the mirror family (Section 2.4) is such a function for $\mathcal{F} = \mathcal{F}_C$: φ is convex and linear on the cones over the cells of $\mathcal{F}_C$, so $H' = (k_0 - 1)\varphi + h_C$, $k_0 \geq 1$, has the maximal domains of linearity of h_C ; and H has integral slopes because φ does, Δ being reflexive, and h_C does by choice. The vertices of $\nabla^{H'}$ are lattice points: the vertex with inner normal cone $\text{cone}(\omega)$, ω a maximal cell, is $-m_\omega$, where $m_\omega \in M$ is the slope of H' on $\text{cone}(\omega)$. The mirror base is $B_Y = \partial \nabla^H$, $\nabla^H = \{m \in M_{\mathbb{R}} : \langle m, n \rangle \geq -H(n) \ \forall n\}$. It is the same construction with the roles of $(\Delta^\circ, \mathcal{T}, \check{h})$ and $(\Delta, \mathcal{F}, H)$ exchanged: $(\Delta, \mathcal{F}, H)$ is the Newton side and $(\Delta^\circ, \mathcal{T}, \check{h})$ the fan side. Its faces are $F_\omega^* = \omega^\vee + \nabla_\omega^{H'}$, $\omega \in \mathcal{F}$, with $\nabla_\omega^{H'}$ the face of $\nabla^{H'}$ with inner normal cone $\text{cone}(\omega)$; its decomposition $\mathcal{P}_Y$ is defined by the heights $\check{h}$ on the lower points and 0 on the upper vertices; a cell c of $\mathcal{P}_Y$ has a face label $\omega_Y(c) \in \mathcal{F}$ and a lower support $\tau_Y(c) \in \mathcal{T}$; the regions are the $\tilde{R}_u^*$, with charts $M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\mathbb{R}u$; the facet F_n^* carries the lattice $n^\perp \cap M$; and the discriminant Γ_Y and the local system Λ_Y are defined as above. Lemmas 4.2–4.4 hold for B_Y with the same proofs, which use only the strict convexity of the Newton-side function ($\check{h}'$, respectively H') and of the fan-side function (h , respectively $\check{h}$). The subdivision of $\tilde{\Sigma}_Y$ that defines $\mathcal{P}_Y$ is good in the sense of [Gro05, Def. 3.8], by the argument above with the vertices of $\nabla^{H'}$ in place of those of $\nabla^{\check{h}'}$. So, by [Gro05, Prop. 3.14] with the roles exchanged, $(B_Y, \mathcal{P}_Y)$ is the dual intersection complex of Gross's toric degeneration $X'_Y \subset X(\tilde{\Sigma}'_Y)$ [Gro05, Thm. 3.10] attached to the good subdivision $\tilde{\Sigma}'_Y$ that defines $\mathcal{P}_Y$. Its equation is $ts + s^0 = 0$, with s^0 the section given by the lattice point $(0, 0)$ and s a general section with Newton polytope $\{(n, l) : n \in \Delta, l \geq H'(n)\}$; the members $\{\sum_n c_n t^{H(n)} x^n = 0\}$ of the mirror family with $c_0 = 1$ have this form, with $s = \sum_{n \neq 0} c_n t^{H'(n)} x^n$, since $H = H' + \varphi$, $\varphi = 1$ on $\partial \Delta$ and $H(0) = 0$. Since $H(0) = 0$, the polytope ∇^H is the region on which the term $n = 0$ of the tropical polynomial $m \mapsto \min_n (H(n) + \langle m, n \rangle)$ attains the minimum, and B_Y is the target of Yamamoto's tropical contraction of the tropical hypersurface [Yam21, Thm. 1.2(2), (5.36), Prop. 5.4]; we do not use the contraction. We write $B_X(\mathcal{F})$ and $B_Y(\mathcal{F})$ when the fan-side subdivision must be specified. Thus $B'_\sigma = B_X(\Sigma')$, and the mirror base of this paper is $B_Y = B_Y(\mathcal{F}_C)$ with the height H of Section 2.4.

4.2. The complex of dual cell pairs.

Definition 4.5 (dual cell pairs). A *dual cell pair* is a pair $(\tau, \omega) \in \mathcal{T} \times \mathcal{F}$ with $\langle u, n \rangle = -1$ for all vertices u of τ and n of ω ; equivalently $\omega \subseteq \tau^\vee$, or $\tau \subseteq \omega^\vee$. The set DP of dual cell

pairs is ordered by $(\tau, \omega) \preceq (\tau', \omega')$ if $\tau \supseteq \tau'$ and $\omega \supseteq \omega'$. We put $\dim \mathbf{p} = 3 - \dim \tau - \dim \omega$ for $\mathbf{p} = (\tau, \omega)$; since $\dim \omega \leq \dim \tau^\vee \leq 3 - \dim \tau$, $\dim \mathbf{p} \geq 0$.

In the language of Batyrev duality, (τ, ω) is a dual cell pair if and only if τ and ω lie in a pair of mutually dual faces of Δ° and Δ .

Definition 4.6 (labels). A point $b \in B_X$ lies in the open simplex of a unique chain $c_0 \subsetneq \cdots \subsetneq c_k$ of $\mathcal{P}_X$; put $\text{lab}_X(b) = (\tau(c_k), \omega(c_0))$, the face label of the top and the lower support of the bottom. For $b \in B_Y$ in the open simplex of a chain $c_0 \subsetneq \cdots \subsetneq c_k$ of $\mathcal{P}_Y$, put $\text{lab}_Y(b) = (\tau_Y(c_0), \omega_Y(c_k))$. For $\mathbf{p} \in \text{DP}$ let $C_X(\mathbf{p}) = \text{lab}_X^{-1}(\mathbf{p})$ and $C_Y(\mathbf{p}) = \text{lab}_Y^{-1}(\mathbf{p})$.

Proposition 4.7 (the complex of dual cell pairs). (1) lab_X and lab_Y take values in DP .

- (2) The sets $C_X(\mathbf{p})$, $\mathbf{p} \in \text{DP}$, are the open cells of a regular CW decomposition of B_X with face poset $(\text{DP}, \preceq)$: each closure $\overline{C_X(\mathbf{p})} = \bigcup_{\mathbf{q} \preceq \mathbf{p}} C_X(\mathbf{q})$ is a subcomplex of $\text{Bar } \mathcal{P}_X$ and a piecewise linear ball of dimension $\dim \mathbf{p}$, with boundary sphere $\bigcup_{\mathbf{q} \prec \mathbf{p}} C_X(\mathbf{q})$. In particular every $C_X(\mathbf{p})$ is nonempty.
- (3) The same holds for the $C_Y(\mathbf{p})$ on B_Y , with the same poset.
- (4) There is a piecewise linear homeomorphism $\mathcal{L}: B_X \rightarrow B_Y$ with $\mathcal{L}(C_X(\mathbf{p})) = C_Y(\mathbf{p})$ for all $\mathbf{p} \in \text{DP}$, and any two such homeomorphisms are isotopic through homeomorphisms with this property.
- (5) $C_X(\tau, \omega) \subseteq \text{ri } F_\tau$ and $C_Y(\tau, \omega) \subseteq \text{ri } F_\omega^*$. Moreover, for every $\tau \in \mathcal{T}$ and every vertex n of $\mathcal{F}$, $\text{ri } F_\tau = \bigcup_\omega C_X(\tau, \omega)$ and $\tilde{R}_n \cap \text{ri } F_\tau = C_X(\tau, n)$; in particular $\text{int } F_u = \bigcup_\omega C_X(u, \omega)$ and $\tilde{R}_n = \bigcup_\tau C_X(\tau, n)$. Likewise on B_Y , with the roles of τ and ω exchanged. Hence $\mathcal{L}(\text{int } F_u) = \tilde{R}_u^*$ and $\mathcal{L}(\tilde{R}_n) = \text{int } F_n^*$: $\mathcal{L}$ exchanges the open facets with the regions.

We call the decomposition of (2) and (3) the *complex of dual cell pairs* of B_X and of B_Y , and a homeomorphism as in (4) a *label-preserving cellular homeomorphism*. On a face F_τ the cells $C_X(\tau, \omega)$ correspond to the cells of the polyhedral complex dual to the regular subdivision $\mathcal{F}|_{\tau^\vee}$, that is, of the tropical hypersurface of $m \mapsto \min_{a \in \tau^\vee \cap N} (h(a) + \langle m, a \rangle)$ [MS15, §3.1], refined by the normal fan of $\nabla_\tau^{h'}$ and truncated by a level set of a gauge function of $\nabla_\tau^{h'}$, which makes the unbounded cells bounded (Lemmas C.1 and C.3). The cells of $\mathcal{P}_X$ are in general not unions of cells of the complex of dual cell pairs: a maximal cell $c \subset F_u$ with lower support an edge $[n, n']$ has its barycentre in $C_X(u, [n, n'])$, while the open simplex of the chain $(n + w \subsetneq c)$, for a vertex $n + w$ of c , lies in $C_X(u, n)$.

Pairs of dual cells were used by Haase and Zharkov [HZ02, HZ03, HZ05], who realise two mutually dual integral affine structures [HZ02, §2.4], [HZ03, §2.2] on a sphere subdivided by the pairs of barycentres of simplices lying in mutually dual faces (the analogue of the dual cell pairs of Definition 4.5) of coherent triangulations of the boundaries of a polar pair of reflexive polytopes [HZ02, §2.1]; for nef-partitions they work with central subdivisions induced by height functions, which need not be triangulations [HZ05, §1, §2.2]. Proposition 4.7 gives the corresponding cell structures on Gross's bases B_X and B_Y themselves, for a fan-side subdivision that need not be a triangulation, and Proposition 4.11 the duality of the lattices. Posets of pairs of dual cells also underlie the mirror isomorphism of tropical homology groups of Matessi and Renaudineau [MR25, Thm. 1.1, §2]. A label-preserving cellular homeomorphism is a piecewise linear homeomorphism matching two cell decompositions; it is not a discrete Legendre transform in the sense of [GS06, §1.4], which requires a polarization.

Remark 4.8 (the discrete Legendre transform). In the proof of [Gro05, Thm. 3.26], Gross takes $\tilde{h} \equiv 0$ on $\tilde{\Sigma}_{\text{up}}$ and $\tilde{h} = h$ on $N_{\mathbb{R}} \times \{0\}$ and subdivides each cone of $\tilde{\Sigma}$ coherently by

these values [Gro05, Rem. 3.18]; this is the subdivision $\tilde{\Sigma}'$ fixed above. Gross's h is integral, so that $\tilde{h}$ has rational slopes. Here h may be taken with rational slopes: the functions that induce $\mathcal{F}$ and the same $\tilde{\Sigma}'$ form a nonempty set cut out by finitely many linear equations and strict linear inequalities with rational coefficients, so they include one with rational slopes, and B_X , its decomposition and Λ_X depend only on $\tilde{h}$ and $\tilde{\Sigma}'$. For suitable positive integers m_0, m_1 the function $m_0\tilde{h} + m_1\tilde{\varphi}$ is then integral and strictly convex on the same $\tilde{\Sigma}'$, and $m_0\tilde{h} + (m_1 - 1)\tilde{\varphi}$ is convex; here $\tilde{\varphi}$, the function of [Gro05, §3] defined by Epi, is linear on each cone of $\tilde{\Sigma}$. So the pair $(B_X, \mathcal{P}_X)$ is polarized by $m_0\tilde{h} + m_1\tilde{\varphi}$, and [Gro05, Thms. 3.21, 3.25] identify its discrete Legendre transform with the base on $\partial\nabla^H$, $H = m_0h + m_1\varphi$, built from the same $\mathcal{F}$, carrying a decomposition that Gross constructs from the polarization, through the normal fan of a Minkowski sum [Gro05, §3, the construction before Lemma 3.19, and Thm. 3.21]. The decomposition $\mathcal{P}_Y$ defined by the heights $(\tilde{h}, 0)$, the one attached to the degeneration of the mirror family, need not be that decomposition (compare [Gro05, §3, after Rem. 3.18]), and the complex of dual cell pairs avoids comparing the two. Propositions 4.7 and 4.11 give for the cells of the complex of dual cell pairs the statements that the discrete Legendre transform gives for cells, and they apply to $\mathcal{P}_X$ and $\mathcal{P}_Y$ alike.

Proof of Proposition 4.7. (1) For a chain $c_0 \subsetneq \cdots \subsetneq c_k$ we have $\tau(c_0) \supseteq \tau(c_k)$, and $\omega(c_0) \subseteq \tau(c_0)^\vee \subseteq \tau(c_k)^\vee$ by Lemma 4.3(2); so $(\tau(c_k), \omega(c_0))$ is a dual cell pair. On B_Y the argument is the same with the roles exchanged.

(2) and (3) are proved in Appendix C.2. In outline: fix $\mathbf{p} = (\tau, \omega)$. On the face F_τ the cells of $\mathcal{P}_X$ are in inclusion-reversing bijection with the cells of a polyhedral complex $\mathcal{R}_\tau$ in the dual of the direction space of F_τ , the common refinement of the complex dual to $\mathcal{F}|_{\tau^\vee}$ and of the normal fan of $\nabla_\tau^{\tilde{h}'}$ (Lemma C.1). Under this bijection the simplices of $\text{Bar } \mathcal{P}_X$ with label $\preceq \mathbf{p}$ form the order complex of the cells of $\mathcal{R}_\tau$ contained in the cell $D(\omega)$ dual to ω , and those with label $\prec \mathbf{p}$ the order complex of its cells in $\partial D(\omega)$ together with its unbounded cells (Lemma C.2). Truncation by a function that is linear on the normal fan of $\nabla_\tau^{\tilde{h}'}$ realises the first order complex as a linear triangulation of a convex polytope of dimension $\dim \mathbf{p}$, with the second on its boundary (Lemma C.3).

(4) Let $|\text{Ord}(\text{DP})|$ be the geometric realisation of the order complex of DP, and $\text{Ord}^{\preceq \mathbf{p}}$, $\text{Ord}^{\prec \mathbf{p}}$ the order complexes of $\{\mathbf{q} \preceq \mathbf{p}\}$ and $\{\mathbf{q} \prec \mathbf{p}\}$. We construct a piecewise linear homeomorphism $\Psi_X: |\text{Ord}(\text{DP})| \rightarrow B_X$ with $\Psi_X(|\text{Ord}^{\preceq \mathbf{p}}|) = \overline{C_X(\mathbf{p})}$ for all $\mathbf{p}$, by induction on $\dim \mathbf{p}$. For $\dim \mathbf{p} = 0$ the cell $C_X(\mathbf{p})$ is a point, and $\Psi_X(\mathbf{p})$ is that point. If Ψ_X has been constructed on the union of the $|\text{Ord}^{\preceq \mathbf{q}}|$ with $\dim \mathbf{q} < k$ and $\dim \mathbf{p} = k$, then Ψ_X maps $|\text{Ord}^{\prec \mathbf{p}}| = \bigcup_{\mathbf{q} \prec \mathbf{p}} |\text{Ord}^{\preceq \mathbf{q}}|$ homeomorphically onto $\bigcup_{\mathbf{q} \prec \mathbf{p}} \overline{C_X(\mathbf{q})} = \partial \overline{C_X(\mathbf{p})}$, a piecewise linear $(k-1)$ -sphere; $|\text{Ord}^{\preceq \mathbf{p}}|$ is the cone from the vertex $\mathbf{p}$ over $|\text{Ord}^{\prec \mathbf{p}}|$, and $\overline{C_X(\mathbf{p})}$ is piecewise linearly the convex polytope of Lemma C.3, the cone from an interior point over its boundary; so Ψ_X extends by coning. The extensions over different cells agree on overlaps, which lie in lower cells. In the same way we obtain Ψ_Y , and $\mathcal{L} = \Psi_Y \circ \Psi_X^{-1}$ has the required property. The uniqueness up to isotopy is proved in Appendix C.3.

(5) A point of $C_X(\tau, \omega)$ lies in the open simplex of a chain whose top c_k has face label τ , hence in $\text{ri } c_k \subseteq \text{ri } F_\tau$. Conversely, a point $b \in \text{ri } F_\tau$ lies in the open simplex of a chain whose top c_k satisfies $b \in \text{ri } c_k \subseteq \text{ri } F_{\tau(c_k)}$; since the relative interiors of the faces partition B_X , $\tau(c_k) = \tau$, and $\text{lab}_X(b) = (\tau, \omega(c_0))$. This gives $\text{ri } F_\tau = \bigcup_\omega C_X(\tau, \omega)$; $\tilde{R}_n = \bigcup_\tau C_X(\tau, n)$ is the definition of $\tilde{R}_n$; and as $C_X(\tau', n) \subseteq \text{ri } F_{\tau'}$, which is disjoint from $\text{ri } F_\tau$ for $\tau' \neq \tau$, also

$\tilde{R}_n \cap \text{ri } F_\tau = C_X(\tau, n)$. On B_Y the roles of the two labels are exchanged: $\text{ri } F_\omega^* = \bigcup_\tau C_Y(\tau, \omega)$ and $\tilde{R}_u^* = \bigcup_\omega C_Y(u, \omega)$. Applying $\mathcal{L}$ gives the last statement. $\square$

The decomposition $\mathcal{P}_X$ depends on the height $\check{h}$ only through its geometry, not through its combinatorics.

Lemma 4.9 (independence of the height). *Let $\check{h}_1, \check{h}_2$ be integral heights on $\mathcal{T}$ with $\check{h}_i$ and $\check{h}_i - \check{\varphi}$ strictly convex on the fan over $\mathcal{T}$, and let $\mathcal{P}_X^i$ be the decomposition of $B_X^i = \partial \nabla^{\check{h}_i}$ built from $\check{h}_i$, with the same fan-side subdivision $\mathcal{F}$ and function h . Then $\omega + \nabla_{\tau'}^{\check{h}_1} \mapsto \omega + \nabla_{\tau'}^{\check{h}_2}$ is a well-defined bijection from the cells of $\mathcal{P}_X^1$ onto those of $\mathcal{P}_X^2$ which preserves inclusions, dimensions, lower supports and face labels. Consequently the barycentric subdivisions are isomorphic simplicial complexes, and the simplexwise linear map $\phi: B_X^1 \rightarrow B_X^2$ sending each barycentre to the barycentre of the corresponding cell is a piecewise linear homeomorphism that preserves the labels lab_X of Definition 4.6, hence the open facets, the regions $\tilde{R}_n$, the discriminant Γ_X and its legs with their lower supports.*

Proof. Fix $\tau \in \mathcal{T}$. By Lemmas 4.3(2) and C.1(3), $c \mapsto c^*$ is an inclusion-reversing bijection from the cells of $\mathcal{P}_X^i$ in F_τ onto the cells of the complex $\mathcal{R}_\tau$ in $M_{\mathbb{R}}/L_\tau$, with $\dim c + \dim c^* = 3 - \dim \tau$, and the cell $c = \omega + \nabla_{\tau'}^{\check{h}_i}$ corresponds to $c^* = D(\omega) \cap \text{cone}_\tau(\tau')$. The complex $\mathcal{R}_\tau$ is the common refinement of the complex $\mathcal{D}_\tau$ of the $D(\omega)$, which is defined by $\mathcal{F}$ and the values of h on $\tau^\vee \cap N$, and of the fan of the images of the cones $\text{cone}(\tau')$, $\tau' \supseteq \tau$, which is defined by $\mathcal{T}$ (Lemma C.1(1),(2)); neither depends on $\check{h}_i$. Indeed, every cell of $\mathcal{R}_\tau$ is such an intersection, and conversely, if $\zeta \in \text{ri } D(\omega) \cap \text{ri } \text{cone}_\tau(\tau')$, then $(\zeta, 1)$ selects exactly $\hat{\omega}$ in $\hat{A}_\tau$ and exactly $\nabla_{\tau'}^{\check{h}_i}$ in $\nabla_{\tau'}^{\check{h}_i}$, hence exactly the bounded face $\hat{\omega} + \nabla_{\tau'}^{\check{h}_i} \times \{0\}$ of the sum, so $\zeta \in \text{ri } c^*$ for $c = \omega + \nabla_{\tau'}^{\check{h}_i}$. Conversely, if $\zeta \in \text{ri } c^*$, then $(\zeta, 1)$ selects exactly $\hat{c} = \hat{\omega} + \nabla_{\tau'}^{\check{h}_i} \times \{0\}$, and since the face of a Minkowski sum selected by a functional is the sum of the faces selected on the summands, uniquely (proof of Lemma C.1(3),(4)), it selects exactly $\hat{\omega}$ in $\hat{A}_\tau$ and exactly $\nabla_{\tau'}^{\check{h}_i}$ in $\nabla_{\tau'}^{\check{h}_i}$; so $\zeta \in \text{ri } D(\omega) \cap \text{ri } \text{cone}_\tau(\tau')$ by Lemma C.1(1),(2). Hence $\text{ri } c^* = \text{ri } D(\omega) \cap \text{ri } \text{cone}_\tau(\tau')$, and a cell of $\mathcal{R}_\tau$ determines the pair (ω, τ') with this property, because the relative interiors of the cells of $\mathcal{D}_\tau$, and those of the cones $\text{cone}_\tau(\tau')$, are pairwise disjoint. So the set of pairs (ω, τ') for which $\omega + \nabla_{\tau'}^{\check{h}_i}$ is a cell in F_τ , and the inclusions and dimensions of these cells, are the same for $i = 1, 2$; the face label is read from the recession cone of c^* (Lemma C.1(5)). Every cell lies in a facet, and two cells $c \subseteq c'$ lie in a common facet F_u , u a vertex of $\tau(c')$; taking $\tau = u$ gives the bijection with its properties. The maps obtained from different facets agree: a cell c lies in every facet F_u with u a vertex of $\tau(c)$, and the pair (ω, τ') with $c = \omega + \nabla_{\tau'}^{\check{h}_1}$ is determined by c (Lemma 4.3(2)), so every such facet assigns to c the same cell $\omega + \nabla_{\tau'}^{\check{h}_2}$. A chain of cells goes to a chain, so the barycentric subdivisions are isomorphic; $\text{Bar } \mathcal{P}_X^i$ is a triangulation of B_X^i , so ϕ is a piecewise linear homeomorphism, and it preserves the labels of the chains. The last assertions follow from the definitions of the open facets, the regions and Γ_X through the labels (Lemma 4.4(2)). $\square$

4.3. The discriminants and the flat pairing. A loop *around an arc* $C_X(\tau, \omega)$, $\tau = [u, u']$, $\omega = [n, n']$, is a loop in a small neighbourhood of a point of the arc which passes once through each of the cells whose closures contain the arc. These are the cells $C_X(\mathbf{p})$ with $\mathbf{p} \succ (\tau, \omega)$, that is, $\mathbf{p} = (\tau', \omega') \neq (\tau, \omega)$ with $\tau' \subseteq \tau$ and $\omega' \subseteq \omega$: four three-cells and four two-cells. Since the complex of dual cell pairs is a regular cell decomposition of a three-manifold with face poset

(DP, $\preceq$) (Proposition 4.7(2)), a small circle linking the arc meets these eight cells in a cyclic order in which a three-cell and a two-cell are consecutive exactly when the two-cell lies in the boundary of the three-cell, that is, when their pairs are comparable. The comparability graph of the eight pairs is the cycle $C_X(u, n)$, $C_X([u, u'], n)$, $C_X(u', n)$, $C_X(u', [n, n'])$, $C_X(u', n')$, $C_X([u, u'], n')$, $C_X(u, n')$, $C_X(u, [n, n'])$, which is therefore the cyclic order in which they surround the arc.

- Proposition 4.10** (the discriminants). (1) $\Gamma_X = \bigcup \{C_X(\tau, \omega) : \dim \tau \geq 1, \dim \omega \geq 1\}$ and $\Gamma_Y = \bigcup \{C_Y(\tau, \omega) : \dim \tau \geq 1, \dim \omega \geq 1\}$. Hence $\mathcal{L}(\Gamma_X) = \Gamma_Y$ and $\mathcal{L}(B_X \setminus \Gamma_X) = B_Y \setminus \Gamma_Y$.
- (2) Γ_X is a graph. Its edges, the arcs, are the one-cells $C_X(\tau, \omega)$ with τ and ω edges; each arc is a union of legs, consecutive legs meeting at a barycentre that lies on exactly two legs. Its vertices are the zero-cells $C_X(\tau, \omega)$ with τ a triangle and ω an edge, of valency three, and those with τ an edge and ω a two-cell of $\mathcal{F}$ in a two-face of Δ , of valency the number of edges of ω . For $\mathcal{F} = \Sigma'$ and $\mathcal{F} = \mathcal{F}_C$ these two-cells are the unimodular triangles and the node parallelograms of the profile σ (Definition 2.3). The same holds for Γ_Y .
- (3) The vertex types are exchanged. A vertex (edge $[u, u']$, unimodular triangle) is of negative type on B_X and of positive type on B_Y ; a vertex (triangle, edge $[n, n']$) is of positive type on B_X and of negative type on B_Y (Definition 3.16). At a node (edge $[u, u']$, node parallelogram) the monodromies of the four arcs have the common covector $u' - u$ on B_X , as at a negative node, and the common vector $u' - u$ on B_Y , as at a positive node [CBM14, Ex. 6.1, 6.2].
- (4) Let γ be a loop around the arc $([u, u'], [n, n'])$, based in $C_X(u, n)$ and passing through $C_X(u, n)$, $C_X(u', n)$, $C_X(u', n')$, $C_X(u, n')$ in this order. Then parallel transport in Λ_X along γ and in Λ_Y along $\mathcal{L} \circ \gamma$ is

$$\begin{aligned} T_\gamma^X(x) &= x + \langle u' - u, x \rangle (n - n') \quad (x \in u^\perp \cap N), \\ T_{\mathcal{L}\gamma}^Y(\zeta) &= \zeta + \langle \zeta, n' - n \rangle (u' - u) \quad (\zeta \in n^\perp \cap M). \end{aligned}$$

Both are nontrivial transvections.

Proposition 4.11 (the flat pairing). On $C_X(u, n) \subseteq \text{int } F_u$ the lattice Λ_X is $u^\perp \cap N$, and on $C_Y(u, n) \subseteq \text{int } F_n^*$ the lattice Λ_Y is $n^\perp \cap M$. There is a unique pairing of local systems

$$\langle \cdot, \cdot \rangle_{\mathcal{L}} : \mathcal{L}^* \Lambda_Y \otimes \Lambda_X \longrightarrow \mathbb{Z} \quad \text{on } B_X \setminus \Gamma_X$$

which on every $C_X(u, n)$ is the restriction of $\langle \cdot, \cdot \rangle : (n^\perp \cap M) \times (u^\perp \cap N) \rightarrow \mathbb{Z}$. It is perfect and flat. Consequently $\mathcal{L}^* \Lambda_Y \cong \Lambda_X^\vee$, and $T_{\mathcal{L}\gamma}^Y = (T_\gamma^X)^{-\top}$ for every loop γ in $B_X \setminus \Gamma_X$.

Proposition 4.11 is the analogue, for the label-preserving cellular homeomorphism $\mathcal{L}$, of the duality of the lattice local systems of a base and its discrete Legendre transform, whose linear holonomies satisfy $\tilde{\rho}_{\tilde{B}}(\gamma) = {}^t \tilde{\rho}_B(\gamma^{-1})$ [GS06, Prop. 1.50(1)]: the tangent lattice of one base is the cotangent lattice of the other, although $\mathcal{L}$ is not a discrete Legendre transform.

Proof of Proposition 4.10(1),(2). (1) By Lemma 4.4(2), Γ_X is the union of the open simplices whose label $(\tau(c_k), \omega(c_0))$ has $\dim \tau \geq 1$ and $\dim \omega \geq 1$. On B_Y the defining condition is that the lower support $\tau_Y(c_0)$ of the bottom and the face label $\omega_Y(c_k)$ of the top have dimension at least one; this is the same set of labels.

(2) If $\dim \tau \geq 1$ and $\dim \omega \geq 1$, then $\dim(\tau, \omega) \leq 1$. The one-cells are the pairs of edges; the zero-cells are the pairs (triangle, edge) and (edge, two-cell), as $\mathcal{T}$ is a triangulation. By Proposition 4.7(2) each arc is a one-ball whose two end points are zero-cells, and the arcs at

a zero-cell (τ, ω) are the pairs of edges (τ_1, ω_1) with $\tau_1 \subseteq \tau$, $\omega_1 \subseteq \omega$, all of which are dual cell pairs: three for a triangle τ , and one for each edge of a two-cell ω . A leg $[b_e, b_f]$ is the closed simplex of a chain $e \subsetneq f$ with label $(\tau(f), \omega(e))$, a pair of edges, so each arc is a union of legs; since the arc is an open one-cell and the legs of other arcs lie in other cells, a point of the arc at which legs meet, a barycentre in the interior of a two-face, lies on exactly two legs. A two-cell ω paired with an edge τ lies in $\tau^\vee$, a face of Δ of dimension at most two. By admissibility and Definition 2.3, the two-cells of Σ' in the two-faces of Δ are unimodular triangles and node parallelograms, and $\mathcal{F}_C$ has the same two-cells there (Proposition 3.26). The argument for Γ_Y is the same. $\square$

Proof of Proposition 4.11. By Propositions 4.7(5) and 4.10(1) and Lemma 4.4(2), $B_X \setminus \Gamma_X$ is covered by the open sets $\text{int } F_u$ and $\tilde{R}_n$. The open facets are pairwise disjoint, the regions are pairwise disjoint, and $\text{int } F_u \cap \tilde{R}_n = C_X(u, n)$, an open three-ball, nonempty if and only if $\langle u, n \rangle = -1$; there are no triple intersections. On $\text{int } F_u$ the local system Λ_X is the constant lattice $u^\perp \cap N$ of the facet chart; on $\tilde{R}_n$ it is the constant lattice $N/\mathbb{Z}n$, through the derivative of π_n , since all local charts there are restrictions of π_n (Lemma 4.4(1)); on $C_X(u, n)$ the two are related by the isomorphism $\pi_{un}^X: u^\perp \cap N \rightarrow N/\mathbb{Z}n$. On B_Y the roles are exchanged: $\mathcal{L}$ maps $\text{int } F_u$ onto $\tilde{R}_u^*$, where Λ_Y is the constant lattice $M/\mathbb{Z}u$, and $\tilde{R}_n$ onto $\text{int } F_n^*$, where Λ_Y is $n^\perp \cap M$; on $C_Y(u, n)$ they are related by $\pi_{un}^Y: n^\perp \cap M \rightarrow M/\mathbb{Z}u$.

On $\text{int } F_u$ define $\langle \cdot, \cdot \rangle_{\mathcal{L}, u}: M/\mathbb{Z}u \times (u^\perp \cap N) \rightarrow \mathbb{Z}$ by $\langle \bar{\zeta}, x \rangle_{\mathcal{L}, u} = \langle \zeta, x \rangle$, which is well defined because $\langle u, x \rangle = 0$. It is perfect: u is primitive, as a lattice point of the boundary of the reflexive polytope Δ° , so $u^\perp \cap N$ is a saturated sublattice of rank three, restriction $M \rightarrow \text{Hom}(u^\perp \cap N, \mathbb{Z})$ is surjective, and its kernel is $\mathbb{Q}u \cap M = \mathbb{Z}u$. On $\tilde{R}_n$ define $\langle \cdot, \cdot \rangle_{\mathcal{L}, n}: (n^\perp \cap M) \times N/\mathbb{Z}n \rightarrow \mathbb{Z}$ in the same way; it is perfect because n is primitive. On $C_X(u, n)$, for $\zeta \in n^\perp \cap M$ and $x \in u^\perp \cap N$,

$$\langle \pi_{un}^Y \zeta, x \rangle_{\mathcal{L}, u} = \langle \zeta, x \rangle = \langle \zeta, \pi_{un}^X x \rangle_{\mathcal{L}, n}.$$

So the local pairings agree on the overlaps of the cover and glue to a perfect morphism of local systems $\langle \cdot, \cdot \rangle_{\mathcal{L}}$, which is unique because $B_X \setminus \Gamma_X$ is connected. Flatness gives $\langle T_{\mathcal{L}\gamma}^Y \zeta, T_{\mathcal{L}\gamma}^X x \rangle_{\mathcal{L}} = \langle \zeta, x \rangle_{\mathcal{L}}$ for every loop γ , that is, $T_{\mathcal{L}\gamma}^Y = (T_{\mathcal{L}\gamma}^X)^{-\top}$. $\square$

Proof of Proposition 4.10(4). The cells $C_X(u, n)$, $C_X([u, u'], n)$, $C_X(u', n)$ lie in $\tilde{R}_n$, where transport is the identity of $N/\mathbb{Z}n$; so transport from $C_X(u, n)$ to $C_X(u', n)$ sends $x \in u^\perp \cap N$ to the element $x + \langle u', x \rangle n$ of $u'^\perp \cap N$ congruent to x modulo n . From $C_X(u', n)$ to $C_X(u', n')$ the loop stays in $\text{int } F_{u'}$, and transport is the identity. From $C_X(u', n')$ to $C_X(u, n')$ it is $y \mapsto y + \langle u, y \rangle n'$, and back to $C_X(u, n)$ it is the identity. Using $\langle u, x \rangle = 0$ and $\langle u, n \rangle = -1$,

$$T_{\mathcal{L}\gamma}^X(x) = x + \langle u', x \rangle n + (\langle u, x \rangle + \langle u', x \rangle \langle u, n \rangle) n' = x + \langle u' - u, x \rangle (n - n').$$

On B_Y , $\mathcal{L}\gamma$ passes through the same labels. From $C_Y(u, n)$ to $C_Y(u', n)$ it stays in $\text{int } F_n^*$ (identity); from $C_Y(u', n)$ to $C_Y(u', n')$ it stays in $\tilde{R}_{u'}^*$, giving $\zeta \mapsto \zeta + \langle \zeta, n' \rangle u'$; then the identity in $\text{int } F_{n'}^*$; and from $C_Y(u, n')$ to $C_Y(u, n)$, in $\tilde{R}_u^*$, $\eta \mapsto \eta + \langle \eta, n \rangle u$. With $\langle \zeta, n \rangle = 0$ and $\langle u', n \rangle = -1$ the composite is $\zeta + \langle \zeta, n' - n \rangle (u' - u)$. Both maps are transvections, since $\langle u' - u, n - n' \rangle = 0$, and they are nontrivial because $u' - u$ is not a multiple of u and $n' - n$ is not a multiple of n . For $\mathcal{F} = \Sigma'$ the formula for $T_{\mathcal{L}\gamma}^X$ is that of Lemma 3.12, which is [Gro05, Prop. 3.15] for hypersurfaces. $\square$

Proof of Proposition 4.10(3). By (4), the arc $([u, u'], [n, n'])$ has $T^X - 1 = \pm(n' - n) \otimes \langle u' - u, \cdot \rangle$ and $T^Y - 1 = \pm(u' - u) \otimes \langle \cdot, n' - n \rangle$. At a zero-cell $([u, u'], \omega)$ all arcs contain the edge $[u, u']$ and

share $u' - u$, a covector on B_X and a vector on B_Y ; at a zero-cell $(\tau, [n, n'])$ they share $n' - n$, a vector on B_X and a covector on B_Y . We check the primitivity and saturation conditions of Definition 3.16 on B_Y , with base cell $C_Y(u_1, n)$, $\Lambda_Y = n^\perp \cap M$ and $\Lambda_Y^\vee = N/\mathbb{Z}n$.

- At (triangle $\{u_1, u_2, u_3\}$, $[n, n']$): the common covector is the class of $n' - n$ in $N/\mathbb{Z}n$, which is primitive because $n' - n$ is primitive and $\langle u_1, n \rangle = \langle u_1, n' \rangle = -1$, so that $\{0, n, n'\}$ is a unimodular triangle and $\mathbb{Z}n + \mathbb{Z}n'$ is saturated. The vectors $u_j - u_i$ lie in $n^\perp \cap M$, sum to zero with orientations, and span a saturated rank-two sublattice of $(n' - n)^\perp$, because $\{0, u_1, u_2, u_3\}$ is a face of a unimodular simplex. So the vertex is of negative type.
- At $([u, u'], \text{triangle } \{n_a, n_b, n_c\})$: the common vector $u' - u$ is primitive, as an edge vector of the unimodular triangulation $\mathcal{T}$. The covectors are the classes of $n_b - n_a$, $n_c - n_b$, $n_a - n_c$ in $N/\mathbb{Z}n_a$. The triangle is unimodular in its plane, on which $\langle u, \cdot \rangle = -1$, so $\{n_a, n_b, n_c\}$ is a basis of the saturated rank-three lattice $N \cap \mathbb{R}\{n_a, n_b, n_c\}$; hence the classes of $n_b - n_a$ and $n_c - n_a$ span a saturated rank-two sublattice of $N/\mathbb{Z}n_a$, contained in $(u' - u)^\perp$. So the vertex is of positive type.

The same two computations with the roles of (u, M) and (n, N) exchanged show that on B_X the first vertex is of positive type and the second of negative type; for $\mathcal{F} = \Sigma'$ this is also Lemma 3.17. At a node $([u, u'], P)$ the four arcs share $u' - u$, a covector on B_X and a vector on B_Y . $\square$

Remark 4.12 (orientation of loops). For one and the same loop the monodromies of Λ_X and Λ_Y are inverse transposes of each other, not transposes; formulas for the two sides that are transposes of each other refer to loops that are inverse to each other under $\mathcal{L}$. The types of Definition 3.16 do not depend on the orientation of the loops, since reversing it replaces each d_i by $-d_i$, respectively each α_i by $-\alpha_i$; but a statement comparing a monodromy on B_X with one on B_Y must use the inverse transpose.

5. RELATIVE TROPICAL 2-CYCLES

This section proves Theorem C: on the base $B = B'_\sigma$ every balanced network is the boundary network of a relative tropical 2-cycle, and the node coefficients of balanced networks form exactly the lattice K of relations among the circuits. The proof is combinatorial topology of the pair (B, Γ) , a three-sphere and a graph in it, with the coefficients of Definition 2.4. It has two parts.

The *boundary level*, $\rho(\text{Bal}(\Gamma)) = K$ (Section 5.3), is a statement about flows. A balanced network splits into one integral 1-chain for each edge ε of the subdivision Σ' of the two-skeleton of Δ , supported on the legs whose lower support is ε . The balance conditions at the negative vertices and nodes say that these 1-chains assemble into a cellular 1-chain on $\partial\Delta$, the node coefficients are read off from its boundary, and $H_1(\partial\Delta; \mathbb{Z}) = 0$ produces a balanced network for every $k \in K$.

The *filling*, $\partial Z(\mathfrak{R}_0) = \text{Bal}(\Gamma)$ (Section 5.5), moves a balanced network, by 2-chains supported in discs transverse to the cells of B , to a 1-cycle in $B \setminus \Gamma$ with coefficients in the local system Λ . The class of that cycle is computed from the cover of $B \setminus \Gamma$ by the open facets and the vertex regions of B , and it lies in the span of the meridians of Γ (Lemma 5.17). The proof of that lemma has two parts. A Mayer–Vietoris argument over a regular neighbourhood of Γ , whose homology relative to its frontier is computed in Section 5.4, shows that meridians generate $H_1(B \setminus \Gamma; \mathbb{Z})$. A computation on the boundary spheres of the facets of B , which uses the cells of B indexed by dual cell pairs, the connectedness of the links of vertices in the triangulated faces of Δ° , $H^1(S^2) = 0$, and the fact that the vertices of every simplex of the

unimodular triangulation $\mathcal{T}$ are part of a basis of M , then passes from constant coefficients to the coefficients Λ . A meridian with an invariant coefficient bounds a disc that crosses Γ once.

Section 5.2 shows that the tropical 2-cycles of Castaño-Bernard and Matessi [CBM14, Def. 7.2], triangulated, are relative tropical 2-cycles with the same node coefficients. Section 5.4 computes the homology of a regular neighbourhood of Γ with coefficients in $\iota_*\Lambda$ and deduces a normal form of relative tropical 2-cycles near Γ , which the lift of Section 8 uses. Throughout, Δ is admissible with a profile σ satisfying (Proj), and $(\mathcal{T}, \check{h})$ is fixed as in Section 2.3; all homology is integral unless coefficients are named.

5.1. The coefficient system. *The cells indexed by dual cell pairs.* We use the cell decomposition of B by dual cell pairs of Section 4. For a simplex τ of $\mathcal{T}$ and a cell ω of Σ' put

$$\begin{aligned}\tau^\vee &= \{n \in \Delta : \langle u, n \rangle = -1 \text{ for all vertices } u \text{ of } \tau\}, \\ \omega^\vee &= \{m \in \Delta^\circ : \langle m, n \rangle = -1 \text{ for all vertices } n \text{ of } \omega\};\end{aligned}$$

$\tau^\vee$ is the face of Δ dual to the smallest face of Δ° containing τ , and likewise for $\omega^\vee$. We write $u^\vee$ for $\{u\}^\vee$, and $\text{lin}(\omega) \subset N_{\mathbb{R}}$ for the linear span of the differences of points of ω , so that $\text{lin}(\omega) \cap N$ is the lattice of integral vectors parallel to ω . A *dual cell pair* is a pair (τ, ω) with $\omega \subseteq \tau^\vee$, equivalently $\tau \subseteq \omega^\vee$, that is, $\langle u, n \rangle = -1$ for every vertex u of τ and every vertex n of ω (Definition 4.5). For a cell c of $\mathcal{P}$ let $\omega(c) \in \Sigma'$ be its lower support and $\tau(c) \in \mathcal{T}$ its face label, so that $F_{\tau(c)}$ is the smallest face of $\nabla^{\check{h}}$ containing c . For a dual cell pair (τ, ω) let $C(\tau, \omega) \subset B$ be the union of the open simplices of the barycentric subdivision $\mathfrak{K}_0$ of $\mathcal{P}$ whose chains $c_0 \subsetneq \cdots \subsetneq c_k$ have $\tau(c_k) = \tau$ and $\omega(c_0) = \omega$; this is the cell $C_X(\tau, \omega)$ of Definition 4.6. The following facts are Propositions 4.7(2),(4),(5) and 4.10, Lemma 4.4 and Lemma 3.17.

- (D1) The sets $C(\tau, \omega)$ are the open cells of a regular cell decomposition of B ; $C(\tau, \omega)$ is nonempty, of dimension $3 - \dim \tau - \dim \omega$, and $C(\tau', \omega')$ lies in its closure if and only if $\tau' \supseteq \tau$ and $\omega' \supseteq \omega$. Each closed cell is a piecewise linear ball and a subcomplex of $\mathfrak{K}_0$.
- (D2) $\text{int } F_u = \bigcup_{\omega} C(u, \omega)$, the vertex region is $\tilde{R}_n = \bigcup_{\tau} C(\tau, n)$, and $\text{int } F_u \cap \tilde{R}_n = C(u, n)$. The open facets are pairwise disjoint, the regions are pairwise disjoint, and together they cover $B \setminus \Gamma$. On $\text{int } F_u$ the local system Λ is the constant lattice $u^\perp \cap N$ (the facet chart), on $\tilde{R}_n$ it is $N/\mathbb{Z}n$ (the region chart), and on $C(u, n)$ the two are identified by $\pi_{un}: u^\perp \cap N \rightarrow N/\mathbb{Z}n$, $x \mapsto x \bmod n$, an isomorphism because $\langle u, n \rangle = -1$.
- (D3) $\Gamma = \bigcup \{C(\tau, \omega) : \dim \tau \geq 1, \dim \omega \geq 1\}$. It is a graph. Its *arcs* are the one-cells $C(\tau, \omega)$ with τ and ω edges; each arc is a union of legs, all with lower support ω and vanishing vector $\pm d_\omega$, $d_{[n, n']} = n' - n$, and it is the interior of an embedded segment whose two end points are distinct zero-cells. The zero-cells in Γ , the Γ -*vertices*, are: the *positive vertices* $C(\tau, \varepsilon)$, τ a triangle and ε an edge, with three arcs; the *negative vertices* $C(\tau, \omega)$, τ an edge and ω a unimodular triangle, with three arcs; and the nodes $p_q = C(\tau, q)$, τ an edge and q a node parallelogram, with four arcs. The arcs at a Γ -vertex $C(\tau, \omega)$ are the $C(\tau_1, \omega_1)$ with $\tau_1 \subseteq \tau$, $\omega_1 \subseteq \omega$ edges.
- (D4) Let $A = C([u, u'], [n, n'])$ be an arc. A small loop around A passes through $C(u, n)$, $C([u, u'], n)$, $C(u', n)$, $C(u', [n, n'])$, $C(u', n')$, $C([u, u'], n')$, $C(u, n')$, $C(u, [n, n'])$ in this cyclic order, and parallel transport along it, based in $C(u, n)$, is $x \mapsto x + \langle u' - u, x \rangle(n - n')$ on $u^\perp \cap N$.

- (D5) There is a piecewise linear homeomorphism $\mathcal{L}$ of B onto the boundary of a convex four-polytope $\nabla^H \subset M_{\mathbb{R}}$ that carries $\tilde{R}_n$ onto the interior of a facet of ∇^H and $\bigcup_{\tau} C(\tau, \omega)$ onto the relative interior of a face of ∇^H of dimension $3 - \dim \omega$.

We use (D5) only to know that each $\tilde{R}_n$ is an open three-ball (Lemma 5.16) and that the closure of $\bigcup_{\tau} C(\tau, \varepsilon)$, for an edge ε of Σ' , is a closed two-disc whose boundary circle is the union of the closures of the sets $\bigcup_{\tau} C(\tau, \omega)$, $\omega \supset \varepsilon$ a two-cell (Lemma 5.14(1)). In (D5), ∇^H is the polytope of the mirror base $B_Y(\Sigma')$ of Section 4, for a height H that is piecewise linear on the fan over Σ' with integral slopes and has $H - \varphi$ strictly convex on that fan. Such an H exists: the functions that are strictly convex on the fan over Σ' , with the cones over the cells of Σ' as their maximal domains of linearity, form a set cut out by finitely many linear equations and strict linear inequalities with rational coefficients, nonempty by (Proj); it contains a function with rational slopes, a positive integral multiple h of which has integral slopes, and $H = h + \varphi$ has the required properties, since φ has integral slopes, Δ being reflexive.

The two-cells of Σ' in two-faces of Δ are the unimodular triangles and the node parallelograms of the profile (Definition 2.3); this is why only these occur in (D3). At a hexagon on the A_2 branch the three node parallelograms share pairwise the edges $[o, w]$ through the centre, and the three nodes are pairwise joined by the arcs $C(G^\circ, [o, w])$; these three arcs form a cycle of Γ whose only vertices are nodes. At a pentagon, and at a hexagon on the A_1 branch, an interior edge $[o, w]$ likewise gives one arc joining the Γ -vertices of the two cells containing it. No argument below assumes that an arc ends at a trivalent vertex.

Stalks. For $x \in \Gamma$ let $\text{Inv}(x)$ be the group of flat sections of Λ over $V \setminus \Gamma$, for a small open ball V around x in which Γ is the cone from x over finitely many points. For an arc A we write $\text{Inv}(A)$ for the common value at its points.

- Lemma 5.1** (stalks). (1) *Let $V \subseteq B$ be open and $y \in N$ with $\langle u, y \rangle = 0$ for every u such that $\text{int } F_u$ meets V . Then y in the facet charts and $y \bmod n$ in the region charts define a flat section of Λ over $V \setminus \Gamma$.*
- (2) *If $x \in C(\tau, \omega) \subseteq \Gamma$, then $\text{Inv}(x) = \tau^\perp \cap N$, embedded as in (1). Thus $\text{Inv}(A) = [u, u']^\perp \cap N$, of rank two and containing d_A , on an arc $A = C([u, u'], [n, n'])$; $\text{Inv}(p) = \text{lin}(\omega) \cap N$, of rank two and spanned by the edge vectors of ω , at a negative vertex or node $p = C([u, u'], \omega)$; and $\text{Inv}(p) = \mathbb{Z}d_\varepsilon$ at a positive vertex $p = C(\tau, \varepsilon)$. If A is an arc at the Γ -vertex p , then $\text{Inv}(p) \subseteq \text{Inv}(A)$ as sublattices of N .*
- (3) *For every Γ -vertex $C(\tau, \omega)$ the vertices of τ are part of a basis of M .*

Proof. (1) By (D2) the local data agree on the overlaps $C(u, n)$, since $\pi_{un}(y) = y \bmod n$.

(2) If V is a small ball around x , the facets whose interiors meet V are the F_u with $u \in \tau$, so every $y \in \tau^\perp \cap N$ is a flat section by (1). Conversely let y be a flat section over $V \setminus \Gamma$, and let $u \in \tau$. Its value in the facet chart of F_u is a vector $y_u \in u^\perp \cap N$; since $\text{int } F_u \cap V$ is connected, y_u is one vector. For every arc $C([u, u_1], \omega_1)$ at x with $u \in [u, u_1] \subseteq \tau$, invariance under the loop of (D4) gives $\langle u_1 - u, y_u \rangle (n - n') = 0$ with $n \neq n'$, hence $\langle u_1, y_u \rangle = 0$. By (D3), if $\tau = [u, u']$ this gives $y_u \perp u'$, and if $\tau = \{u, u_1, u_2\}$ it gives $y_u \perp u_1, u_2$; so $y_u \in \tau^\perp$, and y is the section of (1) defined by y_u . The rank of $\tau^\perp \cap N$ is $3 - \dim \tau$, since the vertices of τ lie in the hyperplane $\langle \cdot, n \rangle = -1$ and are linearly independent. On an arc $\langle u, d_A \rangle = \langle u', d_A \rangle = 0$. At a negative vertex or node, $[u, u']^\perp \cap N$ is saturated of rank two and contains the edge vectors of ω , which span the lattice $\text{lin}(\omega) \cap N$ because ω is a unimodular triangle or a unit parallelogram; two saturated lattices of the same rank, one containing the other, are equal. At a positive vertex $\tau^\perp \cap N$ has rank one and contains the primitive vector d_ε (the edges of Σ'

in faces of dimension at most two are primitive). The inclusion $\text{Inv}(p) \subseteq \text{Inv}(A)$ is $\tau^\perp \subseteq \tau_1^\perp$ for $\tau_1 \subseteq \tau$.

(3) τ is a simplex of the unimodular triangulation $\mathcal{T}$ (Hypothesis (U)), so the cone over τ is a face of a cone generated by a basis of M . $\square$

By Lemma 5.1(2), a flat section near a point of Γ is represented by one vector of N in every facet chart near that point. We use this identification throughout: all the lattices $\text{Inv}(\cdot)$ are sublattices of N , and so are the groups $F(s)$ of simplices meeting Γ .

Lemma 5.2 (the coefficient system). *Let $\mathfrak{K}$ be a triangulation of B in which Γ is full, and s a simplex of $\mathfrak{K}$.*

- (1) *If $s \cap \Gamma = \emptyset$, then $F(s)$ is the group of flat sections of Λ over a neighbourhood of $\bar{s}$, free of rank three.*
- (2) *If $s \cap \Gamma \neq \emptyset$, then $s_\Gamma := \bar{s} \cap \Gamma$ is a face of s , and $F(s) = \bigcap_{x \in s_\Gamma} \text{Inv}(x)$. This is $\text{Inv}(A)$ if s_Γ lies in an arc A , and otherwise the intersection of the stalks of the Γ -vertices in s_Γ .*
- (3) *The restriction maps $F(s) \rightarrow F(s')$ are injective; for simplices meeting Γ they are inclusions of sublattices of N .*

Proof. (2) The first assertion is fullness. Choose the neighbourhood V of $\bar{s}$ as a regular neighbourhood of $\bar{s}$ that meets Γ in a regular neighbourhood of s_Γ in Γ . Then $V \setminus \Gamma$ is connected, and its fundamental group is generated by meridians of Γ at points of s_Γ together with, at each Γ -vertex in s_Γ , meridians of the germs at that vertex of the arcs through it. Indeed, if $s_\Gamma \neq \emptyset$, the pair $(V, V \cap \Gamma)$ is a cone pair, except when s_Γ is an edge whose two end points are Γ -vertices, when it is the union of two cone pairs glued along a disc that meets Γ in one point; the complement of a cone-like tree in a three-ball has free fundamental group generated by meridians, and by van Kampen's theorem so has the complement of Γ in two such pairs glued along a disc meeting Γ once. So the flat sections over $V \setminus \Gamma$ are the vectors invariant under these meridians, which is $\bigcap_{x \in s_\Gamma} \text{Inv}(x)$, since the meridians at a point x are those that define $\text{Inv}(x)$. By Lemma 5.1(2) the stalk is constant along an arc and drops to $\text{Inv}(p)$ at a Γ -vertex p . (1) and (3) hold because the neighbourhoods may be chosen nested and connected off Γ . $\square$

Subdivision and balanced networks. If $\mathfrak{K}_1$ is a subdivision of $\mathfrak{K}$ in which Γ is full, each simplex s_1 of $\mathfrak{K}_1$ lies in a simplex s of $\mathfrak{K}$, and restriction gives $F(s) \rightarrow F(s_1)$. The *subdivision operator* $\text{sd}: C_*(\mathfrak{K}; F) \rightarrow C_*(\mathfrak{K}_1; F)$, which replaces a simplex by the sum of the simplices of $\mathfrak{K}_1$ in it, each with the restricted coefficient, is a chain map. It preserves the chains on Γ with coefficients in $\mathcal{D}$, so it carries $Z(\mathfrak{K})$ into $Z(\mathfrak{K}_1)$ and commutes with taking boundary networks. The group of coefficients may grow under subdivision: a simplex of $\mathfrak{K}_1$ near a positive vertex p that does not contain p has coefficients $\text{Inv}(A)$ or Λ , not $\text{Inv}(p)$.

Every edge E of $\mathfrak{K}$ in Γ lies in the closure $\bar{A}$ of an arc A , and $d_E = \pm d_A$. At a vertex of $\mathfrak{K}$ interior to an arc the coefficient group is the torsion-free group $\text{Inv}(A)$, so the balance condition says that the two edges at that vertex carry the same multiple of d_A . Hence a balanced network is the same as an integer c_A for each arc A , with A oriented and carrying the vector $c_A d_A$, such that the vectors of the arcs at each Γ -vertex p , oriented away from p , sum to zero in $\text{Inv}(p) \subset N$. In particular $\text{Bal}(\Gamma)$ does not depend on the triangulation, and sd induces the identity on it.

5.2. Tropical 2-cycles in the sense of Castaño-Bernard and Matessi. We refer to the clauses of [CBM14, Def. 7.2] by their content. The following dictionary gives their letters

in the arXiv version and their numbers in the published version, followed by the conditions used here.

- Embedding (a; i):** S is a tropical domain [CBM14, Def. 7.1] and $j: (S, \partial S) \rightarrow (B, \Gamma)$ is an embedding.
- Discriminant-intersection (b; ii):** $j^{-1}(\Gamma) = \partial S \cup \{\mathbf{r}_1, \dots, \mathbf{r}_r\}$, where the $\mathbf{r}_k$ are smooth points of S mapped to points of arcs. Put $S_0 = S_{\text{sm}} \setminus \{\mathbf{r}_k\}$.
- Primitive-field (c; iii):** v is a primitive integral parallel vector field along $j(S_0)$.
- Boundary-vertex (d; iv):** $j(p)$ is a negative vertex if and only if p is a boundary vertex of S .
- Node-passage (e; v):** At a negative node the two edges of ∂S follow two opposite legs.
- Positive-node (f; vi):** This clause concerns positive nodes; B has none (Proposition 3.18).
- Interior-edge balancing (g; vii):** Along an interior edge the fields v_1, v_2, v_3 of the adjacent sheets span a rank-two subspace and satisfy $s_1 v_1 + s_2 v_2 + s_3 v_3 = 0$. Here $s_k = \pm 1$ according as the orientation of the k -th sheet induces the chosen orientation of the edge.
- Boundary-monodromy (h; viii):** Along an edge of ∂S , $\ker v$ is the monodromy-invariant subspace of the cotangent lattice.
- Crossing (i; ix):** At $\mathbf{r}_k$ the monodromy on $\ker v$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- Interior-vertex (j; x):** At an interior vertex the six values of v span $T_p B$.

Castañõ-Bernard and Matessi do not name the group in which the conjugacy in the crossing clause is taken; we read it as conjugacy in $GL_2(\mathbb{Z})$. In [CBM14, §7.7, Step 6] the crossing clause enters through an integral basis of $\ker v$ in which the monodromy has this form, with no condition on the orientation of the basis, and in Step 7 through the identification of v with the class e_3 , and so its invariance. Lemma F.4, the proof of Lemma 6.5(3), and the classes of cycles in Conjecture G and Theorem E depend on this reading. Since conjugacy in $SL_2(\mathbb{Z})$ implies conjugacy in $GL_2(\mathbb{Z})$, the class of cycles under the stricter reading is contained in the class used here; so Theorem E as stated covers both readings, while under the stricter reading Conjecture G asks for more. The definition [CBM14, Def. 7.2] is stated for a tropical conifold [CBM14, Def. 6.3], whose three-valent vertices are of the negative or positive type of [CBM14, Ex. 2.3, 2.4]; at the three-valent vertices of Γ we prove only that the monodromies are of these types (Definition 3.16, Lemma 3.17), not that the germs are integral affine isomorphic to those of [CBM14, Ex. 2.3, 2.4], and in the boundary-vertex clause a negative vertex means a vertex of negative type in the sense of Definition 3.16. The sheets of S are oriented, and j is assumed piecewise linear. The readings of the crossing and boundary-vertex clauses above and the piecewise linearity of j are the three conventions of Definition 2.5. The clauses refer only to Γ , Λ and the vertex types; the polarization of $(B, \mathcal{P})$ enters none of them, and the tropical 2-cycles for the placement of the node vertex in Proposition 3.18(2) and for the barycentric placement correspond bijectively with the same node coefficients (Lemma 3.20). We use the barycentric placement.

Passages. Let $q \in Q$ with $p_q \in j(\partial S)$. By the node-passage clause the two edges of ∂S at p_q follow one of the two passages at q , the $b - a$ passage or the $c - b$ passage of Section 3.4 (Figure 1), whose vanishing vector $b - a$, respectively $c - b$, is fixed by the monodromy of all four legs.

Lemma 5.3. *Along an edge of ∂S lying on a leg with vanishing vector d , $v = \pm d$.*

Proof. The monodromy around the leg is a transvection $x \mapsto x \pm \langle \alpha, x \rangle d$ (Lemma 3.12), so it acts on covectors by $\xi \mapsto \xi \pm \xi(d)\alpha$ and its invariant covectors are those vanishing on d . By the boundary-monodromy clause, $\ker v$ is this set, so v is proportional to d ; both are primitive. $\square$

Choose a triangulation $\mathfrak{K}$ of B in which $j(S)$ and Γ are subcomplexes, each triangle of $j(S)$ lies in one sheet, and Γ is full; a derived subdivision of a triangulation with the first two properties has all three [RS72, Ch. 3]. Put

$$z_S := \sum_s v_s [s],$$

the sum over the triangles s of $j(S)$, oriented by their sheets, with v_s the value of v on s .

Lemma 5.4 (tropical 2-cycles are relative tropical 2-cycles). (1) $v_s \in F(s)$ for every triangle s of $j(S)$, and $z_S \in Z(\mathfrak{K})$. Its boundary network is $\sum_E v|_E [E]$, over the edges E of $j(\partial S)$ oriented as the boundary of S , and $v|_E = \pm d_E$.
 (2) $\rho_q(z_S) = 0$ if $p_q \notin j(\partial S)$. If $p_q \in j(\partial S)$, let $\tilde{v}_{\text{in}}$ and $\tilde{v}_{\text{out}}$ be the canonical lifts of v on the edges of ∂S entering and leaving p_q ; then $\tilde{v}_{\text{out}} - \tilde{v}_{\text{in}} = \rho_q(z_S) \text{circ}_q$ and $\rho_q(z_S) \in \{\pm 1\}$.

We write $\rho(S) := \rho(z_S)$ and call it the vector of node coefficients of S . By (2) it is read off from a collar of ∂S and does not depend on $\mathfrak{K}$; it plays the role of the orientation signs that [CBM14, §9.5] attach at the nodes, which motivate the coefficients $\epsilon_{S_1 S_2}(p)$ of [CBM14, Def. 8.1].

Proof. (1) By Lemma 5.2(2) we must show that v_s is invariant under the monodromy at every point of $\bar{s} \cap \Gamma$, which lies in $j(\partial S) \cup \{\mathfrak{r}_k\}$ by the discriminant-intersection clause. At a point of a leg on ∂S , $v = \pm d$ by Lemma 5.3, and d is invariant. At $\mathfrak{r}_k$ the sheet is a punctured disc on which v is parallel, so v is invariant under the loop around $\mathfrak{r}_k$ in the sheet. By the crossing clause the monodromy along this loop is nontrivial, so the loop is a nonzero power of a meridian of the leg, which acts by a transvection $x \mapsto x + \langle \alpha, x \rangle d$; invariance gives $\langle \alpha, v \rangle = 0$, that is, $v \in \text{Inv}(A)$. At a negative vertex $p = j(p')$, the sheet containing s is bounded near p by a boundary edge on a leg with vanishing vector d_i , so $v_s = \pm d_i$, and $d_i \in \text{Inv}(p)$ by Lemma 5.1(2). At a node on ∂S the part of S_0 near p_q is connected (by the discriminant-intersection and node-passage clauses) and v equals $\pm$ the vanishing vector of the passage, which lies in $\text{Inv}(p_q) = \text{lin}(q) \cap N$. At a positive vertex on ∂S , ∂S uses two of the three legs, which have the same vanishing vector d (Lemma 3.17), so $v = \pm d$ on the half-disc of S at p , and $\text{Inv}(p) = \mathbb{Z}d$. Off Γ there is nothing to check.

The boundary of z_S vanishes on edges interior to a sheet (the two triangles induce opposite orientations and carry the same v), on interior edges of S by the interior-edge balancing clause, and on edges through the points $\mathfrak{r}_k$, which are interior to a sheet. What remains is $\sum_E v|_E [E]$ over the edges of $j(\partial S)$, with $v|_E = \pm d_E$ by Lemma 5.3.

(2) If $p_q \notin j(\partial S)$, ∂z_S vanishes near p_q . Otherwise the canonical lift of ∂z_S is $\sum_E \tilde{v}|_E [E]$, and at p_q the edge entering contributes $\tilde{v}_{\text{in}}$ and the edge leaving $-\tilde{v}_{\text{out}}$ to $\partial \partial z_S(p_q)$; by Definition 2.6, $\tilde{v}_{\text{out}} - \tilde{v}_{\text{in}} = \rho_q(z_S) \text{circ}_q$. For the $b - a$ passage, entering along ab and leaving along dc with $v = \epsilon(b - a)$, the jump is $\epsilon(e_c - e_d) - \epsilon(e_b - e_a) = -\epsilon \text{circ}_q$; for the $c - b$ passage, entering along bc with $v = \epsilon'(c - b)$, it is $\epsilon'(e_d - e_a) - \epsilon'(e_c - e_b) = \epsilon' \text{circ}_q$. The other orientations are the same computation. Since j is an embedding, ∂S meets p_q once. $\square$

Reversing the orientation of S together with $v \mapsto -v$ leaves $\rho(S)$ unchanged. The boundary networks of the chains z_S are special: every $|c_A| \leq 1$, at a node exactly one opposite pair of

legs or none is used, at a positive vertex two legs or none, and at a negative vertex three legs or none. Relative tropical 2-cycles have no such restriction, and Theorem C needs the extra room; whether the chains z_S alone suffice is Conjecture G.

5.3. Node coefficients of balanced networks. Let $\Sigma'_{\leq 2}$ be the subdivision of the two-skeleton of Δ induced by Σ' . Its edges are the edges of Δ and the edges $[o, w]$ inside the pentagons and hexagons, and they are the lower supports of legs. Fix an orientation of each edge $\varepsilon = [n, n']$ of $\Sigma'_{\leq 2}$, put $d_\varepsilon = n' - n$, $\tilde{e}_\varepsilon = e_{n'} - e_n \in \mathbb{Z}^R$, and let Γ_ε be the union of the arcs $C(\tau, \varepsilon)$ and the positive vertices $C(\tau, \varepsilon)$. Its closure $\bar{\Gamma}_\varepsilon$ adds the end points of the arcs $C(\tau, \varepsilon)$ that are negative vertices or nodes, the points $C(\tau, \omega)$ with $\omega \supset \varepsilon$ a two-cell; every negative vertex or node $C(\tau, \omega)$ with $\varepsilon \subset \omega$ arises in this way. $\bar{\Gamma}_\varepsilon$ is connected: for an edge ε of Δ it is the graph dual to the triangulation of the polygon $\varepsilon^\vee$ by $\mathcal{T}$, with one extra end for each boundary edge; for $\varepsilon = [o, w]$ inside a pentagon or hexagon G , $\varepsilon^\vee = G^\circ$ is a single edge of $\mathcal{T}$ (it has lattice length one), and $\bar{\Gamma}_\varepsilon$ is one closed arc.

Every leg lies in exactly one Γ_ε . So every chain $\beta \in C_1(\Gamma; \mathcal{D})$ on $\mathfrak{K}_0$ can be written uniquely as $\beta = \sum_\varepsilon \eta_\varepsilon \otimes d_\varepsilon$ with $\eta_\varepsilon \in C_1(\bar{\Gamma}_\varepsilon; \mathbb{Z})$ supported on the legs of Γ_ε , and its canonical lift is $\tilde{\beta} = \sum_\varepsilon \eta_\varepsilon \otimes \tilde{e}_\varepsilon$.

The Δ -side complex. Let $\mathcal{X}$ be the regular cell decomposition of $\partial\Delta \cong S^3$ whose cells are the points of R , the edges and two-cells of $\Sigma'_{\leq 2}$, and the facets of Δ . (Σ' induces the profile on the two-faces and does not subdivide the edges of Δ , which are primitive.) Its cellular chain complex has $C_0(\mathcal{X}) = \mathbb{Z}^R$ and $H_1(\mathcal{X}) = H_1(S^3) = 0$. Let $\vec{\varepsilon}$ denote the one-cell ε oriented from n to n' , so that $\partial\vec{\varepsilon} = \tilde{e}_\varepsilon$, and let $\text{ev}: \mathbb{Z}^R \rightarrow N$, $e_r \mapsto r$; then $\text{ev} \partial\vec{\varepsilon} = d_\varepsilon$. For a negative vertex or node $p = C(\tau, \omega)$ put

$$\chi_p(\beta) := \sum_{\varepsilon \subset \partial\omega} (\partial\eta_\varepsilon)(p) \vec{\varepsilon} \in C_1(\mathcal{X}).$$

Then $\partial\tilde{\beta}(p) = \sum_\varepsilon (\partial\eta_\varepsilon)(p) \tilde{e}_\varepsilon = \partial\chi_p(\beta)$.

Lemma 5.5 (balanced networks). *Let $\beta = \sum_\varepsilon \eta_\varepsilon \otimes d_\varepsilon \in C_1(\Gamma; \mathcal{D})$.*

- (1) $\beta \in \text{Bal}(\Gamma)$ if and only if (i) $\partial\eta_\varepsilon$ vanishes at every point of Γ_ε , for every ε , and (ii) $\text{ev} \partial\chi_p(\beta) = 0$ for every negative vertex and node p .
- (2) Let $\beta \in \text{Bal}(\Gamma)$. Then $\partial\tilde{\beta}$ vanishes at every point of Γ other than the nodes. At a node p_q , with the edges of q oriented along the cycle $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$,

$$\chi_{p_q}(\beta) = x_q([ab] + [cd]) + y_q([bc] + [da]), \quad \partial\tilde{\beta}(p_q) = (x_q - y_q) \text{circ}_q,$$

for integers x_q, y_q , so that $\rho_q(\beta) = y_q - x_q$.

- (3) If the coefficients of β on the legs at p_q with lower supports $\{a, b\}$ and $\{b, c\}$, oriented away from p_q , are $c_{ab}(b - a)$ and $c_{bc}(c - b)$, then $\rho_q(\beta) = c_{ab} - c_{bc}$.

Proof. (1) At a point x of Γ the legs at x have $d_E \in \text{Inv}(x) \subset N$, and the balance condition $\partial\beta(x) = 0$ in $F(x) = \text{Inv}(x)$ is $\sum_E \pm c_E d_E = 0$ in N . At a point of Γ_ε (a positive vertex or a point interior to an arc) all legs lie in Γ_ε and have $d_E = \pm d_\varepsilon$; since $\text{Inv}(x)$ is torsion free the condition is $(\partial\eta_\varepsilon)(x) = 0$. At a negative vertex or node $p = C(\tau, \omega)$ the legs at p are one in each Γ_ε , $\varepsilon \subset \partial\omega$, and the condition is $\sum_\varepsilon (\partial\eta_\varepsilon)(p) d_\varepsilon = \text{ev} \partial\chi_p(\beta) = 0$.

(2) At points of Γ_ε , $\partial\tilde{\beta} = (\partial\eta_\varepsilon) \otimes \tilde{e}_\varepsilon = 0$ by (1)(i). Since $\eta_\varepsilon \otimes d_\varepsilon$ and $\eta_\varepsilon \otimes \tilde{e}_\varepsilon$ are unchanged when d_ε and η_ε change sign together, we may orient the edges of a two-cell ω along its boundary cycle when we compute at $C(\tau, \omega)$. Write $f_\varepsilon = (\partial\eta_\varepsilon)(p)$. If ω is a unimodular triangle, its three edge vectors sum to zero and any two are independent, so (1)(ii) forces $f_\varepsilon = t$

for all three edges; then $\chi_p(\beta) = t \partial[\omega]$ and $\partial \tilde{\beta}(p) = \partial \chi_p(\beta) = 0$. If $\omega = P_q$, the edge vectors are $b-a, c-b, d-c = a-b, a-d = b-c$, and (1)(ii) reads $(f_{ab} - f_{cd})(b-a) + (f_{bc} - f_{da})(c-b) = 0$; as $b-a$ and $c-b$ are independent, $f_{ab} = f_{cd} =: x_q$ and $f_{bc} = f_{da} =: y_q$. Then

$$\partial \chi_{p_q}(\beta) = x_q(e_b - e_a + e_d - e_c) + y_q(e_c - e_b + e_a - e_d) = (x_q - y_q)(e_b + e_d - e_a - e_c) = (x_q - y_q) \text{circ}_q,$$

and Definition 2.6 gives $\rho_q(\beta) = y_q - x_q$.

(3) A leg oriented away from p_q with coefficient $c(b-a)$ contributes $-c$ to $(\partial \eta_{ab})(p_q)$, so $x_q = -c_{ab}$ and likewise $y_q = -c_{bc}$. $\square$

Part (3) is the formula $\rho_q = c_{ab} - c_{bc}$ of Section 2.3. Balance at a node says that the flow passes straight through: with away orientations, the legs with lower supports $\{a, b\}$ and $\{c, d\}$ carry $c_{ab}(b-a)$ and $-c_{ab}(b-a)$, and similarly for the other pair (Lemma 5.10).

Proposition 5.6. $\rho(\text{Bal}(\Gamma)) = K$.

Proof. $\subseteq$: For $\beta \in \text{Bal}(\Gamma)$, Lemma 5.5(2) and Definition 2.6 give $\sum_q \rho_q(\beta) \text{circ}_q = -\sum_x \partial \tilde{\beta}(x)$, the negative of the augmentation of the boundary $\partial \tilde{\beta}$, which is zero.

$\supseteq$: Let $k \in K$ and $c_1 := \sum_q k_q([bc] + [da])_q \in C_1(\mathcal{X})$, the edges oriented along the boundary cycles of the node parallelograms. Then $\partial c_1 = \sum_q k_q(e_c - e_b + e_a - e_d) = -\sum_q k_q \text{circ}_q = 0$, and since $H_1(\mathcal{X}) = 0$ there is $w = \sum_\omega w_\omega[\omega] \in C_2(\mathcal{X})$ with $\partial w = c_1$, where each two-cell ω is oriented and a node parallelogram q is oriented along its boundary cycle $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$, so that $\partial[q] = [ab] + [bc] + [cd] + [da]$ with the edges oriented along the cycle. For each two-cell ω of $\mathcal{X}$ choose one negative vertex or node $p(\omega) = C(\tau, \omega)$; one exists because the dual edge G° of the two-face $G \supseteq \omega$ contains at least one edge τ of $\mathcal{T}$, and for a node parallelogram q there is exactly one, p_q , because $\ell(G^\circ) = 1$ for squares, pentagons and hexagons. Prescribe the 1-chains

$$\chi_{p(\omega)} := -w_\omega \partial[\omega] \quad (\omega \text{ a triangle}), \quad \chi_{p_q} := -w_q \partial[q] + k_q([bc] + [da]),$$

and $\chi_p := 0$ at the remaining negative vertices (a triangle G with $\ell(G^\circ) = m$ has m of them). Each satisfies $\text{ev } \partial \chi_p = 0$, since $\partial \partial[\omega] = 0$ and $\text{ev}(\text{circ}_q) = b + d - a - c = 0$, and

$$\sum_p \chi_p = -\partial w + c_1 = 0 \quad \text{in } C_1(\mathcal{X}).$$

For an edge ε of $\Sigma'_{\leq 2}$ let $f_{p,\varepsilon}$ be the coefficient of $\vec{\varepsilon}$ in χ_p . The coefficient of $\vec{\varepsilon}$ in $\sum_p \chi_p$ is $\sum_p f_{p,\varepsilon}$, the sum over the negative vertices and nodes of $\bar{\Gamma}_\varepsilon$, and it vanishes. Since $\bar{\Gamma}_\varepsilon$ is connected there is $\eta_\varepsilon \in C_1(\bar{\Gamma}_\varepsilon; \mathbb{Z})$, supported on the legs of Γ_ε , with $\partial \eta_\varepsilon = \sum_p f_{p,\varepsilon}[p]$. Then $\beta := \sum_\varepsilon \eta_\varepsilon \otimes d_\varepsilon$ has $\chi_p(\beta) = \chi_p$ for every p , so it satisfies (i) and (ii) of Lemma 5.5(1) and lies in $\text{Bal}(\Gamma)$, and by Lemma 5.5(2) $\rho_q(\beta) = y_q - x_q = (-w_q + k_q) - (-w_q) = k_q$. $\square$

At a pentagon or hexagon no separate treatment is needed: its cells are two-cells of $\mathcal{X}$, its interior edges $[o, w]$ are one-cells of $\mathcal{X}$, and for such an edge $\bar{\Gamma}_\varepsilon$ is the single arc joining the two Γ -vertices, so the condition that the coefficient of $\vec{\varepsilon}$ in $\sum_p \chi_p$ vanishes is the balance along that arc.

Proposition 5.7 (containment). *For every strict tropical 2-cycle S on B (Definition 2.5), $\rho(S) \in K$.*

Proof. $\partial z_S \in \text{Bal}(\Gamma)$ by Lemma 5.4(1), and $\rho(\text{Bal}(\Gamma)) \subseteq K$. $\square$

Only a collar of ∂S enters this argument. Proposition 5.7 is the combinatorial form of Theorems D and E, which realise the relation $\sum_q \rho_q(S)[\delta_q] = 0$, respectively $\sum_q \rho_q(S)[C_q] = 0$, by chains.

5.4. The neighbourhood of the discriminant. This subsection computes the homology of a regular neighbourhood of Γ , relative to its frontier, with the coefficients F , and deduces a normal form of relative tropical 2-cycles near Γ . The results are used with constant coefficients $\mathbb{Z}$ in Section 5.5 and with the coefficients F in Section 8.

Let $\mathfrak{K}$ be a triangulation of B in which Γ is full, and $\mathfrak{K}'$ its barycentric subdivision, whose simplices are the chains $\langle s_0 \subsetneq \cdots \subsetneq s_r \rangle$ of simplices of $\mathfrak{K}$. Such a simplex meets Γ if and only if $s_0 \subseteq \Gamma$, and then its intersection with Γ is the face spanned by the barycentres b_{s_i} with $s_i \subseteq \Gamma$, an initial segment of the chain. Let U be the union of the closed simplices of $\mathfrak{K}'$ that meet Γ , and ∂U the union of the simplices of U that do not; U is a regular neighbourhood of Γ [RS72, Ch. 3]. For a vertex p of $\mathfrak{K}$ on Γ let B_p be the union of the simplices with $p \in s_0$, the closed star of p in $\mathfrak{K}'$: a three-ball, the cone from p over the two-sphere $\partial B_p = \text{Lk}(p, \mathfrak{K}')$. For an edge E of $\mathfrak{K}$ in Γ let $\bar{D}(E)$ be the union of the simplices with $E \subseteq s_0$: a two-disc, the cone from the midpoint b_E over the circle $\partial \bar{D}(E)$ formed by the chains with $E \subsetneq s_0$, which is the barycentric subdivision of $\text{Lk}(E, \mathfrak{K})$. It meets Γ only in b_E . We orient $\bar{D}(E)$ so that its orientation followed by the direction of E is the orientation of B .

Definition 5.8 (transverse discs). For an edge E of $\mathfrak{K}$ in Γ , with A the arc whose closure contains E , and $v \in \text{Inv}(A)$, the 2-chain $v \cdot \bar{D}(E) \in C_2(\mathfrak{K}'; F)$ is the *transverse disc* at E with coefficient v . Orient each arc once and for all, and its edges accordingly. Let $d_\Gamma: \bigoplus_p \text{Inv}(p) \rightarrow \bigoplus_E \text{Inv}(E)$, the sums over the vertices and edges of $\mathfrak{K}$ in Γ and $\text{Inv}(E) := \text{Inv}(A)$, be $(d_\Gamma(v_p)_p)(E) = v_{o(E)} - v_{t(E)}$, where $o(E)$ and $t(E)$ are the origin and terminus of E . The *group of transverse discs* is $\text{Tr}(\Gamma) := \text{coker } d_\Gamma$.

A transverse disc is a disc crossing Γ once, at an interior point of an arc, and carrying a monodromy-invariant coefficient; it is the local form of a sheet of [CBM14] at one of the points $\mathfrak{x}_k$ of the discriminant-intersection clause. Contracting the vertices interior to arcs, at which d_Γ identifies the summands of the two adjacent edges, gives $\text{Tr}(\Gamma) \cong \text{coker}(\bigoplus_p \text{Inv}(p) \rightarrow \bigoplus_A \text{Inv}(A))$, the sum over the Γ -vertices and the arcs; every arc has two end vertices (D3).

Lemma 5.9 (the neighbourhood of Γ). (1) $H_k(U, \partial U; F) = 0$ for $k \neq 2, 3$, and there is an exact sequence

$$0 \rightarrow H_3(U, \partial U; F) \rightarrow \bigoplus_p \text{Inv}(p) \xrightarrow{d_\Gamma} \bigoplus_E \text{Inv}(E) \rightarrow H_2(U, \partial U; F) \rightarrow 0$$

whose third map sends v in the summand of E to the class of $v \cdot \bar{D}(E)$. In particular every relative 2-cycle of $(U, \partial U; F)$ is homologous relative to ∂U to a sum of transverse discs, and $H_2(U, \partial U; F) \cong \text{Tr}(\Gamma)$.

(2) The sequence

$$0 \rightarrow \text{Tr}(\Gamma) \rightarrow H_2(U, \partial U \cup \Gamma; F, \mathcal{D}) \xrightarrow{\partial} \text{Bal}(\Gamma) \rightarrow 0,$$

whose second map is induced by inclusion and whose third map takes the boundary on Γ , is exact.

(3) Part (1) holds, with the same proof, for the constant coefficient system $\mathbb{Z}$, with $\text{Inv}(p) = \text{Inv}(E) = \mathbb{Z}$.

Here $C_*(\partial U \cup \Gamma; F, \mathcal{D}) := C_*(\partial U; F) \oplus C_*(\Gamma; \mathcal{D})$, a subcomplex of $C_*(U; F)$ because $\partial U \cap \Gamma = \emptyset$; its relative 2-cycles are the 2-chains on U whose boundary is a chain on ∂U plus a chain on Γ with coefficients in $\mathcal{D}$, and the latter is their Γ -boundary.

Proof. The blocks. Every simplex of U is a face of a simplex meeting Γ , whose s_0 lies in Γ and is therefore a vertex p or an edge $E = [p, p']$ of Γ ; so every simplex of U lies in some B_p . A simplex of $B_p \cap B_{p'}$, $p \neq p'$, has $p, p' \in s_0$; if $s_0 \subseteq \Gamma$ then $s_0 = [p, p'] = E$ and the simplex lies in $\bar{D}(E)$, and otherwise it misses Γ and lies in ∂U . So $\partial B_p = (\partial B_p \cap \partial U) \cup \bigcup_{E \ni p} \bar{D}(E)$, $\bar{D}(E) \cap \partial U = \partial \bar{D}(E)$, and $\bar{D}(E) \cap \bar{D}(E') \subseteq \partial U$ for $E \neq E'$.

Coefficients. A simplex of U with $s_0 = p$ meets Γ in p or in a half-edge $[p, b_E]$, and has coefficient group $\text{Inv}(p)$ by Lemma 5.2(2) and $\text{Inv}(p) \subseteq \text{Inv}(E)$; a simplex with $s_0 = E$ meets Γ in b_E and has $\text{Inv}(E)$. All restriction maps among the simplices of B_p containing p , and among those of $\bar{D}(E)$ containing b_E , are identities.

The cone lemma. Let $C = c * L$ be a simplicial cone and let a coefficient system on C be the constant group I , with identity restrictions, on the simplices containing c . Then $C_*(C, L; I)$ is $I \otimes \tilde{C}_{*-1}(L)$, by $c * s \leftrightarrow s$ and $c \leftrightarrow$ the augmentation, up to sign; so $H_k(C, L; I) \cong I \otimes \tilde{H}_{k-1}(L)$. Hence $H_*(B_p, \partial B_p; F)$ is $\text{Inv}(p)$ in degree 3, generated by the classes $v_p \cdot [B_p]$, and $H_*(\bar{D}(E), \partial \bar{D}(E); F)$ is $\text{Inv}(E)$ in degree 2, generated by the $v \cdot [\bar{D}(E)]$.

Proof of (1). Let $U_1 = \partial U \cup \bigcup_E \bar{D}(E)$. By the block description,

$$C_*(U_1; F)/C_*(\partial U; F) = \bigoplus_E C_*(\bar{D}(E), \partial \bar{D}(E); F),$$

$$C_*(U; F)/C_*(U_1; F) = \bigoplus_p C_*(B_p, \partial B_p; F).$$

So $H_*(U_1, \partial U; F) = \bigoplus_E \text{Inv}(E)$ in degree 2 and $H_*(U, U_1; F) = \bigoplus_p \text{Inv}(p)$ in degree 3, and the exact sequence of the triple $(U, U_1, \partial U)$ reduces to the sequence of (1), with $H_k(U, \partial U; F) = 0$ for $k \neq 2, 3$. The connecting map sends $v_p \cdot [B_p]$ to $v_p \cdot [\partial B_p]$ modulo $C_*(\partial U; F)$, which is $\sum_{E \ni p} \epsilon(p, E) v_p \cdot [\bar{D}(E)]$ with $\epsilon(p, E) = +1$ if $p = o(E)$ and -1 if $p = t(E)$: at $o(E)$ the outward normal of B_p along $\bar{D}(E)$ points along E . So the connecting map is d_Γ .

Proof of (2). The triple $C_*(\partial U; F) \subseteq C_*(\partial U \cup \Gamma; F, \mathcal{D}) \subseteq C_*(U; F)$ gives the exact sequence

$$H_2(\Gamma; \mathcal{D}) \rightarrow H_2(U, \partial U; F) \rightarrow H_2(U, \partial U \cup \Gamma; F, \mathcal{D}) \rightarrow H_1(\Gamma; \mathcal{D}) \rightarrow H_1(U, \partial U; F).$$

$H_2(\Gamma; \mathcal{D}) = 0$ since Γ is a graph, $H_1(\Gamma; \mathcal{D})$ is the group of 1-cycles of $C_*(\Gamma; \mathcal{D})$ on $\mathfrak{R}'$, which is $\text{Bal}(\Gamma)$ (Section 5.1), and $H_1(U, \partial U; F) = 0$ by (1).

(3) The proof of (1) used only that the coefficients are constant, with identity restrictions, on the simplices of each B_p containing p and of each $\bar{D}(E)$ containing b_E . $\square$

Lemma 5.10 (balanced leg data). *At a Γ -vertex p let $\text{Bal}_p \subseteq \mathbb{Z}^{\{\text{arcs at } p\}}$ be the group of integers (c_A) with $\sum_A c_A d_A = 0$ in $\text{Inv}(p)$, the vanishing vectors d_A oriented away from p .*

- (1) *At a negative vertex, with the d_A oriented so that $d_1 + d_2 + d_3 = 0$: $\text{Bal}_p = \mathbb{Z}(1, 1, 1)$.*
- (2) *At a positive vertex, with the d_A oriented to be equal: $\text{Bal}_p = \{c_1 + c_2 + c_3 = 0\}$, with basis $(1, 0, -1)$, $(0, 1, -1)$.*
- (3) *At a node p_q , with $d_{ab} = d_{dc} = b - a$ and $d_{bc} = d_{ad} = c - b$ on the legs with lower supports $\{a, b\}$, $\{c, d\}$, $\{b, c\}$, $\{a, d\}$: $\text{Bal}_p = \{c_{ab} = -c_{dc}, c_{bc} = -c_{ad}\}$, with basis the $b - a$ passage $(c_{ab}, c_{dc}) = (1, -1)$ and the $c - b$ passage $(c_{bc}, c_{ad}) = (1, -1)$.*

Each basis element is the leg datum of a germ of ∂S for a tropical 2-cycle S in the sense of [CBM14]: a boundary vertex of S (a tripod), a turn of ∂S through a positive vertex, and a passage through a node.

Proof. (1) The d_i span $\text{Inv}(p)$ of rank two with $d_1 + d_2 + d_3 = 0$ (Definition 3.16), so $\sum c_i d_i = 0$ forces $c_1 = c_2 = c_3$. (2) All three d_A equal $d \neq 0$ (Lemma 3.17). (3) The equation is $(c_{ab} + c_{dc})(b - a) + (c_{bc} + c_{ad})(c - b) = 0$ with $b - a, c - b$ independent. $\square$

Lemma 5.11 (normal form). *Let $z \in Z(\mathfrak{K})$, and let Θ be a relative 2-cycle of $(U, \partial U \cup \Gamma; F, \mathcal{D})$ on $\mathfrak{K}'$ whose Γ -boundary is $\text{sd}(\partial z)$. Choose one edge E_A in each arc A . There are $w \in C_3(U; F)$, $v_A \in \text{Inv}(A)$ and $y \in C_2(\mathfrak{K}'; F)$ supported on simplices not in $U \setminus \partial U$, such that*

$$\text{sd}(z) - \partial w = \Theta + \sum_A v_A \cdot \bar{D}(E_A) + y.$$

The class of $(v_A)_A$ in $\text{Tr}(\Gamma)$ is determined by z and Θ . In particular $\text{sd}(z) - \partial w$ has the boundary network and the node coefficients of z , and near Γ it consists of Θ and transverse discs.

Proof. Let z_U be the part of $\text{sd}(z)$ on the 2-simplices of U . An edge of U not in ∂U meets Γ , so every 2-simplex containing it lies in U , and ∂z_U agrees with $\partial \text{sd}(z)$ on such edges. Hence z_U is a relative 2-cycle of $(U, \partial U \cup \Gamma; F, \mathcal{D})$ with Γ -boundary $\text{sd}(\partial z)$, and $z_U - \Theta$ has boundary on ∂U : it is a relative 2-cycle of $(U, \partial U; F)$. By Lemma 5.9(1), $z_U - \Theta = \sum_E v_E \cdot \bar{D}(E) + \partial w + c$ with $w \in C_3(U; F)$ and $c \in C_2(\partial U; F)$. To collect the discs of an arc into one, let E, E' be consecutive edges of A with common vertex $p = t(E) = o(E')$. The proof of Lemma 5.9(1) gives $\partial[B_p] = -\bar{D}(E) + \bar{D}(E') + (\text{a chain on } \partial U)$, so for $v \in \text{Inv}(A) = \text{Inv}(p)$

$$v \cdot \bar{D}(E) - v \cdot \bar{D}(E') = -\partial(v \cdot [B_p]) + (\text{a chain on } \partial U).$$

After these moves, $\text{sd}(z) - \partial w' = \Theta + \sum_A v_A \cdot \bar{D}(E_A) + y$ with $y = (\text{sd}(z) - z_U) + c'$, c' a chain on ∂U . The class of (v_A) is the class of $z_U - \Theta$ in $H_2(U, \partial U; F) \cong \text{Tr}(\Gamma)$. $\square$

An admissible Θ always exists: z_U is one, and by Lemma 5.9(2) every balanced network is the Γ -boundary of a relative 2-cycle of $(U, \partial U \cup \Gamma; F, \mathcal{D})$. Two choices of Θ with the same Γ -boundary differ by an element of $\text{Tr}(\Gamma)$, for instance two windings of a sheet about an arc.

Remark 5.12 (the group of transverse discs is free). The map d_Γ is injective and $\text{Tr}(\Gamma)$ is free of rank $2a - 2(n_- + |Q|) - n_+$, where a, n_- and n_+ are the numbers of arcs, negative vertices and positive vertices. To see the integrality assertion, suppose that a rational family $(v_p)_p$ has $d_\Gamma(v_p)_p$ integral. Its classes in $N_\mathbb{Q}/N$ agree along every arc. In a two-face F_τ with $\tau^\vee = G$ a two-face of Δ , the dual graph of the subdivision of G is connected, including its boundary legs. The common class therefore lies in the images of the rational lines generated by all boundary edge vectors of G : at the adjoining positive vertices these are precisely the stalks of Lemma 5.1. Two of those vectors form a basis of the saturated plane lattice of G . For a pentagon or hexagon one may take $w_{i+1} - w_i, w_{i+2} - w_{i+1}$ with $w_i + w_{i+2} = o + w_{i+1}$; the triangle and square cases are immediate. As these two vectors d_1, d_2 form a basis of the saturated plane lattice $\text{span}(d_1, d_2) \cap N$, the images of $\mathbb{Q}d_1$ and $\mathbb{Q}d_2$ in $N_\mathbb{Q}/N$ meet only in 0: if $t_1 d_1 - t_2 d_2 \in N$ with $t_1, t_2 \in \mathbb{Q}$, then $t_1, t_2 \in \mathbb{Z}$. So this common class is zero. The connected dual graph of the triangulation of each $E^\vee$, for an edge E of Δ , joins every remaining positive vertex to such a boundary leg. Thus every v_p is integral. Applied to $(v_p)/m$ for every positive integer m when $d_\Gamma(v_p) = 0$, this also proves injectivity. The image is saturated, and the rank follows from the stalk ranks and the arc presentation above.

Remark 5.13 (scope). The proofs of Lemmas 5.2, 5.9 and 5.11 use only that B is a triangulated three-manifold, that Γ is a graph full in the triangulation, and that F is the system of flat sections of a local system on $B \setminus \Gamma$; they hold verbatim for any such triple. Lemma 5.10 and Remark 5.12 use the stalks of Lemma 5.1. Section 8 applies the former on the base of the

mirror, through a homeomorphism of pairs and an isomorphism of local systems, and uses from this subsection only the surjectivity in Lemma 5.9(1), the moves of Lemma 5.11, and Lemma 5.10.

5.5. Filling. We work on $\mathfrak{K}_0$, the barycentric subdivision of $\mathcal{P}$, in which Γ is full (Lemma 3.13), and prove that every balanced network is a boundary network. Let $\mathfrak{K}_0^\circ \subset \mathfrak{K}_0$ be the subcomplex of simplices disjoint from Γ . Every point $x \in B \setminus \Gamma$ lies in a simplex $s = s_\Gamma * s_0$, with s_Γ the face spanned by the vertices on Γ and $s_0 \neq \emptyset$ the opposite face, and is $(1-t)a + tb$ with $a \in s_\Gamma$, $b \in s_0$, $t \in (0, 1]$; pushing t to 1 is a deformation retraction of $B \setminus \Gamma$ onto $|\mathfrak{K}_0^\circ|$. Simplicial chains on $\mathfrak{K}_0^\circ$ with coefficients F compute $H_*(B \setminus \Gamma; \Lambda)$.

Lemma 5.14 (dual discs and dual paths). (1) *For an edge ε of $\Sigma'_{\leq 2}$, the dual disc D_ε , the closure of $\bigcup_\tau C(\tau, \varepsilon)$, is a closed two-disc and a subcomplex of $\mathfrak{K}_0$. $D_\varepsilon \cap \Gamma = \bar{\Gamma}_\varepsilon$, with Γ_ε in the interior and the negative vertices and nodes of $\bar{\Gamma}_\varepsilon$ on the boundary circle, which is the union of the dual paths D_ω of the two-cells $\omega \supset \varepsilon$.*

(2) *For a two-cell ω of Σ' , the dual path D_ω , the closure of $\bigcup_\tau C(\tau, \omega)$, is the path $b_{C_0} - b_{f_1} - b_{C_1} - \dots - b_{f_k} - b_{C_k}$ of $\mathfrak{K}_0$, where $C_0|f_1|C_1|\dots|f_k|C_k$ is the sequence of two-cells with lower support ω and maximal cells of Lemma 3.14. Its points on Γ are the barycentres of the f_i that separate maximal cells with different facet indices; if ω lies in a two-face G of Δ with $G^\circ = [u_0, \dots, u_m]$, these are the $m \geq 1$ points $C([u_{i-1}, u_i], \omega)$, and if ω lies in the interior of a facet of Δ there are none. Two dual paths meet at most in common end points, the points b_{C_0} , b_{C_k} , which lie in $C(u, \omega_3)$ for three-cells $\omega_3 \supset \omega$.*

(3) *For every simplex s of D_ε , $d_\varepsilon \in F(s)$, and the restriction maps send d_ε to d_ε . So $c \mapsto c \otimes d_\varepsilon$ is a chain map $C_*(D_\varepsilon; \mathbb{Z}) \rightarrow C_*(\mathfrak{K}_0; F)$.*

Proof. (1) By (D5), $\mathcal{L}(D_\varepsilon)$ is a two-dimensional face of the convex polytope ∇^H , a polygon whose edges are the images of the D_ω , $\omega \supset \varepsilon$. By (D1) its cells are the $C(\tau, \omega)$ with $\omega \supseteq \varepsilon$; those in Γ have $\dim \tau \geq 1$, and are the arcs and positive vertices $C(\tau, \varepsilon)$ and the negative vertices and nodes $C(\tau, \omega)$ with $\omega \supsetneq \varepsilon$ a two-cell (a pair (τ, ω) with ω a three-cell has τ a vertex).

(2) The simplices of $\mathfrak{K}_0$ in $\bigcup_\tau C(\tau, \omega)$ are the chains with $\omega(c_0) = \omega$, which consist of the f_i and of the maximal cells $C_1, \dots, C_{k-1}$, whose lower support is ω ; the closure adds b_{C_0} and b_{C_k} . Such a simplex lies in Γ if and only if $\dim \tau(c_k) \geq 1$, that is, $c_k = f_i$ lies in a two-face of ∇^H , which means that f_i separates cells with different facet indices; by Lemma 3.14 there are $m = \ell(G^\circ)$ of them, and none if ω lies in a facet of Δ . A common point of D_ω and $D_{\omega'}$ lies in a cell $C(\tau, \omega'')$ with $\omega'' \supseteq \omega \cup \omega'$, a three-cell.

(3) A small neighbourhood V of a simplex of D_ε meets only cells $C(\tau', \omega')$ with $\tau' \subseteq \tau$ for some (τ, ω) with $\omega \supseteq \varepsilon$, so $\tau' \subseteq \varepsilon^\vee$. Hence every facet F_u whose interior meets V has $u \in \varepsilon^\vee$, so $\langle u, d_\varepsilon \rangle = 0$, and Lemma 5.1(1) makes the vector d_ε a flat section over $V \setminus \Gamma$. It is the same vector of N for all simplices. $\square$

Step 1: pushing a network onto the dual paths. Let $\beta \in \text{Bal}(\Gamma)$, and write $\beta = \sum_\varepsilon \eta_\varepsilon \otimes d_\varepsilon$ as in Section 5.3. By Lemma 5.5(1)(i), η_ε is a relative 1-cycle of $(D_\varepsilon, \partial D_\varepsilon)$. Since D_ε is a disc, $H_1(D_\varepsilon, \partial D_\varepsilon) = 0$, so $\eta_\varepsilon = \partial \zeta_\varepsilon + \xi_\varepsilon$ with $\zeta_\varepsilon \in C_2(D_\varepsilon; \mathbb{Z})$ and $\xi_\varepsilon \in C_1(\partial D_\varepsilon; \mathbb{Z})$. By Lemma 5.14(3), $\zeta := \sum_\varepsilon \zeta_\varepsilon \otimes d_\varepsilon \in C_2(\mathfrak{K}_0; F)$, and

$$b := \beta - \partial \zeta = \sum_\varepsilon \xi_\varepsilon \otimes d_\varepsilon$$

is a 1-cycle of $C_*(\mathfrak{K}_0; F)$ supported on the union of the dual paths.

Lemma 5.15. *Orient each dual path D_ω from b_{C_0} to b_{C_k} . Along D_ω the cycle b has one constant coefficient $y_\omega \in N$; if ω lies in a two-face of Δ , $y_\omega \in \text{lin}(\omega) \cap N$.*

Proof. Interior points of a dual path lie on no other dual path. At a vertex b_{C_j} or b_{f_j} off Γ , both in the interior of one facet F_u , the cycle condition in $F = u^\perp \cap N$ makes the coefficients of the two edges equal. At a point b_{f_i} on Γ both edges meet Γ only in b_{f_i} , so their coefficients lie in $\text{Inv}(b_{f_i}) = \text{lin}(\omega) \cap N$ (Lemma 5.1(2)), and the cycle condition in $F(b_{f_i}) = \text{Inv}(b_{f_i})$ makes them equal as vectors of N . $\square$

Step 2: moving off Γ . For each two-cell ω in a two-face of Δ fix a vertex n_ω of ω . Let $v = b_f$ be a point of D_ω on Γ , where f separates the maximal cells C_- and C_+ (in this order along D_ω) with facet indices $u_- \neq u_+$, and let v' be a vertex of $\mathcal{P}$ in f with lower ray n_ω . The triangles $[b_{v'}, b_f, b_{C_\pm}]$ of $\mathfrak{R}_0$ meet Γ only in b_f : their other open faces lie in $C(\tau(f), n_\omega)$, $C(u_\pm, \omega)$ and $C(u_\pm, n_\omega)$, and their vertices other than b_f are off Γ . So by Lemma 5.2(2) and Lemma 5.15

$$\theta_v := y_\omega([b_{v'}, b_f, b_{C_-}] - [b_{v'}, b_f, b_{C_+}]) \in C_2(\mathfrak{R}_0; F),$$

with boundary

$$\partial\theta_v = y_\omega([b_{C_-}, b_{v'}] + [b_{v'}, b_{C_+}] - [b_{C_-}, b_f] - [b_f, b_{C_+}]).$$

Adding $\partial\theta_v$ to b replaces the path segment $b_{C_-} \rightarrow b_f \rightarrow b_{C_+}$ by $b_{C_-} \rightarrow b_{v'} \rightarrow b_{C_+}$. Let θ be the sum of the θ_v over the points of the dual paths on Γ and $b' := b + \partial\theta$. Then b' is a 1-cycle on $\mathfrak{R}_0^\circ$ with coefficients in Λ , homologous to b in $C_*(\mathfrak{R}_0; F)$.

The complement of Γ . Let $\mathcal{U}_F$ be the union of the open facets and $\mathcal{U}_R$ the union of the vertex regions. By (D2) and (D5), $\mathcal{U}_F$ and $\mathcal{U}_R$ are disjoint unions of open three-balls, they cover $B \setminus \Gamma$, and $\mathcal{U}_F \cap \mathcal{U}_R$ is the disjoint union of the open three-cells $C(u, n)$, one for each dual cell pair (u, n) of a vertex of $\mathcal{T}$ and a vertex of Σ' . Put

$$\mathcal{Z}_G := \left\{ x = (x_{un}) \in \bigoplus_{(u,n)} u^\perp \cap N : \sum_n x_{un} = 0 \text{ for every } u, \sum_u x_{un} \in \mathbb{Z}n \text{ for every } n \right\}.$$

This is the group of 1-cycles of the bipartite graph G_0 with vertices the u and the n and one edge for each dual pair (u, n) , with the coefficients $u^\perp \cap N$ at u and on the edges and $N/\mathbb{Z}n$ at n ; G_0 is the nerve of the cover. Call a vertex of $\mathfrak{R}_0^\circ$ a *facet vertex* if it lies in $\mathcal{U}_F \setminus \mathcal{U}_R$ and a *region vertex* if it lies in $\mathcal{U}_R \setminus \mathcal{U}_F$.

Lemma 5.16 (the cover by facets and regions). *$H_2(B \setminus \Gamma; \Lambda) = 0$, and the connecting map of the Mayer–Vietoris sequence of the cover $\{\mathcal{U}_F, \mathcal{U}_R\}$ is an isomorphism $\text{it}: H_1(B \setminus \Gamma; \Lambda) \rightarrow \mathcal{Z}_G$. Let c be a 1-cycle of $\mathfrak{R}_0^\circ$ with coefficients in Λ all of whose vertices are facet vertices or region vertices. Then an edge of c joining two facet vertices, or two region vertices, lies in one open facet, respectively one region, and an edge joining a facet vertex in F_u to a region vertex in $\tilde{R}_n$ has its interior in $C(u, n)$. The component x_{un} of $\text{it}[c]$ is the sum, over the edges of the second kind at (u, n) , of $+y$ for an edge running from F_u to $\tilde{R}_n$ and $-y$ for an edge running from $\tilde{R}_n$ to F_u , where y is the coefficient of the edge read in the facet chart.*

We call $\text{it}[c]$ the *itinerary* of c : it records, with coefficients, the transitions of c between open facets and regions.

Proof. The Mayer–Vietoris sequence of an open cover holds for homology with local coefficients: as for constant coefficients, it follows from the exact sequence of a pair and from excision, both of which hold for local coefficients [Hat02, §3.H]. Since every component of $\mathcal{U}_F$, $\mathcal{U}_R$ and $\mathcal{U}_F \cap \mathcal{U}_R$ is contractible, their homology with coefficients in Λ is concentrated in degree zero, where it is the group of flat sections: $u^\perp \cap N$ on $\text{int } F_u$ and on $C(u, n)$, and $N/\mathbb{Z}n$

on $\tilde{R}_n$, the maps being the identity and π_{un} (D2). The sequence gives $H_2(B \setminus \Gamma; \Lambda) = 0$ and identifies $H_1(B \setminus \Gamma; \Lambda)$ with the kernel of $\bigoplus_{(u,n)} u^\perp \cap N \rightarrow \bigoplus_u u^\perp \cap N \oplus \bigoplus_n N/\mathbb{Z}n$, which is $\mathcal{Z}_G$.

An edge of $\mathfrak{R}_0^\circ$ is a chain $c_0 \subsetneq c_1$ of cells of $\mathcal{P}$; its open part lies in $C(\tau(c_1), \omega(c_0))$, and its end points in $C(\tau(c_0), \omega(c_0))$ and $C(\tau(c_1), \omega(c_1))$. Since $\tau(c_1) \subseteq \tau(c_0)$ and $\omega(c_0) \subseteq \omega(c_1)$, an edge between two facet vertices lies in one $\text{int } F_u$, an edge between two region vertices lies in one $\tilde{R}_n$, and an edge between a facet vertex in F_u and a region vertex in $\tilde{R}_n$ has its interior in $C(u, n)$. Subdivide the edges of the third kind at their midpoints and write $c = c_F + c_R$, with c_F the part on the edges of the first kind and the halves at facet vertices. Then $\text{it}[c]$ is the class of ∂c_F in $H_0(\mathcal{U}_F \cap \mathcal{U}_R; \Lambda) = \bigoplus_{(u,n)} u^\perp \cap N$; a half running from a facet vertex to a midpoint in $C(u, n)$ with coefficient y contributes $+y$ there, and a half running from a midpoint to a facet vertex contributes $-y$. $\square$

The itinerary of b' . The vertices of b' are facet vertices or region vertices: the b_{C_j} , the b_{f_j} off Γ and the end points b_{C_0} , b_{C_k} lie in open facets and have lower supports of dimension at least one, while the $b_{v'}$ have a one-point lower support and lie in two-faces of ∇^h . A dual path D_ω with ω in a two-face G , after Step 2, runs through $\text{int } F_{u_0}$, $\tilde{R}_{n_\omega}$, $\text{int } F_{u_1}$, $\dots$, $\tilde{R}_{n_\omega}$, $\text{int } F_{u_m}$, with $G^\circ = [u_0, \dots, u_m]$, and its coefficient $y_\omega \in \text{lin}(\omega) \subseteq u_i^\perp$ is the same vector of N in all these facet charts. By Lemma 5.16 its contribution to $\text{it}[b']$ telescopes to $y_\omega([u_0, n_\omega] - [u_m, n_\omega])$, and a dual path inside a facet contributes nothing. Hence

$$\text{it}[b'] = \sum_{\omega} y_\omega([u_0(\omega), n_\omega] - [u_m(\omega), n_\omega]),$$

the sum over the two-cells in two-faces of Δ , where $[u, n]$ denotes the summand of the dual cell pair (u, n) .

The degree map and the meridians. For $x \in \mathcal{Z}_G$ define $\deg(x) \in \mathbb{Z}^{\text{vert } \Sigma'}$ by $\sum_u x_{un} = \deg(x)_n$. From the display, $\sum_u \text{it}[b']_{un} = \sum_{\omega: n_\omega=n} (y_\omega - y_\omega) = 0$, so

$$\deg \text{it}[b'] = 0.$$

For an arc $A = C([u, u'], [n, n'])$ and $v \in \text{Inv}(A) = [u, u']^\perp \cap N$ put

$$m_A(v) := v([u, n] - [u', n] + [u', n'] - [u, n']) \in \mathcal{Z}_G,$$

and let $I_\Gamma \subseteq \mathcal{Z}_G$ be the span of all $m_A(v)$. Each $m_A(v)$ lies in $\ker \deg$. Let $L = [b_e, b_f]$ be a leg in A , with $e \subset f$, let v_1, v_2 be the end points of e and $C_\pm$ the maximal cells containing f , with facet indices $u_\pm$. The link of L in $\mathfrak{R}_0$ is the 4-cycle $\text{Lk}(L) = b_{v_1} - b_{C_+} - b_{v_2} - b_{C_-} - b_{v_1}$ (the chains $v_i \subset e \subset f \subset C_\pm$), whose vertices are region vertices b_{v_i} and facet vertices $b_{C_\pm}$, and the cone $b_f * \text{Lk}(L)$ consists of the triangles $[b_f, b_{v_i}, b_{C_\pm}]$, whose open parts lie in $C(u_\pm, n_i)$ (Figure 2). These triangles meet Γ only in b_f , which is a two-leg point, a negative vertex or a node, with $\text{Inv}(b_f) = [u_-, u_+]^\perp \cap N = \text{Inv}(A)$. So $v \cdot (b_f * \text{Lk}(L)) \in C_2(\mathfrak{R}_0; F)$ for $v \in \text{Inv}(A)$, with boundary $v \cdot \text{Lk}(L)$, the coefficient on each edge being the restriction of v ; and by Lemma 5.16

$$\text{it}[v \cdot \text{Lk}(L)] = \pm m_A(v),$$

since the coefficient read in the facet charts of $F_{u_\pm}$ is v . The cycle $\text{Lk}(L)$ is a meridian of L : it bounds the disc $x * \text{Lk}(L)$, which meets Γ only in an interior point x of L .

Lemma 5.17 (generation by invariant meridians). $\ker \deg = I_\Gamma$.

The lemma says that every degree-zero class in the complement is generated by meridians with invariant coefficients. The proof below has two parts. Part (a), with constant coefficients,

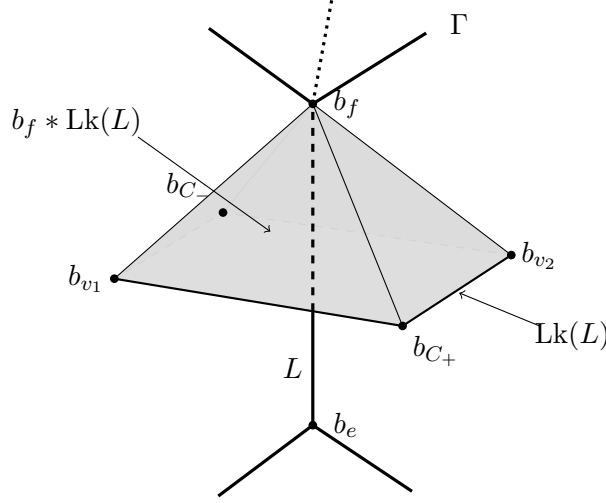


FIGURE 2. A leg $L = [b_e, b_f]$ of Γ , its link $\text{Lk}(L)$ in $\mathfrak{K}_0$, and the cone $b_f * \text{Lk}(L)$. The link passes alternately through the vertex regions of the lower rays of the end points of e and the open facets of the two maximal cells containing f ; with a coefficient $v \in \text{Inv}(A)$ its itinerary is the meridian class $\pm m_A(v)$. The cone meets Γ only at its apex b_f , where v is a flat section.

shows that meridians generate $H_1(B \setminus \Gamma; \mathbb{Z})$; it uses $H_1(B; \mathbb{Z}) = 0$, the fact that transverse discs generate $H_2(U, \partial U; \mathbb{Z})$ (Lemma 5.9(3)), and a radial retraction of the boundary circle of each such disc onto the link of its leg. Part (b) passes to the local system, where the additional obstruction is measured by the quotient of N by the invariant lattices. Its two-cycles split into cochains on the graphs $\Gamma \cap \partial F_u$; since each ∂F_u is a two-sphere, these cochains come from vertex functions, and unimodularity lifts the vertex functions to vectors in N over the integers. This last step is what excludes a finite-index obstruction.

Proof. Let $\mathfrak{X}_0$ be the two-dimensional cell complex obtained from the graph G_0 by attaching one square $\square_A$ for each arc $A = C([u, u'], [n, n'])$ along the edge loop $(u, n), (u', n), (u', n'), (u, n')$. On $\mathfrak{X}_0$ consider the coefficient system $\hat{\Lambda}$ of subgroups of N given by $u^\perp \cap N$ at the vertex u and on the edges (u, n) , by N at the vertex n , and by $[u, u']^\perp \cap N$ on $\square_A$, with inclusions as restriction maps. Then $C_*(\mathfrak{X}_0; \hat{\Lambda})$ is a subcomplex of $C_*(\mathfrak{X}_0) \otimes N$. Its 1-cycles are the x with $\sum_n x_{un} = 0$ and $\sum_u x_{un} = 0$ in N , which is $\ker \deg$ because $n \neq 0$; and with suitable orientations of the squares $\partial(v \cdot \square_A) = m_A(v)$, so its 1-boundaries form I_Γ . The claim is $H_1(\mathfrak{X}_0; \hat{\Lambda}) = 0$. Let $\mathcal{Q} = N/\hat{\Lambda}$. It is 0 at the vertices n ; $N/(u^\perp \cap N) \cong \mathbb{Z}$ by $\langle u, \cdot \rangle$ at u and on the edges (u, n) ; and $N/([u, u']^\perp \cap N) \cong \mathbb{Z}^2$ by $(\langle u, \cdot \rangle, \langle u', \cdot \rangle)$ on $\square_A$, these maps being onto because u, u' are part of a basis of M (Lemma 5.1(3)). The exact sequence of $0 \rightarrow \hat{\Lambda} \rightarrow N \rightarrow \mathcal{Q} \rightarrow 0$ contains

$$H_2(\mathfrak{X}_0; N) \rightarrow H_2(\mathfrak{X}_0; \mathcal{Q}) \rightarrow H_1(\mathfrak{X}_0; \hat{\Lambda}) \rightarrow H_1(\mathfrak{X}_0; N),$$

and we show that the last group vanishes and the first map is onto.

(a) $H_1(\mathfrak{X}_0; N) = H_1(\mathfrak{X}_0; \mathbb{Z}) \otimes N$ since N is free, and $H_1(\mathfrak{X}_0; \mathbb{Z})$ is the quotient of $H_1(G_0; \mathbb{Z})$ by the classes of the attaching loops. Lemma 5.16, with Λ replaced by the constant system $\mathbb{Z}$, identifies $H_1(B \setminus \Gamma; \mathbb{Z})$ with $H_1(G_0; \mathbb{Z})$, and carries the class of $\text{Lk}(L)$, $L \subset A$, to $\pm$ the attaching loop of $\square_A$. So it suffices that the meridians $\text{Lk}(L)$ generate $H_1(B \setminus \Gamma; \mathbb{Z})$. Let

$U \supset \Gamma$ be the neighbourhood of Section 5.4 for $\mathfrak{K} = \mathfrak{K}_0$, and $\mathfrak{E}$ the union of the simplices of $\mathfrak{K}'_0$ disjoint from Γ ; then $B = U \cup \mathfrak{E}$, $U \cap \mathfrak{E} = \partial U$, and $B \setminus \Gamma$ retracts onto $\mathfrak{E}$ as above. In the Mayer–Vietoris sequence $H_1(\partial U) \rightarrow H_1(U) \oplus H_1(\mathfrak{E}) \rightarrow H_1(B) = 0$, a class $(0, \gamma)$ is the image of a class of $H_1(\partial U)$ that vanishes in $H_1(U)$, that is, of a boundary $\partial \vartheta$, $\vartheta \in H_2(U, \partial U; \mathbb{Z})$. By Lemma 5.9(3), ϑ is a sum of discs $\bar{D}(L)$, L a leg, so $H_1(\mathfrak{E})$ is generated by the circles $\partial \bar{D}(L)$. The closed star $L * \text{Lk}(L)$ of L in $\mathfrak{K}_0$ meets Γ only in L , because Γ is full, and the radial projection along the join lines is a deformation retraction of $(L * \text{Lk}(L)) \setminus L$ onto $\text{Lk}(L)$. It maps $\partial \bar{D}(L)$, which lies in $(L * \text{Lk}(L)) \setminus L$, homeomorphically onto $\text{Lk}(L)$: it sends the barycentre of $L * s$ to the barycentre of s , and the segment between two such barycentres onto a segment. Hence $[\partial \bar{D}(L)] = \pm [\text{Lk}(L)]$ in $H_1(B \setminus \Gamma)$.

(b) There are no three-cells, so $H_2(\mathfrak{X}_0; \mathcal{Q})$ is the group of 2-cycles. A 2-chain with coefficients in $\mathcal{Q}$ is a family of integers $p = (p_A^{(u)})$, one for each arc A and each vertex u of τ_A , where $A = C(\tau_A, \omega_A)$; the boundary of $C_*(\mathfrak{X}_0; \mathcal{Q})$ in degree two decouples over u , the component on the edge (u, n) involving only the $p^{(u)}$. Fix u and let $S_u = \partial F_u$, a two-sphere oriented as the boundary of F_u . By (D2), S_u is the union of the cells $C(\tau, \omega)$ with $u \in \tau$ and $\dim \tau \geq 1$, a subcomplex of the complex of dual cell pairs. Its cells in Γ form the graph $\Gamma_u = \Gamma \cap S_u$, whose one-cells are the arcs A with $u \in \tau_A$; its other cells are the $C(\tau, n)$, which group, according to n , into the sets $D_n^u = S_u \cap \tilde{R}_n$. Each nonempty D_n^u is connected: its cells are the $C(u * s, n)$, s a simplex of the link of u in the triangulation of the face $n^\vee$ of Δ° , with $C(u * s, n)$ in the closure of $C(u * s', n)$ when $s \supseteq s'$. That link is connected: $n^\vee$ is a facet of Δ° if n is a vertex of Δ , an edge of lattice length one (with u an end point) if n is the centre of a pentagon or hexagon, and a point if n lies in the interior of a facet of Δ , and the link of a vertex in a triangulated three-polytope is a disc or a sphere. So the D_n^u are the components of $S_u \setminus \Gamma_u$. An arc $A = C([u, u'], [n, n'])$ lies in the closure of exactly two two-cells of S_u , $C([u, u'], n) \subseteq D_n^u$ and $C([u, u'], n') \subseteq D_{n'}^u$, with opposite incidences; the two-face $F_{uu'}$ is oriented oppositely as a face of S_u and of $S_{u'}$, so the orientation of $\square_A$ can be chosen so that, for both u and u' , the coefficient of $\partial(p \square_A)$ on (u, n) is p times the incidence of $C([u, u'], n)$ with A in S_u . With this choice, p is a cycle if and only if, for every u and n , the 1-cochain $p^{(u)}$ on the arcs of Γ_u satisfies

$$\sum_{\text{two-cells } f \subseteq D_n^u} (d\tilde{p}^{(u)})(f) = 0,$$

where $\tilde{p}^{(u)}$ extends $p^{(u)}$ by zero to the one-cells of S_u not in Γ_u and d is the cellular coboundary of S_u .

Consider the exact sequence $0 = H^1(S_u) \rightarrow H^1(\Gamma_u) \rightarrow H^2(S_u, \Gamma_u)$. The relative cellular cochains of (S_u, Γ_u) split as $\bigoplus_n C_n^*$, the cochains on the cells of D_n^u , since the coboundary of a cell $C(\tau, n)$ consists of cells $C(\tau', n)$. Each one-cell of D_n^u lies on exactly two two-cells, both in D_n^u , with opposite incidences, and D_n^u is connected, so the sum of the values on the two-cells identifies C_n^2/dC_n^1 with $\mathbb{Z}$; thus $H^2(S_u, \Gamma_u) = \bigoplus_n \mathbb{Z}$, and the connecting map sends the class of $p^{(u)}$ to the family of sums displayed above. Therefore the u -components of 2-cycles are exactly the coboundaries $p^{(u)} = d\varphi_u$ of functions φ_u on the vertices of Γ_u , that is, the integral combinations $\sum_v \varphi_u(v) d_u[v]$ of the vertex cochains $d_u[v]$, which are ± 1 on the arcs of Γ_u at v and 0 elsewhere (arcs have distinct end points).

For a Γ -vertex $v = C(\tau_v, \omega_v)$ let Σ_v be the sum of the squares $\square_A$ of the arcs at v , with signs making it a 2-cycle of $\mathfrak{X}_0$ with integer coefficients: at a negative vertex or node the arcs are the $C(\tau_v, \varepsilon)$, ε an edge of ω_v , and each edge (u, n) , n a vertex of ω_v , occurs in the two squares of the edges of ω_v at n ; at a positive vertex the arcs are the $C(\tau_1, \omega_v)$, τ_1 an edge of

τ_v , and each edge (u, n) occurs in the two squares of the edges of τ_v at u . For $y \in N$ the image of $y \cdot \Sigma_v$ in $C_2(\mathfrak{X}_0; \mathcal{Q})$ has u -component, for $u \in \tau_v$, equal to $\langle u, y \rangle$ times a cycle supported on the arcs of Γ_u at v with entries ± 1 . The cycles supported on these arcs form a group of rank at most one. Indeed, the arcs of Γ_u at v are the $C(\tau_1, \omega_1)$ at v with $u \in \tau_1$; for such a cycle the condition attached to D_n^u involves exactly the arcs at v whose lower support contains n , each with coefficient ± 1 ; and the graph on the arcs at v in which two are joined when their lower supports share a vertex is connected (a cycle for a negative vertex or node, and two arcs with the same lower support for a positive vertex). So the value on one arc determines the others. As $d_u[v]$ is such a cycle with entries ± 1 , the u -component of the image of $y \cdot \Sigma_v$ is $\epsilon_u(v) \langle u, y \rangle d_u[v]$ with $\epsilon_u(v) = \pm 1$.

Now let p be a 2-cycle, $p^{(u)} = \sum_v \varphi_u(v) d_u[v]$. By Lemma 5.1(3) the map $N \rightarrow \mathbb{Z}^{\tau_v}$, $y \mapsto (\langle u, y \rangle)_{u \in \tau_v}$, is onto, so there are $y_v \in N$ with $\langle u, y_v \rangle = \epsilon_u(v) \varphi_u(v)$ for all $u \in \tau_v$. Then $\sum_v y_v \cdot \Sigma_v \in H_2(\mathfrak{X}_0; N)$ maps to p . So $H_2(\mathfrak{X}_0; N) \rightarrow H_2(\mathfrak{X}_0; \mathcal{Q})$ is onto, and $H_1(\mathfrak{X}_0; \hat{\Lambda}) = 0$. $\square$

Theorem 5.18 (filling). $\partial Z(\mathfrak{K}_0) = \text{Bal}(\Gamma)$.

Proof. $\partial Z(\mathfrak{K}_0) \subseteq \text{Bal}(\Gamma)$ because $\partial\partial = 0$. Let $\beta \in \text{Bal}(\Gamma)$ and let ζ, b, θ, b' be as in Steps 1 and 2. By Lemmas 5.16 and 5.17, $\text{it}[b'] \in \ker \deg = I_\Gamma$, so $\text{it}[b'] = \sum_i m_{A_i}(v_i)$ for finitely many arcs A_i and $v_i \in \text{Inv}(A_i)$. Choose a leg $L_i \subset A_i$ with apex b_{f_i} , and signs so that $\text{it}[v_i \cdot \text{Lk}(L_i)] = m_{A_i}(v_i)$. Then $b' - \sum_i v_i \cdot \text{Lk}(L_i)$ is a 1-cycle on $\mathfrak{K}_0^\circ$ with itinerary zero, hence null-homologous in $B \setminus \Gamma$ (Lemma 5.16), hence the boundary ∂y_0 of a 2-chain y_0 on $\mathfrak{K}_0^\circ$ with coefficients in Λ . Put

$$z := \zeta - \theta + y_0 + \sum_i v_i \cdot (b_{f_i} * \text{Lk}(L_i)) \in C_2(\mathfrak{K}_0; F).$$

Since $\beta = b + \partial\zeta$, $b = b' - \partial\theta$, $b' = \partial y_0 + \sum_i v_i \cdot \text{Lk}(L_i)$ and $\partial(b_f * c) = c$ for a 1-cycle c , we get $\partial z = \beta$. So $z \in Z(\mathfrak{K}_0)$ with boundary network β . $\square$

Proof of Theorem C. By Theorem 5.18 and Proposition 5.6,

$$\rho(Z(\mathfrak{K}_0)) = \rho(\partial Z(\mathfrak{K}_0)) = \rho(\text{Bal}(\Gamma)) = K.$$

For any triangulation $\mathfrak{K}$ of B in which Γ is full, $\partial Z(\mathfrak{K}) \subseteq \text{Bal}(\Gamma)$ (Section 5.1), so $\rho(Z(\mathfrak{K})) \subseteq K$. $\square$

If $\mathfrak{K}$ is a subdivision of $\mathfrak{K}_0$ in which Γ is full, the subdivision operator carries $Z(\mathfrak{K}_0)$ into $Z(\mathfrak{K})$ without changing node coefficients, so $\rho(Z(\mathfrak{K})) = K$ as well.

Remark 5.19 (no torsion). Every step is integral: $H_1(D_\varepsilon, \partial D_\varepsilon) = 0$, $H_1(\mathcal{X}) = 0$, $H^1(S^2) = 0$, $H_1(S^3) = 0$, and the unimodularity of $\mathcal{T}$ in Lemma 5.1(3). No condition on the torsion of $H_2(\hat{X}; \mathbb{Z})$ or of $H_1(B; \iota_* \Lambda)$ enters, and $\rho(Z(\mathfrak{K}_0))$ is K itself, not a sublattice of finite index.

Remark 5.20 (where K enters). The lattice K enters only through Proposition 5.6. Theorem 5.18 holds for every balanced network, whatever its node coefficients, and its proof uses Hypothesis (Proj) only through the existence of B , admissibility through the list of Γ -vertices (D3), and Hypothesis (U) through Lemma 5.1(2),(3). The fillings it constructs consist of sheets $\zeta_\varepsilon \otimes d_\varepsilon$ in the dual discs, with coefficients in $\mathbb{Z}d_\varepsilon$ and containing arcs and positive vertices in their interior; pairs of triangles at the Γ -points of the dual paths, with coefficients in $\text{lin}(\omega)$; cones $b_f * \text{Lk}(L)$ crossing a leg at b_f with an invariant coefficient; and a Λ -valued 2-chain in $B \setminus \Gamma$. The finite computations illustrating the theorem are described in Section 11.4.

Part II. The mirror lift and the relation lattice

6. THE TRANSFER TO THE MIRROR BASE

The relative tropical 2-cycles of Theorem C live on the base B'_σ , whereas the chains of Theorem D are built over the base B_Y of the degeneration of the mirror family. This section constructs a piecewise linear homeomorphism $\mathfrak{h}: B'_\sigma \rightarrow B_Y$ that carries the discriminant of B'_σ onto that of B_Y and the local system of integral tangent vectors of B'_σ onto the dual of that of B_Y , and uses it to transfer relative tropical 2-cycles to B_Y with the same boundary networks and the same node coefficients (Proposition 6.10). The homeomorphism is assembled from three comparisons, each compatible with the complex of dual cell pairs of Section 4: a change of fan-side subdivision (Proposition 6.3), by which Σ' is replaced by $\mathcal{F}_C$; a change of height (Lemma 6.5); and the label-preserving cellular homeomorphism between the two bases built from the same data (Proposition 4.7). Lemmas 6.6 and 6.7 describe the cell structure near a node and near an arc of the discriminant; these are the neighbourhoods in which the local models of Section 8 are placed.

6.1. Change of fan-side subdivision and of height. Three comparisons of bases occur in this paper: a change of fan-side subdivision (Proposition 6.3), a change of height (Lemma 6.5) and the transfer to the mirror base (Section 6.3). Each consists of a piecewise linear homeomorphism of pairs and an isomorphism of local systems, and each carries relative tropical 2-cycles by the following lemma.

Lemma 6.1 (transport of relative cycles). *For $i = 1, 2$ let B_i be a closed piecewise linear three-manifold, $\Gamma_i \subset B_i$ a compact one-dimensional subpolyhedron, Λ_i a local system of free abelian groups of rank three on $B_i \setminus \Gamma_i$, and suppose that Γ_i is the union of finitely many points and open arcs A , each carrying a flat section d_A of Λ_i over $V_A \setminus \Gamma_i$ for a neighbourhood V_A of the closure $\bar{A}$. For triangulations $\mathfrak{K}$ of B_i in which Γ_i is a full subcomplex define F_i , $\mathcal{D}_i(E) = \mathbb{Z}d_A$ for $E \subseteq \bar{A}$, and $Z_i(\mathfrak{K})$ as in Definition 2.4. Let $f: B_1 \rightarrow B_2$ be a piecewise linear homeomorphism with $f(\Gamma_1) = \Gamma_2$, mapping each arc A onto an arc $f(A)$, and $\theta: \Lambda_1 \rightarrow f^*\Lambda_2$ an isomorphism of local systems on $B_1 \setminus \Gamma_1$ with $\theta(d_A) = d_{f(A)}$ for every arc A . Let $\mathfrak{K}$ be a triangulation of B_1 in which Γ_1 is full, $\mathfrak{K}_1^0$ a subdivision of $\mathfrak{K}$ on whose simplices f is linear, and $\mathfrak{K}_1$ a derived subdivision of $\mathfrak{K}_1^0$. Then:*

- (1) *$f(\mathfrak{K}_1)$ is a triangulation of B_2 in which Γ_2 is a full subcomplex; if f is linear on the simplices of $\mathfrak{K}$, the same holds for $f(\mathfrak{K})$, and one may take $\mathfrak{K}_1 = \mathfrak{K}$ below;*
- (2) *θ induces isomorphisms $F_1(s) \rightarrow F_2(f(s))$, compatible with the restrictions and carrying $\mathcal{D}_1(E)$ onto $\mathcal{D}_2(f(E))$, hence an isomorphism of chain complexes $f_\#: C_*(\mathfrak{K}_1; F_1) \rightarrow C_*(f(\mathfrak{K}_1); F_2)$ mapping $C_*(\Gamma_1; \mathcal{D}_1)$ onto $C_*(\Gamma_2; \mathcal{D}_2)$;*
- (3) *the composite $f_* := f_\# \circ \text{sd}: Z_1(\mathfrak{K}) \rightarrow Z_2(f(\mathfrak{K}_1))$, with sd the subdivision operator, is an injective homomorphism, and if $\partial z = \sum_E c_E d_{A(E)}[E]$, $A(E)$ the arc whose closure contains the edge E , then $\partial f_* z = \sum_E c_E \sum_{E' \subseteq E} d_{f(A(E))}[f(E')]$, the inner sum over the edges E' of $\mathfrak{K}_1$ in E with the orientation of E .*

Proof. (1) The subcomplex Γ_1 of $\mathfrak{K}$ is a subcomplex of every subdivision, and in a derived subdivision every subcomplex is full [RS72, Ch. 3]. As f is linear on the simplices of $\mathfrak{K}_1$, it is a simplicial isomorphism onto $f(\mathfrak{K}_1)$, which is therefore a triangulation of B_2 with $\Gamma_2 = f(\Gamma_1)$ full. The derived subdivision is needed: in an arbitrary subdivision on which f is linear, a simplex with all its vertices on Γ_1 need not lie in Γ_1 .

(2) For a simplex s of $\mathfrak{K}_1$, f maps the neighbourhoods V of $\bar{s}$ onto neighbourhoods $f(V)$ of $f(\bar{s})$, all small ones arising in this way, with $f(V \setminus \Gamma_1) = f(V) \setminus \Gamma_2$, and θ maps the flat

sections of Λ_1 over $V \setminus \Gamma_1$ to the flat sections of Λ_2 over $f(V) \setminus \Gamma_2$, compatibly with restriction to smaller sets. By hypothesis it carries d_A to $d_{f(A)}$.

(3) The subdivision operator sends an oriented simplex s of $\mathfrak{K}$ with coefficient $X \in F(s)$ to the sum of the simplices s' of $\mathfrak{K}_1$ of the same dimension in $\bar{s}$, with the induced orientations and the coefficients $X|_{s'}$; the restriction is defined because a neighbourhood of $\bar{s}$ is a neighbourhood of s' . It is a chain map, it maps $C_*(\Gamma_1; \mathcal{D}_1)$ into itself, since the edges of $\mathfrak{K}_1$ in an edge $E \subseteq \bar{A}$ of Γ_1 lie in $\bar{A}$, and it is injective, since restriction of flat sections to a smaller connected set is injective and $V \setminus \Gamma_1$ is connected for small V . So it maps $Z_1(\mathfrak{K})$ injectively into $Z_1(\mathfrak{K}_1)$, with the stated boundary, and (2) applies. $\square$

The hypothesis of Lemma 6.1 concerns neighbourhoods of the closed arcs, including their end points. For the bases of this paper it is verified once, by the following lemma.

Lemma 6.2 (flatness near the closed arcs). *Let $\mathcal{F}$ be a fan-side subdivision, $B_X = B_X(\mathcal{F})$ the base built from $\mathcal{F}$ and an integral height $\check{h}$, and $B_Y = B_Y(\mathcal{F})$ the base built from a height H , as in Section 4.1. Let $\mathbf{p}_e = ([u, u'], [n, n'])$ be a dual cell pair of edges, and let $V_X \subseteq B_X$ and $V_Y \subseteq B_Y$ be the unions of the cells $C_X(\mathbf{p})$, respectively $C_Y(\mathbf{p})$, over the dual cell pairs $\mathbf{p}$ with $\mathbf{p} \succeq z$ for some dual cell pair $z \prec \mathbf{p}_e$.*

- (1) *There are exactly two dual cell pairs $z_{\pm} = (\tau_{\pm}, \omega_{\pm}) \prec \mathbf{p}_e$; they have dimension zero, and $\tau_{\pm} \supseteq [u, u']$, $\omega_{\pm} \supseteq [n, n']$. The closure of the arc $C_X(\mathbf{p}_e)$ is $C_X(\mathbf{p}_e) \cup C_X(z_+) \cup C_X(z_-)$, and V_X is an open neighbourhood of it. The same holds on B_Y .*
- (2) *The vector $n' - n$ in the chart of every open facet $\text{int } F_{u''}$ that meets V_X , together with its class modulo n'' in the chart of every region $\tilde{R}_{n''}$, defines a flat section of Λ_X over $V_X \setminus \Gamma_X$.*
- (3) *The covector $\langle \cdot, n' - n \rangle$, on $n''^{\perp} \cap M$ over every open facet $\text{int } F_{n''}^*$ and on $M/\mathbb{Z}u''$ over every region $\tilde{R}_{u''}^*$ that meets V_Y , defines a flat section of $\Lambda_Y^{\vee}$ over $V_Y \setminus \Gamma_Y$.*

Proof. (1) By Proposition 4.7(2) the closure of a cell $C_X(\mathbf{p})$ is the union of the cells $C_X(\mathbf{q})$, $\mathbf{q} \preceq \mathbf{p}$, and it is a ball of dimension $\dim \mathbf{p}$ whose boundary is the union of the cells with $\mathbf{q} \prec \mathbf{p}$, all of them nonempty. For $\mathbf{p} = \mathbf{p}_e$ the closure is a one-ball, and its boundary, two points, is the disjoint union of the nonempty cells $C_X(\mathbf{q})$, $\mathbf{q} \prec \mathbf{p}_e$, each of dimension less than one. So there are exactly two pairs $z_{\pm} \prec \mathbf{p}_e$, both of dimension zero, and $z_{\pm} \prec \mathbf{p}_e$ means $\tau_{\pm} \supseteq [u, u']$ and $\omega_{\pm} \supseteq [n, n']$. If $\mathbf{q} \not\succeq z_+$ and $\mathbf{q} \not\succeq z_-$, the same holds for every $\mathbf{q}' \preceq \mathbf{q}$, so the closed cell $\overline{C_X(\mathbf{q})}$ does not meet V_X ; hence the complement of V_X is the union of these finitely many closed cells, and V_X is open. It contains $C_X(\mathbf{p}_e)$ and $C_X(z_{\pm})$. On B_Y the argument is the same, with Proposition 4.7(3).

(2) By Lemma 4.4(2), $V_X \setminus \Gamma_X$ is covered by the sets $\text{int } F_{u''} \cap V_X$ and $\tilde{R}_{n''} \cap V_X$. As V_X is a union of cells and $\text{int } F_{u''} = \bigcup_{\omega} C_X(u'', \omega)$ (Proposition 4.7(5)), the open facet $\text{int } F_{u''}$ meets V_X only if $(u'', \omega) \succeq z_+$ or $(u'', \omega) \succeq z_-$ for some ω , that is, only if u'' is a vertex of τ_+ or of τ_- . Since $z_{\pm}$ is a dual cell pair and n, n' are vertices of $\omega_{\pm}$, every vertex u'' of $\tau_{\pm}$ has $\langle u'', n \rangle = \langle u'', n' \rangle = -1$, so $n' - n \in u''^{\perp} \cap N$. On $\text{int } F_{u''}$ the local system Λ_X is the constant lattice $u''^{\perp} \cap N$ of the facet chart, and on $\tilde{R}_{n''}$ it is the constant lattice $N/\mathbb{Z}n''$ of the region chart (proof of Proposition 4.11); so the prescribed values are flat sections over the pieces. On an overlap $C_X(u'', n'') = \text{int } F_{u''} \cap \tilde{R}_{n''}$ the two charts are related by the reduction $u''^{\perp} \cap N \rightarrow N/\mathbb{Z}n''$ modulo n'' (Lemma 4.4(1)), which carries $n' - n$ to its class. So the pieces glue to a flat section over $V_X \setminus \Gamma_X$.

(3) On B_Y the roles are exchanged: Λ_Y is the constant lattice $n''^{\perp} \cap M$ on $\text{int } F_{n''}^*$ and $M/\mathbb{Z}u''$ on $\tilde{R}_{u''}^*$, the two being related on $C_Y(u'', n'')$ by the reduction modulo u'' , and $\tilde{R}_{u''}^* =$

$\bigcup_{\omega} C_Y(u'', \omega)$ meets V_Y only if u'' is a vertex of τ_+ or of τ_- (Proposition 4.7(5)). For such u'' , $\langle u'', n' - n \rangle = 0$ as in (2), so $\langle \cdot, n' - n \rangle$ is a homomorphism $M/\mathbb{Z}u'' \rightarrow \mathbb{Z}$; on $n''^\perp \cap M$ it is defined for every n'' ; and it is compatible with the reductions modulo u'' . So it defines a flat section of the dual local system $\Lambda_Y^\vee$ over $V_Y \setminus \Gamma_Y$. $\square$

At an end point of the arc, Lemma 6.2 contains the following statements about monodromy, which are recorded for comparison (Proposition 4.10(3),(4)); we state them for $\mathcal{F} = \Sigma'$ and $\mathcal{F} = \mathcal{F}_C$, which agree on the faces of Δ of dimension at most two (Proposition 3.26(2)), so that the two-cells in two-faces of Δ are unimodular triangles and node parallelograms. On B_X , at a negative vertex $([u, u'], \omega)$, ω a unimodular triangle, and at a node $([u, u'], P)$, the monodromies around all arcs at the vertex are transvections $x \mapsto x \pm \langle u' - u, x \rangle d$ with the common covector $u' - u$, and they fix the edge vector $n' - n$ of ω , respectively of P (for $\mathcal{F} = \Sigma'$ this is Lemma 3.17, and at a node the invariance of $b - a$ and $c - b$ noted before Lemma 3.19). At a positive vertex $(\{u_1, u_2, u_3\}, [n, n'])$ the three arcs have the common vanishing vector $n' - n$ and covectors $u_j - u_i \in (n' - n)^\perp$, so each of their transvections fixes $n' - n$. On B_Y the types are exchanged, and for $\mathcal{F} = \mathcal{F}_C$ the invariant covectors at the vertices of each type, listed in Proposition 6.9(3), contain $\langle \cdot, n' - n \rangle$. The lemma gives more than invariance under the loops around the arcs: a flat section over the complement of the discriminant in a whole neighbourhood of the closed arc.

Consequently, for every base $B_X(\mathcal{F})$ the discriminant, with its arcs and the sections $d_A = n' - n$ of Lemma 6.2(2), satisfies the hypothesis of Lemma 6.1 on the sections d_A , and so does the discriminant of $B_Y(\mathcal{F})$ with the dual local system and the sections $\langle \cdot, n' - n \rangle$ of Lemma 6.2(3); for the image of the latter under a piecewise linear homeomorphism Υ of B_Y this follows by transport, since Υ maps neighbourhoods of the closed arcs onto neighbourhoods of the closed arcs. In particular d_A lies in $F(E)$ for every edge $E \subseteq \bar{A}$ of a triangulation in which the discriminant is full, including the edges that end at a vertex of the discriminant.

When Lemma 6.1 is applied below, the identity $\theta(d_A) = d_{f(A)}$ is checked at one point of $B_1 \setminus \Gamma_1$ near A . This suffices. Let V be the connected component containing $\bar{A}$ of $V_A \cap f^{-1}(V_{f(A)})$, an open neighbourhood of $\bar{A}$ because $f(\bar{A}) = f(A)$. Then $V \setminus \Gamma_1$ is connected, since a subpolyhedron of codimension two does not separate a connected piecewise linear manifold, and $\theta(d_A)$ and $f^*d_{f(A)}$ are flat sections of $f^*\Lambda_2$ over it; if they agree at one point, they agree on $V \setminus \Gamma_1$.

Proposition 6.3 (change of fan-side subdivision). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fan-side subdivisions which agree on the two-skeleton of $\partial\Delta$: a cell of either contained in a face of Δ of dimension at most two is a cell of the other. Write C^i , Γ_X^i , Λ_X^i for the cells, discriminant and local system of $B_X(\mathcal{F}_i)$; the underlying space $\partial\nabla^{\check{h}}$ is the same.*

- (1) *The dual cell pairs (τ, ω) with $\dim \tau \geq 1$ are the same for $\mathcal{F}_1$ and $\mathcal{F}_2$.*
- (2) *If ω is not contained in a face of Δ of dimension at most two, then every dual cell pair (τ, ω) has $\tau = \{u_G\}$, where G is the facet of Δ containing ω and u_G the vertex of Δ° dual to G .*
- (3) *There is a piecewise linear homeomorphism g of $\partial\nabla^{\check{h}}$, preserving every face F_τ and isotopic to the identity through face-preserving homeomorphisms, such that $g(C^1(\tau, \omega)) = C^2(\tau, \omega)$ for $\dim \tau \geq 1$. In particular $g(\Gamma_X^1) = \Gamma_X^2$, g maps the arc $C^1([u, u'], [n, n'])$ onto the arc $C^2([u, u'], [n, n'])$ and each vertex of Γ_X^1 onto the vertex of Γ_X^2 of the same dual cell pair, and there is an isomorphism $\kappa: \Lambda_X^1 \rightarrow g^*\Lambda_X^2$ that*

is the identity of $u^\perp \cap N$ on every open facet $\text{int } F_u$ and carries the vanishing vector $n' - n$ of each arc to that of its image.

- (4) If h_1 and h_2 agree on the lattice points of the two-skeleton, g can be taken to be the identity on the two-skeleton of $\partial \nabla^h$.

Proof. (1) If $\dim \tau \geq 1$, then $\omega \subseteq \tau^\vee$, a face of Δ of dimension at most two, where the two subdivisions agree. (2) The smallest face of Δ containing ω is a facet G , and $\tau^\vee$ is a proper face of Δ containing ω , so $\tau^\vee = G$; the only point $m \in \Delta^\circ$ with $\langle m, \cdot \rangle = -1$ on G is u_G .

(3) By Proposition 4.7(5), the two-skeleton $\bigcup_{\dim \tau \geq 1} F_\tau$ is the union of the cells $C^i(\tau, \omega)$ with $\dim \tau \geq 1$; these form a subcomplex of each complex of dual cell pairs, with the same indexing set and order by (1). The inductive construction in the proof of Proposition 4.7(4), applied to these two regular CW structures on the same space, gives a piecewise linear homeomorphism g of the two-skeleton with $g(C^1(\tau, \omega)) = C^2(\tau, \omega)$; it preserves every F_τ with $\dim \tau \geq 1$, which is the union of the cells $C^i(\tau', \omega)$ with $\tau' \supseteq \tau$. Extend g over each facet, a convex three-ball whose boundary it preserves, by coning from an interior point; the cone of a piecewise linear map is piecewise linear. Faces of dimension zero are points and are fixed; on the edges, the two-faces and the facets in turn, g is isotopic to the identity relative to the lower-dimensional faces by the Alexander trick, extended by coning as in Appendix C.3. By Propositions 4.10(1),(2), $g(\Gamma_X^1) = \Gamma_X^2$, with arcs and vertices as stated. Define κ on each $\text{int } F_u$, which g preserves, as the identity of $u^\perp \cap N$. A point b of the two-skeleton outside Γ_X^1 lies in some $C^1(\tau, n)$ with $\dim \tau \geq 1$, so $b \in \tilde{R}_n^1$ and $g(b) \in C^2(\tau, n) \subseteq \tilde{R}_n^2$. On a connected neighbourhood $U \subseteq \tilde{R}_n^1 \cap g^{-1}(\tilde{R}_n^2)$ of b both local systems are the constant $N/\mathbb{Z}n$, and on $U \cap \text{int } F_u$ the identity of $u^\perp \cap N$ corresponds to the identity of $N/\mathbb{Z}n$ under π_{un}^X on both sides. So κ extends over U as the identity of $N/\mathbb{Z}n$, compatibly on overlaps, and is an isomorphism of local systems on $\partial \nabla^h \setminus \Gamma_X^1$. Near the arc $C^i([u, u'], [n, n'])$ the vanishing vector is represented in the chart of F_u by $n' - n \in u^\perp \cap N$ on both bases (Proposition 4.10(4)), and κ is the identity there. On both bases $n' - n$ is a flat section over the complement of the discriminant in a neighbourhood of the closed arc (Lemma 6.2(2)); since the two sections correspond under κ in the chart of F_u , they correspond on such a neighbourhood (the last paragraph after Lemma 6.2), so (g, κ) satisfies the hypotheses of Lemma 6.1.

(4) By Lemma 4.3(2), the decomposition of a face F_τ with $\dim \tau \geq 1$ depends only on the restriction of h_i to $\tau^\vee \cap N$, which lies in the two-skeleton of $\partial \Delta$. So the two decompositions, their barycentric subdivisions and their cells agree on the two-skeleton, and g can be taken to be the identity there. $\square$

The vertices of $\mathcal{F}_2$ in the interior of a facet G of Δ only change the cells $C^2(u_G, \omega)$ inside the open facet $\text{int } F_{u_G}$, by (2); there Λ_X^2 is the constant facet lattice, and κ is its identity.

We apply Proposition 6.3 with $\mathcal{F}_1 = \Sigma'$ and $\mathcal{F}_2 = \mathcal{F}_C$. The pulling refinement $\mathcal{F}_C$ is a fan-side subdivision, induced by the height h_C , and it agrees with Σ' on the two-skeleton (Proposition 3.26). We build $B_X(\mathcal{F}_C)$ with the function h_C , and fix the resulting piecewise linear homeomorphism $g_C: B'_\sigma \rightarrow B_X(\mathcal{F}_C)$ and the isomorphism $\kappa_C: \Lambda \rightarrow g_C^* \Lambda_X^{\mathcal{F}_C}$. Since g_C maps the vertex p_q of the discriminant, the zero-cell $C([u, u'], P_q)$, to the zero-cell of the same dual cell pair, and each arc to the arc with the same lower support, the node coefficients of Definition 2.6, read from the lower supports of the legs at p_q , are preserved by the transport $(g_C)_*$ of Lemma 6.1.

Remark 6.4. If every lattice point of $\partial \Delta$ is a vertex of Σ' , no point is pulled and $\mathcal{F}_C = \Sigma'$; if in addition $B_X(\mathcal{F}_C)$ is built with the same function h as B'_σ , then g_C can be taken to be the

identity. This is the case for the subdivisions of the examples X_9 , X_{19} and X_{20} of Section 11 (Remark 3.27(1)). Nothing in this paper uses it.

Lemma 6.5 (change of height). *Let $\check{h}_1$ and $\check{h}_2$ be two integral heights on the same unimodular triangulation $\mathcal{T}$ of $\partial\Delta^\circ$, both with $\check{h}_i$ and $\check{h}_i - \check{\varphi}$ strictly convex on the fan over $\mathcal{T}$, and let $B_i = \partial\nabla^{\check{h}_i}$ be the base $B_X(\Sigma')$ built from $\check{h}_i$, with the same function h in $\tilde{\Sigma}'$. Write $C^i(\tau, \omega)$ for the cells of its complex of dual cell pairs, Γ_i for its discriminant and Λ_i for its local system. Then:*

- (1) *there is a piecewise linear homeomorphism $\phi: B_1 \rightarrow B_2$ with $\phi(C^1(\tau, \omega)) = C^2(\tau, \omega)$ for every dual cell pair (τ, ω) ; it maps each open facet $\text{int } F_u$ and each region $\tilde{R}_n$ onto the open facet and the region with the same index, Γ_1 onto Γ_2 , each vertex p_q of the discriminant onto p_q , each arc onto the arc with the same dual cell pair, and each vertex of Γ_1 onto a vertex of Γ_2 of the same type;*
- (2) *there is an isomorphism of local systems $\kappa: \Lambda_1 \rightarrow \phi^*\Lambda_2$ on $B_1 \setminus \Gamma_1$ that is the identity of $u^\perp \cap N$ on every $\text{int } F_u$ and the identity of $N/\mathbb{Z}n$ on every $\tilde{R}_n$;*
- (3) *the transport ϕ_* of Lemma 6.1 carries relative tropical 2-cycles on B_1 to relative tropical 2-cycles on B_2 with the same integers c_E in their boundary networks and the same node coefficients; and $(S, j, v) \mapsto (S, \phi \circ j, \kappa(v))$ is a bijection from the tropical 2-cycles of [CBM14, Def. 7.2] on B_1 to those on B_2 , which preserves piecewise linearity of j and the node coefficients.*

In particular, relative tropical 2-cycles and tropical 2-cycles on the base of any such height on $\mathcal{T}$ are carried to cycles of the same kind on the base of the normalised height of Section 3.3, with the same node coefficients.

Proof. (1) The homeomorphism of Lemma 4.9 has these properties. Alternatively: the set of dual cell pairs and its order depend only on $\mathcal{T}$ and Σ' , and Proposition 4.7, which uses only the strict convexity of $\check{h}_i$ and $\check{h}_i - \check{\varphi}$, applies to both heights. The inductive construction in its proof gives piecewise linear homeomorphisms $\Psi_i: |\text{Ord}(\text{DP})| \rightarrow B_i$ with $\Psi_i(|\text{Ord}^{\leq \mathbf{p}}|) = \overline{C^i(\mathbf{p})}$, and $\phi = \Psi_2 \circ \Psi_1^{-1}$ preserves the cells. The facets, regions, arcs and vertices of the discriminant are unions of cells described by their dual cell pairs (Propositions 4.7(5) and 4.10(1),(2)); the vertex p_q of the discriminant is the zero-cell $C^i([u, u'], P_q)$, the barycentre of the carrying cell, the unique cell with face label $[u, u']$ and lower support P_q (Lemma 3.14); and the type of a three-valent vertex is determined by its dual cell pair (Proposition 4.10(3)).

(2) The open facets and the regions cover $B_i \setminus \Gamma_i$, with $\text{int } F_u \cap \tilde{R}_n = C^i(u, n)$ and no triple intersections (proof of Proposition 4.11). On $\text{int } F_u$ the lattice Λ_i is $u^\perp \cap N$, on $\tilde{R}_n$ it is $N/\mathbb{Z}n$, and on $C^i(u, n)$ the two are related by the projection $u^\perp \cap N \rightarrow N/\mathbb{Z}n$, on both bases. So the identities on the pieces agree on the overlaps and define κ .

(3) The pair (ϕ, κ) satisfies the hypotheses of Lemma 6.1: the vanishing vector $n' - n$ of the arc of $([u, u'], [n, n'])$ is, on both bases, a flat section over the complement of the discriminant in a neighbourhood of the closed arc (Lemma 6.2(2)); in the chart of F_u it is $n' - n \in u^\perp \cap N$ on both bases and κ is the identity there, so κ carries the one section to the other on such a neighbourhood (the last paragraph after Lemma 6.2). The integers c_E are therefore unchanged, and the node coefficients, read from the lower supports of the legs at the node points (Definition 2.6), are unchanged by (1). The clauses of [CBM14, Def. 7.2] refer to the topology of S and j , to Γ_i with its vertex types and nodes, and to Λ_i and its monodromy. With the crossing clause read as conjugacy in $GL_2(\mathbb{Z})$ (Section 5.2), an orientation of B_i enters only through Definition 3.16, whose types do not depend on it

(Remark 4.12), and through the cyclic indexing of the three sheets at an interior edge in the interior-edge balancing clause of [CBM14, Def. 7.2], which does not affect the balancing condition there. By (1) and (2), ϕ and κ preserve all of these, and κ intertwines parallel transport, so $(S, \phi \circ j, \kappa(v))$ satisfies the clauses if and only if (S, j, v) does; $\phi \circ j$ is piecewise linear if j is. For a tropical 2-cycle, triangulated, the node coefficients are those of the relative cycle it defines (Section 5.2), and these are unchanged. $\square$

6.2. The cells near a node and near an arc of the discriminant. In this subsection $B_X = B_X(\mathcal{F}_C)$, B_Y is the mirror base, and $\mathcal{L} = \Psi_Y \circ \Psi_X^{-1}$ is the label-preserving cellular homeomorphism constructed in the proof of Proposition 4.7(4), which we fix. Since g_C preserves the faces of $\partial \nabla^{\tilde{h}}$ and the cells $C(\tau, \omega)$ with $\dim \tau \geq 1$ (Proposition 6.3(3)), statements about B'_σ that involve only these are carried over by g_C .

Let $q \in Q$ be a node, with node parallelogram $P = abcd$ in the two-face G of Δ , labelled as in Section 2.2, and $G^\circ = [u, u']$, an edge of $\mathcal{T}$ because $\ell(G^\circ) = 1$. Then $([u, u'], P)$ is a dual cell pair of dimension zero; its zero-cell in B'_σ is the vertex p_q of the discriminant, the barycentre of the carrying cell, the unique cell of $\mathcal{P}$ with face label $[u, u']$ and lower support P (Lemma 3.14). Write $\mathfrak{q} = ([u, u'], P)$ for this dual cell pair; its zero-cells are indexed by the node q . Put $f_1 = b - a = c - d$ and $f_2 = d - a = c - b$. The statements below concern only the cells of $\mathcal{T}$ contained in $[u, u']$ and the cells of $\mathcal{F}_C$ contained in P ; they do not depend on whether G is a square, a pentagon or a hexagon, nor, for a hexagon, on the branch of the profile.

To a face ω of P assign $(\epsilon_1(\omega), \epsilon_2(\omega)) \in \{-, 0, +\}^2$, its position in the directions f_1 and f_2 : $a \mapsto (-, -)$, $b \mapsto (+, -)$, $c \mapsto (+, +)$, $d \mapsto (-, +)$, $ab \mapsto (0, -)$, $bc \mapsto (+, 0)$, $cd \mapsto (0, +)$, $da \mapsto (-, 0)$, $P \mapsto (0, 0)$. To $\tau \in \{u, [u, u'], u'\}$ assign $\epsilon_3(u) = -, \epsilon_3([u, u']) = 0, \epsilon_3(u') = +$. In $\mathbb{R}^3$ with coordinates (y_1, y_2, y_3) let $\mathcal{O}_\epsilon = \{y : \text{sign } y_i = \epsilon_i\}$ for $\epsilon \in \{-, 0, +\}^3$, the relatively open coordinate orthants of all dimensions, and $O = \{|y_1| + |y_2| + |y_3| \leq 1\}$. Put $\text{St}_q^X = \Psi_X(\text{st } \mathfrak{q})$ and $\text{St}_q^Y = \Psi_Y(\text{st } \mathfrak{q})$, where $\text{st } \mathfrak{q}$ is the closed star of the vertex $\mathfrak{q}$ in $\text{Ord}(\text{DP})$; these are closed neighbourhoods of the zero-cells indexed by q , and $\mathcal{L}(\text{St}_q^X) = \text{St}_q^Y$. Finally let $\mathcal{W}_q$ be the union of the cells $C_Y([u, u'], \omega)$, $\omega \subseteq P$: the node, the four arcs at it and the four two-cells between them.

- Lemma 6.6** (the node star). (1) *The dual cell pairs $\mathfrak{p} \succeq \mathfrak{q}$ are the 27 pairs (τ, ω) with τ equal to u , $[u, u']$ or u' and ω a face of P : the node; four arcs, $\tau = [u, u']$ and ω an edge; four two-cells, $\tau = [u, u']$ and ω a vertex; two one-cells, $\omega = P$ and τ a vertex; eight two-cells, τ a vertex and ω an edge; and eight three-cells, τ and ω vertices. The map $\epsilon(\tau, \omega) = (\epsilon_1(\omega), \epsilon_2(\omega), \epsilon_3(\tau))$ is an isomorphism of this poset onto the face poset of the coordinate fan: $\mathfrak{p} \preceq \mathfrak{p}'$ if and only if $\mathcal{O}_{\epsilon(\mathfrak{p})} \subseteq \overline{\mathcal{O}_{\epsilon(\mathfrak{p}')}}$; and $\dim \mathfrak{p}$ is the number of nonzero entries of $\epsilon(\mathfrak{p})$.*
- (2) *There are piecewise linear homeomorphisms $\Phi_q^X : \text{St}_q^X \rightarrow O$ and $\Phi_q^Y : \text{St}_q^Y \rightarrow O$ with $\Phi_q^X(C_X(\mathfrak{p}) \cap \text{St}_q^X) = \mathcal{O}_{\epsilon(\mathfrak{p})} \cap O = \Phi_q^Y(C_Y(\mathfrak{p}) \cap \text{St}_q^Y)$ for all $\mathfrak{p} \succeq \mathfrak{q}$, and $\Phi_q^Y \circ \mathcal{L} = \Phi_q^X$ on St_q^X . Any two homeomorphisms $\text{St}_q^Y \rightarrow O$ with this property are isotopic through such homeomorphisms.*
- (3) *Consequently $\Phi_q^Y(\mathcal{W}_q \cap \text{St}_q^Y) = \{y_3 = 0\} \cap O$; Φ_q^Y maps $\tilde{R}_u^* \cap \text{St}_q^Y$ and $\tilde{R}_{u'}^* \cap \text{St}_q^Y$ onto $\{y_3 < 0\} \cap O$ and $\{y_3 > 0\} \cap O$, the open facets $\text{int } F_a^*, \dots, \text{int } F_d^*$ onto the four quadrants $\{(\text{sign } y_1, \text{sign } y_2) = (\epsilon_1, \epsilon_2)\}$ of the vertices, and the edge F_P^* of ∇^H onto the y_3 -axis. On B_X , Φ_q^X maps $F_{uu'} \cap \text{St}_q^X$ onto $\{y_3 = 0\} \cap O$ and $\text{int } F_u, \text{int } F_{u'}$ onto the two half-balls.*

Proof. (1) The pairs $\succeq \mathbf{q}$ are the (τ, ω) with $\tau \subseteq [u, u']$ and $\omega \subseteq P$, and every such pair is dual, being contained in a dual cell pair. The faces of the parallelogram P are the products of faces of two segments, and the faces of the segment $[u, u']$ are its end points and itself; in each factor, inclusion of faces corresponds to $\epsilon_i \in \{0, \epsilon'_i\}$ with 0 for the larger face, which is the order of the coordinate fan. The number of nonzero entries is $3 - \dim \tau - \dim \omega$.

(2) The closed star of $\mathbf{q}$ in $\text{Ord}(\text{DP})$ consists of the chains containing $\mathbf{q}$ and their subchains; since $\mathbf{q}$ is minimal in $\{\mathbf{p} \succeq \mathbf{q}\}$ and has $\dim \mathbf{q} = 0$, it is the cone $\mathbf{q} * \text{Ord}(\text{DP}_{\succ \mathbf{q}})$. The nonzero sign vectors, ordered as in (1), form the face poset of the boundary of the cross-polytope O , the face of ϵ being $\text{conv}\{\epsilon_i e_i : \epsilon_i \neq 0\}$; so ϵ induces a simplicial isomorphism of $\text{Ord}(\text{DP}_{\succ \mathbf{q}})$ with the barycentric subdivision of ∂O , whose cone is a piecewise linear homeomorphism $E_q : |\text{st } \mathbf{q}| \rightarrow O$. The open simplices of $\text{st } \mathbf{q}$ whose chain has top $\mathbf{p}$ are mapped onto $\mathcal{O}_{\epsilon(\mathbf{p})} \cap O$, and Ψ_X maps their union onto $C_X(\mathbf{p}) \cap \text{St}_q^X$, because open simplices of chains with top $\mathbf{p}$ lie in $\Psi_X^{-1}(C_X(\mathbf{p}))$. So $\Phi_q^X = E_q \circ \Psi_X^{-1}$ and $\Phi_q^Y = E_q \circ \Psi_Y^{-1}$ have the stated properties, and $\Phi_q^Y \circ \mathcal{L} = \Phi_q^X$. Two homeomorphisms with the property differ by a homeomorphism of O preserving every $\mathcal{O}_\epsilon \cap O$, which is isotopic to the identity through such homeomorphisms (Appendix C.3).

(3) The cells of $\mathcal{W}_q$ are those with $\tau = [u, u']$, that is, $\epsilon_3 = 0$. By Proposition 4.7(5), $\tilde{R}_u^* \cap \text{St}_q^Y$ is the union of the cells $C_Y(u, \omega)$, that is, $\epsilon_3 = -$; $\text{int } F_a^* \cap \text{St}_q^Y$ is the union of the cells $C_Y(\tau, a)$, that is, $(\epsilon_1, \epsilon_2) = (-, -)$; and $F_P^* \cap \text{St}_q^Y$ is the union of the cells $C_Y(\tau, P)$, that is, $\epsilon_1 = \epsilon_2 = 0$. On B_X , $F_{uu'}$ and $\text{int } F_u$ are the unions of the cells with $\tau = [u, u']$, respectively $\tau = u$, by Proposition 4.7(5). $\square$

The arcs at the node map under Φ_q^Y to the four half-axes in $\{y_3 = 0\}$: the arc over ab to $\{y_1 = y_3 = 0, y_2 < 0\}$, over dc to $y_2 > 0$, over da to $y_1 < 0$ and over bc to $y_1 > 0$. The four *standard halves* at q are the sets $(\Phi_q^Y)^{-1}(\{y_1 = 0, \pm y_3 \geq 0\})$ and $(\Phi_q^Y)^{-1}(\{y_2 = 0, \pm y_3 \geq 0\})$; by (1) and (3), $(\Phi_q^Y)^{-1}(\{y_1 = 0, y_3 \geq 0\})$ is the union of the closed cells with sign vectors $(0, *, +)$ and $(0, *, 0)$, that is, the closure of $C_Y(u', ab) \cup C_Y(u', P) \cup C_Y(u', cd)$, and similarly for the other three. In the node model of Section 7, the tropical chart T_q of Lemma 7.20(4) has the same quadrant and half-space description. That lemma compares the chart with scaled logarithmic coordinates on the hypersurface up to $O(1/L)$; the cell identification here concerns the tropical bases.

Let $\mathbf{p}_e = ([u, u'], [n, n'])$ be a dual cell pair of edges, so that $C_Y(\mathbf{p}_e)$ is an arc of Γ_Y (Proposition 4.10(2)). To the faces of $[n, n']$ assign $\epsilon_1(n) = -$, $\epsilon_1(n') = +$, $\epsilon_1([n, n']) = 0$, and to the faces of $[u, u']$ assign $\epsilon_2(u) = -$, $\epsilon_2(u') = +$, $\epsilon_2([u, u']) = 0$. For $\epsilon \in \{-, 0, +\}^2$ let $\mathcal{O}_\epsilon^{(2)} = \{(x_1, x_2) : \text{sign } x_i = \epsilon_i\}$. In $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) let $V = \text{conv}((\diamond \times \{0\}) \cup \{\pm e_3\})$, where $\diamond = \{|x_1| + |x_2| \leq 1\}$, and $V^\circ = V \setminus \{\pm e_3\}$. Put $\text{St}_e^X = \Psi_X(\text{st } \mathbf{p}_e)$ and $\text{St}_e^Y = \Psi_Y(\text{st } \mathbf{p}_e)$, with $\text{st } \mathbf{p}_e$ the closed star of the vertex $\mathbf{p}_e$ in $\text{Ord}(\text{DP})$; then $\mathcal{L}(\text{St}_e^X) = \text{St}_e^Y$.

Lemma 6.7 (the neighbourhood of an arc). (1) *The dual cell pairs $\mathbf{p} \succeq \mathbf{p}_e$ are the nine pairs (τ, ω) with $\tau \subseteq [u, u']$ and $\omega \subseteq [n, n']$. The map $\epsilon(\tau, \omega) = (\epsilon_1(\omega), \epsilon_2(\tau))$ is an isomorphism of this poset onto the face poset of the coordinate fan of $\mathbb{R}^2$, and $\dim \mathbf{p} - 1$ is the number of nonzero entries of $\epsilon(\mathbf{p})$. There are exactly two dual cell pairs $z_\pm \prec \mathbf{p}_e$; they have dimension zero, and $C_Y(z_\pm)$ are the end points of the arc.* (2) *There are piecewise linear homeomorphisms $\Phi_e^X : \text{St}_e^X \rightarrow V$ and $\Phi_e^Y : \text{St}_e^Y \rightarrow V$ with $\Phi_e^Y \circ \mathcal{L} = \Phi_e^X$, mapping $C_Y(z_\pm)$ to $\pm e_3$, the arc $C_Y(\mathbf{p}_e)$ onto the open segment $\{x_1 = x_2 = 0, |x_3| < 1\}$, and $C_Y(\mathbf{p}) \cap \text{St}_e^Y$ onto $\{x \in V^\circ : (x_1, x_2) \in \mathcal{O}_{\epsilon(\mathbf{p})}^{(2)}\}$ for every $\mathbf{p} \succ \mathbf{p}_e$; likewise on B_X . The set St_e^Y is a neighbourhood of the open arc.*

- (3) Consequently Φ_e^Y maps $\text{int } F_n^* \cap \text{St}_e^Y$ and $\text{int } F_{n'}^* \cap \text{St}_e^Y$ onto $\{x_1 < 0\} \cap V^\circ$ and $\{x_1 > 0\} \cap V^\circ$, $\tilde{R}_u^* \cap \text{St}_e^Y$ and $\tilde{R}_{u'}^* \cap \text{St}_e^Y$ onto $\{x_2 < 0\} \cap V^\circ$ and $\{x_2 > 0\} \cap V^\circ$, the two-cells $C_Y(u, [n, n'])$ and $C_Y(u', [n, n'])$ onto $\{x_1 = 0, x_2 < 0\} \cap V^\circ$ and $\{x_1 = 0, x_2 > 0\} \cap V^\circ$, and $C_Y([u, u'], n)$, $C_Y([u, u'], n')$ onto $\{x_2 = 0, x_1 < 0\} \cap V^\circ$ and $\{x_2 = 0, x_1 > 0\} \cap V^\circ$.
- (4) Suppose that an end of the arc is a node q , so that $[n, n']$ is an edge of the node parallelogram P , labelled so that $n' - n \in \{f_1, f_2\}$. For $\mathbf{p} \succeq \mathbf{p}_e$ the sign vector of $\mathbf{p}$ in Lemma 6.6 is obtained from $\epsilon(\mathbf{p}) = (\epsilon'_1, \epsilon'_2)$ by inserting the constant sign of the arc: it is $(\epsilon'_1, -, \epsilon'_2)$, $(\epsilon'_1, +, \epsilon'_2)$, $(-, \epsilon'_1, \epsilon'_2)$, $(+, \epsilon'_1, \epsilon'_2)$ for the arcs over ab , dc , ad , bc . In particular, on $\text{St}_e^Y \cap \text{St}_q^Y$ the standard halves containing the arc, on the sides $y_3 \geq 0$ and $y_3 \leq 0$, are the closures in $\text{St}_e^Y \cap \text{St}_q^Y$ of $C_Y(u', [n, n'])$ and $C_Y(u, [n, n'])$, which Φ_e^Y maps to $\{x_1 = 0, x_2 \geq 0\}$ and $\{x_1 = 0, x_2 \leq 0\}$.

Proof. (1) The pairs $\succeq \mathbf{p}_e$ are the (τ, ω) with $\tau \subseteq [u, u']$ and $\omega \subseteq [n, n']$, all dual, being contained in a dual cell pair. The faces of a segment are its two end points and itself, and in each factor inclusion corresponds to $\epsilon_i \in \{0, \epsilon'_i\}$ with 0 for the larger face; $\dim \mathbf{p} = 3 - \dim \tau - \dim \omega$ is one more than the number of end points among τ, ω . By Proposition 4.7(2), $\overline{C_Y(\mathbf{p}_e)}$ is a one-ball whose boundary is the union of the nonempty cells $C_Y(z)$, $z \prec \mathbf{p}_e$; these have $d_z = 0$, so they are points, and there are exactly two of them.

(2) This is a join construction like that of Lemma 6.6(2); the details are in Appendix C.3.

(3) By Proposition 4.7(5), $\text{int } F_n^* = \bigcup_\tau C_Y(\tau, n)$ and $\tilde{R}_u^* = \bigcup_\omega C_Y(u, \omega)$; within St_e^Y these are the cells with $\epsilon_1 = -$, respectively $\epsilon_2 = -$. The other statements are (2) for the individual cells.

(4) The sign vector of Lemma 6.6 is $(\epsilon_1(\omega), \epsilon_2(\omega), \epsilon_3(\tau))$ for the faces ω of P . For $\omega \subseteq [a, b]$ the second entry is $-$ and the first is $\epsilon_1(\omega)$ of the present lemma, with $(n, n') = (a, b)$; the other three arcs are the same, with $(n, n') = (d, c)$, (a, d) , (b, c) . The standard half $\{y_1 = 0, y_3 \geq 0\}$ meets $\text{St}_e^Y \cap \text{St}_q^Y$, near the arc over ab , in the closure of $\mathcal{O}_{(0, -, +)}$, the cell of $(u', [a, b])$, and similarly in the other cases; now apply (3). $\square$

Lemmas 6.6 and 6.7 are statements about the cells $C_Y(\mathbf{p})$ and hold for their images under any homeomorphism Υ of B_Y : with $\Phi_q^\Upsilon := \Phi_q^Y \circ \Upsilon^{-1}$ and $\Phi_e^\Upsilon := \Phi_e^Y \circ \Upsilon^{-1}$ they hold for the cells $\Upsilon(C_Y(\mathbf{p}))$. In Section 8 they are used in this form for the model complex.

6.3. Transfer of relative tropical 2-cycles. The lift of Section 8 is built over B_Y , after the discriminant of B_Y has been moved by a piecewise linear homeomorphism Υ^M of B_Y to the positions over which the local models of the mirror sit (Lemma 8.5). We therefore define the transfer for an arbitrary piecewise linear homeomorphism Υ of B_Y . Of Lemma 8.5 only the fact that Υ^M is such a homeomorphism is used here; the cells, the discriminant and the local system of the model complex are then the transported ones of Definition 6.8.

Definition 6.8 (the transfer). Let Υ be a piecewise linear homeomorphism of B_Y . Put $C^\Upsilon(\mathbf{p}) := \Upsilon(C_Y(\mathbf{p}))$ for $\mathbf{p} \in \text{DP}$, $\Gamma^\Upsilon := \Upsilon(\Gamma_Y)$ and $\Lambda^\Upsilon := \Upsilon_* \Lambda_Y$, and let

$$\mathfrak{h} := \Upsilon \circ \mathcal{L} \circ g_C : B'_\sigma \longrightarrow B_Y,$$

with g_C of Section 6.1 and the label-preserving cellular homeomorphism $\mathcal{L} : B_X(\mathcal{F}_C) \rightarrow B_Y$ fixed in Section 6.2. Let

$$\kappa_{\mathfrak{h}} : \Lambda \longrightarrow \mathfrak{h}^*(\Lambda^\Upsilon)^\vee \quad \text{on } B'_\sigma \setminus \Gamma$$

be the composite of $\kappa_C : \Lambda \rightarrow g_C^* \Lambda_X^{\mathcal{F}_C}$ (Proposition 6.3(3)), of the isomorphism $\Lambda_X^{\mathcal{F}_C} \rightarrow (\mathcal{L}^* \Lambda_Y)^\vee$, $x \mapsto \langle \cdot, x \rangle_{\mathcal{L}}$, given by the flat perfect pairing of Proposition 4.11, and of the

transport by Υ . For an arc $A = C^\Upsilon([u, u'], [n, n'])$ of Γ^Υ put $d_A^* := \langle \cdot, n' - n \rangle$, a flat section of $(\Lambda^\Upsilon)^\vee$ over the complement of Γ^Υ in a neighbourhood of the closed arc $\bar{A}$ (Lemma 6.2(3), Proposition 6.9(2)). For a triangulation $\mathfrak{K}^*$ of B_Y in which Γ^Υ is full, let $F^*(s)$ be the group of flat sections of $(\Lambda^\Upsilon)^\vee$ over $V \setminus \Gamma^\Upsilon$, V a small neighbourhood of $\bar{s}$, let $\mathcal{D}^*(E) := \mathbb{Z}d_A^*$ for an edge $E \subseteq \bar{A}$ of Γ^Υ , and let $Z(\mathfrak{K}^*; F^*, \mathcal{D}^*)$ be the group of chains $z^* \in C_2(\mathfrak{K}^*; F^*)$ with $\partial z^* \in C_1(\Gamma^\Upsilon; \mathcal{D}^*)$, as in Definition 2.4.

For a triangulation $\mathfrak{K}$ of B'_σ in which Γ is full, choose a subdivision $\mathfrak{K}_1^0$ of $\mathfrak{K}$ on whose simplices $\mathfrak{h}$ is linear and a derived subdivision $\mathfrak{K}_1$ of $\mathfrak{K}_1^0$. The *transfer* of $z \in Z(\mathfrak{K})$ is $\mathfrak{h}_* z := \mathfrak{h}_\#(\text{sd } z)$, the transport of Lemma 6.1 for the pair $(\mathfrak{h}, \kappa_\mathfrak{h})$.

The group $F^*(s)$ is the analogue on B_Y of the coefficient group $F(s)$ of Definition 2.4, for the dual local system: the coefficients of a transferred cycle are integral cotangent vectors of B_Y , invariant under the local monodromy. In Section 8, $\Upsilon = \Upsilon^M$, and we write $C^M(\mathfrak{p})$, Γ^M and Λ^M for $C^\Upsilon(\mathfrak{p})$, Γ^Υ and Λ^Υ . For $x \in \Gamma^\Upsilon$ let $\text{Inv}^*(x)$ be the group of flat sections of $(\Lambda^\Upsilon)^\vee$ over $V \setminus \Gamma^\Upsilon$ for a small ball V about x , and for an arc A let $\text{Inv}^*(A)$ be its value at the points of A .

- Proposition 6.9** (the transferred coefficients). (1) *The homeomorphism $\mathfrak{h}$ maps every cell $C(\mathfrak{p})$ of B'_σ with $\mathfrak{p} = (\tau, \omega)$, $\dim \tau \geq 1$, onto $C^\Upsilon(\mathfrak{p})$. In particular $\mathfrak{h}(\Gamma) = \Gamma^\Upsilon$; $\mathfrak{h}$ maps the arc of $([u, u'], [n, n'])$ onto the arc of the same dual cell pair, the vertex p_q of the discriminant onto the zero-cell $C^\Upsilon([u, u'], P_q)$, the negative vertices of Γ (edge, triangle) onto the vertices of Γ^Υ of positive type, and the positive vertices (triangle, edge) onto the vertices of negative type. At a point of an open facet $\text{int } F_u$, where $\Lambda = u^\perp \cap N$ and $\Lambda^\Upsilon = M/\mathbb{Z}u$ at the image point, $\kappa_\mathfrak{h}(x) = \langle \cdot, x \rangle$.*
- (2) *On the arc $A = C^\Upsilon([u, u'], [n, n'])$: $u' - u$ is a flat section of Λ^Υ near A ; $d_A^* = \langle \cdot, n' - n \rangle$ is a flat section of $(\Lambda^\Upsilon)^\vee$ over the complement of Γ^Υ in a neighbourhood V of the closed arc $\bar{A}$, and $\kappa_\mathfrak{h}(n' - n) = d_A^*$ over $\mathfrak{h}^{-1}(V) \setminus \Gamma$ for a suitable such V ;*

$$\text{Inv}^*(A) = \{X \in (\Lambda^\Upsilon)^\vee : X(u' - u) = 0\},$$

of rank two, with d_A^* primitive in it; and the sublattice of Λ^Υ fixed by the monodromy around A is $\{n, n'\}^\perp \cap M = \ker d_A^*$, of rank two.

- (3) *At the vertices of Γ^Υ :*
- *at a node $C^\Upsilon([u, u'], P)$, $P = abcd$: $\text{Inv}^* = \{X : X(u' - u) = 0\} = \text{Inv}^*(A)$ for each of the four arcs A at it, with basis $\langle \cdot, b - a \rangle$, $\langle \cdot, c - b \rangle$;*
 - *at a vertex $C^\Upsilon([u, u'], \{n_a, n_b, n_c\})$ of positive type: $\text{Inv}^* = \{X : X(u' - u) = 0\}$, of rank two, containing d_A^* for the three arcs A at it;*
 - *at a vertex $C^\Upsilon(\{u_1, u_2, u_3\}, [n, n'])$ of negative type: $\text{Inv}^* = \mathbb{Z}\langle \cdot, n' - n \rangle$, the common value of d_A^* on the three arcs at it, up to sign.*

In each case $\kappa_\mathfrak{h}$ maps the stalk of $\iota_* \Lambda$ at the corresponding point of Γ onto this group.

- (4) *For a 1-cycle $\sum_A c_A d_A^*[A]$ of $C_*(\Gamma^\Upsilon; \mathcal{D}^*)$, with $d_A^* = \langle \cdot, n'_A - n_A \rangle$, define its canonical lift $\sum_A c_A (e_{n'_A} - e_{n_A})[A] \in C_1(\Gamma^\Upsilon; \mathbb{Z}^R)$ and its node coefficients ρ_q^* by the rule of Definition 2.6 at the zero-cell $C^\Upsilon([u, u'], P_q)$. Equivalently, with the coefficients oriented away from the node and written $c_{ab}\langle \cdot, b - a \rangle$ and $c_{bc}\langle \cdot, c - b \rangle$ on the arcs over ab and bc , $\rho_q^* = c_{ab} - c_{bc}$.*

Proof. (1) The first statement combines Proposition 6.3(3) for g_C , Proposition 4.7(4) for $\mathcal{L}$ and the definition of $C^\Upsilon(\mathfrak{p})$; the vertex types are those of Proposition 4.10(3), and Υ carries the monodromies of Λ_Y to those of Λ^Υ . The map g_C preserves $\text{int } F_u$, where κ_C is the identity

of $u^\perp \cap N$, and $\mathcal{L}(\text{int } F_u) = \tilde{R}_u^*$, where $\Lambda_Y = M/\mathbb{Z}u$ and the flat pairing is $\langle \bar{\zeta}, x \rangle_{\mathcal{L}} = \langle \zeta, x \rangle$ (proof of Proposition 4.11).

(2) Transport by Υ does not change monodromies, so it suffices to argue on B_Y near the arc $C_Y([u, u'], [n, n'])$. In the chart of $C_Y(u, n)$, $\Lambda_Y = n^\perp \cap M$, which contains $u' - u$ because $\langle u, n \rangle = \langle u', n \rangle = -1$. By Proposition 4.10(4) the monodromy is $T(\zeta) = \zeta + \langle \zeta, n' - n \rangle (u' - u)$. So $T(u' - u) = u' - u$, as $\langle u' - u, n' - n \rangle = 0$; the fixed sublattice is $\{\zeta \in n^\perp \cap M : \langle \zeta, n' - n \rangle = 0\} = \{n, n'\}^\perp \cap M$, of rank two because n, n' are linearly independent; and a covector X is invariant under the dual action if and only if $X(T\zeta) = X(\zeta)$ for all ζ , that is, $\langle \zeta, n' - n \rangle X(u' - u) = 0$ for all $\zeta \in n^\perp \cap M$, that is, $X(u' - u) = 0$, since $\langle \cdot, n' - n \rangle$ does not vanish on $n^\perp \cap M$. The covector $\langle \cdot, n' - n \rangle$ vanishes on $u' - u$, hence is flat, and it is primitive in $N/\mathbb{Z}n$ (proof of Proposition 4.10(3)). Near the closed arc: $\langle \cdot, n' - n \rangle$ is a flat section of $\Lambda_Y^\vee$ over the complement of Γ_Y in a neighbourhood of the closure of the arc $C_Y([u, u'], [n, n'])$ of $B_Y = B_Y(\mathcal{F}_C)$ (Lemma 6.2(3)), and transport by Υ carries it to such a section of $(\Lambda^\Upsilon)^\vee$ near $\bar{A}$; and $n' - n$ is a flat section of Λ over the complement of Γ in a neighbourhood of the closed arc of $B'_\sigma = B_X(\Sigma')$ with the same dual cell pair (Lemma 6.2(2)), which $\kappa_{\mathfrak{h}}$ carries to a flat section near $\bar{A}$, as $\mathfrak{h}$ maps that closed arc onto $\bar{A}$ by (1). By (1), $\kappa_{\mathfrak{h}}(n' - n) = \langle \cdot, n' - n \rangle$ on the side of F_u , so the two sections agree on a neighbourhood of $\bar{A}$ (the last paragraph after Lemma 6.2).

(3) At a node or at a vertex of positive type, all arcs share the vector $u' - u$ (Proposition 4.10(3)) and have monodromies $\zeta \mapsto \zeta + \langle \zeta, t_A \rangle (u' - u)$ with t_A the edge vectors of P , respectively of the triangle; as in (2) the invariant covectors are those vanishing on $u' - u$. At a node, in the chart of $C_Y(u, a)$, $(\Lambda_Y)^\vee = N/\mathbb{Z}a$ and the covectors vanishing on $u' - u$ form the image of $(u' - u)^\perp \cap N$. The plane of G is at lattice distance one from the origin and the triangle abd is unimodular in it, so $\{a, b - a, d - a\}$ is a basis of $(u' - u)^\perp \cap N$, and the classes of $b - a$ and $d - a = c - b$ form a basis of the invariants. At a vertex of negative type the three arcs share the covector $\langle \cdot, n' - n \rangle$ and have monodromies $\zeta \mapsto \zeta + \langle \zeta, n' - n \rangle (u_j - u_i)$; a covector is invariant if and only if it vanishes on the $u_j - u_i$, which span a saturated rank-two sublattice of $n^\perp \cap M$ (proof of Proposition 4.10(3)); its annihilator in $N/\mathbb{Z}n$ is saturated of rank one and contains the primitive class of $n' - n$. The last statement holds because $\kappa_{\mathfrak{h}}$ is an isomorphism of local systems and $\mathfrak{h}$ maps small balls about a point of Γ onto neighbourhoods of its image.

(4) The equivalence of the two descriptions is the computation of Section 5.3 for Definition 2.6, which uses only the labels n_A, n'_A of the arcs at the node. $\square$

Proposition 6.10 (transfer of relative tropical 2-cycles). *Let $\mathfrak{K}$ be a triangulation of B'_σ in which Γ is full, and $\mathfrak{K}_1$ as in Definition 6.8. Then $\mathfrak{h}(\mathfrak{K}_1)$ is a triangulation of B_Y in which Γ^Υ is full, and*

$$\mathfrak{h}_*: Z(\mathfrak{K}) \longrightarrow Z(\mathfrak{h}(\mathfrak{K}_1); F^*, \mathcal{D}^*)$$

is an injective homomorphism; if $\mathfrak{h}$ is linear on the simplices of $\mathfrak{K}$ and $\mathfrak{K}_1 = \mathfrak{K}$, it is an isomorphism. If the boundary network of z is $\sum_E c_E d_E[E]$, the boundary network of $\mathfrak{h}_*z$ is $\sum_E c_E \sum_{E' \subseteq E} d_{\mathfrak{h}(A(E))}^* [\mathfrak{h}(E')]$, with the same integers c_E , and

$$\rho^*(\mathfrak{h}_*z) = \rho(z).$$

Proof. By Proposition 6.9(1), $\mathfrak{h}$ is a piecewise linear homeomorphism with $\mathfrak{h}(\Gamma) = \Gamma^\Upsilon$ mapping arcs onto arcs, and by Proposition 6.9(2), $\kappa_{\mathfrak{h}}(d_E) = \kappa_{\mathfrak{h}}(n' - n) = d_{\mathfrak{h}(A)}^*$ for a leg E with lower support $\{n, n'\}$ on the arc A of $([u, u'], [n, n'])$ (Proposition 4.10(2)), as flat sections over the complement of the discriminant in neighbourhoods of the closed arcs, on which d_E and $d_{\mathfrak{h}(A)}^*$

are defined by Lemma 6.2(2),(3). So $(\mathfrak{h}, \kappa_{\mathfrak{h}})$ satisfies the hypotheses of Lemma 6.1, which gives all but the last statement. The node coefficients of z are computed from the integers c_E on the legs at p_q and the lower supports of these legs; those of $\mathfrak{h}_*z$ from the same integers on the arcs at $\mathfrak{h}(p_q) = C^{\mathcal{T}}([u, u'], P_q)$ and the same labels (Proposition 6.9(1),(4)). Subdivision replaces each leg by edges with the same coefficient, which does not change the boundary of the canonical lift at the node. $\square$

Different choices of $\mathfrak{K}_1^0$ and $\mathfrak{K}_1$ give transfers with the same boundary networks, up to subdivision, and the same node coefficients; no orientation of B'_σ or of B_Y enters the definition. By Proposition 6.9(3) the transferred coefficient system has, at the vertices of $\Gamma^{\mathcal{T}}$, the stalks of $\iota_*\Lambda$ at the corresponding vertices of Γ , with the types exchanged: the rank-one stalks, which sit at the positive vertices of B'_σ , sit on B_Y at the vertices of negative type, and the rank-two stalks of the negative vertices of B'_σ at the vertices of positive type.

Remark 6.11 (tropical 2-cycles and the lift of Castaño-Bernard and Matessi). Let (S, j, v) be a strict tropical 2-cycle (Definition 2.5), triangulated, so that it defines a relative tropical 2-cycle z with coefficient v on the triangles of its smooth part (Section 5.2). The coefficients of $\mathfrak{h}_*z$ are the covectors $\kappa_{\mathfrak{h}}(v)$, and $\ker \kappa_{\mathfrak{h}}(v) \subseteq \Lambda^{\mathcal{T}}$ is a rank-two subbundle along the image of the smooth part. It is the bundle $\ker v \subseteq T^*B$ of [CBM14, Def. 7.2], written in the affine structure of B_Y . In [CBM14, §7] the lift $\tilde{S}$ of S lies in the torus fibration over B'_σ with fibres $T_b^*B/\Lambda_b^{\vee}$, with fibre $(\ker v)_b/(\ker v)_b \cap \Lambda_b^{\vee}$ over a smooth point; by Proposition 4.11 this is the torus fibration over B_Y with fibres $\Lambda_Y \otimes U(1)$, and the fibre of the lift over the image point is the two-torus $\ker \kappa_{\mathfrak{h}}(v) \otimes U(1)$. This is the fibre of the chain of Theorem D over the regular part.

Remark 6.12 (why a transfer map is needed). A relative tropical 2-cycle on B'_σ is not read directly as an object on B_Y , for two reasons. First, the polyhedral decompositions need not correspond. An inclusion-reversing bijection $c \mapsto \check{c}$ from the cells of $\mathcal{P}_X$ onto those of $\mathcal{P}_Y$ with $\tau_Y(\check{c}) = \tau(c)$ and $\omega_Y(\check{c}) = \omega(c)$ would map the legs of the arc $C_X(\tau, \omega)$, the chains $e \subsetneq f$ with $\omega(e) = \omega$ and $\tau(f) = \tau$, bijectively onto the legs of $C_Y(\tau, \omega)$, the chains $f' \subsetneq e'$ with $\tau_Y(f') = \tau$ and $\omega_Y(e') = \omega$, so corresponding arcs would consist of the same number of legs. These numbers are determined by mixed subdivisions of different faces with different heights, of F_τ with h on $\tau^{\vee}$ for $C_X(\tau, \omega)$ and of F_ω^* with $\check{h}$ on $\omega^{\vee}$ for $C_Y(\tau, \omega)$; the complex of dual cell pairs needs no such coincidence. Second, the transfer of a tropical 2-cycle is not a tropical 2-cycle of B_Y in the sense of [CBM14, Def. 7.2]: its coefficients are covectors; its boundary vertices, which lie at negative vertices of B'_σ , lie at vertices of positive type of B_Y ; and at a node met by its boundary, which is of positive type on B_Y by Proposition 4.10(3), its boundary continues along one line of the discriminant, whereas the positive-node clause of [CBM14, Def. 7.2] requires the boundary to turn from one line to the other at a positive node [CBM14, Fig. 14(3)]. The transfer $\mathfrak{h}_*$ records the data from which the lift is built, in the affine structure of B_Y , without asking them to be a tropical object there.

7. THE LOCALIZED MIRROR AND MORSE CHARTS AT THE NODES

This section prepares the analytic side of the mirror lift. We fix the members Y_t of Theorem A and study them near the maximally degenerate limit by comparing monomials in units of $L = \log(1/t)$. Three results are proved. First, for t small and every coefficient vector in a window around $\mathcal{Z}$, the member Y_t is smooth and isotopic in $X_{\mathcal{T}}$ to a smooth compact submanifold W^* , the *localized mirror*, on which every stratum of the tropical picture near the discriminant has an exact algebraic model (Lemma 7.11 and Proposition 7.12). Second,

near the node point (Definition 7.13) of each $P \in Q$ the defining function, in the trivialisation of the anticanonical bundle by the monomial $\gamma_a x^a$, has exactly one critical point; it is nondegenerate, its critical value differs by $O(t^{\theta_0/2})$ from $-\kappa_P = 1 - \mu_P^{-1}$, and on a polydisc around it W^* is exactly the conifold smoothing $\{xy + z_1 z_2 = \kappa_P\}$ (Lemmas 7.14 and 7.15). This defines the vanishing sphere δ_q . Third, a sphere isotopic to δ_q is the boundary, over a Lefschetz thimble, of a four-chain Π_q lying over a standard half-plane through the node of the base B_Y of Section 4; we orient δ_q so that this boundary is $\rho_q \delta_q$ for every standard half at q (defined in Section 7.5), with either orientation and either sign of its coefficient (Definition 7.23 and Lemma 7.24). Section 8 builds the chains of Theorem D from these node pieces and from pieces along the legs of the discriminant.

The localized mirror is modelled on the tropical localizations of Mikhalkin [Mik04] and Yamamoto [Yam16, Def. 3.1]: each monomial is multiplied by a cut-off that vanishes where the monomial is far from maximal, so that near each stratum only the monomials of one cell of the tropical picture survive. Their results assume a unimodular subdivision; the mirror subdivision contains the node parallelograms and is not unimodular, so their results do not apply here, and the smoothness of the interpolating family and the isotopy are proved directly. The estimates are comparisons of tropical deficits. The proofs of the one-cell lemma, of Lemma 7.11 and of the lemmas on the Morse chart are in Appendix D.

Standing assumptions. In this section and in Sections 8 and 9 the polytope Δ is admissible, the profile σ satisfies (Proj), the pair $(\mathcal{T}, \check{h})$ is fixed as in Section 2.3, and $\mathcal{F}_C$ is the pulling refinement of Σ' , with its height h_C (Section 3.5). We use three of the properties proved in Proposition 3.26: h_C induces $\mathcal{F}_C$, and every maximal cell of the induced subdivision of $(\partial\Delta \cap N) \cup \{0\}$ contains 0 (item (1)); $\mathcal{F}_C$ agrees with Σ' on the two-skeleton, so that its cells in a two-face of Δ are the cells of σ (Definition 2.3), each without lattice points other than its vertices (item (2)); and the *height condition* of item (4), restated in coordinates in Definition 7.6. The subdivision Σ' itself need not satisfy the height condition, which is why the mirror family is built on $\mathcal{F}_C$.

Constants. The constants of Sections 7 and 8 depend on data chosen in a fixed order. First the *configuration* is fixed: Δ , σ , $\mathcal{T}$ and $\check{h}$, $\mathcal{F}_C$ and its height h_C , the lattice vectors w_u of Section 7.1, and the integer k_0 . A constant is said to *depend only on the configuration* if it depends on these data alone. Then a compact set $\mathcal{C}_0 \subset \mathcal{Z}$, a radius r and a size η_0 of the perturbation of the node binomials are chosen, then a threshold t_0 , and finally the coefficient vector c and $t < t_0$. The principal constants are these.

constant	role	fixed in
<i>Configuration and lattice constants</i>		
h_C	strictly convex height inducing $\mathcal{F}_C$	Section 3.5
k_0	multiple of φ in the height H	(7.3), Lemma 7.7(4)
θ_0	threshold of the one-cell lemma	Lemma 7.5
$\mathfrak{g}_P, \mathfrak{g}_\omega$	apex gap of a node, vertex gap of a triangle	Definition 7.6
$\mathcal{E}_P, \mathcal{E}_{\max}, e_{\max}$	exponent constants	Definition 7.6
θ_e	maximum on the face E_e of the smallest deficit of a monomial other than $0, n, n'$	Definition 7.9
<i>Coefficient bounds and Morse radii</i>		
$C_\eta, C_{\mathbf{R}}$	ratio of the largest to the smallest $ \eta_P $; bound of $\ \xi\ $ by $\max_P \eta_P $	Lemma 7.2; after Definition 7.3
r_1, r	radii of the Morse polydisc; $r_1 = \min(1/4, (20/C_{\mathbf{R}})^{1/2})$	Lemma 7.14, Definition 7.3
η_0	size of the perturbation of the node binomials, $\eta_0 \leq r^2/20$	Definition 7.3
<i>Localization and model pieces</i>		
$\theta_1, \theta_2, \theta_3$	cut-off constant; sizes in the torus directions and of the first segments of the model arcs	Definition 7.9, Section 7.2
θ_P	reach of the node neighbourhood, $\mathfrak{g}_P/2 - \theta_1$	after (7.3)
θ_4	size of the germs at the model points	Definition 8.1(2)
$\varsigma_{\max}, \varsigma_0$	depth of the pieces	Lemma 8.14; Section 8
<i>Small-parameter threshold</i>		
t_0	threshold on t , below the threshold t_1 of Lemma 7.14	Lemmas 7.11, 7.14, 7.15

These constants exist for every admissible Δ and every profile satisfying (Proj), with the conventions of Remark 9.12 when there are no nodes. The height h_C exists by the regularity of $\mathcal{F}_C$. The constant θ_0 is the minimum of finitely many positive values of linear programs (Lemma 7.5). The gaps and the exponent constants are read off from the lattice data, the vectors w_u and the heights, and k_0 is any integer above the explicit bound of Lemma 7.7(4), which involves only lattice data and h_C . The constants C_η and $C_{\mathbf{R}}$ come from the linear algebra of the vectors r_P (Lemma 7.2); $\theta_e, \theta_1, \theta_2, \theta_3, \theta_P$ and r_1 are explicit; θ_4 and $\varsigma_{\max}$ are bounds over the finitely many model data of Section 8; and t_0 is the minimum of finitely many thresholds, each given by an explicit inequality in Appendix D.

7.1. The mirror member.

Toric conventions. The vertices $u \in \partial\Delta^\circ \cap M$ of $\mathcal{T}$ span the rays of a complete unimodular fan in $M_{\mathbb{R}}$, and $X_{\mathcal{T}}$ is the corresponding smooth projective toric fourfold, with Cox coordinates Z_u [CLS11]. For $n \in \mathcal{A} = \Delta \cap N$ the Cox monomial

$$x^n := \prod_u Z_u^{e_u(n)}, \quad e_u(n) := \langle u, n \rangle + 1 \geq 0,$$

is a section of the anticanonical bundle, and $x^n/x^m = \chi^{n-m}$ is a character of the dense torus $T = M \otimes \mathbb{C}^*$. The face $u^\vee = \{n \in \Delta : \langle u, n \rangle = -1\}$ of Δ consists of the lattice points with $e_u(n) = 0$; it is a facet when u is a vertex of Δ° . The origin has $e_u(0) = 1$ for every u . On the chart $U_c \cong \mathbb{C}^4$ of a maximal cone $\mathfrak{c}$ the Cox coordinates $Z_u, u \in \mathfrak{c}$, are coordinates and the others are set to 1; a point lies on the orbit of a cone $\mathfrak{c}'$ exactly when the Z_u with $u \in \mathfrak{c}'$ vanish there. We write $\text{Log}: T \rightarrow M_{\mathbb{R}}$ for the map with $\langle \text{Log } x, n \rangle = \log |\chi^n(x)|$. Cells of $\mathcal{F}_C$ are denoted ω , as in Section 4.

For every edge $[u, u']$ of $\mathcal{T}$ that lies in a face of Δ° of dimension at most two we fix an order (u, u') of its end points and a lattice vector $w_u \in N$ with $\langle u, w_u \rangle = 1$ and $\langle u', w_u \rangle = 0$ (the fan over $\mathcal{T}$ is unimodular; for the dual edge of a node, compare Lemma 3.2).

Heights and the member. The height is $H = k_0\varphi + h_C$ of Section 2.4, with h_C as in Section 3.5: strictly convex on the fan over $\mathcal{F}_C$, inducing $\mathcal{F}_C$, with integral slopes and $h_C(0) = 0$; and φ , equal to 1 on $\partial\Delta$ and linear on the cones over the faces of Δ , has integral slopes since Δ is reflexive. So $H(0) = 0$, $H(n) = k_0 + h_C(n)$ for $n \in \partial\Delta \cap N$, and H is integral at lattice points; the integer $k_0 \geq 1$ is fixed below. A subset $I \subseteq \mathcal{A}$ is *cellular* if the lifted points $(n, H(n))$, $n \in I$, lie on a common lower face of their convex hull.

Lemma 7.1. *For $k_0 \geq 0$ the lower faces are the polytopes $\text{conv}(0, \omega)$ and ω , $\omega \in \mathcal{F}_C$, and their faces. A set $I \subseteq \mathcal{A}$ is cellular if and only if $I \subseteq \{0\} \cup \omega$ for some cell ω of $\mathcal{F}_C$.*

Proof. H is convex, with the cones over the cells of $\mathcal{F}_C$ as its maximal domains of linearity, since φ is convex and linear on the cones over the faces of Δ and h_C is strictly convex on the finer fan over $\mathcal{F}_C$. The lifted lattice points lie on the graph of H , which over Δ is the lower hull of their convex hull, because a convex function is at most any convex combination of its values. The domains of linearity of H on Δ are the $\text{conv}(0, \omega)$. $\square$

The integer k_0 is fixed subject to one lower bound, (7.3) below; by Lemma 7.7(4) this bound holds for every $k_0 \geq \max(2e_{\max}, 6, 1 - \min_{\partial\Delta \cap N} h_C)$. The *mirror member* is the member $Y_{t,c}$ of Section 2.4,

$$Y_t = \{F_t = 0\} \subset X_{\mathcal{T}}, \quad F_t = \sum_{n \in \mathcal{A}} \gamma_n x^n, \quad \gamma_n = c_n t^{H(n)}, \quad 0 < t < 1,$$

and we put $L := \log(1/t)$. The coefficient vector c is real with the *sign pattern* $c_0 > 0$ and $c_n < 0$ for $n \neq 0$.

Node binomials and the window. Let $P = abcd$ be a node parallelogram, labelled as in Section 2.2. Since H is linear on the cone over P and $a + c = b + d$, its node binomial ratio is $\mu_P = c_a c_c / (c_b c_d) = \gamma_a \gamma_c / (\gamma_b \gamma_d)$. The set $\mathcal{Z}$ of Section 2.4 is nonempty: it contains the vector with $c_0 = 1$ and $c_n = -1$ for $n \neq 0$. Fix a compact set $\mathcal{C}_0 \subset \mathcal{Z}$. As in Section 2.4, every real c with the sign pattern is uniquely $c = c^{(0)} e^\xi$ with $c^{(0)} \in \mathcal{C}_0$ and $\xi \in \text{span}_{\mathbb{R}}\{r_P\}$, and we put

$$\eta_P := \log \mu_P = r_P \cdot \xi.$$

Lemma 7.2. *Assume $Q \neq \emptyset$. The map $\xi \mapsto (\eta_P)_P$ has image $V = \{\eta : \sum_P k_P \eta_P = 0 \text{ for every relation } \sum_P k_P r_P = 0\}$, and no coordinate η_P vanishes identically on V . Consequently there are a direction $\bar{\xi} \in \text{span}\{r_P\}$, which we fix once for the matrix $\mathbf{R}$ with rows r_P , and a constant $C_\eta \geq 1$, depending only on $\mathbf{R}$ and $\bar{\xi}$, such that for every $\eta_0 > 0$ the multiple ξ of $\bar{\xi}$ with $\max_P |\eta_P| = \eta_0$ satisfies $\eta_0 / C_\eta \leq |\eta_P| \leq \eta_0$ for all P .*

Proof. The image is the column space of the matrix $\mathbf{R}$ with rows r_P , which is the annihilator of its left kernel. The coordinate η_P vanishes on it if and only if e_P lies in the left kernel, that is $r_P = 0$, which never holds. So the complement in the image of the hyperplanes $\{\eta_P = 0\}$ is dense; take $\bar{\xi}$ with $\mathbf{R}\bar{\xi}$ in it and $C_\eta = \max_P |r_P \cdot \bar{\xi}| / \min_P |r_P \cdot \bar{\xi}|$. $\square$

Definition 7.3 (the window). Fix $r \in (0, r_1]$, with r_1 as in Lemma 7.14, and $\eta_0 \in (0, r^2/20]$. A coefficient vector $c = c^{(0)} e^\xi$ with $c^{(0)} \in \mathcal{C}_0$ and $\xi \in \text{span}_{\mathbb{R}}\{r_P\}$ is *in the window* if $\eta_0 / C_\eta \leq |\eta_P| \leq \eta_0$ for every node. For $c^{(0)} \in \mathcal{C}_0$ the *complex neighbourhood* $U_{\mathbb{C}} = U_{\mathbb{C}}(c^{(0)})$ is the set of $c^{(0)} e^\xi$ with $\xi \in \mathbb{C}^A$, $\|\xi\| \leq 1$ and $|\eta_P(\xi)| \leq \eta_0$ for every node, and $U_{\mathbb{C}}^+$ is defined in the same

way with the bounds 2 and $2\eta_0$. Here $\|\xi\| := \max_n |\xi_n|$, and $\|\cdot\|_*$ denotes the dual norm, so that $\|r_P\|_* = 4$.

In the notation of Theorem A, $C = C_\eta$ and $\eta_1 = r_1^2/20$; for $\eta_0 \leq \eta_1$ we take $r = r_1$. Both depend only on the matrix $\mathbf{R}$ with rows r_P and on the direction $\bar{\xi}$ of Lemma 7.2. Every vector in the window lies in $U_{\mathbb{C}}$: for ξ in the span of the r_P one has $\|\xi\| \leq C_{\mathbf{R}} \max_P |\eta_P|$, with $C_{\mathbf{R}}$ depending only on $\mathbf{R}$ (the map $\xi \mapsto \mathbf{R}\xi$ is injective on the span of the rows), and $C_{\mathbf{R}}\eta_0 \leq 1$, because $\eta_0 \leq r^2/20 \leq r_1^2/20$ and $r_1^2 \leq 20/C_{\mathbf{R}}$ (Lemma 7.14). The union of the sets $U_{\mathbb{C}}^+(c^{(0)})$, $c^{(0)} \in \mathcal{C}_0$, is compact, so $|c_n|$ and $|c_n|^{-1}$ are bounded on it by a constant depending on $\mathcal{C}_0$. Constants called C , and the thresholds on t , depend on the configuration and on $\mathcal{C}_0$ through these bounds, and on r and η_0 where this is indicated; we do not repeat the dependence on $\mathcal{C}_0$. Lemmas 7.11(2), 7.14 and 7.15(2) are stated for $c \in U_{\mathbb{C}}^+$, as Section 9 requires (Lemma 9.1); their proofs use only these two-sided bounds on $|c_n|$, the bound $|\eta_P| \leq 2\eta_0$ and the standing choices $r \leq 1/4$, $\eta_0 \leq r^2/20$. No threshold is uniform on all of $\mathcal{Z}$: when $|\log |c_n^{(0)}||$ is comparable to $\theta_0 L$ the coefficients move the tropical picture, and the estimates below fail.

When every singular two-face of Δ is a unit square, $\mathcal{Z}$ is the set of real points, with the sign pattern, of the locus to which Batyrev and Kreuzer specialise the mirror family [BK10, §3], on which the face polynomial of every square is a product of two binomials. At a hexagon this is different.

Remark 7.4 (the face polynomial of a hexagon). Let G be a hexagon, in lattice coordinates of its plane with $o = 0$ and $w_0, \dots, w_5$ as in Section 2.2; write its face polynomial as $f_G = c_o + \sum_j c_{w_j} \chi^{w_j}$. The node parallelograms have their diagonal through o , so $\mu_P = c_o c_{w_{j+1}} / (c_{w_j} c_{w_{j+2}})$ for $P = \{o, w_j, w_{j+1}, w_{j+2}\}$. The analogues of the specialisation of [BK10] are the product loci below; we write $\mathbf{a}_i, \mathbf{b}_i, \mathbf{e}_i$ for the coefficients of the factors.

- (1) A_1 branch, $P_1 = \{o, w_0, w_1, w_2\}$, $P_2 = \{o, w_3, w_4, w_5\}$ (the other A_1 subdivisions are rotations). For $f_G = (\mathbf{a}_0 + \mathbf{a}_1 \chi^{w_0} + \mathbf{a}_2 \chi^{w_1})(\mathbf{b}_0 + \mathbf{b}_1 \chi^{w_3} + \mathbf{b}_2 \chi^{w_4})$ one finds $c_o = \sum_i \mathbf{a}_i \mathbf{b}_i$,

$$(c_{w_0}, \dots, c_{w_5}) = (\mathbf{a}_1 \mathbf{b}_0, \mathbf{a}_2 \mathbf{b}_0, \mathbf{a}_2 \mathbf{b}_1, \mathbf{a}_0 \mathbf{b}_1, \mathbf{a}_0 \mathbf{b}_2, \mathbf{a}_1 \mathbf{b}_2),$$

and $\mu_{P_1} = \mu_{P_2} = 1 + (\mathbf{a}_0 \mathbf{b}_0 + \mathbf{a}_2 \mathbf{b}_2) / (\mathbf{a}_1 \mathbf{b}_1)$. A real product has real factors after rescaling (a trinomial whose Newton polygon is a unimodular triangle is irreducible); the sign conditions $c_{w_j} < 0$ then force all $\mathbf{a}_i$ to have one sign and all $\mathbf{b}_i$ the other, so $c_o = \sum_i \mathbf{a}_i \mathbf{b}_i < 0$: no real point of the product locus has the sign pattern, and in particular none lies in $\mathcal{Z}$.

- (2) A_2 branch, parallelograms P_j , $j \in \{0, 2, 4\}$ (the odd subdivision is symmetric). For $f_G = (\mathbf{a}_0 + \mathbf{a}_1 \chi^{w_0})(\mathbf{b}_0 + \mathbf{b}_1 \chi^{w_2})(\mathbf{e}_0 + \mathbf{e}_1 \chi^{w_4})$, since $w_0 + w_2 + w_4 = 0$, $c_o = T_1 + T_2$ with $T_1 = \mathbf{a}_0 \mathbf{b}_0 \mathbf{e}_0$, $T_2 = \mathbf{a}_1 \mathbf{b}_1 \mathbf{e}_1$, and $\mu_{P_j} = 1 + T_2 / T_1 \neq 1$ for all three j (for the odd subdivision $1 + T_1 / T_2$). So the product locus misses $\mathcal{Z}$. Its real points do not have the sign pattern either: the factors are again real after rescaling, their Newton segments being distinct, and the conditions $c_{w_j} < 0$ force $T_1, T_2 < 0$, so $c_o < 0$. Its curve is three curves meeting pairwise transversally in one point each; the three points coincide exactly when $c_o = 0$.
- (3) For t small and c in the window, if the face polynomial $\sum_{n \in G} \gamma_n \chi^n$ of F_t is such a product and $|c_n|^{\pm 1}$ are bounded, then $|\mu_P - 1| \leq Ct$ for every node parallelogram P of G . On the A_1 branch, $(\mathbf{a}_0 \mathbf{b}_0) / (\mathbf{a}_1 \mathbf{b}_1) = \gamma_{w_1}^2 \gamma_{w_3} / (\gamma_{w_0} \gamma_{w_2}^2)$ and $(\mathbf{a}_2 \mathbf{b}_2) / (\mathbf{a}_1 \mathbf{b}_1) = \gamma_{w_1}^2 \gamma_{w_5} / (\gamma_{w_0}^2 \gamma_{w_2})$ are bounded multiples of t to the powers $H(w_1) + H(w_3) - H(w_2) - H(o)$ and $H(w_1) + H(w_5) - H(w_0) - H(o)$, using $H(w_0) + H(w_2) = H(o) + H(w_1)$

(linearity of H on the cone over P_1). As $w_1 + w_3 = w_2 + o$, the midpoint of $[w_1, w_3]$ lies on the edge $[o, w_2]$ of a cell, while w_1, w_3 lie in no common cell; so strict convexity makes the first exponent a positive integer, and likewise the second, with $w_1 + w_5 = w_0 + o$. On the A_2 branch $T_2/T_1 = \gamma_{w_1}\gamma_{w_3}\gamma_{w_5}/(\gamma_{w_0}\gamma_{w_2}\gamma_{w_4})$ is a bounded multiple of $t^{H(w_0)+H(w_2)+H(w_4)-3H(o)}$, by linearity of H on the cones over the P_j ; as $w_0 + w_2 + w_4 = 3o$ and H is convex but not affine on $\text{conv}(w_0, w_2, w_4)$, which meets the interiors of the three P_j , the exponent is again a positive integer.

So the product loci lie within $O(t)$ of $\mathcal{Z}$ in the coordinates η_P , outside the window for t small. The node condition used below involves only the four monomials of P and the origin (Lemma 7.14): it is $\varepsilon_P = 0$, close to $\mu_P = 1$, and not a factorisation of f_G .

Deficits and the one-cell threshold. For $x \in X_{\mathcal{T}}$ and $n \in \mathcal{A}$ the *deficit* of n at x is

$$d_n(x) := \frac{1}{L} \log \frac{\max_m |\gamma_m x^m(x)|}{|\gamma_n x^n(x)|} \in [0, \infty],$$

a ratio of sections of one bundle, hence a function; it is infinite where x^n vanishes. On the torus, with $p = \text{Log } x/L$,

$$(7.1) \quad d_n(x) = d_n^{\text{tr}}(p) + O(1/L), \quad d_n^{\text{tr}}(p) := \max_{m \in \mathcal{A}} (\langle p, m \rangle - H(m)) - (\langle p, n \rangle - H(n)).$$

So d_n is, up to $O(1/L)$, the tropical deficit of n at the tropical point p : the amount by which the monomial n falls short of the maximal one, in units of L . We write $\text{val}_n(p) := \langle p, n \rangle - H(n)$, and for $I \subseteq \mathcal{A}$ put $\gamma^*(I) := \min_{p \in M_{\mathbb{R}}} \max_{n \in I} d_n^{\text{tr}}(p)$.

In Sections 7–9 we use the convention $p = -m$ for the tropical variable $m \in M_{\mathbb{R}}$ and maxima of valuations (Section 2.1). Let $\text{Trop} \subset M_{\mathbb{R}}$ be the tropical hypersurface of (Δ, H) , the corner locus of $p \mapsto \max_n \text{val}_n(p)$. Since $H(0) = 0$, the set on which the origin is maximal is $-\nabla^H$, and the union of the bounded cells of Trop is its boundary $-B_Y$ (Section 4). We identify B_Y with this union of bounded cells by $m \mapsto -m$. Then the face F_{ω}^* of B_Y is the cell of Trop on which the origin and the lattice points of ω are the maximal monomials; the unbounded cells of Trop are called *tentacles*. For a vertex u of $\mathcal{T}$, the one-parameter subgroup of u multiplies the Cox coordinate Z_u by $s \in \mathbb{C}^*$, fixes the other Cox coordinates of a chart containing u , and moves Log by $(\log |s|)u$; so lowering $|Z_u|$ moves the tropical position along $-u$.

Lemma 7.5 (one-cell lemma). *$\gamma^*(I) = 0$ if and only if I is cellular. Put $\theta_0 := \min\{\gamma^*(I) : I \text{ not cellular}\} > 0$. For L larger than a constant depending on the configuration and on $\mathcal{C}_0$ (through a bound on $|\log |c_m/c_n||$ over the union of the sets $U_{\mathbb{C}}^+(c^{(0)})$, $c^{(0)} \in \mathcal{C}_0$), and for every c in that union, the set $\{n : d_n(x) < 3\theta_0/4\}$ is cellular for every $x \in X_{\mathcal{T}}$.*

The proof is in Appendix D.1; each $\gamma^*(I)$ is the value of a linear program, and the minimum is over finitely many sets. We call

$$\mathcal{C}(x) := \{n \in \mathcal{A} : d_n(x) < \theta_0/2\}$$

the *core* at x . It is cellular, and every monomial n outside it satisfies $|\gamma_n x^n(x)| \leq t^{\theta_0/2} \mathbf{m}(x)$, where $\mathbf{m}(x) := \max_m |\gamma_m x^m(x)|$.

Node data and the gap lemma.

Definition 7.6. Let $P = abcd$ be a node parallelogram in the two-face G of Δ , and $G^{\circ} = [u, u']$, an edge of $\mathcal{T}$ of lattice length one with $u^{\vee} \cap u'^{\vee} = G$. Let $\omega_1 \subseteq u^{\vee}$ and $\omega_2 \subseteq u'^{\vee}$ be the three-cells of $\mathcal{F}_C$ containing P . The function $e_{u'}$ is affine on ω_1 , nonnegative there,

and vanishes exactly on $\omega_1 \cap u^\vee = \omega_1 \cap G = P$. The height condition (Proposition 3.26) says that every lattice point of ω_1 not in P has $e_{u'} = 1$, and every lattice point of ω_2 not in P has $e_u = 1$. We call the lattice points of $\omega_1 \cup \omega_2$ not in P the *apexes* of P , and put $\mathcal{A}_P := \{0\} \cup \omega_1 \cup \omega_2$. By Lemma 7.1 the cellular sets containing P are the subsets of $\{0\} \cup \omega_1$ and of $\{0\} \cup \omega_2$.

The *apex gap* of P is $\mathbf{g}_P := \min(H(v) + H(v') - H(0) - \ell_P(v + v'))$, over apexes $v \in \omega_1$, $v' \in \omega_2$, where ℓ_P is the linear function equal to H on the cone over P ; the point $v + v'$ lies in the plane of G . For a unimodular triangle ω of $\mathcal{F}_C$ in a two-face with dual edge $[u, u']$, the tropical cell dual to $\text{conv}(0, \omega)$ is a segment; its *vertex gap* $\mathbf{g}_\omega$ is twice the smallest tropical deficit, at the midpoint of that segment, of a lattice point $n \notin \{0\} \cup \omega$ lying in a cell of $\mathcal{F}_C$ that contains ω .

Exponents at a node. Let $w_u \in N$ be the vector of Section 7.1 for the edge $G^\circ = [u, u']$, so that $\langle u, w_u \rangle = 1$ and $\langle u', w_u \rangle = 0$, and put $w_{u'} := -a - w_u$. Since $\langle u, a \rangle = \langle u', a \rangle = -1$, we have $\langle u, w_{u'} \rangle = 0$ and $\langle u', w_{u'} \rangle = 1$. For $n \in N$ the vector $n - a - \langle u, n - a \rangle w_u - \langle u', n - a \rangle w_{u'}$ is orthogonal to u and u' , hence lies in the lattice of the plane of G , of which $b - a, d - a$ is a basis because P is a unit parallelogram. As $\langle u, n - a \rangle = e_u(n)$ and $\langle u', n - a \rangle = e_{u'}(n)$, every $n \in N$ is uniquely

$$(7.2) \quad n - a = \alpha_n(b - a) + \beta_n(d - a) + e_u(n)w_u + e_{u'}(n)w_{u'}, \quad \alpha_n, \beta_n \in \mathbb{Z},$$

and the origin has $\alpha_0 = \beta_0 = 0$. Put $|n|_P := |\alpha_n| + |\beta_n|$ and $A_P := \max |n|_P$ over the apexes. The *exponent constant* of P is

$$\mathcal{E}_P := \max \left(2 + A_P, \max_{n \in \mathcal{A} \setminus \mathcal{A}_P} (|n|_P + A_P + (\max(e_u(n), e_{u'}(n)) - 1)^+(2 + A_P)) \right),$$

where $x^+ := \max(x, 0)$, and $\mathcal{E}_{\max} := \max_P \mathcal{E}_P$. It depends only on the lattice data at P and on the choice of w_u , not on the heights. We write $e_{\max} := \max_{u, n} e_u(n)$, over the vertices u of $\mathcal{T}$ and $n \in \mathcal{A}$.

We require

$$(7.3) \quad \mathbf{g}_P \geq \theta_0 + 1 + 2\theta_1, \quad \mathbf{g}_\omega \geq \theta_0 + 1, \quad k_0 + \min_{\partial \Delta \cap N} h_C \geq 1,$$

for all node parallelograms P and all triangles ω as above, where $\theta_1 \leq \min(\theta_0, 1)$ is the cut-off constant of Section 7.2. The first condition is used in two ways: the apexes have deficit at least $\mathbf{g}_P/2 \geq (1 + \theta_0)/2$ on the Morse polydisc (proof of Lemma 7.14), and the node neighbourhood of Section 7.5 reaches $\theta_P := \mathbf{g}_P/2 - \theta_1 \geq (1 + \theta_0)/2$ along the segment J_P of the next lemma. By that lemma, (7.3) is a lower bound on k_0 .

Lemma 7.7 (the gaps). *Let G be a two-face of Δ with $G^\circ = [u, u']$, let ω be a node parallelogram or a unimodular triangle of $\mathcal{F}_C$ in G , and let $\omega_1 \subseteq u^\vee$ and $\omega_2 \subseteq u'^\vee$ be the three-cells of $\mathcal{F}_C$ containing ω .*

- (1) *The tropical cell dual to $\text{conv}(0, \omega)$ is a segment $J_\omega = \{p_\omega + s(u' - u) : s_- \leq s \leq s_+\}$. On J_ω a lattice point n of $\omega_1 \setminus \omega$ has tropical deficit $e_{u'}(n)(s_+ - s)$, with $e_{u'}(n) \geq 1$, and a lattice point n of $\omega_2 \setminus \omega$ has tropical deficit $e_u(n)(s - s_-)$, with $e_u(n) \geq 1$. The length $|J_\omega| := s_+ - s_-$ satisfies $|J_\omega| = k_0 + |J_\omega|^{(0)}$, where $|J_\omega|^{(0)} > 0$ is the length of the corresponding segment for the height h_C alone.*
- (2) *For a node parallelogram, $\mathbf{g}_P = |J_P|$; at each point of J_P all apexes of ω_1 have the same tropical deficit, and so do all apexes of ω_2 , and at the midpoint both deficits equal $\mathbf{g}_P/2$; and $\mathbf{g}_P = k_0 + \mathbf{g}_P^{(0)}$, where $\mathbf{g}_P^{(0)} > 0$ is the apex gap for the height h_C .*

For a triangle, $\mathfrak{g}_\omega = m_\omega |J_\omega|$, where $m_\omega \geq 1$ is the smallest of the exponents $e_{u'}(n)$, $n \in \omega_1 \setminus \omega$, and $e_u(n)$, $n \in \omega_2 \setminus \omega$, over lattice points.

- (3) $\theta_0 \leq \mathfrak{g}_P/2$ for every node parallelogram P , and $\theta_0 \leq \mathfrak{g}_\omega - |J_\omega|/(2e_{\max})$ for every triangle ω as above.
- (4) Consequently (7.3) holds whenever $k_0 \geq \max(2e_{\max}, 6, 1 - \min_{\partial\Delta \cap N} h_C)$.

Proof. (1) A cell of $\mathcal{F}_C$ containing ω lies in a face of Δ containing G , that is in G , $u^\vee$ or $u'^\vee$. In G the only such cell is ω , which is a maximal cell of the subdivision σ of G and hence of $\mathcal{F}_C|_G$ (Proposition 3.26 and Definition 2.3); in $u^\vee$ it is the unique three-cell ω_1 of $\mathcal{F}_C|_{u^\vee}$ containing the boundary two-cell ω , and likewise ω_2 in $u'^\vee$. So by Lemma 7.1 the lower faces containing $\text{conv}(0, \omega)$ are $\text{conv}(0, \omega)$, $\text{conv}(0, \omega_1)$ and $\text{conv}(0, \omega_2)$. By the inclusion-reversing duality between the lower faces of $\text{conv}\{(n, H(n))\}$ and the cells of Trop, the cell dual to $\text{conv}(0, \omega)$ is a segment whose end points are dual to $\text{conv}(0, \omega_1)$ and $\text{conv}(0, \omega_2)$. On it the origin and the points of ω are maximal, with value 0, so it lies on the line $\{p : \langle p, n \rangle = H(n), n \in \omega\}$; since ω spans the linear span of G , and $\langle u' - u, n \rangle = 0$ for $n \in G$, this line is $p_\omega + \mathbb{R}(u' - u)$ for any point p_ω of it. For $n \in u^\vee$ we have $\langle u' - u, n \rangle = (e_{u'}(n) - 1) + 1 = e_{u'}(n)$, and for $n \in u'^\vee$, $\langle u' - u, n \rangle = -e_u(n)$. Let s_+ be the parameter of the end dual to $\text{conv}(0, \omega_1)$; there every lattice point of ω_1 is maximal, because H is linear on the cone over ω_1 . Since the maximal value is 0 all along J_ω , a lattice point n of ω_1 has tropical deficit $-(\langle p_\omega, n \rangle + s e_{u'}(n) - H(n)) = e_{u'}(n)(s_+ - s)$ on J_ω . If $n \notin \omega$, then $n \notin u'^\vee$, since $\omega_1 \cap u'^\vee \subseteq \omega_1 \cap G = \omega$; so $e_{u'}(n) \geq 1$. The statement for ω_2 is proved in the same way, with s_- the other end.

For the length, choose lattice points $v \in \omega_1 \setminus \omega$ and $v' \in \omega_2 \setminus \omega$ and put $e = e_{u'}(v)$, $e' = e_u(v')$. The deficits vanish at the ends, so $H(v) - \langle p_\omega, v \rangle = e s_+$ and $H(v') - \langle p_\omega, v' \rangle = -e' s_-$, whence

$$ee' |J_\omega| = e' H(v) + e H(v') - \langle p_\omega, w \rangle, \quad w := e' v + e v'.$$

Since $\langle u' - u, w \rangle = e' e - ee' = 0$, w lies in the span of G , where $\langle p_\omega, \cdot \rangle$ is the linear function ℓ_ω equal to H on the cone over ω . The right-hand side is therefore linear in H . For $H = \varphi$: $\varphi(v) = \varphi(v') = 1$, and on the span of $G \subseteq u^\vee$ the function ℓ_ω is $-\langle u, \cdot \rangle$, with $\langle u, v \rangle = -1$ and $\langle u, v' \rangle = e' - 1$; so the right-hand side equals $e' + e + e'(-1) + e(e' - 1) = ee'$. For $H = k_0 \varphi + h_C$ this gives $|J_\omega| = k_0 + |J_\omega|^{(0)}$, where $|J_\omega|^{(0)}$ is given by the same formula for h_C ; by Lemma 7.1 with $k_0 = 0$ it is the length of a segment whose two ends are dual to the distinct lower faces $\text{conv}(0, \omega_1) \neq \text{conv}(0, \omega_2)$, hence positive.

(2) For a node parallelogram every apex has $e_{u'} = 1$ on ω_1 and $e_u = 1$ on ω_2 (the height condition), so by (1) the apexes of ω_1 all have deficit $s_+ - s$ and those of ω_2 all have $s - s_-$. The computation of (1) with apexes v, v' and $e = e' = 1$ reads $|J_P| = H(v) + H(v') - H(0) - \ell_P(v + v')$ for every pair of apexes; so $\mathfrak{g}_P = |J_P|$, the apexes have deficit $\mathfrak{g}_P/2$ at the midpoint, and $\mathfrak{g}_P = k_0 + \mathfrak{g}_P^{(0)}$ with $\mathfrak{g}_P^{(0)} = |J_P|^{(0)} > 0$. For a triangle, the cells containing ω are ω , ω_1 and ω_2 , so the lattice points $n \notin \{0\} \cup \omega$ in a cell containing ω are those of $(\omega_1 \cup \omega_2) \setminus \omega$, and by (1) their deficits at the midpoint are $e_{u'}(n)|J_\omega|/2$ and $e_u(n)|J_\omega|/2$; so $\mathfrak{g}_\omega = m_\omega |J_\omega|$.

(3) Let $v \in \omega_1 \setminus \omega$ and $v' \in \omega_2 \setminus \omega$ be lattice points, with e, e' as in (1). The set $I = \omega \cup \{v, v'\}$ is not cellular: by Lemma 7.1 a cell containing it would lie in a proper face of Δ containing G , that is in G , $u^\vee$ or $u'^\vee$, but $v \notin u'^\vee$ and $v' \notin u^\vee$. The points of ω have deficit 0 on J_ω , so by (1)

$$\theta_0 \leq \gamma^*(I) \leq \min_{s_- \leq s \leq s_+} \max(e(s_+ - s), e'(s - s_-)) = \frac{ee'}{e + e'} |J_\omega|.$$

For a node parallelogram, with apexes v, v' , this is $\theta_0 \leq |J_P|/2 = \mathfrak{g}_P/2$. For a triangle, choose v and v' with e and e' smallest on their sides, so that $\min(e, e') = m_\omega$; then

$$\mathfrak{g}_\omega - \theta_0 \geq |J_\omega| \left(\min(e, e') - \frac{ee'}{e + e'} \right) = |J_\omega| \frac{\min(e, e')^2}{e + e'} \geq \frac{|J_\omega|}{2e_{\max}}.$$

(4) By (1) and (2), $\mathfrak{g}_P > k_0$ and $\mathfrak{g}_\omega \geq |J_\omega| > k_0$. So by (3), $\mathfrak{g}_\omega - \theta_0 \geq |J_\omega|/(2e_{\max}) > k_0/(2e_{\max}) \geq 1$ when $k_0 \geq 2e_{\max}$, and, since $\theta_1 \leq 1$, $\mathfrak{g}_P - \theta_0 - 2\theta_1 \geq \mathfrak{g}_P/2 - 2 > k_0/2 - 2 \geq 1$ when $k_0 \geq 6$. The bound is not circular: $e_{\max}$ depends only on the lattice data, and $\theta_1 \leq 1$ by its definition, whereas θ_0 and the gaps may grow with k_0 (Remark 7.8). The last condition of (7.3) is $k_0 \geq 1 - \min h_C$. $\square$

Remark 7.8 (the size of θ_0). The construction compares θ_0 with no absolute constant: it compares $\theta_1 \leq \min(\theta_0, 1)$, and θ_0 enters (7.3) only through Lemma 7.7(3). So it needs neither a bound on θ_0 nor its growth with k_0 , and both occur. If some facet $u^\vee$ contains a non-cellular set I (for instance when $\mathcal{F}_C$ subdivides $u^\vee$), then along $p' - \lambda u$, $\lambda \rightarrow \infty$, only the monomials of $u^\vee$ stay maximal, and $\gamma^*(I)$ is bounded by a quantity depending on $h_C|_{u^\vee}$ alone, so θ_0 is bounded independently of k_0 . If every facet is a cell of $\mathcal{F}_C$, the cellular sets are those of the height φ , and $\theta_0 \geq k_0 \min_I \gamma^*(I; \varphi) - 2 \max |h_C|$ grows linearly in k_0 .

7.2. The localized mirror and the smoothness of the interpolating family. We multiply every monomial by a cut-off that vanishes where the monomial is far from maximal. Our rule differs from that of [Yam16, Def. 3.1], a product of cut-offs of the pairwise differences of the logarithms of the monomials: the cut-off of a monomial is a function of its smoothed maximal deficit, and the monomial of the origin is never cut.

Definition 7.9 (the faces E_e and their interior polygons). Let $e = [n, n']$ be an edge of $\mathcal{F}_C$ in the two-skeleton of $\partial\Delta$. Its *tropical cell* $E_e \subset M_{\mathbb{R}}$ is the polygon $\{p : d_0^{\text{tr}}(p) = d_n^{\text{tr}}(p) = d_{n'}^{\text{tr}}(p) = 0\}$ on which $0, n, n'$ are the maximal monomials; it is dual to the lower face $\text{conv}(0, n, n')$, and it is the face F_e^* of B_Y . Put $f_e := \min_{m \notin \{0, n, n'\}} d_m^{\text{tr}}$ on E_e and $\theta_e := \max_{E_e} f_e$, which is positive because at relative interior points of E_e only $0, n, n'$ are maximal. The *cut-off constant* is $\theta_1 := \min(\theta_0, \min_e \theta_e, 1)$, over all such edges e . The *interior polygon* of e is $\Omega_e := \{p \in E_e : f_e(p) > 3\theta_1/4\}$, a nonempty open convex subset of E_e , since $\theta_1 \leq \theta_e$ and f_e is a minimum of affine functions.

The arcs of Γ_Y in the relative interior of E_e are the arcs $C_Y([u, u'], e)$ with $[u, u']$ an edge of $\mathcal{T}$ in $e^\vee$ (Propositions 4.7(5) and 4.10). When e lies in the interior of a two-face G there is one such arc for each edge of $\mathcal{T}$ in G° , and only one if G contains a node parallelogram, since then G° has length one; when e lies on an edge E of Δ , $e^\vee = E^\vee$ is a triangulated two-face of Δ° and the arcs meet at the vertices $C_Y(\tau, e)$, τ a triangle of $\mathcal{T}$, which are of negative type on B_Y (Proposition 4.10(3)); we call them *turning points*.

Put $\theta_2 := \theta_1/(16 \max(\mathcal{E}_{\max}, D))$, where D is the largest length of a difference $n - n'$ of lattice points of Δ for a fixed norm on $N_{\mathbb{R}}$; tropical distances in $M_{\mathbb{R}}$ are measured in the dual norm, so that $|\langle p - p', n - n' \rangle| \leq D|p - p'|$. Thus $\mathcal{E}_P \theta_2 \leq \theta_1/16$ for every node and $D\theta_2 \leq \theta_1/16$; θ_2 bounds, in the torus directions, the sets of Lemma 7.18 on which the weights are exactly 0 or 1. Put $\theta_3 := \min(\theta_2/4, c_M/(8e_{\max}))$, where c_M is the smallest norm of a nonzero vector of M ; it bounds the lengths of the level segments with which the model arcs of Section 8 begin. Fix a smooth nonincreasing $\beta: \mathbb{R} \rightarrow [0, 1]$ with $\beta = 1$ on $(-\infty, \theta_1/4]$ and $\beta = 0$ on $[\theta_1/2, \infty)$.

Definition 7.10 (the localized mirror). The *smoothed deficit* of n is

$$\tilde{d}_n := \frac{1}{2L} \left(\log \sum_m |\gamma_m x^m|^2 - \log |\gamma_n x^n|^2 \right).$$

Put $\psi_0 \equiv 1$ and $\psi_n := \beta(\tilde{d}_n)$ for $n \neq 0$, and for $s \in [0, 1]$

$$F^{(s)} := \sum_{n \in \mathcal{A}} \tilde{\psi}_n \gamma_n x^n, \quad \tilde{\psi}_n := (1-s)\psi_n + s, \quad W_s := \{F^{(s)} = 0\}.$$

The *localized mirror* is $W^* := W_0$; and $W_1 = Y_t$.

Since $d_n \leq \tilde{d}_n \leq d_n + \log |\mathcal{A}|/(2L)$, a monomial with $\psi_n \neq 0$ lies in the core, and the maximal monomial at any point has $\psi_n = 1$. Each ψ_n is a smooth function on $X_{\mathcal{T}}$: it depends only on the moduli of the monomials and vanishes identically near the zero set of x^n . So $F^{(s)}$ is a smooth section of the anticanonical bundle, in general not holomorphic. The compact torus $M \otimes U(1)$ multiplies each summand by the same character as x^n .

Lemma 7.11 (smoothness of the interpolating family). *There is a threshold $t_0 = t_0(r, \eta_0, \mathcal{C}_0) > 0$ such that for $0 < t < t_0$:*

- (1) *for every coefficient vector in the window and every $s \in [0, 1]$, $F^{(s)}$ has real rank two along W_s ; so W_s is a smooth compact submanifold of real codimension two, and $W_1 = Y_t$ is a smooth hypersurface;*
- (2) *at $s = 1$, for every $c \in U_{\mathbb{C}}^+$, the singular points of $Y_{t,c}$ are exactly the points q_P of Lemma 7.14 whose critical value ε_P vanishes.*

The proof is in Appendix D.2. It runs through the core at a point x of W_s . By Lemma 7.5 the core is cellular; let $\mathcal{H}(x)$ be the smallest face of Δ containing $\mathcal{C}(x) \setminus \{0\}$. Since the two-cells of $\mathcal{F}_C$ in the two-faces of Δ are unimodular triangles and node parallelograms, at least one of the following holds (Lemma D.3).

- (A) $\mathcal{C}(x)$ is affinely independent. A derivative along the torus action, dual to the simplex $\mathcal{C}(x)$, picks out one core monomial of size at least $\max_m |\gamma_m x^m|/(2|\mathcal{A}|)$.
- (B) $\mathcal{H}(x)$ is a facet $u^\vee$. Then the origin dominates every other monomial of positive Z_u -exponent by a factor $t^{3\theta_0/4}$, and $\partial F^{(s)}/\partial Z_u$ is the derivative of the origin monomial up to a small relative error: W_s is a graph over the Z_u -direction. This is where $\psi_0 \equiv 1$ enters. If the origin were cut, W_0 would be singular at the square faces $\square$ of the prisms adjacent to a node at which the binomial ratio $\mu_\square$ equals 1; for ξ in the window, $\log \mu_\square$ is a combination of several η_P and does vanish for some ξ .
- (C) $\mathcal{C}(x) \setminus \{0\}$ is a node parallelogram P . This case is treated by Lemmas 7.14 and 7.15.

In each case the cut-offs contribute to the derivative through their own derivatives, which are $O(1/L)$ in logarithmic units and carried by monomials in transition; a regularity criterion of Wirtinger type (Lemma D.2) shows that they cannot cancel the holomorphic part. Part (1) uses the window only in case (C) near the node point, where the lower bound comes from $|\kappa_P| \geq \eta_0/(2C_\eta)$.

Proposition 7.12. *For $t < t_0$ and c in the window there is an isotopy $(\Psi_s)_{s \in [0,1]}$ of $X_{\mathcal{T}}$ with $\Psi_0 = \text{id}$ and $\Psi_s(W^*) = W_s$. In particular W^* and Y_t are diffeomorphic.*

Proof. By Lemma 7.11(1) and the implicit function theorem applied to $(s, x) \mapsto F^{(s)}(x)$, the set $\bigcup_s \{s\} \times W_s$ is a compact submanifold of $[0, 1] \times X_{\mathcal{T}}$ submersing onto $[0, 1]$. By Ehresmann's theorem it is a trivial bundle, and the isotopy extension theorem gives Ψ_s [Hir76, Ch. 8]. $\square$

Transversality to the torus orbits is not used, and it fails in general: at the square faces $\square$ above, when $\mu_\square = 1$, the initial form of $\square$ is singular, and W_s is not transverse to the orbits near the tropical cell dual to $\square$. Smoothness holds by case (B).

7.3. The Morse chart at a node. Fix a node parallelogram $P = abcd$ with the data of Definition 7.6. On the affine chart $U_{u,u'} \cong \mathbb{C}^2 \times (\mathbb{C}^*)^2$ of the cone spanned by u, u' we use the characters $Z := \chi^{w_u}$, $Z' := \chi^{w_{u'}}$, $\chi_1 := \chi^{b-a}$ and $\chi_2 := \chi^{d-a}$. They are coordinates, because $(b-a, d-a, w_u, w_{u'})$ is a basis of N , and Z, Z' are the Cox coordinates $Z_u, Z_{u'}$ times units, vanishing to first order along $D_u, D_{u'}$. By (7.2), $x^n/x^a = \chi^{n-a} = \chi_1^{\alpha_n} \chi_2^{\beta_n} Z^{e_u(n)} Z'^{e_{u'}(n)}$. The function $f^{(s)} := F^{(s)}/(\gamma_a x^a)$ is therefore a polynomial in Z, Z' and a Laurent polynomial in χ_1, χ_2 with smooth coefficients. By the height condition the monomials of $\mathcal{A}_P$ contribute

$$1 + \tilde{\chi}_1 + \tilde{\chi}_2 + \mu_P \tilde{\chi}_1 \tilde{\chi}_2 + A Z + A' Z' + \mathbf{b} Z Z', \quad \tilde{\chi}_1 := \frac{\gamma_b}{\gamma_a} \chi_1, \quad \tilde{\chi}_2 := \frac{\gamma_d}{\gamma_a} \chi_2,$$

where A (resp. A') is the sum over the apexes of ω_2 (resp. ω_1), a Laurent polynomial in χ_1, χ_2 , and $\mathbf{b} = \gamma_0/\gamma_a$ is a constant, since $\alpha_0 = \beta_0 = 0$ (in $f^{(s)}$ each term carries its weight). This choice of $w_u, w_{u'}$, which makes the origin the monomial $Z Z'$ up to a constant, is used throughout.

Definition 7.13 (Morse coordinates). Put

$$x := \mu_P^{1/2} \tilde{\chi}_1 + \mu_P^{-1/2}, \quad y := \mu_P^{1/2} \tilde{\chi}_2 + \mu_P^{-1/2}, \quad \kappa_P := e^{-\eta_P} - 1,$$

so that $1 + \tilde{\chi}_1 + \tilde{\chi}_2 + \mu_P \tilde{\chi}_1 \tilde{\chi}_2 = xy - \kappa_P$ exactly. The *node point* is $x = y = 0$. Let $|A|_{\text{mon}}, |A'|_{\text{mon}}$ be the largest moduli of a single term of A, A' at the node point, and $\lambda_P := |\mathbf{b}|/(|A|_{\text{mon}}|A'|_{\text{mon}})$ there; then $\lambda_P \geq C^{-1}t^{-\mathfrak{q}_P}$. The *normalised coordinates* are

$$\hat{Z} := Z/\nu_Z, \quad \hat{Z}' := Z'/\nu_{Z'}, \quad \nu_Z := \frac{\lambda_P^{-1/2}}{|A|_{\text{mon}}}, \quad \nu_{Z'} := \frac{\lambda_P^{-1/2}}{|A'|_{\text{mon}}},$$

with constants satisfying $\nu_Z \nu_{Z'} = 1/|\mathbf{b}|$. The *Morse polydisc* is $\mathbb{D}_P := \{|x| < r, |y| < r, |\hat{Z}| < 1, |\hat{Z}'| < 1\}$.

The normalisation places the node point at the centre of the range of $|Z|/|Z'|$ on which both apex sums are negligible: on $\mathbb{D}_P$ one has $|AZ|, |A'Z'| \leq 2^{A_P} |A| \lambda_P^{-1/2} \leq C t^{\mathfrak{q}_P/2}$, while $|\mathbf{b} Z Z'| \leq C$. The *Morse bidisc* of P is the closed bidisc $\mathbb{B}_P := \{|x| \leq r, |y| \leq r\}$ in the torus coordinates (χ_1, χ_2) , the projection of the closure of $\mathbb{D}_P$.

Lemma 7.14 (Morse chart at a node). Put $r_1 := \min(1/4, (20/C_{\mathbf{R}})^{1/2})$, with $C_{\mathbf{R}}$ the constant defined after Definition 7.3; it depends only on the matrix $\mathbf{R}$ of Lemma 7.2. For $r \leq r_1$ there is $t_1 = t_1(r, \mathcal{C}_0) > 0$ such that for $t < t_1$, every $c \in U_{\mathbb{C}}^+$ and every $s \in [0, 1]$:

- (1) on the polydisc $2\mathbb{D}_P$ with doubled radii, $\tilde{\psi}_n = 1$ for $n \in P \cup \{0\}$ and $\tilde{\psi}_n = s$ for every other n . Hence $f^{(s)} = f^{(0)} + s(f^{(1)} - f^{(0)})$ is holomorphic on $2\mathbb{D}_P$, and

$$f^{(0)} = xy - \kappa_P + \vartheta_P \hat{Z} \hat{Z}', \quad f^{(1)} - f^{(0)} = a \hat{Z} + a' \hat{Z}' + R,$$

where $\vartheta_P := \mathbf{b}/|\mathbf{b}|$, $a = A\nu_Z$, $a' = A'\nu_{Z'}$, and

$$\sup_{2\mathbb{D}_P} |a|, \sup_{2\mathbb{D}_P} |a'| \leq C \lambda_P^{-1/2}, \quad \sup_{2\mathbb{D}_P} |R| \leq C t^{\theta_0/2};$$

- (2) $f^{(s)}$ has exactly one critical point $q_P(s)$ in $\mathbb{D}_P$; it is nondegenerate, lies within $C(r)t^{\theta_0/2}$ of the node point, and its critical value is $\varepsilon_P(s) = -\kappa_P + O(t^{\theta_0/2})$;
- (3) if c is in the window and $C(r)t^{\theta_0/2} < \eta_0/(4C_{\eta})$, then $\varepsilon_P(s) \neq 0$ for all $s \in [0, 1]$.

The proof is in Appendix D.3. On the polydisc only the four monomials of P and the origin survive in W^* , so $W^* \cap \mathbb{D}_P$ is the exact conifold smoothing $\{xy + \vartheta_P \hat{Z} \hat{Z}' = \kappa_P\}$; the family W_s turns the apex monomials and all others back on linearly, and they are too small

to move the critical value off $-\kappa_P$. The critical value comes from the perturbation off $\mathcal{Z}$ and not from the apex monomials. The proof uses only the height condition, that is, that the apexes have exponent one, and not the shape of the cells of $\mathcal{F}_C$ adjacent to P ; so the lemma holds whatever these cells are. Besides pyramids and prisms they include cells with seven or eight vertices, a node parallelogram together with three or four apexes; at the nodes of a hexagon, for instance, eight-vertex cells occur. We write $q_P := q_P(1)$ and $\varepsilon_P := \varepsilon_P(1)$.

Since $-\kappa_P = 1 - e^{-\eta_P}$, the critical value is $\varepsilon_P = \eta_P + O(\eta_P^2) + O(t^{\theta_0/2})$. Critical points and critical values of F_t off Y_t depend on the trivialisation of the anticanonical bundle; ours are those of $f^{(1)} = F_t/(\gamma_a x^a)$, the trivialisation by $\gamma_a x^a$. Other normalisations, for instance $\mu_P = 1 + \eta_P$, or another local trivialisation in place of $\gamma_a x^a$, multiply ε_P by a factor that is nonzero near $\mathcal{Z}$ and depends only on the node (to first order a sign, or the value at q_P of the ratio of the two trivialisations). Such a factor does not change the loci $\{\varepsilon_P = 0\}$, the vanishing spheres, or their orientations, since Definition 7.23 below applies to every argument of κ_P (Remark 7.25); in particular it does not change the relation lattice of the δ_q .

Put $\mathcal{U}_P := \{x \in U_{u,u'} : \mathcal{C}(x) \setminus \{0\} = P\}$, the set of points at which the core, with the origin removed, consists of the vertices of P . It contains $\mathbb{D}_P$ (proof of Lemma 7.14), and the sets $\mathcal{U}_P$ of distinct nodes are disjoint. This applies also to two node parallelograms that share an edge, as in an A_2 subdivision of a hexagon.

Lemma 7.15 (no other critical points near a node). *For $t < t_0(r, \eta_0, \mathcal{C}_0)$:*

- (1) *if c is in the window, every point of $W_s \cap \mathcal{U}_P$ is a regular point of $F^{(s)}$, for every $s \in [0, 1]$;*
- (2) *if $c \in U_{\mathbb{C}}^+$, every singular point of $Y_{t,c}$ in $\mathcal{U}_P$ lies in $\mathbb{D}_P$.*

The proof (Appendix D.3) compares monomials with equal exponents in (Z, Z') , whose ratio depends only on (χ_1, χ_2) . Either the origin dominates the Z - or the Z' -derivative, or all terms depending on Z, Z' are small and the torus derivatives of the four-term polynomial $xy - \kappa_P$ are bounded below. At $s = 1$ the second alternative pins every critical point to $Z \approx -A'/b$, $Z' \approx -A/b$, $x \approx y \approx 0$, which lies in $\mathbb{D}_P$.

Definition 7.16 (the vanishing sphere). For c in the window, $\delta_q \subset Y_t \cap \mathbb{D}_P$ is the vanishing sphere of the Morse point q_P of $f^{(1)}$ for the straight path of values from ε_P to 0: in holomorphic coordinates ζ centred at q_P with $f^{(1)} = \varepsilon_P + \sum_i \zeta_i^2$, it is $\{\zeta \in (-\varepsilon_P)^{1/2} \mathbb{R}^4 : \sum_i \zeta_i^2 = -\varepsilon_P\}$. It is well defined up to isotopy in $Y_t \cap \mathbb{D}_P$. Its orientation is fixed in Definition 7.23. (Compare the three-spheres over edges of [GM18, §4.4].)

Lemma 7.17 (transport of the sphere). *Let $\Sigma_s \subset W_s \cap \mathbb{D}_P$ be the vanishing sphere of $q_P(s)$ for $f^{(s)}$ and the straight path from $\varepsilon_P(s)$ to 0. For c in the window, $\Psi_1(\Sigma_0)$ is isotopic to $\delta_q = \Sigma_1$ in Y_t , with Ψ_1 as in Proposition 7.12.*

Proof. On $\mathbb{D}_P$ the $f^{(s)}$ form a holomorphic family, smooth in s , with one Morse point each and $\varepsilon_P(s) \neq 0$ (Lemma 7.14(3)). The holomorphic Morse lemma with parameters [AGZV85, §§6.2, 9.6] gives Morse coordinates depending smoothly on s , so Σ_s can be chosen to depend smoothly on s ; its radius $|\varepsilon_P(s)|^{1/2} \leq (2\eta_0)^{1/2} \leq r/3$ keeps it inside $\mathbb{D}_P$. Then $\Psi_s^{-1}(\Sigma_s)$ is a smooth family of embedded spheres in W^* from Σ_0 to $\Psi_1^{-1}(\Sigma_1)$, and isotopy extension gives the claim. $\square$

That $\varepsilon_P(s)$ does not vanish along the family is the reason for the lower bound $|\eta_P| \geq \eta_0/C_\eta$ in the window. For $c \in U_{\mathbb{C}}$ the loci $\{\varepsilon_P = 0\}$ are smooth hypersurfaces, since $d\varepsilon_P = e^{-\eta_P} d\eta_P + O(t^{\theta_0/2})$ and the $d\eta_P$ are nonzero. For the estimate of $d\varepsilon_P$: at $s = 1$ the critical

value ε_P is holomorphic in c , and by Lemma 7.14(2) on $U_{\mathbb{C}}^+$, $\varepsilon_P + \kappa_P = O(t^{\theta_0/2})$ there; every point of $U_{\mathbb{C}}$ is the centre of a polydisc in ξ , of polyradius $\min(1/2, \eta_0/(2 \max_P \|r_P\|_*))$, contained in $U_{\mathbb{C}}^+$, so the Cauchy estimates bound $d(\varepsilon_P + \kappa_P)$ by $C(\eta_0)t^{\theta_0/2}$ on $U_{\mathbb{C}}$. These loci are the loci D_q of Section 9.1.

The node neighbourhood. Let P be a node parallelogram. We say that (χ_1, χ_2) is *within* ϵ of the node point if $|\log |\mu_P \tilde{\chi}_1|| \leq \epsilon L$ and $|\log |\mu_P \tilde{\chi}_2|| \leq \epsilon L$. With $\theta_P = \mathbf{g}_P/2 - \theta_1$, the node neighbourhood is

$$\mathcal{N}_P := \{x \in U_{u,u'} : (\chi_1, \chi_2) \text{ within } \theta_2 \text{ of the node point,} \\ |\hat{Z}| \leq t^{-\theta_P}, |\hat{Z}'| \leq t^{-\theta_P}, |\hat{Z}\hat{Z}'| \leq t^{-\theta_2}\}.$$

Tropically, $\mathcal{N}_P$ consists of the points within θ_2 of the region below the part of the edge J_P of Lemma 7.7 at distance at most θ_P from its midpoint, that is, up to distance θ_1 from its ends; it extends to distance θ_2 in the torus directions.

7.4. Where the localized mirror is given by exact models. On the following sets the weights of the localized mirror are exactly 0 or 1, so that W^* is there given by an algebraic model: near the legs, from the nodes and vertices to the legs, at the nodes, and at the vertices of positive type. Section 7.5 uses item (2); Section 8 uses all of them.

Lemma 7.18 (exactness of the weights). *For t small, with $c \in U_{\mathbb{C}}$:*

- (1) *(legs) Let $e = [n, n']$, let x be a point of the chart of the cone of an edge $[u, u']$ of $\mathcal{T}$ in $e^\vee$ whose tropical position lies within θ_2 of Ω_e , and let x' have the same torus coordinates, with $|Z(x')| \leq |Z(x)|$ and $|Z'(x')| \leq |Z'(x)|$. Then at x' the weights are $\psi_0 = \psi_n = \psi_{n'} = 1$ and $\psi_m = 0$ for all other m . The same holds at the turning points, in the chart of the three-cone of the triangle, with the three Cox coordinates of the cone lowered.*
- (1') *(from a node or a vertex to its legs) Let ω be a node parallelogram or a unimodular triangle of $\mathcal{F}_C$ in a two-face G , $[u, u']$ an edge of $\mathcal{T}$ in G° , with Cox coordinates Z, Z' , and e an edge of ω . Let m be the midpoint of J_ω if ω is a node parallelogram, and a point of the segment $J_\omega^\flat$ of (3) if ω is a triangle; let $m' \in E_e$ with $|m' - m| \leq \theta_2/4$, $p_e \in \Omega_e$, and $I := [m, m'] \cup [m', p_e]$. Let $x \in U_{u,u'}$ be a point whose tropical position is obtained from a point within θ_2 of I by lowering $\log |Z|$ and $\log |Z'|$. Then at x the origin and the two monomials of e have weight one, and every monomial outside $\{0\} \cup \omega$ has weight zero.*
- (2) *(nodes) On the node neighbourhood $\mathcal{N}_P$ every apex and every monomial outside $\mathcal{A}_P$ has weight 0, and the monomials of P have weight 1.*
- (3) *(vertices of positive type) Let $\omega = [n_a, n_b, n_c]$ be a unimodular triangle of $\mathcal{F}_C$ in a two-face G , with $G^\circ = [u_0, u_k]$ and $u_0, \dots, u_k$ its lattice points in order; write $J_\omega = \{p_\omega + s(u_k - u_0) : s_- \leq s \leq s_+\}$ as in Lemma 7.7(1), applied with $(u, u') = (u_0, u_k)$, and $s_{\text{mid}} := (s_+ + s_-)/2$. Let $[u, u']$ be an edge of $\mathcal{T}$ in G° , with Cox coordinates Z, Z' . The set $J_\omega^\flat$ of points of J_ω at which every lattice point of $(\omega_1 \cup \omega_2) \setminus \omega$ has tropical deficit at least $3\theta_1/4$ is a segment containing the points with $|s - s_{\text{mid}}| \leq 1/(4e_{\text{max}})$. At the points of $U_{u,u'}$ whose tropical position is within θ_2 of a point of $J_\omega^\flat$, or is obtained from such a point by lowering $\log |Z|$ and $\log |Z'|$, the weights of $0, n_a, n_b, n_c$ are one and all other weights are zero.*
- (4) *Let $x \in W^* \cap D_{u'}$, and let $I(x)$ be the set of monomials that do not vanish on $D_{u'}$ and have positive weight at x . If $I(x)$ is affinely independent, or c is in the window and $I(x) \subseteq P$ for a node parallelogram $P \subset u'^\vee$, then W^* is transverse to $D_{u'}$ at*

x. In particular W^* is transverse to D_u and $D_{u'}$ (resp. D_{u_i}) on the regions of (1)–(3), wherever it meets them; on the regions of (2), and of (1') when ω is a node parallelogram, this requires c to be in the window.

The proof is in Appendix D.4. Every lattice point of Δ has nonnegative Cox exponents, so lowering $|Z|$ and $|Z'|$ at fixed torus coordinates multiplies each monomial by a factor at most one and leaves n and n' unchanged; hence weights that are exactly 0 or 1 at a point remain so after lowering. In (1') the deficits along I are affine, which interpolates between the node or vertex and Ω_e .

The set $\mathcal{N}_P$ contains $\mathbb{D}_P$ (Appendix D.4), and by (2) only the origin and the monomials of P have positive weight on it. So the node neighbourhoods of distinct nodes $P \neq P'$ are disjoint: P' has a vertex $v \notin P$; on $\mathcal{N}_{P'}$ the monomial v has weight 1, while on $\mathcal{N}_P$ every monomial other than 0 and those of P has weight 0. This holds also for two node parallelograms that share an edge, as in an A_2 subdivision of a hexagon.

7.5. The node piece and the orientation of the vanishing sphere. In this subsection c is in the window and $P = abcd$ is a node parallelogram. We describe W^* near the node point as a space fibred by circles over a three-dimensional model base, identify the model base with a neighbourhood of the node of B_Y , and exhibit a sphere isotopic to δ_q as the boundary of a four-chain lying over a standard half-plane, the node piece. The orientation of δ_q is then fixed so that this boundary is $\rho_q \delta_q$. Section 8 uses the node pieces with multiplicities. We work on W^* in the coordinates of Definition 7.13.

The circle action. On the node neighbourhood $\mathcal{N}_P$ of Section 7.3, by Lemma 7.18(2),

$$W^* \cap \mathcal{N}_P = \{g_P + \mathfrak{b}ZZ' = 0\}, \quad g_P := 1 + \tilde{\chi}_1 + \tilde{\chi}_2 + \mu_P \tilde{\chi}_1 \tilde{\chi}_2 = xy - \kappa_P.$$

Let S_α^1 act by $(Z, Z') \mapsto (e^{i\vartheta}Z, e^{-i\vartheta}Z')$, $\vartheta \in \mathbb{R}$; it is the circle of the cocharacter $\pm(u' - u)$ of T , the common vector of the monodromies of the four legs at the node of B_Y (Proposition 4.10(3)). It preserves $|Z|$ and $|Z'|$, hence all weights, and every monomial surviving on $\mathcal{N}_P$ has equal exponents in Z and Z' . So S_α^1 acts on $W^* \cap \mathcal{N}_P$. Put $\hat{\mu} := |\hat{Z}|^2 - |\hat{Z}'|^2$.

Lemma 7.19 (the quotient). *The map $\pi: (Z, Z', \chi_1, \chi_2) \mapsto (\hat{\mu}, \chi_1, \chi_2)$ identifies $(W^* \cap \mathcal{N}_P)/S_\alpha^1$ with a subset of $\mathbb{R} \times (\mathbb{C}^*)^2$. Its fibre over $(\hat{\mu}, \chi_1, \chi_2)$ is a circle, except when $\hat{\mu} = 0$ and $g_P(\chi_1, \chi_2) = 0$, where it is a point.*

Proof. Given (χ_1, χ_2) and $\hat{\mu}$, the equation fixes $\hat{Z}\hat{Z}' = -g_P/\vartheta_P$, and $|\hat{Z}|^2 - |\hat{Z}'|^2 = \hat{\mu}$ then fixes $|\hat{Z}|$ and $|\hat{Z}'|$. The solutions form one S_α^1 -orbit, which is a point exactly when $\hat{Z} = \hat{Z}' = 0$. $\square$

The node model. The node model base is $\mathfrak{B}_q = \mathbb{R}^3$ with coordinates

$$s_1 := \log |\mu_P \tilde{\chi}_1|, \quad s_2 := \log |\mu_P \tilde{\chi}_2|, \quad \hat{\mu},$$

and $\pi_q := (s_1, s_2, \hat{\mu}): W^* \cap \mathcal{N}_P \rightarrow \mathfrak{B}_q$. The fibre angles are $\varphi_1 = \arg \tilde{\chi}_1$, $\varphi_2 = \arg \tilde{\chi}_2$ and the angle ϑ_α of S_α^1 . The cell structure at the corresponding node of B_Y is described by the tropical coordinates (y_1, y_2, α_q) below. Lemma 7.20(4) compares these coordinates with the scaled logarithmic coordinates of the hypersurface.

Lemma 7.20 (the node model). (1) *The moduli of the four monomials of P are proportional to μ_P , e^{s_1} , e^{s_2} , $e^{s_1+s_2}$. When $|s_1|, |s_2| > |\eta_P|$, the largest monomial is a , b , c or d according as the signs of (s_1, s_2) are $(-, -)$, $(+, -)$, $(+, +)$ or $(-, +)$.*
 (2) *$\hat{\mu} < 0$ exactly where $|\hat{Z}| < |\hat{Z}'|$, that is towards the divisor D_u , and $\hat{\mu} > 0$ towards $D_{u'}$.*

- (3) The S_α^1 -orbits collapse exactly where $\hat{Z} = \hat{Z}' = 0$, equivalently where $\hat{\mu} = 0$ and $g_P = 0$. Their (s_1, s_2) -coordinates lie in the amoeba $\mathcal{A}_P := \{(s_1, s_2) : g_P = 0 \text{ for some phases}\}$, which satisfies

$$\mathcal{A}_P \subseteq \{|s_1| \leq |\eta_P| + \log 3\} \cup \{|s_2| \leq |\eta_P| + \log 3\}.$$

- (4) (tropical chart) Let m_P be the midpoint of J_P , and put

$$y_1 := \text{val}_b - \text{val}_a, \quad y_2 := \text{val}_d - \text{val}_a, \quad \alpha_q := \langle \cdot - m_P, w_u - w_{u'} \rangle,$$

and $T_q := (y_1, y_2, \alpha_q) : M_{\mathbb{R}} \rightarrow \mathbb{R}^3$. There is $\varrho_q > 0$ such that T_q maps a neighbourhood of m_P in B_Y piecewise linearly and homeomorphically onto $[-\varrho_q, \varrho_q]^3$. It maps $\text{int } F_a^*, \dots, \text{int } F_d^*$ into the open quadrants of (y_1, y_2) with the sign patterns of (1), the two-faces $F_{ab}^*, F_{bc}^*, F_{cd}^*, F_{da}^*$ into the half-planes $\{y_1 = 0, y_2 < 0\}, \{y_2 = 0, y_1 > 0\}, \{y_1 = 0, y_2 > 0\}, \{y_2 = 0, y_1 < 0\}$, and the edge $F_P^* = J_P$ into the α_q -axis. At a point x of $W^* \cap \mathcal{N}_P$ in the dense torus,

$$T_q(\text{Log } x/L) = (s_1, s_2, \log |\hat{Z}| - \log |\hat{Z}'|)/L + O(1/L).$$

Proof. (1) Relative to $\gamma_a x^a$ the four monomials are $1, \tilde{\chi}_1, \tilde{\chi}_2, \mu_P \tilde{\chi}_1 \tilde{\chi}_2$, of moduli μ_P^{-1} times $\mu_P, e^{s_1}, e^{s_2}, e^{s_1+s_2}$, and $|\log \mu_P| = |\eta_P|$. Comparison of their logarithms gives the four labels whenever both $|s_i| > |\eta_P|$. (2) is the definition of $\hat{\mu}$, with $Z = Z_u$ and $Z' = Z_{u'}$ up to units. (3) The orbit collapses exactly where $\hat{Z} = \hat{Z}' = 0$, which on W^* means $g_P(\chi_1, \chi_2) = 0$. Outside the two strips in (3), one of the four monomials has modulus strictly greater than three times the modulus of each of the other three, by the same logarithmic comparison. The triangle inequality then excludes a zero of g_P . (4) The linear parts $b - a, d - a, w_u - w_{u'}$ of the three coordinates and the linear part a of val_a are linearly independent, since $(b - a, d - a, w_u, w_{u'})$ is a basis of N and $a = -w_u - w_{u'}$. At m_P only the origin and the monomials of P are maximal (Lemma 7.7(2); the other lattice points have positive deficit there), and $H(a) + H(c) = H(b) + H(d)$ because H is linear on the cone over P , so $\text{val}_c = \text{val}_a + y_1 + y_2$. Hence near m_P

$$B_Y = \{\max(\text{val}_a, \text{val}_b, \text{val}_c, \text{val}_d) = 0\} = \{\text{val}_a = -\max(0, y_1, y_2, y_1 + y_2)\},$$

the graph of a piecewise linear function of (y_1, y_2, α_q) ; this gives the homeomorphism, and the faces are read off from which of $\text{val}_a, \dots, \text{val}_d$ attain the maximum. For the last statement: $\chi_1 = \chi^{b-a}$ and $|\gamma_b/\gamma_a| = t^{H(b)-H(a)}|c_b/c_a|$, so $\log |\mu_P \tilde{\chi}_1| = L y_1 (\text{Log } x/L) + O(1)$, and likewise for s_2 . Further $\log |\hat{Z}| - \log |\hat{Z}'| = \langle \text{Log } x, w_u - w_{u'} \rangle - \log(\nu_Z/\nu_{Z'})$ and $\nu_Z/\nu_{Z'} = |A'|_{\text{mon}}/|A|_{\text{mon}}$. By Lemma 7.7(2) the apexes of ω_1 and of ω_2 have the same deficit $\mathfrak{g}_P/2$ at m_P ; since the terms of A carry Z and those of A' carry Z' , this says $\log |A'|_{\text{mon}} - \log |A|_{\text{mon}} = L \langle m_P, w_u - w_{u'} \rangle + O(1)$ at the node point, whose torus coordinates are those of m_P up to $O(1)$. $\square$

The four *standard halves* of the node model base are $\{s_1 = 0, \hat{\mu} \geq 0\}$ and $\{s_1 = 0, \hat{\mu} \leq 0\}$ (the $b - a$ passage, along the legs over ab and dc) and $\{s_2 = 0, \hat{\mu} \geq 0\}$ (the $c - b$ passage, along bc and da). The corresponding standard halves in the tropical chart are the sets $T_q^{-1}(\{y_1 = 0, \pm \alpha_q \geq 0\})$ and $T_q^{-1}(\{y_2 = 0, \pm \alpha_q \geq 0\})$: the parts of $F_{ab}^* \cup F_P^* \cup F_{cd}^*$, respectively of $F_{bc}^* \cup F_P^* \cup F_{da}^*$, near m_P on one side of $\{\alpha_q = 0\}$. In Section 8 the discriminant of B_Y is moved so that its node at P is m_P and its legs leave m_P along $\{\alpha_q = 0\}$ (Lemma 8.5), and the transferred cycle is put in a normal form in which, near m_P , it is an integral combination of these four sets. The node piece below is the lift of one standard half.

The surface $\mathcal{C}$ and the node piece. Consider the $b - a$ passage. On its legs $|\mu_P \tilde{\chi}_1| = 1$, and there the curve $\{g_P = 0\}$ follows the branch $\{x = 0\}$. Write $\kappa_P = |\kappa_P|e^{i\phi}$ and let

$$\mathcal{T}_q := \{(x, y) : y = e^{i\phi} \bar{x}, |x|^2 \leq |\kappa_P|\}$$

be the Lefschetz thimble of $xy - \kappa_P$ at its critical point for the straight path of values, a disc bounded by the circle $\{xy = \kappa_P, |x| = |y|\}$. Let $\mathcal{C}_P$ be the part of $\{g_P = 0\}$ over $\mathcal{N}_P$ tending to the branch $\{x = 0\}$: near the node point it is $\{xy = \kappa_P, |x| \leq |y|\}$, and outside the Morse bidisc it is the curve $\{g_P = 0\}$ along the two legs of the passage. Put $\mathcal{C} := \mathcal{C}_P \cup \mathcal{T}_q$. The coordinate y identifies $\mathcal{C}$ with $\mathbb{C} \setminus \{\mu_P^{-1/2}\} \cong \mathbb{C}^*$, the omitted point being $\tilde{\chi}_2 = 0$, and its two ends run along the two legs.

Definition 7.21 (the node piece). Fix $\theta' \in (0, \theta_2]$, and let $\bar{\mu} > 0$ be a smooth function of (χ_1, χ_2) with $\bar{\mu} \leq t^{-2\theta_P}/4$. Replace $\mathcal{C}$ by its part over the points (χ_1, χ_2) within θ' of the node point, still denoted $\mathcal{C}$. Put

$$\Pi_q^+ := \{p \in W^* \cap \mathcal{N}_P : (\chi_1, \chi_2)(p) \in \mathcal{C}, 0 \leq \hat{\mu}(p) \leq \bar{\mu}(\chi_1, \chi_2)\},$$

and define Π_q^- in the same way with $-\bar{\mu} \leq \hat{\mu} \leq 0$. For the $c - b$ passage replace the branch $\{x = 0\}$ by $\{y = 0\}$, and parametrise $\mathcal{C}$ by x . Section 8 takes $\theta' = \theta_4$ and a specific $\bar{\mu}$.

Lemma 7.22 (the node piece). (1) Over $\mathcal{C}_P$, Π_q^+ is the divisor piece $W^* \cap D_{u'} \cap \{(\chi_1, \chi_2) \in \mathcal{C}_P, |\hat{Z}|^2 \leq \bar{\mu}\} = \{Z' = 0, (\chi_1, \chi_2) \in \mathcal{C}_P, |\hat{Z}|^2 \leq \bar{\mu}(\chi_1, \chi_2)\}$, and Π_q^- is $W^* \cap D_u$ over $\mathcal{C}_P$, with $|\hat{Z}'|^2 \leq \bar{\mu}$.
(2) The map $(y, \hat{\mu}, \vartheta_\alpha) \mapsto p$, with $\vartheta_\alpha = \arg \hat{Z}$, is a homeomorphism from $\{(y, \hat{\mu}) : y \in \mathcal{C}, 0 < \hat{\mu} \leq \bar{\mu}(y)\} \times S^1$ onto $\Pi_q^+ \cap \{\hat{\mu} > 0\}$ which restricts to diffeomorphisms over the two closed pieces $\mathcal{C}_P$ and $\mathcal{T}_q$ of $\mathcal{C}$; likewise for Π_q^- , with $\vartheta_\alpha = -\arg \hat{Z}'$ and $\hat{\mu} \in [-\bar{\mu}, 0)$. In both cases ϑ_α is the angle of S_α^1 .
(3) $\Pi_q^\pm$ is a topological four-manifold, piecewise smooth, whose boundary consists of the outer part $\{\hat{\mu} = \pm \bar{\mu}\}$, the ends over the two legs, and the three-sphere

$$\Sigma_q := \pi^{-1}(\{0\} \times \mathcal{T}_q) = \{(x, y) \in \mathcal{T}_q, |\hat{Z}| = |\hat{Z}'|, \vartheta_P \hat{Z} \hat{Z}' = \kappa_P - xy\}.$$

Over $\{0\} \times \mathcal{C}_P$ the circles collapse without creating boundary.

(4) Σ_q is isotopic in $W^* \cap \mathbb{D}_P$ to the vanishing sphere Σ_0 of Lemma 7.17.

Proof. (1) On $\mathcal{C}_P$ we have $g_P = 0$, so $\mathbf{b}ZZ' = 0$; with $\hat{\mu} \geq 0$ this forces $Z' = 0$ and $|\hat{Z}|^2 = \hat{\mu}$. Conversely every point of $W^* \cap D_{u'}$ over $\mathcal{C}_P$ has $\hat{\mu} = |\hat{Z}|^2 \geq 0$.

(2) By Lemma 7.19, for $\hat{\mu} > 0$ the fibre of π over $(\hat{\mu}, \chi_1, \chi_2)$ is a free S_α^1 -orbit on which $\hat{Z} \neq 0$, containing exactly one point with $\arg \hat{Z} = \vartheta_\alpha$ for each ϑ_α . The point of $\mathcal{C}$ with coordinate y is $(\kappa_P/y, y)$ over $\mathcal{C}_P$ and $(e^{i\phi}\bar{y}, y)$ over $\mathcal{T}_q$; these agree on the neck circle $|y|^2 = |\kappa_P|$, so the map is continuous, and it is smooth on each closed piece but only continuous across the neck. The inverse is continuous, and smooth over each piece, because $|\hat{Z}|$ and $|\hat{Z}'|$ are smooth functions of $\hat{\mu}$ and $|\hat{Z}\hat{Z}'| = |g_P|$ where $|\hat{Z}|^2 \geq \hat{\mu} > 0$. For $\hat{\mu} < 0$ the same holds with $\hat{Z}' \neq 0$ and $\vartheta_\alpha = -\arg \hat{Z}'$, since S_α^1 acts on $\hat{Z}'$ by $e^{-i\vartheta}$.

(3) Over $\mathcal{C}_P$, by (1), Π_q^+ is the bundle over $\mathcal{C}_P$ of the closed discs $|\hat{Z}|^2 \leq \bar{\mu}$, and the points with $\hat{\mu} = 0$ are the centres of the discs, which are interior. Over the interior of $\mathcal{T}_q$ we have $g_P \neq 0$, so $Z, Z' \neq 0$, and Π_q^+ is a circle bundle over $\{(y, \hat{\mu}) : y \in \mathcal{T}_q, 0 \leq \hat{\mu} \leq \bar{\mu}(y)\}$ whose boundary over $\hat{\mu} = 0$ is the circle bundle over $\mathcal{T}_q$ collapsing over $\partial\mathcal{T}_q$, a three-sphere. The two descriptions meet over $\partial\mathcal{T}_q$, where $\mathcal{C}$ has a corner; this is the only failure of smoothness.

(4) With $z_1 := \hat{Z}$ and $z_2 := \vartheta_P \hat{Z}'$, holomorphic coordinates on $\mathbb{D}_P$, we have $f^{(0)} = xy + z_1 z_2 - \kappa_P$ exactly, and $\Sigma_q = \{(x, y) \in \mathcal{T}_q, z_1 z_2 = \kappa_P - xy, |z_1| = |z_2|\}$. On $\mathcal{T}_q$, $\kappa_P - xy = e^{i\phi}(|\kappa_P| - |x|^2)$, so $|z_1|^2 = |\kappa_P| - |x|^2$ and $z_2 = e^{i\phi} \bar{z}_1$. Hence

$$\Sigma_q = \Sigma^1 := \{y = e^{i\phi} \bar{x}, z_2 = e^{i\phi} \bar{z}_1, |x|^2 + |z_1|^2 = |\kappa_P|\},$$

the standard vanishing sphere of $xy + z_1 z_2 - \kappa_P$ for the straight path, which is Σ_0 up to isotopy in $W^* \cap \mathbb{D}_P$. $\square$

The coordinate $\tilde{\chi}_2$ identifies $\mathcal{C}$ with an annulus in $\mathbb{C}^*$. Away from the Morse bidisc, the fibre of Π_q^+ over a point $(s_2, \hat{\mu})$ of the standard half with $\hat{\mu} > 0$ is the circle $\{|\tilde{\chi}_2| = \text{const}\}$ of $\mathcal{C}$ times the S_α^1 -orbit: the affine two-torus of [CBM14, Lemma 5.14], spanned by S_α^1 and the rotation of $\tilde{\chi}_2$. Over $\mathcal{C}_P$, Π_q^+ is the set of points of $W^* \cap D_{u'}$ over $\mathcal{C}_P$ with $|\hat{Z}|^2 \leq \bar{\mu}$, and Π_q^- the set of points of $W^* \cap D_u$ with $|\hat{Z}'|^2 \leq \bar{\mu}$. Section 8 chooses $\bar{\mu}$ so that, along the two legs of the passage, the node piece agrees with the divisor pieces defined there by the same inequality where the two overlap. Figure 3 shows the node piece schematically.

Orientations. The torus fibre over a point of an open facet $\text{int } F_n^*$ of B_Y is $\Lambda_Y \otimes U(1)$ with $\Lambda_Y = n^\perp \cap M$ (Proposition 4.11). The coefficients of a transferred cycle on B_Y are flat sections of the dual local system $\Lambda_Y^\vee$ (Section 4). For an oriented sheet S_0 in the complement of the discriminant, carrying a primitive covector v^* , put $\Lambda_S := \ker v^* \subset \Lambda_Y$; the torus $\mathbb{T}(v^*) := \Lambda_S \otimes U(1)$ is oriented by $\text{or}(\Lambda_S) \wedge n = \text{or}(B_Y)$ for any n with $v^*(n) > 0$, and the lift of S_0 is oriented as $S_0 \times \mathbb{T}(v^*)$. We identify base directions with fibre directions through the complex structure: for a character χ that is a coordinate, the base function $\log |\chi|$ corresponds to the fibre angle $\arg \chi$. In the node model the pairs are (s_1, φ_1) , (s_2, φ_2) and $(\hat{\mu}, \vartheta_\alpha)$; the last is oriented like the complex coordinate $\hat{Z}$ where $Z' = 0$. Let $o_q^{\text{ch}} \in \{\pm 1\}$ be the sign of the chart T_q of Lemma 7.20(4) relative to the orientation of B_Y . It is also the sign of the chart $(s_1, s_2, \hat{\mu})$ transported by Lemma 7.20(4): s_1, s_2 and $\log |\hat{Z}| - \log |\hat{Z}'|$, which has the sign of $\hat{\mu}$, are L times the coordinates of T_q up to $O(1)$.

The node coefficients are those of Definition 2.6; the transfer to B_Y leaves them unchanged (Section 6). On B_Y they are read off as follows: with the boundary coefficients oriented away from q , the covector is $c_{ab}\langle \cdot, b - a \rangle$ on the leg over ab and $c_{bc}\langle \cdot, c - b \rangle$ on the leg over bc , and $\rho_q = c_{ab} - c_{bc}$.

Definition 7.23 (orientation of δ_q). Orient the sphere Σ^1 of Lemma 7.22(4) by o_q^{ch} times the orientation of its projection to $\mathbb{C}_{(x, z_1)}^2$, the round sphere $|x|^2 + |z_1|^2 = |\kappa_P|$ oriented as the boundary of the ball. Orient Σ_q, Σ_0 and, through Lemma 7.17, δ_q accordingly.

The convention is formed from the labelling of P , the order of u, u' and the vector w_u ; of these it depends only on the diagonal $\{a, c\}$ of P chosen by the labelling.

- (i) The relabelling $abcd \rightarrow adcb$, which keeps the diagonal, exchanges y_1 and y_2 , hence reverses o_q^{ch} , and exchanges x and y ; it leaves κ_P, z_1, z_2 and the set Σ^1 unchanged. On Σ^1 we have $y = e^{i\phi} \bar{x}$, and $(x, z_1) \mapsto (e^{i\phi} \bar{x}, z_1)$ reverses the orientation of $\mathbb{C}^2$ and preserves the round ball, so the projections to (x, z_1) and to (y, z_1) induce opposite orientations. The two sign changes cancel.
- (ii) Exchanging the order of u, u' , with the old $w_{u'}$ as the new w_u , exchanges Z and Z' and reverses α_q , hence o_q^{ch} . It leaves $f^{(0)}$ and Σ^1 unchanged, and the new z_1 is $\bar{\vartheta}_P z_2 = \bar{\vartheta}_P e^{i\phi} \bar{z}_1$ on Σ^1 ; as in (i), the projection orientation is reversed, and the two sign changes cancel.

- (iii) Replacing w_u by $w_u + k$, with k a lattice vector of the plane of G , changes α_q by $2\langle \cdot, k \rangle$, a linear combination of y_1 and y_2 up to a constant, so the chart changes by a linear map of determinant one and o_q^{ch} is unchanged. It multiplies z_1 by a holomorphic unit and z_2 by its inverse; the differential of this change at the critical point is complex linear in z_1 , and the tangent planes of the two thimbles there lie in the connected family of maximal positive planes of the proof of Lemma 7.26, so the orientation is unchanged.

Together with Lemma 7.26, which treats the relabelling $abcd \rightarrow bcda$, these determine the effect of every relabelling of P , since the two relabellings generate the symmetry group of the parallelogram. The convention does not depend on the passage: the two passages have the same thimble $\mathcal{T}_q$ and the same sphere Σ_q . Changing the chosen diagonal of P reverses it (Lemma 7.26).

Lemma 7.24 (sign independence). *Let a sheet near q be one of the four standard halves, with either orientation, carrying the covector $\pm\langle \cdot, b - a \rangle$ for the $b - a$ passage or $\pm\langle \cdot, c - b \rangle$ for the $c - b$ passage, and let Π_q be the corresponding node piece, oriented as the lift of the sheet. Then the part of $\partial\Pi_q$ over $\mathcal{T}_q$ is $\rho_q \Sigma_q$, where $\rho_q = \pm 1$ is the node coefficient of the boundary of the sheet.*

Proof. Reductions. Replacing the orientation of B_Y by that of the chart multiplies both the lift and Σ_q by o_q^{ch} , so we may assume $o_q^{\text{ch}} = 1$; then $\Sigma_q = \Sigma^1$ (Lemma 7.22(4)) carries the orientation $[S^3]$ of its projection to (x, z_1) . Reversing the orientation of the sheet reverses its lift and its boundary, hence the sign of ρ_q ; changing the sign of the covector reverses $\mathbb{T}(v^*)$, hence the lift, and changes the sign of every coefficient on the boundary, hence of ρ_q . So both sides change sign together, and it suffices to treat, for each of the four halves, the sheet with the frame written below and the covector $+\langle \cdot, b - a \rangle = ds_1$, respectively $+\langle \cdot, c - b \rangle = \langle \cdot, d - a \rangle = ds_2$.

Orientation classes are written as ordered frames; (y) stands for the complex orientation of the coordinate y . Since $s_2 + i\varphi_2 = \log(\mu_P \tilde{\chi}_2)$ is a holomorphic function of y , $(s_2, \varphi_2) = (y)$ as orientations, and likewise $(s_1, \varphi_1) = (x)$. On Σ^1 the projection to (x, z_1) is $(y, \vartheta_\alpha) \mapsto (e^{i\phi} \bar{y}, (|\kappa_P| - |y|^2)^{1/2} e^{i\vartheta_\alpha})$ in the $b - a$ passage and $(x, \vartheta_\alpha) \mapsto (x, (|\kappa_P| - |x|^2)^{1/2} e^{i\vartheta_\alpha})$ in the $c - b$ passage. The second map has degree +1 onto $[S^3]$: at $x = 0$ its differential sends $(\partial_{x_1}, \partial_{x_2}, \partial_{\vartheta_\alpha})$ to $(\partial_{x_1}, \partial_{x_2}, |\kappa_P|^{1/2} \partial_{\text{Im } z_1})$, and $(\partial_{\text{Re } z_1}, \partial_{x_1}, \partial_{x_2}, \partial_{\text{Im } z_1})$ is a positive frame of $\mathbb{C}^2$ with the outward normal first. Since $y \mapsto e^{i\phi} \bar{y}$ reverses orientation, the first map has degree -1 . Over $\mathcal{T}_q$, the angle $\vartheta_\alpha = -\arg \hat{Z}'$ used where $\hat{\mu} < 0$ differs from $\arg \hat{Z}$ by the phase of $\hat{Z}\hat{Z}'$, a function of y (resp. x) alone, so these degrees hold for both halves.

The $b - a$ passage, half $\hat{\mu} \geq 0$, frame $(s_2, \hat{\mu})$. With the outward normal $-\partial_{\hat{\mu}}$ first, the boundary has the direction ∂_{s_2} : it arrives at q along the leg over ab (where $s_2 < 0$), with away-oriented coefficient $c_{ab} = -1$, and leaves along dc ; so $\rho_q = -1$. The kernel of ds_1 is spanned by the directions of s_2 and $\hat{\mu}$, and $(s_2, \hat{\mu}, s_1)$ is an even permutation of $(s_1, s_2, \hat{\mu})$, so $\mathbb{T}(v^*)$ is oriented by $(\varphi_2, \vartheta_\alpha)$ and the lift by $(s_2, \hat{\mu}, \varphi_2, \vartheta_\alpha) = -(y, \hat{\mu}, \vartheta_\alpha)$; by Lemma 7.22(2) the coordinates $(y, \hat{\mu}, \vartheta_\alpha)$ cover $\Pi_q^+ \cap \{\hat{\mu} > 0\}$. Since $(-\partial_{\hat{\mu}}, y, \vartheta_\alpha) = -(y, \hat{\mu}, \vartheta_\alpha)$, the boundary at $\hat{\mu} = 0$ is oriented by (y, ϑ_α) , that is, by $-[S^3] = \rho_q \Sigma_q$.

The $b - a$ passage, half $\hat{\mu} \leq 0$, frame $(s_2, \hat{\mu})$. The outward normal is $+\partial_{\hat{\mu}}$, so the boundary has the direction $-\partial_{s_2}$ and $\rho_q = +1$. The lift is $-(y, \hat{\mu}, \vartheta_\alpha) = -(\partial_{\hat{\mu}}, y, \vartheta_\alpha)$, so the boundary is $-(y, \vartheta_\alpha) = [S^3] = \rho_q \Sigma_q$.

The $c - b$ passage, half $\hat{\mu} \geq 0$, frame $(s_1, \hat{\mu})$. The legs are over da (where $s_1 < 0$) and bc ; the boundary has the direction ∂_{s_1} , so it leaves along bc with $c_{bc} = 1$ and $c_{ab} = 0$, and

$\rho_q = -1$. Since $(s_1, \hat{\mu}, s_2)$ is an odd permutation of $(s_1, s_2, \hat{\mu})$, $\mathbb{T}(v^*)$ is oriented by $-(\varphi_1, \vartheta_\alpha)$ and the lift by $-(s_1, \hat{\mu}, \varphi_1, \vartheta_\alpha) = (x, \hat{\mu}, \vartheta_\alpha)$. With the outward normal $-\partial_{\hat{\mu}}$ the boundary is $-(x, \vartheta_\alpha) = -[S^3] = \rho_q \Sigma_q$.

The $c - b$ passage, half $\hat{\mu} \leq 0$, frame $(s_1, \hat{\mu})$. Now $\rho_q = +1$, the lift is $(x, \hat{\mu}, \vartheta_\alpha)$, and the outward normal $+\partial_{\hat{\mu}}$ gives the boundary $(x, \vartheta_\alpha) = [S^3] = \rho_q \Sigma_q$. $\square$

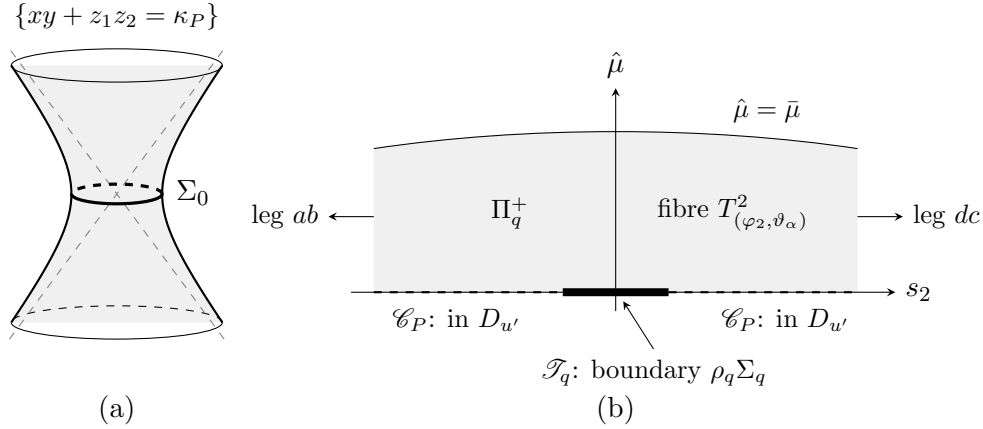


FIGURE 3. The Morse chart at a node. (a) Schematic real picture of $W^* \cap \mathbb{D}_P = \{xy + z_1 z_2 = \kappa_P\}$, two dimensions lower: in coordinates ζ with $xy + z_1 z_2 = \sum_i \zeta_i^2$, the vanishing sphere Σ_0 is the waist $\{\zeta \in \kappa_P^{1/2} \mathbb{R}^4 : \sum_i \zeta_i^2 = \kappa_P\}$. It transports to a sphere isotopic to δ_q (Lemma 7.17); the dashed lines are the cone of the singular fibre $\kappa_P = 0$. (b) The node piece Π_q^+ of the $b - a$ passage over the standard half $\{s_1 = 0, \hat{\mu} \geq 0\}$, drawn in the coordinates $(s_2, \hat{\mu})$. Over the interior the fibre is a two-torus (the angle φ_2 of $\tilde{\chi}_2$ and the angle ϑ_α of S_α^1). Over $\hat{\mu} = 0$ the circles S_α^1 collapse along $\mathcal{C}_P$ (dashed), where Π_q^+ lies in the divisor $D_{u'}$ and has no boundary; over the thimble $\mathcal{T}_q$ (thick) they do not, and the boundary there is the three-sphere Σ_q , which transports to a sphere isotopic to δ_q , with multiplicity ρ_q (Lemma 7.24).

Remark 7.25 (the sign of κ_P). The window allows both signs of η_P , hence of κ_P , and the proof above covers every argument ϕ of κ_P . The convention of Definition 7.23 is also compatible with moving κ_P around 0. In the family $\{xy + z_1 z_2 = \kappa\}$, $\kappa = |\kappa|e^{i\phi}$, the sphere $\Sigma^1(\kappa)$ is carried along a path of κ by the complex-linear maps $(x, y, z_1, z_2) \mapsto e^{i\phi/2}(x, y, z_1, z_2)$, which preserve the orientation of the projection to (x, z_1) . A full turn of κ acts on H_3 of the fibre by the Picard–Lefschetz transformation [Lam81], which fixes $[\delta]$, because $\langle \delta, \delta \rangle = 0$ for the skew intersection form on H_3 of a six-manifold. So the orientations of the δ_q for different coefficient vectors in $U_{\mathbb{C}}$ agree under transport.

Lemma 7.26 (change of diagonal). *Replace the labelling $abcd$ of P by $bcda$, which chooses the other diagonal, and keep the order of u, u' . Then σ_q^{ch} is unchanged, and the orientation of δ_q given by Definition 7.23 is reversed. Since circ_q , the coordinate λ_q of K and ρ_q change sign as well, $\rho_q \delta_q$ and the classes $\sum_q \lambda_q [\delta_q]$, $\lambda \in K$, do not depend on the choice of diagonals.*

The proof is in Appendix D.

8. THE LIFT OF RELATIVE TROPICAL 2-CYCLES

This section proves Theorem D. Fix an admissible Δ with a profile σ satisfying (Proj), the configuration of Section 7.1, a triangulation $\mathfrak{K}$ of $B = B'_\sigma$ in which Γ is full, and a relative tropical 2-cycle $z \in Z(\mathfrak{K})$. We construct an integral 4-chain on the localized mirror W^* of Definition 7.10 whose boundary is $\sum_q \rho_q(z) \Sigma_q$, where $\Sigma_q \subset W^*$ is a three-sphere isotopic to the vanishing sphere, and carry it to Y_t by the isotopy of Proposition 7.12. The construction has four steps.

- (1) *Transfer and normal form* (Section 8.1). The cycle z is carried to the mirror base B_Y by the label-preserving cellular homeomorphism of Section 6, and the discriminant of B_Y is moved onto the positions over which W^* has exact local models (the *model complex*). Near the discriminant the transferred cycle is then replaced, without changing its boundary network, by a sum of finitely many fixed surfaces with integer coefficients: the *model germs* at the vertices of the discriminant and along its arcs, taken with the multiplicities read off from the boundary network, and one transverse disc on each arc, with an arbitrary monodromy-invariant coefficient. What is left lies at positive distance from the discriminant. This is the normal form of Lemma 5.11, applied on B_Y , and it is the only place where z enters: near the discriminant only its boundary network and the coefficients of the transverse discs remain.
- (2) *Pieces* (Sections 8.2–8.4). Each model germ and each transverse disc has an explicit lift near the discriminant, a four-dimensional *piece* of W^* , made of subsets of toric divisors, of the conifold smoothing at a node, and of torus orbits through the positive real locus. At a node the pieces of the two passages contain the same sphere Σ_q , and Lemma 7.24 computes their boundary there.
- (3) *The regular part* (Sections 8.5 and 8.6). The rest of the normalised chain is lifted over an abstract 2-complex $\mathcal{S}$: the disjoint union of its *summands* (the germs, the discs, the remainder), glued only along the curves where their boundaries cancel. Over $\mathcal{S}$ the lift is a bundle of two-tori with integer multiplicities, completed by 3-chains in the fibres over the edges where several summands meet. Summands that meet transversally in B_Y are not identified in $\mathcal{S}$; their lifts are separate chains, which add in W^* . The map to W^* is built cell by cell in regions of W^* that are aspherical, with fundamental group the local lattice of the base.
- (4) *The fibre class* (Proposition 8.26). The boundary of the resulting chain is $\sum_q \rho_q \Sigma_q$ plus closed 3-cycles in fibres over finitely many points. Each is a multiple of the class of a fibre three-torus, and pairing with the positive real locus of W^* , which meets a fibre torus once and misses every Σ_q , shows that the multiples sum to zero.

The construction follows the local lifts, torus bundles and edge fillings of Castaño-Bernard and Matessi [CBM14, §7]. Here the exact local equations occur at prescribed positions, and relative cycles may have arbitrary multiplicities and invariant crossing coefficients. Moving the discriminant and putting the cycle in normal form address the first issue; Proposition 8.26 removes the residual fibre boundary in the second. The principal objects used below are as follows.

term	meaning
model complex	the base cell complex after the discriminant has been moved to the positions of the exact local equations
model vertex	a positive-type vertex of B_Y , the image of a negative-type vertex of B'_σ

turning point	a negative-type vertex of B_Y , the image of a positive-type vertex of B'_σ
model germ	one of the fixed local surfaces, with its coefficient, used in the normal form near the discriminant
summand	a connected part with constant total coefficient, with the tripod convention of Definition 8.12
junction	a curve along which boundaries of adjacent summands cancel; transverse intersections are not junctions
piece and collar	a local lift near the discriminant and its connection to the lift of the regular part
region	an open subset of the localized mirror homotopy equivalent to the torus of the local lattice

For strict cycles, the local models have multiplicity 0 or ± 1 . Under the position conditions of Remark 8.29, their correspondence with [CBM14] is given by the following table.

part of the lift	constructed in	counterpart in [CBM14]
over the sheets: torus bundle	Section 8.6	[CBM14, §7, (46)]
over interior edges	Section 8.6	[CBM14, §7.7, Step 2]
node piece	Lemmas 7.22, 8.15	[CBM14, Lemma 5.14]
pieces at the model vertices	Lemma 8.14(2)	[CBM14, §7.5; §7.7, Steps 2–3]
pieces at turning points	Lemma 8.14(3)	[CBM14, Lemma 7.6]
pieces at crossings	Lemma 8.16(2)	[CBM14, §7.7, Step 6]
interior vertices	Proposition 8.26	replaces [CBM14, §7.7, Step 5]

Throughout the section c is a real coefficient vector with $c_0 > 0 > c_n$ for $n \neq 0$, in the window of Definition 7.3, with $r \leq r_1$ and $\eta_0 \leq r^2/20$, and $t < t_0(r, \eta_0, \mathcal{C}_0)$; these are the members of Theorem A (Section 7.1). We use the notation of Section 7: $L = \log(1/t)$, the valuations val_n , the deficits, the weights ψ_n , the cut-off constants $\theta_0, \dots, \theta_3$ and D , the constants $e_{\max}$ and c_M , the node data of Definition 7.6, and the identification of B_Y with the union of the bounded cells of the tropical hypersurface $\text{Trop} \subset M_{\mathbb{R}}$. The lattices Λ_Y, Λ^M are lattices of integral tangent vectors of B_Y , contained in M ; relative cycles on B_Y have coefficients in the dual local system, the covectors, whose values on $\text{int } F_n^*$ are the classes in $N/\mathbb{Z}n$ of the elements of N .

8.1. Moving the discriminant to the local models; a normal form for the transferred cycle. We first move the discriminant to the positions of the exact local models and transfer the cycle to the mirror base. The normal-form lemma then separates its boundary germs, transverse discs and a remainder disjoint from the discriminant.

The faces E_e and the model positions. The local models of W^* sit over definite points of B_Y : the Morse chart over the midpoint of the edge J_P of Lemma 7.7, the models at the vertices of positive type over the segments J_ω^b of Lemma 7.18(3), and the leg models over the interior polygons Ω_e of Definition 7.9. In general the discriminant Γ_Y does not pass through these positions (Remark 8.6). We move it there by a homeomorphism of B_Y that preserves the complex of dual cell pairs of Proposition 4.7; Section 6 uses only the combinatorics of the cells of that complex, not their positions inside the faces of B_Y .

We use the faces E_e , the interior polygons Ω_e and the turning points of Definition 7.9 and the paragraph after it.

For every edge $[u, u']$ of $\mathcal{T}$ in the dual edge G° of a two-face G , or in a two-face $E^\vee$ of Δ° dual to an edge E of Δ , we take the vector $w_u \in N$ of Section 7.1, with $\langle u, w_u \rangle = 1$ and $\langle u', w_u \rangle = 0$, put $w_{u'} := -n_0 - w_u$ for a lattice point n_0 of the face $[u, u']^\vee$ of Δ , and

write $\alpha_{uu'} := \langle \cdot, w_u - w_{u'} \rangle$; at a node these are the vectors of (7.2), with $n_0 = a$. At the tropical position of a point of the torus, $L\alpha_{uu'} = \log |Z| - \log |Z'|$ for the characters $Z = \chi^{w_u}$, $Z' = \chi^{w_{u'}}$, which are the Cox coordinates of u and u' times units on the chart $U_{u,u'}$. Since $\langle u' - u, w_u - w_{u'} \rangle = -2$, on every polygon E_e with $e \subseteq [u, u']^\vee$ the level lines of $\alpha_{uu'}$ are parallel lines transverse to $u' - u$, and the edges J_ω of E_e for the two-cells $\omega \supset e$ with $\omega^\vee \ni u, u'$ are parallel to $u' - u$.

- Definition 8.1** (model positions). (1) The *model node* of a node parallelogram P is the midpoint m_P of J_P . For a unimodular triangle ω of $\mathcal{F}_C$ in a two-face G , let $u_0, \dots, u_k$ be the lattice points of G° in order, and let $J_\omega^b \subseteq J_\omega$ be the segment of Lemma 7.18(3), which contains the points $p_\omega + s(u_k - u_0)$ with $|s - s_{\text{mid}}| \leq 1/(4e_{\text{max}})$. The *model vertices* of ω are points $m_{\omega,1}, \dots, m_{\omega,k}$ of J_ω^b at mutual distances at least $4\theta_3$, in the order in which the zero-cells $C_Y([u_{i-1}, u_i], \omega)$ lie on J_ω . They exist, since that segment has length at least $k c_M / (2e_{\text{max}}) \geq 4(k-1)\theta_3$.
- (2) Let e be an edge of $\mathcal{F}_C$ in the two-skeleton with $\dim e^\vee \geq 1$, $\omega \supset e$ a two-cell of a two-face G , and m a model node or vertex of ω , with edge $[u, u']$ of $\mathcal{T}$. A *leaf path* from m is a path in E_e made of two segments $[m, m']$ and $[m', p_e]$, where $[m, m']$ lies on the level line of $\alpha_{uu'}$ through m and has positive length at most θ_3 , and $p_e \in \Omega_e$. Since $\theta_3 \leq \theta_2/4$, it is a path of Lemma 7.18(1'). Let $\theta_4 \in (0, \theta_2/2]$ be a constant, depending only on the configuration and the leaf paths, such that at every model node and vertex the points of the leaf paths whose torus coordinates are within $2\theta_4$ of those of m lie on the first segments.
- (3) A *model arc system* of e is a family of piecewise linear arcs in E_e , each meeting every line parallel to the direction $u' - u$ of its edge $[u, u']$ of $\mathcal{T}$ at most once, as follows. If e lies in the interior of a two-face G with k edges of $\mathcal{T}$ in G° , it consists of k disjoint arcs; the i -th joins the i -th model points on the two edges $J_{\omega_\pm}$ of E_e , where $\omega_\pm$ are the two two-cells of G containing e , and consists of a leaf path from each of them and an arc in Ω_e . If e lies on an edge E of Δ , it is a plane graph properly embedded in E_e , isomorphic, as a plane graph with its leaves on ∂E_e in their cyclic order, to $\Gamma_Y \cap E_e$, the graph dual to the triangulation of $E^\vee$ by $\mathcal{T}$ together with one leaf for each edge of $\mathcal{T}$ on $\partial E^\vee$ (Proposition 4.10(2)); its leaves are the model points on the edges J_ω of E_e , its leaf edges begin with leaf paths, and its interior vertices and the remaining parts of its edges lie in Ω_e , each edge dual to $[u_i, u_j]$ being transverse to $u_j - u_i$.

In (3) the graph is a tree exactly when $E^\vee$ has no lattice point in its interior; this fails on each of the examples X_9 , X_{19} and X_{20} of Section 11. When e lies in the interior of a two-face and both two-cells $\omega_\pm$ are node parallelograms, the arc joins two model nodes. This occurs at a pentagon (the edge from the centre shared by its two parallelograms) and at a hexagon on the A_2 branch, where the three rhombi share pairwise the edges $[o, w_{j_0}]$, $[o, w_{j_0+2}]$, $[o, w_{j_0+4}]$ and the three arcs between their model nodes form a cycle of Γ^M without vertices of valency three. Nothing below distinguishes these arcs from others.

Model arc systems. We construct model arc systems as homothetic copies of $\Gamma_Y \cap E_e$, joined to the model points by leaf paths. The arcs, vertices and leaves of $\Gamma_Y \cap E_e$ itself, in their actual positions, are called *original*. Fix e as in Definition 8.1(3). In the picture of Trop the face E_e is $-(e^\vee + \nabla_e^{H'})$, where $e^\vee + \nabla_e^{H'}$ is the face F_e^* of $\partial \nabla^H$ and $\nabla_e^{H'}$ is a polygon parallel to $e^\vee$. By Lemma 4.3(2) for B_Y , the cells of $\mathcal{P}_Y$ in E_e are the sets $-(F_1(\zeta) + F_2(\zeta))$, $\zeta \in N_{\mathbb{R}}$, where $F_1(\zeta)$ is the cell of $\mathcal{T}$ in $e^\vee$ on which $a \mapsto \langle a, \zeta \rangle + \check{h}(a)$ is minimal and $F_2(\zeta)$ is the face of $\nabla_e^{H'}$ on which ζ is minimal; $F_1(\zeta)$ is the lower support of the cell, and only the class of ζ

modulo $\mathbb{R}n + \mathbb{R}n'$ matters. For a cell $\mathfrak{c}$ we write $b_{\mathfrak{c}}$ for the average of its vertices, the vertex of the barycentric subdivision.

Lemma 8.2 (original arcs). *Let $\tau = [u_i, u_j]$ be an edge of $\mathcal{T}$ in $e^\vee$, and let $y \in N_{\mathbb{R}}$ satisfy $\langle u_j - u_i, y \rangle = 0$ and $y \notin \mathbb{R}n + \mathbb{R}n'$. Then the closure of the arc $C_Y(\tau, e)$ is a polygonal arc along which $\langle \cdot, y \rangle$ is strictly monotone. In particular it meets every line parallel to $u_j - u_i$ at most once.*

The proof is in Appendix E.

The construction. Fix $o_e \in \Omega_e$ and $\mathfrak{t}_e \in (0, 1/2]$ with $\mathfrak{t}_e(E_e) \subset \Omega_e$ for the homothety $\mathfrak{t}_e(p) := o_e + \mathfrak{t}_e(p - o_e)$. For an edge J of E_e let y_J be the affine function on the plane of E_e with $y_J = 0$ on J and $y_J(o_e) = 1$, so that $y_J \geq 0$ on E_e and $y_J \circ \mathfrak{t}_e = 1 - \mathfrak{t}_e + \mathfrak{t}_e y_J$. Let $S_J := \{o_e + t(b - o_e) : b \in \text{ri } J, t > 0\}$, an open convex sector; the sectors of distinct edges are disjoint. A model point m on $J = J_\omega$, with edge $[u, u']$ of $\mathcal{T}$, lies in $\text{ri } J \subset S_J$, and the level line of $\alpha_{uu'}$ through m is transverse to J . Choose $\eta_J \in (0, 1/4]$ so small that for every model point m on J the point m' of that level line with $y_J(m') = \eta_J$ satisfies $|m' - m| \leq \theta_3$ and $m' \in S_J$. The leaf path of m in E_e is $[m, m'] \cup [m', \mathfrak{t}_e(\bar{m})]$, where $\bar{m}$ is the original leaf $C_Y([u, u'], \omega)$ corresponding to m (the i -th model point to the i -th original leaf). It is a leaf path in the sense of Definition 8.1(2), since $\mathfrak{t}_e(\bar{m}) \in \Omega_e$.

Lemma 8.3 (leaf paths). *The segments of the leaf paths in E_e are pairwise disjoint, except that the two segments of one path share the point m' . Each leaf path meets ∂E_e only at its model point, and $\mathfrak{t}_e(E_e)$ only at its end $\mathfrak{t}_e(\bar{m})$.*

The proof is in Appendix E.

Lemma 8.4 (model arc systems). *For every triangle s_Δ of $\mathcal{T}$ in $e^\vee$ let $\mathfrak{c}_{s_\Delta}$ be the polygon of the proof of Lemma 8.2, let $\mathfrak{c}$ run over the three edges of $\mathfrak{c}_{s_\Delta}$ at which the closed arcs $\overline{C_Y(\tau, e)}$, $\tau \subset s_\Delta$, end with the segments $[b_{\mathfrak{c}}, b_{\mathfrak{c}_{s_\Delta}}]$, and let v_{s_Δ} be the centroid of the triangle with vertices the three points $b_{\mathfrak{c}}$. Let Γ'_e be $\Gamma_Y \cap E_e$ with each segment $[b_{\mathfrak{c}}, b_{\mathfrak{c}_{s_\Delta}}]$ replaced by $[b_{\mathfrak{c}}, v_{s_\Delta}]$, and $\mathcal{G}_e$ the union of $\mathfrak{t}_e(\Gamma'_e)$ and of the leaf paths in E_e . Then $\mathcal{G}_e$ is a model arc system of e , and the correspondence which maps $C_Y(s_\Delta, e)$ to $\mathfrak{t}_e(v_{s_\Delta})$ and each original leaf to its model point is an isomorphism of plane graphs from $\Gamma_Y \cap E_e$ onto $\mathcal{G}_e$, preserving the rotation at every vertex and the cyclic order of the leaves on ∂E_e (the points $\mathfrak{t}_e(\bar{m})$ being points of degree two). At every interior vertex $\mathfrak{t}_e(v_{s_\Delta})$ the three angles between consecutive edges are less than π .*

The proof is in Appendix E.

The model complex.

Lemma 8.5 (the model complex). *Fix model positions and model arc systems. There are a regular cell decomposition $\{C^M(\mathfrak{p})\}_{\mathfrak{p} \in \text{DP}}$ of B_Y with face poset $(\text{DP}, \preceq)$ and a piecewise linear homeomorphism Υ^M of B_Y such that:*

- (1) $\Upsilon^M(C_Y(\mathfrak{p})) = C^M(\mathfrak{p})$ for every dual cell pair $\mathfrak{p}$; Υ^M maps every face F_ω^* of B_Y onto itself, and it is isotopic to the identity through homeomorphisms with this property;
- (2) the zero-cells $C^M([u, u'], \omega)$, ω a two-cell of a two-face, are the model nodes and model vertices, the zero-cells $C^M(\tau, e)$, τ a triangle of $\mathcal{T}$, are the interior vertices of the model arc systems, and the arcs $C^M([u, u'], e)$ are the arcs and edges of the model arc systems;

- (3) the local system $\Lambda^M := \Upsilon_*^M \Lambda_Y$ is the constant lattice $n^\perp \cap M$ on $\text{int } F_n^*$ and $M/\mathbb{Z}u$ on $R_u^M := \bigcup_\omega C^M(u, \omega)$, glued on $R_u^M \cap \text{int } F_n^* = C^M(u, n)$ by the projection $n^\perp \cap M \rightarrow M/\mathbb{Z}u$, as Λ_Y is (Proposition 4.11).

The proof is in Appendix E.

We write $\Gamma^M := \Upsilon^M(\Gamma_Y)$ for the *model discriminant*; its vertices are the model nodes, the model vertices and the turning points $C^M(\tau, e)$, and we call them the Γ^M -vertices. The model vertices are of positive type and the turning points of negative type on B_Y (Proposition 4.10(3)).

Remark 8.6 (the original positions). The node $C_Y([u, u'], P)$ of Γ_Y is the barycentre of the cell of $\mathcal{P}_Y$ on J_P with lower support $[u, u']$. If $\check{h}(u) \neq \check{h}(u')$ this cell is a segment of length one at an end of J_P , and the node lies at distance $(g_P - 1)/2$ from m_P , where the Morse chart of W^* sits. This is why the model complex is needed.

Charts at the model discriminant. At a model node m_P we use the tropical chart $T_q = (y_1, y_2, \alpha_q)$ of Lemma 7.20(4), with $\alpha_q = \alpha_{uu'} - \alpha_{uu'}(m_P)$. Along the model arcs we use the following charts.

- Lemma 8.7** (tropical charts along the model arcs). (1) Let ω be a two-cell of a two-face G , $[u, u']$ an edge of $\mathcal{T}$ in G° and m its model point on J_ω . Near m , $C^M(u, \omega)$ and $C^M(u', \omega)$ are the parts of J_ω on which $\alpha_{uu'}$ is smaller, respectively larger, than at m .
- (2) Let A be an arc of the model arc system of $e = [n, n']$, with edge $[u, u']$ of $\mathcal{T}$, and let $n_y \in N$ complete $n' - n$ to a basis of the lattice $\{x \in N : \langle u, x \rangle = \langle u', x \rangle = 0\}$. Then $(w_u, w_{u'}, n' - n, n_y)$ is a basis of N , and the level lines of $y := \langle \cdot, n_y \rangle$ in E_e are parallel to $u' - u$, so A is the graph of a piecewise linear function $\bar{\alpha}$ over an interval of values of y : $\alpha_{uu'} = \bar{\alpha}(y)$ on A . Put $y_e := \text{val}_{n'} - \text{val}_n$, $\alpha_e := \alpha_{uu'} - \bar{\alpha}(y)$ and $T_e := (y_e, \alpha_e, y)$. Near the part of A in the interior of E_e , T_e is a piecewise linear homeomorphism onto an open subset of $\mathbb{R}^3$, mapping $\text{int } F_n^*$, E_e and $\text{int } F_{n'}^*$ into $\{y_e < 0\}$, $\{y_e = 0\}$ and $\{y_e > 0\}$, A onto a segment of the y -axis, and the cells $C^M(u, e)$, $C^M(u', e)$ near A into $\{y_e = 0, \alpha_e < 0\}$ and $\{y_e = 0, \alpha_e > 0\}$.

The proof is in Appendix E.

So at every model node and model vertex the side $\alpha_{uu'} > \alpha_{uu'}(m)$ is the side of u' , the side of the tentacles of the divisor $D_{u'}$. The model nodes are the midpoints of pairwise distinct edges J_P of B_Y ; this includes the two nodes of a pentagon and the three nodes of an A_2 hexagon, which share the edge $[u, u']$ of $\mathcal{T}$. We take the ϱ_q of Lemma 7.20(4) so small that the cubes $T_q^{-1}([- \varrho_q, \varrho_q]^3)$ are pairwise disjoint and disjoint from fixed neighbourhoods of the other Γ^M -vertices.

Lemma 8.8 (model cells near the model discriminant). There is $r_S \in (0, \theta_4/4]$, depending only on the configuration, the model positions and Υ^M , such that:

- (1) at each model node m_P , $T_q^{-1}(\{y_1 = 0, \alpha_q \geq 0\}) \cap T_q^{-1}([-r_S, r_S]^3)$ is the intersection with that cube of the closure of $C^M(u', ab) \cup C^M(u', P) \cup C^M(u', cd)$, and it is bounded on Γ^M by the model arcs of the legs over ab and dc ; likewise $\{y_2 = 0, \alpha_q \geq 0\}$ with bc and da , and $\{\alpha_q \leq 0\}$ with u in place of u' ;
- (2) near each model vertex $m = m_{\omega, i}$, $\omega = [n_a, n_b, n_c]$, with edge $[u, u'] = [u_{i-1}, u_i]$ of $\mathcal{T}$ and edges e_1, e_2, e_3 of ω , the closure of $C^M(u', \omega) \cup \bigcup_k C^M(u', e_k)$ is the union of three closed sectors, one in each E_{e_k} , meeting along the half-edge $C^M(u', \omega)$ of J_ω , the k -th bounded by that half-edge and the model arc $C^M([u, u'], e_k)$; likewise with u ;

- (3) near each turning point $C^M(\tau, e)$, $\tau = [u_1, u_2, u_3]$, the closure of $C^M(u_i, e)$ is the sector of E_e between the model arcs of $[u_i, u_j]$ and $[u_i, u_k]$, of angle less than π ;
- (4) near each model arc A in the interior of E_e , in the chart T_e , the closures of $C^M(u', e)$ and $C^M(u, e)$ are $\{y_e = 0, \alpha_e \geq 0\}$ and $\{y_e = 0, \alpha_e \leq 0\}$.

The proof is in Appendix E.

The transferred cycle. We apply Definition 6.8 with $\Upsilon = \Upsilon^M$: $\mathfrak{h} = \Upsilon^M \circ \mathcal{L} \circ g_C : B'_\sigma \rightarrow B_Y$, $\kappa_{\mathfrak{h}} : \Lambda \rightarrow \mathfrak{h}^*(\Lambda^M)^\vee$, the coefficient groups $F^*(s)$ and $\mathcal{D}^*(E) = \mathbb{Z}d_A^*$, $d_A^* = \langle \cdot, n' - n \rangle$ on the arc A of $([u, u'], [n, n'])$. For $z \in Z(\mathfrak{K})$ the *transferred cycle* is

$$z^M := \mathfrak{h}_* z \in Z(\mathfrak{h}(\mathfrak{K}_1); F^*, \mathcal{D}^*),$$

a relative cycle on the triangulation $\mathfrak{h}(\mathfrak{K}_1)$ of B_Y , in which Γ^M is full. By Proposition 6.10 its boundary network $b^M := \partial z^M$ has the integers of ∂z , and $\rho^*(z^M) = \rho(z)$; we write ρ for ρ^* and $\text{Bal}(\Gamma^M)$ for the group of balanced networks on Γ^M , the 1-cycles of $C_*(\Gamma^M; \mathcal{D}^*)$. By Proposition 6.9(2),(3) the stalks Inv^* are: on an arc, $\{v : v(u' - u) = 0\}$, of rank two, with basis d_A^* and $g_A^* := \langle \cdot, n_y \rangle$ (Lemma 8.16); at a model node, the same group, with basis $\langle \cdot, b - a \rangle$, $\langle \cdot, c - b \rangle$; at a model vertex, of positive type, a group of rank two containing the covectors d_A^* of its three arcs; at a turning point $C^M(\tau, e)$, of negative type, $\mathbb{Z}d_e^*$, $d_e^* := \langle \cdot, n' - n \rangle$ for $e = [n, n']$, the common covector of its three arcs up to sign.

Lemmas 5.9 and 5.11 hold on B_Y , for Γ^M and the coefficients F^* , $\mathcal{D}^*$, on every triangulation in which Γ^M is full: their proofs use only a triangulated three-manifold, a full graph and the flat sections of a local system (Remark 5.13). Equivalently, for a triangulation $\mathfrak{K}^*$ of B_Y refining $\mathfrak{h}(\mathfrak{K}_1)$ the map $\mathfrak{h}^{-1}$ is linear on the simplices of $\mathfrak{K}^*$, and Lemma 6.1 applied to $(\mathfrak{h}^{-1}, \kappa_{\mathfrak{h}}^{-1})$ identifies the chain complexes, coefficient systems and stalks on $\mathfrak{K}^*$ with those on the triangulation $\mathfrak{h}^{-1}(\mathfrak{K}^*)$ of B'_σ , in which Γ is full.

Lemma 8.9 (balanced leg data on B_Y). *At a Γ^M -vertex p let Bal_p be the lattice of balanced leg data: the vectors (c_A) of coefficients on the arcs A at p , oriented away from p , with $\sum_A c_A d_A^* = 0$ in $\text{Inv}^*(p)$. Then: at a model vertex, with the d_A^* oriented so that $d_1^* + d_2^* + d_3^* = 0$, $\text{Bal}_p = \mathbb{Z}(1, 1, 1)$; at a turning point, with the d_A^* oriented equal, $\text{Bal}_p = \{c_{12} + c_{13} + c_{23} = 0\}$; at a model node, with $d_{ab}^* = d_{dc}^* = \langle \cdot, b - a \rangle$ and $d_{bc}^* = d_{da}^* = \langle \cdot, c - b \rangle$, $\text{Bal}_p = \{c_{ab} = -c_{dc}, c_{bc} = -c_{da}\}$. A balanced network restricts at p to an element of Bal_p .*

Proof. At a model vertex the three d_A^* are $\pm \langle \cdot, e_k \rangle$ for the three edges e_k of a unimodular triangle, which span a lattice of rank two and satisfy one relation; at a turning point they are equal up to sign; at a node $\langle \cdot, b - a \rangle$ and $\langle \cdot, c - b \rangle$ are independent. The last statement is the balance of a balanced network at p , a relation in $F^*(p) = \text{Inv}^*(p)$. This is Lemma 5.10, transported by $\mathfrak{h}$ and $\kappa_{\mathfrak{h}}$, with the vertex types exchanged. $\square$

Model germs. All germs below are fixed once, after the configuration, the model positions and Υ^M , and before z and t . They lie in N_0 , the r_0 -neighbourhood of Γ^M , where $r_0 \leq r_S/4$ is so small that N_0 lies in the union of the cubes $T_q^{-1}([-r_S, r_S]^3)$, the neighbourhoods of the model vertices and turning points of Lemma 8.8(2),(3), and the domains of the charts T_e . Each germ is a piecewise linear surface with boundary on Γ^M and, off Γ^M , on the curves where it is truncated, carrying a coefficient that is a flat section of $(\Lambda^M)^\vee$ over it. The *away-oriented leg data* of a germ at a Γ^M -vertex p are the coefficients c_A of its boundary $\sum_A c_A d_A^*[A]$ on the arcs A at p , each oriented away from p .

- (M1) *Node halves.* At a model node m_P the four *standard halves* are $T_q^{-1}(\{y_1 = 0, \pm \alpha_q \geq 0\})$ (the $b - a$ passage, along the legs over ab and dc) and $T_q^{-1}(\{y_2 = 0, \pm \alpha_q \geq 0\})$

(the $c - b$ passage, along bc and da). We use the side $\alpha_q \geq 0$. The *unit germ* g_{ab} is the $b - a$ half with the orientation of the frame (y_2, α_q) and the covector $-\langle \cdot, b - a \rangle$; the unit germ g_{bc} is the $c - b$ half with the orientation of the frame (y_1, α_q) and the covector $+\langle \cdot, c - b \rangle$. The boundary of g_{ab} runs from the leg over ab to the leg over dc , and that of g_{bc} from the leg over da to the leg over bc ; so their away-oriented leg data are $(c_{ab}, c_{bc}) = (1, 0)$ and $(0, 1)$, and $\rho_q(g_{ab}) = 1$, $\rho_q(g_{bc}) = -1$. Both covectors lie in $\text{Inv}^*(m_P)$.

- (M2) *Tripods*. At a model vertex $m = m_{\omega, i}$ the unit germ is the union of the three sectors of Lemma 8.8(2) on the side of u' . Orient the edges e_k of ω cyclically, so that $e_1 + e_2 + e_3 = 0$, orient the sector in E_{e_k} so that its boundary leaves m along its model arc (and hence arrives at m along the half-edge $C^M(u', \omega)$), and give it the covector $\langle \cdot, e_k \rangle$, which is d_A^* for its arc. The three sectors induce the same orientation on the half-edge, their covectors sum to zero there, and the away-oriented leg data are $(1, 1, 1)$.
- (M3) *Turning germs*. At a turning point $C^M(\tau, e)$, $\tau = [u_1, u_2, u_3]$, $e = [n, n']$, let A_{ij} be the arc of $[u_i, u_j]$ and $C(u_i)$ the sector of Lemma 8.8(3) between A_{ij} and A_{ik} . Fix an orientation o_e of E_e . With every sector oriented by o_e and carrying d_e^* , the leg data (c_{12}, c_{13}, c_{23}) of the three sectors sum to zero, and each sector has data ± 1 on its two arcs and 0 on the third. We label τ so that $C(u_1) = (1, -1, 0)$, $C(u_2) = (-1, 0, 1)$, $C(u_3) = (0, 1, -1)$. The *turning germs* are $\text{Turn}(u_1) := C(u_1)$ and $\text{Turn}(u_3) := C(u_3)$ with orientation o_e and $\text{Turn}(u_2) := C(u_2)$ with orientation $-o_e$, all with covector d_e^* ; their leg data are $(1, -1, 0)$, $(1, 0, -1)$, $(0, 1, -1)$. The chain $\text{Turn}(u_1) - \text{Turn}(u_2) + \text{Turn}(u_3)$, the full disc of E_e about the vertex with orientation o_e , has zero leg data.
- (M4) *Exit sheets and reference sheets*. Orient each arc A from one end p_0 to the other end p_1 and fix, in the part of A in Ω_e , five points in this order: $y_{A,0}^*$, y_A^0 , $\mathfrak{x}_A$, y_A^1 , $y_{A,1}^*$, and take r_0 at most a quarter of their mutual distances, of their distances from $A \setminus \Omega_e$ and of their distances from the ends p_0, p_1 of A , so that the stretches of length at most r_0 about $y_{A,p}^*$ below lie in Ω_e and away from the balls $B_{p_0}^0$, $B_{p_1}^0$ of Section 8.5, whose radius is at most r_0 . A unit germ at an end p of A that uses A continues along A as its *exit sheet*: in the chart T_e , the half-plane $\{y_e = 0, \alpha_e \geq 0\}$ (the closure of $C^M(u', e)$) or $\{y_e = 0, \alpha_e \leq 0\}$ (that of $C^M(u, e)$), according to its side at p , from p to y_A^0 if $p = p_0$ and to y_A^1 if $p = p_1$. The *reference side* of every arc is $\alpha_e \geq 0$. Fix a convex polygon $P \subset \mathbb{R}^2$, symmetric about 0, with gauge ϱ_P . An exit sheet on the side $\alpha_e \leq 0$ is *rotated* to the reference side at $y_{A,p}^*$: in the chart it is the half-plane $\alpha_e \leq 0$ up to $y_{A,p}^*$, the normal half-disc $\{y = y_{A,p}^*, y_e \geq 0, \varrho_P(y_e, \alpha_e) \leq r_0\}$ there, and the half-plane $\alpha_e \geq 0$ after it. All half-planes are truncated at $\varrho_P \leq r_0$. The *reference sheet* of A with coefficient $c_A d_A^*$ is the half-plane $\alpha_e \geq 0$ over $[y_A^0, y_A^1]$, oriented so that its boundary on A runs in the direction of A .
- (M5) *Transverse discs*. At $\mathfrak{x}_A$, the normal disc $\mathfrak{N}_A := \{y = y(\mathfrak{x}_A), y_e^2 + \alpha_e^2 \leq r_0^2\}$ in the chart T_e , cooriented by A : (orientation of $\mathfrak{N}_A$, direction of A) is the orientation of B_Y .

At an end p of A the exit sheets on a given side come from one germ type: at a turning point the side u_i of the arc of $[u_i, u_j]$ belongs to $\text{Turn}(u_i)$, and at a model node or vertex each leg is used by one germ type, on the side u' . An arc joining two model nodes receives one exit sheet from each end, cut at y_A^0 and y_A^1 . The rotation through the half-plane $y_e > 0$ rather

than through $y_e < 0$ is a convention; the other choice changes the chain by a transverse disc with coefficient in $\mathbb{Z}d_A^*$.

By Lemma 8.9 the unit germs at p form a $\mathbb{Z}$ -basis of Bal_p : g_{ab}, g_{bc} at a model node, the tripod at a model vertex, $\text{Turn}(u_1), \text{Turn}(u_2)$ at a turning point. For $b \in \text{Bal}(\Gamma^M)$ write $b_p = \sum_g m_g(b) \partial g$ at every Γ^M -vertex p , with integers $m_g(b)$ (g running over the unit germs of the basis at p).

Definition 8.10 (the chain of germs). For $b \in \text{Bal}(\Gamma^M)$ the *chain of germs* $\text{Std}^M(b)$ is the sum over all Γ^M -vertices p of $\sum_g m_g(b) g$, each unit germ with its exit sheets, cut transversally at y_A^0 or y_A^1 , plus, on every arc A , the reference sheet with covector $c_A d_A^*$ between y_A^0 and y_A^1 , where $c_A d_A^*[A]$ is the part of b on A .

Along A the exit sheets arriving from each end carry, on the reference side, covectors summing to $c_A d_A^*$ with the orientation of the reference sheet, by the balance of b along the arc; so the transverse cut segments cancel. The rest of the boundary of $\text{Std}^M(b)$ off Γ^M lies where the germs are truncated (at $\varrho_P = r_0$ along the arcs, and on their outer frontier near the Γ^M -vertices); this set Θ_{tr} , where the germs are cut off, is compact and disjoint from Γ^M . So $\text{Std}^M(b)$ is a relative cycle of $(N_0, \Theta_{\text{tr}} \cup \Gamma^M)$ with boundary network b . It is linear in b .

Where sheets of $\text{Std}^M(b)$ coincide as sets, namely sheets on the reference side of one arc, their coefficients add.

The normal form on B_Y . Let $z^M = \mathfrak{h}_* z$ and $b^M = \partial z^M$. Choose a triangulation $\mathfrak{K}_2^0$ of B_Y refining $\mathfrak{h}(\mathfrak{K}_1)$, in which Γ^M , the germs (M1)–(M5) and their junction segments are subcomplexes, and which is so fine near Γ^M that the simplicial neighbourhoods below lie in the interior of N_0 and are disjoint from the set Θ_{tr} where the germs are cut off. Let $\mathfrak{K}_2$ be a derived subdivision of $\mathfrak{K}_2^0$, in which Γ^M is full and the germs are subcomplexes; choose its derived points off the points $\mathfrak{r}_A$, so that $\mathfrak{r}_A$ lies in the interior of an edge E_A of $\mathfrak{K}_2$ in A ; and let $\mathfrak{K}_2'$ be a derived subdivision of $\mathfrak{K}_2$ whose new vertex on E_A is $\mathfrak{r}_A$. Put $U := N(\Gamma^M, \mathfrak{K}_2')$, the union of the closed simplices meeting Γ^M . Then $\text{sd } z^M$ lies on $\mathfrak{K}_2'$, and $\text{Std}^M(b^M) \cap U$ is a relative cycle of $(U, \partial U \cup \Gamma^M)$ on $\mathfrak{K}_2'$ with boundary network b^M : its coefficients lie in the stalks, since near a Γ^M -vertex only the germs of that vertex occur and along an arc only covectors in $\text{Inv}^*(A)$; and an edge of U not in ∂U meets Γ^M , so that all 2-simplices at it lie in U and intersecting with U creates boundary only on ∂U , while the set Θ_{tr} where the germs are cut off lies outside U . By Lemma 5.11, with the edge E_A chosen in each arc A ,

$$(8.1) \quad \text{sd } z^M - \partial w = \text{Std}^M(b^M) \cap U + \sum_A v_A \bar{D}(E_A) + y, \quad v_A \in \text{Inv}^*(A),$$

with w a 3-chain in U , $v_A \bar{D}(E_A)$ the transverse disc at E_A in $\mathfrak{K}_2'$ (Definition 5.8), centred at $\mathfrak{r}_A$, and y supported in $B_Y \setminus \text{int } U$. Of Lemma 5.9 only the surjectivity in part (1) enters, applied to the chain $\text{sd } z^M|_U - \text{Std}^M(b^M) \cap U$, which has zero boundary on Γ^M .

We then make two local modifications. Each replaces a chain C by a chain C' with the same boundary on Γ^M and adds a chain R supported off Γ^M with $\partial R = \partial C - \partial C'$ off Γ^M ; it preserves the boundary network.

- *Straightening the transverse discs.* Let $N_{\varepsilon_A}(\mathfrak{r}_A)$ be a piecewise linear regular neighbourhood of $\mathfrak{r}_A$ of diameter at most ε_A , for instance a closed star in a derived subdivision in which $\bar{D}(E_A)$ and Γ^M are subcomplexes. By regular-neighbourhood theory [RS72, Ch. 3] the triple $(N_{\varepsilon_A}, N_{\varepsilon_A} \cap \bar{D}(E_A), N_{\varepsilon_A} \cap \Gamma^M)$ is unknotted (a ball, a proper disc and a proper arc meeting it once transversally), so $\partial N_{\varepsilon_A}$ minus the

two points of Γ^M is an open annulus in which $\partial N_{\varepsilon_A} \cap \bar{D}(E_A)$ is essential. Let $\mathfrak{N}_A^{\varepsilon'_A}$ be the normal disc of (M5) of radius $\varepsilon'_A \leq r_0$, so small that its solid cylinder $T_e^{-1}(\{y_e^2 + \alpha_e^2 \leq \varepsilon'^2_A, |y - y(\mathfrak{r}_A)| \leq \varepsilon'_A\})$ lies in $\text{int } N_{\varepsilon_A}$. Replace $v_A(\bar{D}(E_A) \cap N_{\varepsilon_A})$ by $v_A \mathfrak{N}_A^{\varepsilon'_A}$ plus v_A times an embedded annulus in $N_{\varepsilon_A} \setminus (\Gamma^M \cup \text{cylinder})$ between the two essential circles; v_A extends over it because it lies in a punctured neighbourhood of an arc point.

- *Smoothing the rotations.* On a stretch $\mathfrak{a}_{A,p} \ni y_{A,p}^*$ of length at most r_0 replace every rotated exit sheet by the union of the rays $\{(\varrho \cos \Theta(y), \varrho \sin \Theta(y), y) : \varrho \geq 0\}$ in the coordinates (y_e, α_e, y) , truncated at $\varrho_P \leq \varrho_0$, with Θ smooth and monotone, $\Theta = -\pi/2$ near the start and $\Theta = \pi/2$ near the end of the stretch, passing through $\Theta = 0$; add the strip on the polygonal cylinder $\{\varrho_P = \varrho_0\}$ over the stretch between the two outer curves, closed up at its ends.

The sizes ε_A and ϱ_0 are chosen so that the neighbourhoods $N_{\varepsilon_A}(\mathfrak{r}_A)$ and the solid cylinders $\{\varrho_P \leq \varrho_0\}$ over the stretches lie in $\text{int } U$, and each meets no summand other than the sheets along its own arc.

Lemma 8.11 (normal form of the transferred cycle). *The result is a relative cycle*

$$(8.2) \quad z^{\natural} = \text{Std}^M(b^M) + \sum_A v_A \mathfrak{N}_A + y^{\natural}, \quad v_A \in \text{Inv}^*(A),$$

with boundary network b^M and $\rho(z^{\natural}) = \rho(z)$, where $\text{Std}^M(b^M)$ now has smooth rotations and is truncated at ∂U , $\mathfrak{N}_A$ is $\mathfrak{N}_A^{\varepsilon'_A}$ continued by the annulus and by $\bar{D}(E_A)$ outside $N_{\varepsilon_A}(\mathfrak{r}_A)$, and $y^{\natural}$ is y plus the strips of the smoothing. Let $r_{\natural} > 0$ be the distance, in a fixed metric on B_Y , from Γ^M to the union of $\text{supp } y^{\natural}$ and of the parts of the transverse discs outside the normal discs $\mathfrak{N}_A^{\varepsilon'_A}$. Within distance $r_{\natural}$ of Γ^M , $z^{\natural}$ is exactly $\text{Std}^M(b^M) + \sum_A v_A \mathfrak{N}_A^{\varepsilon'_A}$: germs in the positions of Lemma 8.8 and normal discs at the points $\mathfrak{r}_A$.

Proof. The two modifications preserve the boundary network. The strips of the smoothing lie at distance at least ϱ_0/C_P from Γ^M , C_P the equivalence constant of ϱ_P and the Euclidean norm; the annuli lie in compact subsets of $N_{\varepsilon_A} \setminus \Gamma^M$; y lies off $\text{int } U$; and the transverse discs meet Γ^M only at their centres. So $r_{\natural} > 0$, with $r_{\natural} \leq \min(\varepsilon'_A, \varrho_0/C_P)$. $\square$

The homology class of z plays no role: what follows needs only a relative cycle with the boundary network of z , and Lemma 8.11 provides one of the required shape.

The surfaces near Γ^M . Near Γ^M the normal form (8.2) is a finite sum of surfaces with integer coefficients:

- (S1) for each Γ^M -vertex p and each unit germ g at p , the surface of g (its component at p in N_0 together with its exit sheets, rotated and smoothed where required, up to y_A^0 or y_A^1), with coefficient $m_g(b^M)$ times the covector of g ;
- (S2) the reference sheets, with coefficients $c_A d_A^*$;
- (S3) the transverse discs $v_A \mathfrak{N}_A$.

Surfaces of this list coincide as sets only for sheets on the reference side of one arc; there we add their coefficients.

Definition 8.12 (summands and junctions). First add the coefficients of coincident sheets in (S1)–(S3). Except at a tripod, a *summand* is a maximal connected part of one surface, or of coincident sheets, on which the total coefficient is constant. At a tripod, its three sectors

and their exit strips form one summand. The half-edge $C^M(u', \omega)$ is its interior branching curve, and the three coefficients $m\langle \cdot, e_k \rangle$ balance with the orientations of (M2).

Two summands are *adjacent* along a *junction segment* if they meet along a curve across which the total coefficient on one side of the arc changes. The half-edge of a tripod is excluded from this definition.

The following observations locate the junction segments. Where a germ continues as its own exit strip, including at the start of a rotation, the coefficient is the same on both sides, so the transverse segment is interior to one summand; thus the part of a unit germ at its vertex and its exit strips belong to one summand up to the first place where another sheet joins them. Junction segments occur only where a rotated exit strip reaches a reference side that already carries a strip: inside the polygonal cylinder at the far end of the stretch, and outside it where the piecewise linear normal half-disc of the rotation at $y_{A,p}^*$ meets the reference-side strip. Where a rotated strip reaches an empty reference side, it continues there; where the exit strips on the reference side meet the reference sheet, at y_A^0 and y_A^1 , they already total $c_A d_A^*$ with the orientation of the reference sheet, by the balance of b^M along A . In both cases the total coefficient does not change and the curve is interior to one summand. Along a junction segment every adjacent coefficient is a multiple of d_A^* , and the multiples sum to zero with the orientation signs, because $z^\natural$ is a relative cycle with boundary only on Γ^M . Inside the stretch, after $y_{A,p}^*$, the reference-side sheet carries one coefficient inside the cylinder $\{\varrho_P < \varrho_0\}$, where the smoothed strip is still rotating, and another outside it, where the piecewise linear rotated sheet has already arrived; they meet the correction strip of the smoothing along the seam $\{\varrho_P = \varrho_0\}$, at distance at least $\varrho_0/C_P \geq r_\natural$ from Γ^M .

Summands may also meet transversally: a transverse disc and a sheet, the two node halves along $C^M(u', P)$, and exit sheets of different germs meeting only on Γ^M . Such meetings are not junctions, and the summands are lifted separately (Section 8.6).

8.2. Local lifts along the discriminant and at its three-valent vertices. We lift the boundary germs in the binomial and pair-of-pants models of the localized hypersurface. The positive real section fixes compatible base points for the torus orbits, and Lemma 7.18 places these orbits in regions where the equations are exact.

The positive section.

Lemma 8.13 (the positive locus). *For t small and every real c with the sign pattern:*

- (1) *the positive real locus $\Sigma^+ := Y_t \cap (\mathbb{R}_{>0})^4$ is a smooth strictly convex three-sphere in the torus (compare [AGIS20, §3.1]);*
- (2) *for $s \in [0, 1]$ the positive locus Σ_s^+ of W_s is a smooth closed three-manifold, the Σ_s^+ form a smooth family, and $\text{Log}(\Sigma_0^+)/L$ is a star-shaped sphere within $O(1/L)$ of B_Y . Radial projection therefore defines a section $\mathfrak{s}^\vee: B_Y \rightarrow \Sigma_W^+ := \Sigma_0^+ \subset W^*$.*

Proof. (1) On the positive part of the torus, $F_t/(\gamma_0 x^0) = 1 - \sum_{n \neq 0} |\gamma_n/\gamma_0| e^{\langle w, n \rangle}$ with $w = \text{Log } x$. The sum is strictly convex in w , because the $n \neq 0$ span $N_{\mathbb{R}}$, and proper, because 0 is interior to their convex hull; its value at $w = 0$ is $O(t^{k_0 + \min h_C}) < 1$. So $\{\sum = 1\}$ is a smooth strictly convex sphere.

(2) Complex conjugation preserves $F^{(s)}$, since the coefficients are real and the weights depend only on moduli; so at a real point $dF^{(s)}$ maps real tangent vectors to $\mathbb{R}$ and imaginary ones to $i\mathbb{R}$, and real rank two (Lemma 7.11(1)) forces $dF^{(s)}|_{\mathbb{R}^4} \neq 0$. Hence Σ_s^+ is a regular level set, and the family is smooth. It does not meet the toric boundary: on the nonnegative part of a divisor the origin monomial vanishes and the others are nonpositive, the maximal

one having weight one. Along a ray $w = \lambda w_0$, at a zero of $F^{(0)}$ the origin monomial lies between $\mathbf{m}$ and $|\mathcal{A}|\mathbf{m}$, $\mathbf{m}$ the largest monomial; so the deficit of the origin is $O(1/L)$, w/L lies within $O(1/L)$ of B_Y , and $|w| = O(L)$. A monomial $n \neq 0$ of positive weight has $d_n \leq \tilde{d}_n < \theta_1/2 \leq 1/2$, so $|\gamma_n x^n| \geq t^{1/2}\mathbf{m}$, that is $\langle w, n \rangle \geq L(H(n) - \frac{1}{2}) - O(1) \geq \frac{L}{2} - O(1)$, since $H(n) \geq 1$ by (7.3). The function $\mathcal{R}(\lambda) := \sum_{n \neq 0} \psi_n |\gamma_n / \gamma_0| e^{\langle w, n \rangle}$ equals 1 at a crossing; with the weights held constant its derivative with respect to $\log \lambda$ is at least $L/2 - O(1)$ there, and the derivatives of the weights are $O(1)$ (Lemma D.1(2)) and multiply monomials in transition, of total size $O(t^{\theta_1/4})$ relative to the origin. So each ray crosses Σ_0^+ exactly once, transversally. $\square$

For a primitive covector v in the stalk of $(\Lambda^M)^\vee$ at a point of $B_Y \setminus \Gamma^M$, the quotient of the fibre torus $\Lambda^M \otimes U(1)$ by $\mathbb{T}(v) := \ker v \otimes U(1)$ is the circle of the character $\chi^{\tilde{v}}$ for any lift $\tilde{v} \in N$ of v , and its phase is 0 on Σ_W^+ . So the union of the $\mathbb{T}(v)$ -orbits through the points of $\mathfrak{s}^\vee$ over a surface with coefficient v is contained in $\{\arg \chi^{\tilde{v}} = 0\}$.

Where the localized mirror is given by exact models. We use Lemma 7.18: on the sets listed there the weights of the localized mirror are exactly 0 or 1. For node parallelograms, item (1') covers the stretches of the model arcs from a model node to Ω_e , along the leaf paths, and for triangles those from a model vertex.

Normalised coordinates along the model arcs. Let A be an arc of the model arc system of $e = [n, n']$, with edge $[u, u']$ of $\mathcal{T}$ and n_y as in Lemma 8.7(2). On the chart $U_{u, u'}$ we use $Z = \chi^{w_u}$, $Z' = \chi^{w_{u'}}$, $\chi_1 = \chi^{n' - n}$ and $\chi_2 = \chi^{n_y}$. The normalised coordinates along A are $\hat{Z} = Z/\nu_Z$ and $\hat{Z}' = Z'/\nu_{Z'}$, where $\nu_Z, \nu_{Z'} > 0$ are smooth functions of $\log |\chi_1|$ and $\log |\chi_2|$ such that $|\gamma_0 x^0| = |\gamma_n x^n| |\hat{Z} \hat{Z}'|$, and $\nu_Z/\nu_{Z'}$ is a function of $\log |\chi_2|$ alone with $\frac{1}{L} \log(\nu_Z/\nu_{Z'}) = \bar{\alpha}(\log |\chi_2|/L) + O(1/L)$; where the torus coordinates are within $2\theta_4$ of those of a model node they are the constants of Definition 7.13, with the reference monomial a in place of n , and near a model vertex, for the reference monomial n_0 of the definition of $w_{u'}$, they are constants with $|\hat{Z}| = |\hat{Z}'|$ at the vertex; on these sets $\bar{\alpha}$ is constant, so the two requirements agree. Put $\hat{\mu} := |\hat{Z}|^2 - |\hat{Z}'|^2$ and

$$\mathcal{S}_G := \sum_m \psi_m |\gamma_m x^m| / |\gamma_n x^n|,$$

the sum over the lattice points m of the face $[u, u']^\vee$ of Δ , the monomials free of Z and Z' . Changing the reference monomial n to n'' , with the same ratio $\nu_Z/\nu_{Z'}$, multiplies $|\hat{Z}|^2$, $|\hat{Z}'|^2$ and $\mathcal{S}_G$ by the same function of (χ_1, χ_2) . Where only $0, n, n'$ have positive weight, W^* is the leg model

$$(8.3) \quad 1 + \kappa_e \chi_1 + \mathbf{b} Z Z' = 0, \quad \kappa_e = \gamma_{n'}/\gamma_n > 0, \quad |\mathbf{b}| \nu_Z \nu_{Z'} = 1,$$

with $\mathbf{b}$ a monomial in the torus coordinates, and $\mathcal{S}_G = 1 + |\kappa_e \chi_1|$. Its critical locus is $\{Z = Z' = 0, \chi_1 = -1/\kappa_e\}$, and the rank-two torus $\mathbb{T}_d := (\{n, n'\}^\perp \cap M) \otimes U(1)$, the exact torus of the leg, acts on it; it contains S_α^1 , and $\{n, n'\}^\perp \cap M = \ker d_A^*$ is the sublattice of Λ^M fixed by the monodromy around the arc.

The leg model base is the (x_1, x_2) -plane, $x_1 = \log |\kappa_e \chi_1|$ and $x_2 = \hat{\mu}$, times the leg coordinate y . As at a node, n is the larger of n, n' where $x_1 < 0$ and n' where $x_1 > 0$, and $x_2 < 0$ (resp. $x_2 > 0$) is the side of D_u (resp. $D_{u'}$). At a point of the torus, $T_e(\log x/L) = (x_1, \log |\hat{Z}/\hat{Z}'|, \log |\chi_2|)/L + O(1/L)$. The model map

$$E_t(y_e, \alpha_e) := (Ly_e, (1 + e^{Ly_e}) \sinh(L\alpha_e))$$

is a smooth homeomorphism of $\mathbb{R}^2$ with $E_t(y_e, \alpha_e) = L(y_e, 2\alpha_e) + O(L^2(y_e^2 + \alpha_e^2))$ near 0; it takes the tropical coordinates (y_e, α_e) to the coordinates (x_1, x_2) of the leg model base, in the sense of Lemma 8.22.

The pieces.

Lemma 8.14 (pieces along the arcs, at vertices and at turning points). *Let $0 < \varsigma \leq \varsigma_{\max}$ for the constant $\varsigma_{\max} \leq \theta_2/4$ of the proof, which depends only on the configuration and the model arc systems, and t small.*

- (1) Arcs. *Let $\mathfrak{a}$ be a compact stretch of a model arc A of $e = [n, n']$, contained in Ω_e , and $\Theta: \mathfrak{a} \rightarrow \mathbb{R}$ smooth. For the sheet $\{(\varrho \cos \Theta(y), \varrho \sin \Theta(y), y) : 0 \leq \varrho \leq \varrho_0, y \in \mathfrak{a}\}$ in the chart T_e , $\varrho_0 \geq \varsigma$, put*

$$\Pi_{\mathfrak{a}, \varsigma} := \mathbb{T}_d \cdot \{\mathbf{x}(E_t(\varrho \cos \Theta(y), \varrho \sin \Theta(y)), y) : 0 \leq \varrho \leq \varsigma, y \in \mathfrak{a}\},$$

where $\mathbf{x}(x_1, x_2, y)$ is the point of W^* with $\log |\chi_2| = Ly$, $\arg \chi_2 = 0$, $\chi_1 = -e^{x_1}/\kappa_e$, $\hat{\mu} = x_2$ and $\hat{Z} \geq 0$. Then $\Pi_{\mathfrak{a}, \varsigma}$ is a smooth four-manifold fibred over the sheet, with fibres $\mathbb{T}_d$ collapsing to circles over the arc and no boundary over the arc, and on it the phase of $\chi^{n'-n}$ is π . Where $\Theta = \pi/2$ it is the divisor piece $\{Z' = 0, \chi_1 = -1/\kappa_e, |\hat{Z}|^2 \leq 2 \sinh(L\varsigma)\}$, and where $\Theta = -\pi/2$ the corresponding part of $W^* \cap D_u$.

Along a part of a model arc on which the sheet is the closure of $C^M(u', e)$ (resp. $C^M(u, e)$), the piece is the divisor piece of depth ς : the set of points x of $W^* \cap D_{u'}$ over the stretch and on the branch of the arc with $|\hat{Z}(x)|^2 \leq \mathcal{S}_G(x) \sinh(L\varsigma)$ (resp. the same with D_u and $\hat{Z}'$). Here the branch is the curve $\{g_e = 0\}$, g_e the weighted sum of the monomials of $[u, u']^\vee$ divided by $\gamma_n x^n$, on which the two monomials of e are the largest; on the node neighbourhood it is $\mathcal{C}_P$, and where the torus coordinates are within θ_4 of those of the node point the divisor piece is the node piece $\Pi_q^\pm$ over $\mathcal{C}_P$ with $\bar{\mu} = \mathcal{S}_G \sinh(L\varsigma)$. Divisor pieces defined near different model points agree where they overlap, and they agree with $\Pi_{\mathfrak{a}, \varsigma}$ where $\Theta = \pm\pi/2$. A divisor piece is a smooth four-manifold, fibred over the sheet with fibre S_α^1 times the circle of the branch, collapsing to that circle over the arc; on it the phase of $\chi^{n'-n}$ is within $O(\eta_0^{1/2})$ of π . It is exact: by Lemma 7.18(2) on the node neighbourhood, by (1') along the leaf paths, by (1) along the arc in Ω_e and at the turning points, and by (3) near a model vertex.

- (2) Model vertices. *Over $m = m_{\omega, i}$, $\omega = [n_a, n_b, n_c]$, with edge $[u, u']$ of $\mathcal{T}$, $W^* = \{\mathfrak{b}ZZ' + \hat{f}(\chi_1, \chi_2) = 0\}$ with $\hat{f}$ the weighted trinomial of ω . The vertex piece Π_v is the set of points of $W^* \cap D_{u'} = \{Z' = 0, \hat{f} = 0\}$ whose torus coordinates are within θ_4 of those of m and with $|\hat{Z}|^2 \leq \mathcal{S}_G \sinh(L\varsigma)$: a smooth pair of pants times a disc, as in [CBM14, §7.5]. It is invariant under S_α^1 ; along each tentacle its level sets of the leg coordinate and of $|\hat{Z}|^2/\mathcal{S}_G$ are S_α^1 times the circle of the tentacle, and its outer part is S_α^1 times the truncated pair of pants $\{\hat{f} = 0\}$ near m , which plays the role of the filling $\mathcal{Q} = \ell \times \mathcal{Q}'$ of [CBM14, §7.7, Steps 2–3] with $\ell = S_\alpha^1$. Along the tentacles it agrees with the divisor pieces of the three arcs, and its boundary consists of the outer part $\{|\hat{Z}|^2 = \mathcal{S}_G \sinh(L\varsigma)\}$ and the ends of the tentacles, where it continues as those divisor pieces.*
- (3) Turning points. *Over $C^M(\tau, e)$, $\tau = [u_1, u_2, u_3]$, $W^* = \{\mathfrak{b} z_1 z_2 z_3 + \gamma_n(1 + \kappa_e \chi_1) = 0\}$ in the Cox coordinates z_i of the u_i . The turning piece $\Pi_t(\text{Turn}(u_1))$ is the union, near the vertex, of the divisor pieces of depth ς of the arcs of $[u_1, u_2]$ and $[u_1, u_3]$ in $W^* \cap D_{u_1} = \{z_1 = 0, \chi_1 = -1/\kappa_e\} \cong \mathbb{C}_{z_2, z_3}^2$; it is swept by the exact torus rotating*

z_2, z_3 , and near each of the two arcs it is the divisor piece of the common ray u_1 . It is the piece of [CBM14, Lemma 7.6]. The pieces of $\text{Turn}(u_2)$, $\text{Turn}(u_3)$ lie in D_{u_2} , D_{u_3} .

The proof is in Appendix E.

8.3. Node pieces with multiplicity. The two elementary passages through a node lift to chains whose boundary is the oriented vanishing sphere. Lemma 7.24 determines the sign; arbitrary integral multiplicities are then handled by addition of chains.

Let $P = abcd$ be a node parallelogram and q its node. We use the node neighbourhood $\mathcal{N}_P$, the circle S_α^1 , the node model and the node pieces $\Pi_q^\pm$ of Section 7.5, with the truncation parameter $\theta' = \theta_4$ of Definition 7.21 and, for $0 < \varsigma \leq \varsigma_{\max}$,

$$\bar{\mu} := \mathcal{S}_G \sinh(L\varsigma),$$

where $\mathcal{S}_G$ is the function of Section 8.2 for the reference monomial a . On $\mathcal{N}_P$ only the four monomials of P among those free of Z and Z' have positive weight, so $\mathcal{S}_G = 1 + |\tilde{\chi}_1| + |\tilde{\chi}_2| + |\mu_P \tilde{\chi}_1 \tilde{\chi}_2|$ is a smooth function of (χ_1, χ_2) , at most $Ct^{-2\theta_2}$ within θ_4 of the node point; since $\varsigma \leq \theta_2/4$ and $2\theta_2 + \theta_2/4 < 1/2 \leq \theta_P$, the bound $\bar{\mu} \leq t^{-2\theta_P}/4$ of Definition 7.21 holds for t small. With this choice the node piece and the divisor pieces along the two legs of its passage are the same subset of W^* where they overlap: by Lemma 7.22(1), over $\mathcal{C}_P$ the node piece Π_q^+ is $\{Z' = 0, (\chi_1, \chi_2) \in \mathcal{C}_P, |\hat{Z}|^2 \leq \mathcal{S}_G \sinh(L\varsigma)\}$, which is the divisor piece of depth ς of Lemma 8.14(1) along either leg of the passage; that divisor piece is defined by the same inequality, which does not depend on the reference monomial, and with the same ratio $\nu_Z/\nu_{Z'}$, which is constant within $2\theta_4$ of the node point.

We write $\Pi_q^\varsigma(g_{ab})$ and $\Pi_q^\varsigma(g_{bc})$ for Π_q^+ , with this $\bar{\mu}$, over the branch of the $b - a$ and of the $c - b$ passage, oriented as the lift of the unit germs g_{ab} and g_{bc} of (M1): by the orientation of the germ followed by that of its torus (Section 7.5).

Lemma 8.15 (node pieces with multiplicity). *For integers c_{ab}, c_{bc} the part of $\partial(c_{ab} \Pi_q^\varsigma(g_{ab}) + c_{bc} \Pi_q^\varsigma(g_{bc}))$ over the thimble $\mathcal{T}_q$ is $(c_{ab} - c_{bc}) \Sigma_q$, with Σ_q oriented by Definition 7.23.*

Proof. Both pieces contain $\Sigma_q = \pi^{-1}(\{0\} \times \mathcal{T}_q)$ (Lemma 7.22(3)). The thimble $\mathcal{T}_q$ is symmetric in x and y , so it is the same disc for the two passages, and Σ_q and its orientation do not depend on the passage (Definition 7.23). The unit germs are standard halves carrying the covectors $-\langle \cdot, b - a \rangle$ and $+\langle \cdot, c - b \rangle$, with node coefficients $\rho_q(g_{ab}) = 1$ and $\rho_q(g_{bc}) = -1$ (Section 8.1); by Lemma 7.24 the part of $\partial \Pi_q^\varsigma(g)$ over $\mathcal{T}_q$ is $\rho_q(g) \Sigma_q$ for each of them. The statement follows by adding. $\square$

In the normal form the summands containing the node halves of g_{ab} and g_{bc} carry c_{ab} and c_{bc} times the covectors of the unit germs, where $b_{m_P}^M = c_{ab} \partial g_{ab} + c_{bc} \partial g_{bc}$ in the basis of Lemma 8.9; both passages may occur, with any integer coefficients, and then $c_{ab} - c_{bc} = \rho_q(b^M)$. The two pieces overlap in W^* , over $\mathcal{T}_q$ and along the S_α^1 -orbits over the half-axis $C^M(u', P)$; as chains they add, and no matching between them is required. Lemma 7.24 is applied only to the two unit germs, and the multiplicities enter by adding. It covers both halves, both orientations and both signs of the covector for either passage, so the other choices of side in (M1), which change the normal form by transverse discs, give the same boundary over the thimble.

8.4. Crossings with an arbitrary invariant coefficient. A transverse disc $v_A \mathfrak{N}_A$ of the normal form may carry any nonzero $v_A \in \text{Inv}^*(A)$: a non-primitive covector, a covector whose kernel meets the vanishing direction with any multiplicity, or a multiple of d_A^* itself. The following lemma gives its piece in all cases. We use the chart $U_{u,u'}$ with the coordinates $Z = \chi^{w_u}$, $Z' = \chi^{w_{u'}}$, $\chi_1 = \chi^{n'-n}$, $\chi_2 = \chi^{n_y}$ of Section 8.2, the dual basis $(e_Z, e_{Z'}, e_{\chi_1}, e_{\chi_2})$ of M to the basis $(w_u, w_{u'}, n' - n, n_y)$ of N , and, for $\varsigma > 0$, the normal disc $\mathfrak{D}_\varsigma := T_e^{-1}(\{y = y(\mathfrak{r}_A), y_e^2 + \alpha_e^2 \leq \varsigma^2\}) \subset B_Y$, the disc $D_\varsigma := E_t(\{y_e^2 + \alpha_e^2 \leq \varsigma^2\})$ in the leg model base, and the Lefschetz neighbourhood $U_{D_\varsigma} := \{(Z, Z', \chi_1) : 1 + \kappa_e \chi_1 + \mathfrak{b} Z Z' = 0, (x_1, x_2) \in D_\varsigma\}$, with $\chi_2 = e^{Ly(\mathfrak{r}_A)}$ fixed. Since $\mathfrak{b}$ depends on $\arg \chi_2$, the fibres over different values of $\arg \chi_2$ are identified by the rotation $\hat{e}_2$ of Lemma 8.16(1) below, which preserves W^* on the exact region, fixes χ_1 , and moves χ_2 and Z together.

Lemma 8.16 (crossings). *Let $v \in \text{Inv}^*(A)$, $v \neq 0$. Write $v = mv_0$ with $m \geq 1$ and v_0 primitive, and $v_0 = k_g g_A^* + k_d d_A^*$. Put $\Lambda_{v_0} := \ker v_0 \cap \Lambda^M$ and $\mathbb{T}(v_0) = \Lambda_{v_0} \otimes U(1)$.*

- (1) *$u' - u \in \Lambda_{v_0}$, and $\Lambda_{v_0} = \langle u' - u, \ell_0 \rangle$ with $\ell_0 := k_g \hat{e}_1 - k_d \hat{e}_2$, where $\hat{e}_1 := a_1 e_Z + e_{\chi_1}$, $\hat{e}_2 := a_2 e_Z + e_{\chi_2}$ and $n = -w_u - w_{u'} + a_1(n' - n) + a_2 n_y$. Put $k_1 := \langle \ell_0, n' - n \rangle = k_g$ and $k_2 := \langle \ell_0, n_y \rangle = -k_d$; then $\gcd(k_1, k_2) = 1$. The monodromy of Λ^M around A preserves Λ_{v_0} and acts on it, in the basis $(u' - u, \ell_0)$, by $\begin{pmatrix} 1 & k_g \\ 0 & 1 \end{pmatrix}$.*
- (2) *If $k_g \neq 0$, the set*

$$\Pi(v_0) := \{(Z, Z', \chi_1, \chi_2) : (Z, Z', \chi_1) \in U_{D_\varsigma}, \log |\chi_2| = Ly(\mathfrak{r}_A), \chi_2^{k_g} \chi_1^{k_d} \in \mathbb{R}_{>0}\}$$

is, through this identification, a connected $|k_g|$ -sheeted cover of U_{D_ς} , a smooth four-manifold without boundary over $\mathfrak{r}_A$, with fibre of type $I_{|k_g|}$ over $\mathfrak{r}_A$ and fibre $\mathbb{T}(v_0)$ over the other points of $\mathfrak{D}_\varsigma$; over $\partial \mathfrak{D}_\varsigma$ it is the union of the $\mathbb{T}(v_0)$ -orbits through the section of phase 0.

- (3) *If $k_g = 0$, then $v_0 = \pm d_A^*$ and $\mathbb{T}(v_0) = \mathbb{T}_d$, the exact torus of the leg. It is generated by the rotation $u' - u$, which rotates Z and Z' oppositely, and by $\hat{e}_2$, which rotates χ_2 and Z ; it fixes χ_1 , and it acts freely at every point with $Z, Z', \chi_2 \neq 0$. The set $\Pi(v_0) := \mathbb{T}_d \cdot \mathfrak{s}^\vee(\mathfrak{D}_\varsigma)$ is homeomorphic to $\mathfrak{D}_\varsigma \times T^2$, and its boundary lies over $\partial \mathfrak{D}_\varsigma$, where it is the union of the $\mathbb{T}_d$ -orbits through $\mathfrak{s}^\vee$.*

The crossing piece of $v \mathfrak{N}_A$ is $m \Pi(v_0)$, with $\varsigma = \varsigma_0$, oriented as the lift of the disc: the orientation of $\mathfrak{N}_A$ followed by that of $\mathbb{T}(v_0)$.

The proof is in Appendix E.

For a strict tropical 2-cycle of [CBM14], the covector at a crossing is primitive and $|k_1| = 1$ (the primitive-field and crossing clauses of [CBM14, Def. 7.2]); then (2) is the piece of [CBM14, §7.7, Step 6]. For relative cycles no splitting of v is needed: when $k_g \neq 0$ the piece $\Pi(v_0)$ is used directly, for every $|k_g| \geq 1$, and the only additional case is $k_g = 0$.

8.5. Joining the local lifts to the positive real section. We truncate the local lifts and join their boundary tori to the positive real section by collar paths. The estimates below keep these paths in the regions used to extend the lift over the regular part.

Truncation near the discriminant. Depths are defined per summand of the normal form (8.2) near Γ^M . Fix for every Γ^M -vertex p a closed ball B_p^0 of radius $r_B \leq r_0$ about p , inside the neighbourhoods of Lemma 8.8(1)–(3) and meeting Γ^M only in the arcs at p . Inside B_p^0 : a node half has depth $|\alpha_q|$; a tripod at a model vertex m with edge $[u, u']$ has depth $\alpha_{uu'} - \alpha_{uu'}(m)$; a turning germ has depth equal to the smaller of the offsets from its two arcs, which are affine coordinates near the turning point, since the angle between the arcs there is less than

π (Lemma 8.4). Outside the balls: a sheet along an arc (exit strip, reference sheet) has depth $|\alpha_e|$, the offset from its arc in the chart T_e ; a rotating stretch and a transverse disc have depth $(y_e^2 + \alpha_e^2)^{1/2}$. The remaining points of a summand, outside the balls and off these sheets, stretches and discs (for instance the parts of tripod and turning-germ sectors far from their arcs), are given depth $+\infty$, so that they belong to the regular part. Let r_{far} be the distance from Γ^M of the set of these points. It is positive: these points lie in the germs, which are fixed before z , and outside the balls B_p^0 every point of the germs close enough to Γ^M lies on an exit strip, a reference sheet, a rotating stretch or a transverse disc. These descriptions agree where they overlap at the points of depth at most $2\varsigma_0$ (with ς_0 as below): at the model nodes and vertices exactly, since the model arcs begin with level segments of $\alpha_{uu'}$, where $\bar{\alpha}$ is constant; at the ends of the stretches since $\Theta = \pm\pi/2$ there; and near ∂B_p^0 at a turning point because there a point at offset at most $2\varsigma_0$ from one arc is at offset at least $c r_B - C \cdot 2\varsigma_0$ from the other, where $c > 0$ depends only on the angles between the arcs at the turning points and $C \geq 1$ is the largest ratio of the gradient norms of the two offsets there. Put $c_B := c/(C+1)$; if $\varsigma_0 \leq c_B r_B/8$, then $c r_B - 2C\varsigma_0 > 2\varsigma_0$, so the smaller offset is the offset from the arc of the strip.

Definition 8.17 (depth). For a summand g and $0 < \varsigma \leq 2\varsigma_0$, $g_{\leq \varsigma}$ is the set of points of g of depth at most ς , the *inner boundary* $\underline{C}_{\varsigma}^{(g)}$ is its frontier in g away from the junction segments, and the *regular part* of g is $g_{\geq \varsigma} := g \setminus g_{\leq \varsigma}$. The inner boundary of a tripod has a trivalent point on the half-edge $C^M(u', \omega)$.

Fix a metric on B_Y and a common Lipschitz constant $C_T \geq 1$ of the inverse charts T_e^{-1} , T_q^{-1} , of the inverses of the charts $p \mapsto (\ell(p), \alpha_{uu'}(p) - \alpha_{uu'}(m))$ of B_Y near the model vertices m (with ℓ as in the next paragraph, in the dual norm of Section 7.2), and of the inverse depth parametrisations near the model vertices and turning points. Fix

$$\varsigma_0 \leq \min(\varsigma_{\max}/2, r_S/4, r_{\natural}/(4C_T), r_{\text{far}}/(4C_T), c_B r_B/8).$$

A point at depth at most $2\varsigma_0$ then lies within $2C_T\varsigma_0 \leq \min(r_{\natural}, r_{\text{far}})/2$ of Γ^M , so it lies at distance at least $r_{\text{far}}/2$ from every point of infinite depth, and a point of finite depth in the closure of the set of points of infinite depth has depth at least $r_{\text{far}}/C_T \geq 4\varsigma_0$; and every piece of depth at most $2\varsigma_0$, and every collar strip below, lies over the region where $z^{\natural}$ consists exactly of the germs and the normal discs (Lemma 8.11), inside the polygonal cylinders of the smoothed rotations.

Base maps. The local pieces of depth ς (Lemmas 8.14 and 8.16, and the node pieces Π_q^{ς} of Section 8.3) lie over the model surfaces: for a summand g the base map π_g sends a point x to the point of the model surface with leg coordinate $y(x) := \log |\chi_2(x)|/L$ (for the arcs of a node passage we take $n_y = d - a$, respectively $n_y = b - a$, according to the passage of g , so that y is the coordinate y_2 , respectively y_1 , of Lemma 7.20(4) up to an additive constant), and with the tropical coordinates $(y_e, \alpha_e) = E_t^{-1}(\log |\kappa_e \chi_1(x)|, \hat{\mu}(x))$ on $\Pi_{a, \varsigma}$ and on the crossing pieces, and the offset $\frac{1}{L} \text{arsinh}(\hat{\mu}(x)/S_G(x))$ from the arc on the divisor pieces, the node pieces and the vertex pieces. Near a model vertex, the position on the three sectors and the half-edge is given by a fixed Lipschitz retraction (for instance a piecewise linear one), onto the tripod of the three arcs, of a neighbourhood of it in the plane of the torus coordinates of ω , which contains the image of the pair of pants; near a turning point, the offsets from the two arcs are given by the moduli of the two Cox coordinates z_2, z_3 . Where two of these descriptions overlap they differ by $O(1/L)$, and π_g is obtained by interpolating them with a partition of unity in the tropical coordinates; since $E_t^{-1}(0, x_2) = (0, \frac{1}{L} \text{arsinh}(x_2/2))$ and $S_G = 2$ on the branch over Ω_e , the

descriptions agree to this order. At a model vertex we fix $r_\rho \in (0, \min(\theta_4/4, \theta_2/(11C_T)))$, with C_T as above. In the torus coordinates $\ell := (\text{val}_{n_b} - \text{val}_{n_a}, \text{val}_{n_c} - \text{val}_{n_a})$ of $\omega = [n_a, n_b, n_c]$, which vanish on J_ω , we take the retraction to send a point near the k -th arc with $|\ell|_\infty \geq r_\rho/2$ to the point of that arc with the same value of the coordinate y of Lemma 8.7(2) for the arc, and the partition of unity to use the vertex description alone over the points of the three sectors with $|\ell|_\infty \leq r_\rho$. Where the vertex description and the description along a tentacle overlap, they then give the same leg coordinate and the same offset, so π_g maps the pieces of a tripod into its three sectors and its half-edge.

The piece of a summand. For a summand g and $0 < \varsigma \leq 2\varsigma_0$ let $\Pi^\varsigma(g)$ be the set of points x of the local pieces of depth ς along g with $\pi_g(x) \in g$: the node piece Π_q^ς over the branch of its passage near a model node, the vertex piece Π_v near a model vertex, the turning piece in D_{u_i} near a turning point if g contains $\text{Turn}(u_i)$, the divisor pieces along its exit strips and reference sheets, the pieces $\Pi_{a,\varsigma}$ along its rotating stretch, and the crossing piece if g is a transverse disc. The local pieces agree where they overlap (Lemma 8.14(1) and Section 8.3), so $\Pi^\varsigma(g)$ is one subset of W^* , each point counted once, and π_g is a proper continuous map of it onto $g_{\leq \varsigma}$, so it induces a homeomorphism from the quotient of $\Pi^\varsigma(g)$ by its fibres onto $g_{\leq \varsigma}$. Over the points of g at positive depth away from the node cubes and from the points of a tripod with $|\ell|_\infty < r_\rho/2$ the fibre is a two-torus, collapsing to a circle over the points of the arcs other than the crossing points $\mathfrak{x}_A$ (Lemma 8.16); the fibres near a model node are those of the node pieces (Section 8.3). Near the half-edge of a tripod the fibres depend on the retraction, and nothing below uses them. If the coefficient of g is $m v_0$ with $m \in \mathbb{Z}$ and v_0 primitive (for a tripod see below), the *piece* of g is $m [\Pi^\varsigma(g)]$, with $\Pi^\varsigma(g)$ oriented as the lift of g : the orientation of g followed by that of $\mathbb{T}(v_0)$. The piece of $-v_0$ is minus that of v_0 , since the set is the same and the torus orientation is reversed. The *outer part* of the piece is $\pi_g^{-1}(C_\varsigma^{(g)}) \cap \Pi^\varsigma(g)$. For a summand containing the node half of g_{ab} , respectively g_{bc} , the part of the piece over the node cube is $c_{ab} \Pi_q^\varsigma(g_{ab})$, respectively $c_{bc} \Pi_q^\varsigma(g_{bc})$, restricted to base points in g ; the thimble lies over $m_P \in g$. A tripod summand of multiplicity m carries $m \langle \cdot, e_k \rangle$ on its k -th sector; its piece is $m \Pi_v$ near the vertex, continued by the divisor pieces of its exit strips, oriented as the lift over each sector: over the k -th sector by the orientation of that sector followed by that of $\mathbb{T}(\langle \cdot, e_k \rangle)$. These three orientations are the restrictions of one orientation of the connected four-manifold Π_v , by the orientations of (M2) and the sign convention of [CBM14, §7.7, Step 2]; Π_v is counted once; at a turning point the pieces of the summands containing $\text{Turn}(u_1)$ and $\text{Turn}(u_2)$, the latter oriented by $-o_e$ as in (M3), lie in the different divisors D_{u_1} and D_{u_2} and are separate chains. For example, a turning point with away-oriented leg data $(c_{12}, c_{13}, c_{23}) = (1, 1, -2)$, which arises from a positive vertex of B'_σ with two turns ending on the same leg, is $-\text{Turn}(u_1) + 2 \text{Turn}(u_2)$ in the basis $(\text{Turn}(u_1), \text{Turn}(u_2))$, or equally $\text{Turn}(u_2) + \text{Turn}(u_3)$, the two differing by the full disc $\text{Turn}(u_1) - \text{Turn}(u_2) + \text{Turn}(u_3)$, which has zero leg data; either way the lift consists of separate turning pieces whose exit sheets run along the common arc on opposite sides, one of them rotated to the reference side. Since the summands are separate, their inner boundaries and pieces may cross in B_Y and in W^* ; nothing below refers to such crossings.

Collar paths. In the exact models a point is determined by $\hat{\mu}$, the moduli and phases of the torus coordinates χ_1, χ_2 , and its position on its S^1_α -orbit, which is free where $\hat{\mu} \neq 0$ or $g_e \neq 0$ (Lemma 7.19, and the same computation for the leg model and, as in Lemma 8.19(1), for the vertex model; where two monomials are in transition, their weights depend on $|\hat{Z}\hat{Z}'|$ with logarithmic derivatives $O(1/L)$, and $|\hat{Z}\hat{Z}'|$ is the fixed point of a contraction). Near a turning point the moduli of the Cox coordinates z_2, z_3 take the place of $\hat{\mu}$. For a covector v with lift

$\tilde{v}$, the $\mathbb{T}(v)$ -orbit of a point x is the set of points with the same $\hat{\mu}$, the same moduli of the torus coordinates and the same phase of $\chi^{\tilde{v}}$; it is a two-torus, and where the model has the exact torus $\mathbb{T}_d$ and $v \in \mathbb{Z}d_A^*$ it is the $\mathbb{T}_d$ -orbit of x . Every piece of Lemma 8.14, and every node piece, has phase π for $\chi^{\tilde{v}}$, v its covector, up to $O(\eta_0^{1/2})$ near the nodes and up to $O(t^{c r_\rho})$ on the sectors of a tripod where $|\ell|_\infty \geq r_\rho$, with $c > 0$ depending only on the configuration; there the fibres of the pieces are graphs over $\mathbb{T}(v)$ -orbits. The $\mathbb{T}(v)$ -orbits through $\mathfrak{s}^\vee$ have phase 0.

On the strip $g_{\leq 2\varsigma_0} \cap g_{\geq \varsigma_0}$ of a summand g the lift consists of *collar paths*, except that near the half-edge of a tripod their second and third stages are replaced by one chain (Lemma 8.19). Over the transverse segment from $b_0 \in C_{\varsigma_0}^{(g)}$ to $b_1 \in C_{2\varsigma_0}^{(g)}$ the collar path of a point runs in three stages. First it runs through the pieces of depths between ς_0 and $2\varsigma_0$ over that segment. Then, from a point x of the piece of depth $2\varsigma_0$ over b_1 , at fixed $\hat{\mu}$, fixed moduli of the torus coordinates and fixed position on the S_α^1 -orbit, it turns the phase of $\chi^{\tilde{v}}$ from its value at x to 0 along one of the two arcs of the circle joining them, keeping the phase of χ^{n_y} fixed, n_y as in Lemma 8.7(2) (for a node passage, the phase of the other node coordinate). Finally, at these phases, $\hat{\mu}$ and the logarithms of the moduli of the torus coordinates move linearly to their values at $\mathfrak{s}^\vee(b_1)$. For the point at which every phase, including the position on the S_α^1 -orbit, is that of the positive real locus, this last stage is a path in Σ_W^+ ending at $\mathfrak{s}^\vee(b_1)$. Since the fibre of the piece over b_1 is a graph over a $\mathbb{T}(v)$ -orbit, the collar paths of its points end on the $\mathbb{T}(v)$ -orbit through $\mathfrak{s}^\vee(b_1)$, which they cover with degree one; where the model has the exact torus, the second and third stages are paths of $\mathbb{T}(v)$ -orbits. The conventions in the second stage matter only where the fibres are not exact $\mathbb{T}_d$ -orbits, that is, near the nodes, along the leaf paths and on the sectors of a tripod; there the collar lies on the side $\hat{\mu} > 0$, the position on the S_α^1 -orbit is measured by $\arg \hat{Z}$, and n_y is the one used by the base map. Where the fibres are exact $\mathbb{T}_d$ -orbits, two choices of the fixed phase or of the trivialisation of S_α^1 differ at each stage by an element of $\mathbb{T}_d$, so the sets swept do not depend on them; at a turning point the second stage fixes the phases of z_2 and z_3 , and where the two ends of an arc prescribe different n_y the change is made over a divisor piece in Ω_e . The two choices of the arc differ by the fibre loop γ_v with $\langle v, \gamma_v \rangle = 1$. The collar path of a summand with coefficient $m v_0$ is m times that of v_0 . Around a crossing no phase is turned, since the crossing piece has phase 0: the collar path runs in Σ_W^+ from the positive point of the piece over b_0 to $\mathfrak{s}^\vee(b_1)$, and the union of the torus orbits through these paths is the crossing piece over the strip.

Orientation of the collar paths. Consider the equivalence relation on the arcs of Γ^M generated by declaring equivalent the two arcs of a turning germ and the two legs of one passage at a model node. All arcs of an equivalence class carry the same covector d_A^* up to sign: through a turning point all arcs of E_e have $d_A^* = \pm \langle \cdot, n' - n \rangle$, and through a node passage $b - a = c - d$. So a class determines a set of edges of $\mathcal{F}_C$, closed under the translations $e \mapsto e + (d - a)$ and $e \mapsto e + (b - a)$ across node parallelograms, and a translation preserves the vector of an edge. Orient these edges compatibly and let $d_{cl} := n' - n$ for the oriented edge $[n, n']$; no cycle of translations reverses the orientation. At an A_2 hexagon the two legs of a node that lie on node-to-node arcs are adjacent and belong to different passages, so an equivalence class enters the hexagon along an outer leg, passes two of its nodes and leaves along an outer leg; it does not run around the cycle of node-to-node arcs. On every arc of a class, and for every summand other than a transverse disc, the geometric collar path turns the phase of $\chi^{d_{cl}}$ from its value near π to 0 in the positive direction; for a summand with covector $m(\pm d_A^*)$ the collar is $\pm m$ times this geometric family, which does not depend on the

sign, since a positive turn of the phase of $\chi^{d_{cl}}$ is a negative turn of that of $\chi^{-d_{cl}}$ along the same paths.

Lemma 8.18 (collar paths). (1) *Along every sector or arc of a summand, and through a turning germ or a node passage, the direction of the collar paths is constant.*
 (2) *At a model vertex the three directions are independent: near the half-edge of a tripod of multiplicity m the collar is m times the chain $\mathcal{C}_v$ of Lemma 8.19, which joins the collars of the three sectors.*
 (3) *At every junction segment the collars of the adjacent summands cancel.*
 (4) *Around a crossing no phase is turned, and the collar runs in the positive locus.*

Proof. (1) The strip cells along an arc are rectangles (segment $\times$ collar path) filled by the explicit family; a change of direction would leave the loop class $\pm\gamma_v \neq 0$ on the boundary of a rectangle. Through a turning point and through a node passage the character is the same on both arcs, by the choice of orientations above. The two legs of a passage, say the $b - a$ passage, are joined inside the node piece. On it, at the depths in $[\varsigma_0, 2\varsigma_0]$ and including over the neck, the coordinate x is $O(|\kappa_P|^{1/2})$, since $|x| \leq |\kappa_P|^{1/2}$ on $\mathcal{C}$; so $\tilde{\chi}_1 = -\mu_P^{-1} + \mu_P^{-1/2}x$ has phase $\pi + O(|\kappa_P|^{1/2})$. For $\hat{\mu} > 0$ a point is determined up to the free action of S_α^1 by $(\hat{\mu}, \chi_1, \chi_2)$ (Lemma 7.19), so the turn of the phase of $\tilde{\chi}_1$ from $\pi + O(|\kappa_P|^{1/2})$ to 0 in the positive direction, at fixed $\hat{\mu}$ and fixed moduli, and the subsequent linear motion of $\hat{\mu}$ and of the logarithms of the moduli, are defined pointwise and continuously over the whole strip at the node, and the direction is constant through the node. The same holds for the $c - b$ passage with y and $\tilde{\chi}_2$. The second circle of the node, the rotation of $\tilde{\chi}_2$ for the $b - a$ passage or of $\tilde{\chi}_1$ for the $c - b$ passage, lies in the fibre torus of the germ, so no half-turn correction arises. (2) The three sectors of a tripod meet only along its half-edge, and near it their collars are replaced by the chain $\mathcal{C}_v$ of Lemma 8.19, whose boundary contains the collars of the three sectors over the points c_k where $|\ell|_\infty = r_\rho$, with whatever directions these have. So no relation among the three directions is needed. (3) The collars on the two sides of a junction segment are given by the same formula, with the same direction and covectors that are multiples of d_A^* summing to zero. (4) The crossing pieces have phase 0 (Lemma 8.16), so the collar paths there run in Σ_W^+ , as described above. $\square$

Near the half-edge of a tripod the vertex model has no exact torus, and the fibres of the vertex piece over points of the sectors close to the half-edge need not be graphs over $\mathbb{T}(v)$ -orbits, so the second and third stages of the collar paths are not used there. They are replaced by one explicit chain in a region with a product structure.

Lemma 8.19 (the collar at a tripod). *Let g be a tripod summand of multiplicity m at the model vertex $m_v = m_{\omega,i}$, $\omega = [n_a, n_b, n_c]$, with edge $[u, u']$ of $\mathcal{T}$, half-edge $h = C^M(u', \omega)$, and let b_1 be the point where $C_{2\varsigma_0}^{(g)}$ meets h . Put $\chi_1 := \chi^{n_b - n_a}$ and $\chi_2 := \chi^{n_c - n_a}$, and use the normalised coordinates for the reference monomial n_a ; near m_v the localized mirror is $\vartheta \hat{Z} \hat{Z}' + \hat{f} = 0$ with $\hat{f} = 1 + \kappa_1 \chi_1 + \kappa_2 \chi_2$, $\kappa_1, \kappa_2 > 0$ and $|\vartheta| = 1$, and $\mathcal{S}_G = 1 + |\kappa_1 \chi_1| + |\kappa_2 \chi_2|$. Write $\ell_j := \frac{1}{L} \log |\kappa_j \chi_j|$, $\zeta := \frac{1}{L} \log \hat{\mu}$ where $\hat{\mu} > 0$, $\Lambda_L(\ell) := \frac{1}{L} \log(1 + e^{L\ell_1} + e^{L\ell_2})$, and let $K \subset \mathbb{R}^3$ be the compact convex set*

$$K := \{(\zeta, \ell) : |\ell|_\infty \leq 2r_\rho, \quad \Lambda_L(\ell) + \frac{1}{2}\varsigma_0 \leq \zeta \leq 4\varsigma_0 + 2r_\rho + \frac{1}{L} \log 3\}.$$

Assume $\varsigma_0 \leq r_\rho$. Then for t small:

- (1) the map $\Phi_v: x \mapsto (\zeta, \ell, \arg \chi_1, \arg \chi_2, \arg \hat{Z})$ restricts to a diffeomorphism from a compact subset $W_K^* \subset W^*$ onto $K \times T^2 \times S^1$, under which S_α^1 rotates the last factor; so W_K^* is homotopy equivalent to a three-torus and $H_3(W_K^*; \mathbb{Z}) \cong \mathbb{Z}$;
- (2) let S'_v be the set of points of $C_{2\zeta_0}^{(g)}$ with $|\ell|_\infty \leq r_\rho$, with end points c_1, c_2, c_3 on the three sectors. Then W_K^* contains the part A over S'_v of the outer part of the piece of depth $2\zeta_0$, the union A^+ of the $\mathbb{T}(\langle \cdot, e_k \rangle)$ -orbits through the points $\mathfrak{s}^\vee(b)$, $b \in S'_v$ on the k -th sector, the second and third stages B_k of the collar paths of the k -th sector over c_k , and the fibre $\mathbb{T}_{b_1} := \Phi_v^{-1}(\{k_1\} \times T^2 \times S^1)$, where k_1 is the (ζ, ℓ) of $\mathfrak{s}^\vee(b_1)$; all of them are invariant under S_α^1 ;
- (3) there are an integral 2-chain $\mathcal{Q}'(1)$ in $\{k_1\} \times T^2$ with $\partial \mathcal{Q}'(1) = \sum_k s_k G_k$, where G_k is the image of the $\mathbb{T}(\langle \cdot, e_k \rangle)$ -orbit through $\mathfrak{s}^\vee(b_1)$, oriented so that $\Phi_v^{-1}(G_k \times S^1)$ is that orbit with its canonical orientation (Section 8.6), and $s_k = -1$ is the sign with which the k -th sector induces the orientation of the half-edge pointing away from m_v [CBM14, §7.7, Step 2], and an integral 4-chain $\mathcal{C}_v$ in W_K^* , explicit up to the choice of a filling in $\mathbb{T}_{b_1}$, with

$$\partial \mathcal{C}_v = D := A^+ + \Phi_v^{-1}(\mathcal{Q}'(1) \times S^1) - A - \sum_k B_k.$$

Here S'_v carries its orientation as part of the boundary of $g_{\leq 2\zeta_0}$; A is oriented as part of the boundary of the piece, B_k as part of the boundary of the collar of the k -th sector beyond c_k , and A^+ and the filling term as the lift (the orientation of the base followed by that of the fibre); with these orientations D is a cycle. The class of $\mathcal{Q}'(1)$ modulo boundaries in $\{k_1\} \times T^2$ is determined by the condition $[D] = 0$ in $H_3(W_K^*; \mathbb{Z})$.

Over the points of its strip with $|\ell|_\infty \leq r_\rho$ the collar of g is the first stage, m times the vertex piece between depths ζ_0 and $2\zeta_0$ over these points, followed by $m \mathcal{C}_v$; over the rest of the strip it consists of the collar paths of the three sectors.

The proof is in Appendix E. The chain $\mathcal{C}_v$ is obtained by contracting the moduli (ζ, ℓ) linearly to k_1 , which carries D into the single fibre $\mathbb{T}_{b_1}$, and filling the image there; the contraction stays in W_K^* because K is convex. Only the boundary of $\mathcal{C}_v$ enters the later gluing, and its terminal part A^+ and the filling term lie in the regions of Section 8.6 by Lemma 8.22.

By Lemma 8.22 below, the ends of the collar paths of a summand over an edge of its inner boundary curve $C_{2\zeta_0}^{(g)}$ lie in the region $W_{V_s}^*$ associated with the adjacent simplex s of the triangulation $\mathfrak{T}$ (Section 8.6); these regions are defined in Definition 8.20. By the same lemma, the collar paths over a compact subset $\mathcal{W}$ of a regular set V lie in W_V^* once t is below a threshold depending on V , $\mathcal{W}$ and ϵ . The loop classes in the proof of Lemma 8.18(1) are taken in such regions, over a subdivision of the strip into finitely many compact pieces, each in a regular set (the strip is a compact subset of $B_Y \setminus \Gamma^M$, covered by the open stars of Proposition 4.7); these regular sets are chosen after ζ_0 , and their thresholds enter the final choice of t (Section 8.8).

8.6. Extending the lift over the regular part. The local pieces determine boundary data for a cell-by-cell extension over the remaining domain. The essential input is Lemma 8.21: each target region is aspherical, and its fundamental group is the prescribed integral lattice.

Regions. Over the regular part the lift is built in regions of W^* defined by shadows. For a vertex u of $\mathcal{T}$ let $\text{Vis}(u) \subseteq B_Y$ be the union of the facets F_n^* with $n \in u^\vee$, and $\pi_u: M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\mathbb{R}u$ the projection. Then π_u is injective on $\text{Vis}(u)$: if $p \in F_n^*$ and $p + su \in F_n^*$, with

$n, n' \in u^\vee$, then $0 \geq \text{val}_n(p + su) = -s$ and $0 \geq \text{val}_{n'}(p) = s$. The vertex region R_u^M lies in $\text{Vis}(u)$. Let $U_u \subset X_{\mathcal{T}}$ be the union of the dense torus and the open orbit of D_u . The characters χ^m , $m \in u^\perp \cap N$, are regular and nowhere zero on U_u , so $\pi_u(\text{Log } x) := (\log |\chi^m(x)|)_m$ is defined on U_u . For $x \in W^* \cap U_u$ with $\pi_u(\text{Log } x)/L \in \pi_u(\text{Vis}(u))$, the u -shadow $\text{sh}_u(x)$ is the point of $\text{Vis}(u)$ with $\pi_u(\text{sh}_u(x)) = \pi_u(\text{Log } x)/L$; for x in the torus, $\text{Log } x/L = \text{sh}_u(x) + \lambda_u(x)u$. For a vertex n of $\mathcal{F}_C$ fix $m_n \in M$ with $\langle m_n, n \rangle = 1$; the projection $\varpi_n: M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}/\mathbb{R}m_n$ restricts to an affine isomorphism of the hyperplane $\{\text{val}_n = 0\}$. For x in the torus the n -shadow $\text{sh}_n(x)$ is the point of that hyperplane with the same image under ϖ_n as $\text{Log } x/L$, and $\text{Log } x/L = \text{sh}_n(x) + \lambda_n(x)m_n$.

Definition 8.20 (regions). A *regular set* is an open set $V \subseteq B_Y$ such that either its closure is compact in the open star of a cell $C^M(u, \omega)$ and $\pi_u(V)$ is convex (V is of *type* u), or $\bar{V}$ is contained in no vertex region, its closure is compact in the open star of a cell $C^M(\tau, n)$ with $\dim \tau \geq 1$, and $\varpi_n(V)$ is convex (type n). For $\epsilon > 0$ the *region* over V is

$$W_V^* := \{x \in W^* \cap U_u : \text{sh}_u(x) \in V, \lambda_u(x) < \epsilon\} \quad (\text{type } u),$$

$$W_V^* := \{x \in W^* \cap T : \text{sh}_n(x) \in V, |\lambda_n(x)| < \epsilon\} \quad (\text{type } n),$$

where $\lambda_u := -\infty$ on D_u and T is the dense torus.

The open stars of the cells $C^M(u, \omega)$ and $C^M(\tau, n)$ cover $B_Y \setminus \Gamma^M$ (Proposition 4.7); the first lie in R_u^M , the second in $\text{int } F_n^*$.

Lemma 8.21 (regular regions). *Let V be a regular set. There is $\epsilon_V > 0$ such that for $\epsilon \leq \epsilon_V$ and $t < t_V(\epsilon)$:*

- (1) *if V is of type u , then W_V^* is the graph of a smooth function $Z_u = \varphi$ over the set $\text{Log}^{-1}(L\pi_u(V))$ of the torus $(\mathbb{C}^*)^3$ of the other Cox coordinates of a maximal cone containing u ; so $W_V^* \cong T^3 \times \pi_u(V)$, and the inclusion $W_V^* \subset U_u$ induces an isomorphism $\pi_1(W_V^*) \rightarrow H_1(U_u) = M/\mathbb{Z}u$;*
- (2) *if V is of type n , then W_V^* is the graph of a smooth function $\chi^n = \varphi$ over the set $\text{Log}^{-1}(L\varpi_n(V))$ of the torus of the characters in $m_n^\perp \cap N$, and the inclusion $W_V^* \subset T$ induces an isomorphism of $\pi_1(W_V^*)$ onto $n^\perp \cap M \subset M = H_1(T)$;*
- (3) *if V' and V are regular sets with $\bar{V}' \subset V$, and $W_{V'}^*, W_V^*$ are taken with constants $\epsilon' \leq \epsilon$, then $W_{V'}^* \subseteq W_V^*$ for $\epsilon \leq \epsilon_{V',V}$ and $t < t_{V',V}(\epsilon')$, also when V' is of type u and V of type n ; the inclusion induces the parallel transport of Λ^M .*

In particular every region is aspherical, with fundamental group the lattice of flat sections of Λ^M over V .

The proof is in Appendix E.

Lemma 8.22 (inner boundary). *Let $\varsigma := \varsigma_0$, and let x be a point of W^* of one of the following kinds: a point of the piece of depth 2ς of a summand g whose base point $\pi_g(x)$ lies at depth at least ς ; a point of a collar path of g over a point b of its strip, and then $\pi_g(x) := b$; for a tripod summand g , a point of the orbits A^+ of Lemma 8.19 over a point b of S'_v , or of its fibre $\mathbb{T}_{b_1}$, and then $\pi_g(x) := b$, respectively b_1 ; or a point $\mathfrak{s}^\vee(b)$, and then $\pi_g(x) := b$. There are $\varsigma_1 \in (0, \varsigma/2]$ and C , depending only on the configuration, the model arc systems, Υ^M , the model germs and their auxiliary choices (the balls B_p^0 , the retraction at the model vertices and the partition of unity defining the base maps π_g), the bounds on the coefficients over $\mathcal{C}_0$, and ς , such that:*

- (1) *if $L \geq 1/\varsigma_1$, then for every vertex u of $\mathcal{T}$ with $\pi_g(x) \in R_u^M$ the point x lies in U_u , $|\text{sh}_u(x) - \pi_g(x)| \leq C/L$, and $\lambda_u(x) \leq C/L$ if x lies in the torus;*

- (2) for every $\varsigma_2 > 0$ there is L_{ς_2} such that, if $\pi_g(x)$ lies at distance at least ς_2 from the two-skeleton of B_Y and $L \geq L_{\varsigma_2}$, then x lies in the torus and $|\log x/L - \pi_g(x)| \leq C/L$.

Consequently, if V is a regular set and $\mathcal{W} \subset V$ is compact, then every such x with $\pi_g(x) \in \mathcal{W}$ lies in W_V^* , for $t < t_{V,\mathcal{W}}(\epsilon)$.

The proof is in Appendix E.

The abstract domain. Let $\mathcal{S}$ be the finite 2-complex obtained from the disjoint union of

- every summand of (8.2) near Γ^M (the transverse discs $\mathfrak{N}_A^{\varepsilon'}$ included), restricted to depth at least $2\varsigma_0$ and continued inside U up to ∂U ,
- the correction strips and annuli of the two local modifications,
- the simplices of $y^\natural$ outside U ,

by identifying (i) the junction segments of adjacent summands, beyond depth $2\varsigma_0$, and (ii) the curves along which a summand ends on another summand: on ∂U , where it meets y ; on the circles $\partial \mathfrak{N}_A^{\varepsilon'}$ and $\partial(\bar{D}(E_A) \cap N_{\varepsilon_A})$ bounding the annuli; and on the polygonal cylinders $\{\varrho_P = \varrho_0\}$, where it meets a correction strip (this includes the seam of a smoothed rotation, within the stretch after $y_{A,p}^*$). Segments interior to a summand, such as the transverse segments where a germ continues as its own exit strip and the start of a rotation, are not cut. Summands that meet transversally in B_Y (a transverse disc and a sheet along its ray, the two node halves along $C^M(u', P)$, exit sheets of different germs meeting only on Γ^M) are *not* identified. The triangulation $\mathfrak{K}_2$ is chosen so that the finitely many curves of transversal meeting cross ∂U and the boundaries of the replacement neighbourhoods and cylinders transversally in finitely many points. Triangulate $\mathcal{S}$ compatibly, each summand triangulated so that the triangulations agree on the identified curves and $y^\natural$ simplicially, with the inner boundary curves $C_{2\varsigma_0}^{(g)}$ as subcomplexes, whose vertices are the ends of the transverse collar segments issuing from the vertices of a fixed triangulation of $C_{\varsigma_0}^{(g)}$ (the strips $g_{\leq 2\varsigma_0} \cap g_{\geq \varsigma_0}$ themselves are not part of $\mathcal{S}$; they are lifted by the collar paths). The inclusions define a continuous map $j^\natural: \mathcal{S} \rightarrow B_Y \setminus \Gamma^M$. Put on each 2-simplex s of $\mathcal{S}$ the coefficient v_s of its summand, a flat section of $(j^\natural)^*(\Lambda^M)^\vee$ over s .

Lemma 8.23 (the coefficient chain). *$v = \sum_s v_s[s]$ is a 2-chain on $\mathcal{S}$ whose boundary is supported on the inner boundary curves, where it is $\sum_g (\text{coefficient of } g) [C_{2\varsigma_0}^{(g)}]$.*

Proof. At an edge inside a summand, including the transverse segments where a germ continues as its own exit strip and the start of a rotation, the two sides carry the same coefficient. At a tripod edge, on the half-edge of a tripod summand of multiplicity m , the three adjacent coefficients $m\langle \cdot, e_k \rangle$ sum to zero with the orientations of (M2). At a junction segment the adjacent coefficients are multiples of one covector d_A^* summing to zero. At an identified curve of type (ii) the adjacent triangles of $\mathcal{S}$ are exactly the triangles of $z^\natural$ at that edge of B_Y , by the transversality of the meeting curves, and their coefficients sum to zero because $z^\natural$ is a relative cycle with boundary only on Γ^M . $\square$

Since summands that meet transversally are different components of $\mathcal{S}$ near the meeting curve, $\mathcal{S}$ has no edges along which two summands cross: a transverse disc and a sheet on the same arc, the two node halves at a node used by both passages, and the two turning germs of a turning point used twice on one arc are lifted separately. Branching edges inside $y^\natural$, coming from the arbitrary chain off Γ^M , remain; they are treated like any other edge.

Triangulation and regular sets. Distances and neighbourhoods in B_Y are taken in the fixed metric on B_Y . By a Lebesgue number argument for the cover of the compact set $j^\sharp(\mathcal{S}) \subset B_Y \setminus \Gamma^M$ by the open stars of Proposition 4.7, there is $\mathfrak{l} > 0$, depending only on $j^\sharp(\mathcal{S})$, such that the $\mathfrak{l}$ -neighbourhood in B_Y of every subset of $j^\sharp(\mathcal{S})$ of diameter at most $\mathfrak{l}$ lies in the open star of a cell $C^M(u, \omega)$ or $C^M(\tau, n)$. Then let $\mathfrak{T}$ be a subdivision of the triangulation of $\mathcal{S}$, fine enough that the image of every simplex has diameter small compared with $\mathfrak{l}$; so the $\mathfrak{l}$ -neighbourhood of each $j^\sharp(s)$ lies in such an open star. Let V_s be the preimage in $\text{Vis}(u)$ under π_u , or in $\{\text{val}_n = 0\}$ under ϖ_n , of the convex hull of the image under that map of the $\mathfrak{r}_{\dim s}$ -neighbourhood of $j^\sharp(s)$ in B_Y , with radii $\mathfrak{r}_0 < \mathfrak{r}_1 < \mathfrak{r}_2$, each small compared with the next, and $\mathfrak{r}_2$ small compared with $\mathfrak{l}$. We take type u , with π_u , whenever $j^\sharp(s)$ lies in a vertex region R_u^M and the set so formed has compact closure in the open star of a cell $C^M(u, \omega)$; this holds if the $\mathfrak{l}$ -neighbourhood lies in such an open star. Otherwise we take type n , with ϖ_n , for a cell $C^M(\tau, n)$ whose open star contains the $\mathfrak{l}$ -neighbourhood. In that case $\overline{V_s}$ lies in no vertex region: otherwise it would lie in $R_u^M \cap \text{int } F_n^* = C^M(u, n)$ (Lemma 8.5(3)), where π_u and ϖ_n restrict to affine isomorphisms of the hyperplane $\{\text{val}_n = 0\}$, so the set formed with π_u would be the same and of type u . Then V_s is a regular set, $V_s \supset j^\sharp(s)$, and $\overline{V_{s'}} \subset V_s$ for $s' \subsetneq s$; if s is of type u , so is s' , so a set of type n never lies in one of type u . We take ϵ and t so small that Lemmas 8.21 and 8.22 hold for these finitely many sets, with $\mathcal{W}$ the images of the simplices. This uses only that $j^\sharp$ is continuous and $|\mathfrak{T}|$ compact, not that $\mathfrak{T}$ triangulates an embedded surface.

The domain over the regular part. For a simplex s of $\mathfrak{T}$ let Λ_s be the lattice of flat sections of Λ^M over V_s , which is $\pi_1(W_{V_s}^*)$ (Lemma 8.21); for $s' \subset s$ the inclusion $W_{V_{s'}}^* \subseteq W_{V_s}^*$ induces the transport $\Lambda_{s'} \cong \Lambda_s$. We orient $\Lambda_s \otimes \mathbb{R}$ by the orientation of B_Y , and put $\mathbb{T}_s^3 := \Lambda_s \otimes \mathbb{R} / \mathbb{Z}$, a $K(\Lambda_s, 1)$. For a primitive covector v^0 on Λ_s the *canonical torus* is $\mathbb{T}(v^0) := (\ker v^0 \otimes \mathbb{R}) / \ker v^0 \subset \mathbb{T}_s^3$, with the tautological identification $\pi_1(\mathbb{T}(v^0)) = \ker v^0 \subseteq \Lambda_s$, oriented so that $\text{or}(\ker v^0) \wedge n = \text{or}(\Lambda_s \otimes \mathbb{R})$ for $v^0(n) > 0$. Then $[\mathbb{T}(v^0)] \mapsto v^0$ is the restriction of the linear isomorphism $H_2(\mathbb{T}_s^3) = \Lambda^2 \Lambda_s \rightarrow \Lambda_s^\vee$, $a \wedge b \mapsto \det(a, b, \cdot)$: for an oriented basis (a, b) of $\ker v^0$, $\det(a, b, \cdot)$ vanishes on $\ker v^0$, is positive where v^0 is, and is primitive.

Three kinds of edges. An edge e of $\mathfrak{T}$ is a *sheet edge* if the coefficients of all its adjacent 2-simplices are, after transport to Λ_e , integer multiples $\mu_s v_e^0$ of one primitive covector v_e^0 ; by Lemma 8.23, $\sum_s [s : e] \mu_s = 0$ unless e lies on an inner boundary curve. Edges inside a summand and on junction segments are sheet edges. An edge is a *tripod edge* if it lies on the half-edge $C^M(u', \omega)$ of a tripod summand. All other edges, the branching edges of $y^\sharp$ and the edges on identified curves of type (ii) where several summands meet, are *general edges*. We take $\mathfrak{T}$ so fine that every edge meeting an inner boundary curve is a sheet edge or a tripod edge; the identified curves of type (ii) lie at distance at least $r_\sharp$ from Γ^M , hence at depth at least $r_\sharp / C_T \geq 4\zeta_0$.

Fibre complexes. For a 2-simplex s with $v_s \neq 0$ write $v_s = m_s v_s^0$ with $m_s \geq 1$ and v_s^0 primitive, and put $F_s := \mathbb{T}_s := \mathbb{T}(v_s^0)$; if $v_s = 0$, F_s is a point. For $s' \subset s$ the transport maps kernels to kernels and induces an isomorphism of canonical tori; these transports are induced by inclusions of regions, so they compose consistently around every vertex of $\mathfrak{T}$, and no composite of identifications below is a nontrivial automorphism of a torus. For an edge e :

- at a sheet edge, all adjacent $\mathbb{T}_s$ are identified with $F_e := \mathbb{T}(v_e^0)$ by the canonical maps, with the orientation sign of μ_s , so that $m_s [\mathbb{T}_s] = \mu_s [F_e]$; and $\mathcal{Q}_e := 0$;
- at a tripod edge, of a tripod summand of multiplicity m_e , the three adjacent sectors carry the covectors $m_e \langle \cdot, e_k \rangle$, whose kernels meet in a rank-one sublattice ℓ (the three

e_k span a plane). Let $F_e := \mathbb{T}_e^3$, with a cell structure in which the three subtori $\mathbb{T}_s$ are subcomplexes, and $\mathcal{Q}_e := m_e \mathcal{Q}_e^0$, $\mathcal{Q}_e^0 := (\ell \otimes U(1)) \times \mathcal{Q}'_e$, where $\mathcal{Q}'_e$ is a 2-chain in $\mathbb{T}_e^3/(\ell \otimes U(1))$ with $\partial \mathcal{Q}'_e = \sum_k s_k G_k$, the images of the three tori with the signs of the boundary orientations [CBM14, §7.7, Step 2]. On the edge e_0 of the half-edge at the inner boundary, $\mathcal{Q}'_{e_0}$ is the filling fixed in the next paragraph; on the other edges of the half-edge it is any such filling, with the incidence orientation on each edge;

- at a general edge, let $Y_e := \bigvee_{s \supset e} F_s$, realise the labels by a cellular map $\gamma_e: Y_e \rightarrow \mathbb{T}_e^3$, and let F_e be its mapping cylinder, a cell complex containing Y_e and retracting onto $\mathbb{T}_e^3$, labelled by $\text{lab}_e: \pi_1(F_e) \cong \Lambda_e$.

For a vertex p of $\mathfrak{T}$, F_p is the union of the F_e , $e \ni p$, glued along the common F_s ; the labels agree on the F_s and define $\text{lab}_p: \pi_1(F_p) \rightarrow \Lambda_p$, which kills the boundary of every 2-cell, since each 2-cell lies in some F_e .

The tripod edge at the inner boundary. The branching half-edge of each tripod summand meets its inner boundary once, at a vertex p_0 of $\mathfrak{T}$ on an edge e_0 of the half-edge, and its other end lies on ∂U beyond the collar region. We orient e_0 away from p_0 , so that $[e_0 : p_0] = -1$ and the signs s_k of Lemma 8.19(3) are the incidence numbers $[s_k : e_0] = -1$. At p_0 the choices are made in the following order, so that the terminal chains agree.

- (i) $F^\vee$ is fixed on $p_0 \times \mathbb{T}_{e_0}^3$ as a diffeomorphism onto the fibre $\mathbb{T}_{b_1}$ of Lemma 8.19(2), $b_1 := j^{\sharp}(p_0)$, which realises the labels, as Lemma 8.25 requires, sends the base point to $\mathfrak{s}^\vee(b_1)$, carries each $\mathbb{T}_s$ onto the corresponding $\mathbb{T}(v)$ -orbit through $\mathfrak{s}^\vee(b_1)$, and carries the $\ell \otimes U(1)$ -orbits onto the S_α^1 -orbits (here $\ell \otimes U(1)$ is S_α^1 , Lemma 8.14(2)). Such a map, affine in the phase coordinates of Φ_v , exists: $\mathbb{T}_{b_1}$ projects diffeomorphically onto the torus factor of the graph description of $W_{V_{p_0}}^*$ in Lemma 8.21, so its inclusion into $W_{V_{p_0}}^*$ is a homotopy equivalence, and the labels identify Λ_{e_0} with its fundamental group.
- (ii) The filling $\mathcal{Q}'(1)$ in $\{k_1\} \times T^2$ and the chain $\mathcal{C}_v$ of Lemma 8.19(3) are chosen.
- (iii) $\mathcal{Q}'_{e_0}$ is the pull-back of $\mathcal{Q}'(1)$ under the diffeomorphism of quotient tori induced by $F^\vee$. It is a 2-chain with boundary $\sum_k s_k G_k$, and $F_*^\vee \mathcal{Q}'_{e_0} = \Phi_v^{-1}(\mathcal{Q}'(1) \times S^1)$ as chains.

Fillings over general edges. At a general edge, $\sum_{s \supset e} [s : e] v_s = 0$ in $\Lambda_e^\vee$ by Lemma 8.23. Under the linear isomorphism $H_2(F_e) = H_2(\mathbb{T}_e^3) \cong \Lambda_e^\vee$ above, the class of $\sum_{s \supset e} [s : e] m_s [\mathbb{T}_s]$ is $\sum_s [s : e] m_s v_s^0 = \sum_s [s : e] v_s = 0$, so there is a cellular 3-chain $\mathcal{Q}_e$ in F_e with

$$\partial \mathcal{Q}_e = \sum_{s \supset e} [s : e] m_s [\mathbb{T}_s].$$

No common rank-one sublattice of the kernels and no bound on the valency of e is needed.

The domain. Let $\mathcal{D}_{\text{reg}}^\vee$ be the union of the $[s] \times F_s$ over the simplices s of $\mathfrak{T}$, glued along $[s'] \times F_s \subseteq [s'] \times F_{s'}$ for $s' \subset s$, with the fundamental chain

$$[\mathcal{D}_{\text{reg}}^\vee] := \sum_{s \in \mathfrak{T}_2} m_s [s \times \mathbb{T}_s] + \sum_{e \in \mathfrak{T}_1 \setminus \text{inner boundary}} [e \times \mathcal{Q}_e],$$

each $s \times \mathbb{T}_s$ oriented as the product. Over an inner boundary edge e the adjacent triangles lie in one summand g , the other side of the inner boundary being the strip lifted by the collar paths, and the cells $e \times \mathbb{T}_s$ are glued to the ends of the collar paths over e , the union of the $\mathbb{T}_s$ -orbits through $\mathfrak{s}^\vee$ (near a tripod, the orbits A^+ of Lemma 8.19); over the point where the inner boundary meets a tripod edge, the filling $m_e \mathcal{Q}_e^0$ is glued to m_e times the filling term $\Phi_v^{-1}(\mathcal{Q}'(1) \times S^1)$ in the boundary of the chain $\mathcal{C}_v$ of that lemma.

Lemma 8.24 (boundary of the domain). *Let $\tilde{\Xi}_{\text{loc}}$ be the sum of the pieces of depth ς_0 of the summands, with their multiplicities, and of their collars: the collar paths, and near the half-edge of each tripod the chain $m\mathcal{C}_v$ of Lemma 8.19 (Section 8.5). Then $\partial(\tilde{\Xi}_{\text{loc}} + [\mathcal{D}_{\text{reg}}^\vee])$, with $[\mathcal{D}_{\text{reg}}^\vee]$ glued as above, is*

$$\sum_q \rho_q(z) \Sigma_q + \sum_p [p \times z_p], \quad z_p := \begin{cases} \sum_{e \ni p} [e : p] \mathcal{Q}_e, & p \text{ off the inner boundary,} \\ 0, & p \text{ on the inner boundary,} \end{cases}$$

the second sum over the vertices p of $\mathfrak{T}$, where z_p is a 3-cycle of F_p .

Proof. The piece of a summand g is $m[\Pi^{\varsigma_0}(g)]$, a four-manifold with corners mapped properly onto $g_{\leq \varsigma_0}$ by π_g . The orbits collapse to circles over the arcs, and to points over the turning points, without creating boundary (Lemma 8.14(1),(3)), and the crossing pieces have no boundary over $\mathfrak{r}_A$ (Lemma 8.16). The seams inside g , where the description near a vertex passes to the description along an exit strip or where a rotation starts, contribute nothing: $\Pi^{\varsigma_0}(g)$ is one set with one coefficient, the local descriptions being glued along their overlaps and not added; in particular a vertex piece Π_v has, inside the piece of its summand, only its outer part, and continues along its tentacles as the divisor pieces of the same summand. So the boundary of the piece consists of three parts.

- (a) Over the thimbles: for the two summands containing the node halves at m_P , the parts over the node cube are $c_{ab}\Pi_q^{\varsigma_0}(g_{ab})$ and $c_{bc}\Pi_q^{\varsigma_0}(g_{bc})$, and by Lemma 8.15 their boundary over $\mathcal{T}_q$ is $(c_{ab} - c_{bc})\Sigma_q = \rho_q(b^M)\Sigma_q = \rho_q(z)\Sigma_q$.
- (b) Over the junction segments at depth at most ς_0 , which lie inside the polygonal cylinders at the far ends of the rotations: the adjacent summands have pieces over the same subset of W^* (divisor pieces or pieces Π_{a,ς_0} of one arc, which agree across the junction), with coefficients multiples of d_A^* summing to zero with the orientation signs; these faces cancel.
- (c) The outer parts over $C_{\varsigma_0}^{(g)}$. They are matched by the collar cells over the strips $g_{\leq 2\varsigma_0} \cap g_{\geq \varsigma_0}$, which cancel along the junction segments (Lemma 8.18(3)) and end at the unions of the torus orbits through $\mathfrak{s}^\vee$ over the curves $C_{2\varsigma_0}^{(g)}$; these are the cells $e \times \mathbb{T}_s$ over the inner boundary edges. Near the half-edge of a tripod the first stage of the collar paths ends on the part A of the outer part at depth $2\varsigma_0$, and $\partial(m\mathcal{C}_v) = mD$ replaces A by the orbits A^+ and the filling term, while the terms B_k cancel against the collars of the sectors beyond the points c_k (Lemma 8.19(3)).

Pieces of different summands that intersect in W^* (the two node halves, a transverse disc and a sheet) are separate chains, and their intersection contributes no boundary.

Over the regular part, $\partial(\mathfrak{s} \times \mathbb{T}_s) = \partial\mathfrak{s} \times \mathbb{T}_s$, so the faces over an edge e are $e \times \sum_{s \supset e} [s : e] m_s \mathbb{T}_s$. At a sheet edge off the inner boundary this is $e \times (\sum_s [s : e] \mu_s) F_e = 0$. At a tripod edge and at a general edge, $\partial(e \times \mathcal{Q}_e) = \partial e \times \mathcal{Q}_e - e \times \partial \mathcal{Q}_e$, and $e \times \partial \mathcal{Q}_e$ cancels the faces over e by the choice of $\mathcal{Q}_e$ (for tripod edges this is [CBM14, §7.7, Step 2], multiplied by m_e). What remains of the edge terms at a vertex p off the inner boundary is $p \times z_p$, and

$$\partial z_p = \sum_{e \ni p} [e : p] \sum_{s \supset e} [s : e] m_s [\mathbb{T}_s] = \sum_{s \ni p} m_s [\mathbb{T}_s] \sum_e [e : p] [s : e] = 0,$$

because every 2-simplex $s \ni p$ contributes through its two edges at p with opposite signs. At an inner boundary vertex the boundary of the collar must also be included. Along a sheet it cancels the product faces exactly. Where a tripod half-edge meets the inner boundary, its endpoint term $[e : p] \mathcal{Q}_e$ is cancelled by the oppositely oriented filling term of $\partial(m_e \mathcal{C}_v)$,

since $F_*^\vee \mathcal{Q}_e$ is m_e times $\Phi_v^{-1}(\mathcal{Q}'(1) \times S^1)$ by the choice of $\mathcal{Q}_{e_0}^0$ at the tripod edge at the inner boundary. There are no general edges at the inner boundary. Thus no residual chain remains there, giving $z_p = 0$ as stated. At the vertices of a tripod half-edge off the inner boundary, z_p is the difference of the fillings of the two adjacent edges; it need not vanish, and it is kept in the sum over p . $\square$

The extension.

Lemma 8.25 (extension). *There is a continuous map $F^\vee: \mathcal{D}_{\text{reg}}^\vee \rightarrow W^*$ which maps the cells over each simplex $\mathfrak{s}$ of $\mathfrak{T}$ into $W_{V_s}^*$, sends fibre loops to loops representing their labels, is $\mathfrak{s}^\vee \circ j^\sharp$ on the base cells $\mathfrak{s} \times \{\star\}$, and on the inner boundary cells agrees with the collar ends.*

Proof. We extend cell by cell in order of dimension. On 0-cells use $\mathfrak{s}^\vee$, which lies in $W_{V_s}^*$ by Lemma 8.22. On 1-cells, base cells go along $\mathfrak{s}^\vee$ and fibre 1-cells to loops of the labelled classes. A 2-cell extends if its boundary loop is trivial in the abelian group $\Lambda_s = \pi_1(W_{V_s}^*)$: for $\mathfrak{s} \times \{\star\}$, $\mathfrak{s}$ a 2-simplex, the boundary loop lies in $\mathfrak{s}^\vee(j^\sharp \mathfrak{s})$, which is contractible in $W_{V_s}^*$; for $e \times f^1$, the two fibre loops represent the same parallel class (Lemma 8.21(3)); for $p \times f^2$, f^2 a 2-cell of F_p , $\text{lab}_p(\partial f^2) = 0$. Cells of dimension three and four extend because $W_{V_s}^*$ is aspherical (Lemma 8.21). The inner boundary cells, including the cells $p_0 \times \mathbb{T}_{e_0}^3$ fixed in step (i) at the tripod edges, are prescribed by the collars, which satisfy the same loop conditions (Lemma 8.18) and lie in the right regions (Lemma 8.22). $\square$

8.7. The positive real locus and the fibre class. The extension leaves a sum of residual three-cycles, each homologous to a multiple of a torus fibre. Intersection with the positive real three-sphere shows that their total fibre coefficient is zero. This replaces the local argument of [CBM14, §7.7, Step 5], which shows that the lift of a strict cycle is a manifold at its interior vertices: with arbitrary multiplicities and branching the residual cycles at the vertices of $\mathfrak{T}$ need not vanish one by one, and only their total class is controlled.

Put $\check{\Xi}_0 := \check{\Xi}_{\text{loc}} + F_*^\vee[\mathcal{D}_{\text{reg}}^\vee]$, an integral four-chain on W^* . By Lemma 8.24,

$$\partial \check{\Xi}_0 = \sum_q \rho_q(z) \Sigma_q + \sum_p F_*^\vee z_p.$$

Each $F_*^\vee z_p$ lies in an aspherical region $W_{V_p}^* \simeq T^3$, so $[F_*^\vee z_p] = n_p [T_p^3]$ in $H_3(W_{V_p}^*) \cong \mathbb{Z}$, where T_p^3 is a fibre torus over p oriented by the orientation of Λ^M . The monodromy of Λ^M lies in $SL(\Lambda^M)$, since that of Λ_Y does (Proposition 4.10(4)), and overlapping regions are homotopy equivalent to each of their members; so all the $[T_p^3]$ agree in $H_3(W^*)$. We call this class $[T^3]$. It is represented by an orbit of the compact subtorus $(n^\perp \cap M) \otimes U(1)$ through a positive point of a region over a cell $C^M(\tau, n)$ at which only the monomials 0 and n have positive weight.

Proposition 8.26 (no fibre-class boundary). $\sum_p n_p = 0$. *Consequently there is an integral four-chain $\check{\Xi}^W(z)$ on W^* with $\partial \check{\Xi}^W(z) = \sum_q \rho_q(z) \Sigma_q$.*

Proof. Replacing each $F_*^\vee z_p$ by $n_p T_p^3$ changes $\check{\Xi}_0$ by chains inside the regions $W_{V_p}^*$. Then $\sum_q \rho_q \Sigma_q + \sum_p n_p T_p^3$ is a boundary, so $\sum_q \rho_q [\Sigma_q] + \bar{n} [T^3] = 0$ in $H_3(W^*; \mathbb{Z})$ with $\bar{n} = \sum_p n_p$. We pair with the class of the positive locus Σ_W^+ , a closed oriented three-manifold (Lemma 8.13).

- In the part of a region over a cell $C^M(\tau, n)$ where only 0 and n have positive weight, $W^* = \{\gamma_0 x^0 + \gamma_n x^n = 0\}$ is a translate of the subtorus $\ker \chi^n$, invariant under $\mathbb{T}_n := (n^\perp \cap M) \otimes U(1)$, and Σ_W^+ is its positive real part. At a positive point x_b the

orbit $\mathbb{T}_n \cdot x_b$ has tangent space $i\mathbb{R}^3$ and Σ_W^+ has $\mathbb{R}^3$, complementary in $T_{x_b} W^* \cong \mathbb{C}^3$. An element of $\mathbb{T}_n$ that maps x_b to a positive point is the identity, so the intersection is the single point x_b , and $[T^3] \cdot [\Sigma_W^+] = \pm 1$.

- $\Sigma_q \cap \Sigma_W^+ = \emptyset$: Σ_q lies in the Morse polydisc, where $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are within $O(r)$ of $-\mu_P^{-1} < 0$, while on Σ_W^+ they are positive.

Hence $\bar{n} = 0$. For a point p_0 of the regular part and paths γ_p from p_0 to p in the connected set $B_Y \setminus \Gamma^M$, the union of the fibre tori over γ_p , built region by region, is a four-chain with boundary $T_p^3 - T_{p_0}^3$. The chain $\tilde{\Xi}^W(z) := \tilde{\Xi}_0 - \sum_p (\text{correction in } W_{V_p}^*) - \sum_p n_p (\text{union of the tori over } \gamma_p)$ has boundary $\sum_q \rho_q \Sigma_q - \bar{n} T_{p_0}^3 = \sum_q \rho_q \Sigma_q$. $\square$

The positive real locus thus replaces a local analysis of the fibre chains at the vertices of $\mathfrak{T}$: at the interior vertices of a strict tropical 2-cycle, at the branching vertices of the remainder, and at the points where two summands that cross in B_Y reach an identified curve.

8.8. Assembly of the local lifts. We now combine the local and regular chains and transport the result to the holomorphic hypersurface. The preceding fibre-class argument removes the residual boundary, leaving exactly the prescribed sum of vanishing spheres.

Remark 8.27 (where summands meet). At a junction where a rotated exit strip reaches a reference side already carrying a strip, all adjacent coefficients are multiples of d_A^* and sum to zero. The pieces, collars and regular triangles therefore cancel (Lemmas 8.24, 8.18(ii) and 8.23). This includes the three-way junction at the end of a stretch. A germ and its own exit strip, the start of a rotation, and the half-edge of a tripod are interior to one summand; their common boundary is treated within that summand.

Both passages at a node may occur. They give separate node pieces with the same sphere Σ_q and total boundary $(c_{ab} - c_{bc})\Sigma_q$ (Lemma 8.15). Their halves meet along the half-axis $C^M(u', P)$ off Γ^M , but remain separate in $\mathcal{S}$. The same convention applies when a crossing disc meets a sheet: intersection of their images in W^* does not identify their domains.

The remainder $y^\natural$ meets the germs and discs along ∂U , along replacement boundaries, and along the seam $\{\varrho_P = \varrho_0\}$ of a smoothed rotation. These are the identified curves of type (ii) in $\mathcal{S}$, at distance at least $r_{\natural}$ from Γ^M ; the seam coefficients are multiples of d_A^* . Lemma 8.23 gives the cycle condition at their edges. At vertices, the remaining boundaries are closed three-cycles in the fibre complexes F_p . Proposition 8.26 removes their total fibre class by pairing with the positive real locus. On the resolution side the proof instead uses the local fundamental groups (Section 10).

Order of choices. The constants and choices are made in the following order.

- (1) The configuration: Δ , $\mathcal{T}$ and $\tilde{h}$, $\mathcal{F}_C$ and h_C , the vectors w_u , and k_0 subject to (7.3); hence $\theta_0, \dots, \theta_3$, the gaps, the constants $\mathcal{E}_P$, the interior polygons Ω_e and C_η ; then the model positions, the model arc systems, θ_4 , Υ^M , the charts T_q , T_e , the constants r_S , $s_{\max}$ and C_T .
- (2) The model germs: the points $y_{A,0}^*, y_A^0, \mathfrak{r}_A, y_A^1, y_{A,1}^*$ in $A \cap \Omega_e$ on every arc, then r_0 , at most $r_S/4$ and at most a quarter of the mutual distances of these points, of their distances from $A \setminus \Omega_e$ and of their distances from the ends of A ; the polygon P , the unit germs with their orientations, the reference sides and the normal discs; the balls B_p^0 of radius $r_B \leq r_0$ and the constants c_B and r_{far} of Section 8.5; the radius r_ρ , the retraction at the model vertices and the partition of unity defining the base maps π_g ; the orientations of the edges used for the collar paths; the maps $\mathfrak{h}$, $\kappa_{\mathfrak{h}}$.

- (3) A compact set $\mathcal{C}_0 \subset \mathcal{Z}$, $r \leq r_1$ (Lemma 7.14), $\eta_0 \leq r^2/20$, $t < t_0(r, \eta_0, \mathcal{C}_0)$ as in Lemmas 7.11, 7.14 and 7.15, and c as at the beginning of this section.

These do not depend on z . Then, for the given $z \in Z(\mathfrak{K})$:

- (4) $\mathfrak{K}_1$ and $z^M = \mathfrak{h}_* z$; $\mathfrak{K}_2^0, \mathfrak{K}_2, \mathfrak{K}_2'$ and U ; w, v_A and y by (8.1); the two local modifications with $\varepsilon_A, \varepsilon'_A$ and ϱ_0 ; $z^\natural$ and $r_{\natural}$ (Lemma 8.11);
- (5) $\varsigma_0 \leq \min(\varsigma_{\max}/2, r_S/4, r_{\natural}/(4C_T), r_{\text{far}}/(4C_T), c_{B^*B}/8, r_\rho)$, so that $2\varsigma_0$ is small against the radius of the balls B_p^0 ; the complex $\mathcal{S}$, the constant $\mathfrak{l}$ and the triangulation $\mathfrak{T}$, the regular sets V_s , the finitely many regular sets V over the strips (Lemma 8.18(1)), and ϵ so small that Lemma 8.21 applies to all these sets;
- (6) $t' \leq \min(t, t(z))$, where $t(z)$ is a threshold for which Lemmas 8.21 and 8.22 hold for the sets V_s and for the finitely many regular sets over the strips used in Lemma 8.18(1) (the errors $O(1/L)$ of Lemma 8.22 must be small compared with the margins of these sets and with ϵ), and which lies below the thresholds of Lemmas 7.18, 8.13, 8.14, 8.16 and 8.19 (the second and third for the depths $\varsigma \leq 2\varsigma_0$);
- (7) at t' : the pieces of depth ς_0 , the collar paths, the fibre complexes, $F^\vee$ on the cells $p_0 \times \mathbb{T}_{e_0}^3$, the fillings $\mathcal{Q}'(1)$ and the chains $\mathcal{C}_v$ of Lemma 8.19, and then the fillings $\mathcal{Q}_e$ (the tripod edge at the inner boundary), the rest of the map $F^\vee$, and the chain $\tilde{\Xi}^W(z)$ of Proposition 8.26.

In particular ς_0 is chosen after U and $r_{\natural}$, and $t(z)$ after $\mathfrak{T}$; the threshold t_0 is that of Lemmas 7.11–7.15 and does not depend on z .

Proposition 8.28 (the lift, precise form). *Let Δ be admissible with a profile σ satisfying (Proj), and fix the configuration as in Section 7.1. Let $\mathcal{C}_0 \subset \mathcal{Z}$ be compact, $r \leq r_1$, $\eta_0 \leq r^2/20$, $t < t_0(r, \eta_0, \mathcal{C}_0)$, and let c be a real coefficient vector with the sign pattern, in the window of Definition 7.3. Then Y_t is smooth; each node parallelogram P has a unique Morse point q_P in $\mathbb{D}_P$ of the defining function $f^{(1)} = F_t/(\gamma_a x^a)$, in the trivialisation of the anticanonical bundle by $\gamma_a x^a$, with vanishing sphere δ_q oriented by Definition 7.23; and for every triangulation $\mathfrak{K}$ of B'_σ in which Γ is full and every relative tropical 2-cycle $z \in Z(\mathfrak{K})$ there is an integral four-chain $\tilde{\Xi}(z)$ on Y_t , constructed from z and the choices listed above, with*

$$\partial \tilde{\Xi}(z) = \sum_q \rho_q(z) \delta_q.$$

Proof. Smoothness of Y_t and the Morse points are Lemmas 7.11(1) and 7.14. Let $t' \leq \min(t, t(z))$. The transfer, the normal form and the choices of steps 4 and 5 do not depend on t . At t' , the pieces, the collars, the domain and the extension give the chain $\tilde{\Xi}_0$, and Proposition 8.26 gives $\tilde{\Xi}^W(z)$ on W^* with boundary $\sum_q \rho_q(z) \Sigma_q$. Transport it by the diffeomorphism $\Psi_1: W^* \rightarrow Y_{t'}$ of Proposition 7.12. By Lemmas 7.22(4) and 7.17, $\Psi_1(\Sigma_q)$ is isotopic to $\delta_q(t')$ in $Y_{t'}$, with the orientations of Definition 7.23; adding the traces of these isotopies gives a chain on $Y_{t'}$ with boundary $\sum_q \rho_q(z) \delta_q(t')$. Finally the members $Y_{t''}$, $t'' \in [t', t]$, with the same c form a smooth family of smooth hypersurfaces (Lemma 7.11(1) at $s = 1$ for each t''), and the Morse points $q_P(t'')$ vary smoothly with critical values bounded away from 0 (Lemma 7.14(3)); so the isotopy of the family carries $\delta_q(t')$ to $\delta_q(t)$ up to isotopy, with orientation. The image of the chain is $\tilde{\Xi}(z)$. $\square$

Proof of Theorem D. The members Y_t of Theorem A are those of Proposition 8.28: with the constants $\eta_1 = r_1^2/20$ and $C = C_\eta$ of Section 7.1, take $r = r_1$, so that $\eta_0 \leq \eta_1 = r^2/20$; every real c with $c_0 > 0 > c_n$ is $c^{(0)} e^\xi$ for a unique $c^{(0)} \in \mathcal{Z}$ and a unique $\xi \in \text{span}_{\mathbb{R}}\{r_P\}$

(Section 2.4), with $\eta_P = \log \mu_P(c)$, so the conditions of Theorem A on c say that c is in the window. So Proposition 8.28 gives $\tilde{\Xi}(z)$ with $\partial \tilde{\Xi}(z) = \sum_q \rho_q(z)[\delta_q]$ for every $z \in Z(\mathfrak{K})$. For a strict tropical 2-cycle (S, j, v) (Definition 2.5), triangulated so that it is a relative tropical 2-cycle (Section 5.2), the chain $\tilde{\Xi}(S)$ realises $\tilde{S}$ as stated in Remark 8.29. $\square$

8.9. What the construction uses, and the strict case. The proof uses from a relative tropical 2-cycle only three pieces of data: its boundary network, which fixes the multiplicities of the model germs; the coefficients v_A of the transverse discs, which are arbitrary monodromy-invariant covectors; and the remainder off a neighbourhood of the discriminant, which is an arbitrary chain with coefficients in the local system. The strict tropical 2-cycles of [CBM14, Def. 7.2] impose conditions that the construction does not use, as follows. The embeddedness of S (the embedding clause) and the condition that S meets the discriminant only in ∂S and in finitely many crossings (the discriminant-intersection clause) are replaced by the normal form: the germs are embedded surfaces by construction, and summands that cross in B_Y are lifted separately over $\mathcal{S}$. Primitivity of the field (the primitive-field clause) is replaced by pieces and tori with integer multiplicities. The boundary vertices at the negative vertices of B'_σ (the boundary-vertex clause) become tripods with any multiplicity (Lemma 8.9), and the node passages (the node-passage clause) become superpositions of the two unit halves, with both passages allowed (Lemma 8.15). The trivalent interior edges with a common rank-one sublattice (the interior-edge balancing clause) are replaced by fillings $\mathcal{Q}_e$ that exist for any valency. The condition $|k_1| = 1$ at the crossings (the crossing clause) is replaced by monodromy invariance of the disc coefficient (Lemma 8.16), and the interior vertices (the interior-vertex clause) are not used, on either route: Proposition 8.26 disposes of the fibre cycles there. The one condition that is used is the boundary condition (the boundary-monodromy clause): along the discriminant the coefficient is a multiple of the vanishing covector d_A^* , so that the torus of a sheet along an arc is the exact torus $\mathbb{T}_d$, which collapses to a circle over the arc and gives a piece without boundary there (Lemma 8.14(1)). This is the relative condition $\mathcal{D} = \mathbb{Z}d$ of Definition 2.4, and the monodromy invariance of the coefficients along the discriminant, which is part of the definition of $\iota_*\Lambda$, is what the crossing pieces use (Lemma 8.16). Relative tropical 2-cycles are therefore the natural class for the lift: they are the chains whose local lifts exist, they form a group on which the boundary $\sum_q \rho_q(z)[\delta_q]$ of the lift is additive, and their node coefficients already exhaust K (Theorem C).

Remark 8.29 (the strict case). Let (S, j, v) be a strict tropical 2-cycle (Definition 2.5) and $z = [S]$ its triangulation.

Away from the discriminant. Outside U the normal form is the transfer of S itself: y agrees there with $\text{sd } z^M$. Over that part the domain is the bundle of the cells $s \times \mathbb{T}_s$ with $m_s = 1$ and $\mathbb{T}_s = \ker \kappa_{\mathfrak{h}}(v) \otimes U(1)$, the fibre of the object $\tilde{S}$ that Castaño-Bernard and Matessi attach to S [CBM14, §7, (46)], written in the affine structure of B_Y through the flat pairing of Proposition 4.11 (Remark 6.11). The map $F^\vee$ is built cell by cell in the regions $W_{V_s}^*$ (Lemma 8.25), so over this part the chain is the bundle of the two-tori $\ker \kappa_{\mathfrak{h}}(v) \otimes U(1)$ up to homotopy in each region.

Near the discriminant. Since ∂S is an embedded curve passing each vertex of the discriminant at most once, the multiplicities of the model germs are 0 or ± 1 . The chain agrees with the local models of [CBM14] under the following two conditions:

- (C1) every arc of Γ^M carries at most one crossing of the transfer of S ;
- (C2) within U the transfer of S is the sum of the model germs (M1)–(M4) with these multiplicities and of normal discs at its crossings: at a model node it is the standard

half on the side $\alpha_q \geq 0$, at a model vertex the tripod on the side of u' , and along each arc of ∂S it lies on the model sides with the windings of the model germs.

Under (C1) and (C2), the part of $\text{sd } z^M$ on U minus $\text{Std}^M(b^M) \cap U$ is a sum of normal discs at the crossings, homologous relative to ∂U , by moving discs along the arcs, to $\sum_A v_A \bar{D}(E_A)$ with v_A the crossing coefficient of S on A (and $v_A = 0$ on arcs without crossing); so the normal form may be taken with these v_A , which are primitive with $|k_1| = 1$ (the primitive-field and crossing clauses of [CBM14, Def. 7.2]), and all pieces are the local models of [CBM14] listed below. Without (C1) or (C2) the transverse disc on an arc collects the crossings on that arc and the differences of sides and windings, and its piece is that of Lemma 8.16 for the collected coefficient, which need not be a local model of [CBM14]: two crossings with $k_g = 1$ on one arc give, when the sum of their covectors is primitive, $k_g = 2$, a two-sheeted cover with a fibre of type I_2 ; a sheet of S on the side $\alpha_q \leq 0$ at a node gives a coefficient in $\mathbb{Z}d_A^*$ and the piece $\mathbb{T}_d \cdot \mathfrak{s}^\vee(\mathfrak{D}_{s_0})$.

Condition (C2) refers to conventions fixed with the model germs, before z (Section 8.8): the basis $(\text{Turn}(u_1), \text{Turn}(u_2))$ at each turning point, the side $\alpha_q \geq 0$ at the model nodes and the side of u' at the model vertices, and the reference side $\alpha_e \geq 0$ of each arc. With these fixed conventions (C2) fails for many strict cycles: for instance when ∂S turns through the sector $C(u_3)$, which has multiplicities $(-1, 1)$ in the basis $(\text{Turn}(u_1), \text{Turn}(u_2))$, or when a sheet of S lies on the side $\alpha_q \leq 0$ or on the non-reference side of an arc. Any two of $\text{Turn}(u_1), \text{Turn}(u_2), \text{Turn}(u_3)$ form a basis, and the other choices of side and of reference side change the normal form only by transverse discs, so the construction and Theorem D hold for every choice of these conventions, and they may be chosen depending on S . If the conventions are chosen as those used by S (at each turning point a basis containing the turning germ of ∂S , at each model node and model vertex the side of S , on each arc the side of S as reference side), and if (C1) holds and within U the transfer of S is the sum of the germs so chosen and of normal discs at its crossings, then all pieces are the local models of [CBM14], in the correspondence of the table at the beginning of this section. This comparison assumes the stated position of S . We do not use that $\tilde{S}$ is a manifold at the interior vertices of S [CBM14, §7.7, Step 5].

9. PERIODS AND THE RELATION LATTICE

This section proves Theorem A and Corollary B. Theorems C and D give the inclusion $K \subseteq \text{Rel}(\delta)$ and item (2) directly (Section 9.2). The reverse inclusion is a period computation, which follows the mechanism of Greene, Morrison and Strominger [GMS95, §2] inside the polynomial family. Near the degenerate locus the period $\wp_q$ of a holomorphic volume form over δ_q is the critical value ε_P at q_P times a holomorphic function $\mathcal{B}_q$ that has no zero near the locus $\{\varepsilon_P = 0\}$ (Lemma 9.3). A relation among the classes $[\delta_q]$, even a relation modulo torsion, therefore gives the same relation among the functions $\wp_q$ on a whole neighbourhood U of the member, and among their differentials at the point $c^{(0)}$ of $\mathcal{Z}$ below it. There the matrix of differentials is $\text{diag}(\mathcal{B}_q) \mathbf{R}$ up to an error $O(t^{\theta_0/2})$, where $\mathbf{R}$ is the integer matrix with rows r_P , whose left kernel is $K \otimes \mathbb{Q}$. So the relations have rank at most $\text{rank } K$, and saturation of K gives equality over $\mathbb{Z}$ and excludes torsion. The same matrix, with a contraction argument and the holomorphic implicit function theorem, describes the nodal locus (Section 9.3). Section 9.4 replaces the spheres δ_q by pairwise disjoint Lagrangian spheres, and Section 9.5 proves Corollary B. Section 9.7 proves smoothness of the locus with exactly $|Q|$ nodes and extends the integral comparison along every connected component meeting the near-limit construction. Section 9.8 exhibits a component outside this comparison in the X_9

family. Section 9.9 classifies its entire codimension-three four-node locus, and Section 9.10 identifies the reverse transition for general members of that component.

Throughout, the configuration of Section 7.1 is fixed: $\mathcal{T}$ and $\bar{h}$, the pulling refinement $\mathcal{F}_C$ and the height h_C , the vectors w_u , the integer k_0 and $H = k_0\varphi + h_C$. We use the notation of Section 7: the compact set $\mathcal{C}_0 \subset \mathcal{Z}$, the radius $r \leq r_1$ and the bound $\eta_0 \leq r^2/20$ of Definition 7.3, the constant C_η of Lemma 7.2, the sets $U_{\mathbb{C}} \subseteq U_{\mathbb{C}}^+$ of Definition 7.3, the max-norm $\|\xi\| = \max_n |\xi_n|$ on $\mathbb{C}^A$ with $\|r_P\|_* = 4$, and $\eta_P(\xi) = r_P \cdot \xi$, extended complex-linearly. A coefficient vector near $c^{(0)}$ is written $c^{(0)}e^\xi$ with $\xi \in \mathbb{C}^A$. Constants called C depend only on the configuration and on $\mathcal{C}_0$; $C(\eta_0)$ depends in addition on η_0 . All thresholds on t below depend only on these data.

9.1. The neighbourhood U and the periods. Fix t below the threshold of Lemmas 7.11, 7.14 and 7.15, and a member $c = c^{(0)}e^{\xi_c}$ as in Theorem A: $c^{(0)} \in \mathcal{C}_0$, and ξ_c real and in the window. For a node q , with parallelogram P , and a coefficient vector c' at which Lemma 7.14 applies, $q_P(c')$ and $\varepsilon_P(c')$ denote the Morse point of $f^{(1)}$ in $\mathbb{D}_P$ and its critical value. Below, sets of nodes are indexed by the nodes q or by their parallelograms P interchangeably.

Lemma 9.1 (the neighbourhood U). *Put $\eta'_0 := 3\eta_0/2$. The open set*

$$U := \{c^{(0)}e^\xi : \xi \in \mathbb{C}^A, \|\xi\| < 3/2, |\eta_P(\xi)| < \eta'_0 \text{ for every } P\}$$

of complex coefficient vectors has the following properties.

- (1) $U_{\mathbb{C}} \subseteq U \subseteq U_{\mathbb{C}}^+$; in particular U contains c and $c^{(0)}$. The set U is simply connected.
 - (2) The conclusions of Lemmas 7.11(2), 7.14 and 7.15(2) hold for every $c' \in U$.
 - (3) Each ε_P is holomorphic on U , and on all of U
- (9.1)
$$d\varepsilon_P = e^{-\eta_P} r_P \cdot d\xi + O(C(\eta_0) t^{\theta_0/2}).$$
- (4) For $c' \in U$ the singular points of $Y_{t,c'}$ are the points $q_P(c')$ with $\varepsilon_P(c') = 0$, and each of them is an ordinary double point.

Proof. The defining bounds of $U_{\mathbb{C}}$, U and $U_{\mathbb{C}}^+$ are $(1, \eta_0)$, $(3/2, 3\eta_0/2)$ and $(2, 2\eta_0)$, so $U_{\mathbb{C}} \subseteq U \subseteq U_{\mathbb{C}}^+$. The map $\xi \mapsto c^{(0)}e^\xi$ is injective on $\{\|\operatorname{Im} \xi\| < \pi\}$, and $3/2 < \pi$; so U is the image of a convex set under a homeomorphism onto its image, hence contractible. This proves (1). Lemmas 7.11(2), 7.14 and 7.15(2) are stated for every coefficient vector in $U_{\mathbb{C}}^+$, for t below the threshold fixed above, hence they hold on U . This proves (2).

The critical point $q_P(c')$ is nondegenerate, so by the holomorphic implicit function theorem it depends holomorphically on c' , and so does $\varepsilon_P(c') = f^{(1)}(q_P(c'))$. By Lemma 7.14(2) on $U_{\mathbb{C}}^+$, $\varepsilon_P + \kappa_P = O(t^{\theta_0/2})$ on $U_{\mathbb{C}}^+$, where $\kappa_P = e^{-\eta_P} - 1$. Every point of U is the centre of a polydisc in ξ of polyradius $\varrho_* := \min(1/2, \eta_0/8) = \min(1/2, \eta_0/(2\|r_P\|_*))$ contained in $U_{\mathbb{C}}^+$: on it $\|\xi\| < 3/2 + 1/2$ and $|\eta_P| < 3\eta_0/2 + \|r_P\|_*\varrho_* \leq 2\eta_0$. The Cauchy estimates on these polydiscs bound $d(\varepsilon_P + \kappa_P)$ by $C(\eta_0)t^{\theta_0/2}$ on U , and $d\kappa_P = -e^{-\eta_P} r_P \cdot d\xi$. This is (3). By Lemma 7.11(2) on U the singular points of $Y_{t,c'}$ are the $q_P(c')$ with $\varepsilon_P(c') = 0$. At such a point the local defining function $f^{(1)}$ in the ambient chart has a nondegenerate critical point with critical value 0, so the point is an ordinary double point of $Y_{t,c'}$. This is (4). $\square$

For a node q with parallelogram P put $D_q := \{c' \in U : \varepsilon_P(c') = 0\}$, and for $Q' \subseteq Q$ put $D_{Q'} := \bigcap_{q \in Q'} D_q$. By Lemma 9.1(4), $D_{Q'}$ is the locus of members of U with a node at q_P for every $q \in Q'$. By (9.1), and since $|e^{-\eta_P}| \geq e^{-\eta'_0} \geq e^{-1}$ on U while the linear form $r_P \cdot d\xi$ has norm $\|r_P\|_* = 4$, the differential $d\varepsilon_P$ does not vanish on U for t small; so D_q is a smooth complex hypersurface of U , possibly empty. The set $U^\circ := U \setminus \bigcup_q D_q$ is connected, as the

complement of a proper analytic subset of a connected complex manifold, and it contains c , because $\varepsilon_P(c) \neq 0$ in the window (Lemma 7.14(3)). The members of U° are smooth.

Vanishing spheres over U . For $c' \in U \setminus D_q$ let $\delta_q(c') \subset Y_{t,c'}$ be the vanishing sphere of the Morse point $q_P(c')$ for the straight path of values from $\varepsilon_P(c')$ to 0, as in Definition 7.16: in holomorphic coordinates ζ centred at $q_P(c')$ with $f^{(1)} = \varepsilon_P + \sum_i \zeta_i^2$, it is $\{\zeta \in (-\varepsilon_P)^{1/2} \mathbb{R}^4 : \sum_i \zeta_i^2 = -\varepsilon_P\}$. On U ,

$$(9.2) \quad |\varepsilon_P| \leq |\kappa_P| + Ct^{\theta_0/2} \leq e^{\eta'_0} \eta'_0 + Ct^{\theta_0/2} < r^2/6,$$

since $\eta'_0 = 3\eta_0/2 \leq 3r^2/40$; so the sphere lies in $\mathbb{D}_P$. We orient $\delta_q(c')$ by continuous transport from the member c , at which it carries the orientation of Definition 7.23. This is well defined on all of $U \setminus D_q$: since U is simply connected and D_q is a smooth hypersurface, $\pi_1(U \setminus D_q)$ is generated by conjugates of meridians of D_q , and along a meridian ε_P turns once around 0, which carries δ_q to itself with its orientation (Remark 7.25). The arguments below use Definition 7.23 only at c .

Lemma 9.2 (flatness). *The members $Y_{t,c'}$, $c' \in U^\circ$, form a locally trivial fibre bundle over U° , and the classes $[\delta_q(c')] \in H_3(Y_{t,c'}; \mathbb{Z})$, $q \in Q$, are flat sections of the local system of the groups $H_3(Y_{t,c'}; \mathbb{Z})$ over U° .*

Proof. The set $\mathcal{Y}^\circ := \{(c', y) : c' \in U^\circ, y \in Y_{t,c'}\}$ is a closed submanifold of $U^\circ \times X_{\mathcal{T}}$, and the projection to U° is a proper submersion, since every fibre is a smooth hypersurface of the compact manifold $X_{\mathcal{T}}$. By Ehresmann's theorem it is locally trivial, and parallel transport along a path in U° is induced by a diffeomorphism of fibres, well defined up to isotopy. Over a small ball in U° the Morse points, the Morse coordinates of the holomorphic Morse lemma with parameters [AGZV85, §§6.2, 9.6] and the spheres $\delta_q(c')$ vary continuously, with the continuous orientation above; in a trivialisation over the ball the spheres form an isotopy, so their classes correspond under parallel transport. $\square$

Periods. Let $\Theta_T := \chi^{-n_1} d\chi^{n_1} \wedge \cdots \wedge \chi^{-n_4} d\chi^{n_4}$ be the invariant four-form of the dense torus for a basis $n_1, \dots, n_4$ of N , and write $F_t/x^0 = \sum_n \gamma_n \chi^n$. In the chart of a maximal cone, with Cox coordinates $Z_1, \dots, Z_4$, we have $\Theta_T = \pm dZ_1 \wedge \cdots \wedge dZ_4 / (Z_1 \cdots Z_4)$ and $F_t/x^0 = F_\sigma / (Z_1 \cdots Z_4)$, where F_σ is the defining polynomial of $Y_{t,c'}$ in the chart. So $\Theta_T / (F_t/x^0) = \pm dZ_1 \wedge \cdots \wedge dZ_4 / F_\sigma$ has a simple pole along $Y_{t,c'}$ and no other pole. For $c' \in U$ its residue $\Omega_{t,c'}$ is a nowhere vanishing holomorphic three-form on the smooth locus of $Y_{t,c'}$, depending holomorphically on c' . Put

$$\wp_q(c') := \int_{\delta_q(c')} \Omega_{t,c'}, \quad c' \in U \setminus D_q.$$

Lemma 9.3 (periods). *Let q be a node, with parallelogram P . There are a holomorphic function $\mathcal{B}_q$ on U , a sign s_q and a constant C independent of t such that $\wp_q = \mathcal{B}_q \varepsilon_P$ on $U \setminus D_q$ and*

$$|\mathcal{B}_q - 4\pi^2 s_q \mu_P / c_0| \leq C(|\varepsilon_P| + t^{\theta_0/2}) \quad \text{on } U.$$

In particular $\wp_q$ extends holomorphically to U , $\mathcal{B}_q$ is bounded on U uniformly in t , and there is $\varepsilon_* > 0$ independent of t such that, for t small, $\mathcal{B}_q$ has no zero at the points of U with $|\varepsilon_P| \leq \varepsilon_*$, which include D_q .

Proof. Coordinates. Fix $c_* \in U$ and freeze the positive scale $\nu_* := \nu_Z(c_*)$. On a neighbourhood $V \subset U$ of c_* use $\mu_P^{1/2} = \exp(\eta_P/2)$ and put $z_1 := Z/\nu_*$ and $z_2 := \mathfrak{b}(c')\nu_* Z'$. These coordinates depend holomorphically on the point and on c' , and $z_1 z_2 = \mathfrak{b} Z Z'$. At c_* they

agree with $\hat{Z}$ and $\vartheta_P \hat{Z}'$, since $\nu_Z \nu_{Z'} = 1/|\mathfrak{b}|$. The scales defining $\mathbb{D}_P$ are continuous in c' , so, after shrinking V , these charts have a common domain inside $2\mathbb{D}_P$. By Lemma 7.14(1), at c_* the function $f^{(1)}$ differs by $O(t^{\theta_0/2})$ from $f_{\text{lin}} := xy + z_1 z_2 - \kappa_P$, since $a\hat{Z}$ and $a'\hat{Z}'$ are $O(\lambda_P^{-1/2})$ there. Fibre-variable Cauchy estimates give the same bound in C^3 on $\mathbb{D}_P$; on a slightly smaller common domain it holds throughout V . Uniform bounds below are taken at the arbitrary centre c_* , where the original normalisation applies, so their constants are independent of c_* and t . Let ζ^0 be the linear coordinates with $x = \zeta_1^0 + i\zeta_2^0$, $y = \zeta_1^0 - i\zeta_2^0$, $z_1 = \zeta_3^0 + i\zeta_4^0$, $z_2 = \zeta_3^0 - i\zeta_4^0$, so that $f_{\text{lin}} = \sum_i (\zeta_i^0)^2 - \kappa_P$ and $dx \wedge dy \wedge dz_1 \wedge dz_2 = -4d^4\zeta^0$, where $d^4\zeta := d\zeta_1 \wedge \cdots \wedge d\zeta_4$ for any coordinates ζ . Put $\zeta^1 := \zeta^0 - \zeta^0(q_P)$ and $\mathbf{H}(\zeta^1) := \int_0^1 (1-s) \text{Hess } f^{(1)}(s\zeta^1) ds$, the Hessian taken in the coordinates ζ^1 . Since q_P is a critical point of $f^{(1)}$ with value ε_P , Taylor's formula with integral remainder gives $f^{(1)} = \varepsilon_P + (\zeta^1)^\top \mathbf{H}(\zeta^1) \zeta^1$. The Hessian of f_{lin} is $2I$, and $f^{(1)} - f_{\text{lin}}$ is $O(t^{\theta_0/2})$ in C^3 on the ball $\{|\zeta^1| < r/2\}$, which lies in $\mathbb{D}_P$ because $r \leq 1/2$; so $\mathbf{H} = I + O(t^{\theta_0/2})$ in C^1 there. Put $\zeta := \mathbf{H}(\zeta^1)^{1/2} \zeta^1$, with the principal square root of the symmetric matrix $\mathbf{H}$. The critical point is holomorphic in c' in the original toric coordinates, by the implicit function theorem. Thus ζ are jointly holomorphic coordinates on the local parameter chart, centred at q_P , with $f^{(1)} = \varepsilon_P + \mathfrak{q}$, $\mathfrak{q} := \sum_i \zeta_i^2$, and $\zeta - \zeta^1 = O(t^{\theta_0/2})$ in C^1 on the ball, with uniform constants at c_* . After shrinking V and decreasing t , the image contains $|\zeta| < r/\sqrt{5}$.

The form. On the chart $U_{u,u'}$ of Section 7.3 we take $n_1, \dots, n_4 = b - a$, $d - a$, w_u , $w_{u'}$. Since $x^a/x^0 = \chi^a = (ZZ')^{-1}$ by (7.2), $\Theta_T/(F_t/x^0) = \Theta/f^{(1)}$ with $\Theta := \pm \gamma_a^{-1}(\chi_1 \chi_2)^{-1} d\chi_1 \wedge d\chi_2 \wedge dZ \wedge dZ'$. From $\chi_1 = (\gamma_a/\gamma_b)\mu_P^{-1/2}(x - \mu_P^{-1/2})$, $\chi_2 = (\gamma_a/\gamma_d)\mu_P^{-1/2}(y - \mu_P^{-1/2})$, $Z = \nu_* z_1$, $Z' = z_2/(\mathfrak{b}\nu_*)$ and $\mathfrak{b} = \gamma_0/\gamma_a$, with $\gamma_0 = c_0$ because $H(0) = 0$,

$$\Theta = \pm \frac{\mu_P}{c_0} \frac{dx \wedge dy \wedge dz_1 \wedge dz_2}{(1 - \mu_P^{1/2}x)(1 - \mu_P^{1/2}y)} = \mp \frac{4\mu_P}{c_0} \frac{d^4\zeta^0}{(1 - \mu_P^{1/2}x)(1 - \mu_P^{1/2}y)}.$$

Write $\Theta = g d^4\zeta$. Then g is holomorphic in c' and ζ , bounded on the ball uniformly in t , and $g(0) = \mp 4(\mu_P/c_0)(1 + O(t^{\theta_0/2}))$, because q_P lies within $O(t^{\theta_0/2})$ of the node point $x = y = 0$ (Lemma 7.14(2)).

The period. For $|\lambda| < r^2/5$ let $\mathbf{B}_\lambda := \lambda^{1/2}\mathbf{B}$, where $\mathbf{B}$ is the closed unit ball of $\mathbb{R}^4 \subset \mathbb{C}^4$ in the coordinates ζ , oriented by $\mathbb{R}^4$; this does not depend on the choice of the square root, since -1 preserves the orientation of $\mathbb{R}^4$. Its boundary lies in $\{\mathfrak{q} = \lambda\}$, and for $\lambda = -\varepsilon_P$ it is the sphere $\delta_q(c')$; by (9.2) this applies throughout V . Let $s'_q = \pm 1$ compare the orientation of δ_q with the boundary orientation of $\mathbf{B}_{-\varepsilon_P}$; it is locally constant in c' . Put

$$\mathcal{I}(c', \lambda) := \int_{\mathbf{B}_\lambda} g d^4\zeta = \lambda^2 \int_{\mathbf{B}} g(\lambda^{1/2}u) d^4u.$$

In the Taylor expansion of g at $\zeta = 0$ the terms of odd degree integrate to zero over $\mathbf{B}$, so $\mathcal{I}$ is holomorphic in (c', λ) , and $\mathcal{I} = \lambda^2(\frac{\pi^2}{2}g(0) + \lambda\mathcal{I}_1)$, since $\mathbf{B}$ has volume $\pi^2/2$; by the Cauchy estimates $\mathcal{I}_1$ is bounded on $|\lambda| \leq r^2/6$ uniformly in t . On $Y_{t,c'}$ near q_P the residue of $\Theta/f^{(1)}$ is g times the Gelfand–Leray form $d^4\zeta/d\mathfrak{q}$, and the Gelfand–Leray formula [AGZV88, §§7.1, 10.2] gives $\partial_\lambda \mathcal{I}(c', \lambda) = \int_{\partial\mathbf{B}_\lambda} g d^4\zeta/d\mathfrak{q}$. Hence

$$\wp_q = s'_q \partial_\lambda \mathcal{I}(c', -\varepsilon_P) = -s'_q \varepsilon_P (\pi^2 g(0) - \varepsilon_P \mathcal{I}_2(c', -\varepsilon_P)), \quad \mathcal{I}_2 := 3\mathcal{I}_1 + \lambda \partial_\lambda \mathcal{I}_1.$$

The function $\mathcal{B}_q := -s'_q(\pi^2 g(0) - \varepsilon_P \mathcal{I}_2(c', -\varepsilon_P))$ is holomorphic near every point of U , also on D_q , and it equals $\wp_q/\varepsilon_P$ on the dense set $U \setminus D_q$; so these local functions define a holomorphic function on U . The estimate follows from the formula for $g(0)$, with $s_q = \pm s'_q$. The sign

agrees at each centre with the one in the continuously varying original normalisation, so it is locally constant on the connected set $U \setminus D_q$ and hence constant. Since $|\mu_P/c_0|$ is bounded below on U by a constant depending only on $\mathcal{C}_0$ and η_0 , the last statement follows. $\square$

9.2. Relations: proof of Theorem A(1) and (2).

The threshold. We use the choices of Section 7.1: $r = r_1$, $\eta_1 = r_1^2/20$ and $C = C_\eta$ in the statement of Theorem A, so that for $\eta_0 \in (0, \eta_1]$ the members of Theorem A are exactly the members $c = c^{(0)}e^{\xi c}$ with $c^{(0)} \in \mathcal{C}_0$ and ξc real and in the window; here $\log \mu_P(c) = \eta_P(\xi c)$. Let t_0 be the threshold of Theorem D (Proposition 8.28) for r , η_0 and $\mathcal{C}_0$, which lies below those of Lemmas 7.11, 7.14 and 7.15. In the proofs of items (1) and (3) we decrease t_0 finitely many times, each time by a condition that involves only the configuration, $\mathcal{C}_0$ and η_0 ; the result is the t_0 of Theorem A. Fix $t < t_0$, a member c and the neighbourhood U of Lemma 9.1.

Proof of item (2), and $K \subseteq \text{Rel}(\delta)$. Let $\mathfrak{K}_0$ be the barycentric subdivision of $\mathcal{P}$, in which Γ is a full subcomplex (Lemma 3.13), and let $k \in K$. By Theorem C there is $z \in Z(\mathfrak{K}_0)$ with $\rho(z) = k$. By Theorem D, applied to $\mathfrak{K}_0$ and z , there is an integral four-chain $\Xi(z)$ on Y_t with $\partial \Xi(z) = \sum_q \rho_q(z)[\delta_q] = \sum_q k_q[\delta_q]$. This is item (2), and it gives $\sum_q k_q[\delta_q] = 0$ in $H_3(Y_t; \mathbb{Z})$, so $K \subseteq \text{Rel}(\delta)$.

Both theorems are stated with the same objects: the group $Z(\mathfrak{K})$ of Definition 2.4, with coefficients in the sheaf $\iota_*\Lambda$ and boundary in $C_1(\Gamma; \mathcal{D})$, and the node coefficients of Definition 2.6. Theorem D reads ρ_q through the passage formula $\rho_q = c_{ab} - c_{bc}$, which agrees with Definition 2.6 (Section 5.3), and it orients δ_q by Definition 7.23, as Theorem A does, for the same labelling of each parallelogram. So no further compatibility is needed; by Lemma 7.26 the conclusion does not depend on the labelling.

Proof of item (1). Let $K_{\text{tor}} := \{k \in \mathbb{Z}^Q : \sum_q k_q[\delta_q] \in \text{Tors } H_3(Y_t; \mathbb{Z})\}$. Then $K \subseteq \text{Rel}(\delta) \subseteq K_{\text{tor}}$, and item (1) is the statement $K_{\text{tor}} = K$. Let $\mathbf{R}$ be the $|Q| \times |\mathcal{A}|$ integer matrix with rows $r_P = e_a + e_c - e_b - e_d$; its entries lie in $\{0, \pm 1\}$. The points of the set R of Section 2.2 are lattice points of $\partial \Delta$, so $\mathbb{Z}^R \subseteq \mathbb{Z}^A$, and under this inclusion $r_P = -\text{circ}_q$, both for the labelling that orients δ_q (a change of diagonal reverses r_P , circ_q and δ_q together, Lemma 7.26). Hence the left kernel of $\mathbf{R}$ over $\mathbb{Q}$ is $\{k \in \mathbb{Q}^Q : \sum_q k_q \text{circ}_q = 0\} = K \otimes \mathbb{Q}$, and

$$r_{\mathbf{R}} := \text{rank } \mathbf{R} = |Q| - \text{rank } K.$$

Step 1: relations propagate over U . Let $k \in K_{\text{tor}}$. By Lemma 9.2, parallel transport along a path in the connected set U° from c to c' is a group isomorphism $H_3(Y_{t,c}; \mathbb{Z}) \rightarrow H_3(Y_{t,c'}; \mathbb{Z})$ carrying $[\delta_q(c)]$ to $[\delta_q(c')]$; so $\sum_q k_q[\delta_q(c')]$ is a torsion class for every $c' \in U^\circ$. Integration of the closed form $\Omega_{t,c'}$ is a homomorphism $H_3(Y_{t,c'}; \mathbb{Z}) \rightarrow \mathbb{C}$, which vanishes on torsion classes; hence $\sum_q k_q \wp_q = 0$ on U° . By Lemma 9.3 each $\wp_q = \mathcal{B}_q \varepsilon_P$ extends holomorphically to U , and U° is dense in U ; so

$$(9.3) \quad \sum_q k_q \wp_q \equiv 0 \quad \text{on } U, \quad \text{for every } k \in K_{\text{tor}}, \text{ in particular for every } k \in K.$$

Step 2: the differentials at $c^{(0)}$. At the point $c^{(0)} \in U$, where $\xi = 0$, every η_P vanishes, so $\kappa_P = 0$ and $|\varepsilon_P(c^{(0)})| \leq Ct^{\theta_0/2}$ (Lemma 7.14(2)). Since $\mathcal{C}_0$ is compact and $c_0 > 0$ on $\mathcal{Z}$, there are $0 < c_- \leq c_+$ with $c_0^{(0)} \in [c_-, c_+]$ for all $c^{(0)} \in \mathcal{C}_0$. By Lemma 9.3, with $\mu_P(c^{(0)}) = 1$, $|\mathcal{B}_q(c^{(0)})| - 4\pi^2/c_0^{(0)} \leq Ct^{\theta_0/2}$, and $|\mathcal{B}_q| \leq b_+$ on U with b_+ independent of t . So for t small

$$b_- := 2\pi^2/c_+ \leq |\mathcal{B}_q(c^{(0)})| \leq b_+ \quad \text{for every } q.$$

The polydisc in ξ about 0 of polyradius $\varrho_* = \min(1/2, \eta_0/8)$ lies in $U_{\mathbb{C}} \subseteq U$: on it $\|\xi\| \leq 1/2$ and $|\eta_P| \leq \|r_P\|_* \|\xi\| \leq \eta_0/2$. Since $\mathcal{B}_q$ is holomorphic and bounded by b_+ on U , the Cauchy estimates on this polydisc give $|\partial \mathcal{B}_q / \partial \xi_n(c^{(0)})| \leq b_+ / \varrho_*$. Hence, at $c^{(0)}$,

$$d\wp_q = \mathcal{B}_q(c^{(0)}) d\varepsilon_P + \varepsilon_P(c^{(0)}) d\mathcal{B}_q = \mathcal{B}_q(c^{(0)}) r_P \cdot d\xi + O(C(\eta_0) t^{\theta_0/2}),$$

by (9.1) with $\eta_P = 0$. Let $\mathbf{D}(t)$ be the $|Q| \times |\mathcal{A}|$ matrix whose rows are the differentials $d\wp_q(c^{(0)})$, as linear forms in $d\xi$. Then

$$(9.4) \quad \mathbf{D}(t) = \text{diag}(\mathcal{B}_q(c^{(0)})) \mathbf{R} + \mathbf{E}(t), \quad \max_{q,n} |\mathbf{E}(t)_{qn}| \leq C(\eta_0) t^{\theta_0/2}.$$

Step 3: the rank. Choose $r_{\mathbf{R}}$ rows I and $r_{\mathbf{R}}$ columns J of $\mathbf{R}$ with $\det \mathbf{R}_{IJ} \neq 0$; as an integer, $|\det \mathbf{R}_{IJ}| \geq 1$. Then $|\det(\text{diag}(\mathcal{B}) \mathbf{R})_{IJ}| = \prod_{q \in I} |\mathcal{B}_q(c^{(0)})| |\det \mathbf{R}_{IJ}| \geq b_-^{r_{\mathbf{R}}}$, and every entry of $\text{diag}(\mathcal{B}) \mathbf{R}$ has modulus at most b_+ . The determinant is a polynomial of degree $r_{\mathbf{R}}$ in the entries, so, once $C(\eta_0) t^{\theta_0/2} \leq 1$,

$$|\det \mathbf{D}(t)_{IJ} - \det(\text{diag}(\mathcal{B}) \mathbf{R})_{IJ}| \leq r_{\mathbf{R}} \cdot r_{\mathbf{R}}! (b_+ + 1)^{r_{\mathbf{R}}-1} C(\eta_0) t^{\theta_0/2}.$$

We decrease t_0 so that the right-hand side is less than $b_-^{r_{\mathbf{R}}}/2$. Then $\det \mathbf{D}(t)_{IJ} \neq 0$, so $\text{rank } \mathbf{D}(t) \geq r_{\mathbf{R}}$, and the left kernel of $\mathbf{D}(t)$ in $\mathbb{C}^Q$ has dimension at most $|Q| - r_{\mathbf{R}} = \text{rank } K$.

Step 4: conclusion. Let $k \in K_{\text{tor}}$. Differentiating (9.3) at $c^{(0)}$ gives $\sum_q k_q d\wp_q(c^{(0)}) = 0$, so k lies in the left kernel of $\mathbf{D}(t)$. The complex span of K_{tor} therefore has dimension at most $\text{rank } K$, and it contains the complex span of $K \subseteq K_{\text{tor}}$, which has dimension $\text{rank } K$. The two spans agree, and since both K_{tor} and K consist of integral vectors, $K_{\text{tor}} \otimes \mathbb{Q} = K \otimes \mathbb{Q}$. As K is saturated in $\mathbb{Z}^Q$, $K_{\text{tor}} \subseteq (K \otimes \mathbb{Q}) \cap \mathbb{Z}^Q = K$. Hence $K_{\text{tor}} = \text{Rel}(\delta) = K$.

The homomorphism $\mathbb{Z}^Q \rightarrow H_3(Y_t; \mathbb{Z})$, $k \mapsto \sum_q k_q [\delta_q]$, therefore has kernel K , and its image meets the torsion subgroup only in 0. The image is isomorphic to $\mathbb{Z}^Q/K$, which is finitely generated and torsion-free because K is saturated, hence free of rank $|Q| - \text{rank } K$. This proves item (1). $\square$

Remark 9.4. Step 1 uses only that $\sum_q k_q [\delta_q]$ is torsion at one member, and the inclusion $K \subseteq \text{Rel}(\delta)$ is the integral statement of item (2), not a statement modulo torsion. The rank bound of Step 3 is where the relation lattice of the circuits enters: the left kernel of $\mathbf{R}$ over $\mathbb{Q}$ is the space of relations among the classes $[C_q]$ in $H_2(\widehat{X}; \mathbb{Q})$ (Proposition 3.4).

The conclusion extends from the given member to all smooth members of U that have Morse points at every node, as Section 2.6 states.

Proposition 9.5. *For every $c' \in U^\circ = U \setminus \bigcup_q D_q$, including complex coefficient vectors, the member $Y_{t,c'}$ is smooth, and, with $\delta_q(c')$ oriented by transport from c , for $k \in \mathbb{Z}^Q$ the class $\sum_q k_q [\delta_q(c')]$ is a torsion element of $H_3(Y_{t,c'}; \mathbb{Z})$ if and only if $k \in K$, and it is then zero.*

Proof. Smoothness is Lemma 9.1(4). Parallel transport along a path in the connected set U° from c to c' is a group isomorphism carrying $[\delta_q(c)]$ to $[\delta_q(c')]$ for every q (Lemma 9.2). It carries the homomorphism $k \mapsto \sum_q k_q [\delta_q(c)]$ to $k \mapsto \sum_q k_q [\delta_q(c')]$ and the torsion subgroup onto the torsion subgroup; so the kernels, and the preimages of the torsion subgroups, agree. Item (1) at c gives the claim. $\square$

9.3. The nodal locus: proof of Theorem A(3). By Lemma 9.1(4) the members of U with a node at every point q_P , $P \in Q$, are the points of D_Q ; each of them has exactly $|Q|$ singular points, all ordinary double points. Let $I \subseteq Q$ index rows r_P , $P \in I$, that form a

basis of the row space of $\mathbf{R}$, so $|I| = r_{\mathbf{R}} = |Q| - \text{rank } K$, and put $D_I := \bigcap_{P \in I} D_P$. We show that D_I is a nonempty complex submanifold of U of codimension $|I|$, and that $D_Q = D_I$.

In the chart ξ write $\varepsilon_P(c^{(0)}e^\xi) = -\kappa_P(\xi) + E_P(\xi)$ with $\kappa_P = e^{-r_P \cdot \xi} - 1$. By Lemma 7.14(2), $|E_P| \leq Ct^{\theta_0/2}$ on $U_{\mathbb{C}}^+$. The space $\mathbb{C}^A$, its subspaces and $\mathbb{C}^I$ carry the max-norm, and operator norms are taken between these.

(a) $D_I \neq \emptyset$. Let $V := \text{span}_{\mathbb{C}}\{r_P : P \in I\} \subseteq \mathbb{C}^A$ and $A : V \rightarrow \mathbb{C}^I$, $A\xi := (r_P \cdot \xi)_{P \in I}$. In the basis $(r_P)_{P \in I}$ of V the matrix of A is the Gram matrix of linearly independent real vectors, which is invertible; so A is an isomorphism, and $\|A^{-1}\| \leq C_A$ for a constant C_A depending only on $\mathbf{R}$ and I . Put $\varrho_V := \min(1/2, \eta_0/8)$ and $B_V := \{\xi \in V : \|\xi\| \leq \varrho_V\}$. Every point of B_V lies in $U_{\mathbb{C}} \subseteq U$, and so does the polydisc of polyradius $\varrho_* = \min(1/2, \eta_0/8)$ about it: on that polydisc $\|\xi\| \leq 1$ and $|\eta_P| \leq \eta_0$. The Cauchy estimates on these polydiscs give $\sup_{B_V} \|E\| \leq Ct^{\theta_0/2}$ and $\sup_{B_V} \|dE\| \leq C(\eta_0)t^{\theta_0/2}$ for $E := (E_P)_{P \in I}$, with $C \leq C(\eta_0)$. Let t be so small that $C(\eta_0)t^{\theta_0/2} \leq 1/2$. On B_V the equations $\varepsilon_P = 0$, $P \in I$, say $e^{-\eta_P} = 1 + E_P$. Here $|\eta_P| \leq \eta_0/2 < 1$ and $|\log(1 + E_P)| \leq 2|E_P| \leq 1$ for the principal logarithm, so $\eta_P = -\log(1 + E_P)$, no other branch being possible, and the equations are equivalent to the fixed point equation

$$\xi = T(\xi) := A^{-1}(-\log(1 + E(\xi))), \quad \xi \in B_V.$$

For $|w| \leq 1/2$ one has $|\log(1 + w)| \leq 2|w|$ and $|d\log(1 + w)/dw| \leq 2$, and B_V is convex. Hence $\|T(\xi)\| \leq \varrho_t := 2C_A C(\eta_0)t^{\theta_0/2}$, and T is Lipschitz on B_V with constant at most $2C_A \sup_{B_V} \|dE\| \leq \varrho_t$. We decrease t_0 so that $\varrho_t \leq \varrho_V$ and $\varrho_t < 1$. Then T maps the closed ball of radius ϱ_t in V , which lies in B_V , into itself and is a contraction there. Its fixed point ξ^* satisfies $\varepsilon_P(c^{(0)}e^{\xi^*}) = 0$ for $P \in I$, with $\|\xi^*\| \leq \varrho_t$, and $c^{(0)}e^{\xi^*} \in U$. So $D_I \neq \emptyset$.

(b) D_I is a complex submanifold of codimension $|I|$. On U , $|e^{-\eta_P}| \in [e^{-1}, e]$, and the differentials (9.1), $P \in I$, are the rows of the matrix $\text{diag}(e^{-\eta_P})_{P \in I} \mathbf{R}_I$ up to an error bounded entrywise by $C(\eta_0)t^{\theta_0/2}$. The minor argument of Step 3 of Section 9.2, with $e^{-\eta_P}$ in place of $\mathcal{B}_q$ and the bounds e^{-1}, e in place of $b_{\pm}$, shows that these $|I|$ forms are linearly independent at every point of U once t is small. By the holomorphic implicit function theorem D_I is a complex submanifold of U of codimension $|I|$.

For use in (c) we bound η_P on D_I . For $P \in I$, $\varepsilon_P = 0$ gives $|e^{-\eta_P} - 1| = |\kappa_P| = |E_P| \leq Ct^{\theta_0/2}$. On U , $|\eta_P| < \eta'_0 = 3\eta_0/2 < 1$, since $\eta_0 \leq r^2/20 \leq 1/80$; and for $|w| \leq 1$ one has $|e^{-w} - 1| \geq |w| - (e - 2)|w|^2 \geq |w|/4$. Hence $|\eta_P| \leq 4Ct^{\theta_0/2}$ on D_I . For $P \notin I$, $r_P = \sum_{P' \in I} a_{PP'} r_{P'}$ with rational coefficients $a_{PP'}$ depending only on $\mathbf{R}$ and I , so $\eta_P = \sum_{P'} a_{PP'} \eta_{P'}$ and $|\eta_P| \leq Ct^{\theta_0/2}$ on D_I as well. Consequently

$$(9.5) \quad |\varepsilon_P| \leq |\kappa_P| + Ct^{\theta_0/2} \leq Ct^{\theta_0/2} \quad \text{on } D_I, \text{ for every } P \in Q.$$

(c) $D_Q = D_I$. Let $q \notin I$, with parallelogram P . The rows $r_{P'}$, $P' \in I \cup \{P\}$, are linearly dependent, and since the rows indexed by I are independent they satisfy a rational relation k with $k_q \neq 0$ and support in $I \cup \{q\}$. As $r_{P'} = -\text{circ}_{q'}$, $k \in K \otimes \mathbb{Q}$, and a nonzero integer multiple mk lies in K . By item (2) and (9.3), $\sum_p mk_p \wp_p \equiv 0$ on U . On D_I , $\wp_p = \mathcal{B}_p \varepsilon_{P_p} = 0$ for $p \in I$, where P_p is the parallelogram of p , so $k_q \mathcal{B}_q \varepsilon_P = 0$ there. By (9.5), $|\varepsilon_P| \leq \varepsilon_*$ on D_I once t is small, so $\mathcal{B}_q$ has no zero on D_I (Lemma 9.3). Hence $\varepsilon_P = 0$ on D_I , that is $D_I \subseteq D_q$. So $D_I \subseteq D_Q \subseteq D_I$, and $D_Q = D_I$ is a nonempty complex submanifold of U of codimension $|Q| - \text{rank } K$.

Finally, by Proposition 3.4(3) the diagonals can be chosen so that the small resolution $\hat{X} \rightarrow X'_\sigma$ is projective, and for every projective small resolution $|Q| - \text{rank } K$ is the relative Picard number of $\hat{X} \rightarrow X'_\sigma$ (Proposition 3.4(2); changing the diagonal at q changes only the

sign of the image of $[C_q]$ in $\mathbb{Q}^R$, and by Lemma 7.26 it leaves items (1) and (2) unchanged). This proves item (3), and with it Theorem A. $\square$

9.4. Lagrangian representatives. Castaño-Bernard and Matessi construct Lagrangian torus fibrations over integral affine three-manifolds with singularities [CBM09], and Lagrangian submanifolds associated with tropical varieties are constructed in [Mik19, Mat21a, Mat21b]; here the Lagrangian spheres are built directly in the Kähler manifolds Y_t . Fix a toric Kähler form $\omega_{\mathcal{T}}$ on the smooth projective toric variety $X_{\mathcal{T}}$, and write $\omega_{c'}$ for its restriction to a smooth member $Y_{t,c'}$, a Kähler form.

Lemma 9.6. *Let c be a member as in Theorem A, with neighbourhood U , and let $Q' \subseteq Q$ be a set of nodes with $D_{Q'} \neq \emptyset$. Then there are pairwise disjoint embedded Lagrangian three-spheres $L_P \subset (Y_t, \omega_c)$, $P \in Q'$, each smoothly isotopic in Y_t to δ_P .*

The lemma controls the isotopy class of L_P , without a bound on its position in the Morse polydisc: for a fixed Kähler form the equivalence constant between $\omega_{\mathcal{T}}$ and the flat metric of the rescaled Morse chart grows like a negative power of t , and at the member c itself the thimbles need not stay in the polydiscs.

Proof. Step 1: a thimble bound near $D_{Q'}$. Fix $b_0 \in D_{Q'}$ and a compact neighbourhood $\mathcal{V} \subset U$ of b_0 . By Lemma 7.14, which holds on $\mathcal{V}$ including where $\varepsilon_P = 0$, each $f_P := f^{(1)}$ has on $\mathbb{D}_P$ a single critical point q_P , and it is nondegenerate. By the holomorphic Morse lemma with parameters [AGZV85, §§6.2, 9.6] there are holomorphic coordinates ζ centred at q_P , depending continuously on the member, with $f_P = \varepsilon_P + \sum_i \zeta_i^2$ exactly. Put $\Phi_P := \operatorname{Re}(e^{-i \arg \varepsilon_P} f_P)$; then $|d\Phi_P|_0 = 2|\zeta|$, where $|\cdot|_0$ is the flat norm of the coordinates ζ . By compactness there are $\varpi_0 > 0$ and $C \geq 1$, uniform on $\mathcal{V}$, such that the ball $B_P := \{|\zeta| \leq \varpi_0\}$ lies in the chart and in $\mathbb{D}_P$, and $C^{-1}|\cdot|_0 \leq |\cdot|_{\omega} \leq C|\cdot|_0$ on B_P , where $|\cdot|_{\omega}$ is the norm of $\omega_{\mathcal{T}}$. Along a flow line of $-\nabla_{\omega} \Phi_P$, Φ_P decreases at rate $|d\Phi_P|_{\omega} \geq C^{-1}|d\Phi_P|_0$ per unit ω -length, and the ω -length needed to cross the shell $\varpi/2 \leq |\zeta| \leq \varpi$ is at least $\varpi/(2C)$; so crossing it costs a drop of Φ_P of at least $c_H \varpi^2/(4C^2)$ with $c_H = 2$, while the drop from q_P to the level $f_P = 0$ is $|\varepsilon_P|$. Hence, if $|\varepsilon_P| < \Phi_0 := c_H \varpi_0^2/(4C^2)$, the descending manifold of q_P between the levels ε_P and 0 lies in the ball of radius $2C(|\varepsilon_P|/c_H)^{1/2}$, and every flow line from q_P reaches the level 0 there, q_P being the only critical point.

Step 2: Lagrangian spheres near $D_{Q'}$. Choose $b_1 \in \mathcal{V} \cap U^{\circ}$ with $0 < |\varepsilon_P(b_1)| < \Phi_0$ for $P \in Q'$; this is possible because the ε_P are continuous, vanish at b_0 , and $\bigcup_q D_q$ is nowhere dense. By the Cauchy–Riemann equations the gradient flow of Φ_P for the Kähler metric preserves $\operatorname{Im}(e^{-i \arg \varepsilon_P} f_P)$, so by Step 1 the descending manifold is the Lefschetz thimble of q_P for the straight path from ε_P to 0, contained in B_P . By [Sei03, §1.3, Lemmas 1.13–1.14], whose argument is local along the thimble, it is an embedded Lagrangian ball, and its boundary $L_P(b_1)$ is an embedded Lagrangian sphere in Y_{t,b_1} . In our setting the Lagrangian property is seen directly: by the Cauchy–Riemann equations the gradient of Φ_P is a Hamiltonian vector field of $\operatorname{Im}(e^{-i \arg \varepsilon_P} f_P)$, so its flow preserves $\omega_{\mathcal{T}}$; it carries the points of the thimble to q_P as time tends to infinity and contracts the tangent spaces of the thimble; hence these tangent spaces are isotropic, and the thimble is Lagrangian. On the thimble $\operatorname{Im}(e^{-i \arg \varepsilon_P} f_P) = 0$, so its boundary is the regular level $\{\Phi_P = 0\}$ of Φ_P on it, an isotropic three-sphere in Y_{t,b_1} , hence Lagrangian; the flow lines reach this level by Step 1. Seidel credits the definition of the thimble to a suggestion of Donaldson; compare [Don00]. The polydiscs are disjoint, so the $L_P(b_1)$ are. Replacing $\omega_{\mathcal{T}}$ on B_P by $(1-\tau)\omega_{\mathcal{T}} + \tau \zeta^* \omega_{\text{std}}$, $\tau \in [0, 1]$, where ζ is the holomorphic Morse chart of Step 1 and ω_{std} the standard Kähler form of $\mathbb{C}^4$, keeps the estimates of Step 1

with the same constant C , since $C \geq 1$, and gives a smooth isotopy in $\mathbb{D}_P \cap Y_{t,b_1}$ from $L_P(b_1)$ to the Morse-chart sphere $\delta_P(b_1)$.

Step 3: symplectic parallel transport. Choose a smooth path γ in the connected set U° from b_1 to c . Every $Y_{t,\gamma(s)}$ is smooth, so $\mathcal{Y} = \{(s, y) : y \in Y_{t,\gamma(s)}\}$ is a manifold submersing onto $[0, 1]$ with compact fibres. Let $\omega_{\mathcal{Y}}$ be the pullback of $\omega_{\mathcal{T}}$. Since the fibres are symplectic, their $\omega_{\mathcal{Y}}$ -orthogonal complement is a line field transverse to them; let $\hat{\partial}_s$ be the lift of ∂_s in it, whose flow ϕ_s is defined on $[0, 1]$. Then $\mathcal{L}_{\hat{\partial}_s} \omega_{\mathcal{Y}} = d(\iota_{\hat{\partial}_s} \omega_{\mathcal{Y}})$, and $\iota_{\hat{\partial}_s} \omega_{\mathcal{Y}}$ vanishes on the fibres, so each ϕ_s restricts to a symplectomorphism between the fibres. Put $L_P := \phi_1(L_P(b_1))$: these are pairwise disjoint embedded Lagrangian spheres in (Y_t, ω_c) .

Step 4: the isotopy to δ_P . Along γ , $\varepsilon_P \neq 0$ and Lemma 7.14 holds, so by the holomorphic Morse lemma with parameters the Morse-chart spheres $\delta_P(\gamma(s))$ form a smooth family. Two holomorphic Morse charts at q_P differ by a biholomorphism g preserving $\sum_i \zeta_i^2$; the maps $\zeta \mapsto g(\lambda\zeta)/\lambda$, $\lambda \in (0, 1]$, preserve it as well and tend to $dg(0) \in O(4, \mathbb{C})$, and $SO(4, \mathbb{C})$ is connected while $O(4, \mathbb{R})$ preserves the real sphere, so the sphere is well defined up to isotopy. Pulling the spheres $\delta_P(\gamma(s))$ back by ϕ_s gives an isotopy in Y_{t,b_1} from $\delta_P(b_1)$ to $\phi_1^{-1}(\delta_P(c))$. Composing with the isotopy of Step 2 and applying ϕ_1 gives a smooth isotopy in Y_t from L_P to δ_P . We orient L_P by transport along this isotopy from $\delta_P = \delta_P(c)$, which carries the orientation of Definition 7.23; along γ the spheres $\delta_P(\gamma(s))$ carry the orientation by transport from c of Section 9.1. $\square$

By Theorem A(3), $D_Q \neq \emptyset$, so $D_{Q'} \supseteq D_Q$ is nonempty for every $Q' \subseteq Q$, and Lemma 9.6 applies with $Q' = Q$: the spheres δ_q , $q \in Q$, are simultaneously isotopic to pairwise disjoint Lagrangian spheres L_q , with $[L_q] = [\delta_q]$ in $H_3(Y_t; \mathbb{Z})$.

9.5. Smoothings: proof of Corollary B. The equivalence of (a) and (b) is Namikawa's theorem [Nam02, Thm. 2.5], which we apply after verifying its hypotheses. Namikawa calls a projective threefold with only terminal singularities, trivial canonical class and $H^1(\mathcal{O}) = 0$ a *Calabi–Yau threefold* [Nam02, §0.2].

Lemma 9.7. *X'_σ is a Calabi–Yau threefold in the sense of [Nam02]: it is a normal projective threefold whose singular points are the $|Q|$ nodes, with $\omega_{X'_\sigma} \cong \mathcal{O}_{X'_\sigma}$ and $H^1(X'_\sigma, \mathcal{O}) = H^2(X'_\sigma, \mathcal{O}) = 0$. The small resolution $\hat{X} \rightarrow X'_\sigma$ is an isomorphism off the curves $C_q \cong \mathbb{P}^1$.*

Proof. The toric variety P'_σ of (Proj) is projective, and it is Gorenstein: its rays are generated by lattice points of $\partial\Delta$, so the support function φ of $-K$ is linear on each of its cones. The morphism $P'_\sigma \rightarrow \mathbb{P}_\Delta$ is crepant for the same reason. So the total transform of the anticanonical hypersurface X is an anticanonical divisor of P'_σ . It has no exceptional component: such a component would map onto the closure of a torus orbit of $\mathbb{P}_\Delta$ contained in X , while the Δ° -regular hypersurface X meets every orbit transversally and contains none [Bat94, §3]. Hence $X'_\sigma \in |-K_{P'_\sigma}|$ is a Cartier divisor, and it is projective. By adjunction $\omega_{X'_\sigma} \cong \mathcal{O}_{X'_\sigma}$. The singular points of X'_σ are the nodes, one for each $q \in Q$ (Section 3.2); ordinary double points are terminal. As a Cartier divisor in the Cohen–Macaulay variety P'_σ , X'_σ is Cohen–Macaulay, and with finitely many singular points it is normal. From $0 \rightarrow \omega_{P'_\sigma} \rightarrow \mathcal{O}_{P'_\sigma} \rightarrow \mathcal{O}_{X'_\sigma} \rightarrow 0$, Serre duality on the four-dimensional Cohen–Macaulay projective variety P'_σ , and $H^j(P'_\sigma, \mathcal{O}) = 0$ for $j > 0$ [CLS11, §9.2], we get $H^i(P'_\sigma, \omega_{P'_\sigma}) \cong H^{4-i}(P'_\sigma, \mathcal{O})^\vee = 0$ for $i \leq 3$, hence $H^0(X'_\sigma, \mathcal{O}) = \mathbb{C}$ and $H^i(X'_\sigma, \mathcal{O}) = 0$ for $i = 1, 2$; in particular X'_σ is connected, hence irreducible. The last sentence is the description of $\hat{X}$ in Section 3.2. $\square$

Lemma 9.8. *Let $L \subseteq \mathbb{Z}^Q$ be a subgroup. The following are equivalent: (i) some $\lambda \in L$ has $\lambda_q \neq 0$ for every q ; (ii) some $\lambda \in L \otimes \mathbb{Q}$ does; (iii) some $\lambda \in L \otimes \mathbb{C}$ does; (iv) for no q is L contained in the coordinate hyperplane $\{\lambda_q = 0\}$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) holds because $L \subseteq L \otimes \mathbb{Q} \subseteq L \otimes \mathbb{C}$, and (iii) $\Rightarrow$ (iv) because $L \subseteq \{\lambda_q = 0\}$ implies $L \otimes \mathbb{C} \subseteq \{\lambda_q = 0\}$. For (iv) $\Rightarrow$ (i) choose $\ell^{(p)} \in L$ with $\ell_p^{(p)} \neq 0$ for each $p \in Q$. For $m \in \mathbb{Z}^Q$, the q -th coordinate of $\sum_p m_p \ell^{(p)}$ is a linear form in m whose coefficient of m_q is $\ell_q^{(q)} \neq 0$. The union of the $|Q|$ kernels of these nonzero forms does not contain $\mathbb{Z}^Q$, so some $\lambda = \sum_p m_p \ell^{(p)}$ has every coordinate nonzero. $\square$

Proof of Corollary B. (a) $\Leftrightarrow$ (b). By Lemma 9.7, X'_σ is a Calabi–Yau threefold in Namikawa’s sense whose singular points are the nodes $q \in Q$, and $\widehat{X} \rightarrow X'_\sigma$ is a small resolution, not necessarily projective, that is an isomorphism off the curves $C_q \cong \mathbb{P}^1$. For these data [Nam02, Thm. 2.5] states that X'_σ is smoothable by a flat deformation if and only if some relation $\sum_q \lambda_q [C_q] = 0$ in $H_2(\widehat{X}; \mathbb{C})$ has every $\lambda_q \neq 0$. (The theorem adds a third equivalent condition, maximality of X'_σ , which we do not use. Friedman’s criterion [Fri86, Rem. (4.5), Cor. (4.7)] gives the sufficiency of such a relation under the hypotheses of [Fri86, Thm. (4.4)], which include that the exceptional curve classes span $H^2(\widehat{X}, \Omega_{\widehat{X}}^2)$; Namikawa’s theorem does not require that hypothesis, and gives both directions for flat smoothings.) The classes $[C_q]$ are rational, and by Proposition 3.4 the relations among them in $H_2(\widehat{X}; \mathbb{Q})$ form $K \otimes \mathbb{Q}$; the relations in $H_2(\widehat{X}; \mathbb{C}) = H_2(\widehat{X}; \mathbb{Q}) \otimes \mathbb{C}$ therefore form $K \otimes \mathbb{C}$. By Lemma 9.8 applied to $L = K$, a relation with every coefficient nonzero exists over $\mathbb{C}$ if and only if one exists over $\mathbb{Q}$. This is (b).

(b) $\Leftrightarrow$ (c). By Proposition 3.4 and Lemma 9.8, (b) holds if and only if some $\lambda \in K$ has every $\lambda_q \neq 0$. By Theorem A(1), $K = \text{Rel}(\delta)$, the lattice of relations among the classes $[\delta_q]$ in $H_3(Y_t; \mathbb{Z})$; so this holds if and only if (c) holds. Conditions (b) and (c) do not depend on the diagonals: a change of diagonal at q changes the sign of $[C_q]$ (Proposition 3.4(2)) and of δ_q (Lemma 7.26), which preserves the existence of a relation with every coefficient nonzero. By Proposition 9.5 the same equivalence holds at every member of U° .

The symplectic statement. By Theorem A(3) and Lemma 9.6 with $Q' = Q$ (Section 9.4), the spheres δ_q are smoothly isotopic to pairwise disjoint embedded Lagrangian spheres $L_q \subset (Y_t, \omega_c)$, $q \in Q$, with $[L_q] = [\delta_q]$; this part is unconditional and does not use (a)–(c). The manifold Y_t is a compact Kähler, in particular symplectic, six-manifold. If (c) holds, $\sum_q k_q [L_q] = 0$ in $H_3(Y_t; \mathbb{Z})$ with every $k_q \neq 0$, which is the hypothesis of [STY02, Thm. 2.9] for the $n = |Q|$ spheres L_q : a relation in $H_3(\cdot; \mathbb{Z})$ among n disjoint embedded Lagrangian three-spheres in a symplectic six-manifold, with every coefficient nonzero. That theorem gives a symplectic structure on one of the $2^{|Q|}$ conifold transitions of Y_t in the spheres L_q in which the exceptional curves are symplectic. In the proof of [STY02, Thm. 2.9] the choice at the sphere L_q , an orientation of L_q together with one of the two small resolutions, is determined by the sign of k_q . $\square$

Remark 9.9 (forced nodes). By Lemma 9.8, (a)–(c) fail exactly when some node q is *forced*: every $\lambda \in K$ has $\lambda_q = 0$. By Theorem A(1) this is the case if and only if δ_q occurs with nonzero coefficient in no relation among the classes $[\delta_{q'}]$, that is, if and only if $[\delta_q]$ does not lie in the span of the other classes in $H_3(Y_t; \mathbb{Q})$. The proof of [Nam02, Thm. 2.5] shows that a forced node of X'_σ survives every flat deformation. Whether (b) holds is a finite linear-algebra test on the circuits circ_q ; Section 11.4 records its outcome on the list.

The next corollary uses the smoothing criterion of [Wue26b] in the following special case, which we restate in the present notation.

Theorem 9.10 ([Wue26b, Thm. 4.7], for nodes and degree-seven points). *Let Z be a connected normal projective threefold with $\omega_Z \cong \mathcal{O}_Z$ and $H^1(Z, \mathcal{O}_Z) = 0$, whose singular points are ordinary double points and points analytically isomorphic to the anticanonical cone over the del Pezzo surface of degree seven. Let $\hat{Z} \rightarrow Z$ be the crepant resolution given by a small resolution at each node q , with exceptional curve C_q , and by the exceptional surface $E_p \cong \text{Bl}_2\mathbb{P}^2$ over each cone point p . Put $R_q := \mathbb{Q}$ for a node, mapped to $H_2(\hat{Z}; \mathbb{Q})$ by $1 \mapsto [C_q]$, and $R_p := \mathbb{Q}(e_1 - e_2) \subset H_2(E_p; \mathbb{Q})$ for a cone point, the line spanned by the (-2) -classes of E_p orthogonal to K_{E_p} , mapped by the inclusion of E_p . Then Z has a convergent one-parameter smoothing if and only if the kernel of $\bigoplus_q R_q \oplus \bigoplus_p R_p \rightarrow H_2(\hat{Z}; \mathbb{Q})$ contains a vector with every component nonzero.*

In [Wue26b] the statement is made for a choice of a reduced local deformation component at each singular point; at a node and at a degree-seven point there is only one, a line, so the choice is unique, and the subspace selected at a cone point is the line R_p ([Wue26b, §1, §2.1]). Every summand is then of rank one, and the condition of [Wue26b, Thm. 4.7] is that every component is nonzero.

Corollary 9.11 (smoothings of X with dP_7 points). *Let Δ be admissible with no hexagonal two-face, so that the profile σ is unique, and assume (U) and (Proj). Let Y_t be a member as in Theorem A. Then the following are equivalent:*

- (a) X admits a smoothing;
- (b) X'_σ admits a smoothing;
- (c) the vanishing spheres satisfy a relation $\sum_{q \in Q} k_q [\delta_q] = 0$ in $H_3(Y_t; \mathbb{Z})$ with $k_q \neq 0$ for every $q \in Q$, including both nodes of every pentagon.

In (c), with the diagonals through the centres chosen after Definition 2.3, $k_{q_1} + k_{q_2} = 0$ at the two nodes q_1, q_2 of each pentagon.

Proof. X satisfies the hypotheses of Theorem 9.10. The argument of Lemma 9.7, with $\mathbb{P}_\Delta$ in place of P'_σ , shows that X is a connected normal projective threefold with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. By Section 3.2 its singular points are the nodes x_G of the squares and, at each pentagon G , a point analytically isomorphic to the vertex of U_G , the anticanonical cone over the toric surface of G , which is dP_7 .

Smoothings over a germ. A smoothing in our sense, a flat deformation over a germ of complex space whose general fibre is smooth, gives a convergent one-parameter smoothing, that is, a flat proper family over a disc with smooth fibres off the centre, by [Wue26c, Cor. 6.7]: a compact threefold with trivial canonical sheaf, $H^1(\mathcal{O}) = 0$, only nodes and anticanonical cones over dP_6 or dP_7 as singularities, and a crepant resolution small over the nodes with an exceptional del Pezzo surface over each cone point, has a smoothing over some base germ only if it has a convergent one-parameter smoothing. Its proof applies the curve selection lemma to the complement of the discriminant in the base of a miniversal deformation. X satisfies these hypotheses by the above and by the resolution $\hat{X} \rightarrow X$ below. The converse is trivial. So (a) holds if and only if the criterion of Theorem 9.10 holds for $Z = X$.

The resolution. Let $\hat{X}$ be the small resolution of X'_σ with the chosen diagonals, which pass through the centres at the pentagons. Near the fibre over a pentagon point it is the toric variety of the star subdivision of the cone over G , that is, a neighbourhood of the zero section of $\text{Tot}(K_E)$ with $E = D_o \cong \text{Bl}_2\mathbb{P}^2$; at the squares it is a small resolution of the node.

So $\widehat{X} \rightarrow X$ is the resolution of Theorem 9.10. By Proposition 3.9(1), $C_{q_1} = e_1$ and $C_{q_2} = e_2$ are two disjoint (-1) -curves of E . The (-2) -classes of E orthogonal to K_E are $\pm(e_1 - e_2)$ in every marking, so the line R_p does not depend on the marking, and its image in $H_2(\widehat{X}; \mathbb{Q})$ is spanned by $[C_{q_1}] - [C_{q_2}]$.

The dictionary. Let V_X be the kernel of Theorem 9.10 for $Z = X$, and define $\Phi: V_X \rightarrow \mathbb{Q}^Q$ by $\Phi(\mathbf{k})_q = \mathbf{k}_q$ at a square and $\Phi(\mathbf{k})_{q_1} = \mathbf{k}_p$, $\Phi(\mathbf{k})_{q_2} = -\mathbf{k}_p$ at a pentagon p , where $\mathbf{k}_p(e_1 - e_2)$ is the component of $\mathbf{k}$ at p . Then $\sum_q \Phi(\mathbf{k})_q [C_q]$ is the image of $\mathbf{k}$ in $H_2(\widehat{X}; \mathbb{Q})$, which vanishes. By Proposition 3.4(2), $\Phi(V_X) \subseteq K \otimes \mathbb{Q}$. Conversely, every $\lambda \in K \otimes \mathbb{Q}$ satisfies $\lambda_{q_1} + \lambda_{q_2} = 0$ at each pentagon (Remark 3.10), so it equals $\Phi(\mathbf{k})$ for the $\mathbf{k}$ with $\mathbf{k}_q = \lambda_q$ at the squares and $\mathbf{k}_p = \lambda_{q_1}$ at the pentagons. So Φ is an isomorphism onto $K \otimes \mathbb{Q}$. The components of $\mathbf{k}$ are all nonzero if and only if every coordinate of $\Phi(\mathbf{k})$ is nonzero, both nodes of each pentagon included. By Lemma 9.8, (a) is therefore equivalent to the existence of $\lambda \in K$ with every $\lambda_q \neq 0$.

Conclusion. By Corollary B and Lemma 9.8, the same condition is equivalent to (b). By Theorem A(1), $K = \text{Rel}(\delta)$, so it is equivalent to (c). The tie holds on $\text{Rel}(\delta) = K$. $\square$

9.6. The factoring locus and the nodal locus. Batyrev and Kreuzer specialised the mirror family, for polytopes whose singular two-faces are squares, to the locus where the face polynomial of each square is a product of two binomials [BK10, §3]; we call its analogue for all node parallelograms, $\mu_P = 1$ for every P , the *factoring locus*. They expected the specialised members to have one conifold singularity at the point q_j of each of their local affine toric charts U_{σ_j} , and derived it from a local equation in which every monomial other than those of the square is divisible by the product of the two normal coordinates; they remark that this argument does not show that these are all the singular points [BK10, §3]. In the node chart of Section 7.3, where the normal coordinates are Z and Z' , that local equation omits the terms $AZ + A'Z'$ of the apexes, which are linear in one normal coordinate (for dual edges of lattice length one, the case considered here). These terms are present: the three-cells ω_1, ω_2 containing P have apexes, and by Proposition 3.26(4) every apex has exponent one in exactly one of Z , Z' and exponent zero in the other. Within U the factoring locus is $\{c' \in U : \eta_P(c') = 0 \text{ for every } P\}$, because $\mu_P = e^{\eta_P}$ and $|\eta_P| < 1$ on U ; its points with real coefficients of our sign pattern form $\mathcal{Z} \cap U$. The nodal locus D_Q lies close to it: by the bound on the η_P in the proof of item (3)(b), $|\eta_P| \leq Ct^{\theta_0/2}$ on D_Q for every P .

On the factoring locus $\kappa_P = 0$, so $\varepsilon_P = E_P$ in the notation of Section 9.3. Lemma 7.14(2) bounds E_P by $Ct^{\theta_0/2}$ without forcing it to vanish, whereas Lemma 9.1(4) requires $\varepsilon_P = 0$ for the member to be singular near q_P . Among the terms that move the critical value are those of the apexes of the cells next to P ; at a pentagon or hexagon node the monomials of the other lattice points of the two-face, which are constant in Z and Z' , move it as well. At a square node, in the node chart at $\kappa_P = 0$, the monomials of $\mathcal{A}_P$ give $xy + \mathbf{b} ZZ' + AZ + A'Z'$, with the apex sums A and A' . In the local model where these are replaced by $A_0 + A_1x$ and $A'_0 + A'_1y$, with constants A_0, A_1, A'_0, A'_1 and $A_1A'_1 \neq \mathbf{b}$, the critical point is $Z = -A'_0/(\mathbf{b} - A_1A'_1)$, $Z' = -A_0/(\mathbf{b} - A_1A'_1)$, $x = -A'_1Z'$, $y = -A_1Z$, and the critical value is

$$-\frac{A_0A'_0}{\mathbf{b} - A_1A'_1},$$

which is nonzero whenever the apex sums do not vanish at the node point. Thus, in this local model at a square node, the factoring condition gives a node only if an apex sum vanishes at the node point. In the algebraic family, the members of U with a node at every q_P are those of D_Q . For these, Corollary B settles the analogue of the question left open in [BK10, §3]: the

spheres δ_q satisfy a relation with every coefficient nonzero exactly when X'_σ is smoothable. The corollary produces a symplectic conifold transition, by [STY02, Thm. 2.9], and not a projective small resolution.

Theorem A(3) is the mirror half of Morrison's count (Section 2) at the members of U . Proposition 9.13 extends this count along every connected nodal component meeting the construction. The further assertion in [Mor99, §3.1], that the nodal mirror members have a small resolution mirror to a smoothing of X'_σ , is proved for X_9 in Theorem 9.25, in the sense of Batyrev families up to birational equivalence. The assertion for the whole admissible class remains open.

Remark 9.12 (polytopes without nodes). If $Q = \emptyset$, then $K = 0$ and the window of Definition 7.3 imposes no condition on binomial ratios. The constructions of Sections 7–9 then need no node-indexed constants: one puts $C_\eta = C_{\mathbf{R}} = 1$, $\mathcal{E}_{\max} = 0$ and $\xi_c = 0$, so that $c = c^{(0)}$, and keeps the conditions on the triangle gaps, the heights, the edges and the three-valent vertices. Only cases (A) and (B) of the proof of Lemma 7.11 occur, and they use only bounds on the moduli of the coefficients, so compactness of $\mathcal{C}_0$ gives one threshold on t for the smoothness of every member of $U_{\mathbb{C}}^+$, and $U^\circ = D_\emptyset = U$. Theorem A then holds with no vanishing spheres, item (3) with codimension and relative Picard number zero, and Corollary B with the trivial smoothing of the smooth threefold X'_σ . The constructions that do not involve nodes apply as stated.

9.7. Continuation along the nodal locus. Retain the hypotheses of Theorem A, and write

$$P = X_{\mathcal{T}}, \quad L = -K_P, \quad V = H^0(P, L) \setminus \{0\}, \quad Y_f = \{f = 0\} \subset P.$$

Thus P is a smooth projective toric fourfold and L is nef, big and basepoint-free. Its sections are indexed by $\Delta \cap N$. For any fixed $t > 0$, the change of coefficients $a_n = c_n t^{H(n)}$ identifies this family with the coefficient family in Section 2.4. We use all of V , including sections with some zero coefficients. We do not take a quotient by automorphisms of P .

Put $\nu = |Q|$ and $r = \nu - \text{rank } K$. Let $S_\nu \subset V$ consist of reduced hypersurfaces with exactly ν distinct ordinary double points and no other singularities. We give this locally closed locus its reduced analytic structure; below we also identify its local defining ideal.

A marking labels the nodes by Q . An orientation marking chooses a generator of $H_3(L_q; \mathbb{Z})$ for each link L_q of a node. These choices form a finite covering of S_ν once its local manifold structure is established. The chosen generator induces an orientation of the vanishing sphere in any sufficiently small smooth deformation. A relation has *full support* if every coefficient is nonzero.

Proposition 9.13 (componentwise continuation). *Let S be a connected component of S_ν containing a nodal member constructed in Theorem A(3). Choose at that member the marking and orientations obtained from the near-limit construction, and let $\tilde{S}$ be the connected component of the marking and orientation cover containing that choice. Then:*

- (1) S is smooth of codimension r in V .
- (2) For $s \in \tilde{S}$ and any sufficiently nearby smooth member Y_b , let $\delta_q(b)$ be its local vanishing spheres. The integral map

$$d_b : \mathbb{Z}^Q \longrightarrow H_3(Y_b; \mathbb{Z}), \quad e_q \longmapsto [\delta_q(b)]$$

satisfies

$$\ker d_b = d_b^{-1}(\text{Tors } H_3(Y_b; \mathbb{Z})) = K.$$

In particular, its image is free of rank r and meets the ambient torsion subgroup only in zero.

- (3) *Every $k \in K$ bounds an integral singular 4-chain in Y_b . Such a chain can be obtained by transport of a bounding chain supplied by the near-limit tropical construction, with the usual prism corrections for choices of fundamental cycles.*
- (4) *The spheres have pairwise disjoint Lagrangian representatives for the restriction of a fixed toric Kähler form. The equivalence of smoothability of X'_σ with a full-support relation among these spheres, and the symplectic-transition conclusion of Corollary B, hold at every such Y_b .*

On S itself, the relation assertion is an equality of sub-local systems of the signed permutation local system generated by the nodes. It does not give a path-independent marking of individual nodes. Monodromy on the whole of $H_3(Y_b)$ need not be a signed permutation. The signed permutation assertion concerns the local generators only.

The proof uses the existing near-limit comparison as an input. Lemma 9.17 proves that S_ν is smooth at every point, including components not meeting the construction, and identifies its local defining ideal through independent vanishing-cycle periods. The comparison with K then continues along each reached component. Proposition 9.19 shows that there can be components which do not meet the construction. The argument does not construct a projective small resolution of the mirror. Transported chains need not admit tropical descriptions at their endpoints.

More precisely, the imported near-limit results provide a member $f_0 \in S_\nu$, a coefficient neighbourhood U_0 in which the only possible singularities are the marked Morse points, and a connected smooth complement U_0° . At a known smooth member of this complement, the relation lattice and the preimage of ambient torsion are both K , and every element of K has a tropical bounding chain. The original resolution-side criterion identifies smoothability of X'_σ with the existence of a full-support vector in K . These are the inputs from Theorem A, Lemma 9.1, Proposition 9.5 and Corollary B.

Completeness of polynomial deformations. For a face F let $\ell^*(F)$ denote its number of relative interior lattice points. The correction term for the Newton polytope Δ is

$$(9.6) \quad c(\Delta) = \sum_{\dim G=2} \ell^*(G)\ell^*(G^\circ) = 0.$$

Indeed, the triangles and squares have no interior lattice points, while the dual edges of the pentagons and hexagons have lattice length one. The dual edge length of a triangular face is immaterial here.

Lemma 9.14. *For every reduced anticanonical hypersurface $Y \subset P$, $H^1(Y, T_P|_Y) = 0$. If Y has only nodes, the Kodaira–Spencer map*

$$H^0(P, L) \longrightarrow \text{Ext}_Y^1(\Omega_Y, \mathcal{O}_Y)$$

of the coefficient family is surjective.

Proof. The cohomology of a smooth projective toric variety is generated by algebraic cycle classes, so $H^j(P, \Omega_P^i) = 0$ for $i \neq j$. Also $H^j(P, \mathcal{O}_P) = 0$ for $j > 0$, and nef line bundles have no higher cohomology. These are the standard toric cohomology results; see [CLS11, Thms. 9.2.3, 9.4.11]. Since $T_P \otimes K_P \simeq \Omega_P^3$, the restriction sequence gives

$$(9.7) \quad H^i(P, T_P) \simeq H^i(Y, T_P|_Y), \quad i = 0, 1.$$

This holds for every anticanonical Cartier divisor, independently of its defining coefficients.

We compute $h^1(P, T_P)$ using a smooth Δ -regular member Z . For any anticanonical member Y , adjunction gives $\omega_Y \simeq \mathcal{O}_Y$. The sequence $0 \rightarrow K_P \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_Y \rightarrow 0$, Serre duality on the fourfold P , and the preceding toric vanishing give

$$H^0(Y, \mathcal{O}_Y) = \mathbb{C}, \quad H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0.$$

In particular, Y is connected. If Y has only nodes, it is normal and hence irreducible; thus the nodal members are projective Calabi–Yau threefolds in the usual sense. For the smooth member Z we obtain $K_Z \simeq \mathcal{O}_Z$ and $H^2(\mathcal{O}_Z) = 0$. Hence $H^0(T_Z) = H^0(\Omega_Z^2) = 0$ and $h^1(T_Z) = h^{2,1}(Z)$. The restriction sequence of L gives

$$h^0(Z, L|_Z) = \ell(\Delta) - 1, \quad H^1(Z, L|_Z) = 0.$$

Taking cohomology in the normal sequence of $Z \subset P$ and using (9.7) yields

$$(9.8) \quad h^1(P, T_P) = h^{2,1}(Z) - \ell(\Delta) + 1 + h^0(P, T_P).$$

The toric Euler sequence [CLS11, Thm. 8.1.6], or equivalently the root description of toric vector fields, gives

$$(9.9) \quad h^0(P, T_P) = 4 + \sum_{\dim F=3} \ell^*(F).$$

For completeness, a nonconstant monomial section of a ray divisor D_ρ is indexed by a lattice point m with $\langle m, v_\rho \rangle = -1$ and $\langle m, v_{\rho'} \rangle \geq 0$ for every other ray. Such a v_ρ must be a vertex of Δ° , and m is an interior lattice point of its dual facet in Δ . Conversely every such point satisfies these inequalities for all the boundary rays: its pairing can equal -1 only at that vertex, and the other integral pairings are nonnegative. Summing $h^0(\mathcal{O}_P(D_\rho))$ and subtracting the rank of $\text{Pic}(P)$ in the Euler sequence proves (9.9).

Batyrev’s formula [Bat94, Thm. 4.3.7] is

$$h^{2,1}(Z) = \ell(\Delta) - 5 - \sum_{\dim F=3} \ell^*(F) + c(\Delta).$$

Equations (9.6)–(9.9) therefore give $H^1(P, T_P) = 0$, proving the first assertion.

For a reduced nodal hypersurface the conormal sequence

$$0 \longrightarrow L^{-1}|_Y \xrightarrow{df} \Omega_P|_Y \longrightarrow \Omega_Y \longrightarrow 0$$

is exact on the left as well: df is nonzero at the generic smooth point of every component, and $\mathcal{O}_Y$ is reduced. Its global Ext sequence contains

$$H^0(Y, L|_Y) \longrightarrow \text{Ext}_Y^1(\Omega_Y, \mathcal{O}_Y) \longrightarrow H^1(Y, T_P|_Y) = 0.$$

Finally $H^0(P, L) \rightarrow H^0(Y, L|_Y)$ is surjective because $H^1(P, \mathcal{O}_P) = 0$. The composite is the claimed Kodaira–Spencer map. $\square$

The rank of the node-smoothing map. Fix $f \in S_\nu$ and write $Y = Y_f$. Choose mutually disjoint small conical neighbourhoods of its nodes, and let W be their compact exterior. Its oriented boundary is a disjoint union $L_\partial = \coprod_{q \in Q} L_q$ with $L_q \simeq S^2 \times S^3$. Write the integral inclusion maps as

$$(9.10) \quad a : H_3(L_\partial; \mathbb{Z}) \longrightarrow H_3(W; \mathbb{Z}), \quad b : H_2(L_\partial; \mathbb{Z}) \longrightarrow H_2(W; \mathbb{Z}).$$

The orientation marking identifies the source of a with $\mathbb{Z}^Q$. Use the boundary intersection pairing to choose the dual basis in the source of b . In this section ranks are computed over $\mathbb{Q}$.

Lemma 9.15. *One has $\text{rank } a + \text{rank } b = \nu$. If Y_u is a sufficiently small smooth deformation of Y , then*

$$(9.11) \quad 0 \longrightarrow H_3(W; \mathbb{Z}) \xrightarrow{j} H_3(Y_u; \mathbb{Z}) \longrightarrow \ker b \longrightarrow 0.$$

Under suitable compatible signs, its vanishing-cycle map is $d_u = ja$. In particular,

$$(9.12) \quad \ker d_u = \ker a, \quad d_u^{-1}(\text{Tors } H_3(Y_u; \mathbb{Z})) = a^{-1}(\text{Tors } H_3(W; \mathbb{Z})).$$

Proof. The boundary map $H_3(W, L_\partial; \mathbb{Q}) \rightarrow H_2(L_\partial; \mathbb{Q})$ is adjoint to a by Poincaré–Lefschetz duality and the intersection pairing on L_∂ . Its image is $\ker b$ by the exact sequence of the pair. Thus $\dim \ker b = \text{rank } a$, giving the rank identity.

Replace each conical neighbourhood by its Milnor fibre F_q . The fibre is a disc cotangent bundle of S^3 , and $H_3(L_q; \mathbb{Z}) \rightarrow H_3(F_q; \mathbb{Z})$ is an isomorphism. One way to see the integral assertion is to trivialise T^*S^3 : the map is the projection $S^3 \times S^2 \rightarrow S^3$. Also $H_2(F_q; \mathbb{Z}) = 0$. Mayer–Vietoris for $Y_u = W \cup \coprod_q F_q$ therefore has the segment

$$\mathbb{Z}^Q \xrightarrow{(a, -1)} H_3(W; \mathbb{Z}) \oplus \mathbb{Z}^Q \longrightarrow H_3(Y_u; \mathbb{Z}) \longrightarrow \mathbb{Z}^Q \xrightarrow{b} H_2(W; \mathbb{Z}).$$

The cokernel of $(a, -1)$ identifies with $H_3(W; \mathbb{Z})$ by $(w, v) \mapsto w + a(v)$. This proves (9.11) and $d_u = ja$ after orienting the local core spheres compatibly. The group $\ker b$ is free, being a subgroup of $\mathbb{Z}^Q$. Consequently j identifies the torsion subgroup of its source with that of its target, giving (9.12). $\square$

Let

$$\lambda : \text{Ext}_Y^1(\Omega_Y, \mathcal{O}_Y) \longrightarrow \bigoplus_{q \in Q} T_q^1 \simeq \mathbb{C}^Q$$

be the global-to-local node-smoothing map. The choice of coordinates on the one-dimensional T_q^1 affects it by nonzero diagonal factors only.

Lemma 9.16. *The rank of λ equals the rank of the span of the vanishing cycles in a nearby smoothing. Both equal $\text{rank } a$.*

Proof. Let $\pi : \bar{Y} \rightarrow Y$ be the blow-up of all nodes. This is a smooth projective threefold, with exceptional quadrics $E_q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of normal bundle $\mathcal{O}_{E_q}(-1, -1)$. Denote their ruling classes in $H_2(\bar{Y}; \mathbb{Q})$ by A_q, B_q , and put $D_q = A_q - B_q$. The node-smoothing sequence of Rollenske–Thomas [RT09, §2 and (3.2)] identifies the image of λ with the kernel of the map sending the q th generator to D_q in

$$H^{2,2}(\bar{Y}) / \langle A_q + B_q : q \in Q \rangle.$$

Here curve classes are identified with degree-four cohomology classes. All hypotheses hold: Y has trivial dualizing sheaf by adjunction, has only ordinary double points, and its blow-up is Kähler because it is projective. No small resolution is used. The span of the D_q has zero intersection with the span of the $A_q + B_q$: pairing with each exceptional divisor gives zero on every D_q and -2 on its own sum, with zero on the other sums. Hodge decomposition on $\bar{Y}$ therefore shows

$$(9.13) \quad \text{rank } \lambda = \nu - \dim_{\mathbb{Q}} \langle D_q : q \in Q \rangle.$$

We identify the last dimension without assuming that a small resolution is Kähler. Topologically $\bar{Y}$ is obtained from W by attaching the disc bundles of $\mathcal{O}_{E_q}(-1, -1)$. The map from

$H_2(L_q; \mathbb{Q})$ to $H_2(E_q; \mathbb{Q})$ is injective with image $\mathbb{Q}(A_q - B_q)$, by the circle-bundle sequence. After compatible choices of sign, Mayer–Vietoris gives

$$H_2(\bar{Y}; \mathbb{Q}) = \frac{H_2(W; \mathbb{Q}) \oplus \bigoplus_q H_2(E_q; \mathbb{Q})}{\{(b(v), -D(v)) : v \in \mathbb{Q}^Q\}}.$$

The map $D : \mathbb{Q}^Q \rightarrow \bigoplus_q H_2(E_q; \mathbb{Q})$ is injective. It follows that $\sum v_q D_q$ vanishes in $H_2(\bar{Y}; \mathbb{Q})$ exactly when $b(v) = 0$. Thus the dimension in (9.13) is $\text{rank } b$. Lemma 9.15 now gives $\text{rank } \lambda = \nu - \text{rank } b = \text{rank } a = \text{rank } d_u$. $\square$

Local equations of the nodal locus. Choose a sufficiently small coefficient neighbourhood $U \subset V$ of $f \in S_\nu$. The holomorphic Morse lemma with parameters gives one holomorphic critical point near each node and a critical-value function $\varepsilon_q : U \rightarrow \mathbb{C}$. In suitable local coordinates the hypersurface equation is

$$(9.14) \quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = \varepsilon_q(u).$$

Multiplying the original local equation by a unit changes the coordinates and the critical value, but not its vanishing locus or the rank of their differentials at f . Compactness of the complement of the node neighbourhoods ensures, after shrinking U , that these are all possible singular points. Each is a node whenever its critical value vanishes.

The derivative $d\varepsilon_q(f)$ is, up to a nonzero factor, evaluation of a section at the node. Basepoint-freeness of L makes each such derivative nonzero. A tangent direction outside their finitely many kernels gives a simultaneous smoothing. The set

$$U^\circ = U \setminus \bigcup_{q \in Q} \{\varepsilon_q = 0\}$$

is nonempty and connected after choosing a small ball for U . Over it, the oriented local vanishing cycles are single-valued as homology classes. A loop of the local parameter in (9.14) acts by the antipodal map on S^3 , of degree $+1$; the disjoint local charts give the same conclusion for arbitrary loops in U° .

Choose the relative holomorphic volume form Ω_u by adjunction from the defining section and a fixed trivialization of $K_P \otimes L$. The periods

$$p_q(u) = \int_{\delta_q(u)} \Omega_u$$

extend holomorphically over U , and

$$(9.15) \quad p_q = h_q \varepsilon_q, \quad h_q(f) \neq 0.$$

Indeed, in the Morse chart the residue is the residue of $g(z, u) dz_1 \cdots dz_4 / (\sum z_i^2 - \varepsilon_q(u))$ with $g(0, f) \neq 0$. Scaling $z = \sqrt{\varepsilon_q} x$ contributes a factor ε_q . The integrals of the odd-degree Taylor terms of g vanish under the antipodal map, and the even-degree terms give a convergent power series in ε_q . The leading integral is a nonzero constant times $g(0, f)$. Shrink U so that every h_q is a unit.

Lemma 9.17. *The locus S_ν is smooth near f , with codimension equal to $c = \text{rank } a$. Its local ideal is $(\varepsilon_q : q \in Q)$ and is generated by c independent periods.*

Proof. Lemma 9.14 and Lemma 9.16 imply $\text{rank } d\varepsilon(f) = c$: the differential of the critical-value map is the composite of the Kodaira–Spencer map with λ , followed by nonzero diagonal factors. Equation (9.15) gives $\text{rank } dp(f) = c$ as well.

Fix $u \in U^\circ$ and put $R_{\mathbb{Q}} = \ker(d_u \otimes \mathbb{Q}) \subset \mathbb{Q}^Q$. This space has dimension $\nu - c$ and is independent of u by parallel transport of the local cycles. For $k \in R_{\mathbb{Q}}$ one has

$$(9.16) \quad \sum_q k_q p_q = 0$$

on U° , hence on U by holomorphic extension. Its complexification is therefore contained in the space of row relations of $dp(f)$. Both spaces have dimension $\nu - c$, so they are equal. Choose an index set $I \subset Q$ of size c whose rows in $dp(f)$ are independent. For each $q \notin I$, the corresponding row dependence belongs to $R_{\mathbb{Q}} \otimes \mathbb{C}$; equation (9.16) therefore expresses p_q as a constant linear combination of the p_i , $i \in I$, throughout U . Thus

$$(\varepsilon_q : q \in Q) = (p_q : q \in Q) = (p_i : i \in I).$$

The last generators have independent differentials at f . The implicit function theorem proves the assertion, including reducedness of this local zero scheme. By the choice of U , its points are exactly those with all ν nodes present and no other singularities, so it is $S_\nu \cap U$. $\square$

The smoothness statement follows from completeness of polynomial deformations, the rank calculation, and the period identities. It applies at every point of S_ν , including components for which no comparison with K has yet been obtained.

Integral continuation and the symplectic consequence.

Proof of Proposition 9.13. Lemma 9.17 gives S_ν its local manifold structure. The universal set of nodes is a finite unramified covering of degree ν . Locally along this locus, the parametrized Morse coordinates trivialize the singularities and their links. The exterior of small node neighbourhoods is a proper smooth family of compact manifolds with boundary. Ehresmann's theorem, also for the boundary, supplies local differentiable trivializations. Choices of sufficiently small neighbourhoods agree up to isotopy and collar identifications. Equivalently, the conical collars identify $H_3(W; \mathbb{Z})$ with $H_3(Y_{\text{reg}}; \mathbb{Z})$, where Y_{reg} is the smooth locus of the nodal hypersurface. Consequently $H_3(L_\partial; \mathbb{Z})$ and $H_3(W; \mathbb{Z})$ are local systems, with the inclusion-induced morphism a in (9.10).

Ordering the nodes and orienting their rank-one link groups kills the finite signed permutation monodromy of the source. Its connected covering $\tilde{S}$ through the initial marking surjects onto S . On this cover the source is the constant group $\mathbb{Z}^Q$. Both $\ker a$ and $a^{-1}(\text{Tors } H_3(W; \mathbb{Z}))$ are constant subgroups of that source: transport in the target preserves the inclusion map and its torsion subgroup. At the near-limit starting point, Proposition 9.5 and Theorem A, together with (9.12), identify both subgroups with K . They therefore equal K throughout $\tilde{S}$. Equation (9.12) proves part (2) for every sufficiently nearby smooth deformation of each member. The rank c in Lemma 9.17 is now r throughout S , proving part (1).

Here is also an integral chain description. Represent the marked link generators by cycles β_q . Since $a(k) = 0$ for $k \in K$, there is an integral 4-chain C_W in W with $\partial C_W = \sum_q k_q \beta_q$. In the local Milnor fibre, β_q and the vanishing sphere represent the same generator of $H_3(F_q; \mathbb{Z})$, so an integral 4-chain C_q has boundary $[\delta_q] - \beta_q$. Then

$$\partial \left(C_W + \sum_q k_q C_q \right) = \sum_q k_q [\delta_q].$$

To retain a specific chain from the tropical construction, take a path in $\tilde{S}$ from the initial member to the desired one. First transport that chain inside U_0° from its constructed smooth member to a sufficiently small smoothing of f_0 ; connectedness of U_0° and the flatness of the

marked spheres allow this. This initial step uses the original neighbourhood U_0 , which need not equal the smaller neighbourhood of f_0 used below. Cover this compact path by finitely many coefficient neighbourhoods as above and subdivide it accordingly. At consecutive overlaps choose smooth members arbitrarily near the intervening nodal member. Within each neighbourhood the complement of the discriminant is path connected, so these choices can be joined to give a path of smooth hypersurfaces from the initial smoothing to the target one. Along each piece the local cycles identify with the marked link generators. Differentiable parallel transport carries the initial integral chain to the final fibre. Local sphere isotopies and prism chains adjust its boundary to the specified fundamental cycles. This proves part (3); no uniqueness of the resulting chain is asserted.

Fix a toric Kähler form on P . At a nodal member and a sufficiently small simultaneous smoothing, the local holomorphic Morse functions have Lefschetz thimbles contained in mutually disjoint node neighbourhoods. Their intersections with the smooth fibre are embedded Lagrangian 3-spheres, smoothly isotopic to the local Milnor spheres. The containment follows by taking the critical values small compared with a fixed ball in each Morse chart; on that ball the Kähler metric is uniformly comparable with the Euclidean metric. This is precisely the local argument of Lemma 9.6 and does not use its near-limit estimates. A path in U° transports the spheres to any other smooth member of the chosen neighbourhood. The restriction of the fixed ambient Kähler form gives symplectic parallel transport on this proper smooth family, preserving disjointness and the vanishing homology classes.

The original resolution-side criterion in Corollary B says that X'_σ is smoothable exactly when K contains an integral vector with every coordinate nonzero. Part (2) transfers this criterion to the mirror spheres. Smith–Thomas–Yau [STY02, Thm. 2.9] then provides a symplectic conifold surgery in one of the choices of local small resolutions, with symplectic exceptional curves. This proves part (4). $\square$

Remark 9.18 (Scope and boundary cases). The statement concerns sufficiently small smoothings of members of a reached nodal component. It does not assert a canonical collection of spheres on every smooth member of the whole coefficient space. One may transport a configuration farther along a chosen path of smooth members, but that is additional path-dependent data. Scalar multiplication acts freely on V and preserves S_ν ; the same smoothness and codimension statement holds in $\mathbb{P}H^0(P, L)$. A coarse quotient by ambient automorphisms is a different question. If $\nu = 0$, the locus is open and the relation assertions are empty. Colliding nodes, additional singularities and nonreduced hypersurfaces lie outside the stated locus.

9.8. Two components of the four-node locus for X_9 . The reached-component restriction in Proposition 9.13 is necessary.

Use the smooth projective toric fourfold $P = X_\mathcal{T}$ specified by the X_9 triangulation of Section 11.1 and Appendix G.1. Its fan has 161 rays and 813 maximal cones; this is the smooth triangulation used for the example in the current manuscript. Put

$$V = H^0(P, -K_P) \setminus \{0\}.$$

Let $S_4 \subset V$ be the locus where Y_f has exactly four nodes and no other singularities. The hypersurfaces in this locus are required to be reduced. By Lemmas 9.16 and 9.17 and the local-system argument of Proposition 9.13, S_4 is smooth, its local codimension is the rank of the vanishing-cycle map, and that rank is locally constant along S_4 . The near-limit construction gives a member for which the rank is three and the marked integral relation lattice is $\mathbb{Z}(1, -1, -1, 1)$.

Proposition 9.19. *The locus S_4 has at least two connected components. One has codimension three and meets the near-limit construction. Another has codimension four and does not meet that construction. For a sufficiently small smoothing of any member of the latter component, the four vanishing spheres are linearly independent over $\mathbb{Q}$. Consequently their integral relation lattice and the preimage of ambient torsion are both zero.*

We prove the assertion by an explicit member. The Newton polytope Δ_9 has precisely the following twelve lattice points. The first nine are its vertices. In the standard character lattice, write $f_* = \sum_{i=0}^{11} c_i \chi^{n_i}$, with

i	n_i	c_i
0	(1, 0, 0, 0)	1
1	(0, 1, 0, 0)	1
2	(-1, -1, 0, 0)	668
3	(0, 0, 1, 0)	324
4	(0, 0, 0, 1)	324
5	(-4, -2, -1, 0)	1944
6	(-4, -2, 0, -1)	1944
7	(3, 1, 1, 1)	108
8	(2, 1, 1, 1)	432
9	(-1, 0, 0, 0)	-21
10	(-2, -1, 0, 0)	2496
11	(0, 0, 0, 0)	0

The support assertion can be checked by testing the integer points of $[-4, 3] \times [-2, 1] \times [-1, 1]^2$ against the polar inequalities. All boundary coefficients are nonzero.

For torus coordinates t_1, t_2, t_3, t_4 , make the unimodular monomial change

$$(9.18) \quad x = t_1, \quad y = t_1 t_2, \quad z = t_1^2 t_2 t_3, \quad w = t_1^2 t_2 t_4.$$

Multiplying f_* by the torus unit $xy = t_1^2 t_2$ gives

$$(9.19) \quad \begin{aligned} F &= y^2 + (x^2 - 21)y + 668x + 2496 \\ &\quad + 324(z + w) + 1944(z^{-1} + w^{-1}) + 108(x + 4)zw. \end{aligned}$$

The four torus singularities. The singular points of (9.19) in the torus are the following two pairs:

$$(9.20) \quad \begin{aligned} x &= 5, \quad y = -2, \quad z + w = 18, \quad zw = -6; \\ z &= w = \xi, \quad \xi^2 = -3, \quad x = -4 + 3\xi, \quad y = 16 + 12\xi. \end{aligned}$$

Here the first pair has $z = 9 \pm \sqrt{87}$ and $w = 9 \mp \sqrt{87}$. All four points are distinct and all coordinates are nonzero.

To prove completeness, subtract two logarithmic derivatives:

$$zF_z - wF_w = 324(z - w) \left(1 + \frac{6}{zw} \right).$$

On the branch $zw = -6$, the equation reduces to

$$h(x, y) = y^2 + (x^2 - 21)y + 20x - 96 = 0.$$

The equations $h = h_x = h_y = 0$ require a multiple root of the y -discriminant

$$(x^2 - 21)^2 - 4(20x - 96) = (x - 5)^2(x^2 + 10x + 33).$$

The only multiple root is $x = 5$, giving $y = -2$. An individual logarithmic derivative then gives $z + w = 18$.

On the branch $z = w$, the equations $F_z = F_y = 0$ give

$$x = 18z^{-3} - 3z^{-1} - 4, \quad y = -\frac{x^2 - 21}{2}.$$

After substitution, the numerators of F_x and F , up to nonzero rational constants, are respectively

$$\begin{aligned} P_{11} &= 4z^{11} + 24z^9 + 3z^8 + 4z^7 - 17z^6 - 48z^5 - 18z^4 + 144z^3 + 108z^2 - 216, \\ Q_{13} &= 48z^{13} - 27z^{12} + 576z^{11} - 18z^{10} + 1712z^9 + 213z^8 \\ &\quad + 288z^7 - 576z^6 - 1728z^5 - 648z^4 + 3456z^3 + 2592z^2 - 3888. \end{aligned}$$

Their monic greatest common divisor is $z^2 + 3$. An exact certificate is

$$P_{11} = (z^2 + 3)P_9, \quad Q_{13} = (z^2 + 3)^2Q_9,$$

where

$$\begin{aligned} P_9 &= 4z^9 + 12z^7 + 3z^6 - 32z^5 - 26z^4 + 48z^3 + 60z^2 - 72, \\ Q_9 &= 48z^9 - 27z^8 + 288z^7 + 144z^6 - 448z^5 - 408z^4 + 384z^3 + 576z^2 - 432, \end{aligned}$$

and the nonzero resultant values are

$$\text{Res}(P_9, Q_9) = 2^{59}3^{13}13^229^231 \cdot 9311, \quad \text{Res}(P_9, z^2 + 3) = 3^7 \cdot 13 \cdot 31.$$

This gives exactly the second pair in (9.20).

The Hessian determinants of F in the coordinates (x, y, z, w) are

$$(9.21) \quad \det \text{Hess } F = \begin{cases} 2^6 3^9 \cdot 87, & \text{at either point of the first pair,} \\ -2^6 3^9 (21 + 16\xi), & \text{at the second pair, } \xi^2 = -3. \end{cases}$$

They are nonzero, so each singularity is an ordinary double point.

Compactification and the boundary. It remains to exclude singular points on the toric boundary of P . For a cone τ with ray generators r_j , let

$$E_0(\tau) = \{i : \langle n_i, r_j \rangle = -1 \text{ for every } r_j \in \tau\}.$$

These are the exponents surviving on the orbit of τ . If they are affinely independent, the equation and its logarithmic tangent derivatives cannot vanish simultaneously, since all their coefficients are nonzero. The complete cone calculation for the fixed triangulation is

$\dim \tau$	cones	independent E_0	remaining
1	161	157	4
2	974	971	3
3	1626	1626	0
4	813	813	0

The four relevant rays, in the accompanying data's ordering, are

$$r_5 = (-1, 2, -1, 1), \quad r_6 = (-1, 2, 1, -1), \quad r_8 = (1, -1, -1, -1), \quad r_{11} = (0, 1, -1, -1).$$

All seven remaining cases are listed below. A normal derivative is computed in a smooth toric chart, then restricted to the orbit; a singleton support gives a nonzero monomial and hence a unit there.

	τ	$E_0(\tau)$	exclusion or additional calculation
(9.22)	(5)	0, 2, 3, 6, 7	n_0 outside the affine span of the others
	(6)	0, 2, 4, 5, 7	n_0 outside the affine span of the others
	(8)	1, 3, 4, 5, 6, 8, 9, 10	$x = 0$ in (9.19)
	(11)	2, 3, 4, 5, 6, 7, 8, 10	$y = 0$ in (9.19)
	(5, 11)	2, 3, 6, 7	normal derivative supported on n_0
	(6, 11)	2, 4, 5, 7	normal derivative supported on n_0
	(8, 11)	3, 4, 5, 6, 8, 10	normal derivative supported on n_9

The first two cases are excluded by taking an affine-linear combination of the equation and tangent derivatives which retains only the nonzero n_0 -term. The last three normal derivatives have respective coefficients 1, 1, -21 .

For the remaining two divisor orbits, (9.18) identifies the chart of $\langle r_8, r_{11} \rangle$ with $\mathbb{C}_x \times \mathbb{C}_y \times (\mathbb{C}^*)^2_{z,w}$, and its anticanonical equation is precisely (9.19). On either divisor orbit a singular point would satisfy

$$zw = -\frac{167}{27}, \quad (z - w) \left(324 + \frac{1944}{zw} \right) = 0.$$

The second factor in parentheses is $1620/167 \neq 0$; hence $z = w$ and $z^2 = -167/27$. On $y = 0$, the other normal equation and either logarithmic derivative give

$$x^2 = 21, \quad (x + 4)z = -\frac{987}{167}.$$

The number x is real and z is nonzero and purely imaginary, whereas the last right-hand side is nonzero real, a contradiction. On $x = 0$, an individual logarithmic derivative instead gives $z = -987/668$, contradicting $z^2 = -167/27$. At $x = y = 0$, the derivative $F_y = -21$ already excludes a singularity. Thus there are no boundary singularities.

The finite checks in this section consist only of the lattice-point list, the unimodular cone determinants, the orbit supports and their affine ranks, and the indicated normal derivative supports. They are reproduced from the fixed example's fan in the accompanying exact calculation (<https://github.com/bwuebben/calabi-yau-smoothability>, directory `paper6; examples/verify_x9_components.py`). The two remaining exclusions above are over $\mathbb{C}$. In particular, Y_{f_*} has exactly the four nodes displayed earlier. It is reduced: a nonreduced divisor would be singular along a positive-dimensional subset of its support.

Smoothing rank and distinct components. The coefficient directions indexed by n_{10}, n_2, n_3, n_4 are the functions $1, x, z, w$ after the trivialization used in (9.19). Their evaluations at the four nodes are linearly independent. Indeed, suppose

$$\alpha + \beta x + \gamma z + \delta w = 0$$

at all four points. Subtracting the values at the first pair gives $\gamma = \delta$. The second pair then gives $\alpha = 4\beta$ and $\gamma = -3\beta/2$. Substitution into either point of the first pair gives $-18\beta = 0$. All four coefficients therefore vanish.

Proof of Proposition 9.19. At an ordinary double point, the differential of its local smoothing parameter is, up to a nonzero factor, evaluation of the coefficient variation at that point. The preceding calculation gives rank four. Lemmas 9.16 and 9.17 imply that the four vanishing

classes are independent over $\mathbb{Q}$, and that S_4 has codimension four near f_* . A nonzero integral combination cannot be zero or torsion, since either would vanish after tensoring with $\mathbb{Q}$.

The rank of the vanishing-cycle map is locally constant along S_4 , as follows from the exterior-of-the-nodes local system used in the componentwise continuation proof. It is therefore four on the connected component containing f_* . On every component meeting the near-limit construction it is three. These components are distinct, proving all assertions. $\square$

Remark 9.20. This proves failure of coverage for the full four-node locus in this fixed toric ambient. It does not classify all its components, nor determine whether every component having the rank-one full-support relation lattice meets the near-limit construction. The original X'_9 remains smoothable; the counterexample shows why its mirror relation criterion must retain the reached-component hypothesis.

9.9. The codimension-three component for X_9 . We retain the toric fourfold P , the full coefficient space V and the exact four-node locus S_4 of Section 9.8. Thus zero coefficients are allowed, but additional singularities are excluded. Number the lattice points as in (9.17), and put

$$(9.23) \quad \mathcal{D} = \left\{ c \in \mathbb{C}^{12} : \text{rank} \begin{pmatrix} c_2 & c_{10} & c_5 & c_6 \\ c_7 & c_8 & c_4 & c_3 \end{pmatrix} \leq 1 \right\}.$$

This irreducible determinantal variety has dimension nine. The special form of Δ_9 permits a classification of the part of S_4 carrying a relation, as well as a projective small resolution on that entire part.

Theorem 9.21. *The locus S_4 has exactly one connected component of codimension three, namely $\mathcal{D} \cap S_4$. It meets the near-limit nodal construction of Theorem A. After marking and orienting the nodes by transport from that construction, the integral relation lattice and the preimage of ambient torsion are both $\mathbb{Z}(1, -1, -1, 1)$. Every other connected component of S_4 has codimension four and has zero relation lattice and zero preimage of torsion. Every member of $\mathcal{D} \cap S_4$ admits a projective small resolution.*

The theorem does not count the codimension-four components; Proposition 9.19 shows that at least one exists. Its proof uses the compact toric boundary, an evaluation-rank argument, and a surface through the four nodes. The finite fan comparisons below use the triangulation in Appendix G.1 and its accompanying data.¹

The compact boundary and a common affine chart. Write r_j for the primitive ray generators of the fan of P , in the order of the accompanying triangulation. For a cone τ , in addition to $E_0(\tau)$ from Section 9.8, define

$$E_{1,j}(\tau) = \{i : \langle n_i, r_j \rangle + 1 = 1, \langle n_i, r_k \rangle + 1 = 0 \text{ for } k \in \tau, k \neq j\}.$$

These are the supports of the first normal derivatives along the orbit of τ . Smoothness of the maximal cones makes this an ordinary coordinate-derivative calculation.

Lemma 9.22. *For every $c \in S_4$, the coefficients $c_0, c_1, c_3, c_4, c_5, c_6, c_7$ are nonzero. Every singular point lies in*

$$U_{\langle r_8, r_{11} \rangle} = \mathbb{C}_x \times \mathbb{C}_y \times (\mathbb{C}^*)^2_{z,w}, \quad r_8 = (1, -1, -1, -1), \quad r_{11} = (0, 1, -1, -1),$$

¹The program `examples/verify_x9_global.py` reproduces the boundary supports, singular ideals, surface fan and polytope comparisons used in Sections 9.9–9.10.

where the local equation is

$$(9.24) \quad \begin{aligned} F &= c_1 y^2 + q(x)y + c_2 x + c_3 z + c_4 w + c_5/z + c_6/w + c_7 xzw + c_8 zw + c_{10}, \\ q(x) &= c_0 x^2 + c_{11}x + c_9. \end{aligned}$$

Proof. The character change (9.18) is unimodular, and its pairings with r_8, r_{11} are the x, y coordinate valuations. Multiplying the Laurent equation by xy gives (9.24) on this chart.

The following cones have $E_0(\tau) = \{i\}$ and every $E_{1,j}(\tau)$ empty:

i	τ	$\dim O(\tau)$
0	(19, 120)	2
1	(136, 141)	2
3	(112, 134, 155)	1
4	(121, 144, 149)	1
5	(127, 149, 150)	1
6	(8, 132, 157)	1

If the indicated coefficient vanishes, the whole positive-dimensional orbit is singular. This proves the first six nonvanishing assertions.

Suppose $c_7 = 0$. If $c_8 = 0$ also, the cone (136, 154, 156) has $E_0 = \{7, 8\}$ and all first-normal supports empty, again giving a singular curve. If $c_8 \neq 0$, the four maximal cones

$$(9, 80, 114, 136), \quad (10, 85, 117, 136), \quad (114, 131, 136, 154), \quad (117, 131, 136, 156)$$

give singular fixed points: each has $E_0 = \{7\}$ and all $E_{1,j}$ empty. There is a further singular point on each of the following one-dimensional orbits:

τ	only nonempty first-normal support
(9, 114, 154)	$\{3, 8\}$
(10, 117, 156)	$\{4, 8\}$
(9, 80, 136)	$\{1, 8\}$
(10, 85, 136)	$\{1, 8\}$

Their orbit equations vanish identically, since $E_0 = \{7\}$. The remaining normal derivative is a binomial with nonzero coefficients and distinct characters of $\mathbb{C}^*$, so has a zero on the orbit. At such a zero every first derivative vanishes. These four orbits and the four fixed points are disjoint, giving at least eight singular points. Therefore $c_7 \neq 0$ on S_4 .

Put $I = \{0, 1, 3, 4, 5, 6, 7\}$. For all 3,571 nonzero cones outside $\langle r_8, r_{11} \rangle$, the finite support calculation gives

$$\begin{array}{l|l} \text{an } i \in I \text{ outside the affine span of } E_0(\tau) \setminus \{i\} & 3,555 \\ \text{otherwise, an } E_{1,j}(\tau) = \{i\}, \quad i \in I & 16. \end{array}$$

In the first row an affine-linear functional on the supported exponents isolates the i th monomial. On the orbit it is a linear combination of the equation and its logarithmic tangent derivatives, and cannot vanish because $c_i \neq 0$. In the second row a normal derivative is already a nonzero monomial. Both arguments remain valid when any of the other coefficients vanish. Together with the three nonzero cones (8), (11) and (8, 11) in the stated chart, these exhaust the 3,574 boundary orbits. The dense torus is in the same chart. $\square$

The coefficient-smoothing rank. At a node in this chart the coefficient directions are evaluations of

$$(9.25) \quad x^2y, y^2, x, z, w, z^{-1}, w^{-1}, xzw, zw, y, 1, xy.$$

These are directions in the full coefficient space even when a coefficient of the defining equation is zero. Multiplying a local equation by a unit only rescales the corresponding evaluation row.

Lemma 9.23. *For $c \in S_4$, the coefficient-smoothing map has rank three if and only if $c \in \mathcal{D}$. It never has rank at most two.*

Proof. First, every three distinct singular points have independent evaluation rows. If not, choose a minimally supported relation among at most three rows. Two rows cannot be proportional, since $1, x, y, z, w$ distinguish points, so all three weights are nonzero. The functions $1, y, y^2$ force all three y -values to agree: three distinct values give a Vandermonde matrix, and two values leave a singleton with zero weight. If their common value is nonzero, $1, x, x^2y$ force common x by the same argument. If it is zero, $F_y = q(x) = 0$ gives at most two x -values, and $1, x$ again force common x . Now

$$(9.26) \quad F_x = (2c_0x + c_{11})y + c_2 + c_7zw = 0, \quad c_3z^2 + (c_7x + c_8)(zw)z - c_5 = 0$$

fix $p = zw$ and allow at most two z -values. Since $w = p/z$, three distinct points are impossible. Lemma 9.22 supplies all nonzero coefficients used here.

Suppose now that the four rows have rank three, and let λ be their relation. It has full support by the preceding paragraph. Let $W \subset \mathbb{C}^4$ be the space of evaluation vectors of the functions in (9.25). It has dimension three.

If the four y -values are distinct, $W = \langle 1, y, y^2 \rangle$. The inclusions $x, xy \in W$, by interpolation at these four values, give $x = \alpha + \beta y$. Then $x^2y \in W$ forces $\beta^2 = 0$. Thus x is constant, whereas $F_y = 2c_1y + q(x) = 0$ forces y constant as well, a contradiction. Three distinct y -values are also impossible: the equations for $1, y, y^2$ make the sum of weights in each equal-value group zero, and at least one group is a singleton.

If there are two y -values, the points form two pairs with opposite weights within each pair. The equations for x, xy force common x within each pair. The two x -values differ, because $F_y = 0$ determines y from x . Equation (9.26) fixes $p = zw$ within each pair; its two z -values differ. Write $\Delta z_a, \Delta z_b$ for their differences, S_a, S_b for their products, and λ_a, λ_b for the opposite weights. The functions z, z^{-1} give

$$\lambda_a \Delta z_a + \lambda_b \Delta z_b = 0, \quad \frac{\lambda_a \Delta z_a}{S_a} + \frac{\lambda_b \Delta z_b}{S_b} = 0.$$

Both summands are nonzero, so $S_a = S_b$. Since $\Delta w = -p \Delta z / S$, the equation for w implies a common product p for both pairs. At each point, $zF_z - wF_w = 0$ says

$$(9.27) \quad (c_3 + c_6/p)z^2 = c_5 + c_4p.$$

If the left coefficient is nonzero, the two z -values in either pair are opposite. The quadratic in (9.26) then forces $c_7x + c_8 = 0$ for both pairs, contradicting their distinct x -values. Consequently both sides' coefficients in (9.27) vanish. On $zw = p$, the remaining equation is

$$h(x, y) = c_1y^2 + q(x)y + Lx + M = 0, \quad L = c_2 + c_7p, \quad M = c_{10} + c_8p.$$

We have $h = h_x = h_y = 0$ at two distinct x -values. Therefore $q(x)^2 - 4c_1(Lx + M)$ has two distinct double roots and is the square of a quadratic $s(x)$ with leading coefficient c_0 . But

$$(q - s)(q + s) = 4c_1(Lx + M)$$

has right side of degree at most one and $\deg(q + s) = 2$. Hence $q = s$ and $L = M = 0$. Using both $h = 0$ and $h_y = 0$ now gives $y = 0$, a contradiction to two y -values.

There remains a common value y_0 . If $y_0 \neq 0$, $F_y = 0$ and the quadratic in (9.26) put the points in two pairs with distinct x -values. The equations for $1, x$ give opposite weights in each

pair. The same calculation with z, z^{-1}, w gives common p . Subtracting the two equations $F_x = 0$ yields $2c_0y_0(x_a - x_b) = 0$, again impossible. Thus all four points have $y = 0$.

Now $q(x) = 0$ and $p = -c_2/c_7$ is fixed and nonzero. There must be two distinct x -values, each with two distinct z -values. Equation (9.27) and the quadratic in (9.26) again force $c_3 + c_6/p = c_5 + c_4p = 0$; the equation $F = 0$ gives $c_{10} + c_8p = 0$. With $r = -1/p$ this is precisely

$$c_7 = rc_2, \quad c_8 = rc_{10}, \quad c_4 = rc_5, \quad c_3 = rc_6,$$

so $c \in \mathcal{D}$.

Conversely, at $c \in \mathcal{D} \cap S_4$, the nonvanishing of c_7, c_4, c_5 forces $c_2 \neq 0$. Put

$$(9.28) \quad A = c_2, \quad B = c_{10}, \quad C = c_5, \quad D_0 = c_6, \quad r = c_4/c_5.$$

Then

$$(9.29) \quad F = c_1y^2 + q(x)y + GH, \quad G = 1 + rzw, \quad H = Ax + B + C/z + D_0/w.$$

The points $y = q = G = H = 0$ are singular. Above each root of q , they are given by

$$(9.30) \quad -rD_0z^2 + (Ax + B)z + C = 0, \quad w = -1/(rz).$$

The leading and constant coefficients are nonzero. A repeated root of q , or of (9.30) above it, gives an actual singular point with degenerate Hessian: at that point the differentials of y, q, G, H are dependent. Such a point is excluded by S_4 . Thus there are four distinct points, and

$$(9.31) \quad \det \frac{\partial(q, G, H)}{\partial(x, z, w)} = q'(x)r(C/z - D_0/w) \neq 0.$$

They are nodes by (9.29), and exhaust the singularities. This includes a simple root $x = 0$ when $c_9 = 0$. At the nodes the evaluation functions span $\langle 1, x, z, w \rangle$, with relation $rD_0z + rCw = Ax + B$. The functions $1, x, z$ are independent because x takes two values and z takes two values above each. Their rank is exactly three. $\square$

A compact surface and its small resolution.

Lemma 9.24. *For every $c \in \mathcal{D} \cap S_4$, the closure S_r of $\{y = 0, 1 + rzw = 0\}$ is a smooth compact toric surface in Y_f . Blowing up its ideal gives a projective small resolution of Y_f . The surface has five toric boundary curves and $K_{S_r}^2 = 7$.*

Proof. Set $R = r_{11}$ and $m = (4, 2, 1, 1)$. The character $\chi^m = zw$ on the toric divisor E_R extends to a morphism $E_R \rightarrow \mathbb{P}^1$. Indeed its pairings with the rays of each of the 18 maximal cones of the star of R have one weak sign. The quotient lattice has coordinates

$$M/\mathbb{Z}R \longrightarrow \mathbb{Z}^3, \quad v \longmapsto (v_0, v_1 + v_2, v_1 + v_3).$$

The induced character is $(4, 1, 1)$, so its saturated kernel has coordinates $(a, b) \mapsto (a, b, -4a - b)$. In this basis the fibre fan, in positive cyclic order, is

ray index	5	8	158	6	131
fibre ray	$(-1, 1)$	$(1, -2)$	$(0, 1)$	$(-1, 3)$	$(-1, 2)$.

Its maximal cones are consecutive pairs, each of determinant one. The self-intersections are $(0, 1, -1, -1, -2)$, giving $K^2 = 12 - 5 = 7$. The fibre over $-1/r \in \mathbb{C}^*$ is this smooth projective toric surface; primitivity of the character makes its kernel torus connected. Fibres over nonzero finite values are its torus translates.

The factorization (9.29) shows that this fibre lies in Y_f . It is S_r , and contains all four nodes. At a node, use (9.31) and put $U = y$, $V = -(q + c_1y)$. The local equation and centre become

$$UV = GH, \quad S_r = \{U = G = 0\}.$$

The blowup is the incidence variety

$$U\beta = G\alpha, \quad H\beta = V\alpha, \quad [\alpha : \beta] \in \mathbb{P}^1.$$

On $\beta = 1$ its free coordinates are (G, V, α) , and on $\alpha = 1$ they are (U, H, β) . Thus it is smooth and replaces the node by a $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Away from the nodes, S_r is a Weil divisor in a smooth threefold and hence Cartier, so the blowup is an isomorphism. The ideal blowup is projective over projective Y_f ; therefore the small resolution is projective.

Smallness and adjunction give trivial canonical bundle. The anticanonical restriction sequence and toric vanishing give $H^1(Y_f, \mathcal{O}) = H^2(Y_f, \mathcal{O}) = 0$. The local small resolution has vanishing higher direct images of $\mathcal{O}$: the symmetric powers of the dual of its exceptional normal bundle are sums of $\mathcal{O}(k)$, $k \geq 0$, whose H^1 vanishes, and formal functions applies. Leray gives the same cohomology vanishings on the resolution. It is consequently a smooth projective Calabi–Yau threefold. $\square$

Proof of Theorem 9.21. We first exhibit a nonempty algebraic open subset of $\mathcal{D}$ contained in S_4 . In (9.28) take all boundary coefficients equal to one and $c_{11} = 0$. Its equation is

$$F = y^2 + (x^2 + 1)y + (1 + zw)(x + 1 + z^{-1} + w^{-1}).$$

In $\mathbb{Q}[x, y, z, w, u]$, put $P_* = zwF$. The saturated singularity calculation is

$$(9.32) \quad (P_*, (P_*)_x, (P_*)_y, (P_*)_z, (P_*)_w, uzw - 1) = (y, x^2 + 1, z + w - x - 1, zw + 1, uzw - 1).$$

The ideal is radical of degree four, and adjoining $\det \text{Hess } P_*$ gives the unit ideal. These facts also follow by substituting $w = -1/z$, solving $x^2 + 1 = 0$ and $z^2 - (x + 1)z - 1 = 0$, and using (9.31). The boundary calculation in Lemma 9.22 excludes all other compact singularities for these nonzero essential coefficients. Hence this member belongs to S_4 .

On the parameter open where $r, A, C, D_0, c_0, c_1, c_9$ are nonzero, q has simple roots, and

$$\text{Res}_x(q(x), (Ax + B)^2 + 4rCD_0) \neq 0,$$

the four expected nodes form a finite étale degree-four scheme. The nondegenerate Hessian makes it an open part of the relative singular scheme; its finiteness makes it closed. The complementary singular scheme is proper over the parameter space, as a closed subset of the proper universal compact hypersurface. Removing its closed image gives a Zariski-open subset $\mathcal{D}^{\text{good}} \subset \mathcal{D}$ whose members have precisely these four nodes. A finite singular locus also implies reducedness. Equation (9.32) proves that this open subset is nonempty. It is connected and Euclidean dense in the irreducible variety $\mathcal{D}$.

By Lemmas 9.16, 9.17 and 9.23, $\mathcal{D}^{\text{good}}$ lies in a connected component C of S_4 of dimension nine, and is open in that component. Every minor defining $\mathcal{D}$ vanishes on this open subset, hence on C by the identity theorem. Conversely, every point of $\mathcal{D} \cap S_4$ is a limit of points of $\mathcal{D}^{\text{good}}$ inside S_4 ; connected components are closed in S_4 . Thus $C = \mathcal{D} \cap S_4$. Lemma 9.23 proves that all other components have codimension four.

To meet the actual near-limit construction, choose $\mathcal{C}_0$ to contain the normalized coefficient vector $c^{(0)}$ with origin coefficient +1 and all boundary coefficients −1. This vector lies in $\mathcal{Z}$, since every parallelogram relation has two terms of each sign. Use the height $H = k_0\varphi + h_C$ and coefficients

$$\gamma_i = -\tau^{H(n_i)} \quad (0 \leq i \leq 10), \quad \gamma_{11} = 1, \quad 0 < \tau \ll 1.$$

These correspond to $c^{(0)}$. Linearity of H on the four node parallelograms gives

$$H(n_7) - H(n_2) = H(n_8) - H(n_{10}) = H(n_4) - H(n_5) = H(n_3) - H(n_6).$$

Thus $\gamma \in \mathcal{D}$, with $r > 0$ and $A, B, C, D_0 < 0$. The quadratic $q(x) = \gamma_0 x^2 + x + \gamma_9$ has two distinct positive roots for small τ , since all boundary heights are at least one. Above each, the discriminant $(Ax + B)^2 + 4rCD_0$ is positive and (9.30) has two distinct nonzero roots. These give four nodes. Lemma 9.1(1) puts $c^{(0)}$ in the actual neighbourhood $U \subset U_{\mathbb{C}}^+$. Lemma 7.11(2) says all compact singularities are among the four unique critical points q_P . The four exhibited nodes therefore exhaust them, so all critical values vanish and the member lies in the actual D_Q of Section 9.3. This proves that C is reached.

Lemma 9.24 gives the projective resolutions. It also provides the integral relation directly. Remove small node balls from the compact oriented surface S_r . Each boundary sphere is a primitive generator of the third homology of the node link: it is the circle bundle $\mathcal{O}(-1, -1)$ restricted to a ruling of the exceptional quadric. The punctured surface and Lemma 9.15 therefore give the relation $v = (1, 1, 1, 1)$ after choosing these sphere orientations. The rational kernel has rank one, so for the sphere map d ,

$$\mathbb{Z}v \subseteq \ker d \subseteq d^{-1}(\text{Tors } H_3) \subseteq \mathbb{Q}v \cap \mathbb{Z}^4 = \mathbb{Z}v.$$

All groups in this chain are equal. Proposition 9.13 identifies the signs with $(1, -1, -1, 1)$ using the marking transported from the near-limit construction. On every other component the rational sphere map has rank four, so both its integral kernel and its preimage of torsion are zero. $\square$

9.10. A reverse mirror transition for X_9 . For X_9 the projective resolutions of Theorem 9.21 admit an explicit comparison with the Batyrev mirror family of a smoothing. We first identify the smoothing of the particular nodal partial resolution X'_9 used in this paper.

Theorem 9.25. *The threefold X'_9 of Section 11.1 has a proper flat analytic smoothing whose noncentral fibres are smooth projective anticanonical hypersurfaces from the toric smoothing family of [Wue26b]. For general $c \in \mathcal{D} \cap S_4$, the projective small resolution of Y_f in Theorem 9.21 is birational to a smooth projective member of the Batyrev mirror family of these smoothing fibres.*

Thus the reverse transition is established here in the sense of Batyrev families, allowing birational Calabi–Yau models for the resolved mirror. The theorem concerns a proper analytic smoothing with projective fibres; it does not require a relatively projective total space.

Proof of Theorem 9.25. A simultaneous modification of the degree-seven smoothing.

The degree-seven cone has the height-one polygon

$$\Pi = \text{conv}\{(0, 0), (1, 0), (2, 1), (1, 2), (0, 1)\} = [0, (1, 1)] + \text{conv}\{0, (1, 0), (0, 1)\}.$$

In coordinates (p, q, a, b) on $\mathbb{Z}^4$, its Cayley cone has rays

$$(9.33) \quad \begin{aligned} A_0 &= (0, 0, 1, 0), & A_1 &= (1, 1, 1, 0), \\ B_0 &= (0, 0, 0, 1), & B_1 &= (1, 0, 0, 1), & B_2 &= (0, 1, 0, 1), \end{aligned}$$

with relation $-A_0 + A_1 + 2B_0 - B_1 - B_2 = 0$. Let $\mathcal{C}$ be the affine toric fourfold of this cone. Altmann’s component construction [Alt97, §8.1] gives the reduced smoothing family as the map $\mathcal{C} \rightarrow \mathbb{C}_s$ defined by the difference of its two height characters. The full local base is $\mathbb{C}\{s, w\}/(w^2, sw)$ [Alt97, §9.1(v)]; in particular every map from a reduced disc factors through this reduced line.

Subdivide the cone into

$$(9.34) \quad \langle A_1, B_0, B_1, B_2 \rangle, \quad \langle A_0, A_1, B_0, B_2 \rangle, \quad \langle A_0, A_1, B_0, B_1 \rangle.$$

The determinants are 1, −1, 1. Heights (1, 0, 0, 1, 1) on the five rays give this regular subdivision: on each cone the omitted ray lies three units above its linear height. The resulting toric morphism is

$$\rho : \mathcal{V} = \mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)) \longrightarrow \mathcal{C}.$$

Indeed the relation in (9.33) gives quotient weights (−1, 1, 2, −1, −1) on $(u_0, u_1, v_0, v_1, v_2)$, with stable set $(u_0, v_1, v_2) \neq 0$. Write $[X_0 : X_1 : X_2] = [u_0 : v_1 : v_2]$ and fibre coordinates $U = u_1$, $V = v_0$. The equivalence is

$$(X, U, V) \sim (\lambda X, \lambda^{-1}U, \lambda^{-2}V).$$

The invariant functions are generated by $X_i U$ and $X_i X_j V$. The tautological bundle inclusions give a closed immersion $\mathcal{V} \hookrightarrow \mathbb{P}^2 \times \mathbb{C}^9$, so the map to its affine invariant image is projective. Its fibre over the vertex is exactly the zero section. Off that vertex it is an isomorphism: if $X_i U \neq 0$, the ratios $(X_j U)/(X_i U)$ recover the base; if all these vanish, some $X_i^2 V$ is nonzero and the ratios $(X_i X_j V)/(X_i^2 V)$ recover it. The invariant relations ensure that these opens cover the complement of the vertex, and the same formulas recover the fibre coordinates regularly.

The height characters are $u_0 u_1$ and $v_0 v_1 v_2$. Hence the parameter on $\mathcal{V}$ is

$$(9.35) \quad s = X_0 U - X_1 X_2 V.$$

It is flat over $\mathbb{C}_s$: on affine charts, the nonconstant function s makes the coordinate ring torsion-free over the principal ideal domain $\mathbb{C}[s]$. On $X_1 \neq 0$, with $a = X_0/X_1$, $b = X_2/X_1$, $u = X_1 U$, $v = X_1^2 V$, the equation is $s = au - bv$. Its unique critical point is the chart origin. The $X_2 \neq 0$ chart gives the other critical point, and on $X_0 \neq 0$ the equation is $s = u - pqv$, with no critical point. Thus the central fibre has exactly two nodes, at $[0 : 1 : 0]$ and $[0 : 0 : 1]$ on the zero section, and all noncentral fibres are smooth. The contraction is an isomorphism on every noncentral fibre.

The central fibre is the precise partial resolution used in X'_9 . To see this, intersect the cones in (9.34) with $a = b = 1$. The three cells are

$$\begin{aligned} & \{(1, 1), (2, 1), (1, 2)\}, \\ & \{(0, 0), (1, 1), (1, 2), (0, 1)\}, \quad \{(0, 0), (1, 0), (2, 1), (1, 1)\}. \end{aligned}$$

The affine lattice map $(p, q) \mapsto (1 - p, 1 - q)$ takes them to T_3, P_4, P_1 of Lemma 3.1, and takes the new central ray to the pentagon centre. The central charts are one affine three-space and two normal node charts. Thus the binomial central fibre is already normal and is the toric variety with this intersection fan. Its exceptional divisor is $\mathbb{P}^2$, with precisely the two nodes prescribed by our pentagon subdivision. The total space has trivial canonical bundle, since $\det(\mathcal{O}(-1) \oplus \mathcal{O}(-2)) = K_{\mathbb{P}^2}$, and the central map is crepant.

The original-side smoothing fibres.

The ambient family in [Wue26b, Theorem 11.8] gives a smoothing of the chosen mixed threefold X_9 , which has two nodes and one degree-seven cone point. To keep this precise central hypersurface, let g_0 be its anticanonical section. By [Wue26b, Proposition 11.7], the relative section bundle is locally free of rank 162 and commutes with base change. Trivialize it on a disc with coordinate t , and extend g_0 by $g(t) = g_0 + th$. For a fixed $t_1 \neq 0$, varying h gives every section on the fibre at t_1 . Smooth sections form a dense open subset, so choose h giving a smooth hypersurface there and not proportional to g_0 . Then $g(t)$ is nonzero

throughout the original disc. The fibres of the ambient family are integral, so the fibrewise regular-section criterion makes $\{g(t) = 0\}$ flat over the disc. It is proper. The image of its nonsmooth locus is a proper analytic subset of the disc, since the fibre at t_1 is smooth. After a further shrinking, all noncentral fibres are therefore smooth, while the central fibre remains the chosen X_9 .

Choose the central analytic identification of the cone point compatibly with the partial resolution $X'_9 \rightarrow X_9$. In the toric hypersurface charts, this is the identification obtained by eliminating the transverse torus coordinate. Analytic semiuniversality identifies the actual local family with the pullback of the degree-seven versal family along a classifying arc. The disc is reduced, so its w -coordinate vanishes; the arc therefore factors through the reduced family just described. Its parameter $s(t)$ is nonzero for $t \neq 0$ after shrinking, because the arc is a smoothing. No assumption on $s'(0)$ is needed.

Pull back ρ along this arc and restrict it over a representative of the identified neighbourhood in the total family. Flatness survives base change. Its only exceptional locus maps to the central cone point, so it glues to the identity on the complement of that point. The glued space is Hausdorff: points with distinct target images are separated downstairs, while points with the same target image are separated in one inverse image of the target open cover. Properness is local on the target, and flatness is local on the source. We obtain a proper flat analytic family with central fibre X'_9 and noncentral fibres isomorphic to those of the chosen anticanonical family. The two original nodes are unchanged in the central fibre and are smoothed by the same family.

We recall the toric ambient of those smooth fibres, to fix the mirror comparison. Its fan polytope ∇ has vertices

$$(9.36) \quad \begin{aligned} v_0 &= (-1, -3, -1, 0), & v_1 &= (-1, -3, 0, -1), \\ v_2 &= (-1, -1, 0, 0), & v_3 &= (-1, 0, 0, 0), \\ v_4 &= (0, 1, 0, 0), & v_5 &= (1, -1, 0, 0), & v_6 &= (1, 3, 1, 1). \end{aligned}$$

The maximal cones, as sets of indices, are

$$(9.37) \quad 1456, 1346, 0456, 0346, 0145, 0134, 01256, 1236, 0236, 0123.$$

This is the Gorenstein Fano fourfold computed in [Wue26b]. Its only singular point is the fixed point of 01256, which a general anticanonical member avoids. The relative anticanonical sections form a rank-162 vector bundle, commute with base change and generate the relative line bundle. Consequently the noncentral hypersurfaces occur in the complete anticanonical family, whose Newton polytope is ∇° . Their Batyrev partner has Newton polytope ∇ .

The birational mirror family.

On the dense torus, use the parameters of (9.28) and the unimodular coordinates

$$a = x, \quad b = z/x^2, \quad c = w/x^2, \quad t = y/x.$$

Put

$$(9.38) \quad \begin{aligned} g &= 1 + ra^4bc, \\ h &= A/a + B/a^2 + C/(a^4b) + D_0/(a^4c), \\ k &= c_0a + c_{11} + c_9/a. \end{aligned}$$

Equation (9.29), divided by x^2t , is $c_1t + k + gh/t = 0$. The rational substitution $t = g\zeta$ gives

$$(9.39) \quad c_1g\zeta + k + h/\zeta = 0.$$

It has inverse $\zeta = t/g$ on $g \neq 0$, and hence identifies dense opens of the two hypersurfaces. The nine possible exponents of (9.39), in (a, b, c, ζ) , are

$$(9.40) \quad \begin{aligned} & (0, 0, 0, 1), (4, 1, 1, 1), (1, 0, 0, 0), (-1, 0, 0, 0), (0, 0, 0, 0), \\ & (-1, 0, 0, -1), (-2, 0, 0, -1), (-4, -1, 0, -1), (-4, 0, -1, -1). \end{aligned}$$

Their convex hull has seven vertices and exactly these nine lattice points. The lattice isomorphism

$$(u_1, u_2, u_3, u_4) \mapsto (u_4, u_1 - u_4, u_2, u_3)$$

has determinant -1 and sends its vertices to (9.36). The ten facets have exactly the index sets in (9.37). This identifies its Newton polytope with ∇ , including the lattice, not only the face combinatorics.

The coefficient map is equally explicit. The nine parameters on the open part of $\mathcal{D}$ map to the nine coefficients of (9.39) as

$$(9.41) \quad (c_1, r, c_0, c_{11}, c_9, A, B, C, D_0) \mapsto (c_1, c_1 r, c_0, c_{11}, c_9, A, B, C, D_0),$$

up to the displayed ordering of monomials. This is an isomorphism of coefficient opens, with inverse $r = \text{second}/\text{first}$. In particular the constant coefficient c_{11} is unrestricted. The compact four-node good locus and the inverse image of the Newton-nondegenerate locus are nonempty Zariski-open subsets of the irreducible $\mathcal{D}$, so intersect. For a general member of the latter, Batyrev's maximal projective crepant partial desingularization is a smooth projective threefold by [Bat94, Thm. 4.2.2, Cor. 4.2.3]. Its function field agrees with that of the projective small resolution of Y_f from Lemma 9.24. This proves Theorem 9.25. $\square$

The torus substitution also preserves the logarithmic volume form, since $d \log t = d \log \zeta + d \log g$ and the latter summand involves only a, b, c . This identity on the common open does not assert a comparison of compact quantum invariants. At special points of $\mathcal{D} \cap S_4$, the Laurent polynomial (9.39) may fail to be Newton nondegenerate; the theorem's mirror identification is stated for general members.

Part III. The resolution side

10. GEOMETRIC REALISATION ON THE SMALL RESOLUTION

Theorem E supplements the comparison of relation lattices with a geometric construction on $\widehat{X}$. Proposition 3.5 already determines the integral relations among the exceptional curves. Here a strict tropical 2-cycle specifies a three-dimensional domain and a continuous map into $\widehat{X}$; the degrees of its boundary spheres give the relation. Its complete proof is in Appendix F.

We retain the hypotheses of Theorem E: Δ is admissible and satisfies (U), the profile satisfies (Proj), and $\widehat{X} \rightarrow X'_\sigma$ is a projective small resolution induced by a regular refinement of Σ' . The diagonals of this resolution determine the signs of the node coefficients $\rho_q(S)$. The exceptional curves C_q have their complex orientations. The following notation describes the construction.

object	meaning
$\widehat{B}, \widehat{S}$	the resolved affine base and the cycle moved to it; a node becomes an edge e_q between two negative vertices
$\mathscr{W}$	a smooth localized hypersurface, isotopic to the algebraic small resolution
$\mathscr{W}_U$	a region of $\mathscr{W}$ associated to a small adapted open set U of the base
$\mathfrak{D}(S)$	the circle bundle over the smooth sheets, completed at the edges, vertices and discriminant
F, f_S	the maps of this domain to $\mathscr{W}$ and, after transport, to $\widehat{X}$
$\partial_q \mathfrak{D}$	the boundary sphere over e_q , mapped to C_q with degree $-\rho_q(S)$

10.1. The geometric statement. The domain in Theorem E is a three-dimensional CW complex, with an integral fundamental chain $[\mathfrak{D}]$ given by its oriented top cells. Over the smooth sheets away from a neighbourhood of the discriminant, its fibre is the circle $\mathbb{R}v/\mathbb{Z}v$, oriented by the primitive field v . At an interior edge the three circle fibres are joined by the triangle prescribed by balancing; at an interior vertex the four triangles are filled by a tetrahedron. The remaining local domains complete it near the discriminant (Definition F.13). For each node met by $j(\partial S)$, its distinguished boundary sphere is

$$\partial_q \mathfrak{D} = (e_q \times S^1)/\sim \cong S^2,$$

where the circle is collapsed at both endpoints of e_q . The continuous map $f_S: \mathfrak{D}(S) \rightarrow \widehat{X}$ satisfies

$$\partial[\mathfrak{D}] = \sum_q [\partial_q \mathfrak{D}], \quad f_S(\partial_q \mathfrak{D}) \subset C_q, \quad \deg(f_S|_{\partial_q \mathfrak{D}}) = -\rho_q(S).$$

These properties are the geometric content of Theorem E. For chosen fundamental singular cycles $[C_q]$, the degree identity gives integral singular 3-chains A_q supported in C_q with

$$\partial A_q = (f_S)_*[\partial_q \mathfrak{D}] + \rho_q(S)[C_q].$$

Consequently

$$\Xi(S) = -(f_S)_*[\mathfrak{D}] + \sum_q A_q, \quad \partial \Xi(S) = \sum_q \rho_q(S)[C_q].$$

The corrections turn equality of homology classes on the boundary curves into equality of singular chains. The map and the chain depend on the auxiliary choices; independence of the resulting relative class is not asserted (Remark F.25). For another small resolution, Lemma F.24 gives a bounding chain with the coefficient negated at each flopped curve, and the same relative class on the common exterior of the node neighbourhoods. This comparison concerns chains and their relative classes.

10.2. Mechanisms of the construction. Castaño-Bernard and Matessi construct a completed circle bundle in their topological resolution models [CBM14, §7.5, §7.7]. We construct a map of this domain into a localized hypersurface $\mathscr{W}$ and then into $\widehat{X}$. The localized equation is obtained by cutting off monomials far from the maximum, following [Mik04, Yam16]. It has exact local equations near the relevant toric strata, a positive real section, and aspherical regions $\mathscr{W}_U$ away from the discriminant. Their fundamental groups identify with the integral tangent lattices of $\widehat{B}$, compatibly with parallel transport (Lemma F.12).

Four local arguments are needed to realize the circle domain in this model.

- (1) *Collar paths through real chambers.* The positive real section avoids every toric stratum, whereas the circle of v collapses only on a stratum. A collar path joins the section to the critical locus of the local model, turning the phase of one character by

- π (Definition F.16). Sweeping it by the circle of v gives the lift near a boundary leg. Lemma F.15 identifies the real chambers through which these paths run.
- (2) *Phase corrections at positive vertices.* To join neighbouring collar paths by a disc, their boundary loop must have zero total winding in certain characters (Lemma F.18). This holds directly at nodes and negative vertices. At a positive vertex it is achieved by rotating the paths along the incident legs by an integral multiple of π in a circle action that preserves $\mathscr{W}$ (Lemma F.19(3)).
 - (3) *Discs at changes of side.* When a sheet changes side of a two-face, an explicit switching disc joins the two families of collar paths (Lemma F.19(4)). It supplies the compatibility needed to glue the local maps.
 - (4) *Signed node degrees.* The node map sweeps an arc between the two poles of the exceptional curve by the circle of the boundary field. The circle action and the orientation of the arc give degree $-\rho_q(S)$ (Lemma F.17). Both the global minus sign in $\Xi(S)$ and the change of coefficient under a flop follow from this calculation and the comparison of the two resolutions of the node.

The local maps extend over the regular part cell by cell. A circle fibre represents v in $\pi_1(\mathscr{W}_U)$; balancing makes the boundaries of the triangle pieces null-homotopic, and asphericity permits the higher extensions (Lemma F.21). This is a local homotopy description of the map, without requiring a global torus fibration of $\widehat{X}$. Finally an orbit-preserving ambient isotopy carries $\mathscr{W}$ to $\widehat{X}$ and its exceptional curves to the C_q with degree one (Lemmas F.22 and F.23). Its composition with F is f_S .

The construction uses the whole strict cycle and its balancing data. It does not use a previously known relation among the $[C_q]$ to choose a bounding chain. On the mirror, Theorem D instead realizes the same coefficients by a 4-chain whose regular part is, locally up to homotopy, the bundle of two-tori annihilated by v . The two constructions therefore give geometric representatives of the corresponding relations in different dimensions.

10.3. The example X_9 . The example of Section 11.1 has four nodes and

$$K = \mathbb{Z}(1, -1, -1, 1).$$

For the triangulation and base in the accompanying data, Lemma 11.2 proves that the belt disc S of Figure 4, with constant primitive field $v = -d_0$, is a strict tropical 2-cycle. Its boundary runs along 18 legs through the four nodes and five positive vertices; its interior misses the discriminant. The node coefficients are $(1, -1, -1, 1)$.

Here the regular part of $\mathfrak{D}(S)$ is a circle bundle over the disc. The local construction collapses the circle along the boundary legs, makes the phase corrections at the positive vertices, and leaves four boundary spheres, of degrees $(-1, 1, 1, -1)$ on $(C_{q_1}, C_{q_2}, C_{q_3}, C_{q_4})$. Thus

$$\partial\Xi(S) = [C_{q_1}] - [C_{q_2}] - [C_{q_3}] + [C_{q_4}].$$

The mirror lift of the same disc has

$$\partial\check{\Xi}(S) = [\delta_{q_1}] - [\delta_{q_2}] - [\delta_{q_3}] + [\delta_{q_4}].$$

The first identity is realized on the original small resolution $\widehat{X}_9$, and the second on the mirror hypersurface Y_t . The polytope, belt, and verification of strictness are given in Section 11.1; the full resolution construction applies to this disc without additional hypotheses.

Part IV. Examples

11. EXAMPLES AND FINITE COMPUTATIONS

This section makes the objects of the paper explicit on five hypersurfaces and reports what the computations on the list of admissible polytopes show. Section 11.1 treats X_9 by hand: it has four nodes and $\text{rank } K = 1$, and a single disc through the four nodes realises the generator of K as a tropical 2-cycle, so that Theorems A–E and Corollary B can be followed through one picture. Section 11.2 lists explicit generators of K on X_{19} and X_{20} ; on X_{20} two of the coordinate functionals $\lambda \mapsto \lambda_q$ vanish identically on K , that is, two nodes are *forced* in the sense of Remark 9.9, and Corollary B then says that no relation among the vanishing spheres has all coefficients nonzero. Section 11.3 gives one example on each branch of the dP_6 hexagon. Section 11.4 describes the list, counts the polytopes to which the theorems apply and those on which the equivalent conditions of Corollary B hold, and reports a computational check of Theorem C, not used in its proof, on the bases of 435 polytopes of the list, 364 without and 71 with hexagonal two-faces. Section 11.5 collects the evidence for Conjecture G.

Throughout, the lattice points of Δ that are rays of Σ' are numbered $n_0, n_1, \dots$ as in Appendix G, a node parallelogram is written $q = (a, b, c, d)$ with $n_a + n_c = n_b + n_d$ and chosen diagonal $\{a, c\}$ (through the centre at a pentagon or hexagon), and e_i stands for the basis vector e_{n_i} of $\mathbb{Z}^R$, so that $\text{circ}_q = e_b + e_d - e_a - e_c$. Vectors of $\mathbb{Z}^Q$ are written as combinations of the nodes: $q_1 - q_2$ is the vector with coordinate 1 at q_1 , -1 at q_2 and 0 elsewhere.

11.1. The example X_9 . The polytope Δ_9 of [Wue26b] has nine vertices, listed in Appendix G.1, one dP_7 pentagon with vertices n_3, n_4, n_5, n_6, n_8 and centre $n_{10} = (-2, -1, 0, 0)$, and two unit squares $\{n_2, n_6, n_3, n_7\}$ and $\{n_2, n_5, n_4, n_7\}$; its other two-faces are unimodular triangles. It is admissible, it satisfies (Proj) with the fan Σ' of Appendix G.1, and by Theorem F it satisfies (U), so it has a base $B = B'_\sigma$. So Theorems A–E apply to it.

The nodes and K . The pentagon contributes two node parallelograms and each square one:

$$q_1 = (n_3, n_6, n_{10}, n_8), \quad q_2 = (n_2, n_6, n_3, n_7), \quad q_3 = (n_4, n_5, n_{10}, n_8), \quad q_4 = (n_2, n_5, n_4, n_7),$$

with q_1, q_3 in the pentagon. Their circuits are

$$\begin{aligned} \text{circ}_{q_1} &= e_6 + e_8 - e_3 - e_{10}, & \text{circ}_{q_2} &= e_6 + e_7 - e_2 - e_3, \\ \text{circ}_{q_3} &= e_5 + e_8 - e_4 - e_{10}, & \text{circ}_{q_4} &= e_5 + e_7 - e_2 - e_4. \end{aligned}$$

If $\sum_k \lambda_k \text{circ}_{q_k} = 0$, the coefficients of e_{10} , e_6 and e_7 give $\lambda_3 = -\lambda_1$, $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_2$; conversely $\text{circ}_{q_1} - \text{circ}_{q_2} - \text{circ}_{q_3} + \text{circ}_{q_4} = 0$. Hence

$$K = \mathbb{Z}(q_1 - q_2 - q_3 + q_4).$$

Let $\widehat{X}_9$ be the small resolution of X'_9 with the chosen diagonals $\{a, c\}$ of the parallelograms $q = (a, b, c, d)$ above. No nonzero multiple of $q_1 - q_2 - q_3 + q_4$ has all coefficients nonnegative, so by Proposition 3.4(3) $\widehat{X}_9$ is projective and induced by a regular refinement of Σ' , and by Proposition 3.4(2) the relative Picard number of $\widehat{X}_9 \rightarrow X'_9$ is $|Q| - \text{rank } K = 3$. The relation $\lambda_{q_1} + \lambda_{q_3} = 0$ holds because the centre n_{10} occurs only in the circuits of the two pentagon nodes; this is the tie of Remark 3.10.

A cycle of node parallelograms. Put $d_0 = (4, 2, 1, 1)$. Every node parallelogram has a pair of opposite sides parallel to d_0 :

$$n_3 - n_6 = n_8 - n_{10} = n_4 - n_5 = n_7 - n_2 = d_0.$$

Write $\varepsilon_1 = [n_6, n_3]$, $\varepsilon_2 = [n_{10}, n_8]$, $\varepsilon_3 = [n_5, n_4]$ and $\varepsilon_4 = [n_2, n_7]$. Then q_1 has the sides $\varepsilon_1, \varepsilon_2$, q_3 the sides $\varepsilon_2, \varepsilon_3$, q_4 the sides $\varepsilon_3, \varepsilon_4$ and q_2 the sides $\varepsilon_4, \varepsilon_1$. So the four segments close up into a cycle $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in which consecutive segments are opposite sides of one node parallelogram. The boundary of the disc constructed below runs through the four nodes in this order.

Belt cycles. The disc belongs to the simplest class of tropical 2-cycles.

Definition 11.1 (belt cycles). A *belt cycle* on B is a strict tropical 2-cycle (S, j, v) (Definition 2.5) such that S is a disc with one sheet and no interior edges or vertices, S does not cross Γ , and ∂S is a closed path in Γ along legs with vanishing vector $\pm v$ which passes every node it meets along two opposite legs and meets no negative vertex.

The field of a belt cycle is constant, since S is a disc on which it is parallel; as a relative tropical 2-cycle (Section 5.2) a belt cycle is the chain $v[S]$, with boundary network $v[\partial S]$, and its node coefficients are supported on the nodes of ∂S .

The facet. Let $\mathcal{T}$ be the unimodular triangulation of $\partial\Delta_\circ^\circ$ of the data accompanying the paper (<https://github.com/bwuebben/calabi-yau-smoothability>, directory `paper6`; see Appendix G), and let $u_0 = (0, 1, -1, -1)$ and

$$\begin{aligned} u_1 &= (1, -1, -1, -1), & u_2 &= (0, 0, 1, -1), & u_3 &= (-1, 2, 1, -1), \\ u_4 &= (-1, 2, 0, 0), & u_5 &= (-1, 2, -1, 1). \end{aligned}$$

In $\mathcal{T}$ the vertex u_0 is joined by edges to $u_1, \dots, u_5$, and $[u_0, u_i, u_{i+1}]$ is a triangle of $\mathcal{T}$ for $i = 1, \dots, 5$ (indices modulo 5). Recall from Section 3.3 that the facets F_u of ∇^h correspond to the lattice points u of $\partial\Delta^\circ$ and its two-faces $F_{uu'} = F_u \cap F_{u'}$ to the edges of $\mathcal{T}$, and that Γ lies in the union of the two-faces and meets no facet interior. All six vectors u_i annihilate d_0 . So the five two-faces $F_{u_0 u_i}$ of the facet F_{u_0} , and the five edges $F_{u_0} \cap F_{u_i} \cap F_{u_{i+1}}$, contain the direction d_0 . For an edge $[u, u']$ of $\mathcal{T}$ and an edge $[n, n']$ of Σ' we write $([u, u'], [n, n'])$ for the arc of Γ in $F_{uu'}$ whose legs have lower support $\{n, n'\}$ (Proposition 4.10(2)); its legs have vanishing vector $\pm(n' - n)$ and covector $\pm(u' - u)$ (Lemma 3.12).

The closed path. Let $\partial S \subset \Gamma$ be the following union of arcs:

- from p_{q_1} along $([u_0, u_1], \varepsilon_1)$ and $([u_0, u_5], \varepsilon_1)$ to p_{q_2} ;
- from p_{q_2} along $([u_0, u_5], \varepsilon_4)$, $([u_0, u_4], \varepsilon_4)$ and $([u_0, u_3], \varepsilon_4)$ to p_{q_4} ;
- from p_{q_4} along $([u_0, u_3], \varepsilon_3)$, $([u_0, u_2], \varepsilon_3)$ and $([u_0, u_1], \varepsilon_3)$ to p_{q_3} ;
- from p_{q_3} along $([u_0, u_1], \varepsilon_2)$ back to p_{q_1} .

Consecutive arcs with the same lower support meet at the positive vertices $([u_0, u_i, u_{i+1}], \varepsilon_j)$, where ε_j is their common lower support, one on each of the five edges $F_{u_0} \cap F_{u_i} \cap F_{u_{i+1}}$, and the nodes are $p_{q_1} = ([u_0, u_1], q_1)$, $p_{q_3} = ([u_0, u_1], q_3)$, $p_{q_4} = ([u_0, u_3], q_4)$ and $p_{q_2} = ([u_0, u_5], q_2)$. In all, ∂S consists of 18 legs; it passes through the four nodes and five positive vertices and meets no negative vertex. At each node it follows two opposite legs, those over $\varepsilon_1, \varepsilon_2$ at q_1 , over $\varepsilon_2, \varepsilon_3$ at q_3 , over $\varepsilon_3, \varepsilon_4$ at q_4 and over $\varepsilon_4, \varepsilon_1$ at q_2 . Figure 4 shows the configuration.

The disc. Let S be the disc bounded by ∂S formed by the 26 triangles (b_e, b_f, b_C) of the barycentric subdivision $\mathfrak{R}_0$ of $\mathcal{P}$ listed in the accompanying data (<https://github.com/bwuebben/calabi-yau-smoothability>, directory `paper6`). Its triangles lie in three maximal cells of $\mathcal{P}$, all with facet index u_0 , and S meets the two-faces of ∇^h only along

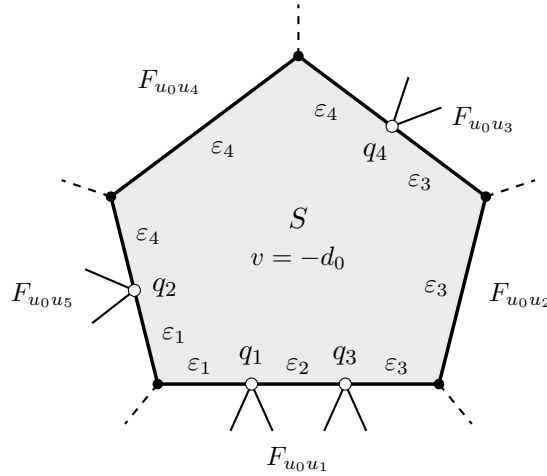


FIGURE 4. The belt cycle of X_9 , schematically: the boundary of the facet F_{u_0} seen along d_0 . The five two-faces $F_{u_0 u_i}$ met by ∂S appear as sides and the five positive vertices of ∂S as corners ($\bullet$); the four nodes are the circles. Thick lines are the 18 legs of ∂S , labelled by their lower supports $\varepsilon_1, \dots, \varepsilon_4$; the two other legs at each node and the third leg at each positive vertex (dashed) leave F_{u_0} and are drawn as stubs. The disc S (shaded) spans ∂S inside F_{u_0} , with the constant field $-d_0$.

∂S . Orient S so that ∂S passes through the nodes in the order q_1, q_2, q_4, q_3 , and give it the constant field $v = -d_0$.

Lemma 11.2. *(S, j, v) is a belt cycle, and $\rho(S) = q_1 - q_2 - q_3 + q_4$. In particular $\rho(S)$ generates K .*

Proof. We check the clauses of [CBM14, Def. 7.2] (Section 5.2). The embedding, discriminant-intersection and primitive-field clauses hold: S is an embedded disc, a tropical domain with one sheet and no interior edges or vertices. Its interior lies in the interior of F_{u_0} , which does not meet Γ , so $j^{-1}(\Gamma) = \partial S$ and S does not cross Γ . On the interior of F_{u_0} the chart is the inclusion into $\text{aff}(F_{u_0})$ with tangent lattice $u_0^\perp \cap N$, which contains the primitive vector d_0 ; so v is a primitive integral parallel field. The boundary-vertex and node-passage clauses hold: ∂S has no vertices, meets no negative vertex, and follows two opposite legs at each node. The boundary-monodromy clause: every leg of ∂S lies in a two-face $F_{u_0 u_i}$ and has vanishing vector $\pm d_0$ and covector $\pm(u_0 - u_i)$. Its monodromy $x \mapsto x \pm \langle u_0 - u_i, x \rangle d_0$ fixes v . At the positive vertices all three legs have the vanishing vector $\pm d_0$, and at the nodes the monodromies of all four legs fix d_0 , the common direction of the two sides used; so v is invariant under every monodromy that ∂S meets. The positive-node, interior-edge balancing, crossing and interior-vertex clauses are void.

For the node coefficients, the boundary network of $v[S]$ is $-d_0[\partial S]$. Along the arcs over $\varepsilon_1, \varepsilon_4, \varepsilon_3, \varepsilon_2$, oriented by ∂S , its canonical lifts (Section 2.3) are $e_6 - e_3$, $e_2 - e_7$, $e_5 - e_4$ and $e_{10} - e_8$. By Definition 2.6, ρ_q is the coefficient of circ_q in the lift after p_q minus the lift before it:

$$\begin{aligned} q_2 : (e_2 - e_7) - (e_6 - e_3) &= -\text{circ}_{q_2}, & q_4 : (e_5 - e_4) - (e_2 - e_7) &= \text{circ}_{q_4}, \\ q_3 : (e_{10} - e_8) - (e_5 - e_4) &= -\text{circ}_{q_3}, & q_1 : (e_6 - e_3) - (e_{10} - e_8) &= \text{circ}_{q_1}. \end{aligned}$$

So $\rho(S) = q_1 - q_2 - q_3 + q_4$. Reversing the orientation of S , or the sign of v , replaces $\rho(S)$ by $-\rho(S)$. $\square$

Everything used here depends on $\mathcal{T}$ and Σ' and not on the height $\check{h}$: by Lemma 4.9, for two heights on $\mathcal{T}$ satisfying the conditions of Hypothesis (U) the decompositions $\mathcal{P}$ correspond cell by cell, preserving lower supports and face labels, and the induced homeomorphism of the bases preserves the facets, Γ and its legs with their lower supports. So S is a belt cycle on the base of every such height, the normalised height of Section 3.3 included, with the same node coefficients.

The theorems on X_9 . Lemma 11.2 proves the node-coefficient statements of Theorem C, $\rho(\text{Bal}(\Gamma)) = \rho(Z(\mathfrak{K}_0)) = K$, for X_9 without the argument of Section 5: the chain $v[S]$ lies in $Z(\mathfrak{K}_0)$, so $K = \mathbb{Z}\rho(S) \subseteq \rho(Z(\mathfrak{K}_0))$, and the reverse inclusion is $\rho(\text{Bal}(\Gamma)) \subseteq K$, the easy half of the boundary level (Section 5.3). The remaining statements take the following form.

- *The mirror lift* (Theorem D). The disc S lifts to an integral 4-chain $\check{\Xi}(S)$ on Y_t with

$$\partial\check{\Xi}(S) = [\delta_{q_1}] - [\delta_{q_2}] - [\delta_{q_3}] + [\delta_{q_4}].$$

Over the interior of S the lift is, locally up to homotopy in each region, the bundle of two-tori $\ker v \otimes U(1)$ annihilated by d_0 (Remark 2.7). Near ∂S it is assembled from pieces cut out by toric divisors of $X_{\mathcal{T}}$, which add no boundary along the 18 legs, and at each of the four nodes it ends on a three-sphere isotopic to $\pm\delta_q$ (Section 8).

- *The relations* (Theorem A(1)). The relation lattice of the four vanishing spheres is exactly $\mathbb{Z}(1, -1, -1, 1)$, and $\sum_q k_q[\delta_q]$ is torsion only if k is a multiple of $(1, -1, -1, 1)$. The classes $[\delta_{q_1}], \dots, [\delta_{q_4}]$ span a free subgroup of rank three of $H_3(Y_t; \mathbb{Z})$.
- *The nodal locus* (Theorem A(3)). Near the given member, the members with a node at each of the four points q_P form a complex submanifold of codimension 3: the four nodes impose three conditions, as Morrison's count predicts for a small resolution of relative Picard number 3.
- *Smoothings* (Corollary B). The generator of K has every coefficient nonzero. So X'_9 has a smoothing, the vanishing spheres satisfy the relation $[\delta_{q_1}] - [\delta_{q_2}] - [\delta_{q_3}] + [\delta_{q_4}] = 0$ with all coefficients nonzero, and one of the 16 conifold transitions of Y_t in Lagrangian spheres isotopic to the δ_q carries a symplectic structure with symplectic exceptional curves.
- *The resolution side* (Theorem E). Section 10.3 describes the circle domain for this same belt disc, its four signed boundary degrees and the resulting 3-chain on $\widehat{X}_9$.

11.2. The examples X_{19} and X_{20} . The polytopes Δ_{19} and Δ_{20} of [Wue26c] have 19 and 20 vertices, one dP_7 pentagon each, and 14 and 17 unit squares; so $|Q| = 16$ and $|Q| = 19$. Both satisfy (Proj). Their data are in Appendices G.2 and G.3. On each, computations find strict tropical 2-cycles (Definition 2.5) whose node coefficients generate K over $\mathbb{Z}$; they illustrate Theorem C and are evidence for Conjecture G on these two polytopes. Table 2 lists them. Their boundary networks and node coefficients are verified from the accompanying data (Appendix G), and so is the shape of the belt cycles (a disc with one field that meets Γ only along its boundary); for the other cycles the interior conditions of [CBM14, Def. 7.2] rest on the computations described in Section 11.5. Besides belt cycles there are cycles whose boundary graph has trivalent vertices; these vertices are negative vertices of Γ , and the boundary graphs are a complete graph on four vertices (X_{19}) and two vertices joined by three paths (X_{20}).

The two examples differ in the conclusion of Corollary B.

	cycle	boundary graph	node coefficients $\rho(S_i)$
X_{19}	S_1	closed path, 4 nodes	$q_2 - q_3 + q_4 - q_7$
	S_2	complete graph on 4 vertices	$q_1 + q_3 - q_4 - q_5 - q_8 - q_9 + q_{11} + q_{12} + q_{14} - q_{15}$
	S_3	complete graph on 4 vertices	$q_2 - q_3 - q_6 + q_8 - q_9 + q_{10} + q_{11} - q_{12} - q_{13} + q_{16}$
X_{20}	S_1	closed path, 3 nodes	$-q_{14} + q_{16} - q_{17}$
	S_2	closed path, 3 nodes	$q_3 - q_4 + q_{14}$
	S_3	closed path, 4 nodes	$-q_4 + q_5 - q_9 + q_{10}$
	S_4	closed path, 3 nodes	$q_2 - q_5 + q_{13}$
	S_5	closed path, 3 nodes	$-q_{13} + q_{16} - q_{19}$
	S_6	two vertices, three paths	$q_4 - q_5 + q_7 - q_8 + q_{11} - q_{12} + q_{15} - q_{18}$

TABLE 2. Tropical 2-cycles found by computation on X_{19} ($|Q| = 16$, $\text{rank } K = 3$) and X_{20} ($|Q| = 19$, $\text{rank } K = 6$) whose node coefficients generate K ; the cycles with a closed path as boundary are belt cycles. For the cycles with trivalent vertices the conditions of [CBM14, Def. 7.2] in the interior rest on the computations of Section 11.5. In the $\mathbb{Z}$ -bases of K of Appendix G the matrices of the $\rho(S_i)$ have all elementary divisors equal to 1.

- On X_{19} no node is forced on K . For instance $\lambda = 3\rho(S_1) + 2\rho(S_2) + \rho(S_3)$, that is

$$\begin{aligned} \lambda = & 2q_1 + 4q_2 - 2q_3 + q_4 - 2q_5 - q_6 - 3q_7 - q_8 \\ & - 3q_9 + q_{10} + 3q_{11} + q_{12} - q_{13} + 2q_{14} - 2q_{15} + q_{16}, \end{aligned}$$

lies in K and has all 16 coefficients nonzero. So X'_{19} has a smoothing, and $\sum_q \lambda_q [\delta_q] = 0$ is a relation among the vanishing spheres of the mirror with every coefficient nonzero.

- On X_{20} the two pentagon nodes q_1 and q_6 occur in no relation: the basis of K in Appendix G.3 has coordinate 0 at both. By Corollary B, X'_{20} has no smoothing, and although $\text{rank } K = 6$, no relation among the 19 vanishing spheres of the mirror has all coefficients nonzero.

In both cases $|Q| - \text{rank } K = 13$, the codimension of the nodal locus in Theorem A(3).

11.3. Hexagon examples. We give one example on each branch of the dP_6 hexagon. On both, the centre o of the hexagon occurs only in the circuits of the nodes of the hexagon, each time with coefficient -1 in the labelling with the diagonal through the centre, so every $\lambda \in K$ satisfies the tie $\sum_{q \text{ at } G} \lambda_q = 0$ at the hexagon G . The data are in Appendix G.4. The subscripts 88 and 154 are the labels of the two polytopes in the accompanying data; the vertex lists in that appendix specify the two polytopes.

X_{88} , *the A_1 branch.* The polytope Δ_{88} has 21 vertices, 22 unit squares, one hexagon and no pentagon. All three A_1 subdivisions of the hexagon satisfy (Proj), and neither A_2 subdivision does. On the A_1 profile of Appendix G.4, X'_σ has $|Q| = 24$ nodes, two of them at the hexagon, and $\text{rank } K = 10$. The two hexagon nodes share only the centre, their carrying cells lie in one two-face of $\nabla^{\tilde{h}}$, and the tie reads $\lambda_{q_{23}} + \lambda_{q_{24}} = 0$. On the resolution side the exceptional surface over the centre is a dP_6 surface, the curves $C_{q_{23}}$ and $C_{q_{24}}$ are two opposite (-1) -curves in its hexagon of lines, and its image in X'_σ is $\mathbb{P}^1 \times \mathbb{P}^1$, with the two nodes at opposite torus-fixed points (Proposition 3.9). No node is forced on K : the vector

$$\begin{aligned} & -q_1 + q_2 - 2q_3 - q_4 + q_5 + q_6 - q_7 + q_8 - 2q_9 + q_{10} - q_{11} + 2q_{12} - 2q_{13} \\ & - q_{14} - q_{15} - q_{16} + q_{17} - q_{18} - q_{19} + q_{20} + q_{21} - q_{22} + q_{23} - q_{24} \end{aligned}$$

lies in K . So X'_{88} has a smoothing, and the nodal locus of the mirror has codimension 14. Computations find ten strict tropical 2-cycles whose node coefficients generate K with index

one: six belt cycles, two cycles whose boundary graph has two trivalent vertices joined by three paths, one whose boundary graph is a complete graph on four vertices, and one whose boundary network was found by integer programming and which crosses Γ transversally at 18 points. The six belt cycles satisfy Definition 11.1: they are discs with one field that meet Γ only along their boundary, as is verified from the accompanying data; for the other four the interior conditions of [CBM14, Def. 7.2] rest on the computations of Section 11.5. Two of the ten pass through the hexagon nodes, with node coefficients $\pm(1, -1)$ there, as the tie requires.

X_{154} , the A_2 branch. The polytope Δ_{154} has 22 vertices, 20 unit squares, one hexagon and no pentagon. No A_1 subdivision of the hexagon satisfies (Proj), and both A_2 subdivisions do; on the A_1 branch two nodes would moreover be forced on K . On the A_2 profile of Appendix G.4, $|Q| = 23$, three nodes lie at the hexagon, and $\text{rank } K = 9$. Among the polytopes of the list that satisfy (Proj) with every hexagon on the A_2 branch and have no forced node, it has the smallest $|Q|$ and the smallest rank of K . The tie reads $\lambda_{q_{21}} + \lambda_{q_{22}} + \lambda_{q_{23}} = 0$, and the projection of K to these three coordinates is the whole lattice $\{\lambda \in \mathbb{Z}^3 : \lambda_1 + \lambda_2 + \lambda_3 = 0\}$. Here the three exceptional curves at the hexagon are three pairwise disjoint (-1) -curves of the dP_6 surface over the centre, whose image in X'_σ is $\mathbb{P}^2$ with the three nodes at its torus-fixed points (Proposition 3.9). No node is forced; the vector

$$\begin{aligned} & -q_1 + q_2 + 2q_3 + q_4 - 2q_5 + 2q_6 - q_7 + q_8 - q_9 - q_{10} + q_{11} + q_{12} \\ & + 2q_{13} + q_{14} - q_{15} - q_{16} + q_{17} + q_{18} - q_{19} + 2q_{20} + q_{21} + q_{22} - 2q_{23} \end{aligned}$$

lies in K , so X'_{154} has a smoothing, which the A_1 branch does not see, and the nodal locus of the mirror has codimension 14.

The discriminant at the A_2 hexagon has a feature that the other examples do not have. The three node parallelograms share pairwise an edge $[o, w]$ through the centre, and their carrying cells lie in one two-face of ∇^h . At each node two of the four legs have lower support $\{o, w\}$ for such an edge, and these two legs are adjacent. The legs with lower support $\{o, w\}$ join the two nodes that share $[o, w]$ by a path of Γ through two-valent points only, with constant vanishing vector $\pm(w - o)$; on B these paths have 2, 4 and 4 legs. The three paths form a cycle of Γ that contains no trivalent vertex. Nine belt cycles, each a disc with one field meeting Γ only along its boundary, have node coefficients generating K with index one. Four of them pass through two of the three hexagon nodes each: the boundary enters along a leg outside the two-face, passes q_i , runs along the path of Γ from q_i to q_j , and leaves along a leg outside the two-face. Their node coefficients at the three hexagon nodes are roots of the tie plane, $\pm(1, 0, -1)$ and $\pm(0, 1, -1)$ up to the order of the nodes, and they span it.

11.4. The 612-polytope collection. The *list* is the following set of 668 vertex sets of reflexive four-polytopes:

- the 77 admissible reflexive four-polytopes with at most nine vertices that have a pentagonal two-face [Wue26c], together with Δ_{19} and Δ_{20} , and the 26-vertex polytope $\Delta_\star$ whose face fan defines the ambient variety of the hypersurface X° in [Wue26c];
- the reflexive four-polytopes Δ for which every edge of Δ and of Δ° is primitive, determined in [Wue26a] from the 473,800,776 reflexive four-polytopes of the classification of Kreuzer and Skarke [KS00];
- the product of two reflexive hexagons, with 36 vertices [Wue26a].

Of these, 613 are admissible (Definition 2.1), and they form 612 polytopes up to $GL_4(\mathbb{Z})$: the only coincidence is that $\Delta_\star$ is equivalent to a polytope of the second item. All 612 occur

among the admissible polytopes of the Kreuzer–Skarke list found in Appendix A. Table 3 gives the counts.

	polytopes (Proj) for some profile	
admissible, no hexagonal two-face	470	
no singular two-face	8	
some singular two-face, $K = 0$	87	
$K \neq 0$	375	364
admissible, some hexagonal two-face	142	83
every hexagon on the A_1 branch		20
every hexagon on the A_2 branch		55
mixed profiles		18

TABLE 3. The 612-polytope collection of admissible polytopes, up to $GL_4(\mathbb{Z})$. For hexagon-free polytopes the profile is unique. In each of the last three rows a polytope is counted if (Proj) holds for some profile of that kind, so a polytope may be counted in several of them.

Hypothesis (Proj). It is a linear feasibility problem over $\mathbb{Q}$ in the heights of the lattice points of $\partial\Delta$ (Section 3.6), which we decide exactly: a feasible profile is exhibited by an integral solution, and an infeasible one is excluded by a Farkas certificate (Proposition 3.29(3)) verified in exact rational arithmetic. For a polytope with h hexagons, whether (Proj) holds for some profile with every hexagon on the A_1 branch, for some with every hexagon on the A_2 branch, and for some mixed profile, is decided by a depth-first search over the 5^h profiles in which a Farkas certificate for a partial choice of subdivisions excludes all its completions. On the hexagon-free polytopes with $K \neq 0$, (Proj) fails on 11 of 375, all with a pentagonal two-face. On the 142 polytopes with a hexagon it holds for some profile on 83 and for none on 59; the branch patterns are counted in Table 3. Where (Proj) fails there is an explicit obstruction: a nonzero effective combination of the proper transforms $\tilde{f}_\epsilon \subset \hat{X}$ of curves in the fibres of $X'_\sigma \rightarrow X$ over the del Pezzo points that is homologous in $H_2(\hat{X}; \mathbb{Q})$ to a combination of the exceptional curves C_q (Proposition 3.29(3)). The polytopes of the list with $K \neq 0$ to which Theorems A–E apply are therefore the $364 + 83 = 447$ polytopes of the last column; on every profile of a hexagonal polytope examined, $K \neq 0$. (Of the 143 admissible vertex sets with a hexagon, $\Delta_\star$ and its equivalent both fail (Proj) on every profile.)

Theorem C on the list. Theorem C is proved in Section 5; the following computation corroborates it and is not used in the proof. On the bases of the 364 hexagon-free polytopes satisfying (Proj) and, in addition, on the bases of X_9 , X_{19} , X_{20} built from the heights in the accompanying data (<https://github.com/bwuebben/calabi-yau-smoothability>, directory `paper6`; 367 bases, with rank K between 1 and 36 and up to 60 nodes) we computed a basis of the group $\text{Bal}(\Gamma)$ of balanced networks, checked $\rho(\text{Bal}(\Gamma)) = K$ by a Smith normal form, and showed, by solving the integral linear systems exactly, that every balanced network is the boundary network of an integral relative tropical 2-cycle on $\mathfrak{R}_0$. On every base $\partial Z(\mathfrak{R}_0) = \text{Bal}(\Gamma)$ and $\rho(Z(\mathfrak{R}_0)) = K$ with index one. On 191 of the bases the balanced networks contained in the boundary of a single facet F_u , which bound inside F_u , have node coefficients spanning a proper sublattice of K , so that the global argument of Section 5 is needed there. The same computation succeeds on 72 bases of 71 polytopes with hexagonal two-faces (for one of these bases only the outcome is reported; see Appendix G): the 71 polytopes with a profile satisfying (Proj) on the A_2 branch or a mixed one, each on one such profile, and in

addition the product of two hexagons on an A_1 profile. It was not run on the 12 polytopes whose profiles satisfying (Proj) are all on the A_1 branch; X_{88} is one of them.

Forced nodes and Corollary B. As in Remark 9.9, a node q is *forced* on K if $\lambda_q = 0$ for every $\lambda \in K$, that is, if the q -th coordinate vanishes on a $\mathbb{Z}$ -basis of K . Since a lattice is not a finite union of sublattices of smaller rank, some $\lambda \in K$ has every $\lambda_q \neq 0$ if and only if no node is forced; this is condition (b) of Corollary B. At a hexagon the forced nodes depend on the profile only through its branch: the three A_1 subdivisions give the same centre-free combination $\pm(e_{w_0} - e_{w_1} + e_{w_2} - e_{w_3} + e_{w_4} - e_{w_5})$ of their two circuits, and the two A_2 subdivisions give the same plane of centre-free combinations of their three circuits, with the same three lines on which one node coefficient vanishes (Remark 3.10). We computed a $\mathbb{Z}$ -basis of K and the forced nodes in exact integer arithmetic for every polytope satisfying (Proj) and every branch pattern on which (Proj) holds. For a polytope with several hexagons we first test the profiles found by the search for (Proj); where each of them has a forced node, an exact search of the same kind over all profiles whose branch pattern has no forced node decides whether one of them satisfies (Proj); for the count by mixed profiles the same search is run over the mixed patterns without forced node whenever the mixed profile found first has a forced node. The result:

- among the 364 hexagon-free polytopes satisfying (Proj), condition (b) holds on 118;
- among the 83 hexagonal polytopes satisfying (Proj), it holds for some profile satisfying (Proj) on 43: on 10 of the 29 with one hexagon and on 33 of the 54 with several. By branch pattern, it holds on 18 of the 20 polytopes with an A_1 profile satisfying (Proj), on 22 of the 55 with an A_2 profile, and on 10 of the 18 with a mixed profile.

So on 161 of the 447 polytopes satisfying (Proj), X'_σ has a smoothing for some profile, and for that profile the vanishing spheres of the mirror satisfy a relation with every coefficient nonzero. On the other 286 every profile satisfying (Proj) has a forced node; by Corollary B, X'_σ then has no smoothing and every relation among the δ_q has a zero coefficient. The data from which these assertions are verified in exact arithmetic are described in Appendix G.

11.5. Evidence for Conjecture G. The computations of this subsection are evidence for Conjecture G and are used in no proof. What can be re-verified from the accompanying data is stated in Appendix G.

Conjecture G asks for more than Theorem C: that strict tropical 2-cycles (Definition 2.5), whose fields are primitive, whose interior edges and crossings with Γ are of the restricted kinds that definition allows, and whose boundaries pass each node at most once along two opposite legs, already have node coefficients generating K . It is the lattice form, for the bases B'_σ , of the conjecture of Castaño-Bernard and Matessi that every good relation among the vanishing cycles is a combination of relations coming from tropical 2-cycles [CBM14, §1, Conj. 8.3]. We tested it on the 364 hexagon-free polytopes of the list satisfying (Proj), each on the base built from a height as in Hypothesis (U).

The evidence separates the boundary of a tropical 2-cycle from its interior. By the boundary-vertex, node-passage and boundary-monodromy clauses of [CBM14, Def. 7.2], the boundary network of a strict tropical 2-cycle has the following form.

Definition 11.3 (networks of CBM type). A balanced network $h = \sum_E c_E d_E[E]$ is of *CBM type* if every $c_E \in \{0, \pm 1\}$, the legs with $c_E \neq 0$ meet every node in none or in two opposite legs, meet every negative vertex in none or all three legs, and meet every other point of Γ in at most two legs. Its negative vertices with three legs are its *trivalent vertices*.

A network of CBM type is a balanced network in the sense of Section 2.3 whose support is a graph of the kind that the boundary of a tropical 2-cycle of [CBM14] can have; its node coefficients lie in K by Theorem C. Conjecture G asserts in particular that networks of CBM type generate K , and in addition that enough of them bound strict tropical 2-cycles. The strict cycles were constructed from a boundary network h in three steps: the germ of the cycle along h is found by an integer program in the coefficients of the triangles of $\mathfrak{R}_0$ that meet the support of h , subject to the clauses of [CBM14, Def. 7.2] along the boundary and with coefficients $0, \pm 1$; the germ is filled by an integral relative tropical 2-cycle, found by an integer program and checked in exact arithmetic; and near each point of Γ where the filling violates a clause of [CBM14, Def. 7.2], the filling is replaced, on the second barycentric subdivision, by a chain with constant coefficient in the lattice of vectors invariant under the local monodromy. Every cycle obtained was checked against every clause of [CBM14, Def. 7.2] by two independent implementations. On the 364 polytopes we find the following.

- (a) *Boundaries.* On all 364 polytopes, the node coefficients of networks of CBM type generate K , as the Smith normal form of their matrix in a basis of K shows. Networks without trivalent vertices, networks with two trivalent vertices joined by three paths and networks whose trivalent vertices form a complete graph on four vertices suffice on 327 polytopes; on the other 37 further networks of CBM type found by integer programming are needed.
- (b) *Belt cycles.* Networks without trivalent vertices generate K on 195 polytopes. On 179 of these, strict tropical 2-cycles whose domain is a disc with one constant field and whose boundary network is a single closed path generate K ; on 171 they can all be taken to be belt cycles (Definition 11.1), while on the other 8 some of the discs used cross Γ transversally at interior points of legs, as [CBM14, Def. 7.2] allows. We checked each of the 3,389 discs of this kind that were constructed: all are discs with one field up to sign, and 28 of them meet Γ off their boundary.
- (c) *Strict cycles.* On 340 polytopes, X_9 , X_{19} and X_{20} among them, the computations find strict tropical 2-cycles whose node coefficients generate K ; the index of the lattice they span in K was computed exactly and is one. For X_9 this is also proved in the text (Lemma 11.2), given the triangulation $\mathcal{T}$ and the triangles of the cycle given in the accompanying data.
- (d) *The remaining polytopes.* On the other 24 polytopes the strict cycles constructed generate a sublattice of K of corank between 1 and 4, and of corank one on 16. For each network of CBM type of a generating set that is not the boundary of a constructed cycle, the construction meets an obstruction of one of three kinds:
 - (1) *a single sheet:* a network without trivalent vertices, with vanishing vector $\pm d$, is not known to bound a disc with field $\pm d$ formed by triangles of $\mathfrak{R}_0$ that lie in facets F_u with $\langle u, d \rangle = 0$ and meet Γ only along the network and at edge points; it may still bound a cycle with several sheets;
 - (2) *the filling:* the germ has an integral filling, but none is known that satisfies, near the points of Γ off the network, the conditions under which the local replacement applies;
 - (3) *the local replacement:* near some point of Γ off the network, or at an interior vertex of the cycle, no constant coefficient of small norm in the invariant lattice removes the violation of the clauses of [CBM14, Def. 7.2].

Each kind occurs, for some network, on 10, 24 and 18 of the 24 polytopes respectively.

- (e) *Trivalent vertices appear to be needed.* Let $k(X)$ be the least k such that the networks of CBM type with at most k trivalent vertices generate K . Then $k(X) = 0$ exactly on the 195 polytopes of (b), and on the other 169 the computations suggest $k(X) = 2, 4, 6, 8$ on 57, 89, 21 and 2 polytopes respectively. The upper bounds are explicit networks; each lower bound $k(X) > k$ with $k \geq 2$ rests on the infeasibility, computed in floating-point arithmetic, of an integer program asking for a network of CBM type with at most k trivalent vertices whose node coefficients lie outside a hyperplane of $K \otimes \mathbb{Q}$ containing those of all networks without trivalent vertices. For X_9 , X_{19} and X_{20} the values so obtained are $k(X) = 0, 4, 2$, in accordance with Table 2.

On the hexagonal polytopes satisfying (Proj), strict cycles constructed in the same way generate K with index one on 19 of the 20 polytopes with an A_1 profile satisfying (Proj) and on 66 of the 71 with an A_2 or mixed profile, each on one profile; on the other six the construction stopped with a sublattice of smaller rank, or did not produce generators of K .

The obstructions of (d) sit in extending a network of CBM type to a strict cycle, not at the boundary level, where (a) holds everywhere; and (e) suggests that a proof of Conjecture G cannot restrict itself to networks with few trivalent vertices. The relative tropical 2-cycles of Theorem C avoid all three obstructions: they allow several sheets, arbitrary branching along edges and crossings of Γ with any invariant coefficient, which is what the filling of Section 5 uses.

APPENDIX A. UNIMODULAR HEIGHTS: PROOF OF THEOREM F

Hypothesis (U) asks for a unimodular triangulation $\mathcal{T}$ of $\partial\Delta^\circ$ refining its faces and an integral height $\check{h}$ on the fan over $\mathcal{T}$ such that $\check{h}$ and $\check{h} - \check{\varphi}$ are strictly convex; Theorem F establishes it for every member of the fixed finite set $\mathcal{L}_{\text{adm}}$ defined below. Proposition 2.8 asserts separately that this set contains every admissible class. Appendix A.1 shows that Hypothesis (U) is equivalent to the existence of a fine, regular, unimodular triangulation of Δ° that is star-shaped from the origin, and that strict convexity of $\check{h} - \check{\varphi}$ implies that of $\check{h}$. Section A.2 explains why admissibility does not yield such a triangulation by a general argument: it constrains the facets of Δ° only along their edges, and lattice polytopes of dimension three need not have unimodular triangulations at all. Section A.3 proves the finite-list theorem using explicit heights verified in exact integer arithmetic and proves the completeness proposition using the classification and admissibility filter. Appendix A.4 distinguishes the evidence for these two claims.

Definition A.1 (the finite list of admissible polytopes). Let $I = \{5, 6, \dots, 33, 36\}$. For $v \in I$, let $\mathcal{R}_v$ be the records in `heights_vNN.jsonl.gz`, with NN the two-digit value of v , in the fixed version of the accompanying data identified in Appendix G. If $V_r \subset \mathbb{Z}^4$ is the vertex array in the field V of record r , put

$$\mathcal{L}_{\text{adm}} = \{[\text{conv}(V_r)]_{GL_4(\mathbb{Z})} : r \in \bigcup_{v \in I} \mathcal{R}_v\}.$$

The vertex arrays determine this set independently of the programs used to find or check it. Admissibility and pairwise inequivalence of its 51,827 records are verified in Step 2 below; completeness is the separate assertion of Proposition 2.8.

Throughout, a piecewise linear function g on a complete simplicial fan is *strictly convex* if it is convex and its maximal domains of linearity are the maximal cones. A *wall* is a common facet of two maximal cones κ, κ' , and the *bend* of g across it is $g(m') - \ell_\kappa(m')$, where ℓ_κ is the linear function equal to g on κ and m' is the generator of κ' not in κ . Bends are

additive in g , and g is strictly convex if and only if every bend is positive; this is the wall criterion of [CLS11, §6.1], where support functions carry the opposite sign, so that convexity here is concavity there. Recall that $\check{\varphi}$ equals 1 on $\partial\Delta^\circ$ and is linear on the cones over the faces of Δ° ; it is convex, its maximal domains of linearity are the cones over the facets, and $\check{\varphi}(m) = \max_v \langle m, -v \rangle$ over the vertices v of Δ .

A.1. Reformulation.

Lemma A.2. *Let $\mathcal{T}$ be a unimodular triangulation of $\partial\Delta^\circ$ refining its faces. The following are equivalent.*

- (a) *There is a piecewise linear function $\check{h}$ on the fan over $\mathcal{T}$, integral on M , such that $\check{h}$ and $\check{h} - \check{\varphi}$ are strictly convex.*
- (b) *There is a real piecewise linear function on the fan over $\mathcal{T}$ that is strictly convex.*
- (c) *There are real heights ω on the vertices of $\mathcal{T}$ such that for every facet F of Δ° the restriction $\mathcal{T}|_F$ is the regular subdivision of $\text{vert}(\mathcal{T}) \cap F$ induced by $\omega|_F$.*

Moreover, for any piecewise linear $\check{h}$ on the fan over $\mathcal{T}$, strict convexity of $\check{h} - \check{\varphi}$ implies strict convexity of $\check{h}$.

Proof. (a) $\Rightarrow$ (b): take $\check{h} - \check{\varphi}$.

(b) $\Rightarrow$ (a): strict convexity is an open condition on the values at the rays, so we may take the strictly convex function g rational, and after scaling integral at the primitive generators of the rays. Put $\check{h} = g + \check{\varphi}$. Then $\check{h} - \check{\varphi} = g$ is strictly convex. Every simplex of $\mathcal{T}$ lies in a facet, so $\check{\varphi}$ is linear on every cone of the fan and bends by ≥ 0 across every wall; hence every bend of $\check{h}$ is at least the corresponding bend of g , and $\check{h}$ is strictly convex. The same computation proves the last assertion, for any triangulation. The function g is integral on M , because a linear function on a unimodular cone that is integral at the generators is integral on the lattice points of the cone, and $\check{\varphi}$ is integral on M because the vertices of Δ are lattice points.

(b) $\Rightarrow$ (c): put $\omega = g|_{\text{vert } \mathcal{T}}$. On the cone over a facet F , g is convex with the cones over the simplices of $\mathcal{T}|_F$ as its maximal domains of linearity. Its restriction to the affine hyperplane of F is therefore a convex piecewise linear function on F whose domains of linearity are exactly the simplices of $\mathcal{T}|_F$ and whose values at their vertices are ω ; this says that $\mathcal{T}|_F$ is the regular subdivision induced by $\omega|_F$.

(c) $\Rightarrow$ (b): let g_0 be linear on each cone of the fan over $\mathcal{T}$ with $g_0 = \omega$ on the vertices. Across a wall inside the cone over a facet F , the bend of g_0 is positive by the regularity of $\mathcal{T}|_F$, since a bend is a local condition, homogeneous of degree one. Across the finitely many walls between the cones over two different facets, $\check{\varphi}$ bends positively, while it does not bend across walls inside a facet cone. So $g_0 + \lambda\check{\varphi}$ is strictly convex for $\lambda \gg 0$. $\square$

Adding a constant to the heights ω on $\partial\Delta^\circ$ adds a multiple of $\check{\varphi}$ to the corresponding piecewise linear function; this is the conversion used below. A triangulation of Δ° is *star-shaped from the origin* if the origin is a vertex of each of its maximal simplices.

Corollary A.3. *Hypothesis (U) holds for Δ if and only if Δ° has a fine, regular, unimodular triangulation that is star-shaped from the origin.*

Proof. A star-shaped triangulation of Δ° consists of the simplices $\text{conv}(0, \tau)$, τ in a triangulation $\mathcal{T}$ of $\partial\Delta^\circ$; each τ lies in a facet, since a convex subset of the boundary of a polytope lies in a facet, so $\mathcal{T}$ refines the faces of Δ° . Since Δ° is reflexive, every facet lies at lattice distance one from the origin, so $\text{conv}(0, \tau)$ is unimodular if and only if τ is unimodular in its facet,

and a unimodular simplex contains no lattice points other than its vertices, so unimodular triangulations are fine. The star-shaped triangulation is regular if and only if it is induced by heights 0 at the origin and ω on $\partial\Delta^\circ$ (subtract an affine function), and the lower envelope of the lifted points is then linear on each $\text{conv}(0, \tau)$ and 0 at the origin; so regularity holds if and only if the piecewise linear function with values ω on the fan over $\mathcal{T}$ is strictly convex. This is (b) of Lemma A.2, which is equivalent to (a). $\square$

A.2. The facets of Δ° . Let v be a vertex of Δ with neighbours $w_1, \dots, w_k$ along edges, and $d_i = w_i - v$, primitive by admissibility. The facet of Δ° dual to v is the lattice three-polytope

$$F_v = \{m \in M_{\mathbb{R}} : \langle m, v \rangle = -1, \langle m, d_i \rangle \geq 0 \ (i = 1, \dots, k)\},$$

the polytope of the anticanonical bundle of $\mathbb{P}_\Delta$ restricted to the toric divisor D_v . Its vertices, edges and two-faces correspond to the facets, two-faces and edges of Δ that contain v .

Proposition A.4. *Let Δ be admissible and let E be an edge of F_v , dual to a two-face $\theta \ni v$ of Δ .*

- (1) *The normal cone of F_v along E is lattice equivalent to the tangent cone of θ at v , in the lattice of the plane of θ . It is unimodular if θ is a triangle, a square or a hexagon, and if θ is a pentagon and v is one of its three corners of index one; at the other two corners of a pentagon it has index 2.*
- (2) *If θ is not a triangle, E has lattice length one.*

Proof. (1) The two-dimensional cone of the fan $\text{Star}(v)/v$ given by θ is $\text{cone}(\theta - v)$ modulo $\mathbb{R}v$. Since θ lies in a facet of Δ at lattice height one, the saturated lattice of $\mathbb{R}\theta$ is $\mathbb{Z}v \oplus L_\theta$, with L_θ the direction lattice of the plane of θ , so L_θ maps isomorphically onto a saturated sublattice of $N/\mathbb{Z}v$, carrying the tangent cone of θ at v to that two-dimensional cone. The index at a corner is the determinant of the two primitive edge vectors there; for the pentagon $\text{conv}\{(1,0), (1,1), (0,1), (-1,0), (0,-1)\}$ these are 1, 1, 1, 2, 2, for the hexagon all are 1. (2) is the condition $\ell(\theta^\circ) = 1$ of Definition 2.1. $\square$

Admissibility is a condition on the two-faces of Δ and on their dual edges, that is, on the edges of the facets F_v and on the normal cones along them; Proposition A.4 records what it says there. It imposes no direct condition on the cones of F_v at its vertices beyond their two-dimensional faces, nor on the two-faces of F_v .

The known sufficient conditions for unimodular triangulations do not apply to these facets. If every facet F_v has lattice width one with respect to each of its facet normals (is *compressed*), then every pulling triangulation of each facet is unimodular [HPPS21, Thm. 2.3], and pulling the boundary lattice points of Δ° in one global order gives heights as in Lemma A.2(c). If the facet normals of a facet form a unimodular matrix, that facet has a regular unimodular triangulation [HPPS21, Thm. 2.4], but the triangulations of two adjacent facets obtained in this way need not agree on the common face. Compressed facets are rare: of the 3,175 facets of the 418 admissible polytopes with at most eight vertices, 46 are compressed. Facets with unimodular normals are common (12,328 of the 13,443 facets of the 612 admissible polytopes of the list of Section 11.4), but their triangulations need not be compatible. The facets are also far from small: among the 3,175 there are facets with 242 lattice points and normalised volume 756, and 2,171 of the 2,444 facets with at most 40 lattice points contain empty lattice tetrahedra of volume greater than one among their lattice points, so that a fine triangulation of such a facet need not be unimodular. For lattice three-polytopes in general, unimodular triangulations need not exist [HPPS21, Ex. 1.2], and their existence is open even for smooth lattice three-polytopes [HPPS21, §1.5.1]; the facets F_v need not be

smooth, since by Proposition A.4 their vertex cones are constrained only through their two-dimensional faces. A general proof of Hypothesis (U) for all admissible Δ would therefore produce compatible regular unimodular triangulations for a class of lattice three-polytopes, the polytopes of $-K_{\mathbb{P}_\Delta}|_{D_v}$, for which no such result is known. We know no conceptual proof, and we prove the theorem by exhaustion.

A.3. Certificates and classification completeness. Steps 1–2 prove Proposition 2.8, using the classification and the original exhaustive filter. Step 3 proves Theorem F by verifying the heights of each member of $\mathcal{L}_{\text{adm}}$ in exact arithmetic; it does not use exclusions made by the filter.

Step 1: the classification. Kreuzer and Skarke classified the reflexive four-polytopes up to $GL_4(\mathbb{Z})$: there are 473,800,776 of them [KS00]. We use a public electronic copy of their list at a fixed revision, identified in the accompanying data together with the digests of its files; its number of rows is the published count. A separate computation, not part of the accompanying data, checked every polytope of the copy reflexive and no two equivalent (by normal forms). Every polytope of the list is treated as Δ ; since the list contains the polar of each of its members, this covers every polar pair in both orders.

Step 2: the admissible polytopes. Admissibility is decided from the vertices and the facet inequalities of Δ , computed with PALP [KS04]. The two-faces of Δ are the maximal vertex sets shared by two facets; each is classified in the lattice of its plane. By Pick’s theorem, a lattice polygon with primitive edges is a unimodular triangle, a unit square, a dP_7 pentagon or a dP_6 hexagon exactly when its number of vertices and twice its area are $(3, 1)$, $(4, 2)$, $(5, 5)$ or $(6, 6)$. For every non-triangular two-face the lattice length of the dual edge is computed from the two facet normals. The result is Table 4: up to $GL_4(\mathbb{Z})$ there are 51,827 admissible polytopes, 40,403 without and 11,424 with a hexagonal two-face, and 741 of them have no singular two-face. Every one of them was checked admissible a second time by an independent program, the 51,827 have pairwise different normal forms, and the 612 admissible polytopes of the list of Section 11.4 all occur among them.

vertices	polytopes	admissible	with hexagon	vertices	polytopes	admissible	with hexagon
5	1,561	5	0	20	21,913,680	3,407	1,557
6	24,189	26	0	21	12,070,919	2,404	1,268
7	177,446	102	0	22	5,826,221	1,567	906
8	834,638	285	1	23	2,450,720	999	623
9	2,867,955	629	7	24	898,929	583	386
10	7,725,801	1,156	25	25	284,696	315	221
11	16,608,387	1,931	57	26	78,468	173	131
12	29,270,253	2,857	123	27	18,417	81	65
13	43,458,000	3,888	225	28	3,781	38	31
14	56,060,584	4,846	345	29	647	14	12
15	64,085,869	5,499	527	30	114	9	7
16	65,615,931	5,824	782	31	23	4	4
17	59,972,682	5,697	1,088	32	8	0	0
18	48,703,033	5,134	1,419	33	2	1	1
19	34,847,821	4,352	1,612	36	1	1	1
total				473,800,776	51,827	11,424	

TABLE 4. Reflexive four-polytopes by number of vertices, the admissible ones, and those among them with a hexagonal two-face.

Step 3: heights. For each admissible Δ we give an integer height $h(m)$ at every nonzero lattice point m of Δ° . It determines a candidate triangulation: in each facet F of Δ° , the lower faces of the convex hull of the lifted points $(m, h(m))$, $m \in F \cap M$. For an integer λ , computed in the check, put $g = h + \lambda$ on $\partial\Delta^\circ \cap M$, extended linearly on the cones over the candidate tetrahedra, and $\check{h} = g + \check{\varphi}$. The following are then checked in exact integer arithmetic:

- (i) every point with a given height is a lattice point of $\partial\Delta^\circ$, and every height is an integer;
- (ii) every candidate tetrahedron lies in a facet hyperplane of Δ° and has determinant ± 1 ;
- (iii) every triangle of a candidate tetrahedron lies in exactly two candidate tetrahedra, and their fourth vertices lie on opposite sides of the hyperplane spanned by the triangle and the origin;
- (iv) of five fixed rays, at least three lie on no wall, that is, in the cone over no triangle of a candidate tetrahedron, and each such ray lies in the cone over exactly one candidate tetrahedron;
- (v) across every wall, the bends of g and of $\check{h}$ are positive integers.

Lemma A.5. *If (i)–(v) hold, the candidate tetrahedra form a unimodular triangulation $\mathcal{T}$ of $\partial\Delta^\circ$ refining its faces, and $\check{h}$ is an integral height on the fan over $\mathcal{T}$ with $\check{h}$ and $\check{h} - \check{\varphi}$ strictly convex.*

Proof. Write $\sigma_T = \text{cone}(T)$ for a candidate tetrahedron T ; by (ii) it is a four-dimensional simplicial cone, and its facets are the cones over the triangles of T , the *walls*. Let Z be the union of the cones over the edges of the candidate tetrahedra and of the intersections of pairs of walls that span different hyperplanes; it is a finite union of cones of dimension at most two, so $M_{\mathbb{R}} \setminus Z$ is connected, and any two points off Z and off the walls are joined by a path that avoids Z and crosses the walls transversally at finitely many points.

The interiors are disjoint. For x off the walls let $N(x)$ be the number of cones σ_T containing x . Let x lie on a wall but not in Z . The walls through x contain x in their relative interiors (the relative boundary of a wall lies in cones over edges) and span a common hyperplane H . A cone containing x on its boundary contains it in exactly one facet (two facets meet in the cone over an edge), which is one of these walls; by (iii) each of these walls is a facet of exactly two cones, lying on opposite sides of H . So near x the same number of cones is gained on each side of H , and N takes the same value on both sides. Hence N is constant off the walls, and by (iv) it equals 1: the interiors of the cones are pairwise disjoint.

Local structure at the faces. We show, for a face $\phi \neq 0$ of some σ_T and a point x in its relative interior, that the cones having ϕ as a face cover a neighbourhood of x , and that no other cone contains x . The second assertion follows from the first: a cone containing x has interior points arbitrarily close to x , which would lie in the interior of a cone having ϕ as a face, so by disjointness it is that cone. If $\phi = \sigma_T$ there is nothing to show. If ϕ is a wall, by (iii) it is a facet of a second cone on the other side of its hyperplane, and the two cones cover a neighbourhood of x ; in particular a wall is never subdivided by the facets of several cones on one side, since (iii) requires the whole wall to be a facet of a cone on each side. If ϕ is the cone over an edge e , project to the plane $M_{\mathbb{R}}/\text{span } \phi$: the cones containing ϕ become sectors of angle less than π , and the walls containing ϕ their boundary rays. By (iii) each such ray bounds exactly two of the sectors, on opposite sides, so the sectors form cycles that turn monotonically around the origin, each covering every direction at least once; their interiors are disjoint, so there is exactly one cycle and it covers every direction once. So the cones containing ϕ cover a neighbourhood of x . If ϕ is a ray $\mathbb{R}_{\geq 0}v$, project to $M_{\mathbb{R}}/\mathbb{R}v$ and intersect with a small sphere: the cones containing ϕ become spherical triangles forming

a closed two-dimensional pseudomanifold L by (iii). By the case of walls and edges already treated, the map from L to the sphere is a local homeomorphism near every point of L , including its vertices, whose links are single circles. So L is a closed surface and the map is a covering of the two-sphere; each component of L maps homeomorphically, and by disjointness of the interiors there is exactly one. So the cones containing ϕ cover a neighbourhood of x .

The fan. The union of the cones is closed, and by the local structure it is open, so it is $M_{\mathbb{R}}$. Let $x \in \sigma_T \cap \sigma_{T'}$, $x \neq 0$, and let ϕ be the face of σ_T containing x in its relative interior; by the local structure ϕ is a face of $\sigma_{T'}$. Applying this to a point x in the relative interior of the convex cone $\sigma_T \cap \sigma_{T'}$, whose carrier face ϕ in σ_T contains $\sigma_T \cap \sigma_{T'}$, gives $\sigma_T \cap \sigma_{T'} = \phi$, a common face. So the cones form a complete simplicial fan.

Conclusion. By (i) and (ii) each tetrahedron lies in a facet of Δ° and is unimodular, so the tetrahedra triangulate every facet, hence every face, of $\partial\Delta^\circ$; a unimodular tetrahedron contains no lattice points besides its vertices, and the tetrahedra cover $\partial\Delta^\circ$, so every lattice point of $\partial\Delta^\circ$ is a vertex. By (v) and the wall criterion, $g = \check{h} - \check{\varphi}$ and $\check{h}$ are strictly convex. The values of g at the vertices are integers, so $\check{h}$ is integral on M as in the proof of Lemma A.2. $\square$

Conditions (i)–(v) hold for the heights of all 51,827 records defining $\mathcal{L}_{\text{adm}}$, with 12,001,172 tetrahedra in all. By Lemma A.5, Hypothesis (U) therefore holds for each of them. This proves Theorem F. $\square$

Step 2 examined every polytope of the classification and its accepted records are exactly those defining $\mathcal{L}_{\text{adm}}$. Together with Step 1 and the positive admissibility checks, this proves Proposition 2.8. The exclusions are an input to this proposition, rather than to the height certificates; Appendix A.4 specifies their verification coverage.

A second program proves Theorem F from the same heights by a different set of sufficient checks, in exact rational arithmetic: (i') the points with given heights are exactly the nonzero lattice points of Δ° , enumerated independently, and lie on its boundary; (ii') in each facet F , every cell of the lower convex hull of the points $(m, h(m))$, $m \in F \cap M$, is a tetrahedron of determinant ± 1 ; (v') for each tetrahedron and each such point m outside its cone, $g(m)$ and $\check{h}(m)$ exceed the values at m of the linear functions that agree with g , respectively $\check{h}$, on the cone. By (ii') each facet is triangulated by the regular subdivision induced by h , and since the heights are global, the triangulations of two facets restrict to the same regular subdivision of their common face; so the tetrahedra form a unimodular triangulation of $\partial\Delta^\circ$ refining its faces, and (v') is strict convexity of g and $\check{h}$ on its fan. This program does not need (iii) and (iv). As a further consistency check it compares the number of tetrahedra with the normalised volume of Δ° .

The heights were found by choosing $h(m)$ close to a large multiple of a positive definite quadratic form in m , first $|m|^2$ and then random forms: the first choice succeeds on 7,231 polytopes, and no polytope needed more than 30 forms. A quadratic height induces a regular triangulation of every facet at once, which is why one height serves all facets; the constant λ makes the bends across walls between different facets positive, as in the proof of Lemma A.2. How the heights were found plays no role in the proof.

A.4. What the proof rests on. The completeness proposition uses the first two inputs below; the finite-list height theorem uses the third. The first is the classification of Kreuzer and Skarke [KS00], itself computer-assisted, which was not recomputed: we use a public copy at a fixed revision whose files agree with digests of that copy (not digests published by Kreuzer and Skarke) and whose number of rows is the published count, and a separate

computation, not part of the accompanying data, checked its members reflexive and pairwise inequivalent.

The second is the identification of the admissible polytopes in Step 2. Let d be the number of nonzero lattice points of Δ that are not vertices; all admissible polytopes found have $d \leq 7$ except two, with $d = 12$ and $d = 20$. Every polytope that the first implementation accepts was checked admissible again by an independent program. A second implementation of the whole test, sharing no code with the first, takes the face lattice of Δ from a different polyhedral library instead of from the vertex-facet incidences computed by PALP, tests the edges of Δ directly, and computes twice the area of a two-face, in the lattice of its plane, as a sum over the triangles of a fan triangulation from one vertex of the greatest common divisors of the 2×2 minors of their edge vectors. It agrees with the first implementation on every polytope on which both were run: all polytopes with 5–9, 24–33 or 36 vertices, all polytopes with 10–23 vertices and $d \leq 7$, and, for each number of vertices, a uniform sample of 2,000 polytopes with $d > 5$. The remaining 431,934,093 polytopes, about 91% of the classification, all with 10–23 vertices and $d > 7$, are excluded by the first implementation alone, so the completeness of the list of 51,827 admissible polytopes rests on these exclusions. The programs and the runs are described in Appendix G.

The third is the verification of (i)–(v), which by Lemma A.5 proves Hypothesis (U) for each polytope found. One program checks (i)–(v) in integer arithmetic, with determinants by fraction-free elimination, and a second checks (i'), (ii') and (v') in exact rational arithmetic; both accept the heights of all 51,827 polytopes.

APPENDIX B. PROOFS FOR SECTION 3

This appendix contains the proofs of the toric and topological facts of Section 3 that are standard in substance: the second homology of the small resolution, the integrality of the relations among the exceptional curves, and the classical description of the exceptional surfaces.

Proof of Lemma 3.3. Let $\mathcal{S}_0$ be a regular fine triangulation of $\partial\Delta$, that is, one whose vertices are all the lattice points of $\partial\Delta$, let P_0 be the projective toric variety of the fan over it, and let $\widehat{X}_0$ be the proper transform of X in P_0 . It is a maximal projective crepant partial resolution in the sense of Batyrev, and a smooth threefold. By [Wue26c, Prop. 8.3] the restrictions of the toric divisors of P_0 span $H^2(\widehat{X}_0; \mathbb{Q})$. That proposition applies because every two-face of Δ is a unimodular triangle or has a dual edge of lattice length one, so that the correction term in Batyrev's formula for $h^{1,1}$ vanishes. The toric divisors of rays outside R restrict to zero, so the D_r , $r \in R$, span $H^2(\widehat{X}_0; \mathbb{Q})$.

Near the preimage of each singular point x_G , both $\widehat{X}$ and $\widehat{X}_0$ are toric resolutions of U_G given by triangulations of G with the same vertex set, the lattice points of G . Both fans contain the subfan consisting of the rays and of the cones over the edges of G , so both toric varieties contain its toric variety as an open subset. The complements are the orbits of the cones that meet the interior of τ_G in dimension two or three; these lie over the vertex of U_G and are compact curves and points. Hence there are compact unions of curves $Z \subset \widehat{X}$ and $Z_0 \subset \widehat{X}_0$, lying over the points x_G , and an isomorphism $\widehat{X} \setminus Z \cong \widehat{X}_0 \setminus Z_0 =: U$, compatible with the maps to X , that carries $D_r \cap (\widehat{X} \setminus Z)$ to $D_r \cap U$ for every $r \in R$. The set Z is a compact analytic subset of the compact complex manifold $\widehat{X}$, hence a subcomplex of a triangulation of $\widehat{X}$ [Loj64, Thm. 2] and locally contractible. The duality $H_i(M, M \setminus K; \mathbb{Z}) \cong H^{n-i}(K; \mathbb{Z})$ for a compact, locally contractible subset K of a closed oriented n -manifold M

[Hat02, Thm. 3.44] gives $H_k(\widehat{X}, \widehat{X} \setminus Z; \mathbb{Z}) \cong H^{6-k}(Z; \mathbb{Z})$, which vanishes for $k \leq 3$ because Z is a polyhedron of dimension at most two. By universal coefficients over the field $\mathbb{Q}$, $H^k(\widehat{X}, \widehat{X} \setminus Z; \mathbb{Q}) \cong \text{Hom}(H_k(\widehat{X}, \widehat{X} \setminus Z; \mathbb{Z}), \mathbb{Q}) = 0$ for $k \leq 3$. The exact sequence of the pair gives

$$(B.1) \quad H^2(\widehat{X}; \mathbb{Q}) \xrightarrow{\cong} H^2(U; \mathbb{Q}),$$

and likewise $H^2(\widehat{X}_0; \mathbb{Q}) \cong H^2(U; \mathbb{Q})$. The two isomorphisms carry D_r to the class of $D_r \cap U$. So the D_r , $r \in R$, span $H^2(\widehat{X}; \mathbb{Q})$, and (3.2) is injective because the evaluation pairing $H_2(\widehat{X}; \mathbb{Q}) \times H^2(\widehat{X}; \mathbb{Q}) \rightarrow \mathbb{Q}$ is perfect. $\square$

Proof of Lemma 3.6. If G is a triangle, it is unimodular and its vertices form a unimodular triple; if G is a unit square, any three of its vertices do. If G is a pentagon or a hexagon, choose j such that P_j is a unit parallelogram (Lemma 3.1(1)). The diagonal $[w_j, w_{j+2}]$ cuts P_j into the triangles $\text{conv}\{w_j, w_{j+1}, w_{j+2}\}$ and $\text{conv}\{o, w_j, w_{j+2}\}$, which are exchanged by the point reflection of P_j in its centre and therefore have equal area. As P_j has normalised area two, w_j, w_{j+1}, w_{j+2} is a unimodular triple.

Let u be a vertex of Δ° with $G \subset u^\vee$, so that $\langle u, \cdot \rangle = -1$ on $\text{aff}(G)$. For $x \in L_G$ we have $x + \langle u, x \rangle y_1 \in L_G \cap u^\perp$, so $L_G = \mathbb{Z}y_1 \oplus (L_G \cap u^\perp)$. The lattice $L_G \cap u^\perp$ is the lattice of translations of $\text{aff}(G) \cap N$, and unimodularity of the triangle means that $y_2 - y_1$ and $y_3 - y_1$ form a basis of it. Hence y_1, y_2, y_3 is a basis of L_G . A basis of a saturated sublattice extends to a basis $y_1, \dots, y_4$ of N , and every basis vector $y_i \wedge y_k$ ($i < k$) of $\Lambda^2 N$ has $i \leq 3$. $\square$

Proof of Lemma 3.7. Put $Y' = T' \setminus W'$ and $\Phi = (\text{incl}, g): Y' \rightarrow T' \times \mathbb{C}^*$. By [Oka80, Thm. (5.3)(ii)], $\pi_1(Y')$ is free abelian of rank $n + 1$ and $\pi_i(Y') = 0$ for $1 < i < n$. The map $H_1(Y') \rightarrow H_1(T')$ is onto, since W' has real codimension two, and a small meridian loop of W' is null-homotopic in T' and is mapped by g with degree one, since $dg \neq 0$ along W' . So Φ_* maps $H_1(Y') \cong \mathbb{Z}^{n+1}$ onto $H_1(T' \times \mathbb{C}^*) \cong \mathbb{Z}^{n+1}$, hence isomorphically, and Φ is an isomorphism on π_1 . As $T' \times \mathbb{C}^*$ is aspherical, Φ is an n -equivalence, so $\Phi_*: H_i(Y') \rightarrow H_i(T' \times \mathbb{C}^*)$ is an isomorphism for $i < n$ and onto for $i = n$ [Hat02, proof of Prop. 4.21].

By the Thom isomorphism of the normal bundle of W' and excision, $H_i(T', Y') \cong H_{i-2}(W')$, and the connecting map $H_i(T', Y') \rightarrow H_{i-1}(Y')$ sends the class of a cycle c to the class of the circle bundle $S(c)$ of the normal bundle over c . In the Künneth decomposition $H_{i-1}(T' \times \mathbb{C}^*) = H_{i-1}(T') \oplus H_{i-2}(T') \otimes H_1(\mathbb{C}^*)$, the first component of $\Phi_*[S(c)]$ is zero, because $S(c)$ bounds the disc bundle in T' , and the second is $\pm j_*[c]$: for $\omega \in H^{i-2}(T')$ and the generator θ of $H^1(\mathbb{C}^*)$, integration over the fibre circles, on which g has degree one, gives $\langle \omega \cup g^* \theta, S(c) \rangle = \pm \langle j^* \omega, c \rangle$, and $H^*(T')$ is free. In the exact sequence

$$H_i(Y') \rightarrow H_i(T') \rightarrow H_i(T', Y') \rightarrow H_{i-1}(Y') \rightarrow H_{i-1}(T')$$

the first map is the composite of Φ_* with the projection, which is onto for $i \leq n$. So for $i \leq n$ the connecting map is injective, with image the kernel of $H_{i-1}(Y') \rightarrow H_{i-1}(T')$; for $i - 1 < n$ this kernel is the preimage under the isomorphism Φ_* of the second Künneth summand. The composite $H_{i-2}(W') \rightarrow H_{i-2}(T') \otimes H_1(\mathbb{C}^*) = H_{i-2}(T')$ is therefore an isomorphism, and it is $\pm j_*$. Put $k = i - 2$. $\square$

Proof of Proposition 3.5(1). The manifold $\widehat{X}$ is compact, complex and of dimension three, hence a closed oriented 6-manifold. Let $D = \widehat{X} \setminus W$. The toric boundary of $\mathbb{P}_\Delta$ is the union of the divisors D_y^Δ of the vertices y of Δ , so D is the union of the surfaces D_y and of the exceptional set of $\widehat{X} \rightarrow X$. Over a square the exceptional set is C_q ; over a pentagon or a hexagon G it is the fibre over x_G in the local toric model, the union of D_o and of the curves

of the diagonals not through o . In the local model each such curve lies in the toric divisor of an endpoint of its diagonal, and $C_q \subset D_a$ at a square. Hence $D = \bigcup_{r \in R} D_r$.

Step A: the map $j_*: H_2(W; \mathbb{Z}) \rightarrow H_2(\hat{X}; \mathbb{Z})$ is zero. The Laurent polynomial of X is nondegenerate with Newton polytope Δ° of dimension four, so the inclusion induces an isomorphism $i_*: H_2(W; \mathbb{Z}) \rightarrow H_2(T; \mathbb{Z}) = \Lambda^2 N$ (Lemma 3.7). Let y be a vertex of Δ , a primitive vector, and choose a splitting $N = \mathbb{Z}y \oplus N_0$. The affine toric variety of the ray through y is $U_y = T \cup O(y) \cong \mathbb{C}_t \times (N_0 \otimes \mathbb{C}^*)$, with $O(y) = \{t = 0\}$, and the factor $\mathbb{C}_t^*$ of T is the one-parameter subgroup of y . The surface $X \cap O(y)$ is the zero set of the face polynomial of the facet $\{y\}^\circ$ of Δ° , a nondegenerate Laurent polynomial on $O(y) \cong N_0 \otimes \mathbb{C}^*$ whose Newton polytope, a translate of that facet, has dimension three. By Lemma 3.7 with $n = 3$, for every $n_0 \in N_0$ there is a loop $\gamma \subset X \cap O(y)$ with class n_0 in $H_1(O(y)) = N_0$. Since X is transverse to $O(y)$ and U_y is smooth, $t|_X$ is a submersion along $X \cap O(y)$, so there are $\epsilon > 0$ and an embedding $\Phi: \gamma \times D_\epsilon \rightarrow X \cap U_y$ with $\Phi(p, s) = (s, \Phi_0(p, s))$ and $\Phi_0(p, 0) = p$. The 3-chain $\Phi(\gamma \times D_\epsilon)$ lies in $\hat{X}$, and its boundary $\tau = \Phi(\gamma \times \partial D_\epsilon)$ lies in W . The maps $h_u(p, \vartheta) = (\epsilon e^{i\vartheta}, \Phi_0(p, u\epsilon e^{i\vartheta}))$, $u \in [0, 1]$, take values in T and join τ to the product of the circle $\{|t| = \epsilon\}$ with γ ; so $i_*[\tau] = \pm y \wedge n_0$ in $H_2(T) = \Lambda^2 N$, and $y \wedge m = y \wedge n_0$ for $m = ay + n_0$. Hence the class of $H_2(W)$ corresponding to $\pm y \wedge n_0$ is a boundary in $\hat{X}$. Take a unimodular triple y_1, y_2, y_3 of some two-face (Lemma 3.6); the y_i are vertices of Δ , and the classes $y_i \wedge n$ span $\Lambda^2 N$. So $H_2(W; \mathbb{Z})$ is spanned by classes that vanish in $H_2(\hat{X}; \mathbb{Z})$.

Step B. By Step A and the exact sequence of the pair $(\hat{X}, W)$, the map $H_2(\hat{X}; \mathbb{Z}) \rightarrow H_2(\hat{X}, W; \mathbb{Z})$ is injective.

Step C: $H_2(\hat{X}, W; \mathbb{Z}) \cong \mathbb{Z}^R$, and the composite with Step B is ev_D . Triangulate $\hat{X}$ so that every D_r is a subcomplex [Loj64, Thm. 2]; that theorem applies because $\hat{X}$, a compact complex manifold, is a real-analytic manifold with countable basis and the D_r are analytic, hence semi-analytic, subsets. Then D is a compact subcomplex, hence locally contractible, and $W = \hat{X} \setminus D$. The duality $H_i(M, M \setminus K; \mathbb{Z}) \cong H^{n-i}(K; \mathbb{Z})$ for a compact, locally contractible subset K of a closed oriented n -manifold M [Hat02, Thm. 3.44] gives

$$H_2(\hat{X}, \hat{X} \setminus D; \mathbb{Z}) \cong H^4(D; \mathbb{Z}).$$

The isomorphism is the direct limit over open neighbourhoods $U \supset D$ of cap products with the image of the fundamental class of $\hat{X}$ (loc. cit., proof of Thm. 3.44); by naturality of the cap product it commutes, for a compact subset $D' \subset D$, with the maps $H_2(\hat{X}, \hat{X} \setminus D) \rightarrow H_2(\hat{X}, \hat{X} \setminus D')$ and $H^4(D) \rightarrow H^4(D')$.

Each D_r is a compact, smooth and connected complex surface. For a centre o , D_o is the orbit closure of the ray through o in the smooth local toric model over x_G , a smooth complete toric surface. For a vertex y , D_y is smooth near the fibres over the points x_G , where it is a toric divisor of the local model, and elsewhere because X is transverse to the orbits of the cones over vertices, edges and unimodular triangles, which are smooth. Its open subset $X \cap O(y)$ is connected by Lemma 3.7, and it is dense: $X \cap D_y^\Delta$ has pure dimension two and meets every smaller orbit in dimension at most one. So D_y is connected. Hence $H^4(D_r; \mathbb{Z}) = \mathbb{Z}$ for every $r \in R$.

Two distinct surfaces D_r meet in a union of compact curves and points, so every intersection $(\bigcup_{s < r} D_s) \cap D_r$, for any ordering of R , has vanishing H^3 and H^4 . The Mayer–Vietoris sequence, applied $|R| - 1$ times, gives $H^4(D; \mathbb{Z}) \cong \bigoplus_{r \in R} H^4(D_r; \mathbb{Z}) = \mathbb{Z}^R$, the r -th component being restriction to D_r . By the naturality above, the r -th component of $H_2(\hat{X}; \mathbb{Z}) \rightarrow H_2(\hat{X}, \hat{X} \setminus D; \mathbb{Z}) \cong \mathbb{Z}^R$ is the map $H_2(\hat{X}) \rightarrow H_2(\hat{X}, \hat{X} \setminus D_r) \cong H^4(D_r) = \mathbb{Z}$.

Through the Thom isomorphism of the normal bundle of the smooth surface D_r , this map is the intersection number with D_r up to sign, $C \mapsto \pm D_r \cdot C$, where the sign is fixed by the orientation conventions for the cap product and the Thom isomorphism and is the same for every r . So the composite of Step B with this isomorphism is $\pm \text{ev}_D$, and ev_D is injective. $\square$

Proof of Proposition 3.9. The fibre over x_G in the local model is the orbit closure of the ray through o , the toric surface of the star of o , whose fan in $\text{span}_{\mathbb{R}}(G)/\mathbb{R}o \cong N_G \otimes \mathbb{R}$ has the rays $w_j - o$ (the isomorphism holds because $\langle u, o \rangle = -1$ and $N_G \subset u^\perp$, so that $N \cap \text{span}_{\mathbb{R}}(G) = N_G \oplus \mathbb{Z}o$) and the two-dimensional cones over the T_j . By Lemma 3.1(1) consecutive rays form lattice bases, so E is smooth, and $f_j^2 = -a_j$ where $(w_{j-1} - o) + (w_{j+1} - o) = a_j(w_j - o)$. For the hexagon $a_j = 1$ for every j ; contracting the disjoint (-1) -curves f_0, f_2, f_4 leaves the fan with rays w_1, w_3, w_5 , which is that of $\mathbb{P}^2$, and gives (3.4). For the pentagon $(a_0, \dots, a_4) = (1, 1, 1, 0, 0)$; contracting f_0 and f_2 leaves the rays w_1, w_3, w_4 of $\mathbb{P}^2$ and gives (3.5). The exceptional curve of P_j is the curve of the wall cone(o, w_{j+1}) (proof of Proposition 3.4(2)), which is $D_o \cap D_{w_{j+1}} = f_{j+1}$. In the three cases this gives the curves listed; in (2), exactly one of $j+1, j+4$ is even, and $f_{2m} = e_{m+1}$, while f_{2m+1} is a line through two of the blown-up points.

The fibre E' is the orbit closure of the ray through o in $U'_{G,\sigma}$: the toric surface whose fan in N_G has the rays $w_j - o$ for the interior edges $[o, w_j]$ of $\sigma|_G$ and the two-dimensional cones over the cells of $\sigma|_G$. The edge vectors of a unit parallelogram form a lattice basis, so the fan is smooth. For the pentagon and for A_2 it has three rays and $E' \cong \mathbb{P}^2$. For A_1 the rays are $w_j - o, w_{j+2} - o, w_{j+3} - o = -(w_j - o)$ and $w_{j+5} - o = -(w_{j+2} - o)$, as $w_i + w_{i+3} = 2o$ in the hexagon; so $E' \cong \mathbb{P}^1 \times \mathbb{P}^1$. The node of a parallelogram is the torus-fixed point of E' of the cone over it. $\square$

APPENDIX C. THE COMPLEX OF DUAL CELL PAIRS

This appendix proves Lemmas 4.2, 4.3 and 4.4 and parts (2) and (3) of Proposition 4.7, and the piecewise linear details of Proposition 4.7(4) and Lemmas 6.6(2) and 6.7(2). Notation is that of Section 4; $\mathcal{F}$ is a fan-side subdivision with function h (Definition 4.1). That pulling a point preserves the regularity of a subdivision, which the construction of $\mathcal{F}_C$ uses, is part of Proposition 3.26.

C.1. Faces, cells and charts.

Proof of Lemma 4.2. (1) A function that is piecewise linear and strictly convex on a complete fan is the negative of the support function of a polytope whose normal fan is that fan [CLS11, §6.1]; since $\check{h}'$ is strictly convex on the fan over $\mathcal{T}$, the polytope is $\nabla^{\check{h}'}$. (2) For a convex piecewise linear g the support function $m \mapsto \min_{\nabla^g} \langle m, \cdot \rangle$ equals $-g$. The function $\check{\varphi}$ is convex, with $\nabla^{\check{\varphi}} = (\Delta^\circ)^\circ = \Delta$. Support functions add under Minkowski sums and determine a polytope, so $\nabla^{\check{\varphi} + \check{h}'} = \Delta + \nabla^{\check{h}'}$. The normal fan of a Minkowski sum is the common refinement of the normal fans, and the fan over $\mathcal{T}$ refines the normal fan of Δ , the fan over the faces of Δ° ; the face of a sum selected by a functional μ is the sum of the faces selected by μ [Zie95, Lecture 7]. For $\mu = \sum_u \lambda_u u$ in the relative interior of cone(τ) and $n \in \Delta$, $\langle \mu, n \rangle \geq -\sum_u \lambda_u$ with equality if and only if $\langle u, n \rangle = -1$ for every vertex u of τ ; so the face of Δ selected by μ is $\tau^\vee$. $\square$

Proof of Lemma 4.3. (1) A vector (y, s) lies in $\tilde{\Sigma}_u$ if and only if $s \geq 0$ and u minimises $f(m) = \langle m, y \rangle + sh'(m)$ on Δ° . For $s = 0$ this says that y lies in the inner normal cone of Δ° at u , which is cone($u^\vee$). For $s = 1$, by the optimality condition for the convex function f on

the convex set Δ° , u is a minimiser if and only if $y \in -\partial\check{h}'(u) + \text{cone}(u^\vee)$; since $\check{h}'$ is convex and positively homogeneous, $-\partial\check{h}'(u) = \{w : \langle m, w \rangle \geq -\check{h}'(m) \ \forall m, \ \langle u, w \rangle = -\check{h}'(u)\} = \nabla_u^{\check{h}'}$. So the slice of $\tilde{\Sigma}_u$ at $s = 1$ is $\nabla_u^{\check{h}'} + \text{cone}(u^\vee)$, which is also the slice of the cone over Cay_u .

(2) Since Cay_u lies in an affine hyperplane not through 0, the fan $\tilde{\Sigma}'$ on $\tilde{\Sigma}_u$ is the cone over the regular subdivision of Cay_u with the heights above, and its mixed cones are the cones over the cells with vertices at both levels. By the Cayley trick [HRS00], [DLRS10, §9.2], a cell C of that subdivision with $C \cap (N_{\mathbb{R}} \times \{0\}) = C_0 \times \{0\}$ and $C \cap (N_{\mathbb{R}} \times \{1\}) = C_1 \times \{1\}$, both nonempty, corresponds to the cell $C_0 + C_1$ of the regular mixed subdivision of $u^\vee + \nabla_u^{\check{h}'} = F_u$ with the same heights, and this is a bijection. So the cells of $\mathcal{P}_X$ in F_u are the cells of that mixed subdivision. For $u \in \tau$, F_τ is the face of F_u selected by any μ in the relative interior of $\text{cone}(\tau)$, and μ selects $\tau^\vee$ in $u^\vee$ and $\nabla_\tau^{\check{h}'}$ in $\nabla_u^{\check{h}'}$ (Lemma 4.2); the lower faces of the lift of F_u lying over F_τ are the lower faces of the lift of F_τ . This gives the description on F_τ . A cell c is the projection of a unique bounded face $\hat{c}$ of $\hat{A}_\tau + \hat{\nabla}_\tau$, and if $(\zeta, 1)$ selects exactly $\hat{c}$, then $\hat{c} = \text{face}_{(\zeta, 1)} \hat{A}_\tau + \text{face}_{(\zeta, 1)} \hat{\nabla}_\tau$. The first summand is the lift $\hat{\omega}$ of a cell $\omega \in \mathcal{F}$, because h is piecewise linear and strictly convex on the fan over $\mathcal{F}$, so that the bounded faces of $\hat{A}_\tau$ are the lifts of the cells of $\mathcal{F}$ in $\tau^\vee$; the second is $\text{face}_\zeta \nabla_\tau^{\check{h}'} \times \{0\}$, a face $\nabla_{\tau'}^{\check{h}'} \times \{0\}$ with $\tau' \supseteq \tau$. The decomposition of a face of a Minkowski sum into faces of the summands is unique. $\square$

Proof of Lemma 4.4. (1) A simplex containing a given one has a smaller bottom cell, whose lower support is contained in $\{n\}$; so the set of simplices defining $\tilde{R}_n$ is closed under passing to larger simplices, and $\tilde{R}_n$ is open. Let b lie in the open simplex of $c_0 \subsetneq \cdots \subsetneq c_k$ with $\omega(c_0) = \{n\}$. Then $b \in \text{ri } c_k \subseteq \text{ri } F_{\tau(c_k)}$, so the facets through b are the F_u with u a vertex of $\tau(c_k)$, and $n \in \tau(c_0)^\vee \subseteq \tau(c_k)^\vee$ by Lemma 4.3(2); that is, $\langle u, n \rangle = -1$ for these u . Near b , $\nabla^{\check{h}}$ coincides with $b + T_b$, where $T_b = \{x : \langle u, x \rangle \geq 0 \ \forall u \in \tau(c_k)\}$ is its tangent cone, and $-n$ lies in the interior of T_b . Since $\langle u, n \rangle = -1$, the point $x + tn$ lies in $b + T_b$ if and only if $t \leq \min_u \langle u, x - b \rangle$; so every line $x + \mathbb{R}n$ meets $\partial(b + T_b)$ in exactly one point, at a parameter that depends continuously and piecewise linearly on the class of x , and π_n is a homeomorphism from $\partial(b + T_b)$ onto $N_{\mathbb{R}}/\mathbb{R}n$. On $\text{aff } F_u$ the projection is affine with linear part $u^\perp \rightarrow N_{\mathbb{R}}/\mathbb{R}n$, and $u^\perp \cap N \rightarrow N/\mathbb{Z}n$ is an isomorphism, with inverse $x \mapsto x + \langle u, x \rangle n$, as $\langle u, n \rangle = -1$.

(2) Let b lie in the open simplex of $c_0 \subsetneq \cdots \subsetneq c_k$. If $\tau(c_k)$ is a vertex u , then $b \in \text{ri } c_k \subseteq \text{int } F_u$; if $\omega(c_0)$ is a vertex n , then $b \in \tilde{R}_n$; otherwise the closed simplex lies in Γ_X . Conversely, the faces of a simplex of Γ_X correspond to subchains, which have a larger bottom and a smaller top, so their chains again have $\dim \omega \geq 1$ at the bottom and $\dim \tau \geq 1$ at the top; such an open simplex lies in no $\text{int } F_u$ and in no $\tilde{R}_n$.

(3) On $\text{int } F_u \cap \tilde{R}_n$ the two charts differ by the integral affine isomorphism of (1). Gross's chart on the interior of a maximal cell is the facet chart of its facet, and his chart at a vertex $n + w$ of $\mathcal{P}_X$ is π_n on the open star of the vertex in $\text{Bar } \mathcal{P}_X$, which lies in $\tilde{R}_n$ because every chain through the vertex starts with it. $\square$

C.2. The cells of the complex of dual cell pairs are balls. Fix $\tau \in \mathcal{T}$ and put $d_\tau = 3 - \dim \tau$, $A = \tau^\vee$ and $\nabla_\tau = \nabla_\tau^{\check{h}'}$. Let $V_\tau = \{x \in N_{\mathbb{R}} : \langle u, x \rangle = 0 \ \forall u \in \tau\}$, of dimension d_τ , with dual $V_\tau^* = M_{\mathbb{R}}/L_\tau$. Both A and ∇_τ lie in affine spaces with direction V_τ , and ∇_τ has dimension d_τ . An element $\zeta \in V_\tau^*$ is a linear function on V_τ , hence an affine function on each of these affine spaces up to an additive constant, so $\text{face}_\zeta Z$, the set of minimisers of ζ on Z , is well defined for $Z \subseteq A$, $Z \subseteq \nabla_\tau$ or $Z \subseteq F_\tau$; likewise $\text{face}_{(\zeta, s)}$ on subsets of $(\text{affine space}) \times \mathbb{R}$.

Write $\hat{A} = \hat{A}_\tau$, $\hat{\nabla} = \hat{\nabla}_\tau$, and for $\omega \in \mathcal{F}$, $\omega \subseteq A$, let $\hat{\omega}$ be its lift, a bounded face of $\hat{A}$. Define $D(\omega) = \{\zeta \in V_\tau^* : \text{face}_{(\zeta,1)} \hat{A} \supseteq \hat{\omega}\}$, $\text{cone}_\tau(\tau') = \text{the image of } \text{cone}(\tau') \text{ in } V_\tau^* \text{ } (\tau' \supseteq \tau)$,

and for a cell c of $\mathcal{P}_X$ in F_τ with lifted face $\hat{c}$ (Lemma 4.3), $c^* = \{\zeta \in V_\tau^* : \text{face}_{(\zeta,1)}(\hat{A} + \hat{\nabla}) \supseteq \hat{c}\}$.

- Lemma C.1** (one face). (1) *The $D(\omega)$, for $\omega \in \mathcal{F}$ with $\omega \subseteq A$, form a polyhedral complex $\mathcal{D}_\tau$ with support V_τ^* ; $\dim D(\omega) = d_\tau - \dim \omega$; $D(\omega')$ is a face of $D(\omega)$ if and only if $\omega' \supseteq \omega$; and $\text{ri } D(\omega) = \{\zeta : \text{face}_{(\zeta,1)} \hat{A} = \hat{\omega}\}$.*
- (2) *The $\text{cone}_\tau(\tau')$, $\tau' \supseteq \tau$, are pointed and form the normal fan of ∇_τ and of F_τ in V_τ^* ; $\text{cone}_\tau(\tau')$ is the normal cone of F_τ at its face $F_{\tau'}$.*
- (3) *$c \mapsto c^*$ is an inclusion-reversing bijection from the cells of $\mathcal{P}_X$ in F_τ onto the cells of a polyhedral complex $\mathcal{R}_\tau$ with support V_τ^* , with $\dim c + \dim c^* = d_\tau$ and $c^* = D(\omega(c)) \cap \text{cone}_\tau(\tau')$ for $c = \omega(c) + \nabla_{\tau'}^{h'}$.*
- (4) *$\text{ri } c^* \subseteq \text{ri } D(\omega(c))$. Hence $c^* \subseteq D(\omega)$ if and only if $\omega(c) \supseteq \omega$, and every $D(\omega)$ is the union of the cells of $\mathcal{R}_\tau$ it contains.*
- (5) *The recession cone of c^* is $\text{cone}_\tau(\tau(c))$. In particular c^* is unbounded if and only if c lies in the relative boundary of F_τ .*

Proof. We use the standard properties of the normal cones $\mathcal{N}_\Pi(\Phi) = \{\ell : \text{face}_\ell \Pi \supseteq \Phi\}$ of the faces Φ of a polyhedron Π in a real affine space W [Zie95, Lecture 7]: they form a fan whose support is the set of linear functions bounded below on Π ; $\dim \mathcal{N}_\Pi(\Phi) = \dim W - \dim \Phi$, also when Π is not full-dimensional; $\mathcal{N}_\Pi(\Phi')$ is a face of $\mathcal{N}_\Pi(\Phi)$ if and only if $\Phi' \supseteq \Phi$; and $\text{ri } \mathcal{N}_\Pi(\Phi)$ consists of the functions selecting exactly Φ .

(1) The recession cone of $\hat{A}$ is $\mathbb{R}_{\geq 0}(0, 1)$, so every $(\zeta, 1)$ is bounded below on it and selects a bounded face, the lift of a cell of $\mathcal{F}|_A$ (Lemma 4.3). A normal cone of a bounded face contains a vector with positive last coordinate and lies in $\{s \geq 0\}$, so its relative interior lies in $\{s > 0\}$ and its slice at $s = 1$ has one dimension less. The statements follow from those for normal cones, with W the product of $\mathbb{R}$ and the affine space with direction V_τ containing A , of dimension $d_\tau + 1$.

(2) By Lemma 4.2 the normal fans of $\nabla^{\tilde{h}}$ and of $\nabla^{\tilde{h}'}$ are the fan over $\mathcal{T}$, and F_τ and ∇_τ are their faces with normal cone $\text{cone}(\tau)$. The normal fan of a face Φ of a polytope Π , in the quotient of the dual space by the span of $\mathcal{N}_\Pi(\Phi)$, consists of the images of the $\mathcal{N}_\Pi(\Phi')$ for the faces $\Phi' \subseteq \Phi$; here the span of $\text{cone}(\tau)$ is L_τ . Since τ is a face of the simplex τ' , $\text{cone}(\tau') \cap L_\tau = \text{cone}(\tau)$, so the image of $\text{cone}(\tau')$ is pointed.

(3) Apply the properties of normal cones to $\Pi = \hat{A} + \hat{\nabla}$, which has dimension $d_\tau + 1$ because ∇_τ has dimension d_τ : the slices at $s = 1$ of the normal cones of its bounded faces form a polyhedral complex with support V_τ^* , inclusion-reversing in the faces, with $\dim c^* = d_\tau - \dim c$. The minimum of $(\zeta, 1)$ on Π is the sum of its minima on $\hat{A}$ and on $\hat{\nabla}$, and a point $x + y$ with $x \in \hat{A}$, $y \in \hat{\nabla}$ attains it if and only if x and y attain theirs. So $\text{face}_{(\zeta,1)} \Pi \supseteq \hat{\omega} + \nabla_{\tau'}^{h'} \times \{0\}$ if and only if $\text{face}_{(\zeta,1)} \hat{A} \supseteq \hat{\omega}$ and $\text{face}_\zeta \nabla_\tau \supseteq \nabla_{\tau'}^{h'}$, that is, $\zeta \in D(\omega) \cap \text{cone}_\tau(\tau')$.

(4) If $\zeta \in \text{ri } c^*$, then $(\zeta, 1)$ selects exactly $\hat{c} = \hat{\omega}(c) + \nabla_{\tau'}^{h'} \times \{0\}$, hence selects exactly $\hat{\omega}(c)$ in $\hat{A}$, so $\zeta \in \text{ri } D(\omega(c))$ by (1). Since the relative interiors of the cells of $\mathcal{R}_\tau$ and of $\mathcal{D}_\tau$ both partition V_τ^* , every $D(\omega)$ is the union of the c^* with $\text{ri } c^* \subseteq D(\omega)$, and $\text{ri } c^* \subseteq D(\omega)$ holds if and only if $D(\omega(c)) \subseteq D(\omega)$, that is, $\omega(c) \supseteq \omega$.

(5) The recession cone of the slice at $s = 1$ of the closed cone $\mathcal{N}_\Pi(\hat{c})$ is its slice at $s = 0$, $\{\zeta : \hat{c} \subseteq \text{face}_{(\zeta,0)} \Pi\}$. As Π projects onto F_τ and $(\zeta, 0)$ does not see the last coordinate, this

is $\{\zeta : c \subseteq \text{face}_\zeta F_\tau\}$, the normal cone of F_τ at the smallest face containing c , which is $F_{\tau(c)}$; by (2) it is $\text{cone}_\tau(\tau(c))$. It is zero if and only if $\tau(c) = \tau$, that is, $c \not\subseteq \partial F_\tau$. $\square$

For $\mathfrak{p} = (\tau, \omega) \in \text{DP}$ let $\mathcal{C}_\mathfrak{p}$ be the set of cells of $\mathcal{R}_\tau$ contained in $D(\omega)$, a polyhedral complex with support $D(\omega)$ by Lemma C.1(4); let $\mathcal{C}_\mathfrak{p}^\partial$ be its subcomplex of cells contained in the relative boundary $\partial D(\omega)$, and $\mathcal{C}_\mathfrak{p}^u$ the set of its unbounded cells. For a poset $\mathcal{C}$, $\text{Ord}(\mathcal{C})$ is its order complex, whose simplices are the chains; the order complex of a subposet is a subcomplex. Let $\text{Bar}^{\preceq \mathfrak{p}}$ and $\text{Bar}^{\prec \mathfrak{p}}$ be the sets of simplices of $\text{Bar } \mathcal{P}_X$ whose label is $\preceq \mathfrak{p}$, respectively $\prec \mathfrak{p}$.

Lemma C.2 (order complexes). *The label of a simplex is $\preceq$ the label of every simplex containing it, so $\text{Bar}^{\preceq \mathfrak{p}}$ and $\text{Bar}^{\prec \mathfrak{p}}$ are subcomplexes and $C_X(\mathfrak{p}) = |\text{Bar}^{\preceq \mathfrak{p}}| \setminus |\text{Bar}^{\prec \mathfrak{p}}|$. The map $c \mapsto c^*$ induces isomorphisms of simplicial complexes*

$$\text{Bar}^{\preceq \mathfrak{p}} \cong \text{Ord}(\mathcal{C}_\mathfrak{p}), \quad \text{Bar}^{\prec \mathfrak{p}} \cong \text{Ord}(\mathcal{C}_\mathfrak{p}^\partial) \cup \text{Ord}(\mathcal{C}_\mathfrak{p}^u).$$

Proof. A face of a simplex corresponds to a subchain, which has a smaller top and a larger bottom, so its face label and its lower support are larger: its label is $\preceq$. A chain has label $\preceq \mathfrak{p}$ if and only if $c_k \subseteq F_\tau$ and $\omega(c_0) \supseteq \omega$, that is, if and only if all its cells lie in $U_\mathfrak{p} = \{c \subseteq F_\tau : \omega(c) \supseteq \omega\}$, since face labels decrease and lower supports increase along a chain. By Lemma C.1(3),(4), $c \mapsto c^*$ is an order-reversing bijection from $U_\mathfrak{p}$ onto $\mathcal{C}_\mathfrak{p}$, and a poset and its opposite have the same order complex. The label of a chain in $U_\mathfrak{p}$ is $\prec \mathfrak{p}$ if and only if $\tau(c_k) \supsetneq \tau$ or $\omega(c_0) \supsetneq \omega$. The first condition says that c_k^* is unbounded (Lemma C.1(5)); since c_k^* is contained in every c_i^* , all cells of the image chain are then unbounded, and conversely. The second says that $D(\omega(c_0))$ is a proper face of $D(\omega)$, that is, $c_0^* \subseteq \partial D(\omega)$ (Lemma C.1(1),(4)); since c_0^* contains every c_i^* , the image chain then lies in $\mathcal{C}_\mathfrak{p}^\partial$, and conversely. $\square$

Lemma C.3 (truncation). *Let W be a finite-dimensional real vector space, $g: W \rightarrow \mathbb{R}_{\geq 0}$ convex and positively homogeneous with $g(x) > 0$ for $x \neq 0$, and $\mathcal{C}$ a finite polyhedral complex in W whose support is a convex polyhedron D of dimension k and such that g is linear on every cell. Choose R larger than the value of g at every vertex of every cell. Then $\text{Ord}(\mathcal{C})$ is isomorphic to a linear triangulation of the convex polytope $D_R = D \cap \{g \leq R\}$, of dimension k , and the isomorphism carries $\text{Ord}(\mathcal{C}^\partial) \cup \text{Ord}(\mathcal{C}^u)$ onto a triangulation of its boundary, where $\mathcal{C}^\partial$ is the subcomplex of cells in ∂D and $\mathcal{C}^u$ the set of unbounded cells. In particular $|\text{Ord}(\mathcal{C})|$ is a piecewise linear k -ball with boundary $|\text{Ord}(\mathcal{C}^\partial) \cup \text{Ord}(\mathcal{C}^u)|$.*

Proof. Let e be a cell and ℓ_e the linear function equal to g on e . For v in the recession cone of e and $x \in e$, $\ell_e(v) = \lim_{t \rightarrow \infty} g(x + tv)/t = g(v)$, which is positive for $v \neq 0$; so e contains no line, has a vertex, and $e_R = e \cap \{\ell_e \leq R\}$ is a polytope. If e is bounded it is the convex hull of its vertices, $g < R$ on e , and $e_R = e$. If e is unbounded, the hyperplane $\{\ell_e = R\}$ separates the vertices of e from points far out along a recession direction and meets $\text{ri } e$; so $e' = e \cap \{\ell_e = R\}$ is a facet of e_R with $\text{ri } e' \subseteq \text{ri } e$. Since the hyperplane contains no vertex, the faces of e_R are the f_R for the faces f of e and the f' for the unbounded faces f of e .

Realise $\text{Ord}(\mathcal{C})$ by sending e to its barycentre $b(e)$ if e is bounded and to a point $b(e) \in \text{ri } e'$ if e is unbounded, and a chain $e_0 < \dots < e_j$ to $\text{conv}\{b(e_0), \dots, b(e_j)\}$. We show by induction on $\dim e$ that the chains with top $\leq e$ give a triangulation of e_R in which the chains of unbounded cells triangulate e' . For bounded e this is the derived subdivision: e is the cone from its barycentre over $\partial e = \bigcup_{f < e} f$. For unbounded e , $b(e)$ lies in the relative interior of the facet e' of e_R , and e' is the only facet of e_R containing it. A polytope is the union of the cones from a point in the relative interior of one facet over the other facets, and

these cones meet along the cones over common faces: a ray from $b(e)$ inside e' leaves e_R through $\partial e'$, which is covered by other facets, and any other ray leaves e' at once and exits through a facet different from e' . The facets of e_R other than e' are the f_R for the facets f of e , so $e_R = b(e) * \bigcup_{f < e} f_R$ and likewise $e' = b(e) * \bigcup_{f < e} \text{unbounded } f'$, and the induction hypothesis applies to the f . The triangulations of the e_R agree on common faces, so they form a triangulation of $\bigcup_e e_R = D_R$. The chains in $\mathcal{C}^\partial$ triangulate $\partial D \cap \{g \leq R\}$, because a cell whose relative interior meets a proper face of D lies in it; and the chains in $\mathcal{C}^u$ triangulate $\bigcup_e e' = D \cap \{g = R\}$, as $g < R$ on a bounded cell.

D_R is convex and compact, as g is bounded below by a positive multiple of a norm. It meets $\{g < R\} \cap \text{ri } D$, near a vertex of a cell, so it has dimension k , and its relative boundary is $(\partial D \cap \{g \leq R\}) \cup (D \cap \{g = R\})$. A linearly triangulated convex k -polytope is a piecewise linear k -ball with boundary the triangulated relative boundary [RS72, Ch. 1–2]. $\square$

Proof of Proposition 4.7(2),(3). (2) Let $\mathbf{p} = (\tau, \omega)$. Apply Lemma C.3 to $\mathcal{C} = \mathcal{C}_{\mathbf{p}}$ in $W = V_\tau^*$, with $g(\zeta) = \langle \zeta, x_0 \rangle - \min_{y \in \nabla_\tau} \langle \zeta, y \rangle$ for a point x_0 in the relative interior of ∇_τ . This g is convex and positively homogeneous, positive off 0 because a nonzero $\zeta \in V_\tau^*$ is not constant on the d_τ -dimensional affine space of ∇_τ , and linear on every cone $\text{cone}_\tau(\tau')$, on which $\min_{\nabla_\tau} \zeta$ is attained at a fixed point of $\nabla_\tau^{\check{h}'}$. Every cell of $\mathcal{R}_\tau$ lies in some $\text{cone}_\tau(\tau')$ (Lemma C.1(3)), so g is linear on the cells of $\mathcal{C}_{\mathbf{p}}$, whose support $D(\omega)$ is a convex polyhedron of dimension $d_\tau - \dim \omega = \dim \mathbf{p}$. The polyhedron $D(\omega)$ contains lines when $\tau^\vee$ has dimension less than d_τ ; this is why the truncation uses g rather than a linear function. By Lemmas C.2 and C.3, $|\text{Bar}^{\preceq \mathbf{p}}|$ is a piecewise linear ball of dimension $\dim \mathbf{p}$ with boundary $|\text{Bar}^{\prec \mathbf{p}}|$. Every open simplex of $\text{Bar } \mathcal{P}_X$ has exactly one label, so $|\text{Bar}^{\preceq \mathbf{p}}| = \bigcup_{\mathbf{q} \preceq \mathbf{p}} C_X(\mathbf{q})$ and $|\text{Bar}^{\prec \mathbf{p}}| = \bigcup_{\mathbf{q} \prec \mathbf{p}} C_X(\mathbf{q})$, and $C_X(\mathbf{p})$ is the interior of the ball. As $\dim \mathbf{q} < \dim \mathbf{p}$ for $\mathbf{q} \prec \mathbf{p}$, this is a regular CW decomposition with face poset $(\text{DP}, \preceq)$.

(3) The arguments of this appendix use only the conclusions of Lemmas 4.2–4.4, which hold for B_Y with face label and lower support exchanged: on a face F_ω^* one uses the complex dual to $(\omega^\vee, \mathcal{T}|_{\omega^\vee}, \check{h})$ in $M_{\mathbb{R}}/L_\omega$ and the normal fan of $\nabla_\omega^{H'}$. The labels lab_Y take values in the same set DP with the same order, since both the definition of a dual cell pair and the order are symmetric in the two factors. $\square$

C.3. Piecewise linear details. *Uniqueness in Proposition 4.7(4).* If $\mathcal{L}_1, \mathcal{L}_2$ both map every $C_X(\mathbf{p})$ onto $C_Y(\mathbf{p})$, $\phi = \mathcal{L}_2^{-1} \mathcal{L}_1$ is a homeomorphism of B_X preserving every cell $C_X(\mathbf{p})$, hence every closed cell. Write $B_X^{(k)}$ for the union of the closed cells of dimension at most k , and identify each closed cell $C_X(\mathbf{p}) = \Psi_X(|\text{Ord}^{\preceq \mathbf{p}}|)$, through Ψ_X , with the cone from the vertex $\mathbf{p}$ over its boundary sphere. We show by induction on k that ϕ is isotopic, through cell-preserving homeomorphisms, to a homeomorphism ϕ_k that is the identity on $B_X^{(k)}$; for $k = 0$ take $\phi_0 = \phi$, which fixes the zero-cells. Suppose ϕ_{k-1} has been found. On each closed k -cell $\bar{e}$, ϕ_{k-1} fixes the boundary sphere, and the Alexander trick [Ale23] in the cone coordinates of $\bar{e}$ gives an isotopy I_s^e of $\bar{e}$ relative to $\partial \bar{e}$ with $I_0^e = \phi_{k-1}|_{\bar{e}}$ and $I_1^e = \text{id}$. These agree on the overlaps, which lie in $B_X^{(k-1)}$, and define an isotopy I_s of $B_X^{(k)}$ preserving every cell. Put $\theta_s = I_s \circ (\phi_{k-1}|_{B_X^{(k)}})^{-1}$, an isotopy of $B_X^{(k)}$ through cell-preserving homeomorphisms with $\theta_0 = \text{id}$, and extend it cellwise in increasing dimension to an isotopy Θ_s of B_X : on a closed cell of dimension $m > k$ on whose boundary sphere Θ_s is defined, let $\Theta_s(\lambda \mathbf{p} + (1 - \lambda)x) = \lambda \mathbf{p} + (1 - \lambda)\Theta_s(x)$ in the cone coordinates. Each Θ_s is a cell-preserving homeomorphism, $\Theta_0 = \text{id}$, and $\Theta_s \circ \phi_{k-1}$ runs from ϕ_{k-1} to $\phi_k = \Theta_1 \circ \phi_{k-1}$, which is the identity on $B_X^{(k)}$. For

$k = 3$ this gives an isotopy Ω_s from ϕ to the identity through cell-preserving homeomorphisms, and $\mathcal{L}_2 \circ \Omega_s$ is an isotopy from $\mathcal{L}_1$ to $\mathcal{L}_2$ through homeomorphisms with the property of (4).

Proof of Lemma 6.7(2). A chain lies in $\text{st } \mathfrak{p}_e$ exactly when its union with $\{\mathfrak{p}_e\}$ is a chain, that is, when it is contained in $\{z_-, z_+, \mathfrak{p}_e\} \cup \text{DP}_{\succ \mathfrak{p}_e}$ and contains at most one of the incomparable $z_{\pm}$. So $\text{st } \mathfrak{p}_e$ is the simplicial join $\{z_-, z_+\} * \{\mathfrak{p}_e\} * \text{Ord}(\text{DP}_{\succ \mathfrak{p}_e})$. By (1), ϵ identifies $\text{DP}_{\succ \mathfrak{p}_e}$ with the nonzero sign vectors, which form the face poset of $\partial \diamond$; so $\text{Ord}(\text{DP}_{\succ \mathfrak{p}_e})$ is isomorphic to the barycentric subdivision of $\partial \diamond$. The join of this triangulation with the vertex 0 and with the two vertices $\pm e_3$ is a triangulation of V : every point of V° with $\pm x_3 \geq 0$ is uniquely $(1-s)(\pm e_3) + sw$ with $s \in (0, 1]$ and $w \in \diamond$, and every $w \neq 0$ is uniquely ty with $t \in (0, 1]$ and $y \in \partial \diamond$. Let $E_e: |\text{st } \mathfrak{p}_e| \rightarrow V$ be the resulting simplicial isomorphism, which sends $z_{\pm}$ to $\pm e_3$ and $\mathfrak{p}_e$ to 0, and put $\Phi_e^X = E_e \circ \Psi_X^{-1}$, $\Phi_e^Y = E_e \circ \Psi_Y^{-1}$; then $\Phi_e^Y \circ \mathcal{L} = \Phi_e^X$. As in the proof of Lemma 6.6, Ψ_Y maps the union of the open simplices of the chains with top $\mathfrak{p}$ onto $C_Y(\mathfrak{p})$. In $\text{st } \mathfrak{p}_e$ a chain with top $\mathfrak{p} \succ \mathfrak{p}_e$ is $c_- \cup c_+$ with $c_- \subseteq \{z_{\pm}, \mathfrak{p}_e\}$ and $\emptyset \neq c_+ \subseteq \text{DP}_{\succ \mathfrak{p}_e}$ with top $\mathfrak{p}$, and the points of its open simplex are the points $(1-s)(\pm e_3) + sty$ with $s, t > 0$ and y in the open simplex of c_+ , which lies in the relative interior of the face of $\epsilon(\mathfrak{p})$; so $(x_1, x_2) = sty \in \mathcal{O}_{\epsilon(\mathfrak{p})}^{(2)}$, and every point of V° with $(x_1, x_2) \in \mathcal{O}_{\epsilon(\mathfrak{p})}^{(2)}$ arises in this way. The chains with top $\mathfrak{p}_e$ in $\text{st } \mathfrak{p}_e$ are $\{\mathfrak{p}_e\}$ and $\{z_{\pm}, \mathfrak{p}_e\}$, whose open simplices form the open segment between $-e_3$ and e_3 . The open star of $\mathfrak{p}_e$ is open in $|\text{Ord}(\text{DP})|$, contains the open arc and lies in $\text{st } \mathfrak{p}_e$. $\square$

Uniqueness in Lemma 6.6(2). Two homeomorphisms with the property differ by a homeomorphism χ of O preserving every $\mathcal{O}_\epsilon \cap O$. These sets are not the cells of a cell decomposition of O , since for $\epsilon \neq 0$ the set $\mathcal{O}_\epsilon \cap O$ contains $\mathcal{O}_\epsilon \cap \partial O = \text{ri face}(\epsilon)$, where $\text{face}(\epsilon) = \text{conv}\{\epsilon_i e_i : \epsilon_i \neq 0\}$. We use instead the regular cell decomposition of O with open cells $\{0\}$, the sets $\mathcal{O}_\epsilon \cap \text{int } O$ for $\epsilon \neq 0$, and the sets $\text{ri face}(\epsilon) \subseteq \partial O$; its closed cells are the simplices $\text{conv}(\{0\} \cup \text{face}(\epsilon))$ and $\text{face}(\epsilon)$. By invariance of domain χ maps $\text{int } O$ onto itself, hence ∂O onto itself, so it preserves every cell of this decomposition. The argument given above for Proposition 4.7(4), with the closed cells, which are simplices, as cones from their barycentres, shows that χ is isotopic to the identity through homeomorphisms preserving every cell, and hence every $\mathcal{O}_\epsilon \cap O$.

APPENDIX D. ESTIMATES FOR THE LOCALIZED MIRROR

This appendix proves Lemmas 7.5, 7.11, 7.14, 7.15, 7.18 and 7.26 of Section 7, together with the auxiliary Lemmas D.1, D.2, D.3, D.4, D.5 and D.6, under the standing assumptions of Section 7 and in its notation. At a point x of a chart U_c the Cox monomials are evaluated in the chart, and we put

$$\mathbf{m}(x) := \max_m |\gamma_m x^m(x)|, \quad \mathfrak{S}(x) := \sum_m |\gamma_m x^m(x)|^2,$$

so that $\mathbf{m} \leq \mathfrak{S}^{1/2} \leq |\mathcal{A}|^{1/2} \mathbf{m}$. The *transition set* $\text{Trans}(x)$ is the set of $n \neq 0$ with $\theta_1/4 \leq \tilde{d}_n(x) \leq \theta_1/2$; outside it each ψ_n is locally constant, equal to 1 or 0. We use $e_{\max}$, $\mathcal{E}_P$ (Definition 7.6), D and θ_2 (Section 7.2), and write $\beta'_{\max} = \max |\beta'|$. Constants C , c depend only on the configuration and on $\mathcal{C}_0$, unless a dependence on r or η_0 is indicated.

D.1. Deficits, cut-offs and a regularity criterion.

Proof of Lemma 7.5. A lower face of $\text{conv}\{(n, H(n))\}$ is the set of lifted points on which $n \mapsto \langle p, n \rangle - H(n)$ is maximal, for some $p \in M_{\mathbb{R}}$. So I is cellular if and only if $d_n^{\text{tr}}(p) = 0$ for

all $n \in I$ and some p , that is, $\gamma^*(I) = 0$. The function $p \mapsto \max_{n \in I} d_n^{\text{tr}}(p)$ is convex, piecewise linear and nonnegative, so its infimum is attained (it is the value of the linear program: minimise γ subject to $\langle p, m - n \rangle - H(m) + H(n) \leq \gamma$ for $n \in I$, $m \in \mathcal{A}$). Hence $\gamma^*(I) > 0$ for non-cellular I , and $\theta_0 > 0$ as a minimum over finitely many sets.

Let c_* bound $|\log |c_m/c_n||$ over the union of the sets $U_{\mathbb{C}}^+(c^{(0)})$, $c^{(0)} \in \mathcal{C}_0$, which is compact. At a torus point, $|d_n(x) - d_n^{\text{tr}}(p)| \leq 2c_*/L$. If $L \geq 8c_*/\theta_0$, the set $I = \{n : d_n(x) < 3\theta_0/4\}$ has $\max_I d_n^{\text{tr}}(p) < \theta_0$, so it is cellular. At a point x of the toric boundary, $d_n(x') \rightarrow d_n(x)$ for torus points $x' \rightarrow x$ (with limit ∞ where $x^n(x) = 0$), because the maximum is attained by a monomial not vanishing at x ; so $\{n : d_n(x) < 3\theta_0/4\} \subseteq \{n : d_n(x') < 3\theta_0/4\}$ for x' close to x , and subsets of cellular sets are cellular. $\square$

Lemma D.1 (derivatives of the cut-offs). *Let x lie in a chart $U_{\mathbb{C}}$.*

- (1) $d_n \leq \tilde{d}_n \leq d_n + \log |\mathcal{A}|/(2L)$, and $|\gamma_n x^n(x)| \leq t^{\theta_1/4} \mathfrak{S}(x)^{1/2}$ for $n \in \text{Trans}(x)$.
- (2) For $\ell \in M_{\mathbb{R}}$ let ∂_{ℓ} be the holomorphic vector field of the torus action with $\partial_{\ell} \chi^m = \langle \ell, m \rangle \chi^m$, and $v_{\ell} := 2 \text{Re } \partial_{\ell}$ the corresponding real field. Then $d\psi_n(Jv_{\ell}) = 0$ and $|d\psi_n(v_{\ell})| \leq \beta'_{\max} |\ell| D/L$.
- (3) Let Z be a holomorphic coordinate near x such that, after division of all monomials by one nowhere vanishing function, each monomial m is $Z^{e(m)}$ times a function independent of Z , with integers $0 \leq e(m) \leq e_{\max}$: a Cox coordinate of the chart, with $e(m)$ the Z -exponent of m , or one of the coordinates Z, Z' of Section 7.3 on $U_{u,u'}$, after division by $\gamma_a x^a$, with $e = e_u$ or $e = e_{u'}$. Moduli of monomials below are taken after this division, and $v = \partial/\partial \text{Re } Z$. Then, where $Z \neq 0$, $\partial_Z \tilde{d}_n = (\bar{e} - e(n))/(2LZ)$ with $\bar{e} = \sum_m p_m e(m)$, $p_m = |\gamma_m x^m|^2/\mathfrak{S}$, and

$$\sum_n |\gamma_n x^n| (|d\tilde{\psi}_n(v)| + |d\tilde{\psi}_n(Jv)|) \leq \frac{2\beta'_{\max}}{L} \left[\left(\sum_{\substack{n \in \text{Trans} \\ e(n)=0}} \frac{|\gamma_n x^n|}{\mathfrak{S}^{1/2}} \right) \mathcal{D}_Z + \sum_{\substack{n \in \text{Trans} \\ e(n) \geq 1}} (e(n) + e_{\max}) \left| \frac{\gamma_n x^n}{Z} \right| \right],$$

where $\mathcal{D}_Z := \sum_m e(m) |\gamma_m x^m|/Z$. Both sides of the inequality are continuous, since every term with $e(m) \geq 1$ is divisible by Z , so it holds also where $Z = 0$.

Proof. (1) $\max_m |\gamma_m x^m|^2 \leq \mathfrak{S} \leq |\mathcal{A}| \max_m |\gamma_m x^m|^2$. If $\tilde{d}_n \geq \theta_1/4$ then $|\gamma_n x^n|^2 \leq t^{\theta_1/2} \mathfrak{S}$. (2) The weights depend only on the moduli, which the compact torus preserves, so $d\psi_n(Jv_{\ell}) = 0$. Since $v_{\ell} \log |\chi^m|^2 = 2\langle \ell, m \rangle$, $v_{\ell} \tilde{d}_n = \frac{1}{L} (\sum_m p_m \langle \ell, m \rangle - \langle \ell, n \rangle)$, of modulus at most $|\ell| D/L$. (3) The common factor cancels in $\tilde{d}_n$, and it multiplies both sides of the inequality by its modulus. After the division, $|\gamma_m x^m|^2$ depends on Z through $|Z|^{2e(m)}$, so $\partial_Z \log |\gamma_m x^m|^2 = e(m)/Z$ where $Z \neq 0$, which gives the formula. For a real function ϕ , $|d\phi(v)| + |d\phi(Jv)| \leq 4|\partial_Z \phi|$, and $d\tilde{\psi}_n = (1-s)\beta'(\tilde{d}_n) d\tilde{d}_n$ vanishes unless $n \in \text{Trans}$ (the origin has constant weight). For $e(n) = 0$, $|\bar{e}| |\gamma_n x^n|/|Z| \leq (|\gamma_n x^n|/\mathfrak{S}^{1/2}) \sum_m e(m) |\gamma_m x^m|/|Z|$ because $p_m \leq |\gamma_m x^m|/\mathfrak{S}^{1/2}$; for $e(n) \geq 1$, $|\bar{e} - e(n)| \leq e(n) + e_{\max}$. $\square$

Lemma D.2 (regularity criterion). *Let $g_1, \dots, g_k$ be holomorphic and $\phi_1, \dots, \phi_k$ smooth real functions near a point x of $\mathbb{C}^4$, and $F = \sum_i \phi_i g_i$. Let v be a real tangent vector at x , and put*

$$h := \sum_i \phi_i(x) dg_i(v), \quad \mathcal{J} := \sum_i |g_i(x)| (|d\phi_i(v)| + |d\phi_i(Jv)|).$$

If $|h| > \mathcal{J}$, then dF_x maps $\text{span}_{\mathbb{R}}(v, Jv)$ onto $\mathbb{C}$; in particular dF_x has real rank two.

Proof. For $w = \lambda v + \mu Jv$, $dg_i(w) = (\lambda + i\mu) dg_i(v)$, so $dF(w) = (\lambda + i\mu)h + \sum_i g_i(x) (\lambda d\phi_i(v) + \mu d\phi_i(Jv))$. The second term has modulus at most $\mathcal{J}|\lambda + i\mu|$. So $dF(w) \neq 0$ for $w \neq 0$, and the $\mathbb{R}$ -linear map $\mathbb{R}^2 \rightarrow \mathbb{C}$ is injective, hence onto. $\square$

On $X_{\mathcal{T}}$ we apply Lemma D.2 in a chart, to $F^{(s)}$ divided by a Cox monomial that does not vanish at x , with $g_n = \gamma_n x^n$ divided by the same monomial and $\phi_n = \tilde{\psi}_n$; at a zero of $F^{(s)}$ this division does not affect the rank.

D.2. Smoothness of the interpolating family: proof of Lemma 7.11. Let $x \in W_s$, and put $\mathcal{C}^\circ(x) := \mathcal{C}(x) \setminus \{0\}$ and $\mathcal{H}(x)$ the smallest face of Δ containing $\mathcal{C}^\circ(x)$.

Lemma D.3 (the cases). *At least one of the following holds: (A) $\mathcal{C}(x)$ is affinely independent; (B) $\mathcal{H}(x)$ is a facet $u^\vee$; (C) $\mathcal{C}^\circ(x)$ is a node parallelogram P , and then $x \in \mathcal{U}_P$.*

Proof. By Lemmas 7.5 and 7.1, $\mathcal{C}(x) \subseteq \{0\} \cup \omega$ for a cell ω of $\mathcal{F}_C$, so $\mathcal{C}^\circ(x) \subseteq \omega \subseteq \partial\Delta$ and $\mathcal{H}(x)$ is a proper face. If it is not a facet, it has dimension at most two, and $\mathcal{C}^\circ(x)$ lies in the cell $\omega \cap \mathcal{H}(x)$ of the subdivision of $\mathcal{H}(x)$ induced by $\mathcal{F}_C$. Since $\mathcal{F}_C$ agrees with Σ' on the two-skeleton (Proposition 3.26), this cell is a cell of σ (Definition 2.3): a point, a primitive segment, a unimodular triangle, a unit square, or one of the parallelograms and triangles of the subdivision of a pentagon or of a hexagon, on either branch. Each of these has no lattice points other than its vertices, and a set of vertices of such a cell is affinely independent unless it is all four vertices of a node parallelogram. Since $0 \notin \text{aff } \mathcal{H}(x)$, $\mathcal{C}(x)$ is affinely independent if and only if $\mathcal{C}^\circ(x)$ is. If $\mathcal{C}^\circ(x) = P$, the monomials of P do not vanish at x , so the Cox coordinates vanishing at x are among those of the vertices $\bar{u}$ of $\mathcal{T}$ with $P \subseteq \bar{u}^\vee$, that is $\bar{u} \in G^\circ = \{u, u'\}$; hence $x \in U_{u,u'}$. $\square$

Case (A). Let n_0 be a maximal monomial at x ; it lies in $\mathcal{C}(x)$ and has $\tilde{\psi}_{n_0} = 1$. Put $f := F^{(s)}/(\gamma_{n_0} x^{n_0}) = \sum_n \tilde{\psi}_n (\gamma_n/\gamma_{n_0}) \chi^{n-n_0}$, regular near x . Since $|f| = 0$ at x and the monomials outside the core contribute at most $|\mathcal{A}|t^{\theta_0/2}$,

$$\sum_{n \in \mathcal{C}(x) \setminus \{n_0\}} \tilde{\psi}_n \frac{|\gamma_n x^n|}{\mathbf{m}} \geq 1 - |\mathcal{A}|t^{\theta_0/2},$$

so some $n_1 \in \mathcal{C}(x) \setminus \{n_0\}$ has $\tilde{\psi}_{n_1} |\gamma_{n_1} x^{n_1}| \geq \mathbf{m}/(2|\mathcal{A}|)$. Since $\mathcal{C}(x)$ is affinely independent, there is $\ell \in M_{\mathbb{R}}$ with $\langle \ell, n - n_0 \rangle = 0$ for $n \in \mathcal{C}(x) \setminus \{n_0, n_1\}$ and $\langle \ell, n_1 - n_0 \rangle = 1$; among the finitely many affinely independent subsets of $\mathcal{A}$ we may take $|\ell| \leq \ell_{\max}$. With $v = v_\ell$ and $g_n = (\gamma_n/\gamma_{n_0}) \chi^{n-n_0}$,

$$|h| \geq \frac{1}{2|\mathcal{A}|} - \ell_{\max} D |\mathcal{A}| t^{\theta_0/2}, \quad \mathcal{J} \leq \frac{\beta'_{\max} \ell_{\max} D}{L} |\mathcal{A}|^{3/2} t^{\theta_1/4}$$

by Lemma D.1(1),(2). For t small, $|h| > \mathcal{J}$, and Lemma D.2 applies. (In particular $v_\ell(x) \neq 0$.)

Case (B). We use the following domination.

Lemma D.4 (domination on a facet). *Let $\mathcal{H}(x) = u^\vee$ be a facet. Then x lies in the chart U_c of every maximal cone $c \ni u$, the Cox coordinates of c other than Z_u do not vanish at x , and every $n \notin u^\vee \cup \{0\}$ satisfies*

$$|\gamma_n x^n / Z_u| \leq t^{3\theta_0/4} |\gamma_0 x^0 / Z_u| \quad \text{at } x,$$

both sides being polynomials in the chart coordinates.

Proof. If $Z_{\bar{u}}(x) = 0$, every monomial not vanishing at x lies in $\bar{u}^\vee$, so $\bar{u}^\vee \supseteq \mathcal{C}^\circ(x)$ and $\bar{u}^\vee \supseteq u^\vee$; as $u^\vee$ is a facet, $\bar{u}^\vee = u^\vee$ and $\bar{u} = u$. This gives the first two assertions. Every cell of $\mathcal{F}_C$ containing $\mathcal{C}^\circ(x)$ lies in a proper face containing $u^\vee$, hence in $u^\vee$; so for $n \notin u^\vee \cup \{0\}$ no set $\mathcal{C}^\circ(x) \cup \{n\}$ or $\mathcal{C}^\circ(x) \cup \{0, n\}$ is cellular.

Assume first $Z_u(x) \neq 0$, and let x_τ be the point with Z_u multiplied by $e^{\tau L}$ and the other chart coordinates fixed; a monomial is multiplied by $e^{\tau L e_u(n)}$, and those of $u^\vee$ are unchanged.

If the maximum at x is attained by a monomial with $e_u \geq 1$, that monomial lies in $\mathcal{C}(x)$ but not in $\mathcal{C}^\circ(x) \subseteq u^\vee$, so it is the origin; then for $n \notin u^\vee \cup \{0\}$ the set $\mathcal{C}(x) \cup \{n\}$ is not cellular, Lemma 7.5 gives $d_n(x) \geq 3\theta_0/4$, and $|\gamma_n x^n| \leq t^{3\theta_0/4} \mathbf{m} = t^{3\theta_0/4} |\gamma_0 x^0|$. Otherwise the maximum $\mathbf{m}_u$ is attained only in $u^\vee$. Let $\tau_* > 0$ be the least τ at which a monomial with $e_u \geq 1$ reaches $\mathbf{m}_u$. For $\tau \leq \tau_*$ the maximum is $\mathbf{m}_u$ and $\mathcal{C}^\circ(x) \subseteq \mathcal{C}(x_\tau)$; at τ_* the monomial reaching $\mathbf{m}_u$ has deficit 0, so it forms a cellular set with $\mathcal{C}^\circ(x)$ and is the origin. By the first case at x_{τ_*} , $|\gamma_n x^n| \leq t^{3\theta_0/4} |\gamma_0 x^0|$ there, and going back from x_{τ_*} to x multiplies the ratio $|\gamma_n x^n|/|\gamma_0 x^0|$ by $e^{-\tau_* L(e_u(n)-1)} \leq 1$ when $e_u(n) \geq 1$; when $e_u(n) = 0$, $n \in u^\vee$. If $Z_u(x) = 0$, points x' near x with $Z_u(x') \neq 0$ and the same other coordinates have $\mathcal{C}^\circ(x) \subseteq \mathcal{C}(x')$ and $\mathcal{H}(x') = u^\vee$, so the inequality holds at x' , and by continuity at x . $\square$

In case (B) apply Lemma D.2 with $v = \partial/\partial \operatorname{Re} Z_u$. The derivative h of that lemma, in which the cut-offs are held constant, is

$$h = \gamma_0 \partial_{Z_u} x^0 + \sum_{n \notin u^\vee \cup \{0\}} \tilde{\psi}_n e_u(n) \frac{\gamma_n x^n}{Z_u}, \quad |h - \gamma_0 \partial_{Z_u} x^0| \leq |\mathcal{A}| e_{\max} t^{3\theta_0/4} |\gamma_0 \partial_{Z_u} x^0|,$$

and $\gamma_0 \partial_{Z_u} x^0 = \gamma_0 \prod_{\bar{u} \in \mathcal{C} \setminus \{u\}} Z_{\bar{u}} \neq 0$ by Lemma D.4. In Lemma D.1(3), $\mathcal{D}_{Z_u}$ is at most $(1 + |\mathcal{A}| e_{\max} t^{3\theta_0/4}) |\gamma_0 \partial_{Z_u} x^0|$; the transition monomials with $e_u = 0$ have $|\gamma_n x^n|/\mathfrak{S}^{1/2} \leq t^{\theta_1/4}$; and those with $e_u \geq 1$ are not the origin, so their contribution is at most $2e_{\max} |\mathcal{A}| t^{3\theta_0/4} |\gamma_0 \partial_{Z_u} x^0|$. Hence $\mathcal{J} \leq (C/L)(t^{\theta_1/4} + t^{3\theta_0/4})|h| < |h|$ for t small. No condition on η or on the binomial ratios of the cells is used.

Case (C) is Lemma 7.15(1), proved in Appendix D.3.

Proof of Lemma 7.11. (1) By Lemma D.3 every point of W_s falls under (A), (B) or (C), and in each case $dF^{(s)}$ has real rank two. The thresholds on t are finitely many and depend on the configuration, on $\mathcal{C}_0$ and, in case (C), on r and η_0 . At $s = 1$ the section is holomorphic, so $W_1 = Y_t$ is a complex hypersurface, smooth by the above.

(2) At $s = 1$ all weights equal 1 and $\mathcal{J} = 0$. The arguments of cases (A) and (B) use only the two-sided bounds on $|c_n|$, so they apply to every $c \in U_{\mathbb{C}}^+$ and show that $Y_{t,c}$ is smooth outside the sets $\mathcal{U}_P$. In $\mathcal{U}_P$, Lemma 7.15(2) places the singular points in $\mathbb{D}_P$, where by Lemma 7.14(2) the only critical point of $f^{(1)}$ is q_P ; it lies on $Y_{t,c}$ exactly when $\varepsilon_P = 0$, and is then a singular point. $\square$

D.3. Near a node. Fix a node P and use the chart (Z, Z', χ_1, χ_2) and the notation of Section 7.3; moduli of monomials are taken relative to $|\gamma_a x^a|$. Being within ϵ of the node point is defined in Section 7.3: $|s_1|, |s_2| \leq \epsilon L$, with $s_1 = \log |\mu_P \tilde{\chi}_1|$ and $s_2 = \log |\mu_P \tilde{\chi}_2|$.

Lemma D.5 (weighted parallelograms). *Let $m_0 = 1$, $m_1 = \tilde{\chi}_1$, $m_2 = \tilde{\chi}_2$, $m_3 = \mu_P \tilde{\chi}_1 \tilde{\chi}_2$ at a point, and let $p_0, \dots, p_3 \in [0, 1]$ with $p_i \geq p_j$ whenever $|m_i| \geq |m_j|$. Put $E = \max_i p_i |m_i|$, $\tilde{g} = \sum_i p_i m_i$, $D_1 \tilde{g} = p_1 m_1 + p_3 m_3$ and $D_2 \tilde{g} = p_2 m_2 + p_3 m_3$. If $\max_i |m_i| \geq e^\epsilon \min_i |m_i|$ with $\epsilon > 0$, and $|\tilde{g}| \leq cE$ with $c = (1 - e^{-\epsilon})/8$, then $\max(|D_1 \tilde{g}|, |D_2 \tilde{g}|) \geq cE$.*

Proof. Suppose $|D_1 \tilde{g}|, |D_2 \tilde{g}| < cE$. Since $p_0 m_0 - p_3 m_3 = \tilde{g} - D_1 \tilde{g} - D_2 \tilde{g}$, we get $|p_0 m_0 - p_3 m_3| < 3cE$, and together with $|p_1 m_1 + p_3 m_3|, |p_2 m_2 + p_3 m_3| < cE$ all four $p_i |m_i|$ lie within $4cE$ of each other; as their maximum is E , each is at least $(1 - 4c)E$. Take i with $|m_i|$ maximal and j with $|m_j|$ minimal. Then $p_i \geq p_j$ and $E \geq p_i |m_i| \geq e^\epsilon p_j |m_j| \geq e^\epsilon (1 - 4c)E = \frac{1}{2}(e^\epsilon + 1)E > E$, a contradiction. $\square$

The weights $\tilde{\psi}_n$ of the four monomials of P satisfy the monotonicity of Lemma D.5, because $\tilde{d}_n$ is a decreasing function of $|\gamma_n x^n|$ at fixed x and β is nonincreasing.

Proof of Lemma 7.14. (1) On $U_{\mathbb{C}}^+$ we have $|\eta_P| \leq 2\eta_0 \leq r^2/10 \leq 1/160$, so $|\mu_P^{\pm 1/2}| \leq e^{1/320} < 1.01$. On $2\mathbb{D}_P$, $|x|, |y| < 2r \leq 1/2$, hence $|\mu_P^{1/2}x|, |\mu_P^{1/2}y| < 0.51$, and $\mu_P\tilde{\chi}_1 = \mu_P^{1/2}x - 1$, $\mu_P\tilde{\chi}_2 = \mu_P^{1/2}y - 1$ have moduli in $[0.49, 1.51]$. So $\chi_1, \chi_2 \neq 0$: the polydisc $2\mathbb{D}_P$ lies in the chart $U_{u,u'}$ (this is where $r \leq r_1 \leq 1/4$ is used), and the four monomials of P , which relative to $\gamma_a x^a$ are $1, \tilde{\chi}_1, \tilde{\chi}_2, \mu_P\tilde{\chi}_1\tilde{\chi}_2$, have moduli within a factor 16 of each other. The origin has modulus $|\hat{Z}\hat{Z}'| \leq 4$. Moving from $x = y = 0$ to $2\mathbb{D}_P$ changes the modulus of $\chi_1^{\alpha_n}\chi_2^{\beta_n}$ by a factor at most $3^{|n|_P}$. Thus each apex has modulus at most $2 \cdot 3^{A_P}\lambda_P^{-1/2} \leq Ct^{\mathfrak{g}_P/2}$.

To bound the other monomials, let z_* have $x = y = 0$, $Z = \nu_Z$ and $Z' = \nu_{Z'}$. Its tropical position is within C/L of the midpoint of J_P : the four monomials of P have bounded nonzero moduli, the origin has modulus one, and the largest apex modulus on each side is $\lambda_P^{-1/2}$. Lemma 7.7(1),(2) therefore places it at the midpoint up to C/L . For $n \notin \mathcal{A}_P$ the set $P \cup \{n\}$ is not cellular: a cell of $\mathcal{F}_C$ containing P lies in G , $u^\vee$ or $u'^\vee$, so it is P , ω_1 or ω_2 , and n lies in none of them. This includes the lattice points of $G \setminus P$, the other vertices of a pentagon or hexagon, because $\omega_1 \cap G = \omega_2 \cap G = P$. At the midpoint the tropical deficits of P vanish, so the deficit of n is at least $\gamma^*(P \cup \{n\}) \geq \theta_0$. Its relative modulus at z_* is consequently at most Ct^{θ_0} . Since the Z, Z' exponents are nonnegative, throughout $2\mathbb{D}_P$ its relative modulus is at most $C3^{|n|_P}2^{e_u(n)+e_{u'}(n)}t^{\theta_0}$. The finite configuration now gives $\mathfrak{m} \leq C$ without an assumption on the remaining monomials. Hence the deficits of P are at most C/L , those of the apexes are at least $\mathfrak{g}_P/2 - C/L \geq (\theta_0 + 1)/2 - C/L$, and those of $n \notin \mathcal{A}_P$ are at least $\theta_0 - C/L \geq 3\theta_0/4$ for L large. Since $\theta_1 \leq \theta_0$, the weights are as stated for t small, the apexes are outside the core, and $\mathbb{D}_P \subseteq \mathcal{U}_P$. The origin term is $\mathfrak{b}\nu_Z\nu_{Z'}\hat{Z}\hat{Z}' = (\mathfrak{b}/|\mathfrak{b}|)\hat{Z}\hat{Z}' = \vartheta_P\hat{Z}\hat{Z}'$, since $\mathfrak{b}$ is a constant, and the monomials of P give $xy - \kappa_P$. The apex terms are $a\hat{Z}$ and $a'\hat{Z}'$ with $|a|, |a'| \leq C\lambda_P^{-1/2}$, and R , the sum of the monomials outside $\mathcal{A}_P$, has $|R| \leq |A|t^{3\theta_0/4}\mathfrak{m} \leq Ct^{\theta_0/2}$.

(2) Put $\hat{Z}_s = \hat{Z} + s a' / \vartheta_P$, $\hat{Z}'_s = \hat{Z}' + s a / \vartheta_P$ and $G_s = xy - \kappa_P - s^2 a a' / \vartheta_P$. Then

$$f^{(s)} = G_s(x, y) + \vartheta_P \hat{Z}_s \hat{Z}'_s + s R,$$

and $\zeta = (x, y, \hat{Z}_s, \hat{Z}'_s)$ are holomorphic coordinates on $2\mathbb{D}_P$, a shift of $(x, y, \hat{Z}, \hat{Z}')$ by at most $C\lambda_P^{-1/2}$, since a, a' are holomorphic functions of (x, y) . So $f^{(s)} = \mathcal{P}(\zeta) - \kappa_P + E_s(\zeta)$ with $\mathcal{P}(\zeta) := xy + \vartheta_P \hat{Z}_s \hat{Z}'_s$ and $E_s := sR - s^2 a a' / \vartheta_P$. By the Cauchy estimates, $\|E_s\|_{C^2} \leq C(r)t^{\theta_0/2}$ on a convex neighbourhood V_P , in the coordinates ζ , of the image of $\mathbb{D}_P$, contained in the image of $\frac{3}{2}\mathbb{D}_P$. The quadratic form $\mathcal{P}$ has gradient $(y, x, \vartheta_P \hat{Z}'_s, \vartheta_P \hat{Z}_s)$, so $|\nabla \mathcal{P}(\zeta)| = |\zeta|$, and its Hessian is a constant invertible matrix H_0 . Hence $\nabla f^{(s)} = H_0 \zeta + \nabla E_s$ on V_P : a zero has $|\zeta| \leq C(r)t^{\theta_0/2}$; two zeros ζ_1, ζ_2 satisfy $|\zeta_1 - \zeta_2| = |H_0(\zeta_1 - \zeta_2)| \leq C(r)t^{\theta_0/2}|\zeta_1 - \zeta_2|$, so there is at most one; and the map $\zeta \mapsto -H_0^{-1}\nabla E_s(\zeta)$ is a contraction of the ball of radius $C(r)t^{\theta_0/2}$ about 0, whose fixed point is a zero. The zero is nondegenerate, since $\text{Hess } f^{(s)} = H_0 + O(C(r)t^{\theta_0/2})$, and the value there is $-\kappa_P + O(|\zeta|^2) + O(t^{\theta_0/2}) = -\kappa_P + O(t^{\theta_0/2})$.

(3) For $|\eta_P| \leq \eta_0 \leq 1$, $|\kappa_P| = |e^{-\eta_P} - 1| \geq |\eta_P|/2 \geq \eta_0/(2C_\eta)$ in the window. $\square$

Lemma D.6 (node comparison). *Let $x \in U_{u,u'}$ with (χ_1, χ_2) within θ_2 of the node point, and assume that the maximum $\mathfrak{m}(x)$ is attained by a monomial of P . For a lattice point n let $m_n := \gamma_n x^n / (\gamma_a x^a)$ be its term in $f^{(1)}$. Then every $n \notin \mathcal{A}_P$ with $e_{u'}(n) \geq 1$ satisfies*

$$|\partial_{Z'} m_n| \leq C e_{\max} t^{\theta_0 - \mathcal{E}_P \theta_2} (|\mathfrak{b}Z| + |A'|_{\text{mon}}) \leq C e_{\max} t^{15\theta_0/16} (|\mathfrak{b}Z| + |A'|_{\text{mon}}),$$

where $|A'|_{\text{mon}}$ is the largest modulus of a single term of A' at (χ_1, χ_2) ; symmetrically for ∂_Z , with $|\mathfrak{b}Z'| + |A|_{\text{mon}}$.

Proof. Both sides are continuous, and if x satisfies the hypotheses and $ZZ' = 0$, so do the nearby points with the same (χ_1, χ_2) and $ZZ' \neq 0$, because the monomials that vanish at x are small near x . So we may assume $ZZ' \neq 0$. Put $\zeta_Z := \log |Z|/L$, $\zeta_{Z'} := \log |Z'|/L$, $\varpi := (\log |\chi_1|, \log |\chi_2|)/L$, and $\text{val}_m := \log |m_m|/L$. By (7.2) and the bounds on the coefficients,

$$\text{val}_m = \alpha_m \varpi_1 + \beta_m \varpi_2 + e_u(m) \zeta_Z + e_{u'}(m) \zeta_{Z'} - H(m) + H(a) + O(1/L),$$

the tropical value of m , relative to a , at the tropical point with coordinates $(\varpi, \zeta_Z, \zeta_{Z'})$ in the basis dual to $(b-a, d-a, w_u, w_{u'})$. Let ϖ^0 be the tropical position of the node point and $\bar{\theta}_2 := \theta_2 + C/L$, so that $|\varpi - \varpi^0|_\infty \leq \bar{\theta}_2$. Replacing ϖ by ϖ^0 at fixed $(\zeta_Z, \zeta_{Z'})$ changes $\text{val}_m - \text{val}_{m'}$ by at most $(|\alpha_m - \alpha_{m'}| + |\beta_m - \beta_{m'}|)\bar{\theta}_2$. Since $\text{val}_a = 0$, the values of a, b, c, d at x are at most $2\bar{\theta}_2$, and the maximum is one of them; so $\text{val}_m(x) \leq 2\bar{\theta}_2$ for all m . The origin has $\text{val}_0 = \zeta_Z + \zeta_{Z'} + H(a) + O(1/L)$, independent of ϖ because $\alpha_0 = \beta_0 = 0$.

Fix apexes $v_1 \in \omega_1$ and $v_2 \in \omega_2$ and put $\lambda_k := 2 + |v_k|_P$, so $2 \leq \lambda_k \leq 2 + A_P$. In the $(\zeta_Z, \zeta_{Z'})$ -plane at ϖ^0 the edge J_P of Lemma 7.7 is the segment $\{\zeta_Z + \zeta_{Z'} = -H(a), \zeta_Z \leq \zeta_Z^*, \zeta_{Z'} \leq \zeta_{Z'}^*\}$: moving along $u' - u$ changes $(\zeta_Z, \zeta_{Z'})$ by $(-1, 1)$ and fixes ϖ , and by Lemma 7.7(1), (2), on J_P the deficits of v_2 and v_1 are $\zeta_Z^* - \zeta_Z$ and $\zeta_{Z'}^* - \zeta_{Z'}$. Its end $p_Z = (\zeta_Z^*, -H(a) - \zeta_Z^*)$ is dual to $\text{conv}(0, \omega_2)$ and $p_{Z'}$ to $\text{conv}(0, \omega_1)$. At the point $x^0 := (\varpi^0, \zeta_Z, \zeta_{Z'})$ the bounds $\text{val}_{v_2}(x), \text{val}_{v_1}(x), \text{val}_0(x) \leq 2\bar{\theta}_2$ give

$$\zeta_Z \leq \zeta_Z^* + \lambda_2 \bar{\theta}_2, \quad \zeta_{Z'} \leq \zeta_{Z'}^* + \lambda_1 \bar{\theta}_2, \quad \zeta_Z + \zeta_{Z'} \leq -H(a) + 2\bar{\theta}_2,$$

up to $O(1/L)$, which we absorb in $\bar{\theta}_2$. This region is nonempty and bounded above in both coordinates, and a function on it that is linear and nondecreasing in ζ_Z and in $\zeta_{Z'}$ attains its maximum at one of the two points $p'_Z := p_Z + (\lambda_2 \bar{\theta}_2, (2 - \lambda_2) \bar{\theta}_2)$, $p'_{Z'} := p_{Z'} + ((2 - \lambda_1) \bar{\theta}_2, \lambda_1 \bar{\theta}_2)$.

Let $i := e_u(n)$ and $j := e_{u'}(n) \geq 1$. By Lemma 7.1 the set $\{0\} \cup P \cup \{v_2, n\}$ is not cellular, as $v_2 \notin \omega_1$ and $n \notin \mathcal{A}_P$; at p_Z its members other than n are maximal, so $d_n^{\text{tr}}(p_Z)$ is the largest deficit on this set at p_Z , and $d_n^{\text{tr}}(p_Z) \geq \gamma^* \geq \theta_0$ by Lemma 7.5. Since the origin is maximal at p_Z , $\text{val}_n(p_Z) - \text{val}_0(p_Z) = -d_n^{\text{tr}}(p_Z)$ up to $O(1/L)$. Likewise $d_n^{\text{tr}}(p_{Z'}) \geq \theta_0$, using v_1 .

If $i \geq 1$, the function $g := \text{val}_n - \text{val}_0 = (i-1)\zeta_Z + (j-1)\zeta_{Z'} + \text{const}$ is nondecreasing, $g(p_Z), g(p_{Z'}) \leq -\theta_0$, and since $\lambda_k \geq 2$, $g(p'_Z) \leq -\theta_0 + (i-1)\lambda_2 \bar{\theta}_2$ and $g(p'_{Z'}) \leq -\theta_0 + (j-1)\lambda_1 \bar{\theta}_2$. Returning from x^0 to x adds at most $|n|_P \bar{\theta}_2$. So

$$\text{val}_n(x) - \text{val}_0(x) \leq -\theta_0 + ((\max(i, j) - 1)(2 + A_P) + |n|_P) \bar{\theta}_2 \leq -\theta_0 + \mathcal{E}_P \bar{\theta}_2,$$

by the definition of $\mathcal{E}_P$. Hence $|m_n| \leq C t^{\theta_0 - \mathcal{E}_P \bar{\theta}_2} |\mathbf{b} ZZ'|$, and

$$|\partial_{Z'} m_n| = j |m_n| / |Z'| \leq C e_{\max} t^{\theta_0 - \mathcal{E}_P \bar{\theta}_2} |\mathbf{b} Z|.$$

If $i = 0$, compare with v_1 : $g' := \text{val}_n - \text{val}_{v_1} = (j-1)\zeta_{Z'} + \text{const}$ at ϖ^0 , and at $p_{Z'}$ the monomials $0, P, v_1$ are maximal, so $g'(p_{Z'}) \leq -\theta_0$; hence $g'(x^0) \leq -\theta_0 + (j-1)\lambda_1 \bar{\theta}_2$, and returning to x adds at most $(|n|_P + |v_1|_P) \bar{\theta}_2$. So $\text{val}_n(x) - \text{val}_{v_1}(x) \leq -\theta_0 + ((j-1)(2 + A_P) + |n|_P + A_P) \bar{\theta}_2 \leq -\theta_0 + \mathcal{E}_P \bar{\theta}_2$, and $|\partial_{Z'} m_n| \leq e_{\max} |m_n| / |Z'| \leq C e_{\max} t^{\theta_0 - \mathcal{E}_P \bar{\theta}_2} |A'|_{\text{mon}}$.

Finally $\mathcal{E}_P \bar{\theta}_2 \leq \theta_1/16 \leq \theta_0/16$. The case of ∂_Z is the same with the roles of (u, ω_1, v_1) and (u', ω_2, v_2) exchanged. $\square$

Proof of Lemma 7.15. Let $x \in \mathcal{U}_P$, so that $\mathcal{C}(x) \setminus \{0\} = P$: the apexes and the monomials outside $\mathcal{A}_P$ are not in the core. Put $O := |\mathbf{b} ZZ'|$ (the origin, of weight one), let $\|AZ\|_1$ and $\|A'Z'\|_1$ be the sums of the moduli of the apex monomials of ω_2 and of ω_1 , and $\gamma := \theta_1/8$. Then $\|AZ\|_1, \|A'Z'\|_1 \leq |\mathcal{A}| t^{\theta_0/2} \mathbf{m}$, and every monomial outside $\mathcal{A}_P$ has modulus at most $t^{3\theta_0/4} \mathbf{m}$ (Lemma 7.5).

(1) Let $x \in W_s$.

Case $O \geq t^\gamma \mathbf{m}$. Apply Lemma D.2 with $v = \partial/\partial \operatorname{Re} Z'$. The derivative h of Lemma D.2, in which the cut-offs are held constant, is $h = \mathbf{b}Z + (\text{weighted } A') + \sum \tilde{\psi}_n \partial_{Z'}(\gamma_n x^n)$ over the monomials outside $\mathcal{A}_P$ with $e_{u'} \geq 1$, so

$$|h| \geq (O - \|A'Z'\|_1 - e_{\max}|\mathcal{A}|t^{3\theta_0/4}\mathbf{m})/|Z'| \geq O/(2|Z'|),$$

since $\gamma < \theta_0/2$. In Lemma D.1(3), $\mathcal{D}_{Z'} \leq (O + \|A'Z'\|_1 + e_{\max}|\mathcal{A}|t^{3\theta_0/4}\mathbf{m})/|Z'| \leq 2O/|Z'|$, and the transition monomials with $e_{u'} \geq 1$ are apexes or lie outside $\mathcal{A}_P$, of total modulus at most $O/2$. So $\mathcal{J} \leq (C\beta'_{\max}/L)O/|Z'| < |h|$ for L large.

Case $O < t^\gamma \mathbf{m}$. Then every monomial other than those of P has modulus at most $2|\mathcal{A}|t^\gamma \mathbf{m}$, and the maximum $\mathbf{m}$ is attained by a monomial of P , which has weight one. Write $f^{(s)} = \tilde{g} + \mathcal{G}$ with $\tilde{g}$ the weighted sum over P ; then $|\mathcal{G}| \leq 2|\mathcal{A}|^2 t^\gamma \mathbf{m}$, and $|\chi_1 \partial_{\chi_1} \mathcal{G}|, |\chi_2 \partial_{\chi_2} \mathcal{G}| \leq Ct^\gamma \mathbf{m}$ for the logarithmic torus derivatives $\chi_1 \partial_{\chi_1}, \chi_2 \partial_{\chi_2}$. By Lemma D.1(2), $\mathcal{J} \leq C\beta'_{\max} t^{\theta_1/4} \mathbf{m}/L$ for these directions.

- If (χ_1, χ_2) is not within θ_2 of the node point, the moduli of the four monomials of P differ by a factor at least $e^{\theta_2 L/2}$, and Lemma D.5, applied with its maximum $E = \mathbf{m}$ (the maximum is a monomial of P of weight one) and $|\tilde{g}| = |\mathcal{G}| \leq c\mathbf{m}$, gives $\max(|\chi_1 \partial_{\chi_1} \tilde{g}|, |\chi_2 \partial_{\chi_2} \tilde{g}|) \geq \mathbf{m}/16$. So one of the two torus directions has $|h| \geq \mathbf{m}/16 - Ct^\gamma \mathbf{m} > \mathcal{J}$.
- If (χ_1, χ_2) is within θ_2 of the node point, the deficits of P are at most $2\theta_2 + C/L < \theta_1/4$, so its weights are one and $\tilde{g} = xy - \kappa_P$. We have $\chi_1 \partial_{\chi_1} \tilde{g} = \mu_P^{1/2} \tilde{\chi}_1 y$ and $\chi_2 \partial_{\chi_2} \tilde{g} = \mu_P^{1/2} \tilde{\chi}_2 x$, and $|xy - \kappa_P| \leq Ct^\gamma \mathbf{m}$ on W_s . If $|y| \geq r$, then, since $|\kappa_P| \leq 4\eta_0 \leq r^2/5$ even on $U_{\mathbb{C}}^+$ and $\mathbf{m} \leq C(1 + |x|)(1 + |y|)$, the inequality $|xy - \kappa_P| \leq Ct^\gamma \mathbf{m}$ gives $|x| \leq r/3$ for t small; so $|\mu_P^{1/2} \tilde{\chi}_1| \geq 1/2$ and $\mathbf{m} \leq C(1 + |y|)$, hence $|\chi_1 \partial_{\chi_1} \tilde{g}| \geq |y|/2 \geq cr\mathbf{m}$. The case $|x| \geq r$ is symmetric. If $|x|, |y| < r$, then $\mathbf{m} \leq C$ and $|xy| \geq |\kappa_P| - Ct^\gamma \geq \eta_0/(4C_\eta)$ in the window, so $\max(|\chi_1 \partial_{\chi_1} \tilde{g}|, |\chi_2 \partial_{\chi_2} \tilde{g}|) \geq \frac{1}{2}(\eta_0/(4C_\eta))^{1/2}$. In each case $|h| > \mathcal{J}$ for $t < t_0(r, \eta_0, \mathcal{C}_0)$.

(2) Let $s = 1$, $c \in U_{\mathbb{C}}^+$, and let $x \in \mathcal{U}_P$ be a singular point of $Y_{t,c}$. All weights are one and $\mathcal{J} = 0$. The first case above and the first item of the second case apply verbatim and show that $O < t^\gamma \mathbf{m}$ and that (χ_1, χ_2) is within θ_2 of the node point; the part of the second item with $|x| \geq r$ or $|y| \geq r$ uses only $|\kappa_P| \leq 4\eta_0$ and shows $|x|, |y| < r$. Since $O < t^\gamma \mathbf{m}$, the maximum is attained by a monomial of P , so Lemma D.6 applies. At x , $\partial_{Z'} f = \mathbf{b}Z + A' + \mathcal{G}_{Z'} = 0$, where $\mathcal{G}_{Z'}$ is the sum of the terms $\partial_{Z'} m_n$, $n \notin \mathcal{A}_P$, and $|\mathcal{G}_{Z'}| \leq Ct^{15\theta_0/16}(|\mathbf{b}Z| + |A'|_{\text{mon}})$ by Lemma D.6. Hence $|\mathbf{b}Z| \leq |\mathcal{A}||A'|_{\text{mon}} + Ct^{15\theta_0/16}(|\mathbf{b}Z| + |A'|_{\text{mon}})$, so $|\mathbf{b}Z| \leq 2|\mathcal{A}||A'|_{\text{mon}}$. Each apex term of A' is a monomial $\chi_1^\alpha \chi_2^\beta$ times a constant with $|\alpha| + |\beta| \leq A_P \leq \mathcal{E}_P$, so moving (χ_1, χ_2) to the node point changes $|A'|_{\text{mon}}$ by a factor at most $t^{-\mathcal{E}_P \theta_2} C \leq t^{-\theta_1/16} C$. Since $\mathbf{b}$ is constant, $|\mathbf{b}|\nu_Z = \lambda_P^{1/2} |A'|_{\text{mon}}(q)$ exactly, where $|A'|_{\text{mon}}(q)$ is the value at the node point, and therefore

$$|\hat{Z}| = \frac{|Z|}{\nu_Z} \leq \frac{2|\mathcal{A}||A'|_{\text{mon}}}{|\mathbf{b}|\nu_Z} \leq C\lambda_P^{-1/2} t^{-\theta_1/16} \leq Ct^{\mathfrak{g}_P/2 - \theta_1/16} < 1,$$

because $\mathfrak{g}_P/2 \geq \theta_0 \geq \theta_1$ by Lemma 7.7(3). Symmetrically $|\hat{Z}'| < 1$. So $x \in \mathbb{D}_P$. $\square$

D.4. Where the localized mirror is given by exact models.

Proof of Lemma 7.18. Throughout, val_m denotes $\frac{1}{L} \log |\gamma_m x^m|$ at the point considered, so that the deficit of m is $\max_k \operatorname{val}_k - \operatorname{val}_m$; by (7.1) and the bounds on the coefficients, val_m is the tropical value $\langle p, m \rangle - H(m)$ at the tropical position p , up to $O(1/L)$ after subtracting

the value of a fixed monomial. Two facts are used repeatedly: lowering $|Z|$ or $|Z'|$ at fixed torus coordinates does not increase $\text{val}_m - \text{val}_n$ when n has $e_u(n) = e_{u'}(n) = 0$, since all Cox exponents are nonnegative; and a monomial whose deficit is below $\theta_1/4 - C/L$ (resp. above $\theta_1/2 + C/L$) has weight one (resp. zero), by Lemma D.1(1).

(1) Let $p_0 \in \Omega_e$ with $|p - p_0| \leq \theta_2$, p the tropical position of x . At p_0 the monomials $0, n, n'$ are maximal and every other monomial has deficit at least $f_e(p_0) > 3\theta_1/4$. Passing from p_0 to p changes each $\text{val}_m - \text{val}_n$ by at most $D\theta_2 \leq \theta_1/16$. Since $u, u' \in e^\vee$, $e_u(n) = e_{u'}(n) = 0$, so passing from x to x' does not increase any $\text{val}_m - \text{val}_n$ and leaves $\text{val}_{n'} - \text{val}_n$ unchanged. So at x' we have $\text{val}_m - \text{val}_n \leq \theta_1/16 + C/L$ for all m , and $\text{val}_m - \text{val}_n \leq -3\theta_1/4 + \theta_1/16 + C/L$ for $m \notin \{0, n, n'\}$. Hence the deficits of n, n' at x' are at most $\theta_1/8 + C/L$ and those of the other $m \neq 0$ at least $\theta_1/2 + \theta_1/8$: the weights are as stated, and $\psi_0 \equiv 1$. At a turning point the same argument applies with the three Cox coordinates of the triangle $\tau \subseteq e^\vee$.

(1') The path I lies in E_e , because $E_e = \{p : d_0^{\text{tr}}(p) = d_n^{\text{tr}}(p) = d_{n'}^{\text{tr}}(p) = 0\}$ is convex and contains m, m' and p_e . On E_e the maximal value is val_0 , which is linear in p , so every deficit d_m^{tr} is affine on E_e . Let m_1 be a lattice point outside $\{0\} \cup \omega$. If $\omega = P$ is a node parallelogram, then $d_{m_1}^{\text{tr}}(m) \geq \theta_0$ if $m_1 \notin \mathcal{A}_P$, because $\{0\} \cup P \cup \{m_1\}$ is not cellular and $0, P$ are maximal at the midpoint m of J_P , and $d_{m_1}^{\text{tr}}(m) = \mathfrak{g}_P/2 \geq \theta_0$ if m_1 is an apex, by Lemma 7.7(2), (3). If ω is a triangle and $m \in J_\omega^b$, then $d_{m_1}^{\text{tr}}(m) \geq 3\theta_1/4$ if $m_1 \in (\omega_1 \cup \omega_2) \setminus \omega$, by the definition of J_ω^b , and $d_{m_1}^{\text{tr}}(m) \geq \theta_0$ otherwise, because $\{0\} \cup \omega \cup \{m_1\}$ is not cellular (proof of (3)). In both cases $d_{m_1}^{\text{tr}}(m) \geq 3\theta_1/4$, hence $d_{m_1}^{\text{tr}}(m') \geq 3\theta_1/4 - D\theta_2/4 \geq 3\theta_1/4 - \theta_1/64$, and $d_{m_1}^{\text{tr}}(p_e) \geq f_e(p_e) > 3\theta_1/4$. By affinity $d_{m_1}^{\text{tr}} \geq 3\theta_1/4 - \theta_1/64$ on I , as $\theta_0 \geq \theta_1$. Now let n be a vertex of e , so $e_u(n) = e_{u'}(n) = 0$, and pass from a point y of I to x . Moving the tropical position by at most θ_2 changes $\text{val}_{m_1} - \text{val}_n$ by at most $D\theta_2 \leq \theta_1/16$; lowering $\log |Z|, \log |Z'|$ does not increase it. At y both vertices of e are maximal. So at x , $\text{val}_{m_1} - \text{val}_n \leq \theta_1/16 + C/L$ for every m_1 , and $\text{val}_{m_1} - \text{val}_n \leq -(3/4 - 1/64 - 1/16)\theta_1 + C/L = -43\theta_1/64 + C/L$ for m_1 outside $\{0\} \cup \omega$. The deficits of the vertices of e are therefore at most $\theta_1/16 + C/L < \theta_1/4$, and those of the monomials outside $\{0\} \cup \omega$ at least $43\theta_1/64 - C/L > \theta_1/2$; the origin has weight one by definition.

(2) First, $\mathbb{D}_P \subseteq \mathcal{N}_P$: on $\mathbb{D}_P$, $|x|, |y| < r$ gives $s_1, s_2 = O(r)$, and $|\hat{Z}|, |\hat{Z}'| < 1$. Let $x \in \mathcal{N}_P$, write val_m relative to $\text{val}_a = 0$, and put $\hat{\zeta}_Z := \log |\hat{Z}|/L$, $\hat{\zeta}_{Z'} := \log |\hat{Z}'|/L$ and $\bar{\theta}_2 := \theta_2 + C/L$. The monomials of P have $\text{val} \leq 2\bar{\theta}_2$, and the origin, whose term in $f^{(0)}$ is $\vartheta_P \hat{Z} \hat{Z}'$, has $\text{val}_0 = \hat{\zeta}_Z + \hat{\zeta}_{Z'} \leq \theta_2$. Since $\hat{\zeta}_Z, \hat{\zeta}_{Z'} \leq \theta_P$ and $\hat{\zeta}_Z + \hat{\zeta}_{Z'} \leq \theta_2$, there is $s \in [-\theta_P, \theta_P]$ with $\hat{\zeta}_Z \leq s + \theta_2$ and $\hat{\zeta}_{Z'} \leq -s + \theta_2$ (take $s = \hat{\zeta}_Z$ if $\hat{\zeta}_Z \geq -\theta_P$, and $s = -\theta_P$ otherwise). Let y be the point of J_P displaced from its midpoint by $(s, -s)$ in the two Cox directions. The normalisation of Definition 7.13 puts the midpoint at $(0, 0)$ up to $O(1/L)$, as in the proof of Lemma 7.14; these errors are included in $\bar{\theta}_2$. At y the monomial a is maximal; by Lemma 7.7(2) the apexes have deficit $\mathfrak{g}_P/2 \mp s \geq \mathfrak{g}_P/2 - \theta_P = \theta_1$; and the monomials outside $\mathcal{A}_P$ have deficit at least $\theta_0 \geq \theta_1$, since $\{0\} \cup P \cup \{m\}$ is not cellular and $0, P$ are maximal on J_P . Passing from y to x raises $\hat{\zeta}_Z, \hat{\zeta}_{Z'}$ by at most $\theta_2 + O(1/L)$ and moves the torus coordinates by at most $\bar{\theta}_2$, so it increases val_m by at most $(e_u(m) + e_{u'}(m) + |m|_P)\bar{\theta}_2 \leq 2\mathcal{E}_P\bar{\theta}_2 \leq \theta_1/8 + C/L$: for $m \notin \mathcal{A}_P$, $e_u(m) + e_{u'}(m) \leq 2 + 2(\max(e_u(m), e_{u'}(m)) - 1)^+$, so $e_u(m) + e_{u'}(m) + |m|_P \leq \mathcal{E}_P + 2 \leq 2\mathcal{E}_P$ by the definition of $\mathcal{E}_P$, and for an apex $e_u + e_{u'} + |m|_P \leq 1 + A_P \leq \mathcal{E}_P$. So at x every apex and every monomial outside $\mathcal{A}_P$ has $\text{val}_m \leq -7\theta_1/8 + C/L$, while $\text{val}_a = 0$. Hence the maximum at x is at most $2\bar{\theta}_2$, the deficits of the monomials of P are at most $4\bar{\theta}_2 < \theta_1/4 - C/L$, as $\theta_2 \leq \theta_1/32$, and those of the apexes and of the monomials outside $\mathcal{A}_P$ are at least $7\theta_1/8 - C/L$.

(3) By Lemma 7.7(1), on J_ω the deficit of a lattice point n of $(\omega_1 \cup \omega_2) \setminus \omega$ is affine in the parameter s , with slope $\pm e(n)$, $e(n) \leq e_{\max}$; so J_ω^b is a segment. At the midpoint $s = s_{\text{mid}}$

these deficits are at least $\mathfrak{g}_\omega/2 \geq (\theta_0 + 1)/2$, by the definition of $\mathfrak{g}_\omega$ and (7.3). So at the points with $|s - s_{\text{mid}}| \leq 1/(4e_{\text{max}})$, which lie in J_ω because $s_+ - s_- = |J_\omega| \geq \mathfrak{g}_\omega/e_{\text{max}} \geq 1/e_{\text{max}}$ by Lemma 7.7(2) and (7.3), they are at least $\mathfrak{g}_\omega/2 - 1/4 \geq \theta_0/2 + 1/4 \geq 3\theta_1/4$, as $\theta_1 \leq \min(\theta_0, 1)$. The cells of $\mathcal{F}_C$ containing ω are $\omega, \omega_1, \omega_2$ (proof of Lemma 7.7(1)), so a lattice point $m \notin \{0\} \cup \omega_1 \cup \omega_2$ forms with $\{0\} \cup \omega$ a set that is not cellular, and has deficit at least $\theta_0 \geq \theta_1$ on J_ω , where 0 and ω are maximal. Moving the tropical position by at most θ_2 changes the deficits by at most $2D\theta_2 \leq \theta_1/8$, and lowering $\log|Z|, \log|Z'|$ does not increase $\text{val}_m - \text{val}_{n_a}$, since $e_u(n_a) = e_{u'}(n_a) = 0$ for the edge $[u, u']$ of $\mathcal{T}$ in G° . So at the points of (3) the deficits of n_a, n_b, n_c are at most $\theta_1/8 + C/L < \theta_1/4$ and all others except the origin are at least $3\theta_1/4 - \theta_1/8 - C/L > \theta_1/2$.

(4) Let $x \in W^* \cap D_{u'}$. The monomials vanishing on $D_{u'}$, and those of weight zero near x , do not contribute to $F^{(0)}$ or to its derivative along $D_{u'}$ at x : the weights are smooth and vanish to infinite order where $\tilde{d}_n = \theta_1/2$. So along $D_{u'}$ near x , $F^{(0)} = \sum_{n \in I(x)} \tilde{\psi}_n \gamma_n x^n$ up to terms vanishing to first order at x , and $F^{(0)}(x) = 0$. The maximal monomial at x among those not vanishing on $D_{u'}$ has weight one, so it lies in $I(x)$. We show that the derivative of $F^{(0)}|_{D_{u'}}$ at x along the torus action has real rank two; the torus vector fields are tangent to $D_{u'}$, so this gives the transversality.

If $I(x)$ is affinely independent, let $n_0 \in I(x)$ be maximal. From $F^{(0)}(x) = 0$, some $n_1 \in I(x) \setminus \{n_0\}$ has $|\tilde{\psi}_{n_1} \gamma_{n_1} x^{n_1}| \geq \mathfrak{m}/|\mathcal{A}|$, where $\mathfrak{m} = |\gamma_{n_0} x^{n_0}|$. Choose ℓ with $\langle \ell, n - n_0 \rangle = 0$ for $n \in I(x) \setminus \{n_0, n_1\}$ and $\langle \ell, n_1 - n_0 \rangle = 1$, with $|\ell| \leq \ell_{\text{max}}$ as in case (A). In Lemma D.2 with $v = v_\ell$ and the functions divided by $\gamma_{n_0} x^{n_0}$, $|h| \geq 1/|\mathcal{A}|$ and $\mathcal{J} \leq \beta'_{\text{max}} \ell_{\text{max}} D |\mathcal{A}|^{3/2} t^{\theta_1/4} / L$ by Lemma D.1(1), (2), which extend by continuity to the points of $D_{u'}$.

Let c be in the window and $I(x) \subseteq P$. If $I(x) \neq P$, then $I(x)$ is affinely independent, and the previous case applies. So let $I(x) = P$. Then $x \in U_{u, u'}$, since the monomials of P do not vanish at x , and $F^{(0)}|_{D_{u'}} = \gamma_a x^a \tilde{g}$ near x , with $\tilde{g}$ the weighted sum of the four monomials of P . If (χ_1, χ_2) is within θ_2 of the node point, the four monomials are within a factor $Ct^{-2\theta_2}$ of the maximum, so they have weight one and $\tilde{g} = g_P = xy - \kappa_P$; its differential $y dx + x dy$ vanishes only at $x = y = 0$, where $g_P = -\kappa_P \neq 0$. Otherwise their moduli differ by a factor at least $e^{\theta_2 L/2}$, and Lemma D.5, with $\tilde{g}(x) = 0$ and $E = \mathfrak{m}$, gives $\max(|\chi_1 \partial_{\chi_1} \tilde{g}|, |\chi_2 \partial_{\chi_2} \tilde{g}|) \geq \mathfrak{m}/16$, against $\mathcal{J} \leq C\beta'_{\text{max}} t^{\theta_1/4} \mathfrak{m}/L$ as in the proof of Lemma 7.15.

On the sets of (1)–(3), $I(x)$ is contained in $\{n, n'\}$, in P , or in ω , by the weights computed there and because the origin vanishes on $D_{u'}$; $\{n, n'\}$ and ω are affinely independent. The same arguments apply to D_u and, at turning points, to D_{u_1} . $\square$

D.5. Change of diagonal: proof of Lemma 7.26.

Proof of Lemma 7.26. Primes denote the objects of the labelling $a'b'c'd' = bcda$. Then $\mu'_P = \mu_P^{-1}$, $\kappa'_P = e^{\eta_P} - 1 = -\mu_P \kappa_P$, $\tilde{\chi}'_1 = \mu_P \tilde{\chi}_2$, $\tilde{\chi}'_2 = \tilde{\chi}_1^{-1}$ and $w'_{u'} = -b - w_u$, so Z is unchanged and Z' becomes $Z' \chi_1^{-1}$; the reference monomial is b , so $f'^{(s)} = f^{(s)}/\tilde{\chi}_1$.

The chart sign. Since $\text{val}_c = \text{val}_a + y_1 + y_2$, we get $y'_1 = y_2$ and $y'_2 = -y_1$, and $\alpha'_q = \alpha_q + \langle \cdot, b - a \rangle$, which is $\alpha_q + y_1$ up to a constant. The linear map $(y_1, y_2, \alpha_q) \mapsto (y_2, -y_1, \alpha_q + y_1)$ has determinant 1, so o_q^{ch} is unchanged.

The coordinates. From Definition 7.13 and $x'y' + z'_1 z'_2 - \kappa'_P = f'^{(0)} = f^{(0)}/\tilde{\chi}_1$ on $\mathbb{D}_P$,

$$x' = y + \mu_P^{1/2} - \mu_P^{-1/2}, \quad y' = \frac{\mu_P x}{\mu_P^{1/2} x - 1}, \quad z'_1 = c_Z z_1, \quad z'_2 = \frac{z_2}{c_Z \tilde{\chi}_1},$$

with a constant $c_Z > 0$, the ratio of the two normalisations of Z .

The convention on the thimble. For $f = xy + z_1 z_2 - \kappa$ the sphere Σ^1 bounds the real four-disc $\{y = e^{i\phi} \bar{x}, z_2 = e^{i\phi} \bar{z}_1, |x|^2 + |z_1|^2 \leq |\kappa|\}$, the thimble, which the projection to (x, z_1) maps onto the ball. So Definition 7.23 orients δ_q as the boundary of the thimble, oriented by o_q^{ch} times the pull-back of the complex orientation of $\mathbb{C}_{(x, z_1)}^2$; for a family of Morse points this orientation is determined by the tangent space of the thimble at the critical point.

A deformation. On $W^* \cap \mathbb{D}_P$ we have $f^{(0)} = xy + z_1 z_2 - \kappa_P$. Put $v := (1 - \mu_P^{1/2} x)^{-1}$, so that $f'^{(0)} = -\mu_P v f^{(0)}$; on $\mathbb{D}_P$, $|\mu_P^{1/2} x| \leq 1/2$, so v takes values in the disc of radius 1 about 1 without 0 and has a principal power v^s . For $s \in [0, 1]$ the function $h_s := v^s f^{(0)}$ has zero set $W^* \cap \mathbb{D}_P$. Its critical points satisfy $x = z_1 = z_2 = 0$ (from the derivatives in y, z_2, z_1 , since $v \neq 0$) and then $y = s\mu_P^{1/2} \kappa_P$: there is exactly one, q_s , with value $-\kappa_P$. Its Hessian is that of $f^{(0)}$ plus $c_s dx^2$, $c_s = s(s-1)\mu_P \kappa_P$; since the matrix $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$ is invertible for every c , q_s is nondegenerate. The tangent space of its thimble for the straight path from $-\kappa_P$ to 0 is

$$T_s = \left\{ v : v_y = e^{i\phi} \bar{v}_x - \frac{c_s}{2} v_x, v_{z_2} = e^{i\phi} \bar{v}_{z_1} \right\},$$

on which the Hessian form is $2e^{i\phi}(|v_x|^2 + |v_{z_1}|^2)$; it is a graph over the (x, z_1) -plane for every s . So the vanishing spheres of q_s form a continuous family in $W^* \cap \mathbb{D}_P$, and orienting each as the boundary of its thimble with the orientation pulled back from $\mathbb{C}_{(x, z_1)}^2$ is continuous in s . At $s = 0$ this is the orientation of Definition 7.23 for $abcd$ (up to the common factor o_q^{ch}). At $s = 1$, $h_1 = -\mu_P^{-1} f'^{(0)}$ has the same level sets as $f'^{(0)}$ along the straight paths, so its vanishing sphere is, up to isotopy, the one of the labelling $bcda$, whose convention pulls back the orientation of $\mathbb{C}_{(x', z'_1)}^2$. The sphere of that convention is linear in the primed coordinates, and the tangent plane T' of its thimble at q_1 need not be T_1 . At q_1 , where $c_1 = 0$, $\text{Hess } f'^{(0)} = -\mu_P \text{Hess } h_1 = -\mu_P \text{Hess } f^{(0)}$, and the path of $f'^{(0)}$ runs from its critical value $-\kappa'_P$ towards 0, in the direction of $\kappa'_P = -\mu_P \kappa_P$; so $\bar{\kappa}'_P \text{Hess } f'^{(0)} = |\mu_P|^2 \bar{\kappa}_P \text{Hess } f^{(0)}$. Hence T_1 and T' are both real four-planes on which $e^{-i\phi} \text{Hess } f^{(0)}$ is real and positive definite; in particular both are maximal positive planes of the real quadratic form $\text{Re}(e^{-i\phi} \text{Hess } f^{(0)})$ at q_1 . After a complex-linear change of coordinates in which $e^{-i\phi} \text{Hess } f^{(0)}$ becomes $\sum_i \zeta_i^2$, the planes of this kind are the images of $\mathbb{R}^4$ under $O(4, \mathbb{C})$, so they form a family $O(4, \mathbb{C})/O(4)$, which is connected since $O(4)$ meets both components of $O(4, \mathbb{C})$; the linear thimbles they contain have their boundary spheres in one level set of the quadratic part, and these spheres move by isotopy along the family. Both thimbles at q_1 are such linear thimbles in one chart. Let $\mathbf{w}' := (x', y', z'_1, z'_2)$, $A := d\mathbf{w}'(q_1)$ and $\mathbf{w} := A^{-1}\mathbf{w}'$. Since $\mathbf{w}'(q_1) = 0$ and $Q' \circ A = -\mu_P Q$ for the quadratic parts Q, Q' of h_1 and $f'^{(0)}$ at q_1 , we have $h_1 = -\mu_P^{-1} f'^{(0)} = -\mu_P^{-1}(Q'(A\mathbf{w}) - \kappa'_P) = Q(\mathbf{w}) - \kappa_P$ on $\mathbb{D}_P$, so $\mathbf{w}$ are holomorphic Morse coordinates of h_1 at q_1 tangent to the identity. In them the thimble of q_1 is the linear thimble with tangent plane T_1 , and the thimble of the primed convention is the linear thimble with tangent plane $T' = A^{-1}T'_{\text{lin}}$, where T'_{lin} is its tangent plane in the coordinates $\mathbf{w}'$. So the orientations of δ_q induced from oriented thimbles are carried along this family of planes by continuity. On each of them the projections to (v_x, v_{z_1}) and to $(v_{x'}, v_{z'_1})$, that is to (v_y, v_{z_1}) up to a complex-linear change, are isomorphisms, because a vector with $v_x = v_{z_1} = 0$, or with $v_y = v_{z_1} = 0$, is isotropic. So the sign relating the two pulled-back orientations is constant on the family, and it suffices to compute it on T_1 . On T_1 (where $c_1 = 0$) the differential of $(x, z_1) \mapsto (x', z'_1)$ is $(v_x, v_{z_1}) \mapsto (e^{i\phi} \bar{v}_x, c_Z v_{z_1})$, which reverses orientation. So the two conventions orient δ_q oppositely. $\square$

APPENDIX E. DEFERRED PROOFS FOR SECTION 8

This appendix contains the proofs of eleven lemmas of Section 8. They use only the statements that precede the lemma in question in Section 8 and the results of Sections 4–7.

Proof of Lemma 8.2. We work in $e^\vee + \nabla_e^{H'}$; negation does not affect the assertions. Write $y(x) := \langle x, y \rangle$ and $y_\tau := y(u_i) = y(u_j)$. The ζ with $F_1(\zeta) = \tau$ satisfy $\langle u_j - u_i, \zeta \rangle = \check{h}(u_i) - \check{h}(u_j)$; modulo $\mathbb{R}n + \mathbb{R}n'$ they form the relatively open cell dual to τ of the tropical curve of $(e^\vee \cap M, \check{h})$, contained in a line. The linear functions on the direction space of $e^\vee + \nabla_e^{H'}$ that vanish on $u_j - u_i$ are the multiples of y , so this cell is $\{\zeta_0 + sy : s \in I\}$ for an open interval I . Put $F(s) := F_2(\zeta_0 + sy)$; the cells with lower support τ are the $\tau + F(s)$, $s \in I$.

(a) Let $s < s'$ in I , $x \in F(s)$, $x' \in F(s')$. Adding $\langle x, \zeta_0 \rangle + sy(x) \leq \langle x', \zeta_0 \rangle + sy(x')$ and $\langle x', \zeta_0 \rangle + s'y(x') \leq \langle x, \zeta_0 \rangle + s'y(x)$ gives $(s' - s)(y(x') - y(x)) \leq 0$, so $\max_{F(s')} y \leq \min_{F(s)} y$. If $F(s) \neq F(s')$, then $y(b_{c'}) < y(b_c)$ for $c = \tau + F(s)$, $c' = \tau + F(s')$: otherwise y would be constant, with one value, on c and c' , and the two inequalities would give $\langle x, \zeta_0 \rangle = \langle x', \zeta_0 \rangle$ for all $x \in F(s)$, $x' \in F(s')$, that is $F(s) = F(s')$.

(b) Along a chain of cells the lower supports increase, so a cell containing $\tau + F(s)$ has lower support τ or a triangle $s_\Delta \supset \tau$ of $\mathcal{T}$; in the second case it is the cell $\mathbf{c}_{s_\Delta} := s_\Delta + F_2(\zeta_{s_\Delta})$ at the end $\zeta_{s_\Delta} = \zeta_0 + s_*y$ of the cell dual to τ that is dual to s_Δ , and $b_{\mathbf{c}_{s_\Delta}}$ is the vertex $C_Y(s_\Delta, e)$. By Definition 4.6 the closure of $C_Y(\tau, e)$ is the polygonal path through the points b_c , $c = \tau + F(s)$, in the order of s , extended at each finite end s_* of I by the segment to $b_{\mathbf{c}_{s_\Delta}}$; at an infinite end the last cell lies in an edge of the polygon and its point b_c is a leaf.

(c) By (a), y decreases strictly along the points b_c as s increases. Let $s_* = \inf I$ be finite, with triangle s_Δ , and w the vertex of s_Δ opposite τ . The value of $a \mapsto \langle a, \zeta_0 + sy \rangle + \check{h}(a)$ at w minus its value at u_i is affine in s with slope $y(w) - y_\tau$, vanishes at s_* and is positive for $s > s_*$ near s_* ; so $y(w) > y_\tau$ and y is minimal on s_Δ exactly along τ . For such s , $F(s)$ is the face of $F_2(\zeta_{s_\Delta})$ on which y is minimal, $c = \tau + F(s)$ is the face of $\mathbf{c}_{s_\Delta}$ on which y is minimal, and $y(b_{\mathbf{c}_{s_\Delta}}) > y(b_c)$. The end $\sup I$ is treated with $-y$. So y decreases strictly along the whole path; the lines parallel to $u_j - u_i$ in the plane of E_e are the level sets of y . $\square$

Proof of Lemma 8.3. A leaf path of a model point on J lies in the convex set S_J , which contains m , m' and $\mathbf{t}_e(\bar{m})$; so leaf paths from different edges are disjoint. The points $o_e + t(b - o_e)$ of S_J with $0 < t \leq 1$ lie in E_e , in ∂E_e only for $t = 1$, and have $y_J = 1 - t$. The leaf path consists of such points with $0 \leq y_J \leq 1 - \mathbf{t}_e$, and $y_J = 0$ only at m ; and $\mathbf{t}_e(E_e) \subset \{y_J \geq 1 - \mathbf{t}_e\}$, while $y_J < 1 - \mathbf{t}_e$ on the path except at its end, because $\eta_J \leq 1/4 < 1 - \mathbf{t}_e$.

Let $m_1, \dots, m_k$ be the model points on J in their order along J , with leaves $\bar{m}_1, \dots, \bar{m}_k$ in the same order, and let s be a linear function of norm one increasing from m_1 to m_k along J . Then $s(m_{i+1}) - s(m_i) \geq 4\theta_3$ and $|s(m'_i) - s(m_i)| \leq \theta_3$, so $s(m'_{i+1}) > s(m'_i)$; and $s(\mathbf{t}_e(\bar{m}_{i+1})) - s(\mathbf{t}_e(\bar{m}_i)) = \mathbf{t}_e(s(\bar{m}_{i+1}) - s(\bar{m}_i)) > 0$. In the affine coordinates (s, y_J) the points m_i , m'_i and $\mathbf{t}_e(\bar{m}_i)$ lie on the lines $y_J = 0$, η_J , $1 - \mathbf{t}_e$, in increasing order of s ; two segments of the same kind are two sides of a convex quadrilateral with strictly ordered ends and are disjoint, and a first and a second segment meet at most on $y_J = \eta_J$, that is, only if they share m'_i . $\square$

Proof of Lemma 8.4. The interior vertices. The three points b_c lie in the relative interiors of three distinct edges of the convex polygon $\mathbf{c}_{s_\Delta}$, so they are not collinear, $v_{s_\Delta} \in \mathbf{ri} \mathbf{c}_{s_\Delta}$, the segments $[v_{s_\Delta}, b_c]$ lie in $\mathbf{c}_{s_\Delta}$ and meet only at v_{s_Δ} , and the three angles at v_{s_Δ} are less than π ; their cyclic order is the order of the edges c along $\partial \mathbf{c}_{s_\Delta}$, which is also their order at $b_{\mathbf{c}_{s_\Delta}}$.

A point of $\mathbf{c}_{s_\Delta}$ lies on Γ_Y exactly when the bottom of its chain has lower support of positive dimension (Definition 4.6), so $\Gamma_Y \cap \mathbf{c}_{s_\Delta}$ is the union of the three replaced segments, and Γ'_e is a plane graph isomorphic to $\Gamma_Y \cap E_e$ with the same rotations and leaves. By the proof of Lemma 8.2(c), the function y for the edge τ with $\mathbf{c} = \tau + F(s)$ is minimal on $\mathbf{c}_{s_\Delta}$ exactly on $\mathbf{c}$; so $y(v_{s_\Delta}) > y(b_c)$, and y is strictly monotone along each closed arc of Γ'_e .

The graph. The closed arcs of Γ'_e meet only at common vertices (Proposition 4.7(3)), and $\mathbf{k}_e$ is injective with $\mathbf{k}_e(E_e) \subset \Omega_e$; with Lemma 8.3, $\mathcal{G}_e$ is a plane graph properly embedded in E_e , with leaves at the model points and all interior vertices and all edges except the leaf paths in Ω_e . *Monotonicity.* Let the arc of $\tau = [u, u']$ end at an original leaf $\bar{m}$ on $J = J_\omega$. Since J is parallel to $u' - u$, it suffices to show that y_J increases strictly from m along the arc. It does so along the leaf path, from 0 through η_J to $1 - \mathbf{k}_e$; along the closed arc of Γ'_e it increases strictly from $y_J(\bar{m}) = 0 = \min_{E_e} y_J$ by Lemma 8.2 and the previous paragraph, and $y_J \circ \mathbf{k}_e = 1 - \mathbf{k}_e + \mathbf{k}_e y_J$. If e lies in the interior of a two-face, the other end lies on the edge J' parallel to J , where $y_{J'}$ is a decreasing affine function of y_J . Edges between interior vertices are images of closed arcs of Γ'_e . For e in the interior of a two-face, $\Gamma_Y \cap E_e$ consists of the k disjoint arcs $\overline{C_Y([u_{i-1}, u_i], e)}$, the i -th joining the i -th original leaves on $J_\omega^\pm$ (Proposition 4.7(3)). The homothety has positive ratio, so it preserves rotations; the model points lie on the same edges J_ω as the original leaves, in the same order. $\square$

Proof of Lemma 8.5. We define Υ^M on the faces F of B_Y by induction on $\dim F$, together with the cells $C^M(\mathbf{p})$ in $\text{ri } F$; by Proposition 4.7(5), $\text{ri } F_\omega^*$ is the union of the cells $C_Y(\tau, \omega)$, $\tau \subseteq \omega^\vee$. On the vertices of B_Y , Υ^M is the identity. Let $F = F_\omega^*$, $\dim F \geq 1$, and suppose Υ^M defined on ∂F , mapping $C_Y(\mathbf{p}) \cap \partial F$ onto $C^M(\mathbf{p}) \cap \partial F$. If $\omega^\vee$ is a vertex u , then $\text{ri } F = C_Y(u, \omega)$; extend by coning from an interior point. If $\dim F = 1$ and $\dim \omega^\vee \geq 1$, then ω is a two-cell of a two-face G , $F = J_\omega$, and its cells of $\mathcal{P}_Y$ form the mixed subdivision of G° , subdivided by $\mathcal{T}$ with the heights $\check{h}$, and $\nabla_\omega^{H'}$; so $\text{ri } F$ is the union of the open segments $C_Y(u_i, \omega)$ and the points $C_Y([u_{i-1}, u_i], \omega)$, in the order $C_Y(u_0, \omega), C_Y([u_0, u_1], \omega), \dots, C_Y(u_k, \omega)$. Let $\Upsilon^M|_F$ fix the ends, map $C_Y([u_{i-1}, u_i], \omega)$ to the i -th model point, and be linear in between. If $\dim F = 2$ and $\dim \omega^\vee \geq 1$, then $\omega = e$ and $F = E_e$; by Propositions 4.7(5) and 4.10(2), $\Gamma_Y \cap F$ is of the same kind as the model arc system of e , with leaves the zero-cells on the edges J_ω , which $\Upsilon^M|_{\partial F}$ maps to the leaves of the model arc system in their cyclic order. Its complementary regions are the cells $C_Y(u, e)$, open discs. Extend $\Upsilon^M|_{\partial F}$ over $\Gamma_Y \cap F$ by a piecewise linear homeomorphism onto the model arc system (Lemma 8.4), and then over each complementary disc: its boundary circle is mapped piecewise linearly and homeomorphically onto the boundary circle of the corresponding complementary disc of the model arc system, and a piecewise linear homeomorphism between the boundaries of two piecewise linear discs extends to a piecewise linear homeomorphism of the discs (identify both with a simplex and cone). If $\dim F = 3$, extend by coning. The isotopy to the identity is obtained face by face, in increasing dimension, by the Alexander trick [Ale23], extended by coning as in the proof of Proposition 6.3(3). For (3): Υ^M maps $\text{int } F_n^*$ onto itself and $\tilde{R}_u^* = \bigcup_\omega C_Y(u, \omega)$ onto R_u^M , and Λ_Y is constant on these open sets, with the stated lattices and gluing (Proposition 4.11). $\square$

Proof of Lemma 8.7. (1) By Lemma 8.5(2) the cells of J_ω next to m are $C^M(u, \omega)$ and $C^M(u', \omega)$. Parametrise $J_\omega = \{p_\omega + s(u_k - u_0)\}$ as in Lemma 7.7(1); since $\langle u' - u, w_u - w_{u'} \rangle = -2$ and $u_k - u_0$ is a positive multiple of $u' - u$, $\alpha_{uu'}$ decreases as s increases. The end $s = s_+$ is the vertex $F_{\omega_1}^*$ of B_Y , $\omega_1 \subseteq u_0^\vee$, which is the zero-cell $C^M(u_0, \omega_1)$ and lies in the closure of $C^M(u_0, \omega)$, since $(u_0, \omega_1) \preceq (u_0, \omega)$, and in no other closure $\overline{C^M(u_j, \omega)}$; so along J_ω the

cells occur in the order $C^M(u_0, \omega), C^M([u_0, u_1], \omega), \dots, C^M(u_k, \omega)$ from that end (proof of Lemma 8.5). So at the model point of $[u, u'] = [u_{i-1}, u_i]$ the cell of u lies on the side of larger s , where $\alpha_{uu'}$ is smaller. (2) $(w_u, w_{u'}, n' - n, n_y)$ is a basis because $\langle u, \cdot \rangle, \langle u', \cdot \rangle$ take the values $(1, 0)$ and $(0, 1)$ on $w_u, w_{u'}$, and $\langle u' - u, n_y \rangle = 0$ gives the level lines. Near the interior of E_e only $0, n, n'$ are maximal, and $B_Y = \{\text{val}_n = -\max(0, y_e)\}$ is the graph of a function of (y_e, α_e, y) , since $n' - n, w_u - w_{u'}, n_y$ and n are linearly independent. The arc is $\{y_e = \alpha_e = 0\}$, and near A it separates E_e into $C^M(u, e)$ and $C^M(u', e)$ (Lemma 8.5(2)), which near the ends of A are adjacent to the cells of u and u' on the edges J_ω ; by (1) and the monotonicity of Lemma 8.4, α_e is negative on the first and positive on the second. $\square$

Proof of Lemma 8.8. (1) By Definition 8.1(2) the model arcs at m_P begin with segments of the level line of $\alpha_{uu'}$ through m_P , which is $\{\alpha_q = 0\}$, in F_{ab}^* and F_{cd}^* . By Lemma 7.20(4) and Lemma 8.7(1), near m_P these arcs separate F_{ab}^* and F_{cd}^* into their parts with $\alpha_q < 0$ and $\alpha_q > 0$, which are the cells of u and of u' , and $J_P \setminus \{m_P\}$ into $C^M(u, P) = \{\alpha_q < 0\}$ and $C^M(u', P) = \{\alpha_q > 0\}$. The other cases are the same. This is the description of the standard halves by closed cells of Lemma 6.6, in the chart T_q . (2) The three two-faces of B_Y containing J_ω are the E_{e_k} ; in each the model arc system has a leaf at m , and near m it separates E_{e_k} into $C^M(u, e_k)$ and $C^M(u', e_k)$, whose closures contain $C^M(u, \omega)$ and $C^M(u', \omega)$ respectively, since $(u', \omega) \preceq (u', e_k)$. (3) The closure of $C^M(u_i, e)$ contains the vertex, since $(\tau, e) \preceq (u_i, e)$, and near it is bounded by the two model arcs whose edges of $\mathcal{T}$ contain u_i ; the angle is less than π by Lemma 8.4. (4) is Lemma 8.7(2). Finally r_S is the smallest of the sizes of the neighbourhoods so obtained, and at most $\theta_4/4$. $\square$

Proof of Lemma 8.14. (1) *Step 1: the piece lies where W^* is the leg model.* In the leg model the point $\mathbf{x}(x_1, x_2, y)$ with $(x_1, x_2) = E_t(y_e, \alpha_e)$ has $\log |\kappa_e \chi_1| = Ly_e, |\hat{Z}\hat{Z}'| = |e^{x_1} - 1| \leq \mathcal{S}_G = 1 + e^{x_1}$ and $|\hat{Z}|^2 - |\hat{Z}'|^2 = x_2$, so $2 \log |\hat{Z}|$ and $2 \log |\hat{Z}'|$ are at most $\log \mathcal{S}_G + L|\alpha_e| + C$. Hence its tropical position is obtained, by lowering $|Z|$ and $|Z'|$, from the point with the same torus coordinates and $|\hat{Z}|^2 = |\hat{Z}'|^2 = \mathcal{S}_G e^{L|\alpha_e|}$, which lies within $C_T(|y_e| + |\alpha_e|) + C/L$ of $\mathbf{a}$, where C_T bounds the Lipschitz constants of the inverse charts T_e^{-1} . We take $\varsigma_{\max} \leq \theta_2/(4C_T)$, and smaller than a quarter of the distance between any two model arcs away from their common ends and of the distance from a model arc to the edges of E_e other than those containing its ends; then for t small the points of $\Pi_{\mathbf{a}, \varsigma}$ lie in the exact region of Lemma 7.18(1), where W^* is the leg model, and the same estimate places the divisor pieces in the exact regions listed in the statement.

Step 2: smoothness and fibres. The curves $\varrho \mapsto E_t(\varrho \cos \Theta, \varrho \sin \Theta)$ are smooth and embedded, with velocity $L(\cos \Theta, 2 \sin \Theta) \neq 0$ at $\varrho = 0$. For $(x_1, x_2) \neq 0$ the point of the section is determined, up to S_α^1 where $\hat{Z} = 0$, by: χ_1 negative real, $\hat{Z} \geq 0$, $\mathfrak{b}\nu_Z \nu_{Z'} \hat{Z} \hat{Z}' = e^{x_1} - 1$ and $\hat{Z}^2 - |\hat{Z}'|^2 = x_2$; it depends smoothly on (x_1, x_2) . Near $\varrho = 0$ both $|\hat{Z}\hat{Z}'|$ and $|\hat{Z}|^2 - |\hat{Z}'|^2$ are of order ϱ , and the S_α^1 -orbits of the section form the graph $\{\hat{Z}' = k(|\hat{Z}|^2) \bar{\hat{Z}}\}$ of a smooth function, a disc through the critical locus; for $\Theta = \pm\pi/2$ it is $\{\hat{Z}' = 0\}$ or $\{\hat{Z} = 0\}$. Sweeping with the rotation of χ_2 and moving y gives a four-manifold fibred over the sheet, whose fibre $\mathbb{T}_d$ collapses to the circle of χ_2 exactly at $\varrho = 0$; no boundary arises there, because the disc is smooth through its centre.

Step 3: the phase and the divisor part. On $\Pi_{\mathbf{a}, \varsigma}$, χ_1 is negative real, so $\chi^{n'-n}$ has phase π . Where $\Theta = \pi/2$, $x_1 = 0$, so $\chi_1 = -1/\kappa_e$, $\hat{Z}' = 0$ and $|\hat{Z}|^2 = x_2 \leq 2 \sinh(L\varsigma) = \mathcal{S}_G \sinh(L\varsigma)$.

Step 4: divisor pieces. Consider a divisor piece on $D_{u'}$ along a stretch from a model node P to Ω_e , say $e = [a, b]$ in the $b - a$ passage. On $D_{u'}$ the origin and the apexes of ω_1 vanish, since their Z' -exponent is one. By Lemma 7.18(1') and (2) every other monomial outside

P has weight zero there, so $F^{(0)}|_{D_{u'}} = \gamma_a x^a g_e$ with $g_e = 1 + \tilde{\chi}_1 + \psi_d \tilde{\chi}_2 + \psi_c \mu_P \tilde{\chi}_1 \tilde{\chi}_2$, and $I(x) \subseteq P$. By Lemma 7.18(4), W^* is transverse to $D_{u'}$ along the piece. On the branch, off the node neighbourhood, $|\tilde{\chi}_2| \leq t^{\theta_2/2}$, and $\partial g_e / \partial \tilde{\chi}_1 = 1 + \psi_c \mu_P \tilde{\chi}_2 + O(t^{\theta_1/4}/L) \neq 0$; so the branch is a graph $\tilde{\chi}_1 = \tilde{\chi}_1(\tilde{Z}, \chi_2)$ with $\tilde{\chi}_1 = -1 + O(t^{\theta_2/2})$. The bound $|\hat{Z}|^2 \leq \mathcal{S}_G \sinh(L\varsigma)$ is a smooth condition with positive $|\hat{Z}|$ -derivative on the branch. Hence the divisor piece is a disc bundle over an annulus in χ_2 , fibred as stated, with phase of χ_1 equal to π up to $O(t^{\theta_2/2})$. Over Ω_e the weights of c, d vanish and the piece is $\{Z' = 0, \chi_1 = -1/\kappa_e, |\hat{Z}|^2 \leq 2 \sinh(L\varsigma)\}$. Over the node neighbourhood the branch is $\mathcal{C}_P$, on which $x = \kappa_P/y$ with $|x| \leq |\kappa_P|^{1/2}$, so $\tilde{\chi}_1 = -\mu_P^{-1}(1 - \mu_P^{1/2}x)$ has phase π up to $O(\eta_0^{1/2})$; there the divisor piece is Π_q^+ over $\mathcal{C}_P$ by Lemma 7.22(1). The stretches from a model vertex to Ω_e , those in Ω_e and those at the turning points are treated in the same way, with Lemma 7.18(1'), (1) and (3), where g_e is a binomial or the trinomial of ω . The defining inequality does not depend on the reference monomial, and $\nu_Z/\nu_{Z'}$ is one function along each arc, which gives the agreement on overlaps.

(2) By Lemma 7.18(3) the model is exact, and $\hat{f}$ is a trinomial of full rank, so $\{\hat{f} = 0\}$ is smooth; by Lemma 7.18(4) the intersection with $D_{u'}$ is transverse near the vertex and along the tentacles, since there the monomials of positive weight that do not vanish on $D_{u'}$ lie in ω and are affinely independent. On a tentacle the cut-offs reduce $\hat{f}$ to a binomial and Π_v to the divisor piece of the arc.

(3) The model is exact by Lemma 7.18(1) for the three-cone. On D_{u_1} the equation reduces to $1 + \kappa_e \chi_1 = 0$. Away from the vertex, where $z_3 \neq 0$, the model is the leg model with $(Z, Z') = (z_1, z_2)$, and Π_t is the divisor piece of that arc. The two inequalities are conditions on $(|z_2|, |z_3|)$, so the union is invariant under the rotations of z_2, z_3 ; it is a four-manifold with corners. $\square$

Proof of Lemma 8.16. (1) Since $\langle u, n \rangle = \langle u', n \rangle = -1$ and $\{u, u'\}^\perp \cap N$ is spanned by $n' - n$ and n_y , we can write $n = -w_u - w_{u'} + a_1(n' - n) + a_2 n_y$. Then $u' - u = e_{Z'} - e_Z$, since it pairs to $-1, 1, 0, 0$ with the basis, and $\Lambda^M = n^\perp \cap M$ on $\text{int } F_n^*$ is $\{z e_Z + z' e_{Z'} + k_1 e_{\chi_1} + k_2 e_{\chi_2} : z + z' = a_1 k_1 + a_2 k_2\}$, with basis $u' - u, \hat{e}_1, \hat{e}_2$. On it d_A^* and g_A^* are the coordinates of $\hat{e}_1$ and $\hat{e}_2$; they vanish on $u' - u$ and form a basis of $\text{Inv}^*(A) = (u' - u)^\perp$. For $v_0 = k_g g_A^* + k_d d_A^*$ primitive, $v_0(x(u' - u) + p_1 \hat{e}_1 + p_2 \hat{e}_2) = k_d p_1 + k_g p_2$, so $\ker v_0 \cap \Lambda^M = \langle u' - u, k_g \hat{e}_1 - k_d \hat{e}_2 \rangle$, with $\langle \ell_0, n' - n \rangle = k_g$ and $\langle \ell_0, n_y \rangle = -k_d$. The monodromy $\lambda \mapsto \lambda + \langle \lambda, n' - n \rangle (u' - u)$ (Proposition 4.10(4), transported by Υ^M) fixes $u' - u$ and sends ℓ_0 to $\ell_0 + k_g(u' - u)$.

(2) The character $\chi_2^{k_g} \chi_1^{k_d} = \chi^{k_g n_y + k_d(n' - n)}$ is fixed by $u' - u$, which does not move χ_1, χ_2 , and by ℓ_0 , which multiplies it by $e^{i(-k_g k_d + k_d k_g)\vartheta} = 1$; so the condition is invariant under $\mathbb{T}(v_0)$. The leg model with χ_2 fixed is exact over D_ς by Lemma 7.18(1) and Step 1 of the proof of Lemma 8.14(1); it is a torus fibration over D_ς with one singular fibre, a pinched torus, over $\mathfrak{r}_A$, and $\pi_1(U_{D_\varsigma}) \cong \mathbb{Z}$, generated by the rotation of χ_1 , because the α -circle dies at the pinch. The rotation $\hat{e}_2$ lies in $\mathbb{T}_d$: it pairs to zero with $n' - n$ and with n , so it fixes the phases of $0, n, n'$ relative to each other and preserves W^* on the exact region; it fixes χ_1 and multiplies χ_2 by $e^{i\vartheta}$ and Z by $e^{ia_2\vartheta}$. Rotating by $-\arg \chi_2$ maps a point of $\Pi(v_0)$ to a point of U_{D_ς} , and the condition becomes $k_g \arg \chi_2 + k_d \arg \chi_1 \in 2\pi\mathbb{Z}$, which has $|k_g|$ solutions $\arg \chi_2$ modulo 2π over each point. So the condition cuts out an $|k_g|$ -sheeted cover, connected because $\gcd(k_g, k_d) = 1$; it is smooth, since $d(k_g \arg \chi_2 + k_d \arg \chi_1)(\hat{e}_2) = k_g \neq 0$. The fibre over a point of D_ς is $\mathcal{S}_\alpha^1$ times the circle $\{k_g \arg \chi_2 + k_d \arg \chi_1 = 0\}$, which is $\mathbb{T}(v_0)$. Over $\mathfrak{r}_A$ the α -circle collapses at the $|k_g|$ points of the ℓ_0 -circle where $\arg \chi_1 = \pi$, so the fibre is of type $I_{|k_g|}$. The phase of $\chi^{k_g n_y + k_d(n' - n)}$, a lift of v_0 , is 0 on $\Pi(v_0)$ and on $\mathfrak{s}^\vee$.

(3) With $k_g = 0$ and $k_d = \pm 1$, $\ell_0 = \mp \hat{e}_2$, so $\mathbb{T}_d$ is generated by $u' - u$ and $\hat{e}_2$, both fixing χ_1 . Their weight matrix on (Z, Z', χ_2) is $\begin{pmatrix} -1 & 1 & 0 \\ a_2 & 0 & 1 \end{pmatrix}$, whose 2×2 minors include -1 , so $\mathbb{T}_d$ acts freely wherever $Z, Z', \chi_2 \neq 0$. It preserves the moduli of all monomials, hence the weights, and multiplies each monomial by a character; on the exact region of Lemma 7.18(1), where only $0, n, n'$ have positive weight and $\mathbb{T}_d \subseteq (\{n, n'\}^\perp \cap M) \otimes U(1)$ fixes their phases, it maps W^* to itself. The positive points over $\mathfrak{D}_\zeta$ lie in that region (Step 1 of the proof of Lemma 8.14(1), and Lemma 8.13(2)); there $\chi_1 > 0$, so $1 + \kappa_e \chi_1 \neq 0$ and $ZZ' \neq 0$, and $\chi_2 \neq 0$, so $\mathbb{T}_d$ acts freely. It preserves tropical positions, so orbits over distinct base points are disjoint, and $\mathbb{T}_d \cdot \mathfrak{s}^\vee(\mathfrak{D}_\zeta) \cong \mathfrak{D}_\zeta \times T^2$. The phase of $\chi^{n'-n}$ is 0 on it, matching the section. $\square$

Proof of Lemma 8.19. (1) Given $(\zeta, \ell, \varphi, \vartheta_\alpha) \in \mathbf{K} \times T^2 \times S^1$, put $\chi_j := \kappa_j^{-1} e^{L\ell_j + i\varphi_j}$, $\hat{\mu} := e^{L\zeta}$ and $\hat{f} := 1 + \kappa_1 \chi_1 + \kappa_2 \chi_2$, and let x be the point with these torus coordinates and with normalised Cox coordinates

$$(E.1) \quad |\hat{Z}|^2 = \frac{1}{2}(\hat{\mu} + (\hat{\mu}^2 + 4|\hat{f}|^2)^{1/2}), \quad \arg \hat{Z} = \vartheta_\alpha, \quad \hat{Z}' = -\hat{f}/(\vartheta \hat{Z}).$$

Then $\vartheta \hat{Z} \hat{Z}' + \hat{f} = 0$ and $|\hat{Z}|^2 - |\hat{Z}'|^2 = \hat{\mu}$. On $\mathbf{K}$, $\hat{\mu} \geq e^{L\zeta_0/2} \mathcal{S}_G \geq \mathcal{S}_G \geq |\hat{f}|$, since $\mathcal{S}_G = e^{L\Lambda_L(\ell)}$; so $\hat{\mu} \leq |\hat{Z}|^2 \leq \hat{\mu} + \mathcal{S}_G$ and $|\hat{Z}'| \leq \mathcal{S}_G/|\hat{Z}|$. Let x^* have the same torus coordinates as x , $|\hat{Z}(x^*)| = |\hat{Z}(x)|$ and $|\hat{Z}'(x^*)| = \mathcal{S}_G/|\hat{Z}(x)|$; then x is obtained from x^* by lowering $\log |Z'|$. At x^* the origin monomial is $\mathcal{S}_G$ times the reference monomial, at least the largest monomial of $[u, u']^\vee$ and at most three times it, the torus coordinates satisfy $|\ell|_\infty \leq 2r_\rho$, with $\ell = 0$ on J_ω , and the offset $\frac{1}{L} \log(|\hat{Z}|/|\hat{Z}'|) = \frac{1}{L} \log(|\hat{Z}|^2/\mathcal{S}_G)$ lies between $\zeta_0/2$ and $4\zeta_0 + 2r_\rho + O(1/L) \leq 6r_\rho + O(1/L)$. So, in the chart of B_Y near m_v whose inverse has Lipschitz constant at most C_T , the tropical position of x^* lies within $C_T(4r_\rho + 6r_\rho) + C/L = 10C_T r_\rho + C/L < \theta_2$ of $m_v \in J_\omega^b$, by the choice of r_ρ , and by Lemma 7.18(3) the weights at x^* , and hence at x , are those of the vertex model. Therefore $x \in W^*$. Conversely, a point of W^* in this region with $\hat{\mu} > 0$ is determined by $(\hat{\mu}, \chi_1, \chi_2, \arg \hat{Z})$: the equation fixes $\hat{Z} \hat{Z}'$, and $\hat{\mu}$ then fixes $|\hat{Z}|$ and $|\hat{Z}'|$, as in Lemma 7.19. So the map Ψ_v defined by (E.1) is a smooth bijection of $\mathbf{K} \times T^2 \times S^1$ onto its image $W_\mathbf{K}^*$, with smooth inverse Φ_v . The circle S_α^1 acts by $(\hat{Z}, \hat{Z}') \mapsto (e^{i\vartheta} \hat{Z}, e^{-i\vartheta} \hat{Z}')$ and fixes $\hat{\mu}$ and the torus coordinates, so it rotates the last factor. As $\mathbf{K}$ is convex, $W_\mathbf{K}^* \simeq T^3$.

(2) The piece of depth $2\zeta_0$ is the vertex piece Π_v , on which $Z' = 0$ and $\hat{f} = 0$, so its outer part is $\{\hat{f} = 0, \hat{\mu} = |\hat{Z}|^2 = \mathcal{S}_G \sinh(2L\zeta_0)\}$. Over S'_v the base map is given by the vertex description, and the retraction moves a point by at most a constant times its distance from the tripod, which is $O(1/L)$ on $\{\hat{f} = 0\}$; so the points of A have $|\ell|_\infty \leq r_\rho + C/L$ and $\zeta = \Lambda_L(\ell) + \frac{1}{L} \log \sinh(2L\zeta_0) = \Lambda_L(\ell) + 2\zeta_0 - \frac{1}{L} \log 2 + o(1/L)$, which lies in $\mathbf{K}$ because $\Lambda_L(\ell) \leq 2r_\rho + \frac{1}{L} \log 3$. For $b \in S'_v$, $|\text{Log } \mathfrak{s}^\vee(b)/L - b| \leq C/L$ (Lemma 8.13(2)); at this positive point $\hat{f} = \mathcal{S}_G$, so $|\hat{Z} \hat{Z}'| = \mathcal{S}_G$, and $|\hat{Z}/\hat{Z}'| = e^{2L\zeta_0 + O(1)}$ since b lies at offset $2\zeta_0$; hence $\zeta = \Lambda_L(\ell) + 2\zeta_0 + O(1/L)$ and $|\ell|_\infty \leq r_\rho + C/L$. The $\mathbb{T}(\langle \cdot, e_k \rangle)$ -orbits through these points and $\mathbb{T}_{b_1}$ have the same (ζ, ℓ) and so lie in $W_\mathbf{K}^*$. At the points c_k , where $|\ell|_\infty = r_\rho$, the third monomial of $\hat{f}$ is at most Ct^{cr_ρ} times the other two on the slice of Π_v , so the slice is a graph over a $\mathbb{T}(\langle \cdot, e_k \rangle)$ -orbit with phase of χ^{e_k} equal to $\pi + O(t^{cr_\rho})$, and B_k is defined. Its second stage has the (ζ, ℓ) of points of A . In its third stage $\hat{\mu}$ lies between its values at the two end points and ℓ lies on the segment between theirs; since these end points have $|\ell_0 - \ell_1|_\infty \leq C/L$ and $\zeta_i \geq \Lambda_L(\ell_i) + 2\zeta_0 - C/L$, and Λ_L is convex and 1-Lipschitz for $|\cdot|_\infty$, the point satisfies $\zeta \geq \min_i \zeta_i \geq \Lambda_L(\ell) + 2\zeta_0 - C'/L$ and $\zeta \leq \max_i \zeta_i$, so it lies in $\mathbf{K}$. The vertex piece, the orbits, the fibre $\mathbb{T}_{b_1}$ and the collar paths are invariant under S_α^1 , the collar paths because they keep the position on the S_α^1 -orbit fixed.

(3) The boundary of A consists of its fibres over c_1, c_2, c_3 . The second stage of B_k begins on the fibre of A over c_k and its third stage ends on the fibre of A^+ over c_k , so these terms cancel in ∂D . The orbits of A^+ over the three arms of S'_v , oriented as part of $\partial g_{\leq 2s_0}$, run towards b_1 , so ∂A^+ contains $+\sum_k \Phi_v^{-1}(G_k \times S^1) = -\sum_k s_k \Phi_v^{-1}(G_k \times S^1)$ over b_1 ; and $\partial(\mathcal{Q}'(1) \times S^1) = \sum_k s_k G_k \times S^1$ cancels it. So D is a cycle. By (2), $D = \Phi_v^{-1}(D' \times S^1)$ for a 2-cycle D' in $\mathbf{K} \times T^2$, the product being taken with the fundamental cycle of the S^1_α -orbits. The G_k are the circles through the phases of $\mathfrak{s}^\vee(b_1)$ in the directions of $\ker\langle \cdot, e_k \rangle$ modulo ℓ , and $\sum_k s_k [G_k] = 0$ in $H_1(T^2)$ [CBM14, §7.7, Step 2]; so there is a 2-chain $\mathcal{Q}'_0$ in $\{k_1\} \times T^2$ with $\partial \mathcal{Q}'_0 = \sum_k s_k G_k$. With $\mathcal{Q}'_0$ in place of $\mathcal{Q}'(1)$ the cycle D' has a class $n[\{k_1\} \times T^2]$ in $H_2(\mathbf{K} \times T^2) \cong \mathbb{Z}$; put $\mathcal{Q}'(1) := \mathcal{Q}'_0 - n[\{k_1\} \times T^2]$, so that $[D'] = 0$. This fixes $\mathcal{Q}'(1)$ modulo boundaries.

Let $h: [0, 1] \times \mathbf{K} \times T^2 \rightarrow \mathbf{K} \times T^2$, $h(s, k, \varphi) = ((1-s)k + sk_1, \varphi)$, which takes values in $\mathbf{K} \times T^2$ as $\mathbf{K}$ is convex, and let $P_h(c) := h_*([0, 1] \times c)$ be its prism operator, so that $\partial P_h(c) + P_h(\partial c) = h_1(c) - c$. Then $\partial P_h(D') = h_1(D') - D'$. The cycle $h_1(D')$ lies in the surface $\{k_1\} \times T^2$ and is homologous there to 0, since $[h_1(D')] = [D'] = 0$ and the inclusion of the fibre is a homotopy equivalence; let E'_1 be a 3-chain in $\{k_1\} \times T^2$ with $\partial E'_1 = h_1(D')$. Put

$$(E.2) \quad \mathcal{C}_v := \Phi_v^{-1}((E'_1 - P_h(D')) \times S^1).$$

Then $\partial \mathcal{C}_v = \Phi_v^{-1}((h_1(D') - h_1(D') + D') \times S^1) = D$, and $\mathcal{C}_v \subset W_{\mathbf{K}}^*$. $\square$

Proof of Lemma 8.21. (1) The open star of $C^M(u, \omega)$ consists of cells $C^M(u, \omega')$, $\omega' \subseteq \omega$, in the relative interiors of the faces $F_{\omega'}^*$, on which exactly the monomials of $\{0\} \cup \omega'$ are maximal. By compactness of $\bar{V}$ there is $\theta_V > 0$ such that at every point of $\bar{V}$ every lattice point outside $\{0\} \cup u^\vee$ has tropical deficit at least θ_V . Let $x \in W^* \cap T$ with $b = \text{sh}_u(x) \in V$ and $\lambda = \lambda_u(x) < \epsilon$, so that the tropical position is $p = b + \lambda u$. For $m \notin u^\vee \cup \{0\}$, $\text{val}_m(p) - \text{val}_0(p) = \text{val}_m(b) - \text{val}_0(b) + \lambda(e_u(m) - 1) \leq -\theta_V + \epsilon e_{\max}$, since $e_u(m) \geq 1$; with $\epsilon \leq \theta_V/(2e_{\max})$ these monomials are at most $t^{\theta_V/2}$ times the origin, up to a bounded factor. The monomials of $u^\vee$ are free of Z_u and the origin is linear in it, so by Lemmas D.2 and D.1, $\partial F^{(0)}/\partial Z_u$ is the derivative of the origin monomial up to a relative error $O(t^{\theta_V/2} + 1/L)$, at every such point and at the points of D_u in their closure. Fix the other Cox coordinates $\mathbf{z}$ of the chart, with $\text{Log } \mathbf{z}/L \in \pi_u(V)$, and let Z_u run over the disc $\{\lambda \leq \epsilon\}$. On it $F^{(0)}(\cdot, \mathbf{z})$ is a C^1 -small perturbation of an affine map with invertible linear part, hence injective; on the boundary circle the origin monomial exceeds the sum of the others, since at $\lambda = 0$ the position lies on B_Y , where the origin is maximal, and raising $|Z_u|$ by $t^{-\epsilon}$ multiplies the origin by $t^{-\epsilon}$ and the monomials of $u^\vee$ by one. So $F^{(0)}(\cdot, \mathbf{z})$ has exactly one zero in the disc, in its interior, and W_V^* is the graph of φ , smooth by the implicit function theorem. In the chart, $U_u = \mathbb{C}_{Z_u} \times (\mathbb{C}^*)^3$, and the cocharacter u acts only on Z_u , so $H_1(U_u) = M/\mathbb{Z}u$.

(2) The open star of $C^M(\tau, n)$ consists of cells $C^M(\tau', n)$, $\tau' \subseteq \tau$, in $\text{int } F_n^*$, where exactly 0 and n are maximal; let $\theta_V > 0$ bound from below the deficits of the other lattice points on $\bar{V}$. At $p = b + \lambda m_n$ with $b \in \bar{V}$ and $|\lambda| < \epsilon$ they are at least $\theta_V - D\epsilon|m_n|$, and $\text{val}_n(p) - \text{val}_0(p) = \lambda$. Write the torus as $\text{Hom}(m_n^\perp \cap N, \mathbb{C}^*) \times \mathbb{C}^*$, with coordinates $\mathbf{z}$ and $w = \chi^n$. For $\epsilon \leq \theta_V/(2D|m_n|)$ and t small, $F^{(0)} = \gamma_0 x^0(1 + (\gamma_n/\gamma_0)\chi^n + E)$ with E small in C^1 relative to the two tied terms (Lemma D.1); so W_V^* is the graph $w = \varphi(\mathbf{z})$, with $|\log|(\gamma_n/\gamma_0)\varphi||$ bounded uniformly in t (equivalently $\log|\varphi| = H(n)L + O(1)$, since $\gamma_n = c_n t^{H(n)}$ and $H(0) = 0$), and for each t the function $(\gamma_n/\gamma_0)\varphi$ stays close to -1 ; so its loops do not wind in w .

(3) For regions of the same type the inclusion is immediate. Let V' be of type u and V of type n . Then $V' \subseteq R_u^M \cap \text{int } F_n^* = C^M(u, n)$ (Lemma 8.5(3)). Let $x \in W_{V'}^*$, in the torus, with $b = \text{sh}_u(x)$ and $\lambda = \lambda_u(x) < \epsilon$. By compactness of $\overline{V'} \subset \text{int } F_n^*$ there is $\theta'_{V'} > 0$ bounding from below, on $\overline{V'}$, the deficits of the points of $u^\vee \setminus \{n\}$. The monomials of $u^\vee$ all change by the same amount along u , so at $p = b + \lambda u$ the monomial n exceeds the origin by $-\lambda$ and every other monomial of $u^\vee$ by at least $\theta'_{V'}$, while the monomials outside $u^\vee$ lie below the origin; since at a point of W^* the largest monomial, of weight one, cannot exceed the sum of the others by a factor larger than $|\mathcal{A}|$, $\lambda \geq -C/L$. The same comparison excludes the points of $W_{V'}^* \cap D_u$: there the origin and the monomials outside $u^\vee \cup \{0\}$ vanish, since they contain Z_u , while n exceeds the other monomials of $u^\vee$ by the factor $t^{-\theta'_{V'}}$ up to a bounded factor. So $W_{V'}^*$ lies in the torus, $\lambda_n(x) = -\lambda$ with $|\lambda_n(x)| < \epsilon$ once $C/L < \epsilon$, and $\text{sh}_n(x)$ lies within $C\epsilon$ of b , hence in V . In the identifications of (1) and (2) the inclusion induces the inverse of the isomorphism $n^\perp \cap M \rightarrow M/\mathbb{Z}u$ induced by $T \subset U_u$, which is the gluing of Λ^M on $C^M(u, n)$ (Lemma 8.5(3)). $\square$

Proof of Lemma 8.22. Except for the points $\mathfrak{s}^\vee(b)$, which are treated by Lemma 8.13(2) below, the points in question lie in the exact models of Lemmas 8.14 and 8.16 and of the collar paths, where W^* is given by $g_e + \vartheta \hat{Z} \hat{Z}' = 0$ in normalised coordinates, g_e the weighted sum of the monomials of $[u, u']^\vee$ divided by $\gamma_n x^n$ and $|\vartheta| = 1$; so $|\hat{Z} \hat{Z}'| = |g_e| \leq \mathcal{S}_G$. At a point of the torus, $T_e(\text{Log } x/L) = (x_1, \log |\hat{Z}/\hat{Z}'|, \log |\chi_2|)/L + O(1/L)$ near an arc, and $T_q(\text{Log } x/L) = (s_1, s_2, \log |\hat{Z}/\hat{Z}'|)/L + O(1/L)$ near a node (Lemma 7.20(4)). Near a model vertex, at a point of $W^* \cap D_{u'}$ the three monomials of $\hat{f}$ sum to zero, so the torus coordinates of the point, divided by L , lie within C/L of the tropical line of $\hat{f}$, which near the model vertex is the tripod of the three arcs; the retraction onto the tripod moves a point p by at most $C \text{dist}(p, \text{tripod})$. Near a turning point the offsets from its two arcs are affine coordinates, and each is computed from the modulus of one Cox coordinate. Finally $\frac{1}{L} \log \mathcal{S}_G$ is, up to $O(1/L)$, the value $\max(0, y_e)$ of $\text{val}_0 - \text{val}_n$ on B_Y .

The side of $D_{u'}$. Suppose that $x \in U_{u, u'}$ has $\hat{\mu}(x) \geq \mathcal{S}_G(x) \sinh(L\alpha_e)$ with $\alpha_e \geq \varsigma/2$. Then $\hat{Z} \neq 0$, so $x \in U_{u'}$. Its u' -shadow is determined by $\log |Z|$, $\log |\chi_1|$, $\log |\chi_2|$, since $\langle u', w_u \rangle = 0$. At the shadow the origin equals the largest monomial of $u'^\vee$, and these monomials are free of Z' , so they have the same values at x and at the shadow; the largest of them is, up to a bounded factor, $\mathcal{S}_G |\gamma_n x^n|$. Hence at the shadow $\log |\hat{Z} \hat{Z}'| = \log \mathcal{S}_G + O(1)$, and its offset from the arc is $\frac{1}{L}(2 \log |\hat{Z}(x)| - \log \mathcal{S}_G) + O(1/L)$. Since $\hat{\mu} \leq |\hat{Z}|^2 \leq \hat{\mu} + \mathcal{S}_G$, this offset is $\frac{1}{L} \log(\hat{\mu}/\mathcal{S}_G) + O(1/L)$, which is $\alpha_e + O(1/L)$ when $\hat{\mu} = \mathcal{S}_G \sinh(L\alpha_e)$. Moreover $\lambda_{u'}(x) = \frac{1}{L} \log(|\hat{Z} \hat{Z}'|/\mathcal{S}_G) + O(1/L) \leq C/L$. The side of D_u is symmetric.

Points on B_Y . Suppose that $|g_e(x)| \geq \gamma \mathcal{S}_G(x)$ with $\gamma > 0$ and $\hat{\mu}(x) = \mathcal{S}_G \sinh(L\alpha_e)$ with $|\alpha_e| \leq 2\varsigma$. Then $\log |\hat{Z} \hat{Z}'| = \log \mathcal{S}_G + O_\gamma(1)$ and $\log |\hat{Z}/\hat{Z}'| = L\alpha_e + O_\gamma(1)$, so x lies in the torus and its tropical position is, up to $O_\gamma(1/L)$, the point of B_Y with coordinates (y_e, α_e, y) ; all its shadows are within $O_\gamma(1/L)$ of that point, with $|\lambda| \leq O_\gamma(1/L)$. For the u -shadow this uses a Lipschitz inverse of π_u on $\text{Vis}(u)$ near $\overline{R_u^M}$: the compact set $\overline{R_u^M}$ lies in the union of the relative interiors of the faces $F_{\omega'}^*$ with $\omega' \subseteq u^\vee$, each point of which has a neighbourhood in B_Y covered by the facets F_m^* , $m \in \omega'$, and u is transverse to each facet F_m^* , $m \in u^\vee$, since $\langle u, m \rangle = -1$. The same holds for the u -shadows of the points $\mathfrak{s}^\vee(b)$ below.

The pieces. For a point of $\Pi_{a, 2\varsigma}$ over (ϱ, y) , $(y_e, \alpha_e) = \varrho(\cos \Theta, \sin \Theta)$, $\varrho \geq \varsigma$, we have $x_1 = Ly_e$, $\mathcal{S}_G = 1 + e^{x_1}$, $\hat{\mu} = \mathcal{S}_G \sinh(L\alpha_e)$ and $|g_e|/\mathcal{S}_G = \tanh(L|y_e|/2)$, and $\pi_g(x) = T_e^{-1}(y_e, \alpha_e, y)$. If $L|y_e| \geq 1$ the previous paragraph applies with $\gamma = 1/3$; this covers the

case $|y_e| \geq \varsigma_1$ of (1), and gives (2) with $L_{\varsigma_2} := C_T/\varsigma_2$. If $|y_e| < \varsigma_1$, then $|\alpha_e| \geq \varsigma/2$, and the first paragraph applies on the side of the sign of α_e ; moreover $\pi_g(x)$, which lies within $C_{T\varsigma_1}$ of E_e at offset at least $\varsigma/2$ from the arc, lies in $R_{u'}^M$ for $\alpha_e > 0$ and in R_u^M for $\alpha_e < 0$, and in no other vertex region, if ς_1 is small: a compact part of $C^M(u', e)$ has a neighbourhood in B_Y contained in $R_{u'}^M$, whose size depends on the slopes of the walls $C^M([u, u'], n)$, $C^M([u, u'], n')$ created by the coning in the proof of Lemma 8.5. For a divisor piece the first paragraph applies with $\hat{\mu} = |\hat{Z}|^2$ and $y_e = O(1/L)$; the node piece over $\mathcal{C}_P$, the vertex pieces and the turning pieces are divisor pieces. On the node piece over the thimble $|g_P| \leq 2|\kappa_P|$, the torus coordinates are within $O(r)$ of the node point, and the first paragraph gives $\text{sh}_{u'}(x) = T_q^{-1}(0, 0, \alpha_q) + O(1/L) = \pi_g(x) + O(1/L)$ on the half-axis $C^M(u', P)$. The crossing pieces with $k_g \neq 0$ are treated as $\Pi_{a, 2\varsigma}$, the argument of χ_1 being arbitrary: $|g_e| = |1 + \kappa_e \chi_1|$ lies between $|e^{x_1} - 1|$ and $\mathcal{S}_G$. On the crossing pieces with $k_g = 0$, $\chi_1 > 0$, so $|g_e| = \mathcal{S}_G$ and the paragraph on points on B_Y applies with $\gamma = 1$.

Collar paths and the section. The first stage of a collar path lies on the pieces of depths between ς and 2ς . In the second stage $\hat{\mu}$ and $|\chi_1|, |\chi_2|$ are those of the piece at depth 2ς and only the phase of $\chi^{\tilde{v}}$ changes. Along an arc $|g_e|$ then stays between $|e^{x_1} - 1|$ and $\mathcal{S}_G$, and the dichotomy used for the pieces applies; near a node or a model vertex the piece lies on the side of $D_{u'}$, with $\hat{\mu} = \mathcal{S}_G \sinh(2L\varsigma)$, and the paragraph on the side of $D_{u'}$ applies, since it uses only $\hat{\mu}$ and the moduli. The point $\mathfrak{s}^{\vee}(b)$ satisfies $|\text{Log } \mathfrak{s}^{\vee}(b)/L - b| \leq C/L$ by Lemma 8.13(2), and the end point of the second stage has, by the preceding paragraphs, its shadows within C/L of b ; so the logarithms of the moduli of the torus coordinates at the two points differ by $O(1)$, and their offsets $\frac{1}{L} \text{arsinh}(\hat{\mu}/\mathcal{S}_G)$ differ by $O(1/L)$. In the third stage the phase of $\chi^{\tilde{v}}$ is 0, $\hat{\mu}$ lies between its values at these two points and the logarithms of the moduli lie between theirs. Along an arc χ_1 is then positive real, so $g_e = \mathcal{S}_G$, and the paragraph on points on B_Y applies with $\gamma = 1$ and $|\alpha_e| \leq 2\varsigma + C/L$, or the paragraph on the side of D_u or $D_{u'}$ applies where $|\alpha_e| \geq \varsigma/2$; near a node or a model vertex, $\hat{\mu} \geq \mathcal{S}_G \sinh(L\varsigma)$ along the stage, and the paragraph on the side of $D_{u'}$ applies. So the points of the collar paths have shadows within C/L of b . For a tripod summand, the points of the orbits A^+ of Lemma 8.19 over $b \in S'_v$ and of its fibre $\mathbb{T}_{b_1}$ have the $\hat{\mu}$ and the moduli of the torus coordinates of $\mathfrak{s}^{\vee}(b)$, respectively $\mathfrak{s}^{\vee}(b_1)$, and arbitrary phases; by Lemma 8.19(1), $\hat{\mu} \geq \mathcal{S}_G e^{L\varsigma_0/2}$ there, and the paragraph on the side of $D_{u'}$ uses only these moduli, so these points too have shadows within C/L of b .

Consequence. If V is of type u , then $\pi_g(x) \in V \subseteq R_u^M$, and by (1) $\text{sh}_u(x) \in V$ and $\lambda_u(x) < \epsilon$ for t small, uniformly for $\pi_g(x) \in \mathcal{W}$; if x is not in the torus, it lies on D_u . If V is of type n , then $\bar{V} \subset \text{int } F_n^*$ lies at positive distance from the two-skeleton, and (2) applies. $\square$

APPENDIX F. CONSTRUCTION OF THE RESOLUTION-SIDE CHAINS

This appendix gives the complete construction and proof of Theorem E, whose statement and geometric meaning are discussed in Section 10. We first resolve the affine base, put the strict cycle in standard position and construct a localized smooth hypersurface $\mathcal{W}$. We then define the three-dimensional domain $\mathfrak{D}(S)$, construct its map into $\mathcal{W}$, compute the node degrees, and transport the map to $\hat{X}$. The assembly is in Appendix F.8; Appendix F.9 records the limit of the comparison with relative tropical cycles. The proofs of the regularity, region, positive-section, fundamental-group and cover lemmas follow in Appendices F.10–F.14.

The parameter of Gross's degeneration is $t = -1/\varkappa$, with $\varkappa > 1$ large. The localized hypersurface $\mathcal{W}$ is generally nonholomorphic; an ambient isotopy carries it to the algebraic small resolution. As usual in toric geometry, σ and σ' also denote cones; the profile σ occurs only as a subscript of X'_σ , B'_σ and P'_σ .

Of the hypotheses we use (Proj) and the pair $(\mathcal{T}, \check{h})$ of Hypothesis (U), with $\check{h}$ normalised as in Section 3.3, so that $\check{h}(m)$ is odd and positive for every $m \in \partial\Delta^\circ \cap M$; the pulling refinement and the mirror family play no role. Throughout, j is piecewise linear.

Inputs from the other parts. The toric resolution data: Proposition 3.4 on the small resolutions (and, through it, Lemma 3.3), Proposition 3.9 on the exceptional surfaces, used in Remark F.26, and Remark 3.10 on the ties. The base and its discriminant: Lemmas 3.12, 3.14 and 3.17, the passages at a node, and Propositions 4.7 and 4.10 on the complex of dual cell pairs, whose proofs do not use the present section. The tropical cycles: the conditions of [CBM14, Def. 7.2] and Lemma 5.3 (Section 5.2), Lemma 5.4, the node coefficients (Section 5.3) and, in a remark, Proposition 5.7. Only for the last assertion of Theorem E and for Appendix F.9: Theorems C and D, Proposition 8.26, and Definition 7.23 of the orientations of the vanishing spheres.

F.1. The resolved base. We compare the discriminants before and after splitting each node parallelogram along a diagonal. The description by dual cell pairs controls both the topology of this modification and the transport of the integral tangent lattices.

We use the complex of dual cell pairs of Section 4. For a fan-side subdivision $\mathcal{F}$ of $\partial\Delta$, a pair (τ, ω) of a simplex τ of $\mathcal{T}$ and a cell ω of $\mathcal{F}$ is *dual* if $\langle u, n \rangle = -1$ for all vertices u of τ and n of ω . The base built from $(\mathcal{T}, \check{h})$ and $\mathcal{F}$ is the disjoint union of open cells $C_B(\tau, \omega)$ of dimension $3 - \dim \tau - \dim \omega$, one for each dual pair, whose closures are balls (Proposition 4.7); the discriminant is the union of the cells with $\dim \tau \geq 1$ and $\dim \omega \geq 1$, a graph whose edges, the *arcs*, are the one-cells $C_B(\tau, \omega)$ with τ and ω edges, each a union of legs (Proposition 4.10). We write F_τ for the face of $\nabla^{\check{h}}$ with inner normal cone $\text{cone}(\tau)$, and $F_u, F_{uu'}$ for its facets and two-faces.

Labels. In this appendix the diagonal of a node parallelogram $P = abcd$ is always the diagonal $\{a, c\}$ that the resolution $\hat{X}$ under consideration induces on it; by Section 2.2 this fixes the circuit circ_q and the node coefficients ρ_q , and a change of diagonal reverses both signs. Projective small resolutions exist, with the diagonal through the centre at every node parallelogram of a pentagon or a hexagon (Proposition 3.4(3)). What this construction needs beyond Proposition 3.4(3) is that the refinement can be taken not to subdivide anything else in the faces of dimension at most two.

Lemma F.1. *Let $\hat{X}$ be a projective small resolution of X'_σ . There is a regular refinement $\mathcal{F}'$ of Σ' , with rays generated by lattice points of $\partial\Delta$, which induces the diagonal of $\hat{X}$ on every node parallelogram and subdivides no other cell of Σ' lying in a face of Δ of dimension at most two; the proper transform of X in the toric variety of $\mathcal{F}'$ is $\hat{X}$.*

Proof. Apply Proposition 3.4(3) with the diagonals of $\hat{X}$ as the chosen diagonals. Since $\hat{X}$ is projective, no nonzero $\lambda \geq 0$ lies in K , and the proof of Proposition 3.4(3) produces a function ψ with $\langle \text{circ}_q, \psi \rangle > 0$ for every q and the refinement induced by $h + \epsilon\psi$, h strictly convex inducing Σ' and $\epsilon > 0$ small. It subdivides each cell ω of Σ' by the regular subdivision of $\psi|_\omega$. The cells of Σ' in faces of dimension at most two are unimodular triangles, primitive segments and node parallelograms (Definition 2.3); the first two have no lattice points besides their vertices and are not subdivided, and each node parallelogram receives the diagonal $\{a, c\}$ of $\hat{X}$, since $\langle \text{circ}_q, \psi \rangle > 0$. The last assertion is part of Proposition 3.4(3). $\square$

Throughout Appendices F.1–F.7, $\hat{X}$ is projective, and we fix $\mathcal{F}'$ as in Lemma F.1, with toric variety $X_{\mathcal{F}'}$. For a small resolution that is not projective the chain is constructed on a projective one and carried over by Lemma F.24. Let $\hat{B}$ be Gross's base (Section 3.3) built

from the same pair $(\mathcal{T}, \check{h})$ and the fan side $\mathcal{F}'$ in place of Σ' , with the subdivision of the upper and lower points given by heights h' on the lower points and 0 on the upper points, h' a strictly convex function inducing $\mathcal{F}'$ (Section 4). It is the same polyhedral sphere $\partial\nabla^{\check{h}}$ as B'_σ , with the same facets and facet charts; only the fan side differs. We write Γ for the discriminant of B'_σ and $\hat{\Gamma}$ for that of $\hat{B}$, and call $\hat{B}$ the *resolved base*. It is the base of Gross's toric degeneration of $\hat{X}$, just as B'_σ is that of X'_σ .

The node in $\hat{B}$. Let $q \in Q$, with parallelogram $P = abcd$ in the two-face G of Δ , and let $G^\circ = [u_-, u_+]$; the vertex p_q of the discriminant lies in the two-face $F_q := F_{u_+u_-}$ of $\nabla^{\check{h}}$. In $\mathcal{F}'$ the parallelogram is replaced by the unimodular triangles $\{a, b, c\}$ and $\{a, c, d\}$, which by Lemmas 3.14 and 3.17 give negative vertices y_0 and y_∞ of $\hat{\Gamma}$ in F_q . They are joined by the arc $e_q = C_{\hat{B}}([u_+, u_-], [a, c])$ (Proposition 4.10), a union of legs with lower support $\{a, c\}$ and vanishing vector $c - a$. The legs with lower supports $\{a, b\}$ and $\{b, c\}$ end at y_0 , those with $\{c, d\}$ and $\{d, a\}$ at y_∞ .

Lemma F.2 (two-faces). *Let B be B'_σ or $\hat{B}$, with discriminant Γ_* and cells $C_B(\tau, \omega)$. Let $F_{uu'}$ be a two-face of $\nabla^{\check{h}}$, $\tau = [u, u']$, and $G = \tau^\vee$ the dual face of Δ ; then $\dim G \leq 2$, and every lattice point of G is a vertex of the subdivision of G .*

- (1) $\text{ri } F_{uu'} \setminus \Gamma_*$ is the disjoint union of the open discs $\tilde{R}_n \cap \text{ri } F_{uu'} = C_B(\tau, n)$, $n \in G \cap N$.
- (2) $\Gamma_* \cap \text{ri } F_{uu'}$ is the union of the arcs $C_B(\tau, [n, n'])$, $[n, n']$ an edge of the subdivision of G , and of the vertices $C_B(\tau, \omega)$, ω a two-cell of the subdivision. The arc $C_B(\tau, [n, n'])$ lies in the closures of $C_B(\tau, n)$ and $C_B(\tau, n')$ and in no other closure $\overline{C_B(\tau, n'')}$.
- (3) The closure of $C_B(\tau, n)$ is a closed piecewise linear disc, the union of the cells $C_B(\tau', \omega)$ with $\tau' \supseteq \tau$ and $n \in \omega$.

Thus $\Gamma_* \cap F_{uu'}$ with its regions is the dual-block decomposition of G with its subdivision, together with the arcs running to $\partial F_{uu'}$. For an edge $F_{\tau'}$ of $\nabla^{\check{h}}$ the same holds with the open intervals $C_B(\tau', n)$ and the points $C_B(\tau', [n, n'])$.

Here $\tilde{R}_n$ is the vertex region of n (Section 3.3), ri denotes the relative interior, and the lattice points of G are its vertices and, for a pentagon or a hexagon, its centre.

Proof. Apply Propositions 4.7 and 4.10 with $\mathcal{F} = \Sigma'$ and with $\mathcal{F} = \mathcal{F}'$. By Proposition 4.7(5), $\text{ri } F_{uu'}$ is the union of the cells $C_B(\tau, \omega)$, and $\tilde{R}_n \cap \text{ri } F_{uu'} = C_B(\tau, n)$, which is nonempty if and only if (τ, n) is a dual cell pair, that is, $n \in G \cap N$. By Proposition 4.10(1), Γ_* is the union of the cells $C_B(\tau', \omega)$ with $\dim \tau' \geq 1$ and $\dim \omega \geq 1$; intersecting with $\text{ri } F_{uu'}$ gives the cells $C_B(\tau, \omega)$ with $\dim \omega \geq 1$, which are the arcs (ω an edge) and the vertices (ω a two-cell) of (2), and their complement in $\text{ri } F_{uu'}$ is the union of the $C_B(\tau, n)$. By Proposition 4.7(2), $C_B(\tau, n)$ is an open cell of dimension $3 - \dim \tau - 0 = 2$, and its closure is the piecewise linear disc $\bigcup_{p \preceq (\tau, n)} C_B(p)$; since $(\tau', \omega) \preceq (\tau, n)$ means $\tau' \supseteq \tau$ and $n \in \omega$, this gives (3) and the incidence statement of (2). The case of an edge is the same with $\dim \tau' = 2$. $\square$

Since Σ' and $\mathcal{F}'$ induce the same subdivision on every face of Δ of dimension at most two except at the node parallelograms, Lemma F.2 gives isomorphic cell structures, with the same vertex regions, on every two-face of B'_σ and $\hat{B}$ away from the node points. Near a node, let U_q be the open star of p_q in the barycentric subdivision of the polyhedral decomposition $\mathcal{P}$ of B'_σ . It lies in $\text{ri } F_q \cup \text{int } F_{u_+} \cup \text{int } F_{u_-}$, and it is the open cone from p_q over its link $\text{lk}_q \cong S^2$, linear on each simplex; for $0 < r \leq 1$ let U_q^r be the part of cone parameter $< r$. The set $\Gamma \cap U_q$ consists of the four legs at p_q , which are radial and end at points $\ell_{ab}, \ell_{bc}, \ell_{cd}, \ell_{da}$ of lk_q , in this cyclic order on the circle $\text{lk}_q \cap F_q$; here ℓ_{ab} is the end of the leg with lower support

$\{a, b\}$. The simplices of the barycentric subdivision correspond to chains of cells of $\mathcal{P}$, and U_q is the union of the open simplices whose chain contains the carrying cell of q ; since no chain contains two different two-cells, distinct U_q are disjoint. For $r < 1$ the closures of the U_q^r are therefore pairwise disjoint as well. This holds even when two closed carrying cells share an edge, as happens at a hexagon on the A_2 branch.

Lemma F.3 (moving to the resolved base). *For every $r_* \in (0, 1)$ there is a piecewise linear homeomorphism g of $\partial\nabla^{\tilde{h}}$ with the following properties.*

- (1) *g preserves every closed face of $\nabla^{\tilde{h}}$ and is the identity on each facet interior outside a collar of its boundary.*
- (2) *$\widehat{U}_q := g(U_q^{r_*})$ is an open ball containing $y_0 \cup e_q \cup y_\infty$, and g maps $\Gamma \setminus \bigcup_q U_q^{r_*}$ onto $\widehat{\Gamma} \setminus \bigcup_q \widehat{U}_q$, the part of each leg at p_q outside $U_q^{r_*}$ onto the outer part of the leg of $\widehat{B}$ with the same lower support, and, on the two-skeleton $\bigcup F_{uu'}$ outside $\bigcup_q U_q^{r_*}$, each vertex region $\widetilde{R}_n$ into the vertex region of the same n .*
- (3) *Off $\bigcup_q U_q^{r_*}$, g carries the local system Λ of B'_σ to that of $\widehat{B}$, by the identity on each facet lattice $u^\perp \cap N$, and it preserves the types of the vertices of the discriminant.*

Proof. We build g skeleton by skeleton.

Edges. On each edge of $\nabla^{\tilde{h}}$ the two discriminants meet the edge in finite sets separating intervals in the same vertex regions, in the same order (Lemma F.2 for the edge), and g is the piecewise linear map matching them.

Two-faces without node points. On a two-face that contains no node point, the two cell structures, with their vertex regions, are isomorphic, compatibly with g on the boundary. We first define g on each arc of the graph inside the two-face as a piecewise linear homeomorphism onto the corresponding arc, matching the end points. Then g is defined on the boundary circle of each closed region, a closed disc (Lemma F.2(3)), as a piecewise linear homeomorphism onto the boundary circle of the corresponding region, and it extends over the region by coning: a piecewise linear homeomorphism between the boundaries of two piecewise linear discs extends to one of the discs (identify both with a simplex and cone).

Two-faces with node points. Let F be a two-face containing node points, and G the dual face of Δ . It contains one node point if G is a square, two if G is a pentagon or a hexagon on the A_1 branch, and three if G is a hexagon on the A_2 branch. For each node point $p_q \in F$ let $F'_q = U_q^{r_*} \cap F$, an open disc around p_q whose boundary meets the four legs in the cyclic order ab, bc, cd, da , and let $\widehat{F}_q$ be a closed regular neighbourhood in F of the tree $y_0 \cup e_q \cup y_\infty$ that meets $\widehat{\Gamma}$ only in the tree and four initial leg segments. Its boundary meets these legs in the same cyclic order; the region of b lies between ab and bc , that of d between cd and da , and the regions of a and c each border e_q .

The closures $\overline{F'_q}$ are pairwise disjoint, since $r_* < 1$. The trees are pairwise disjoint as well: the tree of q is the union of the cells $C_{\widehat{B}}([u, u'], \omega)$ with ω one of the two triangles into which the diagonal of $\widehat{X}$ cuts the parallelogram of q , or that diagonal, and different parallelograms give different cells, which are disjoint (Proposition 4.7). With the diagonals through the centre o this says that the diagonals are pairwise non-adjacent edges of the star subdivision of G , so that no triangle of the star belongs to two trees; at a hexagon on the A_2 branch the three trees use all six triangles, and consecutive trees are joined by the arcs across the edges $[o, w_j]$, $j \equiv j_0 \pmod{2}$. We take the neighbourhoods $\widehat{F}_q$ so small that they are pairwise disjoint; each of them then meets such a joining arc only in an initial segment.

Since $\Gamma \cap U_q$ consists of the four radial legs, F'_q is the open cone from p_q over the circle $\text{lk}_q \cap F$, and it meets the closure of each region in the cone over an arc of that circle: a sector bounded by two radial leg segments, those of ab and bc for the region of b , of bc and cd for that of c , and so on. Likewise $\widehat{F}_q$, a regular neighbourhood of a tree lying on the boundaries of the regions, meets the closure of each region in a disc attached along an arc of its boundary.

By Lemma F.2, $\Gamma \cap F$ and $\widehat{\Gamma} \cap F$ are the dual-block graphs of the two subdivisions of G , together with the arcs to ∂F : a two-cell ω gives the vertex $C_B([u, u'], \omega)$, and an edge of the subdivision gives the arc $C_B([u, u'], [n, n'])$ joining the vertices of the two-cells that contain it, or running to ∂F if only one does. The subdivision induced by $\mathcal{F}'$ differs from that of Σ' only by the diagonal $\{a, c\}$ in each node parallelogram. So the graph of $\widehat{B}$ is obtained from that of B'_σ by replacing each four-valent vertex p_q by the tree $y_0 \cup e_q \cup y_\infty$, where y_0 (the triangle $\{a, b, c\}$) carries the arcs across $[a, b]$ and $[b, c]$, and y_∞ (the triangle $\{a, c, d\}$) those across $[c, d]$ and $[d, a]$; every other vertex, every other arc and every incidence with a region is unchanged. After removing the discs F'_q from F and the interiors of the neighbourhoods $\widehat{F}_q$, the two labelled graphs, with their regions, are therefore isomorphic, the four boundary points of each removed disc being matched in the cyclic order ab, bc, cd, da . We define g on the arcs of the graph in F outside the removed discs, and on each circle $\partial F'_q$ onto $\partial \widehat{F}_q$, as piecewise linear homeomorphisms that match the end points of the arcs and the four leg points; then g is defined on the whole boundary circle of each closed region that remains. Each closed region $C_B([u, u'], n)$ is a closed disc (Lemma F.2(3)), and by the previous paragraph the removed discs meet it in pairwise disjoint discs attached along pairwise disjoint arcs of its boundary, so what remains of it is again a closed disc, on both bases. Hence g extends over $F \setminus \bigcup_q F'_q$ region by region, by coning, and over each F'_q onto $\text{int } \widehat{F}_q$ by coning.

The shape of the graph inside F depends on G . For a square it has one four-valent vertex. For a pentagon it has two four-valent vertices, joined by the arc across the edge that the two parallelograms share, and one negative vertex, the triangle, joined to both. For a hexagon on the A_1 branch it has two four-valent vertices and two negative vertices, forming a cycle of four arcs around the region of the centre o . For a hexagon on the A_2 branch the three node points are joined pairwise by three arcs across the edges $[o, w_j]$, $j \equiv j_0 \pmod{2}$, which bound the region of o in F and form a cycle without trivalent vertices; on $\widehat{B}$ the six triangles of the star subdivision of G give a cycle of six negative vertices, three of whose arcs are the e_q . In every case the argument above applies unchanged, since it only uses the local replacement at each node point.

Facets. On the boundary of a facet F_u , a piecewise linear two-sphere, g fixes every vertex of $\nabla^{\widehat{h}}$ and maps every edge and every two-face of F_u onto itself; in particular it is an orientation-preserving piecewise linear homeomorphism of ∂F_u . We extend it over a collar by a piecewise linear homeomorphism $\mathcal{G}$ of $\partial F_u \times [0, 1]$ that equals g on $\partial F_u \times \{1\}$, the identity on $\partial F_u \times \{0\}$, and maps $\phi \times [0, 1]$ onto itself for every face ϕ of ∂F_u ; it is built on these prisms by increasing dimension. On $v \times [0, 1]$, v a vertex, $\mathcal{G}$ is the identity. On $e \times [0, 1]$, $e \cong [0, 1]$ an edge, it is the Alexander trick: $\mathcal{G}(x, t) = (tg(x/t), t)$ for $x \leq t$ and $\mathcal{G}(x, t) = (x, t)$ for $x \geq t$. The set $\{x \leq t\}$ is the cone from $(0, 0)$ over $e \times \{1\}$, and on the cone over a segment of $e \times \{1\}$ on which $g(x) = \alpha x + \beta$ the map is $(x, t) \mapsto (\alpha x + \beta t, t)$, which is linear; so $\mathcal{G}$ is a piecewise linear homeomorphism of the square, and it is the identity on the vertical sides. On $D \times [0, 1]$, D a two-face, $\mathcal{G}$ is now defined on the boundary sphere $\partial(D \times [0, 1])$ as a piecewise linear homeomorphism onto itself, and we extend it by coning from an interior point c_D : if the boundary map is simplicial from a triangulation K of the sphere onto a triangulation L , the cone map is the simplicial isomorphism from the cone $c_D * K$ onto $c_D * L$, both of which

triangulate the convex cell $D \times [0, 1]$. Finally let c be an interior point of F_u and $0 < \lambda < 1$. Triangulate the two-faces of F_u without new vertices, fix a total order of the vertices of ∂F_u , and triangulate each prism $\phi \times [0, 1]$, ϕ a simplex of the resulting triangulation of ∂F_u , by the staircase triangulation for this order; these triangulations agree on common faces and have no new vertices. Map each vertex (w, t) to $c + (\lambda + (1 - \lambda)t)(w - c)$ and each simplex linearly. As in the proof of Lemma F.4 this is a piecewise linear homeomorphism of $\partial F_u \times [0, 1]$ onto the collar between ∂F_u and $c + \lambda(\partial F_u - c)$. Transport $\mathcal{G}$ to the collar by it, and put g equal to the identity on $c + \lambda(F_u - c)$; this gives (1). Since U_q meets the two-skeleton only in $\text{ri } F_q$, $\widehat{U}_q$ is an open ball with $\widehat{U}_q \cap F_q = \text{int } \widehat{F}_q$, and since $\widehat{\Gamma}$ lies in the two-skeleton, $\widehat{\Gamma} \cap \widehat{U}_q = \widehat{\Gamma} \cap \text{int } \widehat{F}_q$.

Property (3). Both local systems are the facet lattices $u^\perp \cap N$ on facet interiors, and the transition across a two-face at a point of the vertex region of n is $u^\perp \cap N \rightarrow N/\mathbb{Z}n \rightarrow u'^\perp \cap N$ on both bases; g preserves the open facets, and on the two-skeleton outside the node discs it preserves the vertex regions. Vertex types are determined by the lower supports (Lemma 3.17), which g preserves. $\square$

Del Pezzo faces. The construction is local at each node, and the global shape of a two-face G of Δ enters only at two places: Lemma F.3, where a two-face of ∇^h may carry two or three node points, and Remark F.26, which describes the boundary at a del Pezzo point by means of the ties of Remark 3.10. Everything else uses only that a node parallelogram is a unit parallelogram in a two-face G with $\ell(G^\circ) = 1$, that the other cells of Σ' in G are unimodular triangles and primitive segments, and that distinct node points have disjoint stars.

F.2. Standard position near the nodes. Near a node the germ of S may wind around the legs. In a spherical shell around p_q this winding can be traded for finitely many ordinary crossings of Γ , and inside the shell S then has a standard form.

Lemma F.4 (standard position near the nodes). *Let (S, j, v) be a tropical 2-cycle on B'_σ , with j piecewise linear. There are $r_* \in (0, 1)$ and a tropical 2-cycle (S, j', v) such that:*

- (0) $j' = j$ outside $\bigcup_q U_q^{2r_*}$, and $j'(\partial S) = j(\partial S)$ with the same legs at every node and the same boundary fields; hence $\rho(S, j') = \rho(S, j)$;
- (1) for each node q , $j'(S) \cap U_q^{r_*} = \emptyset$ if $p_q \notin j(\partial S)$; otherwise $j'(S) \cap U_q^{r_*}$ is the cone from p_q over a piecewise linear arc $\gamma_q^0 \subset \text{lk}_q$ which joins the ends of the two opposite legs of the passage and lies, apart from its ends, in one of the two open hemispheres of $\text{lk}_q \setminus F_q$, on a side $s_q \in \{\pm 1\}$;
- (2) $U_q^{r_*}$ contains no crossing point $\mathfrak{x}_k$, no interior edge or vertex of S , and no other sheet;
- (3) the closures of the $U_q^{r_*}$ are pairwise disjoint.

Here the passages at a node are those of Sections 3.4 and 5.2: by the node-passage clause, ∂S follows either the legs with lower supports $\{a, b\}$ and $\{c, d\}$ (the $b - a$ passage) or those with $\{b, c\}$ and $\{d, a\}$ (the $c - b$ passage). The side $s_q = +1$ is that of the facet F_u and $s_q = -1$ that of $F_{u'}$, for the ordering (u, u') of F_q fixed in Appendix F.5.

Proof. Choose triangulations of S and of B'_σ , the latter subdividing the barycentric subdivision near each p_q and having p_q as a vertex, for which j is simplicial. Since the star of a vertex is the cone over its link, for small r_0 the set $j(S) \cap U_q^{r_0}$ is the union of the radial segments through $\gamma = j(S) \cap \partial U_q^{r_0}$. If $p_q \notin j(S)$ this set is empty for small r_0 . Otherwise $p_q = j(p)$ with $p \in \partial S$ a smooth boundary point, since a four-valent point of Γ is neither the image of a crossing point $\mathfrak{x}_k$ nor a negative vertex (the discriminant-intersection and boundary-vertex clauses). The crossing points, the interior edges and vertices, and $j(S \setminus V)$

for a neighbourhood V of p are compact and miss p_q , so for small r_0 the set $j(S) \cap U_q^{r_0}$ is the germ of $j(S)$ at p_q , and by the node-passage clause γ is an arc from ℓ_{ab} to ℓ_{cd} (the $b - a$ passage) or from ℓ_{bc} to ℓ_{da} (the $c - b$ passage).

Put $r_* = r_0/2$ and let γ_q^0 be an arc on side s_q with the same ends. Arcs in S^2 with fixed ends are isotopic; choose a piecewise linear isotopy γ_r , $r \in [r_*, r_0]$, from γ_q^0 to γ fixing the ends and transverse to the other two leg ends, which it therefore crosses finitely often. Define j' on $U_q^{r_0}$ as the cone over γ_q^0 for $r \leq r_*$ and as γ_r at level r for $r \in [r_*, r_0]$. Different levels are disjoint, so j' is an embedding, and it agrees with j near $\partial U_q^{r_0}$. The map $(x, r) \mapsto rx$ from $\text{lk}_q \times [r_*, r_0]$ to the shell is not piecewise linear, so we realise the levels differently: triangulate the prisms $\sigma \times [r_*, r_0]$, σ a simplex of lk_q , compatibly and without new vertices (by the staircase triangulations for a fixed order of the vertices of lk_q), send each vertex (w, r) , $r \in \{r_*, r_0\}$, to rw , and extend linearly on each simplex. Since the cone parameter is the linear function equal to 1 on σ on the cone over σ , this map is a level-preserving piecewise linear homeomorphism of each prism onto its frustum, and composing the piecewise linear isotopy $(x, r) \mapsto (\gamma_r(x), r)$ with it makes j' piecewise linear.

The new crossings satisfy the crossing clause. Take the $b - a$ passage, so that $v = \pm(b - a)$ on the passing sheet (Lemma 5.3), and a crossing of the leg with lower support $\{b, c\}$. By Lemma 3.12 the monodromy around this leg is $T(x) = x + \langle \alpha, x \rangle (c - b)$ with $\alpha = u_+ - u_-$, up to inversion, which depends on the orientation of the loop. Since $\langle \alpha, b - a \rangle = 0$, v is invariant and extends across the crossing. On covectors the monodromy is $\xi \mapsto \xi \pm \xi(c - b)\alpha$, and $\ker v$ has lattice $\mathcal{F}_v = \{\xi \in M/\mathbb{Z}u_+ : \xi(b - a) = 0\}$. The map $\mathcal{F}_v \rightarrow \mathbb{Z}$, $\xi \mapsto \xi(c - b)$, is onto, because $b - a$ and $c - b$ span a saturated sublattice of $u_+^\perp \cap N$, and its kernel is spanned by the class of α , which is primitive in $M/\mathbb{Z}u_+$ because $\{0, u_+, u_-\}$ is a unimodular triangle. In a basis $\{\xi, \alpha\}$ with $\xi(c - b) = 1$ the monodromy is $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$, which is conjugate to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $GL_2(\mathbb{Z})$ for either sign. The same holds at the leg with lower support $\{d, a\}$, and for the $c - b$ passage with the roles exchanged. All other clauses of [CBM14, Def. 7.2] are local and hold where $j' = j$; in $U_q^{r_0}$ the only points of $j'(S)$ on Γ are those of ∂S and the new crossings, and the passing sheet has no interior edges or vertices there. Items (2) and (3) hold for r_0 small, since the closures of the U_q^r , $r < 1$, are pairwise disjoint (Appendix F.1), and ρ is read from ∂S and its boundary field, which are unchanged. $\square$

The crossing clause is read, as in Section 5.2, as conjugacy in $GL_2(\mathbb{Z})$. With this reading the orientation of the loop around a leg plays no role in this section, and we fix no convention for it; in particular no statement here depends on the orientation convention for loops in [Gro05, Prop. 3.15]. If the crossing clause were read with a fixed orientation convention, the new crossings might violate it; the construction below never uses the crossing clause, only the other clauses and the invariance of v under the monodromy at the crossings, which holds at the new crossings because $\langle \alpha, b - a \rangle = 0$ (respectively $\langle \alpha, c - b \rangle = 0$). The shell $U_q \setminus U_q^{r_*}$ may contain crossings, other sheets, interior edges and vertices of S , and the winding of the passing sheet.

Definition F.5 (resolved cycle). Let j' , r_* be as in Lemma F.4 and g as in Lemma F.3 for this r_* . The *resolved cycle* $\hat{S} = \hat{j}(S)$ on $\hat{B}$ is defined as follows. Off $\bigcup_q j'^{-1}(U_q^{r_*})$ put $\hat{j} := g \circ j'$, and let v be transported by g . Inside $\hat{U}_q$ replace the cone over γ_q^0 by the *standard half-disc* $\hat{\mathfrak{H}}_q \subset \widehat{\hat{U}}_q$: a piecewise linear disc on side s_q of F_q bounded by the arc $g(r_*\gamma_q^0) \subset \partial \hat{U}_q$, where $r_*\gamma_q^0$ is the arc in which the cone over γ_q^0 meets the level r_* , and by the *boundary path*

formed by the inner segments of the legs of $\widehat{B}$ with lower supports $\{a, b\}$ and $\{c, d\}$ together with $y_0 \cup e_q \cup y_\infty$ (for the $c - b$ passage, the legs with lower supports $\{b, c\}$ and $\{d, a\}$).

Such a disc exists. Let H_q be the closed half-ball $g(\overline{U_q^{r_*}} \cap \overline{F_{u_{s_q}}})$, where $u_{+1} = u$ and $u_{-1} = u'$ for the ordering (u, u') of F_q . Its boundary sphere is the union of the image of the closed hemisphere of side s_q at level r_* and of $g(\overline{U_q^{r_*}} \cap F_q) = \widehat{F}_q$. The arc $g(r_* \gamma_q^0)$ lies in the first part, the boundary path in the second, and they have the same end points, the two leg points on $\partial \widehat{F}_q$; so their union is a simple closed curve on this sphere. It bounds a disc in the sphere, and pushing the interior of that disc into the interior of H_q , relative to its boundary, gives a piecewise linear disc that meets $\widehat{\Gamma}$, which lies in the two-skeleton, only in the boundary path.

Outside $\bigcup_q \widehat{U}_q$ the data $(S, \hat{j}, v)$ satisfy the clauses of [CBM14, Def. 7.2] relative to $\widehat{B}$, by Lemma F.3; the image of the shell meets $\widehat{\Gamma}$ only on the outer segments of the legs at the nodes, which carry the same monodromy. Inside $\widehat{U}_q$, $\hat{S}$ is not a tropical 2-cycle of $\widehat{B}$: its boundary turns at the negative vertices y_0, y_∞ and runs along e_q , whose vanishing vector $c - a$ is not $\pm v$. This is the configuration that produces the exceptional curve. Every crossing point of $\hat{S}$ lies outside $\bigcup_q \widehat{U}_q$.

Normal form near ∂S . The germ of $\hat{S}$ along $\hat{j}(\partial S)$ outside $\bigcup_q \widehat{U}_q$ is normalised by a piecewise linear isotopy of $\hat{j}$ that fixes $\hat{j}(\partial S)$ pointwise, is supported in a small neighbourhood of $\hat{j}(\partial S) \setminus \bigcup_q \widehat{U}_q$ containing no point $\hat{j}(\mathfrak{r}_k)$, and keeps $\hat{S}$ off $\widehat{\Gamma}$ outside $\hat{j}(\partial S)$. Such an isotopy preserves the clauses of [CBM14, Def. 7.2], which near ∂S involve only $\hat{j}(\partial S)$ and the germ of the parallel field v on the complement of $\widehat{\Gamma}$. After it:

- (T1) near every edge of ∂S lying on a leg in a two-face $F_{uu'}$, the sheet of $\hat{S}$ ending on the edge meets $F_{uu'}$ only in $\hat{j}(\partial S)$ and in finitely many arcs, along which the sheet crosses $F_{uu'}$ transversally and which leave $\hat{j}(\partial S)$ at distinct interior points of the edge, the *switching points*;
- (T2) at every negative vertex y , in a two-face $F_{uu'}$, the link of $\hat{S}$ at y is a tripod lying, apart from its three ends, in one of the two open hemispheres of the link sphere lk_y bounded by $\text{lk}_y \cap F_{uu'}$;
- (T3) at every positive vertex p on $\hat{j}(\partial S)$, the link of $\hat{S}$ at p is an arc lying, apart from its ends, in the open facet that contains the two legs used by ∂S .

By (T1), near every point of such an edge other than a switching point the sheet lies on one side of $F_{uu'}$, in F_u or in $F_{u'}$; in particular it does not run inside $F_{uu'}$ along a stretch of the edge, and it changes side exactly at the switching points. By Lemma F.4, (T2) and (T3), the switching points lie outside $\bigcup_q \widehat{U}_q$ and away from the negative and positive vertices.

For (T1): near an edge of ∂S on a leg, away from its end points, both the sheet and $F_{uu'}$ are piecewise linear surfaces containing the edge, and by general position relative to $\hat{j}(\partial S)$ [RS72, Ch. 5] a small isotopy fixing $\hat{j}(\partial S)$ makes them meet outside the edge in finitely many arcs, along which they cross transversally and which leave the edge at distinct points. For (T2): the three leg ends lie on the circle $\text{lk}_y \cap F_{uu'}$, and the link of $\hat{S}$ is a tripod with these ends, meeting $\widehat{\Gamma}$ only in them. Tripods in S^2 with three fixed ends form two isotopy classes relative to the ends, distinguished by the cyclic order of the edges at the centre, and each class is represented in exactly one of the two open hemispheres, since a tripod in a hemisphere has at its centre the cyclic order of its ends seen from that hemisphere. The level-wise isotopy of the proof of Lemma F.4, which here creates no crossings, moves the germ into that hemisphere.

For (T3): the link sphere lk_p meets the edge of $\nabla^{\tilde{h}}$ through p in two points, and the three two-faces through the edge in three arcs joining them, one through each leg end; these arcs bound three lunes, one in each facet. The link of $\hat{S}$ joins the two leg ends used by ∂S in the complement of the third leg end, a disc, so it is isotopic relative to its ends, in that complement, to an arc in the lune between them, and the level-wise isotopy moves the germ accordingly without creating crossings. This is the normalisation that [CBM14, §7.4] makes up to local isotopy.

Finally, choose a closed neighbourhood $\mathcal{N}_\Gamma$ of $\hat{j}^{-1}(\hat{\Gamma}) = \partial S \cup \{\mathfrak{x}_k\}$ in S carrying the local pieces of Appendix F.5, and apply a generic piecewise linear isotopy of $\hat{j}$ relative to $\mathcal{N}_\Gamma \cup \bigcup_q \hat{j}^{-1}(\hat{U}_q)$, supported off $\hat{\Gamma}$, so that off $\mathcal{N}_\Gamma$ the surface $\hat{S}$ avoids the vertices of $\nabla^{\tilde{h}}$, meets its edges transversally in finitely many points (the *edge crossings*) and its two-faces transversally in curves. Such an isotopy preserves the clauses of [CBM14, Def. 7.2], which are local near $\hat{\Gamma}$.

F.3. The localized hypersurface. We choose a localization with exact equations near the strata met by the cycle. Its orbit-preserving isotopy to a holomorphic hypersurface and the topology of its regions over the regular part supply the targets for the chain construction.

Let $\tilde{\mathcal{A}} = \Delta^\circ \cap M$ and $\check{h}_m := \check{h}(m)$, so that $\check{h}_0 = 0$ and $\check{h}_m$ is odd and positive for $m \neq 0$. For $\varkappa > 1$ put

$$f_\varkappa(x) = \sum_{m \in \tilde{\mathcal{A}}} (-\varkappa)^{-\check{h}_m} x^m,$$

the member of Gross's degeneration $\{\sum_m t^{\check{h}(m)} x^m = 0\}$ at $t = -1/\varkappa$, all of whose coefficients are 1. On the positive real torus the constant term is 1 and all other terms are negative. Write $\text{Log}(x) = -\log|x|/\log \varkappa \in N_\mathbb{R}$ for x in the torus $N \otimes \mathbb{C}^*$, and

$$T_m(x) = \log_\varkappa |(-\varkappa)^{-\check{h}_m} x^m| = -\check{h}_m - \langle m, \text{Log } x \rangle.$$

In this section the tropical variable is $y = \text{Log } x \in N_\mathbb{R}$, and the tropical polynomial of the heights $\check{h}_m$ is $y \mapsto \min_m (\check{h}_m + \langle m, y \rangle) = -\max_m T_m$; this is the convention of Section 2.1, with the monomials in M and the variable in $N_\mathbb{R}$. The constant term dominates exactly over the interior of $\nabla^{\tilde{h}}$, and $\hat{B} = \partial \nabla^{\tilde{h}}$ is where it ties with other terms.

Definition F.6 (localized hypersurface). Let $\beta: \mathbb{R} \rightarrow [0, 1]$ be smooth and non-increasing with $\beta = 1$ exactly on $(-\infty, C_1]$ and $\beta = 0$ exactly on $[C_0, \infty)$, where $0 < C_1 < C_0 < \frac{1}{4}$. Put $\beta_m(x) = \prod_{i \in \tilde{\mathcal{A}}} \beta(T_i(x) - T_m(x))$,

$$\tilde{f}(x) = \sum_{m \in \tilde{\mathcal{A}}} \beta_m(x) (-\varkappa)^{-\check{h}_m} x^m, \quad \mathscr{W} = \overline{\{\tilde{f} = 0\}} \subset X_{\mathcal{F}'},$$

and for $s \in [0, 1]$ let V_s be the closure of the zero set of $\sum_m ((1-s) + s\beta_m) (-\varkappa)^{-\check{h}_m} x^m$. The *active set* at x is $A(x) = \{m : \beta_m \not\equiv 0 \text{ on every neighbourhood of } x\}$, for $x \in X_{\mathcal{F}'}$, the functions β_m being defined on the dense torus; since $\tilde{\mathcal{A}}$ is finite, x has a neighbourhood on which $\beta_m \equiv 0$ for every $m \notin A(x)$.

So $V_0 = \hat{X}_\varkappa$, the proper transform of $X_\varkappa = \{f_\varkappa = 0\}$, and $V_1 = \mathscr{W}$; a monomial is active only where it is within C_0 of the maximum of the T_i . The hypersurface $\mathscr{W}$ is the localization of $X_\varkappa$ in the sense of Mikhalkin [Mik04], in the form of [Yam16, Def. 3.1]: near each point it is the zero set of the monomials that are close to maximal there, and it is isotopic to $\hat{X}_\varkappa$ (Lemma F.22). In the notation of [Yam16, §3], with $k_m = t^{\check{h}_m} \in \mathbb{C}\{t\}$, so that $v_m = \text{val}(k_m) = -\check{h}_m$ for Yamamoto's valuation, and with $q_Y = -\varkappa$, the substitution

$t = 1/q_Y$ turns k_m into $q_Y^{v_m} = (-\varkappa)^{-\tilde{h}_m}$, and the function $f_\varkappa$ is f_{q_Y} , $\tilde{f}$ is $\tilde{f}_{q_Y}$ and $\mathscr{W}$ is $\mathscr{W}_{q_Y}$. The leading coefficients c_m of loc. cit. are all 1, so the correction term in Yamamoto's interpolating family $\tilde{f}_{q_Y,s}$ vanishes, and V_s is the member $V_{q_Y,s'}$ of that family for every s' with $d(s') = s$ [Yam16, proof of Thm. 3.9].

Lemma F.7. (1) *For C_0 small, every V_s meets only torus orbits $O(\sigma)$ of $X_{\mathcal{F}'}$ with σ contained in the cone over a face of Δ of dimension at most two.*
 (2) *Every cone of $\mathcal{F}'$ contained in the cone over a face of Δ of dimension at most two is unimodular; so $X_{\mathcal{F}'}$ is smooth along the orbits met by the V_s .*
 (3) *For C_0 small and $\varkappa$ large, for every $s \in [0, 1]$ and every torus orbit $O(\sigma)$ met by V_s , the restriction to $O(\sigma)$ of the defining function of V_s has 0 as a regular value, and its derivatives along the radial and angular fundamental vector fields of $N \otimes \mathbb{C}^*$ span $\mathbb{C}$ at every point of $V_s \cap O(\sigma)$. Hence every V_s is a smooth hypersurface transverse to every torus orbit it meets. Near each orbit the active monomials at a point of $\mathscr{W}$ are the vertices of one simplex of the triangulation of Δ° induced by the heights, and the equation of $\mathscr{W}$ is the localized hyperplane [Yam16, Def. 3.2] in the ratios of the active monomials, independent of the remaining coordinates [Yam16, Thm. 3.9(2),(3)].*

Proof. (1) In Cox coordinates (Z_ρ) , indexed by the rays ρ of $\mathcal{F}'$ with primitive generators n_ρ , the defining function is $\sum_m \beta_m^{(s)} (-\varkappa)^{-\tilde{h}_m} \prod_\rho Z_\rho^{\langle m, n_\rho \rangle + 1}$ with $\beta_m^{(s)} = (1 - s) + s\beta_m$, and on $O(\sigma)$ only the terms with m in the dual face $\sigma^* = \{m \in \Delta^\circ : \langle m, n_\rho \rangle = -1, \rho \in \sigma(1)\}$ survive. If σ is not contained in the cone over a face of dimension at most two, σ^* is a vertex u of Δ° . Approaching $O(\sigma)$, Log tends to infinity in a direction n of the relative interior of σ , and $T_m - T_u \rightarrow -\infty$ for $m \neq u$ because $\langle m - u, n \rangle > 0$; so $\beta_u = 1$ near $O(\sigma)$ and the restriction is a nonvanishing monomial. (2) Such a cone is the cone over a primitive segment or a unimodular triangle at lattice height one (Lemma F.1), hence unimodular. (3) The last sentence is [Yam16, Thm. 3.9(2),(3)]. Yamamoto assumes that all coefficients are nonzero, that the heights are integral, that the subdivision of Δ° induced by the heights is unimodular, which holds because $\mathcal{T}$ is unimodular (Hypothesis (U)), and that the refinement of the normal fan of Δ° is unimodular; statements (2) and (3) of his theorem are local near each orbit met by $\mathscr{W}$, and by (1) and (2) above the charts $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{4-k}$ of the relevant cones are those of his setting. His proof of (3) takes a maximal cone $\tau \supset \sigma$ of the refinement and the basis of $\tau^\vee$; here the maximal cones containing σ need not be unimodular, and we replace that basis by the basis of M dual to a basis of N extending the primitive generators of the unimodular cone σ . His computation uses only the resulting coordinates on $\bigcup_{\sigma' \preceq \sigma} O(\sigma')$. The regularity statement is asserted in the proof of [Yam16, Thm. 3.9] as a consequence of $\det \mathbf{J} \neq 0$ for a 2×2 matrix $\mathbf{J}$, “by the concrete calculation”, which is not written there. We do not rely on that assertion: the calculation is carried out in Appendix F.10, which proves (3), and $\det \mathbf{J} > 0$ there for $\varkappa$ large, uniformly in s and in the point. $\square$

Lemma F.8 (exact local symmetry). *Let x be a point of $\mathscr{W}$, in the dense torus or on a toric stratum, and V an open neighbourhood of x in $X_{\mathcal{F}'}$, invariant under the compact torus $N \otimes U(1)$, such that $\beta_m \equiv 0$ on the part of V in the dense torus for every $m \notin A(x)$. The subgroup $((A(x) - A(x))^\perp \cap N) \otimes U(1)$ of the compact torus preserves $\mathscr{W} \cap V$.*

Proof. Each β_m depends only on the moduli $|x^i|$, so it is invariant under the compact torus. Such V exist: by Definition F.6 some open neighbourhood V_0 of x has $\beta_m \equiv 0$ on its part in the dense torus for every $m \notin A(x)$, and $V = (N \otimes U(1)) \cdot V_0$ has the same property. For λ in the compact torus and y in the dense torus, $\tilde{f}(\lambda y) = \sum_m \beta_m(y) (-\varkappa)^{-\tilde{h}_m} \lambda^m y^m$. If λ^m is

constant on $A(x)$, then $\tilde{f}(\lambda y) = \lambda^{m_0} \tilde{f}(y)$ for y in the part of V in the dense torus, for any $m_0 \in A(x)$; so λ preserves the zero set $\{\tilde{f} = 0\} \cap V$. Since V is open, $\mathcal{W} \cap V$ is the closure in V of this zero set, and λ , a homeomorphism of V , preserves it. $\square$

Lemma F.9 (no positive points on the strata). *The closure of $\mathcal{W} \cap (\mathbb{R}_{>0})^4$ in $X_{\mathcal{F}'}$ meets no toric stratum.*

Proof. On $O(\sigma)$, $\sigma \neq 0$, the constant term vanishes, since its Cox monomial is $\prod_{\varrho} Z_{\varrho}$. At a nonnegative real point of $O(\sigma)$ every surviving term, $m \in \sigma^* \setminus \{0\}$, is a non-positive real number, and the largest active one has $\beta_m = 1$; so the restriction is strictly negative. $\square$

A chain that ends on a toric stratum and starts on the positive real points must therefore leave the positive reals. This is why the local pieces of Appendix F.5 contain collar paths, paths in $\mathcal{W}$ from a point of the positive section of Lemma F.10 to the critical locus of a local model.

Lemma F.10 (the positive section). *For $\varkappa$ large, $\mathcal{W}_{>0} = \mathcal{W} \cap (\mathbb{R}_{>0})^4$ is a smooth hypersurface of $(\mathbb{R}_{>0})^4$ along which $\mathcal{W}$ is smooth, and Log maps it homeomorphically onto a star-shaped sphere $\Sigma_{>0}$ lying within radial distance $2C_0 t_{\max}/(1 - C_0)$ of $\widehat{B}$, where $t_{\max} = 1 + \max_{y \in \nabla^h} |y|$. Radial projection identifies $\Sigma_{>0}$ with $\widehat{B}$. The inverse $\mathfrak{s}: \widehat{B} \rightarrow \mathcal{W}$ of the composite is continuous, and $\text{Log } \mathfrak{s}(b)$ lies on the ray through b .*

The proof is in Appendix F.12; it uses only $\check{h}_m \geq 1$ for $m \neq 0$, the parity of $\check{h}_m$, that all coefficients of $f_{\varkappa}$ have modulus one, and $\varkappa$ large. We call $\mathfrak{s}$ the *positive section*.

Charts of the faces. Let Φ be a facet, a two-face or an edge of ∇^h , and $A_{\Phi} = \{0\} \cup \{u : \Phi \subset F_u\}$ its set of maximizing exponents, of type (a) $\{0, u\}$, (b) $\{0, u, u'\}$ or (c) $\{0, u_1, u_2, u_3\}$. For types (b), (c) a *fan point* of Φ is a vertex n of $\mathcal{F}'$ whose vertex region $\tilde{R}_n$ meets Φ ; it satisfies $\langle a, n \rangle = -1$ for $a \in A_{\Phi} \setminus 0$. Since $\mathcal{T}$ is unimodular, $A_{\Phi} \setminus 0$ extends to a basis of M , so $N = A_{\Phi}^{\perp} \oplus \mathbb{Z}n \oplus C$ for a lattice complement C . A *chart* of (Φ, n) is a linear isomorphism $(\pi_{\parallel}, \pi_{\perp}): N_{\mathbb{R}} \rightarrow (A_{\Phi}^{\perp})_{\mathbb{R}} \times \mathbb{R}^{A_{\Phi} \setminus 0}$ with

$$\pi_{\perp}(y) = (\langle a, y \rangle + \check{h}_a)_{a \in A_{\Phi} \setminus 0}, \quad \pi_{\parallel}(n) = 0,$$

and $\pi_{\parallel}$ a projection onto $(A_{\Phi}^{\perp})_{\mathbb{R}}$; the condition $\pi_{\parallel}(n) = 0$ is imposed in types (b) and (c), and in type (a) $\pi_{\parallel}$ is the projection along a vector n_F with $\langle u, n_F \rangle = -1$. Near Φ the polytope is $\{\pi_{\perp} \geq 0\}$, so in the chart $\nabla^h = (A_{\Phi}^{\perp})_{\mathbb{R}} \times \mathcal{O}_{\perp}$ and $\widehat{B} = (A_{\Phi}^{\perp})_{\mathbb{R}} \times \partial \mathcal{O}_{\perp}$, with $\mathcal{O}_{\perp} = \mathbb{R}_{\geq 0}^{A_{\Phi} \setminus 0}$; $\partial \mathcal{O}_{\perp}$ is a point, two half-lines or three quadrants. The *divisorial ray* from y is $y + \mathbb{R}_{\geq 0} n$; in the chart it is $\{\pi_{\parallel}(y)\} \times (\pi_{\perp}(y) + \mathbb{R}_{\geq 0}(-1, \dots, -1))$, and it runs into the toric divisor D_n . Put $d = |A_{\Phi} \setminus 0|$ and, for $p = (p_1, \dots, p_d)$, $p_{\min} = \min(0, p_1, \dots, p_d)$. In the chart, near Φ , the tropical hypersurface of the heights is $(A_{\Phi}^{\perp})_{\mathbb{R}} \times \{p : \text{the minimum defining } p_{\min} \text{ is attained at least twice}\}$, and the *contraction* of the chart,

$$\varpi(\pi_{\parallel}, p) = (\pi_{\parallel}, p - p_{\min}(1, \dots, 1)),$$

maps it, near Φ , onto $\widehat{B}$, by the identity on $\widehat{B}$ and along the divisorial rays. This is modelled on Yamamoto's tropical contraction [Yam21]; below we use only the chart formula.

Definition F.11 (product-shaped sets and their regions). An open set $U \subset \widehat{B}$ is *product-shaped* at Φ if in a chart of (Φ, n) it is $\{\pi_{\parallel} \in U_{\parallel}, \pi_{\perp} \in U_{\perp}\}$ with $U_{\parallel}$ convex and $U_{\perp} = \partial \mathcal{O}_{\perp} \cap \{|\pi_{\perp}|_{\infty} < r\}$, if $\ell_m/|m| > \ell_*$ on U for every $m \in \check{\mathcal{A}} \setminus A_{\Phi}$, where $\ell_m = \langle m, \cdot \rangle + \check{h}_m$, and

if its distance to $\widehat{\Gamma}$ exceeds ℓ_* . For such U put

$$\Omega_U = \{\pi_{\parallel} \in U_{\parallel}^{\eta}, \pi_{\perp} \in \Omega_{\perp}\}, \quad \Omega_{\perp} = S_{d,r}^{\eta}, \quad d = |A_{\Phi} \setminus 0|,$$

where

$S_{d,r} = \{p : \text{the minimum defining } p_{\min} \text{ is attained at least twice, } p_i - p_{\min} < r \ (1 \leq i \leq d)\}$.

Thus $\{\pi_{\parallel} \in U_{\parallel}, \pi_{\perp} \in S_{d,r}\} = \varpi^{-1}(U)$ is the set of points of the tropical hypersurface of the chart that the contraction maps into U , and Ω_U is its η -neighbourhood in the chart norm. Also $S_{1,r} = \{0\}$; $S_{2,r}$ consists of two truncated positive rays and the negative diagonal ray; and $S_{3,r}$ consists of the truncated boundary of the positive orthant together with the divisorial rays issuing from its three coordinate rays and from its vertex. The *region* over U is $\mathcal{W}_U = \mathcal{W} \cap \text{Log}^{-1}(\Omega_U)$ together with its limit points on the divisor D_n at the ends of the divisorial rays. Distances, and the norms $|y|$ on $N_{\mathbb{R}}$ and $|m|$ on $M_{\mathbb{R}}$, are Euclidean for the standard inner product in lattice coordinates, so $|m| \geq 1$ for $0 \neq m \in M$; the *chart norm* of $y \in N_{\mathbb{R}}$ is $|y|_{\Phi} = \max(|\pi_{\parallel}(y) - \pi_{\parallel}(0)|, |\pi_{\perp}(y) - \pi_{\perp}(0)|_{\infty})$. Here $\eta > C_0$ and $\ell_* = C_{\text{act}}C_0 + (C_{\text{ch}} + 1)\eta$, where $C_{\text{ch}} \geq 1$ bounds the distortion between the chart norms and the Euclidean norm ($|y|_{\Phi} \leq C_{\text{ch}}|y|$ and $|y| \leq C_{\text{ch}}|y|_{\Phi}$) over the finitely many charts in use, and C_{act} is such that active monomials lie within $C_{\text{act}}C_0$ of the maximum.

The regions play, for $\mathcal{W}$, the role that the preimages $U \times T^3$ of small open sets U of the smooth part of the base play for a torus fibration, as in the topological models of [CBM14]; Lemma F.12 makes this precise at the level of homotopy.

Lemma F.12 (regions). *For $\varkappa$ large and every product-shaped U of type (a), (b) or (c):*

- (1) $\mathcal{W}_U \cong (A_{\Phi}^{\perp} \otimes U(1)) \times U_{\parallel}^{\eta} \times L_U$, where L_U is a point in type (a), homotopy equivalent to a circle in type (b), and homotopy equivalent to a two-torus in type (c). Hence $\mathcal{W}_U$ is aspherical, and $\pi_1(\mathcal{W}_U) \cong \Lambda_U$, the lattice $u^{\perp} \cap N$ in type (a) and $N/\mathbb{Z}n$ in types (b) and (c).
- (2) If U' is product-shaped and $\mathcal{W}_{U'} \subset \mathcal{W}_U$, the inclusion induces on π_1 the parallel transport of Λ from U' to U .
- (3) The same holds for the sheared charts $(\pi_{\parallel} + \lambda \circ (\pi_{\perp} - \pi_{\perp}(0)), \pi_{\perp})$ with λ linear and $\lambda(\pi_{\perp}(n) - \pi_{\perp}(0)) = 0$.

The proof is in Appendix F.11. In type (b), L_U is a truncated localized line capped at the divisor D_n ; in type (c), it is a truncated localized plane, $\mathbb{P}^2$ minus four lines, capped at D_n . By Lemma F.10, $\mathfrak{s}(U) \subset \mathcal{W}_U$ once η exceeds C_{ch} times the radial width of Lemma F.10, and loops $\mathfrak{s}(\gamma)$ with γ in a contractible U are contractible in $\mathcal{W}_U$.

F.4. The domain. We construct the three-dimensional domain by attaching circles, triangles and tetrahedra over the cells of the tropical cycle. Balancing cancels the boundaries along interior strata; the remaining boundary components lie over the nodes.

Let $\mathfrak{T}$ be the triangulation of $S \setminus \text{int } \mathcal{N}_{\Gamma}$ of Lemma F.20 below. The cell complex $\mathfrak{D}(S)$ is glued from pieces over the cells of $\mathfrak{T}$ and from local domains over $\mathcal{N}_{\Gamma}$.

Definition F.13 (the domain). The *domain* $\mathfrak{D} = \mathfrak{D}(S)$ is the CW complex obtained by gluing the following pieces.

- Over each cell c of $\mathfrak{T}$ in a sheet, $c \times S^1$, the circle being $\mathbb{R}v/\mathbb{Z}v$, oriented by v .
- Over each cell c of $\mathfrak{T}$ on an interior edge, with fields v_1, v_2, v_3 on the adjacent sheets and $s_1v_1 + s_2v_2 + s_3v_3 = 0$ (the interior-edge balancing clause): $c \times \mathcal{Q}$, where $\mathcal{Q}$ is a 2-simplex with its three vertices identified to one point, whose three edges are

glued to the circles $G_k = \mathbb{R}v_k/\mathbb{Z}v_k$ of the three sheets, and which is oriented so that $\partial\mathcal{Q} = s_1G_1 + s_2G_2 + s_3G_3$ with G_k oriented by v_k . It is the domain of the flat triangle, cf. [CBM14, Lemma 7.7]: the linear map sending the edges of a 2-simplex to s_kv_k , followed by the quotient by the lattice.

- Over an interior vertex, where four interior edges and six sheets meet: a 3-simplex whose four faces are glued to the triangles $\mathcal{Q}$ of the four edges. This is possible because the six fields are the edge vectors, and the four balancing relations the face relations, of the image of a 3-simplex under a linear map, which is nondegenerate by the interior-vertex clause [CBM14, §7.7, Step 4].
- Over $\mathcal{N}_\Gamma$, the local domains of Appendix F.5: a disc times an arc along each leg of ∂S (over the leg pieces and the switching discs), a cone over a triangle at each boundary vertex, a copy of $\mathbb{C} \times \mathbb{R}$ at each positive vertex met by ∂S , a solid torus at each crossing $\mathfrak{x}_k$, and, at each node, the *node domain* $(\mathfrak{H}_q^\circ \times S^1)/\sim$, where $\mathfrak{H}_q^\circ \subset \mathfrak{H}_q$ is the part of $\hat{j}(\mathcal{N}_\Gamma)$ in the standard half-disc, a collar of its boundary path, and the circle is collapsed over the two leg segments.

Each piece is oriented as (sheet) $\times$ (circle oriented by v), and $[\mathfrak{D}]$ denotes the sum of the oriented top cells. For a node q on ∂S let $\partial_q \mathfrak{D} = (e_q \times S^1)/\sim$, the circle collapsed at y_0 and y_∞ , a 2-sphere. The frontier of $\mathcal{N}_\Gamma$ in S is the *inner boundary*.

Over the smooth part of S away from $\mathcal{N}_\Gamma$, $\mathfrak{D}(S)$ is the circle bundle with fibre $\mathbb{R}v/\mathbb{Z}v$ of [CBM14, §7], and over the interior edges and vertices it is their completion by flat triangles and tetrahedra. Each local domain meets the cells over the regular part $S \setminus \text{int } \mathcal{N}_\Gamma$ over its part of the inner boundary.

Lemma F.14 (the domain). $\partial[\mathfrak{D}] = \sum_q [\partial_q \mathfrak{D}]$, the sum over the nodes q with $p_q \in j(\partial S)$, where e_q carries the boundary orientation of $\mathfrak{H}_q$.

Proof. Interior faces cancel in pairs if the local orientations agree across them. Over the sheets this holds because the orientations are products. Along an interior edge, let c be a one-cell of $\mathfrak{T}$ on the edge, oriented as the edge. Then $\partial(c \times \mathcal{Q}) = \partial c \times \mathcal{Q} - c \times \partial \mathcal{Q} = \partial c \times \mathcal{Q} - \sum_k s_k c \times G_k$. The term $\partial c \times \mathcal{Q}$ cancels between adjacent cells of the edge, and $s_k c \times G_k$ is the contribution of c to the boundary of the cylinders over the k -th sheet, because s_k is by definition the sign with which c occurs in the boundary of the k -th sheet; so the three cylinders are cancelled. At an interior vertex the boundary of the 3-simplex is the sum of the four triangles with the signs of the balancing conditions. At the local domains the pieces are glued along the inner boundaries of Appendix F.5, and over the legs, boundary vertices, positive vertices and crossings the circle collapses or closes up without creating boundary. What remains is the part of $\partial \mathfrak{H}_q^\circ \times S^1$ over e_q , where the circle $\mathbb{R}v/\mathbb{Z}v$ does not collapse, because the vanishing vector of e_q is $c - a \neq \pm v$. $\square$

F.5. The local pieces. Each local domain is mapped into $\mathscr{W}$ by an explicit map, built in the exact local models of Lemma F.7(3), whose restriction to the inner boundary with the regular part consists of orbits of the circle $S_v^1 = \mathbb{R}v/\mathbb{Z}v \subset N \otimes U(1)$ through points of the positive section, over base points in the sets V_c of Lemma F.20. Since the positive points miss the strata (Lemma F.9), each piece contains collar paths from a point of $\mathfrak{s}$ to the critical locus of the local model. For a path γ in $\mathscr{W}$ and a character x^m that does not vanish on γ , a *lifted phase* of x^m along γ is a continuous function θ with $x^m(\gamma(\cdot)) \in \mathbb{R}_{>0} e^{i\theta(\cdot)}$; its total change along γ does not depend on the choice. If x^m vanishes only at the end point of γ and its phase is constant near that point, we use the limit.

Adapted coordinates. For each two-face $F_{uu'}$ of $\nabla^{\check{h}}$ we fix an ordering (u, u') and a cocharacter $\nu = \nu_{uu'} \in N$ with

$$\langle u, \nu \rangle = 0, \quad \langle u', \nu \rangle = 1;$$

it exists because $\{0, u, u'\}$ is a unimodular triangle, so that u, u' extend to a basis of M . Let a leg in $F_{uu'}$ have lower support $\{n, n'\}$, and $\sigma = \text{cone}(n, n') \in \mathcal{F}'$. Choose $m_1 \in M$ with $\langle m_1, n \rangle = 1$, $\langle m_1, n' \rangle = 0$ and $\langle m_1, \nu \rangle = 0$ (adjust any solution of the first two conditions by a multiple of $u' - u \in \sigma^\perp$), put $m_2 = -u - m_1$, so that $\langle m_2, n \rangle = 0$, $\langle m_2, n' \rangle = 1$ and $\langle m_2, \nu \rangle = 0$, and let κ generate $\sigma^\perp \cap \nu^\perp \cap M$, which has rank one. Then $(m_1, m_2, u' - u, \kappa)$ is a basis of M with dual basis $(n, n', \nu, \cdot)$, and

$$z_1 = x^{m_1}, \quad z_2 = x^{m_2}, \quad t = x^{u' - u}, \quad w = x^\kappa, \quad z_1 z_2 = x^{-u},$$

are coordinates on $U_\sigma \cong \mathbb{C}^2 \times (\mathbb{C}^*)^2$. Near the leg only $0, u, u'$ are active, and multiplying $\tilde{f}$ by x^{-u} shows that $\mathcal{W} \cap U_\sigma$ is

$$(F.1) \quad \beta_0 z_1 z_2 = \alpha_u + \alpha_{u'} t, \quad \alpha_u = \beta_u \varkappa^{-\check{h}_u}, \quad \alpha_{u'} = \beta_{u'} \varkappa^{-\check{h}_{u'}},$$

where $\beta_0, \alpha_u, \alpha_{u'} \geq 0$ are functions of $|x|$ (the signs because $\check{h}_u, \check{h}_{u'}$ are odd), and β_0 vanishes where the constant term is cut off, in particular near $D_n \cup D_{n'}$. The circle S_v^1 , $v = n' - n$, acts with weights $(-1, 1, 0, 0)$ on (z_1, z_2, t, w) , exactly by Lemma F.8, and freely off $\text{Crit} = \{z_1 = z_2 = 0, t = t_0\}$, $t_0 = -\alpha_u / \alpha_{u'} < 0$. We call Crit the *critical locus* of the leg model: it is the fixed locus of S_v^1 in $\mathcal{W} \cap U_\sigma$, the curve over which the circle of the boundary field collapses.

Sides and chambers. By (T1), near a point e of an edge of ∂S on the leg that is not a switching point, the sheet of $\hat{S}$ ending on the leg lies on the side of F_u or of $F_{u'}$; its *side* at e is $s = +1$ or -1 accordingly. At every point of $\mathcal{W}$ near the leg, $\alpha_u - \alpha_{u'} |t|$ has the sign of $T_u - T_{u'} = \check{h}_{u'} - \check{h}_u - \log_\varkappa |t|$, because $\alpha_{u'} |t| / \alpha_u = \varkappa^{T_{u'} - T_u} \beta_{u'} / \beta_u$ and $\beta_{u'} \leq \beta_u$ if and only if $T_{u'} \leq T_u$. The *chamber* of side s at the leg is

$$\begin{aligned} \text{Ch}_s &= \mathcal{W} \cap U_\sigma \cap \{z_1 z_2 \in s\mathbb{R}_{\geq 0}, t \in \mathbb{R}_{< 0}, w \in \mathbb{R}_{> 0}\} \\ &= \mathcal{W} \cap U_\sigma \cap \{x^{-u} \in s\mathbb{R}_{\geq 0}, t \in \mathbb{R}_{< 0}, w \in \mathbb{R}_{> 0}\}; \end{aligned}$$

it is S_v^1 -invariant, contains $\text{Crit} \cap \{w > 0\}$, and on it $s\beta_0 |z_1 z_2| = \alpha_u - \alpha_{u'} |t|$.

Coefficients near a leg. Put $v = T_0 - T_u$ and $\vartheta = T_u - T_{u'}$. Near the leg these depend only on $|z_1 z_2| = \varkappa^{v - \check{h}_u}$ and on $|t| = \varkappa^{\check{h}_{u'} - \check{h}_u - \vartheta}$; $\vartheta = 0$ exactly where $|t| = |t_0|$, and the points of $\hat{B}$ near the leg on side s have $s\vartheta > 0$. Since only $0, u$ and u' are active near the leg,

$$(F.2) \quad \beta_0 = \beta(-v) \beta(-v - \vartheta), \quad \varkappa^{\check{h}_u} \alpha_u = \beta(v) \beta(-\vartheta), \quad \varkappa^{\check{h}_u} \alpha_{u'} |t| = \varkappa^{-\vartheta} \beta(v + \vartheta) \beta(\vartheta).$$

Lemma F.15 (real points of a chamber). *For $\vartheta' > 0$ put*

$$f_{\vartheta'}(r) = \beta(-r) \beta(-r - \vartheta') \varkappa^r - \beta(r) + \varkappa^{-\vartheta'} \beta(\vartheta') \beta(r + \vartheta').$$

For $\varkappa$ large, every $f_{\vartheta'}$ has exactly one zero $X(\vartheta')$; $X(\vartheta') \in (-C_0, C_0)$, X is continuous, and $X(\vartheta') \rightarrow -C_0$ as $\vartheta' \rightarrow 0$. Near the leg, a point of Ch_s with $z_1 z_2 \neq 0$, $z_2 > 0$ and given w and $|z_1|/|z_2|$ is determined by (v, ϑ) , and those with $s\vartheta \geq 0$ are:

- (1) *for $s\vartheta > 0$, the points with $T_0 - T_{u_s} = X(s\vartheta)$, where $u_{+1} = u$ and $u_{-1} = u'$;*
- (2) *for $\vartheta = 0$, that is $t = t_0$, the points with $|z_1 z_2| \leq \varkappa^{-C_0 - \check{h}_u}$.*

Proof. On Ch_s the equation is $s\beta_0 |z_1 z_2| = \alpha_u - \alpha_{u'} |t|$. For $s = 1$ and $\vartheta \geq 0$ we have $\beta(-\vartheta) = 1$, and by (F.2), after multiplication by $\varkappa^{\check{h}_u}$, the equation reads $f_\vartheta(v) = 0$. For $s = -1$ and $\vartheta = -\vartheta' \leq 0$ we have $\beta(\vartheta) = 1$, and after multiplication by $-\varkappa^{\check{h}_u - \vartheta'}$ it reads

$f_{\vartheta'}(v') = 0$ with $v' = v - \vartheta' = T_0 - T_{u'}$. At $\vartheta = 0$ it reads $\beta(-v)^2 \varkappa^v = 0$, which holds exactly when $v \leq -C_0$. So (1) and (2) follow from the statements on $f_{\vartheta'}$.

Let $\vartheta' > 0$. For $r \leq -C_0$, $\beta(-r) = 0$ and $\beta(r) = 1$, so $f_{\vartheta'}(r) \leq \varkappa^{-\vartheta'} \beta(\vartheta') - 1 < 0$; for $r \geq C_0$, $f_{\vartheta'}(r) = \varkappa^r > 0$. For $r > -C_0$ we have $\beta(-r - \vartheta') \geq \beta(-r) > 0$, and since $\beta' \leq 0$,

$$f'_{\vartheta'}(r) \geq \varkappa^r \log \varkappa \cdot \beta(-r)^2 + \varkappa^{-\vartheta'} \beta(\vartheta') \beta'(r + \vartheta').$$

The last term vanishes unless $C_1 < r + \vartheta' < C_0$ and $\vartheta' < C_0$. Then $r > C_1 - C_0$, so $\beta(-r) \geq \beta(C_0 - C_1) > 0$ and $\varkappa^r > \varkappa^{C_1 - \vartheta'}$, and $f'_{\vartheta'}(r) \geq \varkappa^{-\vartheta'} (\varkappa^{C_1} \log \varkappa \cdot \beta(C_0 - C_1)^2 - \max |\beta'|)$, which is positive for $\varkappa$ large. So $f_{\vartheta'}$ has positive derivative on $(-C_0, \infty)$ and exactly one zero $X(\vartheta') \in (-C_0, C_0)$, which depends smoothly on ϑ' by the implicit function theorem. As $\vartheta' \rightarrow 0$, $f_{\vartheta'}$ converges uniformly on $[-C_0, C_0]$ to $\beta(-r)^2 \varkappa^r$, which is bounded below by a positive constant on $[-C_0 + \eta', C_0]$ for every $\eta' > 0$; so $X(\vartheta') \rightarrow -C_0$. $\square$

Definition F.16 (leg collar paths). Let e be a point of an edge of ∂S on the leg that is not a switching point, and s its side. Over the transverse segment $[0, \varsigma_0]$ of $\hat{S}$ at e , the *leg collar path* $\gamma_e(\varsigma)$ is the path in $\mathscr{W}$ that

- (i) for $\varsigma \in [\varsigma_0/2, \varsigma_0]$ follows $\mathfrak{s}$;
- (ii) then moves along the positive real points of $\mathscr{W}$, with $|z_1|/|z_2|$ and w fixed, changing $|\mathfrak{t}|$ monotonically until $s\vartheta \geq C_0$ (it is constant if this holds at its start);
- (iii) then turns $\arg \mathfrak{t}$ from 0 to π with $|\mathfrak{t}|$, $|z_1|$, z_2 and w fixed;
- (iv) then moves inside Ch_s with $z_2 > 0$ and with w and $|z_1|/|z_2|$ fixed: first through the points of Lemma F.15(1), $s\vartheta$ decreasing to 0, then through those of Lemma F.15(2), $|z_1 z_2|$ decreasing from $\varkappa^{-C_0 - \check{h}_u}$ to 0. It ends at the *point of Crit over e* , the point of Crit (where $z = 0$ and $\mathfrak{t} = \mathfrak{t}_0$) with the value $w = w(\mathfrak{s}(e, \varsigma_0/2))$ fixed along the collar path.

Over a transverse segment, the collar paths play the role of the sections $\check{\sigma}$ with $\check{\sigma}(\partial S)$ in the critical locus that Castaño-Bernard and Matessi use in their local models [CBM14, §7.2, (47)–(48), §7.4, §7.5], and the leg piece below is the orbit of such a section under S_v^1 . The end point depends continuously on e because $\mathfrak{s}$ does. In (ii) the positive real point with given $|\mathfrak{t}|$, w and $|z_1|/|z_2|$ is unique and depends continuously on $|\mathfrak{t}|$: by (F.2) its equation is $\beta_0 \varkappa^v = \varkappa^{\check{h}_u} (\alpha_u + \alpha_{u'} |\mathfrak{t}|)$, whose left side vanishes for $v \leq v^* := \max(-C_0, -C_0 - \vartheta)$ (for $\vartheta < 0$ the factor β_0 also vanishes on $(-C_0, -C_0 - \vartheta]$) and has positive derivative for $v > v^*$, while the right side is positive for $v \leq v^*$, non-increasing in v , and zero for v large. At the end of (ii), $\beta(s\vartheta) = 0$, so by (F.2) $\alpha_{u'} = 0$ if $s = 1$ and $\alpha_u = 0$ if $s = -1$. The coefficients of (F.1) depend only on $|z_1 z_2|$ and $|\mathfrak{t}|$, so they are constant along (iii), and (iii) is explicit: for $s = 1$, $z_1 = \alpha_u / (\beta_0 z_2)$ does not move, and for $s = -1$, $z_1 = \alpha_{u'} \mathfrak{t} / (\beta_0 z_2)$ turns with $\mathfrak{t}$ from the positive to the negative real axis ($\beta_0 > 0$ at these points, since the right side of the equation is positive at the end of (ii)). Since $\beta(s\vartheta) = 0$ there, the equation of the positive real points at the end of (ii) coincides with that of Ch_s , so the end point of (iii) is the point of Lemma F.15(1) with this value of $s\vartheta$, where (iv) starts. By Lemma F.15, (iv) is a continuous path, and (i)–(iv) depend continuously on e along each open interval of the edge between switching points. Along the whole collar path the lifted phase of $\mathfrak{t}$ increases by π , those of z_2 and w do not change, and that of z_1 increases by $(1 - s)\pi/2$ (in (iv), z_1 and z_2 keep their phases until they vanish at the end point); hence the lifted phase of a character x^m changes by

$$(F.3) \quad \pi \left(\langle m, \nu \rangle + \frac{1-s}{2} \langle m, n \rangle \right),$$

and $x^{-u} = z_1 z_2$ ends in $s\mathbb{R}_{\geq 0}$. The *leg piece* is $S_v^1 \cdot \{\gamma_e(\varsigma)\}$, a disc times an arc: the circle collapses at $\varsigma = 0$, on Crit, so the piece has no boundary over the leg. Its outer part consists of S_v^1 -orbits through points of $\mathfrak{s}$.

Switching discs. At a switching point e^* the side changes, and the leg collar paths of the two sides end (iii) with z_1 of opposite signs, so they do not form a continuous family across e^* . For each switching point e^* we choose points e' and e'' of the edge on either side of e^* , with no other switching point between them, and call *switching disc* the closed disc $\mathbb{B}_{e^*} \subset S$ bounded by the segment $[e', e'']$ of ∂S , the transverse segments at e' and e'' , and an arc of the inner boundary. The *switching piece* is $S_v^1 \cdot \Theta_{e^*}$ over $\mathbb{B}_{e^*}$, where $\Theta_{e^*}: \mathbb{B}_{e^*} \rightarrow \mathcal{W}$ is given by Lemma F.19(4): it agrees with $\gamma_{e'}$ and $\gamma_{e''}$ on the transverse segments and with $\mathfrak{s}$ on the inner boundary arc, and maps each point e of $[e', e'']$ to the point of Crit over e . Like a leg piece it is a disc times an arc; the leg pieces are taken over the edge with the interiors of the switching discs removed.

Boundary vertices of S (negative vertices). Let the lower support be the unimodular triangle $\{n_1, n_2, n_3\}$ in $F_{uu'}$, and $\sigma = \text{cone}(n_1, n_2, n_3)$. Let $Z_j = x^{m^{(j)}}$ with $\langle m^{(j)}, n_k \rangle = \delta_{jk}$ and $\langle m^{(j)}, \nu \rangle = 0$; then $Z_1 Z_2 Z_3 = x^{-u}$, $(Z_1, Z_2, Z_3, \mathfrak{t})$ are coordinates on $U_\sigma \cong \mathbb{C}^3 \times \mathbb{C}^*$, and $\mathcal{W}$ is $\beta_0 Z_1 Z_2 Z_3 = \alpha_u + \alpha_{u'} \mathfrak{t}$. On the leg with lower support $\{n_k, n_l\}$ the adapted coordinates may be taken to be $z_1 = Z_k$, $z_2 = Z_l Z_j$, $w = Z_j$, $\{j, k, l\} = \{1, 2, 3\}$. The torus $T^2 = (u, u')^\perp \otimes U(1) = \{\lambda_1 \lambda_2 \lambda_3 = 1\}$ acts exactly; the three legs are the coordinate axes L_j , whose stabilisers G_j are the circles of the three boundary fields. By (T2) the three sheets at the vertex have one side s . The *vertex path* is a path in $\mathcal{W}$ over the segment of the interior edge of S leaving the vertex, from a point of $\mathfrak{s}$ to the point $Z = 0$, $\mathfrak{t} = \mathfrak{t}_0$ of Crit over the vertex, which turns $\arg \mathfrak{t}$ from 0 to π as in (iii) and ends with $|Z_1| = |Z_2| = |Z_3| \rightarrow 0$ and $\arg Z_j = 0$ (side +1) or $\pi/3$ (side -1), so that $Z_1 Z_2 Z_3 = x^{-u}$ ends in $s\mathbb{R}_{\geq 0}$. Let $\mathbb{B}_j$ be the corner of the j -th sheet at the vertex: a closed disc in the sheet bounded by a segment of its leg ending at the vertex, a segment of the interior edge, a transverse segment of the sheet at the other end of the leg segment, and an arc of the inner boundary; the leg segment contains no switching point. On this leg segment the boundary map of $\mathbb{B}_j$ is a path in the closure of the critical curve Crit of the leg, from the point of Crit over the far end of the segment to the point $Z = 0$, $\mathfrak{t} = \mathfrak{t}_0$, which lies in that closure; along it $\mathfrak{t} = \mathfrak{t}_0$, and it is fixed by G_j . The piece is the union of the leg pieces over the rest of the legs, the three sheet pieces $G_j \cdot \Theta_j$ over the $\mathbb{B}_j$, with Θ_j given by Lemma F.19(2), and the cone, over the vertex path, on the triangle $\mathcal{Q} \subset T^2$ with $\partial \mathcal{Q} = s_1 G_1 + s_2 G_2 + s_3 G_3$ as in Definition F.13 [CBM14, §7.5]. The two-cells $\mathcal{Q} \cdot \mathfrak{s}(b)$ over the interior edge leaving the vertex lie in $T^2 \cdot \mathfrak{s}(b)$.

Positive vertices. Let ∂S turn at the positive vertex p on the edge of $\nabla^{\tilde{h}}$ dual to $\{u_1, u_2, u_3\}$, with lower support $\{n, n'\}$, along the legs $\Gamma_{12} \subset F_{u_1 u_2}$ and $\Gamma_{13} \subset F_{u_1 u_3}$; by (T3) the corner of S lies in F_{u_1} . Since $\{0, u_1, u_2, u_3\}$ is a unimodular simplex, there are $\nu_2, \nu_3 \in N$ with $\langle u_1, \nu_k \rangle = 0$ and $\langle u_j, \nu_k \rangle = \delta_{jk}$ for $j, k \in \{2, 3\}$. Choose z_1, z_2 as above with $z_1 z_2 = x^{-u_1}$ and $\langle m_i, \nu_2 \rangle = \langle m_i, \nu_3 \rangle = 0$, and put $\mathfrak{t}_j = x^{u_j - u_1}$. In U_σ , $\mathcal{W}$ is $\beta_0 z_1 z_2 = \alpha_{u_1} + \alpha_{u_2} \mathfrak{t}_2 + \alpha_{u_3} \mathfrak{t}_3$, and Crit = $\{z = 0, \alpha_{u_1} + \alpha_{u_2} \mathfrak{t}_2 + \alpha_{u_3} \mathfrak{t}_3 = 0\}$ is a localized line in the $(\mathfrak{t}_2, \mathfrak{t}_3)$ -torus, a pair of pants whose three ends lie over the three legs of $\hat{\Gamma}$ at p . Along Γ_{12} the leg collar paths of the two-face $F_{u_1 u_2}$ arrive, by (F.3), with lifted phases

$$(\arg \mathfrak{t}_2, \arg \mathfrak{t}_3) = (\epsilon_{12} \pi, \langle u_3 - u_1, \nu_{12} \rangle \pi), \quad \epsilon_{12} = \langle u_2 - u_1, \nu_{12} \rangle = \pm 1,$$

where ν_{12} is the cocharacter of $F_{u_1 u_2}$, and x^{-u_1} ends in $\mathbb{R}_{\geq 0}$, since the side is that of F_{u_1} ; symmetrically along Γ_{13} . In the part of a neighbourhood of Γ_{12} where u_3 is inactive, the circle $\nu_3 \otimes U(1)$ preserves $\mathcal{W}$ (Lemma F.8), commutes with S_v^1 , preserves Crit, and changes,

among $\mathbf{t}_2, \mathbf{t}_3, x^{-u_1}$, only the phase of $\mathbf{t}_3$. Let $\mathbb{B}_p$ be the corner of S at p , a closed disc bounded by segments $[p, e_2] \subset \Gamma_{12}$ and $[p, e_3] \subset \Gamma_{13}$ of the two legs, the transverse segments at e_2 and e_3 , and an arc of the inner boundary; the segments are long compared with ς_0 , so that the collar paths over points of Γ_{12} near e_2 lie in the part where u_3 is inactive, and symmetrically at e_3 . The *slide* along Γ_{12} modifies the collar paths over a segment $[e'_2, e_2] \subset \Gamma_{12}$ in that part: it replaces $\gamma_e(\varsigma)$ by $e^{i\pi k \phi(e)\chi(\varsigma)\nu_3} \gamma_e(\varsigma)$, where ϕ increases from 0 at e'_2 to 1 at e_2 , $\chi = 1$ for $\varsigma \leq \varsigma_0/4$ and $\chi = 0$ for $\varsigma \geq \varsigma_0/2$, and the integer k is chosen so that the lifted phase of $\mathbf{t}_3$ at the end becomes $\epsilon_{13}\pi$, the value with which the collar paths along Γ_{13} arrive. Symmetrically, the slide along Γ_{13} turns $\arg \mathbf{t}_2$ to $\epsilon_{12}\pi$ with ν_2 . After the slides both families arrive at e_2 and e_3 with $(\arg \mathbf{t}_2, \arg \mathbf{t}_3) = (\epsilon_{12}\pi, \epsilon_{13}\pi)$ and $x^{-u_1} \in \mathbb{R}_{\geq 0}$, that is, in the *corner chamber* $\mathscr{W} \cap U_\sigma \cap \{x^{-u_1} \in \mathbb{R}_{\geq 0}, \mathbf{t}_2, \mathbf{t}_3 \in \mathbb{R}_{< 0}\}$, which contains the real arc of Crit joining Γ_{12} to Γ_{13} ; on it x^{-u} , for the first element u of the ordering of $F_{u_1 u_j}$, lies in $s\mathbb{R}_{\geq 0}$, s being the side of the corner sheet at Γ_{1j} . The piece is $S_v^1 \cdot \Theta_p$ over $\mathbb{B}_p$, with Θ_p given by Lemma F.19(3), together with the slid leg pieces; near p , Θ_p may be taken with values in the corner chamber, as in the model of [CBM14, §7.4], where the swept set is homeomorphic to $\mathbb{C} \times \mathbb{R}$.

Crossings. At a crossing $\mathbf{r}_k$ of a leg with covector $\alpha = u' - u$, v is monodromy invariant, that is $v \in (u, u')^\perp$, so S_v^1 acts exactly on the local model (Lemma F.8). The piece is $S_v^1 \cdot \mathfrak{s}(D_k)$ for a small disc D_k around $\hat{j}(\mathbf{r}_k)$ in $\hat{S}$, a solid torus; $\mathfrak{s}$ lies in the dense torus, where S_v^1 acts freely.

Nodes. Let q be a node on ∂S , with parallelogram $P = abcd$ and $F_q = F_{u_+ u_-}$. Near $C_q^\mathscr{W} := \mathscr{W} \cap V(\text{cone}(a, c))$ only the monomials $0, u_\pm$ can be active, because the dual edge $[u_-, u_+]$ has length one, and the constant term is cut off near the strata; hence near $C_q^\mathscr{W}$, $\mathscr{W}$ is $\{\mathbf{t} = \mathbf{t}_0\} \times \widehat{\text{Con}}_q$, where $\mathbf{t} = x^{u' - u}$ for the ordering (u, u') of F_q and $\widehat{\text{Con}}_q$ is the resolved conifold of the two cones $\text{cone}(a, b, c)$, $\text{cone}(a, c, d)$ in $N_q = \text{span}(a, b, c, d) \cap N$; $N = N_q \oplus \mathbb{Z}\nu$ for the cocharacter ν of F_q , and $N_q \otimes U(1)$ acts exactly. So $C_q^\mathscr{W} \cong \mathbb{P}^1$ is the exceptional curve of $\mathscr{W}$ at q . The lattice $\Lambda_f = (u_+, u_-)^\perp \cap N = \mathbb{Z}(b - a) \oplus \mathbb{Z}(c - b)$ is the lattice of differences in N_q . On $C_q^\mathscr{W}$, with coordinate $\zeta = Z_b/Z_d$, the circle S_λ^1 , $\lambda = k_1(b - a) + k_2(c - b)$, acts by $\zeta \mapsto e^{i(k_1 - k_2)\theta} \zeta$; we call $\text{wt}(\lambda) = k_1 - k_2$ its *weight*. The curves $D_{ab} = \{Z_a = Z_b = 0\}$ and D_{bc} meet $C_q^\mathscr{W}$ at $\zeta = 0$, and D_{cd}, D_{da} at $\zeta = \infty$. Let $\mathbb{B}_q \subset \mathfrak{H}_q^\circ$ be a closed disc bounded by the part of the boundary path of $\mathfrak{H}_q$ between points e', e'' of the two leg segments near y_0 and y_∞ , the transverse segments of $\mathfrak{H}_q^\circ$ at e' and e'' , and an arc of the inner boundary. The *node chain* is $S_v^1 \cdot \Theta_q$ over $\mathbb{B}_q$, together with the leg pieces over the rest of the two leg segments, where $\Theta_q: \mathbb{B}_q \rightarrow \mathscr{W}$ is given by Lemma F.19(1): it agrees with the leg collar paths $\gamma_{e'}, \gamma_{e''}$ on the transverse segments and with $\mathfrak{s}$ on the inner boundary, maps e_q to an arc $\gamma_q \subset C_q^\mathscr{W}$ from $\zeta = 0$ to $\zeta = \infty$, and maps the leg segment from e' to y_0 to a path in the closure of the critical curve Crit of that leg, from the point of Crit over e' to the point $\zeta = 0$ of $C_q^\mathscr{W}$, which lies in that closure, and symmetrically the leg segment from y_∞ to e'' to a path from $\zeta = \infty$ to the point of Crit over e'' . Along these paths $\mathbf{t} = \mathbf{t}_0$, and they are fixed by S_v^1 .

Lemma F.17 (node degree). *Let q be a node on ∂S . The node chain restricts on $\partial_q \mathcal{D}$ to a map onto $C_q^\mathscr{W}$ of degree $-\rho_q(S)$, for the complex orientation of $C_q^\mathscr{W}$. Equivalently, let $c_{ab}(b - a)$ and $c_{bc}(c - b)$ be the coefficients of the legs with lower supports $\{a, b\}$ and $\{b, c\}$, oriented away from the node as in Section 5.3, so that, ∂S passing q once along one pair of opposite legs, one of c_{ab}, c_{bc} is 0 and the other is ± 1 ; then the degree is $\text{wt}(v_q) = c_{bc} - c_{ab}$, where $v_q = -(c_{ab}(b - a) + c_{bc}(c - b))$ is the coefficient on e_q (oriented from y_0 to y_∞) determined by balancing at y_0 .*

Proof. On $\partial_q \mathfrak{D}$ the map is $(x, \theta) \mapsto e^{i\theta v} \cdot \gamma_q(x)$, $x \in e_q$, collapsed at the poles, which S_v^1 fixes. Take the $b - a$ passage entering along the leg with lower support $\{a, b\}$, with $v = \epsilon_v(b - a)$. The boundary path of $\mathfrak{H}_q$ runs from that leg through y_0 , along e_q and through y_∞ to the leg with lower support $\{c, d\}$, and γ_q runs from $\zeta = 0$ (on D_{ab}) to $\zeta = \infty$ (on D_{cd}). By Stokes, the orientation of $\partial_q \mathfrak{D}$ is (boundary orientation of e_q) $\times$ (circle), and in polar coordinates $\zeta = re^{i\phi}$ the map is $(r, \theta) \mapsto re^{i\epsilon_v \theta}$, of degree $\epsilon_v = \text{wt}(v)$. Since $c_{ab} = -\epsilon_v$ and $c_{bc} = 0$, this is $\text{wt}(v_q)$, and it equals $-\rho_q$ by the formula $\rho_q = c_{ab} - c_{bc}$ of Section 5.3. For the $c - b$ passage entering along the leg with lower support $\{b, c\}$, with $v = \epsilon'_v(c - b)$, the arc again runs from $\zeta = 0$ to $\zeta = \infty$ and the degree is $\text{wt}(v) = -\epsilon'_v = -\rho_q$. Reversing the orientation of ∂S reverses both γ_q and the sign of ρ_q . The degree depends only on the homotopy class of γ_q relative to the poles, and $C_q^{\mathcal{W}}$ is simply connected, so it does not depend on the choice of γ_q . $\square$

The existence of the maps Θ rests on a computation of local fundamental groups, proved in Appendix F.13. Let (τ, ω) be a dual cell pair, u_0 a vertex of τ and $\tau' = \tau \setminus \{u_0\}$, and let $X_\omega \subset X_{\mathcal{F}'}$ be the open toric subvariety of the cones of $\mathcal{F}'$ contained in $\text{cone}(\omega)$. The *extended tropicalisation* of X_ω is Log on the dense torus and, on the orbit $O(\sigma)$ of a cone $\sigma \subseteq \text{cone}(\omega)$, the logarithm of that orbit, with values in $N_{\mathbb{R}}/\text{span}(\sigma)$; a subset $\Omega \subseteq N_{\mathbb{R}}$ has a preimage under it, consisting of the points whose logarithm lies in Ω , respectively in the image of Ω in $N_{\mathbb{R}}/\text{span}(\sigma)$. The four cases used are: a node ($\tau = [u_+, u_-]$, $\omega = P$, subdivided by its diagonal in $\mathcal{F}'$), a negative vertex (τ an edge, ω a triangle), a positive vertex (τ a triangle, ω an edge) and a switching point (τ and ω edges).

Lemma F.18 (local fundamental groups). *Let $\dim \tau + \dim \omega = 3$, or $\dim \tau = \dim \omega = 1$, and let $N_\omega = \text{span}(\omega) \cap N$. The characters $u - u_0$, $u \in \tau'$, are part of a basis of the saturated lattice $N_\omega^\perp \subset M$; if $\dim \tau + \dim \omega = 3$ they are a basis, and if $\dim \tau = \dim \omega = 1$ we complete them to a basis by a character κ . Let $\nu_u \in N$, $u \in \tau'$, and in the second case $\nu_\kappa \in N$, satisfy $\langle u_0, \cdot \rangle = 0$ and pair with this basis of $N_\omega^\perp$ to the dual basis; so $\langle u', \nu_u \rangle = \delta_{uu'}$ for $u, u' \in \tau'$. Then $N = N_\omega \oplus \bigoplus_{u \in \tau'} \mathbb{Z}\nu_u$, respectively $N = N_\omega \oplus \mathbb{Z}\nu_u \oplus \mathbb{Z}\nu_\kappa$. Let $\Omega = \Omega' \oplus \sum_{u \in \tau'} I_u \nu_u$, respectively $\Omega = \Omega' \oplus I_u \nu_u \oplus I_\kappa \nu_\kappa$, with $\Omega' \subset N_{\omega, \mathbb{R}}$ open and convex, $\Omega' + \text{cone}(\omega) = \Omega'$, and I_u, I_κ bounded open intervals. Assume that only the monomials 0 and $u \in \tau$ are active on $\text{Log}^{-1}(\Omega)$, and that Ω contains the ball of radius $4C_0$ about a point of $\text{ri } F_\tau$. Here, as in Appendix F.14, balls and distances in $N_{\mathbb{R}}$ are Euclidean for the standard inner product in lattice coordinates, so that $|m| \geq 1$ for $0 \neq m \in M$. Let $\mathcal{W}_\Omega$ be the preimage of Ω in $\mathcal{W} \cap X_\omega$ under the extended tropicalisation. Then $\mathcal{W}_\Omega$ is open in $\mathcal{W}$ and path connected, and the characters $\mathbf{t}_u = x^{u-u_0}$, $u \in \tau'$, induce an isomorphism $\pi_1(\mathcal{W}_\Omega) \rightarrow \mathbb{Z}^{\tau'}$, respectively, together with $w = x^\kappa$, an isomorphism $\pi_1(\mathcal{W}_\Omega) \rightarrow \mathbb{Z}^{\tau'} \times \mathbb{Z}$.*

Lemma F.19 (consistency of the collar paths). (1) *At a node q on ∂S , for every arc $\gamma_q \subset C_q^{\mathcal{W}}$ from $\zeta = 0$ to $\zeta = \infty$, the map on $\partial \mathbb{B}_q$ given by the leg collar paths $\gamma_{e'}$, $\gamma_{e''}$, by $\mathbf{s}$ on the inner boundary arc, by the paths in the closures of the critical curves over the leg segments and by γ_q over e_q extends to a continuous map $\Theta_q: \mathbb{B}_q \rightarrow \mathcal{W}$.*

(2) *At a negative vertex, on each of the three sheet corners $\mathbb{B}_j$, the map on $\partial \mathbb{B}_j$ given by the leg collar paths, $\mathbf{s}$, the path in the closure of Crit over the leg segment and the vertex path extends to a continuous map $\Theta_j: \mathbb{B}_j \rightarrow \mathcal{W}$.*

(3) *At a positive vertex, after the slides, the map on the boundary of the corner $\mathbb{B}_p$ given by the leg phase paths, $\mathbf{s}$ and the real arc of Crit extends to a continuous map $\Theta_p: \mathbb{B}_p \rightarrow \mathcal{W}$.*

- (4) At a switching point e^* , the map on $\partial\mathbb{B}_{e^*}$ given by the leg collar paths $\gamma_{e'}$, $\gamma_{e''}$, by $\mathfrak{s}$ on the inner boundary arc and by Crit over $[e', e'']$ extends to a continuous map $\Theta_{e^*}: \mathbb{B}_{e^*} \rightarrow \mathcal{W}$.

Explicitly, in (1) the two families of collar paths end, in the conifold coordinates $z_1 = Z_a Z_b$, $z_2 = Z_c Z_d$, $z_3 = Z_a Z_d$, $z_4 = Z_b Z_c$ taken as characters vanishing on ν , with $(\text{sign } z_1, \text{sign } z_2) = (s_q, +)$ on the leg with lower support $\{a, b\}$ and $(+, s_q)$ on the leg with lower support $\{c, d\}$ for the $b - a$ passage, and with $(\text{sign } z_4, \text{sign } z_3) = (s_q, +)$ on $\{b, c\}$ and $(+, s_q)$ on $\{d, a\}$ for the $c - b$ passage. At side $s_q = +1$ the ends lie in one chamber, and near $C_q^{\mathcal{W}}$ the map Θ_q can be taken with values in the positive real part of the local model $\{t = t_0\} \times \widehat{\text{Con}}_q$, the union of whose S^1 -orbits is the set $\pi^{-1}(P_0)$ of [CBM14, Lemma 4.2], with boundary the exceptional curve (compare [CBM14, Lemma 5.7], which writes the local collar P_0 as the circle orbit $S^1 \cdot \check{\sigma}(S)$ of a section); γ_q is then the positive real arc of $C_q^{\mathcal{W}}$. At side $s_q = -1$ the element $e^{i\pi\lambda}$, with $\lambda = c - b$ for the $b - a$ passage and $\lambda = a - b$ for the $c - b$ passage, acts by $(-1, -1)$ on (z_1, z_2) , respectively (z_4, z_3) , and exchanges the two chambers; Θ_q can be taken with γ_q the positive real arc followed by the half-turn $e^{i\pi\phi(x)\lambda}$ along e_q , ϕ increasing from 0 at y_0 to 1 at y_∞ .

None of these choices (the cocharacters ν , the slides, the switching discs, γ_q and the extensions) affects the boundary computation: $\partial F_*[\mathfrak{D}]$ depends on them only through the maps $F|_{\partial_q \mathfrak{D}}$, whose degrees are given by Lemma F.17.

Proof. In each case the domain $(\mathbb{B}_q, \mathbb{B}_j, \mathbb{B}_p, \mathbb{B}_{e^*})$ is a disc, and the boundary data form a loop in the set $\mathcal{W}_\Omega$ of Lemma F.18 for a suitable Ω . Let the dual cell pair (τ, ω) be

$$([u, u'], P), \quad ([u, u'], \{n_1, n_2, n_3\}), \quad (\{u_1, u_2, u_3\}, [n, n']), \quad ([u, u'], [n, n'])$$

at a node, a negative vertex, a positive vertex and a switching point respectively, with $u_0 = u$, respectively u_1 . Then the ν_u may be taken to be the cocharacters ν , respectively ν_2, ν_3 , of the adapted coordinates, and at a switching point κ is the character of the coordinate w and ν_κ the fourth element of the dual basis of $(m_1, m_2, u' - u, \kappa)$.

Sizes, step 1: the sets. Let $\mathcal{X}_0 \subset \text{ri } F_\tau$ be the compact set $y_0 \cup e_q \cup y_\infty$, $\{y\}$, $\{p\}$ or $\{e^*\}$, and let $\mathcal{X}$ be the union of $\mathcal{X}_0$ with the images under $\hat{j}$ of the arcs of ∂S and of the interior edge in the boundary of the domain. Since $\text{conv } \mathcal{X}_0$ is a compact subset of $\text{ri } F_\tau$, there is $d_0 > 0$ such that on its d_0 -neighbourhood every monomial other than 0 and the vertices of τ lies at least d_0 below the maximum of the T_i . The lengths of these arcs (the corner lengths and the widths of the switching discs), at most ς_1 , and the transverse length ς_0 are chosen with $\mathcal{N}_\Gamma$, with ς_0 small compared with ς_1 and ς_1 small compared with d_0 .

Sizes, step 2: where the boundary data lie. The section $\mathfrak{s}$ over the inner boundary arc and the transverse segments, and the collar paths, lie over the $d_1(\varsigma_0 + C_0 t_{\max})$ -neighbourhood of $\mathcal{X}$ plus $\text{cone}(\omega)$, with d_1 depending only on the adapted coordinates: $\mathfrak{s}$ lies in the band of Lemma F.10, and steps (ii)–(iv) change only $|t|$ and $|z_1 z_2|$, which moves the logarithm by $O(\varsigma_0 + C_0 t_{\max})$ in the direction of ν and otherwise along $\text{cone}(\omega)$.

Sizes, step 3: the set Ω . Let $\tilde{\mathcal{X}}$ be the open $(d_1(\varsigma_0 + C_0 t_{\max}) + 4C_0)$ -neighbourhood of $\text{conv } \mathcal{X}$, an open convex set within $\varsigma_2 = \varsigma_1 + d_1(\varsigma_0 + C_0 t_{\max}) + 4C_0$ of $\text{conv } \mathcal{X}_0$. Let Ω' be the projection of $\tilde{\mathcal{X}}$ to $N_{\omega, \mathbb{R}}$ along the ν_u (and ν_κ), plus $\text{cone}(\omega)$, and I_u and I_κ the projections of $\tilde{\mathcal{X}}$ to the lines of ν_u and ν_κ . The ν_u -component of a point y is $\langle u - u_0, y \rangle$, which is constant on $\text{aff } F_\tau \supset \text{conv } \mathcal{X}_0$, and at a switching point $\mathcal{X}_0$ is a point; so Ω lies within $d_2 \varsigma_2$ of $\text{conv } \mathcal{X}_0 + \text{cone}(\omega)$, with d_2 depending only on the splitting.

Sizes, step 4: the active monomials. For ς_1 small compared with d_0 and C_0 small, only 0 and the vertices of τ are active on the $d_2\varsigma_2$ -neighbourhood of $\text{conv } \mathcal{X}_0$, and this persists on its sum with $\text{cone}(\omega)$, because for $z \in \text{cone}(\omega)$, $u \in \tau$ and $m \in \Delta^\circ$ the difference $T_u - T_m$ changes by $\langle m - u, z \rangle \geq 0$ from y to $y + z$.

Sizes, step 5: conclusion. So Ω satisfies the hypotheses of Lemma F.18, and the boundary data lie in $\mathcal{W}_\Omega$: $\mathfrak{s}$ and the collar paths by the above, and Crit and $C_q^{\mathcal{W}}$ because they lie on the strata of X_ω over $\mathcal{X}$. By Lemma F.18 the loop is null-homotopic in $\mathcal{W}_\Omega$, and the map extends over the disc, if and only if the total change of the lifted phases of the $\mathfrak{t}_u$, and at a switching point of w , around the loop is zero.

The switching points are finitely many and lie away from the vertices and the nodes (Appendix F.2), so we choose the corner lengths, the slide segments and the switching discs so that no corner arc, no slide segment and no arc of ∂S in the boundaries of $\mathbb{B}_q$, $\mathbb{B}_j$ and $\mathbb{B}_p$ contains a switching point, and so that the switching discs are disjoint from the slide segments and from these discs. In particular the collar paths are unslid over every switching disc, so they keep w fixed there, as (4) uses. Moreover the collar paths on the transverse segments bounding these discs then have the side of the corresponding sheet at the vertex or node; at a node this holds because $\mathbb{B}_q \subset \widehat{U}_q$.

(1) Here $\tau' = \{u'\}$ and $\mathfrak{t} = x^{u'-u}$. Around $\partial\mathbb{B}_q$: along $\mathfrak{s}$ the phase of $\mathfrak{t}$ is 0; each leg collar path in F_q uses the cocharacter ν of F_q and increases the lifted phase of $\mathfrak{t}$ by π from $\mathfrak{s}$ to Crit, by (F.3); and on Crit and on $C_q^{\mathcal{W}}$ we have $\mathfrak{t} = \mathfrak{t}_0 < 0$, so the phase is constantly π . Going around, the change is $0 + \pi + 0 - \pi = 0$, whatever the sides and the arc γ_q . The chamber pattern follows from (F.3): on the leg with lower support $\{a, b\}$ the adapted coordinates are $z_1^{\text{leg}} = z_3$, $z_2^{\text{leg}} = z_4$, $w = z_2$, so the collar path ends with $z_2, z_4 > 0$ and z_3 , $z_1 = z_3 z_4 / z_2 \in s_q \mathbb{R}$; on the leg with lower support $\{c, d\}$ the roles of z_1, z_2 and of z_3, z_4 are exchanged; the $c - b$ passage is the same with $w = z_3$ on $\{b, c\}$ and $w = z_4$ on $\{d, a\}$. The element $e^{i\pi(c-b)}$ has weights $\langle (1, 1, 0, 0), c - b \rangle = -1$ and $\langle (0, 0, 1, 1), c - b \rangle = 1$ on z_1, z_2 and 0 on z_3, z_4 (vectors of values on a, b, c, d), and similarly for $e^{i\pi(a-b)}$; both lie in $\Lambda_f \otimes U(1)$, which acts exactly near $C_q^{\mathcal{W}}$; both rotate $C_q^{\mathcal{W}}$ with weight -1 , since $\text{wt}(c - b) = \text{wt}(a - b) = -1$.

(2) Here too $\tau' = \{u'\}$. Around $\partial\mathbb{B}_j$ the lifted phase of $\mathfrak{t}$ changes by 0 along $\mathfrak{s}$, by $+\pi$ along the leg collar path, by 0 on Crit, where it is constantly π , and by $-\pi$ back along the vertex path, which also turns $\arg \mathfrak{t}$ by π . On the interior edge Θ_j is the vertex path for each j , so $G_j \cdot \Theta_j$ and the cone on $\mathcal{Q}$ agree there, T^2 acting exactly and $\partial\mathcal{Q} = s_1 G_1 + s_2 G_2 + s_3 G_3$.

(3) Here $\tau' = \{u_2, u_3\}$. Around $\partial\mathbb{B}_p$ the lifted phases of $(\mathfrak{t}_2, \mathfrak{t}_3)$ change by $(0, 0)$ along $\mathfrak{s}$, by $(\epsilon_{12}, \epsilon_{13})\pi$ along the slid collar path at e_2 from $\mathfrak{s}$ to Crit, by $(0, 0)$ along the real arc of Crit, on which $\mathfrak{t}_2, \mathfrak{t}_3 < 0$, and by $-(\epsilon_{12}, \epsilon_{13})\pi$ back along the slid collar path at e_3 . The total is zero.

(4) Here $\tau' = \{u'\}$, and $\pi_1(\mathcal{W}_\Omega) \cong \mathbb{Z}^2$ is detected by $(\mathfrak{t}, w)$. Around $\partial\mathbb{B}_{e^*}$ the lifted phase of $\mathfrak{t}$ changes by 0 along $\mathfrak{s}$, by $+\pi$ along $\gamma_{e'}$ from $\mathfrak{s}$ to Crit, whatever the side at e' , by 0 along Crit, where $\mathfrak{t} = \mathfrak{t}_0$, and by $-\pi$ back along $\gamma_{e''}$; the total is 0. The lifted phase of w does not change: $w > 0$ on $\mathfrak{s}$ and on Crit over $[e', e'']$, and the collar paths keep w fixed.

For the last statement: by Lemma F.14 the boundary of $[\mathfrak{D}]$ is supported on the spheres $\partial_q \mathfrak{D}$, on which F is $(x, \theta) \mapsto e^{i\theta\nu} \gamma_q(x)$, whose degree does not depend on γ_q (Lemma F.17). $\square$

Without the slides, the lifted phases around $\partial\mathbb{B}_p$ would change by $(\epsilon_{12} - \langle u_2 - u_1, \nu_{13} \rangle, \langle u_3 - u_1, \nu_{12} \rangle - \epsilon_{13})\pi$, which is in general nonzero; the slides are therefore needed, and their amounts are fixed by the cocharacters. At nodes and negative vertices no correction of the phase of $\mathfrak{t}$ is needed, because all collar paths there lie in one two-face and use one cocharacter; the

half-turn at a node on side -1 is part of an explicit choice of the arc γ_q , not a condition for the existence of Θ_q , which holds for every arc. The normalisation $\langle u, \nu \rangle = 0$ is what allows adapted coordinates with $z_1 z_2 = x^{-u}$ and $\langle m_1, \nu \rangle = \langle m_2, \nu \rangle = 0$; it is used for the explicit chamber pattern in Lemma F.19, while the existence of the maps Θ uses only that all collar paths in one two-face turn $\arg t$ by the same $+\pi$.

The pieces satisfy the following *inner boundary condition*:

- (I) on the boundary of $\mathcal{N}_\Gamma$ in S , the local pieces are S_v^1 -orbits through points $\mathfrak{s}(b)$, and on the two-cells $\mathcal{Q} \cdot \mathfrak{s}(b)$ at the interior edges leaving boundary vertices they lie in $T^2 \cdot \mathfrak{s}(b)$; in both cases the base points b lie in the set V_c of the adjacent cell c of $\mathfrak{T}$, so the images lie in $\mathscr{W}_{V_c}$.

F.6. The regular part. Over $S \setminus \text{int } \mathcal{N}_\Gamma$ the map is extended cell by cell, each cell being mapped into the region over an open set of $\widehat{B}$ that is product-shaped, contains the image of the cell, and increases with the cell.

Lemma F.20 (compatible cover). *For the choices listed in Appendix F.14, there are a triangulation $\mathfrak{T}$ of $S \setminus \text{int } \mathcal{N}_\Gamma$, adapted to the faces of ∇^h (each closed cell meets each closed face in one face of the cell), and for each cell c an open set $V_c \subset \widehat{B}$ such that:*

- (1) $\hat{j}(c) \subset V_c$, and V_c is product-shaped at the face Φ_c of lowest dimension met by $\hat{j}(c)$: if Φ_c is a two-face or an edge, in a chart of (Φ_c, n_c) for a vertex n_c whose vertex region contains $\hat{j}(c) \cap \Phi_c$, and if Φ_c is a facet, in a facet chart (Appendix F.3);
- (2) if $c' \subset c$ then $\Omega_{V_{c'}} \subset \Omega_{V_c}$, hence $\mathscr{W}_{V_{c'}} \subset \mathscr{W}_{V_c}$;
- (3) for every $r > 0$ the choices can be made so that V_c lies in the r -neighbourhood of $\hat{j}(c)$.

The proof is in Appendix F.14. Its main point is that near an edge crossing x_0 the cells must be graded (the transverse radius of V_c depends on the lowest face of c) and the ring of cells around x_0 must be a cone from x_0 ; admissibility is reduced to a strict inequality on a compact set, which holds by conicity of $\hat{S}$ and of $\widehat{B}$ at x_0 .

Lemma F.21 (extension). *Given local pieces satisfying (I), the map on $\partial \mathfrak{D}_{\text{reg}}$ extends to a map F on $\mathfrak{D}_{\text{reg}} = \mathfrak{D}|_{S \setminus \text{int } \mathcal{N}_\Gamma}$ that sends the cells over each cell c of $\mathfrak{T}$ into $\mathscr{W}_{V_c}$.*

Proof. Build F by increasing dimension of cells, mapping the cells over each cell c of $\mathfrak{T}$ into the region $\mathscr{W}_{V_c}$. By Lemma F.20(2), the cells over the faces of c are then already mapped into $\mathscr{W}_{V_c}$. On 0-cells use $\mathfrak{s}$. On 1-cells in the base direction use $\mathfrak{s}$; on circle fibres use S_v^1 -orbit arcs through $\mathfrak{s}$ where S_v^1 acts exactly, and otherwise arcs whose loops represent $v \in \Lambda_{V_c} = \pi_1(\mathscr{W}_{V_c})$ (Lemma F.12). A map on the boundary of a 2-cell extends if and only if the boundary loop is trivial in $\pi_1(\mathscr{W}_{V_c})$, which is abelian, so base points do not matter. A section $\times$ section cell is filled by $\mathfrak{s}(c)$, and its boundary is contractible by Lemma F.10. For a section $\times$ fibre cell the two fibre loops represent the same element v of Λ_{V_c} , by Lemma F.12(2) and because v is parallel. For the triangles $\mathcal{Q}$ over interior edges the boundary loop is $s_1 v_1 + s_2 v_2 + s_3 v_3 = 0$, which is the interior-edge balancing clause. At an interior vertex the four triangles are flat, so their union is the boundary of the image of the flat 3-simplex, and the classes agree across the four edges. For cells of dimension at least three the extension exists because $\mathscr{W}_{V_c}$ is aspherical. $\square$

F.7. Transport. An orbit-preserving isotopy carries the constructed chain from the localized model to a holomorphic member. Transport through the nondegenerate family then identifies this member with the chosen small resolution, preserving the exceptional curves.

Lemma F.22. *For $\varkappa$ large there is a homeomorphism Υ_{res}^0 of $X_{\mathcal{F}'}$, isotopic to the identity, smooth on the smooth locus and preserving every torus orbit, with $\Upsilon_{\text{res}}^0(\mathcal{W}) = \widehat{X}_\varkappa$ and, for every q , $\Upsilon_{\text{res}}^0(C_q^\mathcal{W}) = C_q^\varkappa$ with degree one, where $C_q^\varkappa := \widehat{X}_\varkappa \cap V(\text{cone}(a, c))$.*

Proof. Let $\partial_1, \dots, \partial_8$ be the fundamental vector fields of the complex torus $N \otimes \mathbb{C}^*$, regarded as a real Lie group: for a basis $e_1, \dots, e_4$ of N , the radial fields generating $r \mapsto e^r \cdot x$ along $e_j \otimes \mathbb{R}_{>0}$ and the angular fields generating $\theta \mapsto e^{i\theta e_j} \cdot x$. They are smooth on $X_{\mathcal{F}'}$, since the action is algebraic, tangent to every orbit, and they span the tangent space of each orbit, on which the torus acts transitively. Let $\mathcal{G}(x, s)$ be the defining function of V_s in local Cox coordinates. By Lemma F.7(3), at every point (x, s) of $\mathcal{V} = \bigcup_s V_s \times \{s\}$ the real derivatives $d\mathcal{G}(\partial_j)$ span $\mathbb{C}$; by continuity two of them, $d\mathcal{G}(\partial_i)$ and $d\mathcal{G}(\partial_k)$, are $\mathbb{R}$ -linearly independent on a neighbourhood, and there $\mathfrak{X} = \partial_s + a_i \partial_i + a_k \partial_k$, with (a_i, a_k) the smooth solution of $a_i d\mathcal{G}(\partial_i) + a_k d\mathcal{G}(\partial_k) = -\partial_s \mathcal{G}$, is tangent to $\mathcal{V}$. A partition of unity glues these local fields, since the condition is affine in the coefficients, and the coefficients are cut off outside a compact neighbourhood of $\mathcal{V}$ in the smooth locus (Lemma F.7(2)). The field $\mathfrak{X}$ is tangent to $O(\sigma) \times [0, 1]$ for every σ , so its flow preserves orbits; its ∂_s -component is 1, so the flow from $s = 1$ to $s = 0$ exists, and it carries $V_1 = \mathcal{W}$ to $V_0 = \widehat{X}_\varkappa$. (The radial fields are needed: at a positive real point in a cut-off band, $\partial_s \mathcal{G}$ is real and nonzero while the derivatives along the angular fields are purely imaginary.) On $V(\text{cone}(a, c))$ only $u_\pm$ survive, and the restricted defining function depends only on the character $z = x^{u_+ - u_-}$, the cut-offs only on $|z|$; so $V_s \cap V(\text{cone}(a, c))$ is the closure of a level set $\{z = z_s\}$ of the orbit, z_s the unique regular zero of the radial equation, and these curves form an isotopy. The flow preserves $V(\text{cone}(a, c))$, hence maps $C_q^\mathcal{W}$ onto $C_q^\varkappa$; its degree is +1 because Υ_{res}^0 is isotopic to the identity and $D_b \cdot C_q = 1$. $\square$

Lemma F.23. *Let $\mathcal{R}$ be the set of coefficient vectors of Δ° -regular hypersurfaces, a connected Zariski open set. For every X in $\mathcal{R}$ and every path in $\mathcal{R}$ from $X_\varkappa$ to X there is a homeomorphism Υ_{res}^1 of $X_{\mathcal{F}'}$, isotopic to the identity, smooth on the smooth locus and preserving every torus orbit, which carries $\widehat{X}_\varkappa$ to $\widehat{X}$ and $C_q^\varkappa$ to C_q with degree one.*

Proof. First, $X_\varkappa \in \mathcal{R}$. Let τ be a cone of the fan of $\mathbb{P}_\Delta$, the cone over a face of Δ or $\{0\}$, and σ' a cone of $\mathcal{F}'$ with relint $\sigma' \subseteq \text{relint } \tau$. Then $\sigma'^* = \tau^*$, so the restriction of the defining function of $\widehat{X}_\varkappa = V_0$ to $O(\sigma')$ is, up to a unit, the pullback of that of $X_\varkappa$ to $O(\tau)$ under the surjective homomorphism of tori $O(\sigma') \rightarrow O(\tau)$. If τ is $\{0\}$ or the cone over a face of dimension at most two, 0 is a regular value of the former by Lemma F.7(3) at $s = 0$, hence of the latter; if τ is the cone over a facet, V_0 misses $O(\sigma')$ by Lemma F.7(1), hence $X_\varkappa$ misses $O(\tau)$. So $X_\varkappa$ is Δ° -regular. For $f \in \mathcal{R}$ the proper transform $\widehat{X}_f$ is transverse to every orbit of $X_{\mathcal{F}'}$, because each orbit maps to an orbit of $\mathbb{P}_\Delta$ by a surjective homomorphism of tori, which is a submersion. The construction of Lemma F.22 along the path gives Υ_{res}^1 . It preserves $V(\text{cone}(a, c))$, hence carries $C_q^\varkappa$ to C_q . By Lemma F.7(1) and the choice of $\mathcal{F}'$, the proper transform of X in $X_{\mathcal{F}'}$ is the resolution $\widehat{X}$. $\square$

Put $\Upsilon_{\text{res}} = \Upsilon_{\text{res}}^1 \circ \Upsilon_{\text{res}}^0$. The path is one of the choices; the relative class of $\Xi(S)$ may change with it by an element of $H_3(\widehat{X})$.

F.8. Proof of Theorem E. We assemble the local maps and the extension over the regular part, with the orientations fixed above. The degree computation at each node gives the boundary coefficient, and transport yields the chain on the small resolution.

Let (S, j, v) be a tropical 2-cycle on B'_σ , with j piecewise linear, and let $\widehat{X}$ and $\mathcal{F}'$ be as in Appendix F.1, the node coefficients being computed with the diagonals of $\widehat{X}$. The choices are made in the following order, each depending only on the previous ones:

- (1) the orderings and cocharacters ν of the two-faces, the characters κ and the adapted coordinates at the legs, and the cocharacters used for the slides (these are combinatorial);
- (2) the standard position j' , the sides s_q and r_* (Lemma F.4);
- (3) the homeomorphism g and the half-discs $\mathfrak{H}_q$ (Lemma F.3, Definition F.5);
- (4) the normalising isotopy (T1)–(T3), which determines the switching points; then the neighbourhood $\mathcal{N}_\Gamma$, with the switching discs, the corner lengths and the transverse length ς_0 as in the proof of Lemma F.19; and the generic isotopy (Appendix F.2);
- (5) the charts, the triangulation $\mathfrak{T}$, the radii and η (Appendix F.14);
- (6) C_0 small, the cut-off profile β and the constant C_1 , and then $\varkappa$ large, for Lemmas F.7, F.10, F.12, F.15 and F.22;
- (7) the leg collar paths and the slides, the arcs γ_q and the extensions Θ of Lemma F.19 (Appendix F.5);
- (8) the extension F (Lemma F.21);
- (9) the path in $\mathcal{R}$ (Lemma F.23).

The local pieces and Lemma F.21 define a continuous map $F: \mathfrak{D} \rightarrow \mathcal{W}$. Within each local domain the pieces are continuous: the leg collar paths depend continuously on the point of the leg between switching points (Appendix F.5), each switching point lies in a switching disc, and by Lemma F.19 the maps Θ extend the collar paths; the circle actions used (S_v^1 , and T^2 at the negative vertices) act exactly on the relevant parts of $\mathcal{W}$ and fix Crit where the domain collapses the circle. Different local domains meet along transverse segments at points of the legs, where both are given by the same (slid) leg collar paths, and the local domains meet the regular part along the inner boundary (I), which consists of S_v^1 -orbits through points of $\mathfrak{s}$ and of the sets $\mathcal{Q} \cdot \mathfrak{s}(b)$, where Lemma F.21 extends them. By Lemma F.14,

$$\partial F_*[\mathfrak{D}] = \sum_q F_*[\partial_q \mathfrak{D}],$$

and by Lemma F.17, $F_*[\partial_q \mathfrak{D}]$ is a 2-cycle supported on $C_q^{\mathcal{W}}$ of degree $-\rho_q(S)$. A 2-cycle on $C_q^{\mathcal{W}} \cong S^2$ of degree d differs from d times a fundamental cycle by the boundary of a 3-chain in $C_q^{\mathcal{W}}$; adding these chains, we obtain a 3-chain $\Xi_{\mathcal{W}}$ on $\mathcal{W}$, supported in $F(\mathfrak{D}) \cup \bigcup_q C_q^{\mathcal{W}}$, with

$$\Xi_{\mathcal{W}} \equiv -F_*[\mathfrak{D}] \pmod{\text{chains in } \bigcup_q C_q^{\mathcal{W}}}, \quad \partial \Xi_{\mathcal{W}} = \sum_q \rho_q(S) [C_q^{\mathcal{W}}].$$

Put $f_S = \Upsilon_{\text{res}} \circ F$ and $\Xi(S) = (\Upsilon_{\text{res}})_* \Xi_{\mathcal{W}}$. Lemmas F.22 and F.23 preserve the exceptional curves with degree one, so $\deg(f_S|_{\partial_q \mathfrak{D}}) = -\rho_q(S)$. If the transported fundamental cycles differ from the chosen $[C_q]$, correct inside the C_q once more. The resulting chain has the form $\Xi(S) = -(f_S)_*[\mathfrak{D}] + A$, with A supported in the exceptional curves, and

$$\partial \Xi(S) = \sum_q \rho_q(S) [C_q].$$

The chain is constructed from (S, j, v) and the listed choices, and no relation among the $[C_q]$ is used. This proves the domain, map and boundary assertions of Theorem E. In particular $\sum_q \rho_q(S) [C_q] = 0$ in $H_2(\widehat{X}; \mathbb{Z})$, hence in $H_2(\widehat{X}; \mathbb{Q})$, and $\rho(S) \in K$ by Proposition 3.4.

The node coefficients in these assertions are those of the diagonals of $\widehat{X}$. With the diagonals fixed in Section 2.2, which need not be those of $\widehat{X}$, the identity holds after reversing the sign of ρ_q at every node where the two differ, since a change of diagonal reverses the sign of ρ_q and not that of $[C_q]$. The assertion of Theorem E, for an arbitrary small resolution, including the common-exterior relative class, follows from Lemma F.24, applied to $\lambda = \rho(S)$ and the chain $\Xi(S)$.

Lemma F.24 (flops). *Let $\widehat{X}$ and $\widehat{X}'$ be small resolutions of X'_σ whose diagonals differ exactly at the nodes of a set $Q' \subseteq Q$, with exceptional curves C_q and C'_q , and let $\lambda \in \mathbb{Z}^Q$. Put $\lambda'_q = -\lambda_q$ for $q \in Q'$ and $\lambda'_q = \lambda_q$ otherwise. Then $\sum_q \lambda_q [C_q] = 0$ in $H_2(\widehat{X}; \mathbb{Z})$ if and only if $\sum_q \lambda'_q [C'_q] = 0$ in $H_2(\widehat{X}'; \mathbb{Z})$. More precisely, the image of a 3-chain Ξ on $\widehat{X}$ with $\partial \Xi = \sum_q \lambda_q [C_q]$ in $H_3(\widehat{X}, \bigcup_q \widehat{V}_q) \cong H_3(X^\circ, \bigcup_q \partial V_q)$ is also the class of a 3-chain Ξ' on $\widehat{X}'$ with $\partial \Xi' = \sum_q \lambda'_q [C'_q]$, where $\widehat{V}_q$, ∂V_q and X° are as in the proof.*

Proof. Let $V_q \subset X'_\sigma$ be small disjoint closed conical neighbourhoods of the nodes, with boundaries $\partial V_q \cong S^2 \times S^3$, and let $\widehat{V}_q \subset \widehat{X}$ and $\widehat{V}'_q \subset \widehat{X}'$ be their preimages; these are tubular neighbourhoods of C_q and C'_q , with boundary ∂V_q . Both resolutions are isomorphisms over $X^\circ := X'_\sigma \setminus \bigcup_q \text{int } V_q$. By excision, $H_3(\widehat{X}, \bigcup_q \widehat{V}_q) \cong H_3(X^\circ, \bigcup_q \partial V_q) \cong H_3(\widehat{X}', \bigcup_q \widehat{V}'_q)$, and the connecting maps of the two pairs factor as

$$H_3(X^\circ, \bigcup_q \partial V_q) \xrightarrow{\partial} \bigoplus_q H_2(\partial V_q) \longrightarrow \bigoplus_q H_2(\widehat{V}_q), \text{ respectively } \bigoplus_q H_2(\widehat{V}'_q),$$

where $H_2(\partial V_q) \cong \mathbb{Z}$ maps isomorphically onto $H_2(\widehat{V}_q) = \mathbb{Z}[C_q]$ and onto $H_2(\widehat{V}'_q) = \mathbb{Z}[C'_q]$. Let $\ell_q \in H_2(\partial V_q)$ map to $[C_q]$. The class D_b of Section 3.2, b a vertex of the node parallelogram off the diagonal $\{a, c\}$ of $\widehat{X}$, is defined on X° and restricts to the same class on ∂V_q for both resolutions, and $\langle D_b, \ell_q \rangle = D_b \cdot C_q = 1$. For $q \in Q'$ the curve C'_q is the curve of the wall cone (b, d) , and the relation $a + c - b - d = 0$ gives $D_b \cdot C'_q = -1$; so ℓ_q maps to $-[C'_q]$. For $q \notin Q'$, $\widehat{V}'_q = \widehat{V}_q$ and ℓ_q maps to $[C'_q] = [C_q]$. By the exact sequences of the pairs, $\sum_q \lambda_q [C_q] = 0$ in $H_2(\widehat{X})$ if and only if $\sum_q \lambda_q \ell_q$ lies in the image of ∂ , if and only if $\sum_q \lambda'_q [C'_q] = 0$ in $H_2(\widehat{X}')$, the preimage under ∂ being the same class; this also gives the statement about chains. $\square$

A small resolution that is not projective is handled in the same way: by Proposition 3.4(3) and Lemma F.1 there is a projective small resolution induced by a regular refinement of Σ' , the chain is constructed there, and Lemma F.24 carries it to the given one. This persistence under a change of resolution agrees with [CBM14, Rem. 9.4] for relations induced by tropical 2-cycles. Lemma F.24 records the sign change at each flopped curve when every exceptional curve is given its complex orientation.

The last assertion of Theorem E. Label the node parallelograms by the diagonals of $\widehat{X}$, and orient the vanishing spheres δ_q by Definition 7.23 with this labelling. Triangulated, S is a relative tropical 2-cycle z_S with node coefficients $\rho(S)$ (Lemma 5.4), so for the members Y_t of Theorem A Theorem D gives a 4-chain $\Xi(S)$ with $\partial \Xi(S) = \sum_q \rho_q(S) [\delta_q]$. Together with the first assertion,

$$\sum_q \rho_q(S) [C_q] = 0 \text{ in } H_2(\widehat{X}; \mathbb{Z}), \quad \sum_q \rho_q(S) [\delta_q] = 0 \text{ in } H_3(Y_t; \mathbb{Z}),$$

with the same coefficients on both sides. This completes the proof of Theorem E. $\square$

Remark F.25. (1) The part of $\Xi(S)$ over the regular part, $\Upsilon_{\text{res}} \circ F(\mathfrak{D}_{\text{reg}})$, lies in Υ_{res} of the union of the regions $\mathscr{W}_{V_c}$, $c \in \mathfrak{T}$, that is, of the part of $\mathscr{W}$ over the union of the sets Ω_{V_c} together with its limit points on the divisors at the ends of the divisorial rays (Lemma F.21); by Lemma F.20(3) the sets V_c can be taken inside an arbitrarily small neighbourhood of $\hat{S}$ in $\hat{B}$. The rest of $\Xi(S)$ consists of the local pieces over $\mathcal{N}_\Gamma$, whose collar paths along the legs run into the toric strata, and of chains in the exceptional curves.

(2) The relative class of $\Xi(S)$ is not claimed to be independent of the choices; the extensions Θ (for instance the sign of a half-turn), the switching discs, the slides and the path in $\mathcal{R}$ may change it by closed 3-cycles.

(3) The identity $\sum_q \rho_q(S) \text{circ}_q = 0$ is proved combinatorially in Proposition 5.7, from a collar of ∂S alone. Theorem E realises it by a chain built from all of S .

Remark F.26 (root classes at the del Pezzo points). Let G be a pentagon or a hexagon of Δ , with the diagonals through its centre, and let E be the exceptional surface of $\hat{X}$ over the singular point of X on the orbit of $\text{cone}(G)$ (Proposition 3.9). Since $\rho(S) \in K$, the part $\sum_{q \subset G} \rho_q(S)[C_q]$ of $\partial \Xi(S)$ in E is a class in the root lattice of Remark 3.10: a multiple of $e_1 - e_2$ at a pentagon, of ℓ at a hexagon on the A_1 branch, and an element of the A_2 root lattice at a hexagon on the A_2 branch. So $\Xi(S)$ defines a relative class on the complement of small neighbourhoods of the singular points of X , whose boundary at a del Pezzo point lies in that root lattice, and at an ordinary node of X is ρ_q times the generator of $H_2(S^2 \times S^3)$.

F.9. Relative tropical 2-cycles. Theorem E is stated for tropical 2-cycles in the sense of [CBM14]. For the relative tropical 2-cycles of Definition 2.4 the relation itself needs no chain: $\rho(z) \in K$ for every relative cycle z on every triangulation in which Γ is full (Theorem C), so $\sum_q \rho_q(z)[C_q] = 0$ in $H_2(\hat{X}; \mathbb{Z})$ by Proposition 3.5.

Remark F.27. The construction of this section uses the strictness of S mainly through the shape of the domain: one primitive field on each sheet, trivalent interior edges and four-valent interior vertices, and a boundary that passes each node at most once, along one pair of opposite legs. Its ingredients are nevertheless local: the pieces along the legs, at the negative and positive vertices, at the crossings and at the nodes, and the extension over aspherical regions. At a node passed with arbitrary leg coefficients c_{ab} , c_{bc} , balancing at y_0 suggests the degree $c_{bc} - c_{ab}$ on the exceptional curve, which is the formula of Lemma F.17 for a single passage; we have not proved it in general. The crossing pieces $S_v^1 \cdot \mathfrak{s}(D_k)$ are solid tori for every monodromy-invariant field. The mirror side removes the corresponding restrictions by building the regular part over an abstract domain and by removing fibre-class boundary with the positive real locus (Section 8, Proposition 8.26); on $\hat{X}$ we have no analogue of the second step, and we do not extend Theorem E to relative tropical 2-cycles here. Nothing in this paper depends on such an extension.

F.10. Regularity of the family: proof of Lemma F.7(3). *Active sets.* For C_0 small, at every point the active monomials are vertices of one simplex of the triangulation of Δ° induced by the heights, the cone over $\mathcal{T}$ from 0, or, on an orbit $O(\sigma)$, of its restriction to the face σ^* . Indeed, the sets of monomials attaining $\max_i T_i$ at some point are exactly the cells of that triangulation; if a set S of monomials is not contained in a cell, then $\max_i T_i - \min_{m \in S} T_m$ is a convex piecewise linear function on $N_{\mathbb{R}}$ without zeros, hence bounded below by a positive constant, and we take C_0 smaller than these finitely many constants. Since $\mathcal{T}$ is unimodular (Hypothesis (U)), each such simplex is unimodular, so for any vertex m_0 of it the differences $m - m_0$ with its other vertices m are part of a basis of M .

The computation. Fix $s \in [0, 1]$, a cone σ with $O(\sigma)$ met by V_s , and $x \in V_s \cap O(\sigma)$. In Cox coordinates the defining function restricted to $O(\sigma)$ is $\sum_{m \in \sigma^*} \beta_m^{(s)} (-\varkappa)^{-\tilde{h}_m} \prod_{\varrho \notin \sigma(1)} Z_\varrho^{\langle m, n_\varrho \rangle + 1}$ with $\beta_m^{(s)} = (1-s) + s\beta_m$, and on $O(\sigma)$ the factors $\beta(T_i - T_m)$ of β_m with $i \notin \sigma^*$ equal 1, by the argument of Lemma F.7(1). Let $A \subseteq \sigma^*$ be the active set at x , and let $m_1 \in \sigma^*$ maximise $T_m(x)$; then $T_i - T_{m_1} < C_1$ near x for all i , so $\beta_{m_1} = 1$ and $\beta_{m_1}^{(s)} = 1$ near x . A monomial $m \notin A$ has $\beta_m = 0$ near x , hence $T_m \leq T_{m_1} - C_0$ there. If $A = \{m_1\}$, the defining function is the m_1 -term times $1 + O(\varkappa^{-C_0})$, which does not vanish; so there is $m_0 \in A \setminus \{m_1\}$. By the first paragraph the differences $m - m_0$, $m \in A \setminus \{m_0\}$, are part of a basis of M ; they lie in the saturated sublattice $\sigma^\perp$, hence are part of a basis of it, and there is a cocharacter ν of the torus of $O(\sigma)$ with $\langle m_1 - m_0, \nu \rangle = 1$ and $\langle m - m_0, \nu \rangle = 0$ for $m \in A \setminus \{m_0, m_1\}$. Let ∂_{rad} and ∂_{ang} be its radial and angular fundamental vector fields, so that $\partial_{\text{rad}} x^m = \langle m, \nu \rangle x^m$, $\partial_{\text{ang}} x^m = i \langle m, \nu \rangle x^m$, $\partial_{\text{rad}} T_m = \langle m, \nu \rangle / \log \varkappa$ and $\partial_{\text{ang}} T_m = 0$.

Put $\tilde{x}_m = (-\varkappa)^{\tilde{h}_{m_0} - \tilde{h}_m} x^{m - m_0}$, a function on $O(\sigma)$ with $|\tilde{x}_m| = \varkappa^{T_m - T_{m_0}}$, and divide the defining function by the m_0 -term:

$$G = \sum_{m \in \sigma^*} \beta_m^{(s)} \tilde{x}_m = \beta_{m_0}^{(s)} + \tilde{x}_{m_1} + \sum_{m \in A \setminus \{m_0, m_1\}} \beta_m^{(s)} \tilde{x}_m + E, \quad E = (1-s) \sum_{m \in \sigma^* \setminus A} \tilde{x}_m.$$

Let $r = |\tilde{x}_{m_1}| \geq 1$. Then $|\tilde{x}_m| \leq r$ for all m , and $|\tilde{x}_m| \leq \varkappa^{-C_0} r$ for $m \notin A$. The cut-off factors depend only on moduli, so $\partial_{\text{ang}} \beta_m^{(s)} = 0$; each β_m is a product of $|\Delta^\circ \cap M|$ factors $\beta(T_i - T_m)$, and $\partial_{\text{rad}}(T_i - T_m) = \langle i - m, \nu \rangle / \log \varkappa$, so $|\partial_{\text{rad}} \beta_m^{(s)}| \leq C_\beta / \log \varkappa$ with C_β depending only on β , Δ° and ν ; for $m \notin A$, $\beta_m^{(s)} = 1-s$ is constant near x . The functions $\tilde{x}_m$, $m \in A \setminus \{m_1\}$, are annihilated by ∂_{rad} and ∂_{ang} . Hence

$$\partial_{\text{ang}} G = i \tilde{x}_{m_1} + O(\varkappa^{-C_0} r), \quad \partial_{\text{rad}} G = \tilde{x}_{m_1} + O(r / \log \varkappa) + O(\varkappa^{-C_0} r),$$

with constants depending only on β , Δ° and the finitely many cocharacters ν that occur. The real 2×2 matrix $\mathbf{J}$ with columns (Re, Im) of $\partial_{\text{rad}} G$ and $\partial_{\text{ang}} G$ has

$$\det \mathbf{J} = \text{Im}(\overline{\partial_{\text{rad}} G} \partial_{\text{ang}} G) = r^2 (1 + O(1 / \log \varkappa)),$$

which is positive for $\varkappa$ large, uniformly in s and x . Since G is the defining function times a nonvanishing function, at x the derivatives of the defining function along ∂_{rad} and ∂_{ang} span $\mathbb{C}$. This is (3); it also shows that the differential of the defining function is nonzero at x and onto $\mathbb{C}$ on $T_x O(\sigma)$, so V_s is smooth at x and transverse to $O(\sigma)$. In the notation of [Yam16, proof of Thm. 3.9], with $\tilde{x}_1 = \tilde{x}_{m_1} = r_1 e^{i\theta_1}$ and $\partial_{r_1} = r_1^{-1} \partial_{\text{rad}}$, $\partial_{\theta_1} = \partial_{\text{ang}}$, this is $\det \mathbf{J} = |c_1|^2 r_1 (1 + O(1 / \log \varkappa))$ with leading coefficient $c_1 = 1$. $\square$

F.11. The regions: proof of Lemma F.12. Fix a face Φ with set of maximizing exponents $A = A_\Phi$, a fan point n and a chart $(\pi_\parallel, \pi_\perp)$ as in Appendix F.3.

Product-shaped sets lie in $\widehat{B}$. Let $U_\parallel$ be convex and $U_\perp \subset \partial \mathcal{O}_\perp$ connected, and put $\tilde{U} = \{y : \pi_\parallel(y) \in U_\parallel, \pi_\perp(y) \in U_\perp\}$, a connected set. $\tilde{U} \cap \nabla^{\tilde{h}}$ is closed in $\tilde{U}$. It is also open: at a point of $\tilde{U} \cap \nabla^{\tilde{h}}$ where a functional $\ell_m = \langle m, \cdot \rangle + \tilde{h}_m$ with $m \notin A$ vanishes, the point lies in $\widehat{B}$, since some functional of A vanishes there, hence in U ; this is excluded by the condition $\ell_m > |m| \ell_* > 0$ on U , so near such a point $\nabla^{\tilde{h}}$ is cut out by the functionals of A alone. Since it is nonempty, $\tilde{U} \subset \nabla^{\tilde{h}}$, and $\tilde{U} \subset \widehat{B}$ because some functional of A vanishes wherever $\pi_\perp \in \partial \mathcal{O}_\perp$. So a product-shaped set is exactly its model set $\tilde{U}$.

The active set. Let $y \in \Omega_U$ and $m \in \tilde{A} \setminus A$; write $p(y) = \pi_\perp(y)$, so that $T_a = -\ell_a = -p_a$ for $a \in A \setminus 0$, $T_0 = 0$, and $\max_i T_i(y) \geq \max_{a \in A} T_a(y) = -p(y)_{\min}$. Choose y' with $\pi_\parallel(y') \in U_\parallel$,

$p(y') \in S_{d,r}$ and $|y - y'|_\Phi < \eta$. Then $z := \varpi(y')$ lies in U , and $y' = z - p(y')_{\min} n$ lies on the divisorial ray from z (in type (a), $p(y')_{\min} = 0$ and $y' = z$). Along the divisorial ray, ℓ_a decreases at rate 1 for $a \in A \setminus 0$ and ℓ_m at rate at most 1, because $\langle a, n \rangle = -1$ and $\langle m, n \rangle \geq -1$ for $m \in \Delta^\circ$ and the lattice point n of Δ ; hence $\ell_m - p_{\min}$ does not decrease along it, and $\ell_m(y') - p(y')_{\min} \geq \ell_m(z) > |m|\ell_*$. Passing from y' to y changes ℓ_m by at most $|m||y - y'| \leq |m|C_{\text{ch}}\eta$ and $p_{\min}$ by at most $|p(y) - p(y')|_\infty < \eta$, so, as $|m| \geq 1$,

$$T_{\max}(y) - T_m(y) \geq \ell_m(y) - p(y)_{\min} > |m|(\ell_* - (C_{\text{ch}} + 1)\eta) = |m|C_{\text{act}}C_0 \geq C_{\text{act}}C_0,$$

where $T_{\max} = \max_i T_i$. A monomial active at a point x with $\text{Log } x = y$ is within $C_{\text{act}}C_0$ of the maximum of the T_i at x , so m is not active there. Only the chart formula of ϖ enters. Hence on $\text{Log}^{-1}(\Omega_U)$ only monomials of A are active, and by Lemma F.7(3) the equation of $\mathscr{W}$ there depends only on the characters x^a , $a \in A$: if π_A denotes the homomorphism from $N \otimes \mathbb{C}^*$ onto the torus T_A with characters x^a , $a \in A \setminus 0$, then $\mathscr{W} \cap \text{Log}^{-1}(\Omega_U) = \pi_A^{-1}(L) \cap \text{Log}^{-1}(\Omega_U)$, where $L \subset T_A$ is the localized hyperplane in the variables x^a .

The product. Write $N = A^\perp \oplus \mathbb{Z}n \oplus C$ and $N \otimes \mathbb{C}^* = T_\parallel \times T_\perp$ accordingly, $T_\parallel = A^\perp \otimes \mathbb{C}^*$, $T_\perp = (\mathbb{Z}n \oplus C) \otimes \mathbb{C}^*$. The pairing identifies $\mathbb{Z}n \oplus C$ with the dual of $\mathbb{Z}^{A \setminus 0}$, so $\pi_A|_{T_\perp}$ is an isomorphism onto T_A ; it extends over the partial compactification along n , which is the part of the toric chart of the ray n lying over $T_\parallel$. The logarithm splits accordingly: $\pi_\parallel(\text{Log } x)$ depends only on the $T_\parallel$ -component k , and $\pi_\perp(\text{Log } x)$ only on the $T_\perp$ -component, hence only on $\ell = \pi_A(x)$. Therefore

$$\mathscr{W}_U = \{(k, \ell) \in T_\parallel \times \bar{L} : -\log |k| / \log \varkappa \in U_\parallel^\eta, \pi_\perp(\ell) \in \Omega_\perp\} = (A^\perp \otimes U(1)) \times U_\parallel^\eta \times L_U,$$

with $L_U = \{\ell \in \bar{L} : \pi_\perp(\ell) \in \Omega_\perp\}$, $\bar{L}$ including the limit points on D_n .

The capped localized hyperplane and its truncation. Write $z_0 = 1$ and $z_i = (-\varkappa)^{-\tilde{h}_{a_i}} x^{a_i}$ for $A \setminus 0 = \{a_1, \dots, a_d\}$. Thus $p_i = -\log_\varkappa |z_i/z_0|$. The partial compactification at D_n identifies the transverse ambient space with

$$Z_d = \{[z_0 : \dots : z_d] \in \mathbb{P}^d : z_i \neq 0 \text{ for } 1 \leq i \leq d\},$$

and the divisor D_n is $z_0 = 0$. Let $Y_d \subset Z_d$ be the capped localized hyperplane $\sum_{j=0}^d \beta_j z_j = 0$, where $\beta_j = \prod_k \beta(\log_\varkappa |z_k/z_j|)$, with its usual extension over $z_0 = 0$.

For completeness, Y_d has the homotopy type of $(\mathbb{C}^*)^{d-1}$. Apply the regularity calculation of Appendix F.10 to the standard simplex and the family $\sum_j ((1-s) + s\beta_j)z_j = 0$ in $\mathbb{P}^d$. At a zero on each toric orbit, two real torus vector fields have derivatives spanning $\mathbb{C}$. Solving the isotopy equation with these fields and patching by a partition of unity gives a vector field tangent to every toric orbit; compactness of $\mathbb{P}^d$ gives its flow for $0 \leq s \leq 1$. Thus the isotopy preserves every coordinate divisor and restricts to Z_d . The ordinary hyperplane in Z_d projects isomorphically to $\{[z_1 : \dots : z_d] : z_i \neq 0\} \cong (\mathbb{C}^*)^{d-1}$. This also identifies its fundamental group with that of Z_d via inclusion.

We give a deformation retraction which preserves the logarithmic condition defining L_U . For $d = 2, 3$, choose $C_0 < a < \eta$, and set

$$M(z) = \max_{0 \leq j \leq d} |z_j|, \quad \delta_i(z) = \log_\varkappa \frac{M(z)}{|z_i|} \quad (1 \leq i \leq d).$$

The ratios defining δ_i are well defined on Z_d . Define

$$H_t[z_0 : z_1 : \dots : z_d] = [z_0 : z_1 \varkappa^{t(\delta_1 - a)_+} : \dots : z_d \varkappa^{t(\delta_d - a)_+}].$$

The maximum M is unchanged. Every changed monomial has gap at least $a > C_0$ throughout this homotopy, so its cutoff coefficient vanishes; its contribution to each active coefficient is

a factor equal to 1. The other monomials and their coefficients do not change. Consequently H_t preserves Y_d exactly. It is continuous at $z_0 = 0$ and is a strong deformation retraction onto $K_a = \{z \in Y_d : \delta_i \leq a \text{ for all } i\}$. We call H_t the *clamping homotopy*, since at $t = 1$ it raises the modulus of every coordinate z_i lying more than a below the maximum, on the logarithmic scale, to exactly a below it, and K_a its *clamped core*.

On the dense torus the logarithmic map at $t = 1$ is

$$f_a(p)_i = \min\{p_i, p_{\min} + a\}, \quad H_t^{\log}(p) = (1-t)p + tf_a(p).$$

The map f_a is 1-Lipschitz in the sup-norm: both p_i and $p_{\min} + a$ are 1-Lipschitz, and their minimum is 1-Lipschitz. It preserves $S_{d,r}$, since it leaves the minimum and all coordinates attaining it unchanged and only decreases $p_i - p_{\min}$. The same holds for $H_t^{\log}$, which is also 1-Lipschitz. Hence $H_t^{\log}$ preserves $S_{d,r}^{\eta}$. For a point of K_a in the dense torus, the point $q = p_{\min}(1, \dots, 1)$ belongs to $S_{d,r}$ and satisfies $|p - q|_{\infty} \leq a < \eta$. The capped points follow by continuity and the local product structure at $z_0 = 0$. Thus $K_a \subset L_U$, and both Y_d and L_U strongly deformation retract onto K_a . This proves $L_U \simeq (\mathbb{C}^*)^{d-1}$. For $d = 1$ the localized two-term equation has its unique root at $p_1 = 0$, so L_U is a point.

π_1 and naturality. $\mathcal{W}_U$ is a product of a compact torus, a convex set and L_U , so it is aspherical with $\pi_1 = A^{\perp} \oplus \pi_1(L_U)$. In type (a) this is $u^{\perp} \cap N$. In type (b), $\pi_1(L_U) = \mathbb{Z}\lambda$ with λ the loop around the first leg, whose class generates C modulo n ; so $\pi_1 = A^{\perp} \oplus C \cong N/\mathbb{Z}n$. In type (c), $\pi_1(L_U)$ is generated by the loops around the three rays $+e_k$ modulo their relation, $\cong C$, and again $\pi_1 \cong N/\mathbb{Z}n$. Invariantly, in types (b), (c) the inclusion of $\mathcal{W}_U$ into the toric chart of the ray n , whose fundamental group is $N/\mathbb{Z}n$, is an isomorphism on π_1 ; in type (a) the inclusion into the torus induces $\pi_1(\mathcal{W}_U) = u^{\perp} \cap N \subset N$. For $\mathcal{W}_{U'} \subset \mathcal{W}_U$ both maps factor through these inclusions, and the induced map is the composite of the identifications, which is Gross's parallel transport: $u^{\perp} \cap N \rightarrow N/\mathbb{Z}n$ from type (a) to types (b), (c), and the identity of $N/\mathbb{Z}n$ between types (b) and (c). This proves (1) and (2).

Sheared charts. Let $\pi_{\parallel}^{\lambda} = \pi_{\parallel} + \lambda \circ (\pi_{\perp} - \pi_{\perp}(0))$ with $\lambda: \mathbb{R}^{A \setminus 0} \rightarrow (A^{\perp})_{\mathbb{R}}$ linear and $\lambda(\pi_{\perp}(n) - \pi_{\perp}(0)) = 0$. Then $\pi_{\parallel}^{\lambda}(n) = 0$, so divisorial rays are still products in the chart. In the splitting above the set defined with $\pi_{\parallel}^{\lambda}$ is $\{(k, \ell) : -\log|k|/\log \varkappa + \lambda(\tau(\ell)) \in U_{\parallel}^{\eta}, \pi_{\perp}(\ell) \in \Omega_{\perp}\}$, $\tau(\ell) = \pi_{\perp}(\ell) - \pi_{\perp}(0)$. The map $\text{Sh}_{\lambda}(k, \ell) = (k \cdot \exp(\log \varkappa \lambda(\tau(\ell))), \ell)$ is a homeomorphism of $T_{\parallel} \times \bar{L}$, with inverse the same map for $-\lambda$, carrying the product set above onto this set. It extends continuously to the cap, since approaching D_n the vector $\tau(\ell)$ diverges only in the direction $\pi_{\perp}(n) - \pi_{\perp}(0) = (-1, \dots, -1)$, which λ kills. The maps $\text{Sh}_{t\lambda}$ form a homotopy of maps into the toric chart of n , so (1) and (2) hold for the sheared chart. This proves (3). $\square$

F.12. The positive section: proof of Lemma F.10. On the positive real torus, $\tilde{f} = \varkappa^{T_0}(\beta_0 - \phi)$ with $\phi = \sum_{m \neq 0} \beta_m \varkappa^{\tau_m}$ and $\tau_m = T_m - T_0$, because all terms other than the constant one are negative there ($\tilde{h}_m$ odd). Fix a unit vector u and restrict to the ray $y = tu$, $t > 0$. Then $\tau_m(tu) = -\tilde{h}_m - t\langle m, u \rangle$ is affine in t . Put $\bar{\tau}(t) = \max_{m \neq 0} \tau_m(tu)$, a convex function with $\bar{\tau}(0) \leq -1$, tending to $+\infty$ because 0 is an interior point of Δ° .

(i) *Signs away from the band* $|\bar{\tau}| < C_0$. If $\bar{\tau} \leq -C_0$, every $m \neq 0$ has $\beta(T_0 - T_m) = \beta(-\tau_m) = 0$, so $\beta_m = 0$, and every factor of $\beta_0 = \prod_i \beta(\tau_i)$ equals 1; so $\beta_0 - \phi = 1$. If $\bar{\tau} \geq C_0$, the factor $\beta(\tau_{m^*})$ of β_0 vanishes for the maximal m^* , while $\beta_{m^*} = 1$; so $\beta_0 - \phi \leq -\varkappa^{\bar{\tau}} < 0$.

(ii) *The band is an interval.* By convexity, for $t \geq t_-$, where $\bar{\tau}(t_-) = -C_0$, the slope of $\bar{\tau}$ is at least $(1 - C_0)/t_{\max}$. So the band meets the ray in an interval of length at most $2C_0 t_{\max}/(1 - C_0)$.

(iii) *Monotonicity in the band.* If $\beta_m \neq 0$ then $\tau_m \geq \bar{\tau} - C_0 \geq -2C_0$, and then $\tau'_m = -\langle m, u \rangle = (\tau_m + \check{h}_m)/t \geq c := (1 - 2C_0)/t_{\max} > 0$. Each factor $\beta(\tau_i)$ of β_0 is non-constant only where $\tau_i \in (C_1, C_0)$, where $\tau'_i > 0$; so β_0 is non-increasing. Write $\beta_m = \beta(-\tau_m) \prod_{i \neq 0} \beta(\tau_i - \tau_m)$. The first factor is non-decreasing along the ray, and the derivative of the product is bounded by a constant C'_β independent of $\varkappa$; hence $\beta'_m \geq -C'_\beta \beta(-\tau_m)$. The maximal term has $\beta_{m^*} = \beta(-\bar{\tau})$, and every active m has $\tau_m \leq \bar{\tau}$, so $\beta(-\tau_m) \leq \beta(-\bar{\tau})$ and $\varkappa^{\tau_m} \leq \varkappa^{\bar{\tau}}$. Therefore

$$\phi' = \sum_m \varkappa^{\tau_m} (\beta'_m + \beta_m \tau'_m \log \varkappa) \geq \beta(-\bar{\tau}) \varkappa^{\bar{\tau}} (c \log \varkappa - C'_\beta |\check{\mathcal{A}}|) > 0$$

on the band for $\varkappa$ large, since $\beta(-\bar{\tau}) > 0$ there. So $(\beta_0 - \phi)' < 0$ on the band.

Hence on each ray $\tilde{f}$ has exactly one zero on the positive reals, and it is transversal. By the implicit function theorem the zero depends continuously (indeed smoothly) on u , $\mathscr{W}_{>0}$ is a smooth hypersurface of $(\mathbb{R}_{>0})^4$, and $\text{Log}(\mathscr{W}_{>0})$ is a radial graph over the sphere of directions lying in the band.

(iv) *Smoothness of $\mathscr{W}$.* At a point $x \in \mathscr{W}_{>0}$ the derivative of $\tilde{f}$ along the radial direction of the ray is the nonzero real number of (iii). Along the angular one-parameter group $\theta \mapsto e^{i\theta u} x$ the cut-off functions, which depend only on moduli, are constant, and

$$\frac{d}{d\theta} \Big|_{\theta=0} \tilde{f}(e^{i\theta u} x) = \sum_{m \neq 0} \beta_m i \langle m, u \rangle (-\varkappa)^{-\check{h}_m} x^m = i \sum_{m \neq 0} \beta_m \varkappa^{T_m} (-\langle m, u \rangle),$$

since the terms with $m \neq 0$ are negative at x . By (iii) every m with $\beta_m \neq 0$ has $-\langle m, u \rangle \geq c > 0$, and the maximal m^* has $\beta_{m^*} = \beta(-\bar{\tau}) > 0$ on the band; so this derivative is i times a positive number. The two derivatives span $\mathbb{C}$ over $\mathbb{R}$, so $d\tilde{f}$ is onto at x and $\mathscr{W}$ is smooth along $\mathscr{W}_{>0}$.

Since $\nabla^{\check{h}}$ is star-shaped from 0 and $\bar{\tau}$ vanishes on each ray exactly at its point of $\partial \nabla^{\check{h}}$, radial projection identifies $\text{Log}(\mathscr{W}_{>0})$ with $\hat{B}$. $\square$

F.13. Local fundamental groups: proof of Lemma F.18. Let $d = |\tau'| = \dim \tau$. We first treat the case $\dim \tau + \dim \omega = 3$, in which $\dim \omega = 3 - d$ and N_ω has rank $4 - d$, and then the case $\dim \tau = \dim \omega = 1$.

Coordinates. The characters $u - u_0$, $u \in \tau'$, are part of a basis of M , since $\mathcal{T}$ is unimodular, and they vanish on N_ω , since every vertex of τ pairs to -1 with every lattice point of ω . As $N_\omega^\perp$ is saturated, they are part of a basis of it, and a basis if its rank $3 - \dim \omega$ equals d . Pairing with a basis of $N_\omega^\perp$ maps N onto the dual lattice, with kernel N_ω (both lattices being saturated); elements ν_u (and ν_κ) as in the lemma are obtained from preimages of the dual basis by adding multiples of a lattice point n of ω , which has $\langle u_0, n \rangle = -1$ and pairs to zero with $N_\omega^\perp$. Hence $N = N_\omega \oplus \bigoplus_u \mathbb{Z} \nu_u$ in the first case. Accordingly $X_\omega = Y_\omega \times (\mathbb{C}^*)^{\tau'}$, where Y_ω is the smooth toric variety of the cones of $\mathcal{F}'$ in $\text{cone}(\omega)$, regarded in N_ω , and the $\mathfrak{t}_u$ are the coordinates of the second factor; $\text{Log}(y, \mathfrak{t}) = \text{Log}_\omega y + \sum_u (-\log_\varkappa |\mathfrak{t}_u|) \nu_u$. The character $\chi = x^{-u_0}$ vanishes on the ν_u , so it is a function on Y_ω ; it is regular and vanishes to first order along every toric divisor D_n of Y_ω , because $\langle -u_0, n \rangle = 1$ for every lattice point n of ω ; and $|\chi| = \varkappa^{\langle u_0, \text{Log}_\omega y \rangle}$. Let $Y_{\Omega'}$ be the set of $y \in Y_\omega$ whose logarithm, modulo $\text{span}(\sigma)$ on the orbit of a cone $\sigma \subseteq \text{cone}(\omega)$, lies in the image of Ω' ; it is open, because Ω' is open and $\Omega' + \text{cone}(\omega) = \Omega'$. Let $T_I = \{\mathfrak{t} : -\log_\varkappa |\mathfrak{t}_u| \in I_u\}$.

The equation. On $\text{Log}^{-1}(\Omega)$ only 0 and the vertices of τ are active, and the factors $\beta(T_i - T_m)$ of β_m with i inactive equal 1 for active m , since there $T_i \leq \max_j T_j - C_0 < T_m$.

Moreover $T_{u_0} - T_0 = -\check{h}_{u_0} - \log_{\varkappa} |\chi|$ and $T_u - T_{u_0} = \check{h}_{u_0} - \check{h}_u + \log_{\varkappa} |\mathbf{t}_u|$. Multiplying $\tilde{f}$ by x^{-u_0} therefore gives

$$\mathscr{W}_\Omega = \{(y, \mathbf{t}) \in Y_{\Omega'} \times T_I : (\chi(y), \mathbf{t}) \in L\}, \quad L = \left\{ (\chi, \mathbf{t}) : \beta_0 \chi = \alpha_{u_0} + \sum_{u \in \tau'} \alpha_u \mathbf{t}_u \right\},$$

with $\alpha_{u_0} = \beta_{u_0} \varkappa^{-\check{h}_{u_0}}$, $\alpha_u = \beta_u \varkappa^{-\check{h}_u}$ nonnegative functions of $(|\chi|, |\mathbf{t}|)$; at $\chi = 0$, where $\beta_0 = 0$, the equation is $\alpha_{u_0} + \sum_u \alpha_u \mathbf{t}_u = 0$. Let $\Pi(y, \mathbf{t}) = (\chi(y), \mathbf{t})$ and $L_\Omega = \Pi(\mathscr{W}_\Omega)$.

The cap. Put $r_\chi = \varkappa^{-\check{h}_{u_0} - C_0}$; we call $\mathscr{W}_\Omega \cap \{|\chi| < r_\chi\}$ the *cap*. Where $|\chi| < r_\chi$, $T_0 - T_{u_0} < -C_0$, so $\beta_0 = 0$ and α_{u_0}, α_u depend only on $|\mathbf{t}|$. Hence, with $L_0 = \{\mathbf{t} \in T_I : \alpha_{u_0} + \sum_u \alpha_u \mathbf{t}_u = 0\}$ and $Y_J = \{y \in Y_{\Omega'} : |\chi(y)| \in J\}$,

$$\mathscr{W}_\Omega \cap \{|\chi| < r_\chi\} = Y_{[0, r_\chi)} \times L_0, \quad \mathscr{W}_\Omega \cap \{r_\chi/2 < |\chi| < r_\chi\} = Y_{(r_\chi/2, r_\chi)} \times L_0.$$

L_0 is a point if $d = 1$ and a truncated localized line, a pair of pants, if $d = 2$; it is connected. For $d = 2$ this follows from the clamping homotopy of the step on interval boxes below, applied at $\chi = 0$: it retracts L_0 , and also the untruncated localized line in the torus of the same variables, onto the same clamped core; and the untruncated localized line is connected, being isotopic to a line of $\mathbb{P}^2$ minus its three points on the coordinate lines by the isotopy of Appendix F.11.

The factor Y . The part Y° of $Y_{[0, r_\chi)}$ in the dense torus is $(N_\omega \otimes U(1)) \times \{\lambda \in \Omega' : \langle u_0, \lambda \rangle < \log_R r_\chi\}$, the second factor convex, so $\pi_1(Y^\circ) = N_\omega$; likewise $\pi_1(Y_{(r_\chi/2, r_\chi)}) = N_\omega$. The complement $Y_{[0, r_\chi)} \setminus Y^\circ$ is the union of the smooth divisors $D_n \cap Y_{[0, r_\chi)}$, each nonempty because $\Omega' + \text{cone}(\omega) = \Omega'$ and $\langle u_0, n \rangle = -1$. By general position a loop in the manifold $Y_{[0, r_\chi)}$ can be pushed into Y° , and a null-homotopy can be made transverse to the divisors, meeting them in finitely many points of their smooth parts; so $\pi_1(Y_{[0, r_\chi)})$ is the quotient of N_ω by the classes of the meridians of the D_n , which are the orbits of the circles $\mathbb{Z}n \otimes U(1)$. The rays n of a maximal cone of $\mathcal{F}'$ in $\text{cone}(\omega)$ form a basis of N_ω , since that cone is unimodular (Lemma F.7(2)). Hence $Y_{[0, r_\chi)}$ is simply connected.

The part away from the cap. Fix a ray n_0 of Y_ω ; then $\langle -u_0, n_0 \rangle = 1$ and $N_\omega = N_\omega^0 \oplus \mathbb{Z}n_0$ with $N_\omega^0 = N_\omega \cap u_0^\perp$. Writing points of Y° as $(\theta, \lambda) \in (N_\omega \otimes U(1)) \times \Omega'$ and $\theta = (\theta_{N_\omega^0}, \theta_0)$, the phase θ_0 is determined by $\arg \chi$, so

$$\begin{aligned} \mathscr{W}_{>r_\chi/2} &:= \mathscr{W}_\Omega \cap \{|\chi| > r_\chi/2\} \cong (N_\omega^0 \otimes U(1)) \times \mathcal{S}, \\ \mathcal{S} &= \{(\lambda, \ell) \in \Omega' \times L_{>r_\chi/2} : \langle u_0, \lambda \rangle = \log_{\varkappa} |\chi(\ell)|\}, \end{aligned}$$

$L_{>r_\chi/2} = L_\Omega \cap \{|\chi| > r_\chi/2\}$. The projection $\mathcal{S} \rightarrow L_{>r_\chi/2}$ has nonempty convex fibres $\Omega' \cap \{\langle u_0, \cdot \rangle = \log_{\varkappa} |\chi|\}$, and it has a continuous section $\ell \mapsto \lambda(\log_{\varkappa} |\chi(\ell)|)$: choose points $p_j \in \Omega'$, $j \geq 0$, with $\langle u_0, p_j \rangle$ increasing to $\sup_{\Omega'} \langle u_0, \cdot \rangle$, let $\lambda(c)$ run linearly from p_j to p_{j+1} for c between their values, and put $\lambda(c) = p_0 + (\langle u_0, p_0 \rangle - c)n_0$ below, which lies in $p_0 + \text{cone}(\omega) \subseteq \Omega'$. The straight-line homotopy in the fibres retracts $\mathcal{S}$ onto the section, so $\pi_1(\mathscr{W}_{>r_\chi/2}) = N_\omega^0 \times \pi_1(L_{>r_\chi/2})$.

Van Kampen. $\mathscr{W}_\Omega$ is the union of the open sets $\mathscr{W}_{[0, r_\chi)} = Y_{[0, r_\chi)} \times L_0$ and $\mathscr{W}_{>r_\chi/2}$, with connected intersection $Y_{(r_\chi/2, r_\chi)} \times L_0$, whose fundamental group $N_\omega \times \pi_1(L_0)$ maps to $\pi_1(L_0)$ by the projection, and to $N_\omega^0 \times \pi_1(L_{>r_\chi/2})$ by the identity on N_ω^0 , by $n_0 \mapsto \gamma_\chi$, the loop in $L_{>r_\chi/2}$ that turns $\arg \chi$ once with $\mathbf{t}$ fixed in L_0 , and by the inclusion of $\{\chi\} \times L_0$ on $\pi_1(L_0)$. So

$$\pi_1(\mathscr{W}_\Omega) \cong \pi_1(L_{>r_\chi/2}) / \langle \langle \gamma_\chi \rangle \rangle \cong \pi_1(L_\Omega),$$

the second isomorphism by van Kampen for $L_\Omega = L_{>r_\chi/2} \cup (\{|\chi| < r_\chi\} \times L_0)$; both are induced by Π . The same decompositions show that $\mathcal{W}_\Omega$ is path connected, and it is open in $\mathcal{W}$ because $Y_{\Omega'}$ and T_I are open.

Interval boxes. In the notation above, enumerate $\tau' = \{u_2, \dots, u_{d+1}\}$ and use projective monomial coordinates

$$z_0 = \chi, \quad z_1 = \varkappa^{-\check{h}_{u_0}}, \quad z_i = \varkappa^{-\check{h}_{u_i}} \mathbf{t}_{u_i} \quad (i = 2, \dots, d+1).$$

The signs can equivalently be absorbed in these coordinates; they do not affect any modulus below. The image box defining L_Ω is described by

$$c = \log_\varkappa |z_0/z_1| < b_0, \quad b_i = \log_\varkappa |z_i/z_1| \in J_i \quad (i = 2, \dots, d+1),$$

where $b_0 \geq 4C_0$ (possibly $+\infty$) and every J_i contains $[-4C_0, 4C_0]$. The equality of this image box with these intervals follows from the nonempty convex fibres of the projection from Ω' ; $\Omega' + \text{cone}(\omega) = \Omega'$ implies that the c -interval has no lower endpoint. Apply the same homotopy H_t with $a = 2C_0$, clamping all the nonconstant monomials $z_1, \dots, z_{d+1}$ and leaving z_0 unchanged. Writing $h_i = (\delta_i - a)_+$, its action is

$$b_i(t) = b_i + t(h_i - h_1), \quad c(t) = c - th_1.$$

Since $\delta_i = \delta_1 - b_i$, each $b_i(t)$ lies between b_i and 0; hence it remains in J_i . Also $c(t) \leq c$, so the upper bound is preserved. At $t = 1$, all $|b_i| \leq a$ and $c \leq a$, so the clamped core is contained in the image box. Therefore the full capped localized hyperplane and L_Ω both retract onto this core. In particular $L_\Omega \simeq (\mathbb{C}^*)^d$. The characters $\mathbf{t}_u$ do not vanish on L_Ω and define a homomorphism $\pi_1(L_\Omega) \rightarrow \mathbb{Z}^{\tau'}$. It is onto. In the coordinates $\log_\varkappa |\chi|$ and $\log_\varkappa |\mathbf{t}_u|$ the box above is a product of intervals, and each contains the value at the vertex of the tropical hyperplane with a margin of at least $4C_0$, because the projection of the Euclidean $4C_0$ -ball in Ω under a nonzero lattice covector m has half-width $4C_0|m| \geq 4C_0$. For $u \in \tau'$, decreasing $\log_\varkappa |\mathbf{t}_u|$ from its value at the vertex by $2C_0$, with the other coordinates at their values at the vertex, gives a point of the box at which the term of u lies $2C_0$ below the others, which tie. Near that point only the other monomials are active, so L is there the product of the localized hyperplane of the other monomials in the remaining variables (a point if $d = 1$) and an annulus in $\mathbf{t}_u$; the loop that turns $\arg \mathbf{t}_u$ once, with the other coordinates fixed, lies in L_Ω and maps to the basis vector of u . A surjective homomorphism $\mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is an isomorphism. By the previous step the characters $\mathbf{t}_u$ induce an isomorphism on $\pi_1(\mathcal{W}_\Omega)$ as well.

The case $\dim \tau = \dim \omega = 1$. Here $d = 1$, $\tau' = \{u\}$, and $N_\omega^\perp$ has the basis $u - u_0, \kappa$. As above $N = N_\omega \oplus \mathbb{Z}\nu_u \oplus \mathbb{Z}\nu_\kappa$, $X_\omega = Y_\omega \times \mathbb{C}_{\mathbf{t}_u}^* \times \mathbb{C}_w^*$ with $w = x^\kappa$, and $\text{Log}(y, \mathbf{t}_u, w) = \text{Log}_\omega y - \log_\varkappa |\mathbf{t}_u| \nu_u - \log_\varkappa |w| \nu_\kappa$. The character $\chi = x^{-u_0}$ vanishes on ν_u and ν_κ , and the functions $T_0 - T_{u_0}$ and $T_u - T_{u_0}$ depend only on $|\chi|$ and $|\mathbf{t}_u|$; so the equation of $\mathcal{W}$ on $\text{Log}^{-1}(\Omega)$ does not involve w , and

$$\mathcal{W}_\Omega = \mathcal{W}_\Omega^b \times \{w : -\log_\varkappa |w| \in I_\kappa\},$$

where $\mathcal{W}_\Omega^b \subset Y_\omega \times \mathbb{C}_{\mathbf{t}_u}^*$ is defined by the conditions above with $\Omega' \oplus I_u \nu_u$. The preceding steps apply to $\mathcal{W}_\Omega^b$ word for word: Y_ω is the smooth toric surface $\mathbb{C}^2$ of $\text{cone}(\omega)$, in which χ vanishes to first order along both coordinate axes, $N_\omega^0 = N_\omega \cap u_0^\perp$ has rank one, L_0 is a point, and L_Ω is a capped localized line as in type (b) of Appendix F.11. So $\mathcal{W}_\Omega^b$ is open and path connected, with π_1 mapped isomorphically to $\mathbb{Z}$ by $\mathbf{t}_u$, and the annulus contributes a factor $\mathbb{Z}$ detected by w . $\square$

F.14. The compatible cover: proof of Lemma F.20. *Data.* Let $d_S = \text{dist}(\hat{S} \setminus \mathcal{N}_\Gamma, \hat{\Gamma}) > 0$, distances being Euclidean for the standard inner product in lattice coordinates on $N_\mathbb{R}$ (so $|m| \geq 1$ for $0 \neq m \in M$), and chart norms as in Definition F.11. For a two-face or an edge Φ and a fan point n we use the *canonical chart* π_Φ , the projection onto $(A_\Phi^\perp)_\mathbb{R}$ along $J_\Phi(n) = \mathbb{R}n + ((A_\Phi^\perp)_\mathbb{R} + \mathbb{R}n)^\perp$. $J_\Phi(n)$ is a complement of $(A_\Phi^\perp)_\mathbb{R}$ containing n , so π_Φ differs from a lattice chart by a shear killing $\pi_\perp(n) - \pi_\perp(0)$, and Lemma F.12(3) applies. These charts nest: for $\Phi' \supset \Phi$ with the same n , $J_{\Phi'}(n) \subset J_\Phi(n)$, so $\pi_\Phi = \text{pr} \circ \pi_{\Phi'}$ for a linear projection pr . For a facet $F = F_u$ fix one chart, the projection along $\mathbb{R}n_F$ with $\langle u, n_F \rangle = -1$. Let $C_{\text{ch}} \geq 1$ bound the distortion between the chart norms and the Euclidean norm, and let $C_{\text{norm}} \geq 1$ bound, in the chart norms, the linear parts of the transition maps $(\pi_\parallel^{\Phi'}, \pi_\perp^{\Phi'}) \mapsto (\pi_\parallel^\Phi, \pi_\perp^\Phi)$ between nested charts, over the finitely many charts in use; thus $|\text{pr}(x)| \leq C_{\text{norm}}|x|$, and $|\langle u, y \rangle| \leq C_{\text{norm}}|y|_{\Phi'}$ for $u \in A_\Phi \setminus A_{\Phi'}$ and $y \in N_\mathbb{R}$; put $C_* = C_{\text{norm}}C_{\text{ch}}^2$, $\gamma_F = 0$, $\gamma_G = 1$ for two-faces, $\gamma_E = 1 + C_{\text{norm}}(\gamma_G + 1)$ for edges.

Cells and labels. Let $\mathfrak{T}$ be a triangulation of $S \setminus \text{int } \mathcal{N}_\Gamma$ for which $\hat{j}$ is linear on cells and which is adapted to the faces of $\nabla^{\check{h}}$. A cell c has type (a), (b) or (c) according as the face Φ_c of lowest dimension met by $\hat{j}(c)$ is a facet, a two-face or an edge. For types (b), (c), $\hat{j}(c) \cap \Phi_c$ is connected and at distance $\geq d_S$ from $\hat{\Gamma}$, and the region boundaries in the two-skeleton lie in $\hat{\Gamma}$ (Lemma F.2); so it lies in one vertex region, that of a vertex we call n_c . In the triangulation constructed below every cell meets a unique face of lowest dimension (see the first ring, the type-(b) cells near the crossings and the remainder). By (G0) below, every face met by $\hat{j}(c)$ contains Φ_c , since a face not containing Φ_c lies in a facet F_m with $m \notin A_{\Phi_c}$; hence $\Phi_{c'} \supseteq \Phi_c$ for $c' \subset c$. If $c' \subset c$ are both of type (b) or (c), then $n_{c'} = n_c$. If $\Phi_{c'} = \Phi_c$ this holds because $\hat{j}(c') \cap \Phi_c \subseteq \hat{j}(c) \cap \Phi_c$. If c' has type (b) at a two-face $G = F_\tau$ and c type (c) at an edge $E = F_{\tau'} \subset G$, $\tau \subset \tau'$, the set $\hat{j}(c) \cap G \setminus E$ contains $\hat{j}(c') \cap G$ and is a nonempty convex set minus a face, hence connected; it lies in $\text{ri } G \setminus \hat{\Gamma}$, since $\hat{j}(c)$ meets no edge of G other than E , so in one disc $C_{\hat{B}}(\tau, n'')$ of Lemma F.2(1), and $n_{c'} = n''$. The closure of this disc contains $\hat{j}(c) \cap E$, which lies in the interval $C_{\hat{B}}(\tau', n_c)$; by Lemma F.2(3) that closure is a union of cells, so it meets $C_{\hat{B}}(\tau', n_c)$ only if it contains it, that is, only if $n'' = n_c$. Put $h_c = \sup_{\hat{j}(c)} |\pi_\perp|_\infty$ in the chart of (Φ_c, n_c) , and for a radius λ

$$V_c = \{y \in \hat{B} : \pi_\parallel(y) \in (\text{conv } \pi_\parallel \hat{j}(c))^\lambda, \pi_\perp(y) \in \partial \mathcal{O}_\perp, |\pi_\perp(y)|_\infty < \gamma_{\Phi_c}(h_c + \lambda)\}$$

for types (b), (c), and $V_c = \{y \in F : |\pi_F(y) - \pi_F(\text{conv } \hat{j}(c))| < \lambda\}$ for type (a), where $\lambda = \lambda_{\dim c}$.

The condition (G0). For a cell of type (b) or (c) let $V_c^0 = \{y : \pi_\parallel(y) \in \text{conv } \pi_\parallel \hat{j}(c), \pi_\perp(y) \in \partial \mathcal{O}_\perp, |\pi_\perp(y)|_\infty \leq \gamma_{\Phi_c} h_c\}$, the set V_c at $\lambda = 0$; for type (a), $V_c^0 = \text{conv } \hat{j}(c)$. We require of $\mathfrak{T}$:

(G0) for every cell c , every functional $\ell_m = \langle m, \cdot \rangle + \check{h}_m$ with $m \notin A_{\Phi_c}$ is positive on V_c^0 , and $V_c^0 \cap \hat{\Gamma} = \emptyset$.

By compactness the margin $m(c) = \min(\min_m \min_{V_c^0} \ell_m / |m|, \text{dist}(V_c^0, \hat{\Gamma}))$ is positive; note $\text{dist}(y, F_m) \geq \ell_m(y) / |m|$.

Order of choices. After $\hat{S}$, $\mathcal{N}_\Gamma$ and the generic isotopy: the charts and constants above; a triangulation $\mathfrak{T}$ satisfying (G0); then radii $\lambda_0 < \lambda_1 < \lambda_2$ such that $\lambda_k \geq 2C_*\lambda_{k-1}$ and $\lambda_2 < \min_c m(c) / (2C_{\text{ch}}(\gamma_E + 1))$; then $\eta \leq \lambda_0 / (2C_*)$ with $(C_{\text{norm}} + 1)\eta < \min_c m(c) / 2$ and $\eta < \min_c m(c) / (8C_{\text{ch}})$; then $C_0 < \min(\eta / (4C_{\text{ch}}t_{\text{max}}), \min_c m(c) / (4C_{\text{act}}))$, which makes the band of Lemma F.10 narrower than η / C_{ch} and gives $\ell_* = C_{\text{act}}C_0 + (C_{\text{ch}} + 1)\eta \leq C_{\text{act}}C_0 + 2C_{\text{ch}}\eta < \min_c m(c) / 2$; finally $\varkappa$ large. Every constraint is an upper bound.

Admissibility (Lemma F.20(1)). V_c lies within $\delta_c := C_{\text{ch}}(\gamma_{\Phi_c} + 1)\lambda_2 < m(c)/2$ of V_c^0 , and $\ell_* < m(c)/2$. For $m \notin A_{\Phi_c}$ the function $\ell_m/|m|$ is 1-Lipschitz, so $\ell_m/|m| > m(c) - \delta_c > m(c)/2 > \ell_*$ on V_c ; likewise V_c is more than ℓ_* away from $\hat{\Gamma}$. By the first step of Appendix F.11, V_c is a product-shaped subset of $\hat{B}$. It contains $\hat{j}(c)$ by construction. For type (a) (G0) is automatic: each face of the convex facet F is exposed by a functional that is non-negative on F , so it is positive on $\text{conv } \hat{j}(c)$ if $\hat{j}(c)$ misses that face.

Monotonicity (Lemma F.20(2)). Let $c' \subset c$; the lowest face of c' contains that of c . Write $\lambda' = \lambda_{\dim c'} < \lambda = \lambda_{\dim c}$. First, $V_{c'} \subset V_c$ with a margin. If $\Phi_{c'} = \Phi_c$, the charts agree, the hull and h shrink, and $\gamma(h_{c'} + \lambda') \leq \gamma(h_c + \lambda) - \gamma(\lambda - \lambda')$. If c' has type (a) and c type (b) or (c), every $y \in V_{c'}$ is within $C_{\text{ch}}\lambda'$ of $\text{conv } \hat{j}(c') \subset \text{conv } \hat{j}(c)$, so $\pi_{\parallel}(y)$ is within $C_{\text{ch}}^2\lambda'$ of $\text{conv } \pi_{\parallel}\hat{j}(c)$ and $|\pi_{\perp}(y)| \leq h_c + C_{\text{ch}}^2\lambda'$. If c' has type (b) at G and c type (c) at an edge $E \subset G$, take $y \in V_{c'}$ and $w \in \text{conv } \hat{j}(c')$ with $|\pi_G(y) - \pi_G(w)| < \lambda'$. By nesting $|\pi_E(y) - \pi_E(w)| \leq C_{\text{norm}}\lambda'$. Write $r_c := \gamma_{\Phi_c}(h_c + \lambda)$ and $r_{c'} := \gamma_{\Phi_{c'}}(h_{c'} + \lambda')$ for the transverse radii of V_c and $V_{c'}$. Further $|y - w|_G \leq \max(\lambda', r_{c'} + h_{c'}) = r_{c'} + h_{c'}$ in the chart norm of G , and $\pi_{\perp}^G$ is a sub-tuple of $\pi_{\perp}^E$ independent of n , so $h_{c'} \leq h_c$ and the components of $\pi_{\perp}^E(y)$ that are those of $\pi_{\perp}^G(y)$ lie in $[0, r_{c'})$, where $r_{c'} \leq h_c + \lambda'$. The remaining component is $\ell_{u_3}(y) = \ell_{u_3}(w) + \langle u_3, y - w \rangle$, u_3 the element of $A_E \setminus A_G$, with $0 \leq \ell_{u_3}(w) \leq h_c$ and $|\langle u_3, y - w \rangle| \leq C_{\text{norm}}|y - w|_G$ by the choice of C_{norm} . Since $C_{\text{norm}} \geq 1$ and $r_{c'} + h_{c'} \leq 2h_c + \gamma_G\lambda'$,

$$|\pi_{\perp}^E(y)|_{\infty} \leq h_c + C_{\text{norm}}(r_{c'} + h_{c'}) \leq \gamma_E h_c + C_{\text{norm}}\gamma_G\lambda' \leq r_c - \lambda,$$

the last step because $r_c - \lambda = \gamma_E h_c + (\gamma_E - 1)\lambda = \gamma_E h_c + 2C_{\text{norm}}\lambda$ and $\lambda' < \lambda$. This is where the grading $\gamma_G < \gamma_E$ of the transverse radii is needed. In every case the room between $V_{c'}$ and the frontier of V_c is at least $\lambda - C_*\lambda' \geq \lambda/2 \geq C_*\eta$ in the chart norm of c .

Second, $\Omega_{V_{c'}} \subset \Omega_{V_c}$. Let z be within η of $\varpi^{-1}(V_{c'})$ in the chart norm of c' . In the $\pi_{\parallel}$ -directions of the chart of c , z is within $C_{\text{norm}}\eta$ of $\pi_{\parallel}(V_{c'})$; in the $\pi_{\perp}$ -directions of G there is no loss, $\pi_{\perp}^G$ being a sub-tuple of $\pi_{\perp}^E$. In the remaining coordinate π_{u_3} of E the divisorial ray direction has $d\pi_{u_3}(n) = -1$, which absorbs the deviation along the divisorial ray; the transverse deviation is at most $C_{\text{norm}}\eta$, and it is absorbed because ℓ_{u_3} is a foreign functional of c' , so $\ell_{u_3} \geq |u_3|(m(c') - 2C_{\text{ch}}\lambda') > m(c')/2 > (C_{\text{norm}} + 1)\eta$ on $V_{c'}$, since $V_{c'}$ lies within $C_{\text{ch}}(\gamma_G + 1)\lambda' = 2C_{\text{ch}}\lambda' < m(c')/2$ of $V_{c'}^0$. The case of type (a) inside type (b) or (c) is the same, using $\ell_{u'} \geq |u'|m(c')/2 > |u'|C_{\text{ch}}\eta$ on $V_{c'}$ for the other facets u' through Φ_c , which (G0), the bound on λ_2 and the choice $\eta < m/(8C_{\text{ch}})$ give; the transverse deviation of $\ell_{u'}$ is at most $|u'|C_{\text{ch}}\eta \leq C_{\text{ch}}^2\eta$, since $|u'| \leq C_{\text{ch}}$, and the parallel deviation at most $C_{\text{ch}}^2\eta \leq C_*\eta$. By the margin of the first step, $z \in \Omega_{V_c}$.

Existence of a triangulation satisfying (G0). Let x_0 be an edge crossing, on the edge E of $\nabla^{\hat{h}}$. Choose a convex ball U_0 around x_0 that misses $\hat{\Gamma}$ (possible since $\hat{\Gamma} \cap E$ is finite and $x_0 \notin \hat{\Gamma}$ by genericity), the other crossings, $\mathcal{N}_{\Gamma}$ and every face not containing E , and that lies in the open star of x_0 in a triangulation of $\hat{B}$ for which $\hat{j}$ is simplicial and the faces are subcomplexes. Then $\hat{S} \cap U_0$ is a cone from x_0 , and so is $\hat{B} \cap U_0 = \partial C_E \cap U_0$, C_E the cone cut out by the three functionals of A_E . The cone $\hat{S} \cap U_0$ meets each of the three two-faces G_k through E in finitely many rays.

Foreign functionals near x_0 . Let $y \in \partial C_E \cap U_0$ and $m \notin A_E$. Then $\ell_m(x_0) > 0$. If some foreign functional vanished on $[x_0, y]$, let z be its first zero; at z all other foreign functionals and those of A_E are ≥ 0 (C_E is convex), so $z \in \nabla^{\hat{h}}$ lies on the facet F_m , a face not containing E , inside U_0 , a contradiction. So on $\partial C_E \cap U_0$ only the functionals of A_E can vanish, and these are linear, hence invariant under homotheties centred at x_0 .

The first ring. Take a polygon in $\hat{S} \cap U_0$ around x_0 with a vertex on every ray of $\hat{S} \cap \bigcup_k G_k$ and at every crease of the cone, so that each outer edge meets at most one two-face and each cone triangle lies in $\hat{S}$. The first ring consists of the triangles $[x_0, p_i, p_{i+1}]$; it may later be subdivided only by coning from x_0 , so every cell containing x_0 keeps type (c). (Splitting a ring triangle at an interior point of $[x_0, p_i]$ would create a type-(b) cell touching G_k near x_0 whose set V^0 can leave ∇^h .) For a ring cell c and $y \in V_c^0$, $|\pi_{\parallel}(y) - \pi_{\parallel}(x_0)| \leq C_{\text{ch}} R_c$ and $|\pi_{\perp}(y)| \leq \gamma_E h_c \leq \gamma_E C_{\text{ch}} R_c$, where $R_c = \sup_{\hat{j}(c)} |z - x_0| \leq s$ for a ring of radius s . So $|y - x_0| \leq C_{\text{ch}}^2 (\gamma_E + 1)s$, and for s small $V_c^0 \subset \partial C_E \cap U_0$, where (G0) holds by the previous step.

Type-(b) cells near x_0 . Let c' be a type-(b) cell at G with $\hat{j}(c') \subset \frac{1}{2}U_0$, and put $t = \text{dist}(x_0, \hat{j}(c')) > 0$. Since the link of the cone $\hat{S} \cap U_0$ is compact and crosses G transversally, there are $\varphi_1, c_\gamma > 0$ with $\ell_{u_3}(z) \geq c_\gamma |z - x_0|$ for $z \in \hat{S} \cap U_0$ with $|\pi_{\perp}^G(z)| \leq \varphi_1 |z - x_0|$, u_3 the element of $A_E \setminus A_G$. In the chart of G let w_1, w_2 be the vectors with $\pi_{\parallel} = 0$ and $d\pi_{\perp}^G = e_1, e_2$, and $\beta_w = \max_a |\langle u_3, w_a \rangle|$. A point of $V_{c'}^0$ has the form $y_G + \sigma w_a$ with $0 \leq \sigma \leq \gamma_G h_{c'}$ and y_G in the hull of the points $z - \pi_a(z)w_a$, $z \in \hat{j}(c')$. If $h_{c'} \leq \varphi_0 t$, then $\ell_{u_3} \geq (c_\gamma - 2\varphi_0 \beta_w)t > 0$ on $V_{c'}^0$ for $\varphi_0 \leq \min(\varphi_1, c_\gamma/(4\beta_w))$, chosen also so small that $V_{c'}^0 \subset U_0$. Homothety about x_0 commutes with the chart of G , so φ_0 is uniform in t . Then $V_{c'}^0 \subset \partial C_E \cap U_0$ and (G0) holds by the first-zero argument. The condition $h \leq \varphi_0 t$ is met by grading the outer polygon angularly near each ray and the cells along each ray geometrically.

The remainder and subdivision. Triangulate the rest of $\hat{S} \setminus \mathcal{N}_\Gamma$ with the outer polygons of the rings as inner boundary. On this compact remainder, away from the crossings, take cells small compared with their distance to the faces they do not meet and to $\hat{\Gamma}$; then h_c is small too, the lowest face is well defined, and (G0) follows from the first-zero argument started at a point of Φ_c . (G0) survives any adapted subdivision of the cells outside the rings: a subcell with the same lowest face and label has a smaller V^0 ; a type-(b) subcell of a type-(b) cell near x_0 has t at least as large and h at most as large; a type-(a) subcell satisfies (G0) automatically.

Support (Lemma F.20(3)). $\hat{j}(c)$ meets Φ_c , where $\pi_{\perp} = 0$, so $h_c \leq C_{\text{ch}} \text{diam } \hat{j}(c)$, and V_c lies in the r' -neighbourhood of $\hat{j}(c)$ with $r' = O(\text{mesh}(\mathfrak{T}) + \lambda_2)$. The mesh can be decreased by subdivision outside the rings and by shrinking the rings, and λ_2, η, C_0 are bounded only from above. $\square$

APPENDIX G. DATA OF THE EXAMPLES

This appendix lists the data of the examples of Section 11: for X_9, X_{19} and X_{20} the rays of Σ' , the node parallelograms, a $\mathbb{Z}$ -basis of K and the tropical 2-cycles of Lemma 11.2 and Table 2; for X_{88} and X_{154} the vertices, the profile, the node parallelograms and a $\mathbb{Z}$ -basis of K .

Data availability. The accompanying data and programs are available at <https://github.com/bwuebben/calabi-yau-smoothability>, directory `paper6`.

Example cycles. For the five examples, the data contain the triangulations $\mathcal{T}$, the integral heights, the fans Σ' , the polyhedral decompositions of the bases and the triangles of the cycles, together with programs that recompute from them the toric data, the relation lattices, the boundary networks and node coefficients of the cycles, and their Euler characteristics and numbers of interior intersections with Γ ; the data printed below agree with these. The programs do not rebuild the base B'_σ from Σ' and $(\mathcal{T}, h)$, and do not verify every interior condition of [CBM14, Def. 7.2]. For the belt cycles they verify that each is a disc with one

field that meets Γ only along its boundary; for the cycle of X_9 the conditions of [CBM14, Def. 7.2] are proved in Lemma 11.2; for the other cycles, those with trivalent vertices and the cycle of X_{88} whose boundary is a closed path and which crosses Γ at 18 interior points, the interior conditions rest on the computations described in Section 11.5.

The four-node coefficient locus of X_9 . The example data also determine the compact boundary calculations of Sections 9.8–9.10. The accompanying exact programs verify the singular ideals and Hessians of the displayed members, the supports of orbit equations and first normal derivatives on all 3,574 boundary orbits, the coefficient-smoothing ranks, the compact surface fan, the Cayley subdivision and the Newton-polytope identification. These finite comparisons support the written component, integral-topology and analytic-gluing arguments; the programs do not replace those arguments.

Projectivity. For the list of Section 11.4, the data contain the input vertex sets, the integral solutions and Farkas certificates for Hypothesis (Proj), and programs that verify these and the computation of the forced nodes in exact arithmetic.

Unimodular heights and classification. For Theorem F the data contain the height certificates, the two certificate checkers, the published admissibility filter and the classification-file digests; the program of Step 2 of Appendix A.3 is `unimodular/admissible_filter.py`. The run of that step over the whole classification used an earlier copy of this program with the same tests; the published version reproduces the admissible rows of that run on the files for 5–8, 23, 24, 30–33 and 36 vertices. Of the 468,609,901 polytopes with 10–23 vertices, the second implementation of Section A.4, with d as there, tested the 36,648,946 with $d \leq 7$ and samples containing 26,862 with $d > 7$; two runs of it tested 15,637,714 and 18,627,401 polytopes, in overlapping sets; and earlier checks on 7,644,706 polytopes with at most 13 vertices, with the 10,879 admissible polytopes among them matched again to their vertices and face types, were retained. On 135 altered height data (random, negated, constant or rescaled heights, a point removed) the two programs that verify (i)–(v) and (i'), (ii'), (v') reach the same conclusions.

Reported computational evidence. For the computations of Section 11.5, the data contain summaries for each polytope, from which the totals there are recomputed; the bases and cycles themselves are not rebuilt from these summaries. Two outcomes are reported without data from which they can be verified: the count of 327 polytopes in (a) of Section 11.5, which is available only as a total, and one of the 72 hexagonal bases of the computation of Theorem C on the list (Section 11.4).

Fixed version of $\mathcal{L}_{\text{adm}}$. The 30 files of Definition A.1 are `heights_vNN.jsonl.gz` in `paper6/unimodular/certificates/` at public version `000380bcebe1`. The manifest `paper6/MANIFEST.json` at that version records their sizes and SHA-256 digests. The vertex counts are 5–33 and 36; the file for 32 vertices is empty. This immutable version fixes membership of the list even if the accompanying programs are subsequently revised.

Conventions. The rays of Σ' are numbered $n_0, n_1, \dots$; the vertices of Δ come first. A node parallelogram is written $q = (a, b, c, d)$ with $n_a + n_c = n_b + n_d$, in the labelling of Section 2.2: $\{a, c\}$ is the chosen diagonal, which passes through the centre at a pentagon or a hexagon, and $\text{circ}_q = e_b + e_d - e_a - e_c$. A *boundary segment* of a tropical 2-cycle S is a maximal path in ∂S whose interior avoids the negative vertices; it is either closed or joins two negative vertices t_i . A boundary segment is written, in one of its two directions, as the sequence of the lower supports of its arcs and of the nodes it passes. An ordered pair (i, j) stands for the arcs with lower support $\{n_i, n_j\}$ on which the boundary network $\partial(v[S])$, read in the direction of listing, has coefficient $n_j - n_i$; so the boundary field is $v = n_j - n_i$ if the listing follows the boundary orientation of ∂S and $v = n_i - n_j$ otherwise, and $e_{n_j} - e_{n_i}$ is

the corresponding canonical lift (Section 2.3). Consecutive arcs with the same pair, which meet at positive vertices or at edge points, are listed once, and q_k between two pairs marks the passage through the node q_k . A closed boundary segment returns to its first pair after the last node listed. Reversing the direction of listing reverses the order and the signs, so the difference of the lifts after and before q_k does not depend on it; by Definition 2.6, $\rho_{q_k}(S)$ is the coefficient of circ_{q_k} in this difference, and can therefore be read off from the lists below. For the cycles with trivalent vertices, we also give the number of four-valent interior vertices of S , the number of points at which S crosses Γ transversally (the discriminant-intersection clause of [CBM14, Def. 7.2]), and the Euler characteristic of S . Vectors in $\mathbb{Z}^Q$ are written as combinations of the nodes: $q_2 - q_3$ stands for the vector with coordinate 1 at q_2 , -1 at q_3 and 0 elsewhere.

G.1. The example X_9 . Rays of Σ' . The vertices of Δ_9 are $n_0 = (1, 0, 0, 0)$, $n_1 = (0, 1, 0, 0)$, $n_2 = (-1, -1, 0, 0)$, $n_3 = (0, 0, 1, 0)$, $n_4 = (0, 0, 0, 1)$, $n_5 = (-4, -2, -1, 0)$, $n_6 = (-4, -2, 0, -1)$, $n_7 = (3, 1, 1, 1)$, $n_8 = (2, 1, 1, 1)$. The centre of the dP_7 pentagon is $n_{10} = (-2, -1, 0, 0)$. The fan Σ' has one further ray, $n_9 = (-1, 0, 0, 0)$, in the interior of a facet of Δ_9 .

Node parallelograms $q = (a, b, c, d)$, given by the indices of n_a, n_b, n_c, n_d ; an asterisk marks the two parallelograms of the pentagon: $q_1 = (3, 6, 10, 8)^*$, $q_2 = (2, 6, 3, 7)$, $q_3 = (4, 5, 10, 8)^*$, $q_4 = (2, 5, 4, 7)$.

A $\mathbb{Z}$ -basis of K (rank 1): $q_1 - q_2 - q_3 + q_4$.

Generating cycles. The cycles are the following.

- S_1 : a belt cycle with constant field $\pm(4, 2, 1, 1)$, a disc; its boundary passes 5 positive vertices. Boundary segments: $(3, 6) q_2 (7, 2) q_4 (4, 5) q_3 (8, 10) q_1$. Node coefficients: $\rho(S_1) = q_1 - q_2 - q_3 + q_4$.

Index. The node coefficient vector of S_1 is the basis vector of K .

G.2. The example X_{19} . Rays of Σ' . The vertices of Δ_{19} are $n_0 = (1, 0, 0, 0)$, $n_1 = (0, 1, 0, 0)$, $n_2 = (0, 0, 1, 0)$, $n_3 = (0, 0, -1, 0)$, $n_4 = (0, -1, 0, 0)$, $n_5 = (1, -1, -1, 0)$, $n_6 = (0, 0, 0, 1)$, $n_7 = (-1, 1, 1, -1)$, $n_8 = (0, -1, 0, 1)$, $n_9 = (0, 0, -1, 1)$, $n_{10} = (-1, 1, 0, -1)$, $n_{11} = (-1, 0, 1, -1)$, $n_{12} = (-1, 0, 1, 0)$, $n_{13} = (-1, 1, 0, 0)$, $n_{14} = (0, -1, -1, 0)$, $n_{15} = (0, -1, -1, 1)$, $n_{16} = (-1, 0, 0, -1)$, $n_{17} = (-2, 1, 1, -1)$, $n_{18} = (-1, 0, 0, 1)$. The centre of the dP_7 pentagon is $n_{19} = (-1, 0, 0, 0)$.

Node parallelograms $q = (a, b, c, d)$, given by the indices of n_a, n_b, n_c, n_d ; an asterisk marks the two parallelograms of the pentagon: $q_1 = (16, 17, 19, 14)^*$, $q_2 = (1, 7, 17, 13)$, $q_3 = (2, 7, 17, 12)$, $q_4 = (2, 6, 18, 12)$, $q_5 = (15, 18, 19, 14)^*$, $q_6 = (4, 8, 12, 11)$, $q_7 = (1, 6, 18, 13)$, $q_8 = (4, 11, 16, 14)$, $q_9 = (3, 10, 16, 14)$, $q_{10} = (3, 9, 13, 10)$, $q_{11} = (4, 8, 15, 14)$, $q_{12} = (3, 9, 15, 14)$, $q_{13} = (0, 2, 4, 5)$, $q_{14} = (7, 10, 16, 11)$, $q_{15} = (6, 8, 15, 9)$, $q_{16} = (0, 1, 3, 5)$.

A $\mathbb{Z}$ -basis of K (rank 3): $q_2 - q_3 + q_4 - q_7$, $q_2 - q_3 - q_6 + q_8 - q_9 + q_{10} + q_{11} - q_{12} - q_{13} + q_{16}$, $q_1 + q_2 - q_5 - q_7 - q_8 - q_9 + q_{11} + q_{12} + q_{14} - q_{15}$.

Generating cycles. The boundary graphs below have the following negative vertices.

- t_1 : lower support $\{n_{10}, n_{16}, n_{17}\}$, face label $[(0, 0, 0, 1), (1, 0, 1, 0)]$.
- t_2 : lower support $\{n_7, n_{10}, n_{17}\}$, face label $[(0, -1, 0, 0), (0, 0, 0, 1)]$.
- t_3 : lower support $\{n_8, n_{15}, n_{18}\}$, face label $[(0, 0, 0, -1), (1, 1, 0, 0)]$.
- t_4 : lower support $\{n_6, n_8, n_{18}\}$, face label $[(0, 0, -1, -1), (0, 0, 0, -1)]$.
- t_5 : lower support $\{n_2, n_{11}, n_{12}\}$, face label $[(0, 0, -1, 0), (0, 1, -1, 0)]$.
- t_6 : lower support $\{n_1, n_9, n_{13}\}$, face label $[(0, -1, 0, -1), (0, -1, 1, 0)]$.
- t_7 : lower support $\{n_2, n_4, n_{11}\}$, face label $[(-1, 1, -1, 1), (0, 1, -1, 0)]$.

- t_8 : lower support $\{n_1, n_3, n_9\}$, face label $[(-1, -1, 1, 0), (0, -1, 1, 0)]$.

The cycles are the following.

- S_1 : a belt cycle with constant field $\pm(1, 0, 0, 0)$, a disc; its boundary passes 5 positive vertices. Boundary segments: $(17, 7) q_3 (12, 2) q_4 (18, 6) q_7 (13, 1) q_2$. Node coefficients: $\rho(S_1) = q_2 - q_3 + q_4 - q_7$.
- S_2 : boundary graph a complete graph on four vertices; 20 positive vertices on ∂S , 6 four-valent interior vertices, 11 transverse crossings with Γ , $\chi(S) = 5$. Boundary segments: $t_1 (17, 16) q_1 (19, 14) q_5 (18, 15) t_3; t_1 (10, 17) t_2; t_1 (16, 10) q_9 (14, 3) q_{12} (15, 9) q_{15} (8, 6) t_4; t_2 (7, 17) q_3 (2, 12) q_4 (6, 18) t_4; t_2 (10, 7) q_{14} (16, 11) q_8 (14, 4) q_{11} (15, 8) t_3; t_3 (18, 8) t_4$. Node coefficients: $\rho(S_2) = q_1 + q_3 - q_4 - q_5 - q_8 - q_9 + q_{11} + q_{12} + q_{14} - q_{15}$.
- S_3 : boundary graph a complete graph on four vertices; 26 positive vertices on ∂S , 17 four-valent interior vertices, 39 transverse crossings with Γ , $\chi(S) = 5$. Boundary segments: $t_5 (12, 11) q_6 (8, 4) q_{11} (15, 14) q_{12} (9, 3) t_8; t_5 (11, 2) t_7; t_5 (2, 12) q_3 (7, 17) q_2 (1, 13) t_6; t_6 (9, 13) q_{10} (3, 10) q_9 (14, 16) q_8 (4, 11) t_7; t_6 (1, 9) t_8; t_7 (4, 2) q_{13} (5, 0) q_{16} (3, 1) t_8$. Node coefficients: $\rho(S_3) = q_2 - q_3 - q_6 + q_8 - q_9 + q_{10} + q_{11} - q_{12} - q_{13} + q_{16}$.

Index. In the basis above, the node coefficient vectors of $S_1, \dots, S_3$ have coordinates $(1, 0, 0)$, $(-1, 0, 1)$, $(0, 1, 0)$; the elementary divisors of this matrix are $1, 1, 1$, so the vectors $\rho(S_i)$ generate K .

G.3. The example X_{20} . *Rays of Σ' .* The vertices of Δ_{20} are $n_0 = (1, 0, 0, 0)$, $n_1 = (0, 1, 0, 0)$, $n_2 = (0, 0, 1, 0)$, $n_3 = (0, 0, 0, 1)$, $n_4 = (1, -1, -1, -1)$, $n_5 = (-1, 1, 1, 1)$, $n_6 = (0, 0, 0, -1)$, $n_7 = (0, 0, -1, 0)$, $n_8 = (0, -1, 0, 0)$, $n_9 = (-1, 1, 0, 1)$, $n_{10} = (0, 0, -1, -1)$, $n_{11} = (-1, 1, 0, 0)$, $n_{12} = (-1, 0, 1, 1)$, $n_{13} = (0, -1, 0, -1)$, $n_{14} = (-1, 0, 1, 0)$, $n_{15} = (0, -1, -1, -1)$, $n_{16} = (-1, 0, 0, 1)$, $n_{17} = (-1, 0, -1, 0)$, $n_{18} = (-1, -1, 0, 0)$, $n_{19} = (-1, -1, 1, 0)$. The centre of the dP_7 pentagon is $n_{20} = (-1, 0, 0, 0)$.

Node parallelograms $q = (a, b, c, d)$, given by the indices of n_a, n_b, n_c, n_d ; an asterisk marks the two parallelograms of the pentagon: $q_1 = (18, 19, 20, 17)^*$, $q_2 = (13, 15, 18, 19)$, $q_3 = (12, 16, 18, 19)$, $q_4 = (3, 8, 18, 16)$, $q_5 = (4, 8, 18, 15)$, $q_6 = (14, 19, 20, 11)^*$, $q_7 = (6, 13, 19, 14)$, $q_8 = (5, 12, 19, 14)$, $q_9 = (4, 7, 17, 15)$, $q_{10} = (3, 7, 17, 16)$, $q_{11} = (5, 9, 16, 12)$, $q_{12} = (6, 10, 15, 13)$, $q_{13} = (4, 8, 19, 13)$, $q_{14} = (3, 8, 19, 12)$, $q_{15} = (0, 1, 9, 3)$, $q_{16} = (0, 2, 19, 8)$, $q_{17} = (0, 2, 12, 3)$, $q_{18} = (0, 1, 10, 4)$, $q_{19} = (0, 2, 13, 4)$.

A $\mathbb{Z}$ -basis of K (rank 6): $q_{14} - q_{16} + q_{17}$, $q_{13} - q_{16} + q_{19}$, $q_3 - q_4 + q_{14}$, $q_2 - q_5 + q_{13}$, $q_4 - q_5 + q_9 - q_{10}$, $q_4 - q_5 + q_7 - q_8 + q_{11} - q_{12} + q_{15} - q_{18}$.

Generating cycles. The boundary graphs below have the following negative vertices.

- t_1 : lower support $\{n_3, n_9, n_{16}\}$, face label $[(0, 0, 0, -1), (0, 0, 1, -1)]$.
- t_2 : lower support $\{n_4, n_{10}, n_{15}\}$, face label $[(0, 0, 0, 1), (0, 0, 1, 0)]$.

The cycles are the following.

- S_1 : a belt cycle with constant field $\pm(1, 0, -1, 0)$, a disc; its boundary passes 3 positive vertices. Boundary segments: $(8, 19) q_{16} (0, 2) q_{17} (3, 12) q_{14}$. Node coefficients: $\rho(S_1) = -q_{14} + q_{16} - q_{17}$.
- S_2 : a belt cycle with constant field $\pm(0, 1, 0, 1)$, a disc; its boundary passes 5 positive vertices. Boundary segments: $(18, 16) q_4 (8, 3) q_{14} (19, 12) q_3$. Node coefficients: $\rho(S_2) = q_3 - q_4 + q_{14}$.
- S_3 : a belt cycle with constant field $\pm(1, 0, 0, 0)$, a disc; its boundary passes 5 positive vertices. Boundary segments: $(8, 18) q_5 (4, 15) q_9 (7, 17) q_{10} (3, 16) q_4$. Node coefficients: $\rho(S_3) = -q_4 + q_5 - q_9 + q_{10}$.

- S_4 : a belt cycle with constant field $\pm(1, 0, -1, -1)$, a disc; its boundary passes 5 positive vertices. Boundary segments: $(18, 15) q_5 (8, 4) q_{13} (19, 13) q_2$. Node coefficients: $\rho(S_4) = q_2 - q_5 + q_{13}$.
- S_5 : a belt cycle with constant field $\pm(1, 0, -1, 0)$, a disc; its boundary passes 6 positive vertices. Boundary segments: $(8, 19) q_{16} (0, 2) q_{19} (4, 13) q_{13}$. Node coefficients: $\rho(S_5) = -q_{13} + q_{16} - q_{19}$.
- S_6 : boundary graph with two trivalent vertices; 18 positive vertices on ∂S , 8 four-valent interior vertices, 25 transverse crossings with Γ , $\chi(S) = 8$. Boundary segments: $t_1 (9, 16) q_{11} (5, 12) q_8 (14, 19) q_7 (6, 13) q_{12} (10, 15) t_2$; $t_1 (16, 3) q_4 (18, 8) q_5 (15, 4) t_2$; $t_1 (3, 9) q_{15} (0, 1) q_{18} (4, 10) t_2$. Node coefficients: $\rho(S_6) = q_4 - q_5 + q_7 - q_8 + q_{11} - q_{12} + q_{15} - q_{18}$.

Index. In the basis above, the node coefficient vectors of $S_1, \dots, S_6$ have coordinates $(-1, 0, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 0, -1, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, -1, 0, 0, 0, 0)$, $(0, 0, 0, 0, 0, 1)$; the elementary divisors of this matrix are $1, 1, 1, 1, 1, 1$, so the vectors $\rho(S_i)$ generate K .

G.4. The hexagon examples X_{88} and X_{154} . For these two examples the rays listed are the vertices of Δ and the centre of the hexagon; the fan Σ' of a P'_σ as in (Proj) may have further rays, which are in the accompanying data. The hexagon is listed in cyclic order $w_0, \dots, w_5$, and an asterisk marks its node parallelograms. The $\mathbb{Z}$ -bases of K are reduced (by the LLL algorithm) bases of the lattice computed from the circuits.

The example X_{88} . Δ_{88} has 21 vertices, 22 unit squares and one hexagon. The profile is the A_1 subdivision with the parallelograms $\{o, w_1, w_2, w_3\}$ and $\{o, w_4, w_5, w_0\}$; the other two A_1 subdivisions also satisfy (Proj), and their lattices K are identified with this one by matching the pairs of hexagon nodes, up to sign.

Rays. $n_0 = (1, 0, 0, 0)$, $n_1 = (0, 1, 0, 0)$, $n_2 = (1, -1, 0, 0)$, $n_3 = (0, 0, 1, 0)$, $n_4 = (0, 0, 0, 1)$, $n_5 = (0, 0, 1, -1)$, $n_6 = (0, 0, -1, 1)$, $n_7 = (0, 0, 0, -1)$, $n_8 = (0, 0, -1, 0)$, $n_9 = (0, -1, 0, 0)$, $n_{10} = (-1, 1, 0, 0)$, $n_{11} = (-1, 1, 1, 0)$, $n_{12} = (-1, 1, 0, 1)$, $n_{13} = (0, -1, 0, -1)$, $n_{14} = (0, -1, -1, 0)$, $n_{15} = (-1, 0, 0, -1)$, $n_{16} = (-1, 0, -1, 1)$, $n_{17} = (-1, 0, -1, 0)$, $n_{18} = (-1, 0, 0, 1)$, $n_{19} = (-1, 0, 1, -1)$, $n_{20} = (-1, 0, 1, 0)$. The centre of the hexagon is $n_{21} = (-1, 0, 0, 0)$. The hexagon is $n_{15}, n_{19}, n_{20}, n_{18}, n_{16}, n_{17}$.

Node parallelograms $q = (a, b, c, d)$: $q_1 = (11, 1, 0, 3)$, $q_2 = (12, 4, 0, 1)$, $q_3 = (12, 4, 3, 11)$, $q_4 = (13, 7, 0, 2)$, $q_5 = (14, 2, 0, 8)$, $q_6 = (14, 8, 7, 13)$, $q_7 = (15, 7, 1, 10)$, $q_8 = (15, 19, 5, 7)$, $q_9 = (15, 19, 11, 10)$, $q_{10} = (17, 10, 1, 8)$, $q_{11} = (17, 8, 6, 16)$, $q_{12} = (17, 10, 12, 16)$, $q_{13} = (17, 8, 7, 15)$, $q_{14} = (20, 3, 2, 9)$, $q_{15} = (9, 2, 4, 18)$, $q_{16} = (18, 4, 3, 20)$, $q_{17} = (19, 5, 3, 20)$, $q_{18} = (19, 13, 9, 20)$, $q_{19} = (16, 18, 4, 6)$, $q_{20} = (16, 18, 9, 14)$, $q_{21} = (18, 12, 11, 20)$, $q_{22} = (14, 17, 15, 13)$, $q_{23} = (21, 19, 20, 18)^*$, $q_{24} = (21, 16, 17, 15)^*$.

A $\mathbb{Z}$ -basis of K (rank 10): $q_3 - q_{16} + q_{21}$, $-q_1 + q_2 - q_3$, $q_6 - q_{13} - q_{22}$, $q_4 - q_5 + q_6$, $q_7 - q_{10} + q_{13}$, $q_{14} + q_{15} + q_{16}$, $q_2 - q_5 + q_{10} - q_{12} + q_{15} - q_{20}$, $-q_1 + q_4 - q_7 + q_9 + q_{14} + q_{18}$, $q_9 - q_{12} - q_{21} - q_{23} + q_{24}$, $q_8 - q_{11} + q_{13} - q_{16} + q_{17} - q_{19}$.

No forced node. The vector $-q_1 + q_2 - 2q_3 - q_4 + q_5 + q_6 - q_7 + q_8 - 2q_9 + q_{10} - q_{11} + 2q_{12} - 2q_{13} - q_{14} - q_{15} - q_{16} + q_{17} - q_{18} - q_{19} + q_{20} + q_{21} - q_{22} + q_{23} - q_{24}$ lies in K .

The example X_{154} . Δ_{154} has 22 vertices, 20 unit squares and one hexagon. The profile is the A_2 subdivision with the three parallelograms $\{o, w_0, w_1, w_2\}$, $\{o, w_2, w_3, w_4\}$, $\{o, w_4, w_5, w_0\}$; the other A_2 subdivision also satisfies (Proj).

Rays. $n_0 = (1, 0, 0, 0)$, $n_1 = (0, 1, 0, 0)$, $n_2 = (0, 0, 1, 0)$, $n_3 = (0, 0, 0, 1)$, $n_4 = (1, 0, 0, -1)$, $n_5 = (0, 1, -1, 0)$, $n_6 = (0, -1, 1, 0)$, $n_7 = (-1, 0, 0, 1)$, $n_8 = (0, 0, 0, -1)$, $n_9 = (0, 0, -1, 0)$, $n_{10} = (-1, 0, 0, 0)$, $n_{11} = (0, -1, 0, 0)$, $n_{12} = (-1, 0, -1, 1)$, $n_{13} = (0, -1, 1, -1)$, $n_{14} =$

$(-1, -1, 1, 0)$, $n_{15} = (-1, -1, 0, 1)$, $n_{16} = (-1, 0, -1, 0)$, $n_{17} = (0, -1, 0, -1)$, $n_{18} = (-1, -1, 1, -1)$, $n_{19} = (-1, -1, -1, 1)$, $n_{20} = (-1, -1, 0, -1)$, $n_{21} = (-1, -1, -1, 0)$. The centre of the hexagon is $n_{22} = (-1, -1, 0, 0)$. The hexagon is $n_{20}, n_{18}, n_{14}, n_{15}, n_{19}, n_{21}$.

Node parallelograms $q = (a, b, c, d)$: $q_1 = (13, 4, 0, 6)$, $q_2 = (19, 12, 5, 9)$, $q_3 = (19, 15, 6, 11)$, $q_4 = (17, 4, 0, 11)$, $q_5 = (17, 13, 6, 11)$, $q_6 = (17, 4, 9, 21)$, $q_7 = (17, 11, 19, 21)$, $q_8 = (18, 8, 4, 13)$, $q_9 = (18, 10, 7, 14)$, $q_{10} = (12, 7, 1, 5)$, $q_{11} = (16, 10, 1, 5)$, $q_{12} = (16, 10, 7, 12)$, $q_{13} = (16, 5, 8, 20)$, $q_{14} = (20, 18, 10, 16)$, $q_{15} = (18, 13, 6, 14)$, $q_{16} = (19, 15, 7, 12)$, $q_{17} = (20, 17, 4, 8)$, $q_{18} = (21, 9, 5, 16)$, $q_{19} = (20, 18, 13, 17)$, $q_{20} = (21, 16, 12, 19)$, $q_{21} = (22, 20, 18, 14)^*$, $q_{22} = (22, 14, 15, 19)^*$, $q_{23} = (22, 19, 21, 20)^*$.

A $\mathbb{Z}$ -basis of K (rank 9): $q_2 - q_{18} + q_{20}$, $q_1 - q_4 + q_5$, $q_8 - q_{17} + q_{19}$, $q_{10} - q_{11} + q_{12}$, $q_6 + q_{13} + q_{17} + q_{18}$, $q_3 - q_5 - q_{19} - q_{21} + q_{22}$, $-q_5 + q_7 + q_{15} - q_{21} + q_{23}$, $q_9 - q_{12} - q_{20} - q_{21} + q_{23}$, $q_{12} + q_{14} - q_{16} + q_{21} - q_{22}$.

No forced node. The vector $-q_1 + q_2 + 2q_3 + q_4 - 2q_5 + 2q_6 - q_7 + q_8 - q_9 - q_{10} + q_{11} + q_{12} + 2q_{13} + q_{14} - q_{15} - q_{16} + q_{17} + q_{18} - q_{19} + 2q_{20} + q_{21} + q_{22} - 2q_{23}$ lies in K .

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