

TOWARD AN EXOTIC $S^2 \times S^2$ AND AN EXOTIC $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

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ABSTRACT. We study a specified Luttinger filling in the construction of Lidman and Piccirillo whose simple connectivity would yield exotic $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. An earlier version of this manuscript claimed simple connectivity from incorrectly derived group relations; that claim is withdrawn. We derive the complete fundamental-group presentation and express the group as a one-relator quotient of an iterated HNN extension containing the genus-two surface group. The filled group is trivial exactly when the final relation kills a specified fiber commutator. We also determine its universal central extension with cyclic kernel, construct geometric realizations of it, and extend the boundary involution over the filling. Proposed symplectic extensions construct a torus with a negative area–Maslov relation and exact tori detecting a nonzero transfer kernel for a choice of auxiliary handles. These constructions do not establish survival of the central element. The final filling problem remains unresolved.

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1. INTRODUCTION

The purpose of this paper is to give a starting point for further work on a specific four-manifold construction whose fundamental group remains unknown. We record its marked geometry, the complete group presentation, structural reductions, and geometric restrictions on possible proofs. The results separate what the construction already provides from the precise fundamental-group question that prevents its completion. The starting point is Lidman and Piccirillo's construction [LP25] of a spin rational homology four-sphere distinguished from a comparison manifold by smooth sliceness of the figure-eight knot. Their construction also gives closed manifolds with the integral cohomology rings of $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and nonstandard smooth behavior. Identifying their homeomorphism types requires additional information about fundamental groups.

We fix one member of the surgery construction, denoted

$$V = W(+1, -1; 0, 0).$$

The parameters $(p, q; n, m)$ specify the slopes $M(\ell C^m)^p$ and $N(DE^m)^q$ in the peripheral bases defined below; the last two entries thus specify sections of the surgery tori. In the convention of Section 2, the positive cotangent disk meridians are M and N^{-1} , the section longitudes are ℓ and D , and the killed slopes are $M\ell$ and $N^{-1}D$, equivalently $M\ell$ and ND^{-1} . These data complete the marking of the construction; they are not inferred from an unspecified numerical surgery sign. Lidman and Piccirillo explicitly allow choices of parametrization of their tori [LP25, Section 1, footnote 2].

Write $G = \pi_1(V)$ and $Y = \partial V$. The boundary is identified with zero surgery on the square knot. Let σ be the free orientation-reversing boundary involution specified in Section 3, and put

$$Z = V \cup_{\sigma} V, \quad W = V / \sim_{\sigma}.$$

In the definition of W , only boundary points are identified. Thus $Z \rightarrow W$ is a connected double cover. The notation W without parameters denotes this closed manifold, whereas $W(p, q; n, m)$ denotes a compact surgery piece.

Theorem 1.1 (Geometric consequences). *For the specified filling and boundary marking the following hold.*

- (1) V is a spin integral homology S^2 . The double Z is symplectic and has the integral cohomology ring of $S^2 \times S^2$. A surviving fiber F and a doubled section H are genus-two symplectic surfaces forming a basis of $H_2(Z; \mathbb{Z})$, with

$$F^2 = H^2 = 0, \quad F \cdot H = 1.$$

Every nonzero square-zero class is $k[F]$ or $k[H]$ and has minimum genus $|k| + 1$. In particular Z is not diffeomorphic to $S^2 \times S^2$.

- (2) W is a spin rational homology four-sphere with $H_1(W; \mathbb{Z}) = \mathbb{Z}/2$. The figure-eight knot is not smoothly slice in W .
- (3) There is an orientation-preserving diffeomorphism $f : Y \rightarrow Y$ such that $Z_f = V \cup_{f \circ \sigma} V$ has the integral cohomology ring of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and is not diffeomorphic to that manifold. Here f is the boundary map of [LP25, Figure 3], transported through the specified marking of Y .
- (4) σ extends to an orientation-reversing diffeomorphism of V . Consequently either copy of V is a retract of Z . If $H_{\partial} = \text{im}(\pi_1(Y) \rightarrow G)$, then

$$\pi_1(Z) \cong G *_{H_{\partial}} G, \quad \pi_1(Z) = 1 \iff G = 1.$$

The amalgamation uses the actual boundary image, not an assertion that $\pi_1(Y)$ injects into G .

Let $\mathcal{B}$ denote the Kawauchi manifold used in [LP25]: it is obtained from the zero trace of the figure-eight knot by identifying boundary points under the free involution induced by its strongly negative-amphicheiral symmetry. We use the established facts that $\mathcal{B}$ is spin, is a rational homology four-sphere, has $\pi_1(\mathcal{B}) = \mathbb{Z}/2$, and contains a slice disk for the figure-eight knot.

Corollary 1.2 (Conditional homeomorphism statements). *If $G = 1$, then Z is homeomorphic but not diffeomorphic to $S^2 \times S^2$, and Z_f is homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Moreover*

$$W \text{ is homeomorphic to } \mathcal{B} \iff G = 1.$$

Whenever this condition holds, W and $\mathcal{B}$ are distinguished smoothly by sliceness of the figure-eight knot.

The geometric conclusions in the first three parts of Theorem 1.1 follow the construction and arguments of Lidman–Piccirillo. We supply the slope, boundary-marking, and normal-framing details needed for the fixed V . The extension in the fourth part makes the missing group condition necessary as well as sufficient for the stated homeomorphism comparison with $\mathcal{B}$.

Part II gives the complete presentation of G and its reduction to three generators and five relators. It also identifies an intermediate group in which the fiber group does inject. Here is the resulting algebraic form of the remaining question; all the words in the statement are part of its definition.

Theorem 1.3 (The last relation). *Let*

$$\Pi = \langle x, y, r, s \mid [x, y][r, s] = 1 \rangle, \quad d_0 = y^{-1}x^{-1}s, \quad e_0 = x^{-1}s,$$

and define

$$Q_2 = \langle \Pi, B \mid BrB^{-1} = r, BxB^{-1} = y^{-1}, Bd_0B^{-1} = e_0 \rangle.$$

There is an automorphism θ of Q_2 fixing x, y, r, B and sending s to sB^{-1} . Let ϕ be the automorphism of Π interchanging x, r and y, s , and set

$$K = \langle Q_2, A \mid A\theta(g)A^{-1} = \phi(g) \ (g = x, y, r, s) \rangle.$$

Both displayed extensions are injective HNN extensions. In particular Π embeds in K . With

$$m = yB^{-1}x^{-1}y^{-1}B, \quad \lambda = Axm,$$

there is an isomorphism

$$G \cong K / \langle\langle \lambda \rangle\rangle_K$$

preserving the original fiber generators. The element $J = [x, y]$ has infinite order in K and normally generates G after passage to this quotient. Therefore

$$G = 1 \iff J \in \langle\langle \lambda \rangle\rangle_K.$$

Neither membership nor nonmembership in this normal closure is proved.

The commutator J represents the separating fiber curve dividing its two handles.

The further results serve two purposes. Proposition 11.1 computes the entire kernel of restoring the meridian transport before the last filling. Theorems 13.1 and 14.2, with Propositions 15.1 and 16.3, identify a single cyclic central kernel in algebraic, bundle, and

exact symplectic terms. The accompanying limitations specify what these descriptions do not yet control. Appendix C gives another exact presentation as a quotient of a trefoil amalgam; its two remaining relations are displayed in full.

The lower-case word m in this statement maps to the original meridian M only after the final quotient. The attaching word $\lambda = Axm$ is distinct from the section longitude $\ell = Ax$.

The geometric presentation and the final filling. Based transport through a punctured square requires a conjugated meridian correction; the corrections cannot be removed by retaining the unpunctured monodromy relations. We use a corner-excised square and compute all end inclusions in the drilled bundle. The resulting fifteen relations, including both meridian transports, are printed in Section 8.

The remaining question is whether the final filling kills the normally generating fiber commutator. Theorem 1.3 establishes the surface-group injection before that filling; its proof does not extend this injection to the final quotient. Perfectness alone does not decide whether the quotient is trivial.

Conventions. Paths are composed from left to right, diffeomorphisms from right to left, and $[g, h] = ghg^{-1}h^{-1}$. Normal closures are taken in the group indicated by the subscript. Homology has integral coefficients unless otherwise specified. An exotic copy of a manifold means a smooth manifold homeomorphic but not diffeomorphic to it; this paper makes such assertions only under the stated hypothesis $G = 1$.

How to read the paper. The geometric reduction is the organizing principle: simple connectivity of the specified piece would give all three homeomorphism conclusions. Part I proves the geometric inputs without assuming that condition. The extension of the boundary involution makes simple connectivity of the piece necessary as well as sufficient for the even double and the slicing comparison. It also yields restrictions on proper surfaces and a contractible singular orbit space.

Part II is independent of a conjectured answer for the group. It derives every relation, proves the intermediate surface injections, and identifies the last quotient. Readers interested primarily in the group can begin with Theorem 1.3, the full presentation in Section 8, and Section 20. Part III gives two geometric realizations of the universal central extension: a circle bundle and a quotient of the section-exterior group. Its central fiber offers a sufficient test for nontriviality, not a second equivalent test for triviality. The final discussion distinguishes these logical directions and explains the restrictions on several natural approaches.

The working extension in Sections 17, 18, and 19 develops the exact symplectic realization. Its principal construction is a nullhomologous Lagrangian torus whose Liouville class is a negative multiple of its Maslov class. Li's disk-existence theorem then gives a proposed obstruction to finite first Gutt–Hutchings capacity. A simply connected comparison contains a torus with the same area–Maslov relation, so this obstruction alone cannot detect the central meridian. A different construction chooses the auxiliary attaching knots so that their handle cores cap exact Lagrangian punctured tori. Cotangent neighborhoods of the resulting closed tori detect two classes in the kernel of symplectic-cohomology transfer. The discussion of stabilized handles keeps its analytic comparison hypotheses explicit. Finally, an elementary group-algebra criterion reformulates infinite order of the central element as a nonvanishing problem; that nonvanishing remains unproved.

Notation	Meaning
$V, Y = \partial V$	Fixed Luttinger filling and its outer boundary
Z, Z_f, W	Even double, odd regluing, and boundary quotient
F, H	Genus-two fiber and doubled section in Z
Π, G	Abstract surface group and $\pi_1(V)$
Q_2, K	Injective HNN extensions before the last quotient
Q_c, H_β	Meridian vertex group and beta-only filling group
$\mathcal{P}, \mathcal{Z}$	Circle bundle and auxiliary Liouville domain
$\ell = Ax, \lambda = Axm$	Section longitude and final attaching word
$J = [x, y]$	Normal generator of the complete group G
$\widehat{G}, t$	Universal central extension and its central generator
X, Q_X, μ	Section exterior, its group, and section meridian
$\mathcal{O}, \Lambda$	Involution orbit space and its regular-part group

The principal group maps are

$$\begin{aligned} \Pi &\hookrightarrow Q_c \hookrightarrow Q_2 \hookrightarrow K \twoheadrightarrow H_\beta \twoheadrightarrow G, \\ Q_X &\twoheadrightarrow \widehat{G} \twoheadrightarrow G, \quad \pi_1(\mathcal{P}) = \pi_1(\mathcal{Z}) = \widehat{G}. \end{aligned}$$

Here H_β is the group after the beta filling alone. The first line locates the last filling; the second locates the centralization of the section meridian. Only the arrows explicitly drawn as inclusions assert injectivity. The later notation is defined in detail when it is used. In particular X is a manifold and $\mathcal{X}$ is a relator; W identifies only boundary points, whereas $\mathcal{O}$ identifies all orbits of an involution on V .

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Part I. The marked construction and its geometric consequences

2. THE MARKED BUNDLE AND THE SURGERY SLOPES

2.1. The fiber and its monodromies. Let F be the quotient of a regular octagon with directed edge word

$$xyx^{-1}y^{-1}rsr^{-1}s^{-1}.$$

The vertices become a single point p_0 , and the center becomes O . Orient F so that the intersection numbers of x, y and of r, s are $+1$. Number the directed polygon edges $E_1, \dots, E_8$ in this order. The following chord endpoints specify the five-chain a, b, c, d, e . A parameter on an edge is measured from its initial vertex in the polygon boundary traversal.

Curve	Chord endpoints
a	$(E_2, 13/20), (E_4, 7/20)$
b	$(E_1, 13/20), (E_3, 7/20)$
c	$(E_2, 7/20), (E_8, 13/20)$ and $(E_6, 7/20), (E_4, 13/20)$
d	$(E_5, 13/20), (E_7, 7/20)$
e	$(E_6, 13/20), (E_8, 7/20)$

The half-turn ϕ fixes p_0, O and interchanges x, r and y, s as based loops. Choose positive Dehn twists supported away from these points. Their based actions are

$$T_a(x) = x, \quad T_a(y) = yx, \quad T_b(x) = xy^{-1}, \quad T_b(y) = y,$$

and they fix r, s . For example, the a -parallel at the y crossing is based by the y prefix and has word xyx^{-1} . The corresponding b -parallel has word xyx^{-1} . Inserting these parallels with their opposite transverse signs gives the displayed formulas. Hence

$$(1) \quad \phi(x, y, r, s) = (r, s, x, y), \quad \psi(x, y, r, s) = (y^{-1}, yx, r, s), \quad \psi = T_a T_b.$$

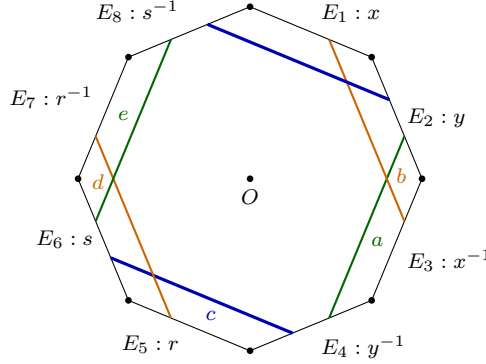


FIGURE 1. The marked fiber. All vertices are identified to p_0 ; paired end-points close the five curves. The half-turn exchanges a, e and b, d and preserves c . The endpoint table, rather than the scale of the drawing, specifies the curves.

2.2. The punctured-torus base. Remove equal quarter-disks of radius less than $1/8$ from the four corners of $[0, 1]^2$, with coordinates (v, w) . Glue $v = 1$ to $v = 0$ using ϕ , and $w = 1$ to $w = 0$ using ψ : explicitly $(z, 1, w) \sim (\phi z, 0, w)$ and $(z, v, 1) \sim (\psi z, v, 0)$. Smooth the boundary. This gives a genus-two bundle R over a once-punctured torus. There is no unremoved corner at which the two transition maps would have to commute. Indeed $\phi\psi(x) = s^{-1}$ whereas $\psi\phi(x) = r$.

At $(1/4, 1/4)$ lift the positive horizontal and vertical base loops a_0, b_0 using p_0 , and set $A = a_0^{-1}$, $B = b_0^{-1}$. The alpha torus T_α is the suspension of c over $w = 1/2$; the beta torus T_β is the suspension of e over $v = 1/2$. They are disjoint, since c and e are disjoint. An invariant fiber area form and a positive base area term make them Lagrangian. The fixed points p_0, O give two disjoint proper punctured-torus sections, denoted Σ, Σ' , avoiding the tori.

This is a marked version of the intrinsic bundle in [LP25, Section 1]. To identify the surface models, take a regular neighborhood of the ordered five-chain. It has four intersection vertices, eight edges, genus two, and two boundary components. Its complement consists of two disks. Match intersection disks and bands equivariantly, preserving cyclic orders. The involution is free on the chain neighborhood and has one fixed point in each complementary disk. The boundary identifications extend equivariantly over these disks by passing to their branched disk quotients. This identifies the ordered curves and p_0, O , and conjugates the two chosen monodromies. It therefore extends over the base disk and its two handles.

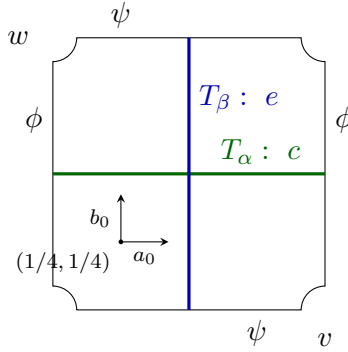


FIGURE 2. Projection to the corner-excised base square. The horizontal circle supports the c suspension and the vertical circle the e suspension. Their projections cross, but the tori are disjoint because $c \cap e = \emptyset$. The rounded arcs become the single boundary circle after opposite sides are identified. The arrows indicate the positive base directions; $A = a_0^{-1}$ and $B = b_0^{-1}$.

2.3. Peripheral paths and section choices. Write $y = y_1 y_2$, where y_1 ends at the c crossing, and write $s = s_1 s_c = s_2 s_3$, where s_1 ends at its c crossing and s_2 at the subsequent e crossing. The half-turn exchanges y_1, s_1 . The two half-arcs d_c, d'_c on c , from the s crossing to the y crossing, satisfy

$$(2) \quad s_1 d_c y_1^{-1} = r^{-1}, \quad s_1 d'_c y_1^{-1} = x.$$

These equalities can be checked directly in the polygon. If w_i is the boundary prefix to vertex i , with $w_1 = 1$, crossing from E_i to its paired edge E_j contributes $g_i = w_i w_{j+1}^{-1}$. The two products are $w_6^{-1} g_4 w_2 = r^{-1}$ and $w_6^{-1} g_6 w_2 = ([x, y][r, s])^{-1} x$.

For the alpha torus use y_1 followed by a short positive vertical interval as a common basing path for its meridian and both longitudes. For the beta torus use s_2 followed by a short positive horizontal interval. Let M, N be the positive puncture boundaries in the (y, b_0) and (s, a_0) transport squares, respectively. The based fiber longitudes, second alpha meridian, and section longitudes are

$$(3) \quad C = (rx)^{-1}, \quad E = sr^{-1}s^{-1}, \quad K_0 = r^{-1}Mr, \quad \ell = Ax, \quad D = BK_0^{-1}.$$

Here the alpha section travels negatively in the base and closes by d'_c . More explicitly, if a_c is positive transport from the s crossing to the y crossing in a rectangle just below the alpha torus, then $s_1 a_c y_1^{-1} = a_0$. Thus $y_1 a_c^{-1} d'_c y_1^{-1} = Ax$. Closing instead by d_c gives $\ell C = Ar^{-1}$; it changes the section by one fiber turn. The two alpha punctures have the same sign, since the half-turn preserves the oriented core and exchanges their tracks.

The rectangle comparing the beta section to b_0 crosses c once. For its positive section b_e it gives $s_2 b_e s_2^{-1} b_0^{-1} = K_0$. Inverting the section yields $D = BK_0^{-1}$. The terminal path s_3^{-1} gives a useful independent comparison: the entire beta triple rebased by s^{-1} is

$$(4) \quad n = s^{-1}Ns, \quad s^{-1}Es = r^{-1}, \quad s^{-1}Ds = B.$$

This terminal rectangle misses c and is fixed by ψ . Accordingly ND^{-1} becomes nB^{-1} , not NB^{-1} .

2.4. Cotangent charts and the fixed filling. Near each core choose area coordinates (θ, t) with form $dt \wedge d\theta$, with θ increasing along C or E . Let $\kappa > 0$ multiply the base area form. The following ordered cotangent coordinates satisfy $\omega = dQ_1 \wedge dY_1 + dQ_2 \wedge dY_2$:

$$(5) \quad \begin{array}{c|cc|cc} & Q_1 & Q_2 & Y_1 & Y_2 \\ \hline \alpha & \theta - v/2 & -v & -t & t/2 - \kappa(w - 1/2) \\ \beta & \theta & -w & -t & \kappa(v - 1/2). \end{array}$$

Direct substitution gives $dt \wedge d\theta + \kappa dv \wedge dw$. At alpha the half-turn is $\theta \mapsto \theta + 1/2$, so Q_1 descends. The section with Q_1 constant and v decreasing is the chosen half-drift section ℓ . At beta the section has θ constant and w decreasing. A push-off with constant momenta in these charts gives the Lagrangian framing and introduces no meridian turn in either section.

Definition 2.1. The positive meridians are the positively oriented boundaries of coordinate disks in the ordered (Y_1, Y_2) planes in (5). Use section coorientations $+C, +E$ and perform meridian-plus-section fillings along the chosen sections. The resulting compact manifold is $V = W(+1, -1; 0, 0)$.

The determinants from the puncture coordinates (t, w) and (t, v) to (Y_1, Y_2) are $+\kappa$ and $-\kappa$. The two positive coordinate meridians are therefore M and N^{-1} , and the slopes are

$$(6) \quad M\ell, \quad N^{-1}D.$$

Since the peripheral loops commute, the second word can be inverted to give ND^{-1} . In the coordinates of Auroux–Donaldson–Katzarkov [ADK03, Section 2.1], ordered as (section, coorientation), these are coefficient $+1$ surgeries with coorientations $+C, +E$. To check the sign, their coordinates are $(x_1, x_2, y_1, y_2) = (Q_2, Q_1, Y_2, Y_1)$. The interchange of the normal coordinates reverses the normal disk orientation. Their shear adds $k\chi(y_1)$ to x_1 on the top edge of the normal square; a positively oriented boundary traverses that edge with y_1 decreasing and acquires $-k$ section turns. With the opposite (Y_1, Y_2) disk orientation, this gives meridian plus k section turns. Here the gluing map runs from the inserted annulus to the exterior, so the killed slope is the image of its meridian; reversing the direction of this map reverses the coefficient convention. Taking $k = 1$ gives (6). Thus these are Luttinger surgeries and preserve the symplectic structure outside their neighborhoods.

The word “positive” in Definition 2.1 matters. The normal orientation induced by the ordered torus coordinates (Q_1, Q_2) is opposite to the coordinate orientation (Y_1, Y_2) , since

$$\omega^2 = -2 dQ_1 \wedge dQ_2 \wedge dY_1 \wedge dY_2.$$

Using the induced-normal meridians with the same numerical meridian-plus-section instruction would instead give $M^{-1}\ell$ and ND . This is a different specification of slopes.

Remark 2.2 (Changes of marking). A simultaneous conjugation of an entire peripheral triple conjugates its slope and preserves its normal closure. Inverting the whole slope also preserves the filling instruction. In contrast, if a new section coordinate is $\ell' = \ell C^n$, the old slope $M\ell^p$ becomes $M(\ell')^p C^{-np}$; dropping the last factor changes the section used for surgery. The general marked family has slopes $M(\ell C^n)^p$ and $N(DE^m)^q$. No equivalence between distinct members of that family is assumed here. All conclusions below concern the particular slopes (6).

3. THE BOUNDARY INVOLUTION AND THE SECTION FRAMINGS

Choose the based boundary loop d of the punctured torus by $[a_0, b_0]d = 1$. For a base loop u the fiber transport acts as $g \mapsto u^{-1}gu$, so transport reverses the order of path concatenation. The boundary monodromy is

$$(7) \quad \rho = \phi\psi^{-1}\phi\psi.$$

The half-turn conjugates T_a to T_e and T_b to T_d . Consequently

$$\psi\rho\psi^{-1} = T_a T_b T_d^{-1} T_e^{-1}.$$

This is the zero-surgery monodromy of the square knot in [LP25, Section 1]. Conjugating once more by T_e^{-1} gives

$$(8) \quad \chi = T_e^{-1} T_d^{-1}, \quad U = T_e^{-1} \psi, \quad U\rho U^{-1} = \delta = \psi\chi.$$

All these twists are the identity near p_0, O .

3.1. Compatible representatives. Use the reflection ϵ of the symmetric five-chain model in [LP25, Figure 2]. It reverses orientation, fixes each of p_0, O , commutes with ϕ , and exchanges a, e and b, d . Put $j = \phi\epsilon$. One concrete realization is the double of a pair of pants: b, d are the inner boundary circles, a, e are doubled seams from the outer circle, and c is the doubled seam between the inner circles. The map j exchanges the two halves. Thus it reverses a, c, e and fixes b, d pointwise.

Choose annular twist representatives satisfying $jT_a j = T_a^{-1}$ and $jT_b j = T_b^{-1}$, and define the right-hand twists by ϕ conjugacy. These choices can be made simultaneously. In annular coordinates an orientation-reversing involution is of the form $(\theta, t) \mapsto (-\theta + c, t)$ or $(\theta, t) \mapsto (\theta + c, -t)$. A twist $(\theta, t) \mapsto (\theta + h(t), t)$ is reversed in the first case for any profile, and in the second if $h(-t) = 1 - h(t)$. Integer end values make it the identity off the collar. Choose the T_a profile zero on a smaller core collar, and make the same choice for T_e by conjugacy. These choices give identities of maps

$$(9) \quad \epsilon^2 = 1, \quad \epsilon\psi\epsilon = \chi, \quad \epsilon\chi\epsilon = \psi.$$

They preserve both half-turn conjugacies and the fixed neighborhoods of the two section points. Average a metric over the finite group generated by ϕ, ϵ and use its area form to choose these annular coordinates and twists. The orientation-preserving finite symmetries and the twists preserve area, while the reflections reverse it. No metric invariant under the Dehn twists is required.

3.2. The boundary map with both seams retained. Realize the δ mapping torus by two pieces $F \times [0, 1] \times \{0, 1\}$, glued by

$$(z, 1, 0) \sim (\chi z, 0, 1), \quad (z, 1, 1) \sim (\psi z, 0, 0).$$

The map $(z, u, i) \mapsto (\epsilon z, u, 1 - i)$ respects both seams by (9). It is a free involution because it covers a half-rotation of the base circle; it reverses the fiber orientation and preserves each section circle.

For a formula in a single mapping torus, write $Y_\delta = (F \times [0, 2]) / ((z, 2) \sim (\delta z, 0))$. The two pieces map to $[z, u]$ and $[\chi^{-1}z, 1 + u]$. On $F \times \mathbb{R}$ the lifted boundary map is

$$(z, t) \mapsto (\chi^{-1}\epsilon z, t + 1).$$

Its square is $(z, t) \mapsto (\delta^{-1}z, t + 2)$, the deck transformation, since $(\chi^{-1}\epsilon)^2 = \delta^{-1}$. Transport through $F_U : Y_\rho \rightarrow Y_\delta$, $[z, t] \mapsto [Uz, t]$, to define the boundary involution σ on $Y = Y_\rho$. Its lifted formula is

$$(10) \quad (z, t) \mapsto (f_\rho z, t + 1), \quad f_\rho = U^{-1}\chi^{-1}\epsilon U, \quad f_\rho^2 = \rho^{-1}.$$

The factor χ^{-1} is part of the formula; reflection alone in a one-seam model would not describe this involution.

Lemma 3.1. *The parallel normal framing along the boundary of either proper section extends over that section. In these framings $D\sigma$ on the normal planes is a constant reflection. Each doubled section in Z has self-intersection zero.*

Proof. The vertical normal bundle of a section is a flat oriented plane bundle with holonomies $D\phi = -I$ and $D\psi = I$. Its boundary holonomy is I . Cap the section by a flat disk using the parallel boundary frame. The resulting flat plane bundle on a torus has Euler number zero: the holonomies $-I, I$ are realized on the trivial bundle by the flat connection $\pi J_0 du$, where J_0 is the infinitesimal rotation and u is the first circle coordinate. Thus the original relative Euler number is zero and the frame extends.

The maps χ, ψ, U are the identity near the section points. Equation (10) therefore has normal derivative $D\epsilon$, the same constant reflection on both halves of the boundary circle. To form the oriented double, reverse both the fiber and base orientations on its second copy. The four-dimensional orientation is unchanged. Identifying its normal plane by a fixed reflection makes the clutching map a constant element of $GL^+(2, \mathbb{R})$. Its winding is zero. The relative Euler numbers on both sections are zero as well, so the doubled normal Euler number is zero. Surgery takes place away from the sections and does not change this calculation. $\square$

4. EXTENSION OVER THE FILLING AND THE GROUP OF THE DOUBLE

Proposition 4.1. *The map σ extends to an orientation-reversing involution $I : V \rightarrow V$. The retraction and fundamental-group assertions of Theorem 1.1(4) follow.*

Proof. Use the representatives just constructed and put $\eta = T_a T_e j$. Twists on opposite handles commute; j inverts each twist; and ϕ exchanges T_a, T_e . Hence

$$(11) \quad \eta^2 = 1, \quad \eta\phi = \phi\eta, \quad \eta\psi\eta^{-1} = \psi^{-1}.$$

On the corner-excised bundle define

$$(12) \quad (z, v, w) \mapsto (\eta z, 1 - v, 1 - w).$$

The identities $\phi\eta\phi = \eta$ and $\psi\eta\psi = \eta$ give compatibility at the two seams. Equal corner excisions and equivariant smoothing make this a smooth involution of R . It reverses four-dimensional orientation and preserves both surgery tori. Near c, e , the map η agrees with j : the supports of T_a, T_e miss c , and the T_e profile is zero near the core e . No disjointness from c is asserted for the supports of T_b, T_d .

In suitable area coordinates on these collars, $\eta(\theta, t) = (-\theta + c_i, t)$. At alpha, substitution in (5) gives

$$Q'_1 = -Q_1 + c_\alpha - \frac{1}{2}, \quad Q'_2 = -Q_2 - 1, \quad \begin{pmatrix} Y'_1 \\ Y'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}.$$

At beta the corresponding formula is

$$Q'_1 = -Q_1 + c_\beta, \quad Q'_2 = -Q_2 - 1, \quad (Y'_1, Y'_2) = (Y_1, -Y_2).$$

Both torus directions reverse, and each constant normal matrix has determinant -1 . There is no longitude-dependent framing turn. With a common basing path transported for each triple, the induced peripheral maps are therefore

$$(M, C, \ell) \mapsto (M^{-1}, C^{-1}, \ell^{-1}), \quad (N, E, D) \mapsto (N^{-1}, E^{-1}, D^{-1}).$$

We next extend the involution itself, including its order-two property. Use invariant elliptical normal disks for the two constant normal involutions above, and choose the angular origin on a fixed axis. In longitude and meridian coordinates the boundary action is then

$$(q_1, q_2, \theta) \mapsto (-q_1 + b_1, -q_2 + b_2, -\theta), \quad q_1, q_2, \theta \in \mathbb{R}/\mathbb{Z}.$$

The ordered peripheral bases are (C, ℓ, M) and (E, D, N^{-1}) . In either basis the selected slope is meridian plus second longitude. Use the linear gluing map

$$S(q_1, q_2, \theta) = (q_1, q_2 + \theta, \theta).$$

It fixes the old longitudes and sends the inserted meridian to the specified slope. It commutes with the displayed affine involution. Consequently the cap map

$$(q_1, q_2, z) \mapsto (-q_1 + b_1, -q_2 + b_2, \bar{z}) \quad \text{on } T^2 \times D^2$$

glues to (12) and squares to the identity.

This linear filling is diffeomorphic, with the outer boundary fixed, to the specified symplectic filling. Indeed the boundary shear of [ADK03, Section 2.1] has the form $(q_1, q_2, \theta) \mapsto (q_1, q_2 + h(\theta), \theta)$, where $h : S^1 \rightarrow S^1$ has degree one. Interpolating a lift of h to θ gives an isotopy of gluing maps to S , preserving the longitude classes and the meridian image. Changing the shape of the normal disk gives the same marked isotopy. Extend this isotopy over an exterior collar, and conjugate the involution of the linear filling by the resulting diffeomorphism. Conjugation preserves its square, so we obtain an involution on the specified filling, not merely an extension of its peripheral homology action.

It remains to identify its outer boundary restriction with the specified σ . Start at the south end of the rounded corner $(1, 1)$. In the induced boundary direction the corners occur in the order $(1, 1), (1, 0), (0, 0), (0, 1)$, with seam crossings $\psi, \phi, \psi^{-1}, \phi^{-1}$. The total transport is ρ ; the first half transports by $\phi\psi$. The base rotation in (12) shifts two corner arcs, so its fiber factor in the continuous boundary marking is $(\phi\psi)^{-1}\eta$. This equals the factor in (10), since

$$\begin{aligned} U^{-1}\chi^{-1}\epsilon U &= \psi^{-1}T_e\chi^{-1}\epsilon T_e^{-1}\psi \\ &= \psi^{-1}T_e\chi^{-1}T_a\chi\epsilon = \psi^{-1}T_aT_e\epsilon = \psi^{-1}\phi\eta. \end{aligned}$$

Here $\epsilon T_e^{-1}\epsilon = T_a$ and T_a commutes with the right-hand twists. Thus $I|_Y = \sigma$ for the chosen representatives.

For completeness, a change from another symmetric T_a profile can be made without changing the surgery marking. Keep T_b fixed and use $\psi_u = T_a(u)T_b$. The target-side isotopy $\psi_u\psi_0^{-1} = T_a(u)T_a(0)^{-1}$ is supported near a . Insert it near the vertical seam, away from $w = 1/2$; its fiber support misses e and the section points. The resulting bundle identification fixes both surgery core collars. The T_e change belongs to the boundary marking, and its induced boundary isotopy extends over an outer collar. Hence the extension statement is unchanged by these representative choices.

Now fold $Z = V_1 \cup_\sigma V_2$ onto V_1 by the identity on V_1 and I^{-1} on V_2 . On a seam point $b \sim \sigma(b)$ the two values agree. This is a retraction, so G injects into $\pi_1(Z)$. Changing the second marking by I^{-1} makes the two boundary homomorphisms identical. Their van Kampen pushout factors through their common image H_∂ and is $G *_{H_\partial} G$. Both amalgamating maps are inclusions by the definition of H_∂ . It follows that $\pi_1(Z) = 1$ exactly when $G = 1$. $\square$

5. HOMOLOGY, SYMPLECTIC SURFACES, AND THE CLOSED QUOTIENTS

5.1. Integral homology of the piece and double. We use the complete presentation proved in Proposition 8.1; its derivation is independent of the present section. We label a transport relation by its stable letter and its underlying fiber generator: for example, (Bs) means $B(K_0^{-1}s)B^{-1} = s$, and (By) means $B(M^{-1}y)B^{-1} = yx$. These are labels for equations, not group words. In the abelianization, (Bs) gives $M = 0$, and (By) gives $x = -M = 0$. The Bx relation gives $y = -x = 0$. The Ax, Ay relations then give $r = s = 0$, and the As relation gives $N = 0$. The two filling relations finally give $A = B = 0$. These are integral eliminations with unit coefficients, proving $H_1(V) = 0$.

The original bundle has Euler characteristic $\chi(F)\chi(\Sigma_{1,1}) = 2$, unchanged by torus surgery. The boundary is connected. In the pair sequence for (V, Y) , the boundary map $H_4(V, Y) \rightarrow H_3(Y)$ is an isomorphism and $H_3(V, Y) = H^1(V) = 0$, so $H_3(V) = 0$. The same sequence gives $H_1(V, Y) = 0$, hence $H^3(V) = 0$. Universal coefficients imply that $H_2(V)$ is torsion free, and Euler characteristic gives its rank as one. A surviving fiber meets Σ once, so it generates $H_2(V)$ primitively, and Σ generates $H_2(V, Y)$. Since $F^2 = 0$, the map $H_2(V) \rightarrow H_2(V, Y)$ is zero. Thus

$$(13) \quad H_2(Y) \xrightarrow{\cong} H_2(V), \quad H_2(V, Y) \xrightarrow{\partial \cong} H_1(Y).$$

Both boundary groups are $\mathbb{Z}$, generated by the fiber and the knot meridian, respectively. The mod-two second homology of V is generated by F , and $w_2(V)$ evaluates on it as $F^2 = 0$. Hence V is spin and, since $H^1(V; \mathbb{F}_2) = 0$, has a unique spin structure.

Mayer–Vietoris gives $H_1(Z) = 0$ and $\chi(Z) = 4$. Integral duality then gives $H_2(Z) = \mathbb{Z}^2$. The fiber and either doubled section have genus two, and Lemma 3.1 gives their matrix

$$(14) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its determinant has absolute value one, so these classes form a basis. This proves the cohomology-ring assertion and shows that Z is spin.

5.2. Symplectic representatives and minimum genus. Before surgery, $R \cup_\sigma R$ is an oriented genus-two surface bundle over a genus-two surface. As in [LP25, Theorem 8 and footnote 3], reverse both fiber and base orientations on the second copy to obtain the oriented bundle description. A fiber area form with a sufficiently large base area term gives a symplectic form [Thu76]. The fixed sections are symplectic, the suspension tori are Lagrangian, and a fiber away from the surgery circles is symplectic. The local Luttinger surgeries on both copies preserve the form near that fiber and the sections. Thus F, H are symplectic in Z .

Proposition 5.1. *If $S = F$ or H and $k \neq 0$, the minimum genus of $k[S]$ in Z is $|k| + 1$. These are all the nonzero square-zero classes.*

Proof. A symplectic tubular neighborhood of S can be identified with $S \times D^2$ with product symplectic form, since its normal Euler number is zero. Choose a smooth map $h : S \rightarrow S^1$ inducing a surjection on fundamental groups. For $k > 0$ and sufficiently small $\varepsilon > 0$, define

$$S_k = \{(u, z) \in S \times D^2 : z^k = \varepsilon h(u)\}.$$

It is a connected unbranched k -fold cover of S : the monodromy cyclically permutes all roots. It represents $k[S]$ and has genus $k + 1$. Since $|z|$ is constant, the disk area form restricts to zero, whereas the surface area form is positive. Thus S_k is symplectic. The symplectic Thom theorem [OS00, Theorem 1.1] makes it genus minimizing. Orientation reversal gives the assertion for negative k . Finally $(a[F] + b[H])^2 = 2ab$, proving the description of all square-zero classes. $\square$

In particular Z has no homologically nonzero square-zero sphere or torus. The essential square-zero spheres in $S^2 \times S^2$ exclude a diffeomorphism with Z . This argument uses no simple-connectivity hypothesis.

5.3. Proper surfaces and handles in the filling. The minimum-genus calculation also constrains the piece V itself. This consequence is useful when trying to turn a formal group identity into a smooth handle cancellation.

Proposition 5.2. *Let $S \subset V$ be a compact connected oriented properly embedded surface of genus g with $b \geq 1$ boundary components. If $m = [S] \cdot [F] \neq 0$, then*

$$2g + b - 2 \geq |m|.$$

Every proper oriented disk or annulus represents zero in $H_2(V, Y; \mathbb{Z})$. The inequality is sharp for either surviving punctured-torus section. Moreover, V admits no smooth handle decomposition consisting only of handles of indices 0, 1, 2.

Proof. The extension I identifies the ordinary oriented double $D(V) = V \cup_{\text{id}} (-V)$ with Z , preserving its first copy. Straighten S in a boundary collar and double it. The resulting oriented surface DS has genus $2g + b - 1$. Its normal bundle has a nonzero section: the normal bundle on S is trivial because $H^2(S; \mathbb{Z}) = 0$, and a section constant along the collar glues to its reflection on the other half. Thus $[DS]^2 = 0$. Keep F in the first copy of V . Then $[DS] \cdot [F] = m$. In the hyperbolic basis F, H , these two equations force $[DS] = m[H]$ when $m \neq 0$. Proposition 5.1 gives $2g + b - 1 \geq |m| + 1$.

Duality identifies $H_2(V, Y; \mathbb{Z})$ with $\text{Hom}(H_2(V; \mathbb{Z}), \mathbb{Z})$, so intersection with F detects its classes. The inequality excludes a nonzero class for a disk or annulus. A section has $(g, b, |m|) = (1, 1, 1)$ and realizes equality.

If a handle decomposition used only indices 0, 1, 2, its 2-handle cocores would be proper disks. Turn the decomposition upside down: the relative handles have indices 2, 3, 4, so the cocores generate $H_2(V, Y) = C_2(V, Y)/\text{im } \partial_3$. All their classes would vanish, contradicting $H_2(V, Y) = \mathbb{Z}$. $\square$

This proposition concerns embedded surfaces. A nullhomotopy of J is a mapped disk and need not be embedded. Resolving its intersections can increase its genus. Thus the proposition neither excludes such a nullhomotopy nor contradicts hypothetical simple connectivity of V . Also $H_2(V) \rightarrow H_2(V, Y)$ is zero, so adding an absolute spherical class does not alter the relative intersection number m .

5.4. The boundary quotient and sliceness. The double cover gives an exact sequence

$$1 \longrightarrow \pi_1(Z) \longrightarrow \pi_1(W) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

Since $H_1(Z) = 0$, its fundamental group is perfect, and its image vanishes in any abelian quotient of $\pi_1(W)$. Hence $H_1(W) = \mathbb{Z}/2$. Also $\chi(W) = 2$. Duality shows that W is a rational homology sphere, with integral groups $H_2(W) = \mathbb{Z}/2$ and $H_3(W) = 0$, and $\dim H_2(W; \mathbb{F}_2) = 2$.

To check spin, consider the image S of Σ in W . Its boundary is identified by a free orientation-preserving half-rotation, so S is a closed nonorientable surface. In the extending normal frame of Lemma 3.1, choose a nonzero vector in the $+1$ eigenspace of the constant reflection $D\epsilon$. Extend this vector using the frame over Σ . It matches at the seam and gives a nowhere-zero normal field on S . Thus $S^2 = 0$ over $\mathbb{F}_2$. The image $\bar{F}$ of an interior fiber also has square zero and meets S once. They form a hyperbolic basis of $H_2(W; \mathbb{F}_2)$, and the Wu formula gives $w_2(W) = 0$. The framing calculation, rather than nonorientability alone, is what establishes the square of S .

Suppose that the figure-eight knot 4_1 bounded a smooth disk D_0 in punctured W . If its relative integral class were zero, its mod-two class would be zero, so the disk would be characteristic in this spin manifold. The relative Rochlin formula [Klu21, Theorem 2], with signature zero, self-intersection zero, and boundary S^3 , would give $\text{Arf}(4_1) = 0$, contrary to $\text{Arf}(4_1) = 1$. This formula does not require the first homology of the four-manifold to vanish.

If the disk class were nonzero, its neighborhood together with the puncture ball would give an embedding of the zero trace $X_0(4_1)$ in W inducing a nonzero map on H_2 . The framing is zero because the integral intersection pairing of W vanishes. The trace is simply connected, so the embedding lifts to Z . A core capped by a genus-one Seifert surface gives a square-zero torus in the lifted trace. Its homology class is nonzero, since its push-forward to W is nonzero, even though the latter class is torsion. This contradicts Proposition 5.1. Thus 4_1 is not slice in W . This is the slicing obstruction of [LP25], with its normal framing and homological hypotheses made explicit for our filling.

5.5. The odd boundary regluing. We use two specific inputs from [LP25, Section 3]. First, the boundary map f_0 in their Figure 3 sends the framed meridian $(\mu, 0)$ to $(\mu', 1)$, where $(\mu', 0)$ is framed homologous to $(\mu, 0)$; framings are measured relative to the Seifert framing in that diagram. Transport this diffeomorphism to Y by the boundary identification of Section 3, so $f = F_U^{-1} f_0 F_U$ in those markings. This is orientation preserving, as is f_0 ; conjugation by the marking also preserves its permutation of the two spin structures. The framing property implies that f interchanges the two spin structures on Y : a single framing twist changes the induced spin structure on the meridian, which generates $H_1(Y; \mathbb{F}_2)$.

The unique spin structures on the two copies of V match under σ , because Z is spin. They therefore fail to match under $f \circ \sigma$. The resulting Z_f is non-spin. As above, $H_1(Z_f) = 0$, $\chi(Z_f) = 4$, and $H_2(Z_f) = \mathbb{Z}^2$. Novikov additivity gives signature zero, since the form of V has zero signature. The unimodular rank-two form is consequently the odd form $\langle 1 \rangle \oplus \langle -1 \rangle$.

Second, we use the relative Heegaard Floer invariants and their pairing from [LP25, Lemmas 9–10], with coefficients $\mathbb{F}_2$. On Y the two structures $\mathfrak{s}_\pm$ with $\langle c_1(\mathfrak{s}_\pm), [F] \rangle = \pm 2$ satisfy $HF_{\text{red}}(Y, \mathfrak{s}_\pm) = \mathbb{F}_2$. The required relative nonvanishing needs a separate argument: the symplectic double Z has $b^+ = 1$, whereas the canonical nonvanishing theorem used below requires $b^+ > 1$.

Lemma 5.3. *The relative invariants of V in the extensions of $\mathfrak{s}_\pm$ are nonzero. This assertion does not require $G = 1$.*

Proof. Write $Z = V_1 \cup_\sigma V_2$ and choose a square-zero symplectic genus-two fiber in the interior of V_1 , disjoint from the surgery neighborhoods. Form a symplectic sum of two copies of Z along these fibers [Gom95]. Denote the result by Z_* , retaining the untouched V_2 from the first copy together with a collar of its boundary Y . The sum formulas give

$$\chi(Z_*) = 2\chi(Z) - 2\chi(F) = 12, \quad \sigma(Z_*) = 0, \quad b^+(Z_*) = 5 + b_1(Z_*) > 1.$$

For its canonical Spin^c structure $\mathfrak{k}$, the canonical nonvanishing theorem [OS04, Theorem 1.1] gives $\Phi_{Z_*, \mathfrak{k}}(1) = 1$ over $\mathbb{F}_2$.

Let $F_Y \subset Y$ be a fiber oriented by the symplectic form. Adjunction gives $\langle c_1(\mathfrak{k}), [F_Y] \rangle = -2$. Its restriction to Y is therefore non-torsion and is one of $\mathfrak{s}_\pm$ in the original marking. Remove a ball from V_2 and one from its complement to obtain a cobordism $W = W_1 \cup_Y W_2$ from S^3 to S^3 , with $W_1 = V_2 \setminus B^4$. Put $\mathfrak{k}_i = \mathfrak{k}|_{W_i}$. For a non-torsion restriction on the cut, the composition formula [OS04, Section 2] is

$$(15) \quad F_{W_2, \mathfrak{k}_2}^+ \Pi_Y^{\text{red}} F_{W_1, \mathfrak{k}_1}^-(\Theta^-) = \sum_{\substack{\mathfrak{s}|_{W_i} = \mathfrak{k}_i \\ i=1,2}} F_{W, \mathfrak{s}}^{\text{mix}}(\Theta^-).$$

Here $\Theta^\pm$ are the standard generators for S^3 , and Π_Y^{red} is the canonical map through reduced Floer homology. This formula requires $b^+(W) > 1$, but does not require positive b^+ on both sides of this non-torsion cut.

We check that the coefficient of Θ^+ on the right cannot cancel. By Mayer–Vietoris the extensions in the sum differ by the image of $H^1(Y; \mathbb{Z}) = \mathbb{Z}$. Its generator maps to $\text{PD}_{Z_*}[F_Y]$. The latter has infinite order, since F_Y has positive symplectic area. Thus the extensions are $\mathfrak{k} + n \text{PD}[F_Y]$, $n \in \mathbb{Z}$. Writing

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(Z_*) - 3\sigma(Z_*)}{4},$$

we have $d(\mathfrak{k}) = 0$ and, since $F_Y^2 = 0$,

$$d(\mathfrak{k} + n \text{PD}[F_Y]) = -2n.$$

The grading of the mixed map therefore permits a Θ^+ coefficient only for $n = 0$. That coefficient is $\Phi_{Z_*, \mathfrak{k}}(1) = 1$. Hence the left side of (15) is nonzero, and in particular the relative invariant $\Pi_Y^{\text{red}} F_{W_1, \mathfrak{k}_1}^-(\Theta^-)$ is nonzero. Conjugation symmetry [OS06, Theorem 3.6] gives the other boundary structure. Finally $H^2(V; \mathbb{Z}) = \mathbb{Z}$, with restriction an isomorphism to $H^2(Y; \mathbb{Z})$, so each of these boundary structures has a unique extension to V . $\square$

Every self-diffeomorphism of Y permutes $\mathfrak{s}_+, \mathfrak{s}_-$. The one-dimensional pairing is nonzero on nonzero elements, so compatible extensions on the two copies give a nonzero mixed invariant of $(Z_f, \mathcal{L}, \mathfrak{t})$ for some $\mathfrak{t}$, where $\mathcal{L} \subset H_2(Z_f; \mathbb{Q})$ is the line spanned by the boundary fiber. This line is nonzero: by (13) the Mayer–Vietoris map into the two copies of $H_2(V; \mathbb{Q})$ is diagonal up to signs, and either fiber survives in its cokernel.

The line must be retained when $b^+ = 1$. In $\mathbb{CP}^2 \# \mathbb{CP}^2$ the only rational square-zero lines are spanned by $h + e$ and $h - e$, where $h^2 = 1$, $e^2 = -1$, and $h \cdot e = 0$. Both have embedded sphere representatives. Their tubular boundaries are separating copies of $S^2 \times S^1$ realizing these lines. The reduced Floer groups vanish in the relevant non-torsion structures, so the corresponding mixed invariants vanish for either line. This is the comparison in [LP25,

proof of Theorem 2]. It excludes a diffeomorphism with Z_f without a hypothesis on its fundamental group.

5.6. The conditional classifications.

Proof of Corollary 1.2. If $G = 1$, van Kampen gives $\pi_1(Z) = \pi_1(Z_f) = 1$. Freedman's classification [Fre82], applied to their respective forms and their vanishing Kirby–Siebenmann invariants, gives the asserted homeomorphisms. The preceding smooth obstructions distinguish both from their standard models.

If $\pi_1(Z) = 1$, the covering sequence gives $\pi_1(W) = \mathbb{Z}/2$. The manifolds W and $\mathcal{B}$ then have the same finite cyclic fundamental group, zero intersection form on $H_2/\text{torsion}$, spin w_2 -type, and Kirby–Siebenmann invariant zero. Hambleton–Kreck's classification [HK93, Theorem C] gives a homeomorphism. That theorem has no lower bound on Euler characteristic. Conversely, if W is homeomorphic to $\mathcal{B}$, its connected double cover corresponds to the trivial subgroup of $\mathbb{Z}/2$, so $\pi_1(Z) = 1$. Proposition 4.1 identifies this condition with $G = 1$. The figure-eight knot is slice in $\mathcal{B}$ and not in W , so they are not diffeomorphic. $\square$

Together these arguments prove Theorem 1.1. They leave the hypothesis in Corollary 1.2 unresolved.

6. THE INVOLUTION ORBIT SPACE

There is a second quotient associated to the boundary extension. Choose the involutive representatives and product filling extensions used in Proposition 4.1, and define

$$\mathcal{F} = \text{Fix}(I), \quad \mathcal{O} = V/\langle I \rangle, \quad \mathcal{O}^{\text{reg}} = \mathcal{O} \setminus p(\mathcal{F}).$$

Here p is the orbit projection. Unlike W , this quotient identifies interior points as well. It is a singular space, and its regular part is the complement of its singular stratum, with the outer boundary retained.

Proposition 6.1 (The orbit space and its regular part). *The fixed set $\mathcal{F}$ is one interior circle, and $\mathcal{O}$ is contractible. If $\Lambda = \pi_1(\mathcal{O}^{\text{reg}})$, there is a split exact sequence*

$$(16) \quad 1 \longrightarrow G \longrightarrow \Lambda \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

The image ι of the nontrivial loop in a normal $\mathbb{RP}^2$ link has order two and normally generates Λ . Furthermore $[\Lambda, \Lambda] = G$. In particular, $G = 1$ if and only if $\Lambda = \mathbb{Z}/2$.

Proof. The base half-turn has three fixed points on the punctured torus: the fourth fixed point of the closed torus was removed at the puncture. The section through p_0 is invariant and contains a fixed point over each. On a fixed fiber the return map reverses orientation, so its nonempty fixed set is one-dimensional. In a filling cap the action inverts both torus coordinates and reflects the disk. Its fixed set is four intervals. These join the exterior fixed set across the equivariant collars. Since the boundary action is free, $\mathcal{F}$ is a nonempty union of k interior circles.

The section quotient is an oriented surface with one boundary component and three simple branch points. Its Euler characteristic satisfies $-1 = 2\chi - 3$, so it is a disk. The two section generators A, B therefore vanish under p_* . This map is onto on fundamental groups: use an admissible equivariant triangulation [Ill78], base at a fixed vertex, and lift an edge loop one edge at a time, translating a lift by I when necessary. Its final vertex must be the starting vertex, since that vertex has a singleton orbit. In the presentation of Proposition 8.1, imposing $A = B = 1$ gives $r = x$, $s = y$, then $K_0 = M = 1$ from the Bs

row, $x = 1$ from the alpha slope, $y = 1$ from Bx , and $N = 1$ from the beta slope. Thus A, B normally generate G , and $\pi_1(\mathcal{O}) = 1$.

Write $S_I = p(\mathcal{F})$. With coefficients $\mathbb{F}_2$, the admissible triangulation gives an exact sequence of chain complexes

$$0 \longrightarrow C_*(\mathcal{O}, S_I) \xrightarrow{\text{tr}} C_*(V) \xrightarrow{p_*} C_*(\mathcal{O}) \longrightarrow 0.$$

The transfer sends a nonfixed cell to the sum of its two lifts; fixed faces cancel in their boundary. On fixed cells p_* is the identity. Since S_I is a union of circles, the pair sequence identifies $H_j(\mathcal{O}, S_I)$ with $H_j(\mathcal{O})$ for $j \geq 3$. Using these identifications and the dimension bound, the associated transfer homology sequence first gives $H_4(\mathcal{O}) = H_3(\mathcal{O}) = 0$ and then an injection $H_2(\mathcal{O}, S_I) \hookrightarrow H_2(V) = \mathbb{F}_2$. The pair sequence, using simple connectivity, gives

$$0 \longrightarrow H_2(\mathcal{O}) \longrightarrow H_2(\mathcal{O}, S_I) \longrightarrow \mathbb{F}_2^k \longrightarrow 0.$$

Consequently $k = 1$ and $\mathcal{O}$ is acyclic over $\mathbb{F}_2$.

The involution acts by -1 on $H_2(V; \mathbb{Z})$: this follows from fiber-orientation reversal on the boundary and the isomorphism $H_2(Y) \rightarrow H_2(V)$. Over a field of characteristic different from two, the orbit chain complex is the coinvariant complex, and taking coinvariants is exact by averaging. The only positive-dimensional homology class of V is negated, so $\mathcal{O}$ is acyclic over every such field as well. Finite generation and universal coefficients give integral acyclicity. Hurewicz and Whitehead now give contractibility.

Removing a smooth circle from a four-manifold does not change its fundamental group: general position moves loops and their homotopies off the circle. The free double cover $V \setminus \mathcal{F} \rightarrow \mathcal{O}^{\text{reg}}$ gives (16). The oriented rank-three normal bundle of the fixed circle is trivial, and I acts by -1 on each normal fiber. The quotient neighborhood is therefore $S^1 \times \text{cone}(\mathbb{RP}^2)$. The nontrivial normal-link loop has order exactly two in Λ : its square dies in the link, whereas its covering monodromy is nontrivial in $S^2 \rightarrow \mathbb{RP}^2$. It supplies the splitting. Reattaching the singular neighborhood kills precisely this loop in van Kampen, giving $\pi_1(\mathcal{O}) = \Lambda / \langle\langle \iota \rangle\rangle$. Finally G is perfect and Λ/G is abelian, whence $[\Lambda, \Lambda] = G$. $\square$

Contractibility of $\mathcal{O}$ does not compute Λ . A nullhomotopy downstairs can meet the singular circle, where the transverse model is $\text{cone}(\mathbb{RP}^2)$ rather than $\mathbb{R}^3$. The general-position argument upstairs therefore does not apply downstairs. Nor does normal generation by an involution make a group cyclic. The remaining question in this description is whether $\Lambda = \langle \iota \rangle$.

7. OTHER UNIT FILLINGS AND CHANGES OF DESCRIPTION

The fixed target is not canonical among the possible parametrizations. It is therefore useful to separate a different description of V from a different filling with the same elementary invariants.

Proposition 7.1. *For $p, q \in \{1, -1\}$ let $V_{p,q} = W(p, q; 0, 0)$, with the same cotangent charts and outer boundary marking. Each $V_{p,q}$ is a spin integral homology S^2 . Its double has the symplectic hyperbolic basis and minimum-genus formula of Theorem 1.1(1). The spin boundary quotient, slicing obstruction, odd-regluing construction, and conditional classification arguments apply to each filling. The boundary involution extends over each of them, so its double is simply connected exactly when the piece is. No one of these four groups is decided here.*

Proof. The complement relations are unchanged. In abelianization they kill M, N, x, y, r, s , and the filling rows impose $pA = qB = 0$. More generally, for integral p, q and these fixed sections,

$$H_1(W(p, q; 0, 0); \mathbb{Z}) = \mathbb{Z}/p \oplus \mathbb{Z}/q, \quad \mathbb{Z}/0 = \mathbb{Z}.$$

For unit coefficients this is zero. The homology, spin, and boundary homology calculations of Section 5 then apply. In the cotangent convention the Luttinger coefficients are $p, -q$; both signs are allowed in the local construction [ADK03]. The fiber and sections miss the surgery neighborhoods, so their symplectic representatives and framings persist. The same double therefore has the hyperbolic basis and the same minimum-genus argument. The quotient spin calculation uses the unchanged section framing. The auxiliary symplectic sum and grading argument of Lemma 5.3 apply to each $V_{p,q}$: its symplectic double contains the same untouched fiber and copy of $V_{p,q}$, and the boundary and cohomology calculations are unchanged. Thus the relative nonvanishing needed for the odd regluing remains valid. Finally the peripheral action of I is $-\text{id}$ on each whole triple, so it preserves every chosen unoriented slope. The cap construction of Proposition 4.1, with $S(q_1, q_2, \theta) = (q_1, q_2 + k\theta, \theta)$ for the relevant $k = \pm 1$, extends it as an involution over every such filling. $\square$

7.1. A reflection and the section index. There is a useful reflection of the marked exterior. Its peripheral dictionary, with a common conjugator for each triple, is

$$(17) \quad \begin{aligned} (M, C, \ell) &\mapsto A(M^{-1}, C^{-1}, \ell C)A^{-1}, \\ (N, E, D) &\mapsto w(N, E^{-1}, D^{-1})w^{-1}, \quad w = M^{-1}r. \end{aligned}$$

Here a parenthesized triple means that each entry is conjugated. For an algebraic check, the following formulas define an involution of the complement group (using $[M, C] = 1$, not of the free group):

$$\begin{array}{c|cccccccc} g & x & y & r & s & M & N & A & B \\ \hline g' & x^{-1} & xM^{-1}y & r^{-1} & M^{-1}rs & xM^{-1}x^{-1} & M^{-1}rNr^{-1}M & A & B^{-1}. \end{array}$$

The geometric reflection has the same peripheral action. With the representatives of Section 4, it is induced by $(z, v, w) \mapsto (\eta z, v, 1 - w)$. At alpha it reverses the fiber direction and preserves the base direction; the half-drift section acquires one fiber winding. Thus the longitude becomes ℓC and the normal meridian is reversed. At beta both torus directions reverse and the normal meridian is preserved. More explicitly, in the alpha coordinates $\Theta = \theta - v/2$, $P_1 = t$, $P_2 = t/2 - \kappa(w - 1/2)$, the map is

$$(\Theta, v, P_1, P_2) \mapsto (-\Theta - v + \theta_0, v, P_1, P_1 - P_2).$$

The increments $(1, 0)$ of C become $(-1, 0)$; those $(0, -1)$ of ℓ become $(1, -1)$. At beta the angular coordinates change by $(\theta, w) \mapsto (-\theta + \theta_1, 1 - w)$ and the momenta are fixed. Constant momenta remain constant in both charts, proving preservation of the Lagrangian framing and the peripheral action. This argument uses the peripheral map; it does not identify the two descriptions on every based generator.

Applying the dictionary to the whole slopes and inverting the first image to normalize its meridian gives

$$(18) \quad (p, q; n, m) \mapsto (-p, -q; 1 - n, m).$$

In particular our filling is carried to $W(-1, +1; 1, 0)$, whose alpha slope is $M(\ell C)^{-1}$. This does not identify it with $W(-1, +1; 0, 0)$. Distinct slopes in the exterior do not, conversely,

prove that their filled manifolds are nondiffeomorphic. The involution I , which inverts all three peripheral directions, is different from this reflection and preserves each filling.

The four unit-sign cases therefore remain separate candidates unless an explicit equivalence is proved. Proposition 7.1 establishes their geometric eligibility, not a reason to expect one of their groups to be easier. Throughout the rest of the paper the target remains $V = V_{+1,-1}$.

Part II. The fundamental group and the remaining filling

8. THE COMPLETE FUNDAMENTAL-GROUP PRESENTATION

We derive the complement group before imposing either filling slope. All peripheral loops use the common paths of Section 2.3. The basic transport identity records where the puncture is based. If a square has lower loop g , upper loop h , positive base loop u , and puncture meridian m_0 based at the lower corner, then

$$(19) \quad guh^{-1}u^{-1} = m_0, \quad u^{-1}gu = (u^{-1}m_0u)h.$$

In particular the correction in the second equation is conjugated.

8.1. The disk block and the two handles. Cut the base into a disk and two one-handles, leaving the crossing of the surgery arcs inside the disk. The fiber collars of c, e are disjoint. Each drilled collar in the disk block is S^1 times a three-ball with a straight proper arc removed. The latter has fundamental group $\mathbb{Z}$; attaching the collar to the remaining fiber introduces its meridian and commutation with the core. The disk block therefore has presentation

$$(20) \quad P_0 = \langle x, y, r, s, M, N \mid [x, y][r, s], [M, C], [N, E] \rangle.$$

Disjointness of the drilled collars supplies no relation $[M, N]$. Each base one-handle is a product of a single-curve complement with an interval; its group is generated by the four fiber loops and the meridian of that curve.

The end inclusions can be read in generators adapted to the crossings. Put

$$f_0 = rx, \quad u = y, \quad v_1 = y^{-1}rs, \quad n = s^{-1}Ns.$$

The loops f_0, r are inverse parallels of c, e with paths missing the opposite core. By (2), the loop v_1 has representative $y_2^{-1}d_c^{-1}s_c$. Its two adjoining segments meet c on the same side; pushing the half-arc to that side makes the representative disjoint from c . It crosses e once with terminal path s_3 . The punctured transport square (19) now determines each correction. For c the positive puncture meridian is M , and the crossing contributes its inverse before the path u . Thus $u \mapsto M^{-1}u$, while f_0, r, v_1, n stay fixed. For e the terminal path is s_3 , so the same comparison places n^{-1} after v_1 : this crossing changes v_1 to v_1n^{-1} and fixes f_0, u, r, n, M . The side of the correction is part of the based-path calculation, rather than an interchangeable convention.

Returning to $x = r^{-1}f_0$, $s = r^{-1}uv_1$, and $N = sns^{-1}$ makes the transported meridians explicit. Across c we obtain

$$s \mapsto r^{-1}M^{-1}uv_1 = K_0^{-1}s, \quad N \mapsto K_0^{-1}NK_0,$$

whereas across e we obtain

$$s \mapsto r^{-1}uv_1n^{-1} = sn^{-1} = N^{-1}s.$$

The opposite ends are identified by the original monodromies: ϕ exchanges (x, y) with (r, s) , and ψ sends (x, y, r, s) to (y^{-1}, yx, r, s) . The alpha meridian target is K_0 , by the half-arc comparison along d_c ; the beta target retains the terminally based meridian N . These computations give all entries of the table below.

Edge generator	Alpha source	Alpha target	Beta source	Beta target
x	x	r	x	y^{-1}
y	y	s	$M^{-1}y$	yx
r	r	x	r	r
s	$N^{-1}s$	y	$K_0^{-1}s$	s
meridian	M	K_0	$K_0^{-1}NK_0$	N

The beta meridian path crosses c and so acquires the displayed conjugation by K_0 . The alpha meridian path misses e ; its target path compares along d_c , giving $r^{-1}Mr = K_0$. These paths explain the meridian entries as well as the four surface entries. Van Kampen imposes $A(\text{source})A^{-1} = (\text{target})$ and $B(\text{source})B^{-1} = (\text{target})$. Injectivity of the edge maps is not needed for this presentation calculation.

One may check directly that the relations of the edge spaces hold under both inclusions. For the alpha source, $[r, N^{-1}s] = [r, s]$ follows from $[N, sr^{-1}s^{-1}] = 1$. For the beta source the transformed surface relation follows from $[M, rx] = 1$. An exact identity making the latter check explicit is

$$(21) \quad [x, M^{-1}y][r, K_0^{-1}s] = (xM^{-1}x^{-1})F_0(xMx^{-1}) \cdot (xM^{-1})[M, C]^{-1}(Mx^{-1}), \quad F_0 = [x, y][r, s].$$

It is an equality of free words after $K_0 = r^{-1}Mr$ and $C = (rx)^{-1}$ are substituted. Thus there are thirteen complement relations: the three in (20) and ten end relations.

Proposition 8.1. *With the abbreviations (3), the group $G = \pi_1(V)$ has generators x, y, r, s, A, B, M, N and precisely the following defining relations:*

$$(22) \quad \begin{aligned} [x, y][r, s] &= 1, & [M, C] &= 1, & [N, E] &= 1, \\ Ax A^{-1} &= r, & Ay A^{-1} &= s, & Ar A^{-1} &= x, \\ A(N^{-1}s)A^{-1} &= y, \\ BxB^{-1} &= y^{-1}, & B(M^{-1}y)B^{-1} &= yx, \\ BrB^{-1} &= r, & B(K_0^{-1}s)B^{-1} &= s, \\ AMA^{-1} &= K_0, & B(K_0^{-1}NK_0)B^{-1} &= N, \\ M\ell &= 1, & ND^{-1} &= 1. \end{aligned}$$

Proof. The preceding decomposition proves the first thirteen relations and their completeness for the complement. Its peripheral triples commute. Indeed $\ell = Ax = rA$ and $AMA^{-1} = r^{-1}Mr$ give $\ell M \ell^{-1} = r(AMA^{-1})r^{-1} = M$. The beta meridian row is $[N, D] = 1$. The remaining commutations $[\ell, C] = 1$ and $[D, E] = 1$ follow from the two x, r swaps and the corrected Bs row with $BrB^{-1} = r$, respectively.

The boundary homomorphism $\pi_1(T^3) \rightarrow \pi_1(T^2 \times D^2)$ is surjective, with kernel generated by the new meridian. Thus van Kampen for each cap adds exactly its slope relation. By Definition 2.1 these are the last two rows of (22). This proves the claimed isomorphism, rather than only a surjection from a shorter relation group. $\square$

No unpunctured replacement for As or By , no relation $[M, N] = 1$, and no relation $M^2 = 1$ has been used. Adding such a relation would require a separate proof in the displayed group.

9. A REVERSIBLE REDUCTION AND A NORMAL GENERATOR

In the free group on A, x, y define the following words:

$$(23) \quad \begin{aligned} z &= A^2, & J &= [x, y], & r &= Ax A^{-1}, & s &= Ay A^{-1}, \\ \ell &= Ax, & M &= \ell^{-1}, & N &= [s, z^{-1}], & B &= s^{-1}Ns, \\ K_0 &= r^{-1}Mr, & E &= sr^{-1}s^{-1}. \end{aligned}$$

In particular $B = z^{-1}s^{-1}zs = A^{-1}y^{-1}A^2yA^{-1}$. Write

$$(24) \quad \begin{aligned} \mathcal{R} &= JAJA^{-1}, & \mathcal{T} &= [A^2, x], \\ \mathcal{X} &= BxB^{-1}y, \\ \mathcal{Y} &= B A x y B^{-1} x^{-1} y^{-1}, \\ \mathcal{S} &= B x A s B^{-1} s^{-1}. \end{aligned}$$

Proposition 9.1. *The dictionary (23) induces an isomorphism*

$$G \cong \langle A, x, y \mid \mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S} \rangle$$

that preserves $J = [x, y]$.

Proof. Use Ax , Ay , and the alpha filling in (22) to eliminate r, s, M . The corrected As row then solves for $N = sA^{-1}y^{-1}A = [s, A^{-2}]$. Keep B free for the moment. Let the beta filling and corrected Bs words be

$$F_\beta = NK_0B^{-1}, \quad \mathcal{S}_0 = BK_0^{-1}sB^{-1}s^{-1}.$$

The exact free identity

$$(25) \quad s^{-1}F_\beta\mathcal{S}_0s = (s^{-1}Ns)B^{-1}$$

allows replacement of F_β by the right side while retaining $\mathcal{S}_0$. It is reversible, since $F_\beta = s(\text{right side})s^{-1}\mathcal{S}_0^{-1}$. The new row eliminates $B = s^{-1}Ns$.

The surface, Ar , Bx , and By rows now become $\mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{Y}$. The corrected Bs word satisfies the free identity

$$(26) \quad \mathcal{S}_0 = (B\mathcal{T}B^{-1})\mathcal{S}.$$

It may be replaced by $\mathcal{S}$ because $\mathcal{T}$ is retained. The remaining peripheral and meridian rows are products of conjugates of these relators. Appendix A prints the exact products, including the conjugators. They prove both directions of the presentation change and account for every row of (22). $\square$

In particular, the formula for B does not come from deleting a corrected transport. It uses the beta filling together with the corrected Bs relation. Likewise (26) cannot be replaced by $\mathcal{S}_0 = \mathcal{S}$ in a group where $\mathcal{T}$ is not known to vanish.

Proposition 9.2. *The group G is perfect and is normally generated by $J = [x, y]$. In G one has $AJA^{-1} = J^{-1}$.*

Proof. Perfectness is the abelianization calculation in Section 5. The inversion of J follows from the surface relation and $r = Ax A^{-1}$, $s = Ay A^{-1}$.

Work in the quotient obtained by imposing $J = 1$. Conjugating $[x, y] = 1$ by B and using the corrected B relations gives

$$[y^{-1}, (BMB^{-1})yx] = 1.$$

Since y commutes with yx , this forces $[y, BMB^{-1}] = 1$. Conjugating back and using $B^{-1}yB = x^{-1}$ gives $[x, M] = 1$. The alpha filling says $M = (Ax)^{-1}$, so $[x, A] = 1$ and $r = x$. As B commutes with r , the Bx row now gives $y = x^{-1}$. Hence A commutes with y too, so $s = y$. The corrected As row gives $N = 1$. The beta filling gives $D = 1$, and the corrected Bs row, written $Ds = sB$, gives $B = 1$. Then $D = BK_0^{-1} = 1$ gives $K_0 = M = 1$. The By row becomes $y = yx$, so $x = y = 1$. Finally $r = s = 1$ and $M\ell = 1$ give $A = 1$. All generators vanish. Thus $G/\langle\langle J \rangle\rangle = 1$, as asserted. $\square$

10. AN INTERMEDIATE GROUP CONTAINING THE SURFACE GROUP

The successive quotients in the proof of Theorem 1.3 are

$$\Gamma \rightarrow \Gamma_1 = \Gamma/\langle\langle \mathcal{X} \rangle\rangle \rightarrow K = \Gamma_1/\langle\langle \mathcal{C} \rangle\rangle \rightarrow G = K/\langle\langle \lambda \rangle\rangle.$$

The word $\mathcal{C}$ and the attaching word λ are defined in (30). The surface group embeds in each of the first three groups. The reductions retain the marking of the surface group

$$\Pi = \langle x, y, r, s \mid [x, y][r, s] = 1 \rangle.$$

The groups introduced below are intermediate algebraic groups; no injection into the final filled group is implicit in their definitions.

10.1. The first of the three remaining relations. Let

$$\Gamma = \langle A, x, y \mid \mathcal{R}, \mathcal{T} \rangle, \quad \Gamma_1 = \Gamma/\langle\langle \mathcal{X} \rangle\rangle_\Gamma.$$

The word B continues to mean the expression in (23), except when it is explicitly introduced as an independent generator in an equivalent HNN presentation. Put

$$Q_0 = \langle \Pi, B \mid [B, r] = 1 \rangle.$$

The substitution $\theta(s) = sB^{-1}$, fixing x, y, r, B , is an automorphism of Q_0 , with inverse $s \mapsto sB$. Indeed, for $\varepsilon = \pm 1$,

$$(27) \quad [x, y][r, sB^\varepsilon] = [x, y][r, s] s[r, B^\varepsilon] s^{-1}.$$

Both maps preserve the only relation affected by the substitution.

There is an equivalent presentation

$$(28) \quad \Gamma = \langle Q_0, A \mid A\theta(g)A^{-1} = \phi(g) \ (g = x, y, r, s) \rangle.$$

To verify it algebraically, eliminate $r = Ax A^{-1}$ and $s = Ay A^{-1}$. The fourth edge relation gives $B = A^{-1}y^{-1}A^2yA^{-1}$. The surface and third edge relations are $\mathcal{R}, \mathcal{T}$. The remaining $[B, r]$ row follows from them by (59)–(60) in the appendix. These eliminations reverse to recover (28).

The group Q_0 is the HNN extension of Π centralizing its infinite cyclic subgroup $\langle r \rangle$, so Π embeds in it. Since θ is an automorphism and ϕ is the handle-interchange automorphism of Π , both surface maps in (28) are injective. The HNN normal form theorem therefore embeds Q_0 in Γ ; we use the standard Bass–Serre description [Ser80].

The word $\mathcal{X} = BxB^{-1}y$ is fixed by θ and its inverse. Hence

$$Q_1 = \langle \Pi, B \mid BrB^{-1} = r, BxB^{-1} = y^{-1} \rangle,$$

and

$$(29) \quad \Gamma_1 = \langle Q_1, A \mid A\theta(g)A^{-1} = \phi(g) \ (g = x, y, r, s) \rangle.$$

Both are injective HNN extensions. For the first, the associated subgroups $\langle r, x \rangle$ and $\langle r, y^{-1} \rangle$ are free on the indicated bases. One direct verification is

$$\Pi = \langle F(x, y, r), s \mid srs^{-1} = [x, y]r \rangle,$$

whose cyclic edge maps inject, so $F(x, y, r)$ embeds by HNN normal forms. The two rank-two subgroups are then free on their displayed generators. The second assertion follows because θ descends to an automorphism of Q_1 . Thus Π embeds in Γ_1 .

10.2. Replacing the last two relations. In words on x, y, r, s, B set

$$(30) \quad \begin{aligned} d_0 &= y^{-1}x^{-1}s, & e_0 &= x^{-1}s, & \mathcal{C} &= Bd_0B^{-1}e_0^{-1}, \\ a_* &= B^{-1}yxB^{-1}x^{-1}, & m &= yB^{-1}x^{-1}y^{-1}B, & \lambda &= Aa_*^{-1} = Axm. \end{aligned}$$

The last equality is free, as is $m = (a_*x)^{-1}$. The following identities hold in the free group on A, x, y, s, B :

$$(31) \quad \begin{aligned} \mathcal{Y} &= B\lambda B^{-1}, \\ \mathcal{S} &= (BxB^{-1})\mathcal{Y}(BxB^{-1})^{-1}\mathcal{X}x\mathcal{C}x^{-1}. \end{aligned}$$

Thus in Γ_1 each pair of relations implies the other, and

$$(32) \quad \langle\langle \mathcal{Y}, \mathcal{S} \rangle\rangle_{\Gamma_1} = \langle\langle \mathcal{C}, \lambda \rangle\rangle_{\Gamma_1}, \quad K = \Gamma_1 / \langle\langle \mathcal{C} \rangle\rangle_{\Gamma_1}, \quad G = K / \langle\langle \lambda \rangle\rangle_K.$$

This is an equality of normal closures, not merely an epimorphism to an easier group.

Both θ and its inverse fix $\mathcal{C}$ freely. For example $d_0 \mapsto d_0B^{-1}$ and $e_0 \mapsto e_0B^{-1}$, whose added factors cancel in $Bd_0B^{-1}e_0^{-1}$. Consequently

$$(33) \quad Q_2 = Q_1 / \langle\langle \mathcal{C} \rangle\rangle_{Q_1} = \langle \Pi, B \mid B(r, x, d_0)B^{-1} = (r, y^{-1}, e_0) \rangle,$$

and K has exactly the HNN presentation in Theorem 1.3.

Lemma 10.1. *The subgroups $\langle r, x, d_0 \rangle$ and $\langle r, y^{-1}, e_0 \rangle$ of Π are free of rank three on the displayed bases.*

Proof. Neither subgroup has finite index. An index- n subgroup of a closed genus-two surface group is the group of a closed surface of genus $n + 1$, whose abelianization needs $2n + 2 \geq 4$ generators. It cannot be generated by three elements. The covers corresponding to our subgroups are therefore noncompact surfaces, and their fundamental groups are free.

In the coordinates x, y, r, s of $H_1(\Pi) = \mathbb{Z}^4$, the two triples have images

$$\begin{aligned} (0, 0, 1, 0), & \quad (1, 0, 0, 0), & \quad (-1, -1, 0, 1), \\ (0, 0, 1, 0), & \quad (0, -1, 0, 0), & \quad (-1, 0, 0, 1). \end{aligned}$$

Each triple is linearly independent. The corresponding free group has rank at least three and is generated by three elements, so its rank is three. A generating triple of a rank-three free group is a basis, by Hopficity of finite-rank free groups. $\square$

Lemma 10.1 proves that (33) is an injective HNN extension. Since θ remains an automorphism, the subsequent A extension also has injective surface edge maps. Thus

$$(34) \quad \Pi \hookrightarrow Q_2 \hookrightarrow K.$$

The nontrivial element $[x, y]$ has infinite order in the surface group and hence in K . Equations (32) and Proposition 9.2 now prove Theorem 1.3.

10.3. The entire kernel of the intermediate quotient. One additional statement holds for the $\mathcal{C}$ step, without restricting the length or form of relations.

Proposition 10.2. *The kernel of $\Gamma_1 \rightarrow K$ is free.*

Proof. First consider $Q_1 \rightarrow Q_2$. Its kernel acts on the B -HNN tree of Q_1 . Every vertex stabilizer is conjugate to Π , which maps injectively into Q_2 . The kernel therefore acts freely on that tree and is a free group.

The kernel of $\Gamma_1 \rightarrow K$ acts on the A -HNN tree of (29). Its vertex stabilizers are conjugates of the free kernel just described. Its edge stabilizers are trivial, because both surface edge groups inject into K . The associated graph of groups has free vertex groups and trivial edge groups, so its fundamental group is a free product of free groups and a free group. It is free. $\square$

The standard two-complexes of the presentations of Γ , Γ_1 , and K used above are also aspherical. We include the identification of these particular complexes, since asphericity of some graph of spaces with the same group would not alone identify a presentation complex.

First introduce a generator J for $[x, y]$ in Γ . Before adjoining y , the group

$$P_0 = \langle A, x, J \mid [A^2, x], AJA^{-1} = J^{-1} \rangle$$

is an injective graph of groups: put $z = A^2$, $v = AJ$, take central vertex $\langle z, x \mid [z, x] \rangle \cong \mathbb{Z}^2$, and attach two cyclic vertices $\langle A \rangle, \langle v \rangle$ along $z = A^2 = v^2$. Adjoin y as a cyclic HNN extension with $xyy^{-1} = J^{-1}x$. Its edge maps inject, since both indicated words have x exponent one. The tree-of-spaces theorem makes this complex aspherical [Hat02, Theorem 1B.11]. Eliminating the defining generator v using $v = AJ$, then z using $z = A^2$, and finally J using the y -HNN row gives the standard Γ complex by elementary homotopy equivalences.

The surface-edge model (28) is independently aspherical by its injective edge maps. The two marked classifying spaces for Γ are homotopy equivalent. Attach the $\mathcal{X}$ disk to both: on the graph-of-spaces side this adds x to the B -edge graph, yielding the rank-two edge model of Γ_1 . The marked equivalence extends across the disk, and the resulting graph of spaces is aspherical. Repeating with the $\mathcal{C}$ disk adds d_0 to that edge graph, giving the rank-three model of K . Its edge maps still inject. This proves the assertion for the actual presentation complexes, without deleting a redundant two-cell from a higher-dimensional model.

Let $N_C = \ker(\Gamma_1 \rightarrow K)$. In the universal cover of the standard K complex, the preimage of the Γ_1 complex is a connected $K(N_C, 1)$. The relative cellular chain complex consists of $\mathbb{Z}[K]$ in degree two, from the newly attached disk. Its exact homology sequence gives

$$(N_C)_{\text{ab}} \cong \mathbb{Z}[K]$$

with the conjugation action. Thus the free kernel in Proposition 10.2 has countably infinite rank. The exponent rows of $\mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{C}$ are $(0, 0, 0), (0, 0, 0), (0, 1, 1), (0, 0, -1)$, and therefore

$$H_1(K) = \mathbb{Z}\langle A \rangle, \quad H_2(K) = \mathbb{Z}^2, \quad H_i(K) = 0 \quad (i \geq 3).$$

These conclusions concern the entire intermediate kernel and the intermediate group. They do not determine $\ker(K \rightarrow G)$.

11. RESTORING THE MERIDIAN TRANSPORT

The HNN extension K retains the four surface identifications across A . Restoring the fifth identification, for the meridian, gives a geometrically familiar intermediate group and a kernel that can be computed completely. This separates a solved normal-subgroup problem from the unresolved final filling.

11.1. A second description of the vertex group. Put $P = rx$ and $k = r^{-1}mr$, where m is the word in (30). There is a reversible presentation

$$(35) \quad \begin{aligned} Q_c &= \langle \Pi, m \mid [m, P] = 1 \rangle = \Pi *_{\langle P \rangle} (\langle P \rangle \times \langle m \rangle), \\ Q_2 &= \langle Q_c, B \mid B(x, m^{-1}y, r, k^{-1}s)B^{-1} = (y^{-1}, yx, r, s) \rangle. \end{aligned}$$

Here m is an independent generator in the second presentation; eliminating it recovers its earlier word. Thus the notation does not impose the final relation $m = (Ax)^{-1}$ prematurely.

We verify the presentation change. The corrected By relation solves for $m = yB^{-1}x^{-1}y^{-1}B$. Put $V_0 = sB^{-1}s^{-1}$. The surface relation and $[B, r] = 1$ give $rV_0r^{-1} = J^{-1}V_0J$. The equation $\mathcal{C} = 1$ says $V_0 = xyB^{-1}x^{-1}$. Using $BxB^{-1} = y^{-1}$ gives

$$J^{-1}V_0J = yB^{-1}x^{-1}y^{-1} = mB^{-1}.$$

Consequently $\mathcal{C} = 1$ is equivalent to $m = rsB^{-1}s^{-1}Br^{-1}$, which is the corrected Bs relation. All substitutions reverse. The remaining peripheral relation follows from the free identity, with $F_0 = [x, y][r, s]$,

$$(36) \quad [x, m^{-1}y][r, k^{-1}s] = wF_0w^{-1}v[m, P^{-1}]^{-1}v^{-1}, \quad w = xm^{-1}x^{-1}, \quad v = xm^{-1}.$$

The four beta equations conjugate its left side by B to F_0 ; hence $[m, P] = 1$. Conversely, the equations in (35) imply the original three equations for Q_2 .

The crossing map

$$\theta_c(x, y, r, s, m) = (x, m^{-1}y, r, k^{-1}s, m)$$

is an automorphism of Q_c , with inverse (x, my, r, ks, m) . Identity (36) verifies preservation of the surface relation; for the inverse the free identity is

$$[x, my][r, ks] = (xmx^{-1})F_0(xm^{-1}x^{-1}) \cdot x[m, P^{-1}]x^{-1}.$$

Both maps fix the peripheral relation. The target map $(x, y, r, s) \mapsto (y^{-1}, yx, r, s)$ is an automorphism of Π . Thus (35) is also an injective HNN extension, and Q_c embeds in Q_2 . The map ϕ extends to an involutive automorphism of Q_c by $\phi(m) = k$: it carries $[m, P^{-1}]$ to its conjugate by r^{-1} , and $\phi^2(m) = P^{-1}mP = m$. The automorphism θ of Q_2 fixes m , since its defining word contains no s .

11.2. A free kernel with an explicit module. Write

$$U = Ax, \quad \delta = [U, m], \quad H_\beta = K / \langle\langle \delta \rangle\rangle, \quad \mathcal{D} = \ker(K \rightarrow H_\beta), \quad \mathcal{N} = \ker(K \rightarrow G).$$

We use H_β , rather than H , for this group to distinguish it from the doubled section. For a subgroup L of a group H_β , $\mathbb{Z}[H_\beta/L]$ denotes the permutation module on its left cosets.

Proposition 11.1 (The meridian-transport kernel). *The group H_β is the fundamental group after the beta filling alone. It has the injective HNN presentation*

$$H_\beta = \langle Q_2, A \mid A\theta(g)A^{-1} = \phi(g) \quad (g = x, y, r, s, m) \rangle.$$

The kernel $\mathcal{D}$ is free of countably infinite rank, and

$$\mathcal{D}_{\text{ab}} \cong \mathbb{Z}[H_\beta / \langle P \rangle]$$

as a module under conjugation. Furthermore

$$\mathcal{D} \subset \mathcal{N}, \quad \mathcal{N}/\mathcal{D} \cong \ker(H_\beta \rightarrow G), \quad G = H_\beta / \langle\langle \lambda \rangle\rangle.$$

Proof. Since $rA = Ax = U$,

$$r(AmA^{-1}k^{-1})r^{-1} = [U, m].$$

Thus the new relation is precisely the missing meridian transport. Both enlarged edge maps are injective by (35). To check the geometric description, perform only the beta filling in the fifteen-row presentation and eliminate $N = D$. The beta equations give $D = sBs^{-1}$ and $[D, E] = 1$; the alpha source $N^{-1}s$ becomes sB^{-1} . The surviving equations are exactly the displayed HNN presentation. Also $\delta = [Um, m] = [\lambda, m]$, so δ lies in the normal closure of λ in K .

Let T_K be the A -HNN tree of K . Since Q_2 injects into H_β , the kernel $\mathcal{D}$ acts freely on T_K and is free. Put $E_0 = \theta(\Pi)$ and $E_1 = \theta(Q_c)$. The graph $\mathcal{D} \backslash T_K$ has vertices H_β/Q_2 and oriented edges H_β/E_0 . The tree of H_β has the same vertices and edges H_β/E_1 . The natural map between these graphs identifies parallel edges and is the identity on vertices. Since the boundary map of a tree is injective,

$$\mathcal{D}_{\text{ab}} = \ker(\mathbb{Z}[H_\beta/E_0] \rightarrow \mathbb{Z}[H_\beta/E_1]).$$

Now $E_1 = \langle E_0, m \mid [m, P] = 1 \rangle$. Its own HNN tree gives the exact augmented chain sequence

$$0 \rightarrow \mathbb{Z}[E_1/\langle P \rangle] \rightarrow \mathbb{Z}[E_1/E_0] \rightarrow \mathbb{Z} \rightarrow 0.$$

Inducing this sequence to H_β proves the asserted module isomorphism. Induction is exact because $\mathbb{Z}[H_\beta]$ is free as a right $\mathbb{Z}[E_1]$ -module. Finally the height map, with A of height one and Q_2 of height zero, shows that $H_\beta/\langle P \rangle$ is countably infinite. $\square$

This result determines an entire kernel, rather than a collection of short relations. It does not determine the larger kernel $\mathcal{N}$. The distinction can be seen in an explicit subgroup. Amalgam normal forms in Q_c , followed by its inclusion in Q_2 , give

$$\langle P, m \rangle \cap \Pi = \langle P \rangle, \quad \langle P, m \rangle \cap \theta(\Pi) = \langle P \rangle.$$

The second equality follows by applying the automorphism θ of Q_2 , which fixes P and m ; it does not require $\theta(\Pi)$ to lie in Q_c . Replacing the stable letter A by $U = Ax$, its edge map fixes P . Britton's lemma therefore gives

$$(37) \quad \langle P, m, U \mid [P, m] = [P, U] = 1 \rangle \cong \mathbb{Z} \times F(m, U) \hookrightarrow K.$$

In particular $\delta \neq 1$ in K . In H_β the same three generators give $\mathbb{Z}^3$: P, m generate the embedded peripheral $\mathbb{Z}^2$, and height separates U from it.

These subgroups give necessary conditions for proposed free-kernel arguments. If $\mathcal{N}$ were free, P would have infinite order in G ; otherwise $\langle \lambda, P^d \rangle \cong \mathbb{Z}^2$ would lie in $\mathcal{N}$ for some $d > 0$. Similarly,

$$(38) \quad \ker(H_\beta \rightarrow G) \text{ free} \implies \langle P, m \rangle \hookrightarrow G.$$

Indeed a nontrivial element w in the kernel on that torus would give $\langle \lambda, w \rangle \cong \mathbb{Z}^2$ in the larger kernel. Neither freeness assertion is established.

12. RESTRICTIONS ON POSSIBLE FILLING ARGUMENTS

The final attaching word has height one, but it is a stable letter times a vertex word. The following restrictions explain why several familiar injection theorems do not immediately settle its effect. They concern the displayed intermediate groups; they are not evidence for triviality of G .

Proposition 12.1. *An almost-malnormal subgroup of K containing a conjugate of a nonzero power of λ or P is all of K . Moreover every action of K or H_β without inversions on a simplicial tree with finite edge stabilizers has a global fixed vertex.*

Proof. Recall that $L \leq K$ is almost malnormal if $L \cap gLg^{-1}$ is finite whenever $g \notin L$. Such a subgroup contains the full centralizer of each of its infinite-order elements. In K we have

$$(39) \quad [U, P] = [m, P] = [\lambda, P] = 1, \quad U^2 = A^2P, \quad [B, r] = 1, \quad x = A^{-1}U, \quad y = Bx^{-1}B^{-1}.$$

The surface injection makes P, r infinite, and height makes A^2, λ infinite. Starting from $\lambda^n \in L$ puts $\lambda, P, U, A^2, A, x, r, B, y$ in L in that order. Starting from P^n omits only the first step. Conjugating L handles the stated conjugates.

For the tree assertion, $\langle P, U \rangle \cong \mathbb{Z}^2$ in either group. If P were hyperbolic, its centralizer would preserve its axis and act by translations: a reflection would conjugate its nonzero translation to its inverse. The homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ would have infinite kernel fixing an edge, contrary to finite edge stabilizers. Hence P fixes a vertex v , uniquely because it has infinite order. Its centralizer fixes v , so U does. Then $A^2 = U^2P^{-1}$ fixes v ; this infinite-order element also has a unique fixed vertex, so A fixes v . Now x, r fix v , and the centralizer of the infinite-order element r contains B . Finally y fixes v . Thus the generators fix v . $\square$

Hyperbolically embedded subgroups are almost malnormal [DGO17, Proposition 4.33]. Consequently the final slope cannot be placed in a proper such subgroup of K to invoke a deep-filling theorem. Taking a nonzero power of the slope does not remove this obstruction. The tree assertion likewise rules out applying a theorem that requires a tree with trivial edge stabilizers on which the attaching word is hyperbolic; compare [ADL11, Theorem 5.11]. It does not conflict with our HNN descriptions, whose edge groups are infinite.

There are two further, distinct issues with the displayed HNN quotient. First, $\lambda = A(xm)$ is cyclically conjugate to $(xm)A$. It is precisely the stable-letter-length-one case excluded in [Che26, Theorem 6.1]. Second, the surface edge group is not malnormal in Q_2 : $m \notin \Pi$ but m centralizes the nontrivial $P \in \Pi$. Applying θ gives the same obstruction for the other surface edge image. A theorem for an equation over the free product $Q_2 * \langle A \rangle$ cannot simply retain the HNN identifications without verifying its hypotheses for them. Eliminating A in the actual quotient imposes those further equations on Q_2 .

12.1. Two obstructions to aspherical decompositions. Aspherical-gluing results for other Luttinger-surgery constructions suggest a geometric approach, but require injective gluing faces and the appropriate boundary path conditions. Baykur's recent treatment [Bay26, Theorem 3, Lemma 6, and Remark 15] does not include this construction. Two natural choices of faces can be excluded here.

First, there is no monodromy-invariant multicurve containing the alpha curve c , even after any finite cover of the base. Direct substitution in the marked beta monodromy gives

$$\psi(J) = J, \quad \psi^6(x) = J^{-1}xJ, \quad \psi^6(y) = J^{-1}yJ, \quad \psi^{6n}(C) = J^{-n}x^{-1}J^n r^{-1}.$$

Split the surface group as $F(x, y) *_{\langle J \rangle} F(r, s)$, with $J = [r, s]^{-1}$ in the second factor. The edge subgroup is maximal cyclic and malnormal in each free factor. The words displayed for $\psi^{6n}(C)$ are cyclically reduced of amalgam length two. If the word for n were conjugate to the word for zero, the conjugacy normal form would give integers k_0, k_1 with

$$J^{-n}x^{-1} = J^{k_0}x^{-1}J^{-k_1}, \quad J^n r^{-1} = J^{k_1}r^{-1}J^{-k_0}.$$

Malnormality first gives $k_1 = 0$, $k_0 = -n$, and then $n = 0$. Their homology classes also exclude conjugacy to an inverse. Applying a power of ψ proves pairwise distinction. Every finite-index base subgroup contains a positive power of the beta loop, so no finite invariant set containing c appears in a finite base cover. This excludes a global fiberwise cut, not cuts over proper subsets of the base or more general hypersurfaces.

Second, consider the exterior S_c of c in the closed fiber mapping torus with monodromy ψ . Its group is precisely the Q_2 presentation in (35), before the A attachment, and

$$H_1(S_c) = \mathbb{Z}\langle B, r, s \rangle, \quad x = y = m = 0, \quad [C] = -[r].$$

Its boundary torus is generated by C, m . After the alpha filling, an annulus in its filling solid torus joins the old meridian to the new core: in longitude and normal-disk coordinates it is $(v, \rho) \mapsto (-v, \rho e^{2\pi i v})$. Taking its product with C extends S_c to a diffeomorphic transverse piece S_1 with boundary on the alpha core torus. An identity-side push-off survives the beta surgery and has the same based generator images.

No compact completion of this entire S_1 across its boundary torus can map injectively on fundamental groups to V while extending that inclusion. Indeed Mayer–Vietoris for $S_1 \cup_T L$ shows that the kernel of $H_1(S_1) \rightarrow H_1(S_1 \cup_T L)$ lies in $\text{im } H_1(T) = \mathbb{Z}\langle r \rangle$. Thus the word

$$\kappa_y = m^{-1}x^{-1}yxms^{-1}$$

has infinite-order homology class $-[s]$ in every such completion. Its image in V is trivial, since the alpha filling gives $A = m^{-1}x^{-1}$ and the retained relation is $AyA^{-1} = s$. The same argument applies before the beta filling, and also to S_1 without a cap. This excludes arbitrary compact caps along the whole boundary torus, not modifications of the transverse piece itself. In particular these two natural decompositions do not supply the injective faces required for an asphericity proof.

12.2. Normal generation is sensitive to the attaching word. Although A normally generates K , this does not prove that λ does. Indeed, setting $A = 1$ in K gives $s = y$, $B = 1$, $\mathcal{X} = xy$ and $\mathcal{C} = y^{-1}$, so the quotient is trivial. There is, however, no endomorphism of K that simultaneously sends A to $g\lambda^\varepsilon g^{-1}$ and J to $gJ^\epsilon g^{-1}$ for a common $g \in K$ and arbitrary signs $\varepsilon, \epsilon \in \{1, -1\}$.

To see the obstruction, the surface relation requires A to invert J . Since A already does so in K , replacing it by $\lambda = A(xm)$ would require $[xm, J] = 1$; inversion of either A or J gives the same requirement. But

$$[J^{-1}, xm] = (J^{-1}x)mJm^{-1}x^{-1}$$

is a reduced alternating word in $\Pi *_{\langle P \rangle} (\langle P \rangle \times \langle m \rangle)$. Its three surface syllables lie outside $\langle P \rangle$: for the outer two use abelianization, and for J use its nontriviality and its zero abelianization. Thus the word is nonidentity in Q_c and K . This excludes the indicated substitution, not every possible endomorphism and not normal generation by λ itself.

12.3. Collapse criteria and two-dimensional representations.

Proposition 12.2. *Any quotient of G satisfying one of*

$$[A^2, y] = 1, \quad [A, r] = 1, \quad [x, r] = 1, \quad M^2 = 1$$

is trivial. For every field k , every homomorphism from G to $\mathrm{PGL}_2(k)$ or $\mathrm{GL}_2(k)$ is trivial.

Proof. If $[A^2, y] = 1$, the corrected As row gives $N = 1$. The beta filling then gives $D = 1$. The corrected Bs row says $Ds = sB$, so $B = 1$ and then $K_0 = M = 1$. The Bx, By rows give $x = y = 1$; the remaining rows give $r = s = A = 1$. If $[A, r] = 1$, then $x = r$ and B commutes with x , so $y = x^{-1}$ and the first case applies. If $[x, r] = 1$, conjugation by B gives $[y, r] = 1$. The formula

$$M = B^{-1}((ByB^{-1})(yx)^{-1})B$$

then gives $[M, r] = 1$. Since $M = (Ax)^{-1}$, this implies $[A, r] = 1$. Finally $M^2 = 1$ implies $(Ax)^2 = 1$, whence $A^2 = (rx)^{-1}$. Conjugating by A gives also $A^2 = (xr)^{-1}$, and the third case applies.

For a projective representation choose matrix lifts U_0, T_0 of A, r . The relation $[A^2, r] = 1$ gives $T_0 U_0^2 T_0^{-1} = c U_0^2$, where determinants imply $c^2 = 1$. If U_0^2 is scalar, the first collapse criterion applies. If it is nonscalar and $c = 1$, its matrix centralizer is the commutative algebra of polynomials in it; hence U_0, T_0 commute, and the second criterion applies. Otherwise $c = -1 \neq 1$. Over an algebraic closure U_0^2 has distinct nonzero eigenvalues $d, -d$, since its trace is zero. In that eigenbasis U_0 is diagonal and T_0 is off-diagonal. Thus $(T_0 U_0)^2$ is scalar. Since $rA = Ax = M^{-1}$, the fourth criterion applies. The argument includes characteristic two in the $c = 1$ case. Finally a linear representation has scalar image after projectivization, and that image is trivial because G is perfect. $\square$

Thus two-dimensional representations cannot detect nontriviality of G . Likewise a bi-orderable image is trivial: conjugation preserves its order but A conjugates J to J^{-1} , forcing $J = 1$ and then the entire image to vanish.

13. THE UNIVERSAL CENTRAL EXTENSION

There is a second precise algebraic question, sufficient for nontriviality but not equivalent to it. It concerns a cyclic central kernel, whose geometric interpretations will occupy Part III. Use the five words $\mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S}$ of (24) and define

$$(40) \quad \widehat{G} = \langle A, x, y, t \mid \mathcal{R}t^{-1}, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S}, [t, A], [t, x], [t, y] \rangle.$$

A central extension is universal if it admits a unique homomorphism over G to every central extension of G .

Theorem 13.1. *The map $\widehat{G} \rightarrow G$ fixing A, x, y and killing t is the universal central extension. Its kernel is the cyclic central subgroup $\langle t \rangle$, naturally isomorphic to $H_2(G; \mathbb{Z})$. Under the dictionary (23), the surface row of the original presentation has value t and the other fourteen rows have value one. No order of t is asserted.*

Proof. The exponent matrix of $\mathcal{X}, \mathcal{Y}, \mathcal{S}$, with columns A, x, y , is

$$C_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \det C_0 = 1.$$

Take any central extension $C \rightarrow D \rightarrow G$ and arbitrary lifts of the three generators. Write $\eta_X, \eta_Y, \eta_S \in C$ for the three corresponding row values. Multiply the lifts of A, x, y , respectively, by

$$\eta_Y^{-1}, \quad \eta_Y \eta_S^{-1}, \quad \eta_X^{-1} \eta_Y^{-1} \eta_S.$$

The exponent matrix shows that the three new row values are one. The values of $\mathcal{R}, \mathcal{T}$ are unchanged, since these words have zero exponent vectors.

Appendix B prints a free identity consisting of nineteen conjugates of the five relators, whose signed multiplicities in their displayed order are $(0, 1, 0, 0, 0)$. Evaluate this identity on the normalized lifts. Every row value is central; the identity therefore forces the value of $\mathcal{T}$ to be one. Assigning t the value of $\mathcal{R}$ gives a homomorphism $\widehat{G} \rightarrow D$ over G .

The kernel of $\widehat{G} \rightarrow G$ is exactly $\langle t \rangle$, since killing t gives the presentation of G . Also $\widehat{G}$ is perfect: the three exponent rows kill A, x, y in its abelianization, and $\mathcal{R}t^{-1}$ then kills t . Two homomorphisms to D over G differ by a homomorphism from $\widehat{G}$ to the abelian central group C . Perfectness makes it trivial, proving uniqueness. The identification of the kernel with $H_2(G; \mathbb{Z})$ is the standard kernel theorem for universal central extensions, or equivalently the Hopf formula. Finally, the products in Appendix A express every original row in the reduced rows. Apart from the surface row, every product has total $\mathcal{R}$ exponent zero. Its central factors thus cancel, proving the last assertion. $\square$

If $t \neq 1$, then $G \neq 1$: otherwise $\widehat{G}$ would be both cyclic and perfect. The converse is not justified. A nontrivial perfect group can have zero second integral homology. The central question must therefore be kept distinct from Question 20.1.

One immediate limitation concerns the standard presentation complex of (24). It has $H_2 \cong \mathbb{Z}^2$, by the displayed exponent matrix and its two zero rows. It cannot be aspherical, since its fundamental group has cyclic H_2 . Thus a proof requiring this particular two-complex to be aspherical cannot succeed. This observation does not exclude other classifying spaces for G and does not determine which additional spherical relations may kill t .

Part III. Geometric interpretations of the central kernel

14. TWO CLOSED FORMS AND A PRIMITIVE CIRCLE BUNDLE

The central generator in (40) can be realized as a circle fiber. This realization translates its possible order into an exact question about spherical classes in V .

Proposition 14.1. *The specified filling V admits closed two-forms ω_1, ω_2 of constant rank two, with*

$$\omega_1^2 = \omega_2^2 = 0, \quad \omega_1 \wedge \omega_2 > 0, \quad \int_F \omega_1 = 1, \quad [\omega_2] = 0.$$

The class $e = [\omega_1]$ is the real image of the integral generator of $H^2(V; \mathbb{Z})$ evaluating as one on F .

Proof. On the original flat surface bundle choose the fiber area form invariant under the half-turn and preserved by the annular twists. It descends to a closed rank-two form Ω_F . A positive base area form gives Ω_B , with complementary kernel. In the adopted alpha chart these forms are

$$(41) \quad \Omega_F = (dQ_1 - \tfrac{1}{2}dQ_2) \wedge dY_1, \quad \Omega_B = dQ_2 \wedge (dY_2 + \tfrac{1}{2}dY_1).$$

Their sum is the cotangent symplectic form. At beta the corresponding forms are $dQ_1 \wedge dY_1$ and $dQ_2 \wedge dY_2$.

Use the surgery shear of [ADK03, Section 2.1] in the coordinates $(x_1, x_2, y_1, y_2) = (Q_2, Q_1, Y_2, Y_1)$. On the upper side of a normal-square annulus it replaces Q_2 by $Q_2 + \chi(Y_2)$, where χ is a step function with derivative supported in the middle third. Its derivative replaces dQ_2 by $dQ_2 + a(Y)dY_2$ on that annulus. The pullbacks in (41) are respectively

$$\Omega_F + \frac{1}{2}a(Y)dY_1 \wedge dY_2, \quad \Omega_B - \frac{1}{2}a(Y)dY_1 \wedge dY_2.$$

Take the normal square $[-r_0, r_0]^2$ with inner square $[-r_0/2, r_0/2]^2$, and choose a smooth function $\rho(Y_1)$ equal to zero for $Y_1 \leq 0$ and one for $Y_1 \geq r_0/2$. The support of χ' lies in $|Y_2| < r_0/3$. The function $f(Y) = \rho(Y_1)\chi'(Y_2)/2$ agrees with $a/2$ throughout the gluing annulus: on its upper and lower strips ρ has the required constant value, and on the lateral strips $\chi' = 0$. The two forms

$$\tilde{\Omega}_F = \Omega_F + f(Y)dY_1 \wedge dY_2, \quad \tilde{\Omega}_B = \Omega_B - f(Y)dY_1 \wedge dY_2$$

therefore extend across the inserted piece. They are closed, their squares vanish, and their product equals $\Omega_F \wedge \Omega_B$: the added normal two-form wedges to zero with each original form. At beta the shear preserves both forms separately. These constructions use precisely the existing two gluing maps and filling slopes.

Finally divide the extended fiber form by its positive integral over F . Since $H_2(V; \mathbb{Z}) = \mathbb{Z}[F]$, the resulting class is the stated integral generator. The base form integrates to zero over F and is consequently exact. $\square$

Such a pair is called a *symplectic pair*: the kernels are complementary foliations, by closedness and constant rank [BK06]. This terminology asserts more than the existence of the symplectic sum, but does not assert that either foliation has simply connected leaves.

Theorem 14.2. *Let $p : \mathcal{P} \rightarrow V$ be the principal circle bundle of Euler class e , with five-dimensional total space. Then $\mathcal{P}$ has the integral homology of S^3 , and $\pi_1(\mathcal{P}) \rightarrow G$ is the universal central extension. The oriented circle fiber corresponds to t up to inversion. A connection α can be chosen so that, with $\beta = p^*\omega_2$,*

$$d\alpha = p^*\omega_1, \quad \alpha \wedge d\alpha \wedge \beta \neq 0, \quad (d\alpha)^2 = \beta^2 = 0,$$

and β is exact.

Proof. The integral homology spectral sequence of this principal bundle has four nonzero entries, at $(0, 0)$, $(0, 1)$, $(2, 0)$, $(2, 1)$, all equal to $\mathbb{Z}$. Its only nonzero differential is cap product with e ,

$$d_2 : H_2(V; \mathbb{Z}) \longrightarrow H_0(V; \mathbb{Z}), \quad [F] \longmapsto 1.$$

It is an isomorphism, leaving precisely $H_0(\mathcal{P}) = H_3(\mathcal{P}) = \mathbb{Z}$. Write $\Pi_{\mathcal{P}} = \pi_1(\mathcal{P})$. The Hopf homology sequence gives $H_1(\Pi_{\mathcal{P}}) = H_2(\Pi_{\mathcal{P}}) = 0$. The homotopy sequence gives a central extension of G by the image of the circle group. Pulling an arbitrary central extension with kernel C back to $\Pi_{\mathcal{P}}$ gives an extension classified by $H^2(\Pi_{\mathcal{P}}; C) = 0$. It splits uniquely, since $\text{Hom}(\Pi_{\mathcal{P}}, C) = 0$. This proves the universal property and identifies the extension with Theorem 13.1.

Over the genus-two fiber F the circle bundle has Euler number one. The product of the two commutators of lifted surface generators is therefore the circle fiber or its inverse. Central changes of the lifts do not change the commutators. The same surface word has value t in (40), proving the marking assertion. A connection with the prescribed curvature exists

because ω_1 represents the Euler class. The remaining identities follow from Proposition 14.1 and the connection splitting. $\square$

The pair (α, β) is a contact-symplectic pair in the terminology of [BK06]. Its existence does not imply that the circle fiber injects on fundamental groups. The exact obstruction is visible in the homotopy sequence:

$$(42) \quad \pi_2(V) \xrightarrow{\langle e, - \rangle} \mathbb{Z} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1.$$

If the image of the first map is $d\mathbb{Z}$, with $d \geq 0$, then

$$(43) \quad \langle t \rangle \cong \mathbb{Z}/d\mathbb{Z},$$

where $d = 0$ means infinite cyclic. Thus a restriction $d \neq 1$ would prove $G \neq 1$. No such restriction has been established. In particular the minimum genus bounds for embedded surfaces do not apply to an arbitrary mapped sphere representing a class in $\pi_2(V)$.

For comparison, the product pair on $S^2 \times D^2$ has primitive circle bundle $S^3 \times D^2$. It satisfies the same rank and volume identities, yet its Hopf circle is nullhomotopic. This example does not have the genus-two geometry of our construction. It shows exactly why the pair and the homology calculation alone cannot prove survival.

15. REMOVING THE SECTION

Let Σ_O be the proper punctured-torus section through the center O of the marked octagon, and set

$$X = V \setminus \text{int}_V \nu \Sigma_O, \quad Q_X = \pi_1(X).$$

Choose the normal disk bundle small enough to miss the two surgery neighborhoods and round the resulting corners. The positive normal meridian, based radially from the initial octagon vertex, is denoted μ .

Proposition 15.1. *The group Q_X has the fourteen relations of Section 8 other than the surface relation, with*

$$\mu = [x, y][r, s], \quad G = Q_X / \langle\langle \mu \rangle\rangle.$$

Its integral homology is

$$H_1(X) = H_3(X) = 0, \quad H_2(X) = \mathbb{Z}^2,$$

where the two generators are normal circle bundles over a basis of $H_1(\Sigma_O)$. Both $\omega_1|_X$ and $\omega_2|_X$ are exact. If

$$C_\mu = \langle\langle [\mu, q] : q \in Q_X \rangle\rangle_{Q_X},$$

then $Q_X/C_\mu \cong \widehat{G}$ over G , with μ corresponding to t up to the common orientation convention.

Proof. The geometric calculation must be repeated in the punctured fiber: simply deleting a relator after a closed-surface simplification would not justify the assertion. Remove a small disk about O . The remaining polygon retracts radially onto its outer boundary, and the punctured fiber retracts onto the rose with edges x, y, r, s . Its positive normal meridian is $[x, y][r, s]$; the oriented boundary of the punctured fiber has the opposite orientation.

Here are the peripheral checks in this free group. Retain the edge parameters of Section 2. The first alpha half-arc runs from $E_6(7/20)$ to $E_4(13/20)$ and returns to the vertex along E_4 . With its initial edge segment it lies in a sector of angular width less than π and retracts onto E_5 backwards, giving r^{-1} . For the second half-arc use the paired segment on E_8 from parameter 1 to $13/20$, then the chord to $E_2(7/20)$ and the returning edge segment. Its sector

contains E_8, E_1, E_2 and gives x . Both homotopies avoid O . Hence the alpha longitude is $C = x^{-1}r^{-1}$ and its section is still $\ell = Ax$. The beta chord from $E_6(13/20)$ to $E_8(7/20)$, with its based segments, retracts onto $E_6E_7E_8$, giving $E = sr^{-1}s^{-1}$. Its transport rectangle still crosses c once, so $D = BK_0^{-1}$.

The disk block now has the free fiber group and the two relations $[M, C] = [N, E] = 1$. All ten end inclusions in the table of Section 8 remain valid: their paths and rectangles lie in these sectors and the two torus collars, disjoint from O . In particular the beta meridian source is still $K_0^{-1}NK_0$, not N . The half-turn sends the four free generators to r, s, x, y , and the twists give (y^{-1}, yx, r, s) by homotopies supported away from O . Neither uses the surface relation. The two filling slopes are unchanged. Van Kampen therefore gives exactly the asserted fourteen rows. Reattaching $\Sigma_O \times D^2$ kills precisely its normal meridian.

Excision identifies $H_j(V, X)$ with $H_{j-2}(\Sigma_O)$. The map $H_2(V) \rightarrow H_2(V, X)$ is intersection with the section, hence an isomorphism. The long exact sequence gives the asserted homology and identifies $H_2(X)$ with $H_1(\Sigma_O)$ by the normal-torus boundary map. It also makes $H_2(X) \rightarrow H_2(V)$ zero. Consequently restriction kills $H^2(V; \mathbb{R})$, proving exactness of both forms.

The Euler class e is dual to the proper section. Its line bundle has a section vanishing simply on Σ_O , so the principal circle bundle is trivial over X . It is also trivial over $\Sigma_O \times D^2$. The transition function has degree one on each normal circle. In the X trivialization, write h for the principal circle. The common boundary maps onto the fundamental group of the section neighborhood times that circle, and its normal kernel generator maps to μh^{-1} . Thus

$$\pi_1(\mathcal{P}) = (Q_X \times \langle h \rangle) / \langle \mu h^{-1} \rangle = Q_X / C_\mu.$$

Theorem 14.2 identifies this group with $\widehat{G}$. □

There is a useful algebraic check on the missing surface row. The same defining eliminations as in Proposition 9.1 give

$$(44) \quad Q_X = \langle A, x, y \mid \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S}, [A^2, J] \rangle.$$

The last relator is needed in this derivation. With $z = A^2$ and u_*, v_*, w_* as in Appendix A, the exact comparison is

$$[N, E] = (u_* \mathcal{T}^{-1} u_*^{-1})(u_*[z, J] u_*^{-1})(v_* \mathcal{T} v_*^{-1}), \quad [B, r] = w_*[N, E]^{-1} w_*^{-1}.$$

All other eliminations use the retained rows. Only after the surface word $\mathcal{R} = JAJA^{-1}$ is made central does $AJA^{-1} = J^{-1}\mathcal{R}$ imply $[A^2, J] = 1$ automatically. Adjoining $t = \mathcal{R}$ then recovers (40). This verifies the same quotient without assuming that a peripheral group survives centralization.

The geometric central-kernel question is therefore exactly

$$(45) \quad \mu \notin C_\mu.$$

Even proving that μ has infinite order in Q_X would not by itself establish (45).

16. A LIOUVILLE DOMAIN WITH HYPERTIGHT BOUNDARY

The exterior has additional symplectic geometry. We give the boundary calculation because it distinguishes noncontractibility in the boundary from the stronger assertion required in the filling or its central quotient. A Liouville domain is a compact exact symplectic manifold whose primitive defines an outward transverse Liouville vector field. A contact form is hypertight if it has no contractible closed Reeb orbit.

Proposition 16.1 (The Liouville exterior and its boundary). *For all sufficiently large $k > 0$, the form $\Omega_k = \omega_1 + k\omega_2$ makes the rounded exterior X a Liouville domain. Its boundary contact structure admits a contact form whose every closed Reeb orbit, including every iterate, represents an infinite-order element of $\pi_1(\partial X)$.*

Proof. The normal bundle of Σ_O has flat rotational holonomies $-I$ and I . On its punctured disk bundle let η be the closed angular connection with normal-circle period one, and I the enclosed fiber area. With p_Σ denoting projection to the section,

$$\omega_1 = dI \wedge \eta, \quad \omega_2 = p_\Sigma^* \sigma_\Sigma,$$

where σ_Σ is its positive area form. Remove the bundle $I < \epsilon$, with $0 < \epsilon < 1$. The outward direction at this new face decreases I .

Let $\tau : Y \rightarrow \mathbb{R}/\mathbb{Z}$ be the original boundary mapping-torus coordinate and $b = \int_{\Sigma_O} \omega_2 > 0$. Choose a primitive λ_2 of ω_2 on V . The pullback of ω_2 to Y is zero, since the base area form restricts to zero on the one-dimensional base boundary. Hence $\lambda_2|_Y$ is closed and has period b on the section boundary. Since $H_1(Y) = \mathbb{Z}$, exact modifications give, simultaneously,

$$\lambda_2|_Y = b d\tau, \quad \lambda_2 = p_\Sigma^* \beta_\Sigma \text{ near } \Sigma_O, \quad d\beta_\Sigma = \sigma_\Sigma, \quad \beta_\Sigma|_{\partial\Sigma_O} = b d\tau.$$

For completeness, first take $\beta_\Sigma = \lambda_2|_{\Sigma_O}$ and adjust its boundary value by an exact form. The difference $\lambda_2 - p_\Sigma^* \beta_\Sigma$ is closed and zero on the section, hence exact on its neighborhood. Remove it there. The remaining exact boundary difference has a primitive zero on that neighborhood and can be removed using a collar supported away from the section.

A primitive λ_1 of $\omega_1|_X$ has normal-circle period $\epsilon - 1$, by Stokes on a punctured fiber of area $1 - \epsilon$ and oriented boundary $-\mu$. A further exact modification thus gives

$$\lambda_1 = (I - 1)\eta + p_\Sigma^* \zeta, \quad d\zeta = 0,$$

on a collar of the normal face. Set $\lambda_k = \lambda_1 + k\lambda_2$. On that face,

$$\lambda_k \wedge d\lambda_k = k(\epsilon - 1)\eta \wedge p_\Sigma^* \sigma_\Sigma > 0$$

with its outward orientation. On the original boundary face,

$$\lambda_k \wedge d\lambda_k = \lambda_1 \wedge \omega_1 + kb d\tau \wedge \omega_1 > 0$$

for sufficiently large k , by compactness. The Liouville vector field is outward on both faces. A monotone rounding preserves the two strict outward inequalities.

We can choose the rounding and primitive more precisely. Near the section boundary take a base area coordinate J_Σ with $J_\Sigma = b$ at its boundary and $\beta_\Sigma = J_\Sigma d\tau$. The flat boundary framing gives $\eta = d\vartheta$ there. Since $\int_{\partial\Sigma_O} \zeta = 0$, remove its exact restriction on this collar. We obtain

$$\lambda_k = (I - 1)d\vartheta + kJ_\Sigma d\tau.$$

These exact modifications preserve contactness for one sufficiently large k . Round along a curve $(I(q), J_\Sigma(q))$ with both derivatives nonnegative and not simultaneously zero, in the region $I < 1$, $J_\Sigma > 0$. On the resulting torus collar put $f = I - 1 < 0$, $g = kJ_\Sigma > 0$. Its Reeb field is

$$(46) \quad \Delta = fg' - f'g < 0, \quad R_{\lambda_k} = \frac{g'\partial_\vartheta - f'\partial_\tau}{\Delta}.$$

It has no q component and is a nonzero linear field on each torus.

Topologically ∂X is the union of the mapping torus $\mathcal{A}$ of the punctured genus-two fiber and $\mathcal{B}_\Sigma = \Sigma_O \times S^1$, meeting in a torus T . Both maps from $\pi_1(T)$ inject: on $\mathcal{A}$, use mapping-torus height and the nontrivial boundary word in the free fiber group; on $\mathcal{B}_\Sigma$, use the nontrivial

boundary word of the punctured torus and its independent circle factor. Amalgam normal forms embed both pieces and T into $\pi_1(\partial X)$.

Choose invariant level tori in the normalized collar. A closed orbit cannot cross them. In $\mathcal{A}$ it has positive mapping-torus degree, since the Reeb direction is the positively oriented suspension direction. In $\mathcal{B}_\Sigma$ it is a nonzero negative multiple of the normal circle, since $R_{\lambda_k} = (\epsilon - 1)^{-1} \partial_\theta$ there. In the collar it has nonzero integral displacement in $\pi_1(T)$ by (46). Each class has infinite order in its piece and hence in the boundary group. The argument includes all iterates. $\square$

This boundary is universally tight. Indeed hypertightness lifts to finite covers. Hofer's theorem [Hof93] would produce a contractible closed Reeb orbit for an overtwisted contact structure on any such closed cover, so all those covers are tight. Residual finiteness of three-manifold groups then places any compact overtwisted disk from the universal cover into a finite cover, a contradiction; see [CP26, Lemma 4.23] for this covering argument. In particular no π_1 -injective torus in this boundary is a *mixed torus* in the sense of [CM25]: such a torus has a virtually overtwisted toric neighborhood, whose universal cover would embed in the universal cover of the boundary. The natural interface T is incompressible and therefore cannot supply the mixed torus required by that splitting theorem.

The two pieces also give the natural spinal open book of this contact boundary, with genus-two pages and punctured-torus spine [LVW26a]. Thus this particular open book does not satisfy the planar-page hypotheses of the corresponding filling classification [LVW26b]. Neither observation rules out every possible contact decomposition. No Weinstein structure on X is asserted here. Most importantly, the injections just proved are into $\pi_1(\partial X)$, not into Q_X or Q_X/C_μ . Hypertightness is not a substitute for either missing injection.

16.1. Why exactness does not give a transverse primitive. One plausible use of the symplectic pair would require a primitive of ω_2 annihilating its kernel foliation. This stronger exactness condition fails for the particular pair constructed above.

Proposition 16.2. *There is no one-form ξ on V with $d\xi = \omega_2$ and $\xi|_{\ker \omega_2} = 0$.*

Proof. On the exterior of the two surgery neighborhoods, Cartan's formula would make ξ invariant along the fibers as well as zero on their tangent spaces. Those fibers are connected. Indeed the two disjoint curves removed from the fiber have independent homology classes $-[x] - [r]$ and $-[r]$; a disconnected complement would supply a nonzero homology relation by taking the boundary of a complementary component. Thus ξ would descend to a primitive of the base area form, using the section to identify its transverse values.

In the beta insertion put $q = Q_2 \in \mathbb{R}/\mathbb{Z}$, $p = Y_2$. A primitive annihilating $\ker \omega_2$ is independent of Q_1, Y_1 . Its values on the upper and lower gluing sides must agree, so the base primitive must satisfy

$$\mathfrak{h}^* \xi = \xi, \quad d\xi = dq \wedge dp, \quad \mathfrak{h}(q, p) = (q + \chi(p), p).$$

Write $\xi = a(q, p)dq + b(q, p)dp$ and let $F(p) = \int_0^1 a(q, p)dq$. Integrating the dp coefficient of the invariance equation gives $\chi'(p)F(p) = 0$, whereas integrating differential exactness gives $F'(p) = -1$. The nonconstant smooth step χ has derivative nonzero on an interval. The two equations contradict each other there. $\square$

This does not contradict ordinary exactness of ω_2 , or the local primitive normalizations used for the Liouville boundary. It rules out a global transverse Liouville field obtained from this pair by the stronger annihilation condition.

16.2. Two handles realizing the central quotient. In fact the central extension has a four-dimensional exact symplectic realization. To distinguish it from the double Z , denote this auxiliary domain by $\mathcal{Z}$.

Proposition 16.3 (A four-dimensional central-extension realization). *The map $\pi_1(\partial X) \rightarrow Q_X$ is surjective, and*

$$C_\mu = \langle\langle [\mu, x], [\mu, y] \rangle\rangle_{Q_X}.$$

Two Weinstein two-handles can be attached to X to give a Liouville domain $\mathcal{Z}$ with

$$\pi_1(\mathcal{Z}) = \widehat{G}, \quad H_1(\mathcal{Z}) = H_3(\mathcal{Z}) = 0, \quad H_2(\mathcal{Z}) = \mathbb{Z}^4.$$

A retained normal-circle Reeb orbit γ represents t^{-1} . The handles may be chosen so that $c_1(T\mathcal{Z}) = 0$.

Proof. The marked x, y can be transported to the original boundary along a base arc in $v, w < 1/2$, missing both surgery tracks. Fix their common basepoint on the original boundary. For the normal-face loop below, compare its radial basing path with a path from this boundary basepoint: radial transport times the same base arc is a square contained in $v, w < 1/2$, disjoint from both surgery tracks. Its boundary homotopy identifies the two basing routes in X . Thus all three loops below are images of loops with this one boundary basepoint, rather than independently conjugated boundary images. On the normal face, follow the A base loop using flat normal transport and close it by the positive inner semicircle from the opposite radial point back to the initial one. Based by the radial arc, this loop is $A[r, s]$: the comparison path retracts to $E_5 E_6 E_7 E_8$. The transport rectangle misses the alpha track, and the beta chord e misses both radial arcs. Thus no additional meridian occurs. The exterior relations give

$$[r, N^{-1}s] = [r, s], \quad A[r, s]A^{-1} = [x, y] = J,$$

using $[N, E] = 1$ and the Ar, As rows. Hence this boundary loop is $U_\partial = JA$. Together with x, y it generates Q_X , since $A = J^{-1}U_\partial$.

For the centralization assertion use the free identity

$$[JA, \mathcal{R}] = \mathcal{R}[A^2, J]\mathcal{R}^{-1}.$$

Thus JA commutes with $\mu = \mathcal{R}$ already in Q_X . Imposing $[\mu, x] = [\mu, y] = 1$ makes μ commute with J , and consequently with $A = J^{-1}(JA)$. Since A, x, y generate the whole group, this centralizes μ against every element. The reverse normal-subgroup inclusion is immediate.

Choose disjoint embedded knot representatives of these two boundary commutators, away from one normal-circle Reeb orbit. Legendrian approximation and the exact two-handle construction [Wei91] give two Weinstein handles. Their attachment kills precisely the stated two conjugacy classes. The contact form near the chosen orbit can be retained, so its class becomes t^{-1} . The relative handle chain complex has $\mathbb{Z}^2$ only in degree two. Together with Proposition 15.1, this gives the stated homology.

Since $H_2(X) \rightarrow H_2(V)$ is zero and $H_1(X) = 0$, restriction gives $c_1(TX) = 0$ integrally. Fix a determinant trivialization and use the induced contact-plane trivialization to measure rotation. Positive and negative stabilizations change a Legendrian rotation number by 1 and -1 , so the two knots can be given rotation zero. Their free homotopy classes and the marked orbit are unchanged. The relative first Chern obstruction of each handle is that rotation number, by the local obstruction computation in the proof of [Gom98, Proposition 2.3]; hence the determinant trivialization extends over both handles. These choices change

auxiliary handle framings, which were not prescribed. They change none of the original two surgery slopes defining V . $\square$

This construction uses a Weinstein attaching cobordism, and does not assert that X or $\mathcal{Z}$ itself is Weinstein. Since $\widehat{G}$ is superperfect, the Hopf sequence makes $\pi_2(\mathcal{Z}) \rightarrow H_2(\mathcal{Z}) = \mathbb{Z}^4$ surjective. Thus $\mathcal{Z}$ is not aspherical. Exactness does not exclude these mapped spherical classes.

16.3. The orbit index and its limitation. The retained orbit has identity linearized return map, so its relevant index is the Robbin–Salamon index, not a nondegenerate Conley–Zehnder index. Normalize a positive full rotation of a symplectic plane to have index two. With the determinant trivialization of Proposition 16.3,

$$(47) \quad \mu_{\text{RS}}(\gamma^n) = -6n \quad (n \geq 1).$$

Here is the framing calculation. A fiber punctured at the section has genus two, Euler characteristic -3 , and oriented boundary $\gamma = -\mu$. Its normal bundle has the product frame n_0 from the base disk. An exact change near the chosen base point arranges that the tangential base term of the primitive vanishes there; the Liouville vector v along γ is then the outward tangent normal of the punctured fiber. If $\mathfrak{d}$ is the determinant trivialization, write $v \wedge n_0 = a\mathfrak{d}$ on γ . The relative Euler number gives $\deg a = -3$. The contact frame e_0 induced by $\mathfrak{d}$ satisfies $n_0 = ae_0$. Pure circle translation preserves n_0 , so in the e_0 frame the return path has winding -3 . This proves (47) and its iterate formula. Since $H^1(\mathcal{Z}; \mathbb{Z}) = 0$, the determinant trivialization is unique up to homotopy. If an iterate contracts in $\mathcal{Z}$, its capping-disk index is the same, independently of the disk, because $c_1(T\mathcal{Z}) = 0$.

The index does not preclude such a disk. A useful comparison is the Brieskorn link

$$\{z_0^6 + z_1^6 + z_2^6 = 0\} \cap S^5.$$

Its periodic contact form has simple Hopf-circle orbits of index -6 and iterate indices $-6n$, computed in the Milnor-filling trivialization [KvK16, Sections 2–3 and 5.3, Proposition 5.9]. The Milnor filling is homotopy equivalent to a wedge of 125 two-spheres and has $c_1 = 0$, so every orbit contracts there. The precise periodic boundary form can be obtained from a boundary form $f\alpha$ by taking the graph $s = \log(C/f)$ in the positive symplectization, with $C > \max f$. Yet the link is a circle bundle over the smooth plane sextic of genus ten; its circle subgroup injects because that base is aspherical. The periodic form is therefore hypertight. This comparison rules out an implication from exact fillability, $c_1 = 0$, hypertightness and the indices (47) alone. It does not have the topology or the complete presentation of our domain.

16.4. What a mixed-torus decomposition would still require. There is also a general homological restriction on applying the splitting theorem of [CM25, Theorem 1.1]. If a connected exact filling L has $H_1(L; \mathbb{Z}) = 0$, splitting it along a mixed torus produces exactly two filling components, and the torus separates its boundary. Indeed the reconstruction attaches one round one-handle

$$(S^1 \times D^1 \times D^2, S^1 \times S^0 \times D^2).$$

Its relative H_1 is $\mathbb{Z}$. In the pair sequence the boundary of its generator is the difference of the two attaching component classes. Since $H_1(L) = 0$, these components must be distinct. A single round handle can connect only two components. Neither can be closed, by Stokes'

theorem for an exact symplectic form. A nonseparating torus would instead give a connected cut-and-capped boundary, a contradiction.

This applies both to X and to the auxiliary $\mathcal{Z}$. For X it complements the universal-tightness obstruction: any mixed torus would have to be separating and fail to be π_1 -injective. If each auxiliary attaching knot is stabilized once more in both signs, its rotation remains zero and the surgery supplies the mixed tori of [CM25, Theorem 1.4]. Applying that theorem twice reduces an exact filling of the resulting boundary to some exact filling of ∂X . It does not identify that filling with the marked X or prove uniqueness of its fundamental group. For the filling constructed here, reattaching the two handles restores exactly $Q_X/\langle\langle[\mu, x], [\mu, y]\rangle\rangle$. The same centralization problem remains.

17. A PROPOSED CAPACITY OBSTRUCTION FROM A LAGRANGIAN TORUS

We use the Liouville form λ_k from Proposition 16.1 and the determinant trivialization $\mathfrak{d}$ from Proposition 16.3. For an oriented Lagrangian torus L , write $m_L \in H^1(L; \mathbb{Z})$ for its Maslov class: it is twice the winding of the complex determinant of an oriented tangent frame relative to $\mathfrak{d}$. Thus a positively oriented circle in the standard complex line has Maslov index two. Since $H^1(X; \mathbb{Z}) = 0$, the homotopy class of $\mathfrak{d}$ is unique.

17.1. The torus and its periods. Write $S = \Sigma_O$. On a collar of its removed normal disk bundle, the formulas in Proposition 16.1 read

$$(48) \quad \begin{aligned} \Omega_k &= dI \wedge \eta + kp_{\Sigma}^* \sigma_{\Sigma}, \\ \lambda_k &= (I - 1)\eta + p_{\Sigma}^* \zeta + kp_{\Sigma}^* \beta_{\Sigma}, \\ d\eta &= d\zeta = 0, \quad d\beta_{\Sigma} = \sigma_{\Sigma}. \end{aligned}$$

The angular form η has period one on the positive normal circle. Fix I_0 strictly inside this collar, with $I_0 < 1$, and set $c = 1 - I_0 > 0$.

Proposition 17.1 (The area–Maslov relation). *For all sufficiently large k , the interior of X contains a nullhomologous Lagrangian torus L with an integral basis μ, ν of $H_1(L)$ satisfying*

$$(49) \quad \begin{array}{c|cc} & \mu & \nu \\ \hline \int_{m_L} \lambda_k & -c & c/3 \\ & 6 & -2. \end{array}$$

In particular,

$$(50) \quad [\lambda_k|_L] = -\frac{c}{6}m_L.$$

The same identity holds in every auxiliary domain $\mathcal{Z}$ of Proposition 16.3 for which the determinant trivialization extends across the handles.

Proof. Choose a compact once-punctured torus $S' \subset \text{int } S$ whose σ_{Σ} -area is

$$a = \frac{c}{3k}.$$

Such a subsurface exists for large k : regular neighborhoods of a two-loop spine of S have arbitrarily small area, and enlarging the boundary through the remaining annulus varies the area continuously. Put $\ell' = \partial S'$, with its boundary orientation, and define

$$L = \{I = I_0\} \cap p_{\Sigma}^{-1}(\ell').$$

This is a torus. Both terms of Ω_k vanish on its tangent planes: dI vanishes, and their projection to S is one-dimensional. The normal circle bundle at radius I_0 over S' is an embedded three-manifold bounded by L , so $[L] = 0$ in $H_2(X; \mathbb{Z})$.

Let μ be its positive normal-circle class. The normal circle bundle over S' is topologically trivial because $H^2(S'; \mathbb{Z}) = 0$. Choose a unit section s_0 and let ν be its boundary lift of ℓ' . These classes form an integral basis. The one-form $s_0^* \eta$ is closed, hence

$$\int_{\nu} \eta = 0, \quad \int_{\ell'} \zeta = 0, \quad \int_{\ell'} \beta_{\Sigma} = a.$$

Equation (48) gives the first row of (49). This use of Stokes' theorem does not require a horizontal section for the flat normal connection.

The normal-circle determinant winding, with a fixed nonzero base vector, is locally constant as the base point varies in S . For the Maslov index of μ , we may therefore evaluate it at an untouched genus-two fiber punctured at radius I_0 . Its oriented boundary is $-\mu$, and its Euler characteristic is -3 . Its complex normal line has the product frame from the base disk. The relative Euler-number calculation in (47) therefore gives determinant winding -3 along $-\mu$. A tangent frame of L along that circle differs from the outward fiber-normal frame and this product normal frame by constant nonzero complex factors. Its Maslov index along $-\mu$ is consequently -6 , giving $m_L(\mu) = 6$.

For ν , use the displaced section S'_{s_0} at radius I_0 . The restriction of Ω_k is $k\sigma_{\Sigma}$, so it is symplectic. The angular vector field is everywhere nonzero and symplectically normal to this section: its contraction with $dI \wedge \eta$ vanishes on all tangent vectors of S'_{s_0} . It therefore frames the normal complex line. Choose a compatible almost complex structure preserving this tangent-normal splitting. Along ν , the angular vector and the oriented boundary tangent span TL . Their determinant winding is the relative Euler number of TS' , namely $\chi(S') = -1$. Thus $m_L(\nu) = -2$. The space of compatible almost complex structures is contractible, so these two adapted computations determine the same Maslov class.

The two rows of (49) prove (50) on an integral basis. The torus and its bounding three-manifold lie in the unchanged interior of X . The Liouville form there is unchanged by the auxiliary handles, and the determinant trivialization extends by hypothesis. All the assertions therefore persist in $\mathcal{Z}$. $\square$

The torus is not exact, since its meridional period is nonzero. For every smooth relative disk $u : (D^2, \partial D^2) \rightarrow (M, L)$, where M is the completion of X or $\mathcal{Z}$, exactness and the determinant trivialization give

$$(51) \quad \int_{D^2} u^* \Omega_k = \int_{\partial D^2} u^* \lambda_k = -\frac{c}{6} \mu_L(u).$$

Here $\mu_L(u)$ is the relative Maslov index of the disk. This identity assumes neither injectivity of $\pi_1(L) \rightarrow \pi_1(M)$ nor nontriviality of t . In particular a nonconstant compatible pseudoholomorphic disk cannot have Maslov index two, since its symplectic area would be $-c/3$.

17.2. The capacity consequence. Use real coefficients for symplectic cohomology and its S^1 -equivariant version. We use the convention of [Li26] for the first Gutt–Hutchings capacity c_1^{GH} . By [Li26, Proposition 5], its finiteness is equivalent to vanishing of the equivariant image of the unit under

$$I : \text{SH}^0(M; \mathbb{R}) \longrightarrow \text{SH}_{S^1}^0(M; \mathbb{R}).$$

It is also equivalent to a cyclic dilation reaching the unit in the sense of that reference.

Corollary 17.2. *For the Liouville forms and rotation-zero auxiliary handles above,*

$$c_1^{\text{GH}}(X) = c_1^{\text{GH}}(\mathcal{Z}) = \infty, \quad \text{SH}^0(X; \mathbb{R}) \neq 0, \quad \text{SH}^0(\mathcal{Z}; \mathbb{R}) \neq 0.$$

In each completion the equivariant image $I(1)$ is nonzero.

Proof. The torus L is closed, oriented, spin, and a $K(\mathbb{Z}^2, 1)$. It is relatively spin with background class zero, and the ambient first Chern class vanishes. If the first capacity were finite, [Li26, Corollary 8] would give a loop on L with Maslov class two and positive symplectic area. In this exact ambient manifold that area is its λ_k -period, contradicting (50). Thus the capacity is infinite. Proposition 5 of the same reference gives $I(1) \neq 0$, which implies that the ordinary symplectic-cohomology unit is nonzero. $\square$

This excludes cyclic dilations reaching the unit. It does not exclude a cyclic quasi-dilation reaching another invertible class. The construction gives no obstruction to a continuous disk bounded by $\gamma = -\mu$: such a disk would have area c and Maslov index -6 , consistently with (51).

17.3. A simply connected comparison. The area–Maslov relation itself does not imply survival of a torus loop in the ambient group. The following comparison also retains nullhomology of the torus.

Proposition 17.3. *There is a simply connected Liouville four-manifold M_0 with $c_1(TM_0) = 0$ and a nullhomologous Lagrangian torus L_0 such that, for some $\tau > 0$,*

$$[\lambda_0|_{L_0}] = -\tau m_{L_0}.$$

In an integral basis the periods are $(-6\tau, 2\tau)$ and the Maslov indices are $(6, -2)$. Moreover $H_2(M_0) = \mathbb{Z}^8$, $H_1(M_0) = H_3(M_0) = 0$, and $c_1^{\text{GH}}(M_0) = \infty$.

Proof. Let F_0 and B_0 be Liouville surfaces of genus two and one, respectively, each with one boundary component. Round the corners of their product $W_0 = F_0 \times B_0$. Each tangent complex line is trivial, so fix a product determinant trivialization on W_0 . Choose interior subsurfaces F'_0, B'_0 of the same genera and scale the area forms so that their areas are 6τ and 2τ . The product

$$L_0 = \partial F'_0 \times \partial B'_0$$

is Lagrangian and bounds $F'_0 \times \partial B'_0$. Take $\mu_0 = -\partial F'_0$ and $\nu_0 = \partial B'_0$. Stokes' theorem gives periods $-6\tau, 2\tau$. Twice the relative Euler numbers gives Maslov indices $6, -2$, proving the stated relation in W_0 .

The boundary inclusion surjects onto $\pi_1(W_0) = F_4 \times F_2$: each factor's generators can be placed over a boundary point of the other factor. Represent the six generators by disjoint boundary knots and choose Legendrian approximations. Positive and negative stabilizations make each rotation zero relative to the chosen determinant trivialization. Attach six Weinstein two-handles along these knots. Van Kampen gives $\pi_1(M_0) = 1$, and the rotation-zero condition extends the determinant trivialization over the handles, as in Proposition 16.3. The torus, its bounding three-manifold, and its Liouville and Maslov classes are unchanged.

The relative handle boundary map

$$H_2(M_0, W_0) = \mathbb{Z}^6 \longrightarrow H_1(W_0) = \mathbb{Z}^6$$

is an isomorphism, since the attaching knots are the six generators. The pair sequence and the Künneth formula give $H_2(M_0) = H_2(W_0) = \mathbb{Z}^8$ and $H_1(M_0) = H_3(M_0) = 0$. The

proof of Corollary 17.2 applies to L_0 and gives infinite first capacity and nonzero symplectic cohomology. $\square$

All torus loops contract in this example. Its six attaching knots kill generators, whereas the two knots in our construction impose centrality of a meridian. Thus the actual attaching data remain essential to a fundamental-group argument.

18. A PROPOSED EXACT-TORUS REALIZATION

The torus in Section 17 lies inside X and is not exact. We now choose the two auxiliary attaching knots so that closed exact tori appear outside X . The construction and its application below are proposed additions awaiting independent verification. They preserve the quotient Q_X/C_μ , but prescribe different Legendrian representatives and auxiliary framings from the stabilized choice in Section 19.

18.1. Exact fillings of commutators. An exact Lagrangian cobordism in the symplectization $(\mathbb{R} \times Y, d(e^s \alpha))$ is cylindrical outside a compact set, and the primitive of $e^s \alpha$ on it is constant on each entire end. In particular the constants agree across the components of a disconnected end. This convention permits concatenation of the elementary cobordisms used below [SVW24, Definition 2.1].

Lemma 18.1. *Let $(Y, \ker \alpha)$ be a cooriented contact three-manifold and let $a, b \in \pi_1(Y)$ be based loops. There is a Legendrian knot representing $[a, b]$, up to conjugacy and inversion, with an embedded orientable exact Lagrangian filling of genus one in the symplectization. Its two band generators project to a, b . Finitely many such fillings may be constructed with disjoint supports, away from a prescribed embedded curve in Y .*

Proof. Start with a standard maximal Legendrian unknot U in a Darboux ball disjoint from the prescribed curve. Choose four points on U in cyclic order, and two disjoint arcs joining the first to the third and the second to the fourth. Closing each arc by a path in the ball, with fixed paths to the basepoint, prescribes the classes a, b . General position in Y permits embedded arcs with these relative homotopy classes, disjoint from the protected curve and from U except at their endpoints.

Prescribe Legendrian endpoint germs satisfying the orientation condition for ambient surgery: if T is the oriented tangent to U and G is the outward tangent of the arc, require $d\alpha(G, T) > 0$ at both endpoints. These are the local conditions of [SVW24, Section 2.2]. Relative Legendrian approximation in disjoint small tubes, fixing the endpoint germs, gives Legendrian arcs with the same relative homotopy classes; see [Etn05, Theorem 2.5 and its arc construction]. The resulting graph is a Legendrian handle graph.

Fill U by the elementary exact Lagrangian zero-handle, and perform the two ambient surgeries. Their traces are orientable exact Lagrangian one-handles, supported near the arcs [DR16, Section 4], [SVW24, Theorem 2.2]. The alternating endpoints make the first surgery split the boundary and the second rejoin it. Thus the connected filling has one positive boundary and Euler characteristic $1 - 2 = -1$. The end-constant primitives of the elementary cobordisms glue; in particular the intermediate two-component end introduces no mismatch of constants. The filling is therefore exact.

Collapsing the initial disk gives a ribbon graph with one vertex and two alternating bands. Its boundary word is $aba^{-1}b^{-1}$, up to orientation and choice of starting point. The disk projects into the Darboux ball, and the band cores follow the prescribed arcs, so the two

band generators have the asserted images. For finitely many pairs, choose disjoint balls and arcs. Locality of the zero-handles and ambient surgeries then gives disjoint supports. $\square$

If the ambient complex determinant line is trivialized, the positive boundary of this filling has rotation zero. Indeed its Maslov class evaluates trivially on the boundary, which is zero in the first homology of the filling. At a cylindrical end that evaluation is twice the Legendrian rotation. This argument does not assert that the whole Maslov class of the filling vanishes.

18.2. The tori and the transfer map. Use the two based pairs (μ, x) and (μ, y) in $\pi_1(\partial X)$ from Proposition 16.3. Apply Lemma 18.1 away from the retained orbit γ . Place both fillings in a sufficiently long positive collar of X , and attach a Weinstein two-handle along each positive boundary. Write $\mathcal{Z}_L$ for this auxiliary domain. Let

$$\tau_X : \mathrm{SH}^*(\mathcal{Z}_L; \mathbb{F}_2) \longrightarrow \mathrm{SH}^*(X; \mathbb{F}_2)$$

be Hamiltonian Viterbo restriction, and let $c_M : H^*(M; \mathbb{F}_2) \rightarrow \mathrm{SH}^*(M; \mathbb{F}_2)$ denote the canonical map. We use the cohomological grading in which c_M has degree zero.

Proposition 18.2. *The domain $\mathcal{Z}_L$ contains disjoint closed exact Lagrangian tori T_x, T_y in the complement of X , and*

$$\pi_1(\mathcal{Z}_L) = \widehat{G}, \quad c_1(T\mathcal{Z}_L) = 0, \quad H_1(\mathcal{Z}_L) = H_3(\mathcal{Z}_L) = 0, \quad H_2(\mathcal{Z}_L) = \mathbb{Z}^4.$$

The tori map to the two standard generators of $H_2(\mathcal{Z}_L, X) = \mathbb{Z}^2$. For

$$K_L = \ker(H^2(\mathcal{Z}_L; \mathbb{F}_2) \longrightarrow H^2(X; \mathbb{F}_2)),$$

the restriction of $c_{\mathcal{Z}_L}$ to $K_L \cong \mathbb{F}_2^2$ is injective, and its image lies in $\ker \tau_X$. In particular,

$$(52) \quad \dim_{\mathbb{F}_2} \ker(\mathrm{SH}^2(\mathcal{Z}_L; \mathbb{F}_2) \longrightarrow \mathrm{SH}^2(X; \mathbb{F}_2)) \geq 2.$$

These detected classes belong to the contractible free-homotopy summand.

Proof. Translation in the symplectization rescales the primitive and places the compact parts of the fillings outside the original X . Extending the collar changes neither its homotopy type nor its symplectic cohomology. The core disk of each new handle caps the corresponding punctured torus. Normalize its primitive to zero at its single positive end, matching the core primitive. This produces disjoint closed exact tori T_x, T_y outside X . Their boundary rotations vanish by the preceding argument, so the determinant trivialization extends across both handles.

Van Kampen imposes exactly $[\mu, x] = [\mu, y] = 1$, giving $\pi_1(\mathcal{Z}_L) = Q_X / C_\mu = \widehat{G}$. The band generators induce the marked maps

$$\begin{aligned} \pi_1(T_x) &= \mathbb{Z}^2 \longrightarrow \widehat{G}, & (m, n) &\longmapsto t^m x^n, \\ \pi_1(T_y) &= \mathbb{Z}^2 \longrightarrow \widehat{G}, & (m, n) &\longmapsto t^m y^n. \end{aligned}$$

No injectivity is asserted. The relative handle sequence gives

$$0 \longrightarrow H_2(X) \longrightarrow H_2(\mathcal{Z}_L) \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

and the stated homology. Each torus uses its own handle core once and the other core zero times, so their classes split this sequence. Evaluation on them consequently identifies K_L with $\mathbb{F}_2^2$.

Choose disjoint exact cotangent neighborhoods $N_j \cong D^*T_j$, disjoint from X , for $j = x, y$. The ambient primitive and the cotangent primitive differ by an exact form: their difference

is closed and has zero periods on the zero section. A cutoff change of primitive therefore makes each N_j a Liouville subdomain without changing X or the external boundary [Rit13, Example 9.1 and Lemma 9.2]. Naturality of the canonical maps gives

$$(53) \quad \tau_j c_{\mathcal{Z}_L}(h) = c_{N_j}(h|_{N_j}), \quad \tau_X c_{\mathcal{Z}_L}(h) = c_X(h|_X),$$

where τ_j is restriction to N_j [Rit13, Theorem 9.5].

For a cotangent neighborhood of a torus, c_{N_j} is injective. Under the cotangent-bundle identification

$$\mathrm{SH}^*(D^*T^2; \mathbb{F}_2) \cong H_{2-*}(\mathcal{L}T^2; \mathbb{F}_2),$$

where $\mathcal{L}T^2$ is the free loop space, it is Poincaré duality followed by inclusion of constant loops [Rit13, Section 14, Remark 14.1]. Evaluation at a fixed time is a left inverse to that inclusion. The restricted ambient determinant trivialization need not be the canonical cotangent one; the two give the same grading on contractible loops, where c_{N_j} takes values. No Maslov-zero assertion about T_j or injection of its fundamental group is used.

For $0 \neq h \in K_L$, its restriction to at least one N_j is nonzero by the evaluation isomorphism. The first equality in (53) implies $c_{\mathcal{Z}_L}(h) \neq 0$, and the second implies $\tau_X c_{\mathcal{Z}_L}(h) = 0$. Canonical images are represented by constant Hamiltonian orbits, proving the final assertion. $\square$

This construction gives specific classes in a transfer kernel; it does not compute all of either symplectic cohomology group. Its exact tori are homologically essential and lie outside X , whereas the non-exact torus of Section 17 is nullhomologous and lies inside X . Neither argument establishes that the marked torus loops survive in $\widehat{G}$. For example, applying Lemma 18.1 to $a = b = 1$ in ∂B^4 and attaching one handle gives a simply connected Liouville domain containing an exact torus with primitive second-homology class. The same cotangent-neighborhood argument detects that class in symplectic cohomology.

19. SYMPLECTIC HOMOLOGY AND THE AUXILIARY HANDLES

There is a further possible calculation for a choice of the two auxiliary handles. Stabilize each rotation-zero attaching Legendrian once positively and once negatively, away from the retained orbit. The rotations and smooth free homotopy classes are unchanged, while the auxiliary framings decrease. Denote the resulting domain by $\mathcal{Z}_{\mathrm{st}}$. It still has

$$\pi_1(\mathcal{Z}_{\mathrm{st}}) = \widehat{G}, \quad c_1(T\mathcal{Z}_{\mathrm{st}}) = 0, \quad H_2(\mathcal{Z}_{\mathrm{st}}) = \mathbb{Z}^4.$$

The Legendrian surgery formula of Bourgeois–Ekhholm–Eliashberg [BEE12, Section 5, Theorem 5.6] allows a Liouville base with a critical Weinstein attaching cobordism. If the anchored Chekanov–Eliashberg algebra of each successive attaching knot has an element a with $\partial a = 1$, its handle contribution is acyclic by [BEE12, Section 7.2]. To interpret the resulting isomorphism in Hamiltonian symplectic homology, we also assume compatibility of the SFT surgery construction with that theory. Under these hypotheses,

$$(54) \quad \mathrm{SH}_*(\mathcal{Z}_{\mathrm{st}}; \mathbb{Q}) \cong \mathrm{SH}_*(X; \mathbb{Q})$$

as graded vector spaces, with all free homotopy classes included. No Weinstein structure on the base X is required for this conditional application.

The local stabilized-knot computation in standard contact three-space gives a short chord with differential one [EN20, Proposition 3.12]. To apply it to the anchored algebra here, one must also exclude disks leaving the Darboux neighborhood and contributions with negative closed-orbit punctures, at both successive attachments. A possible argument uses a sufficiently small chord action, lower action bounds for other asymptotics, and a uniform

monotonicity bound on the surrounding annulus. We do not establish that localization assertion in this revision; consequently (54) is retained with the stated algebraic hypothesis and the comparison assumption above. These are separate analytic issues. The discussion following [Sei13, Theorem 6.13] explicitly assumes equivalence of the SFT and Hamiltonian constructions in deriving the stabilized-handle statement. Identifying the resulting map with Hamiltonian Viterbo restriction requires the corresponding naturality comparison; an abstract isomorphism of graded vector spaces does not provide it.

The domain $\mathcal{Z}_{\text{st}}$ is not identified with $\mathcal{Z}_L$ from Section 18. The Legendrian representatives and auxiliary framings differ. Moreover, (52) uses $\mathbb{F}_2$, whereas (54) uses $\mathbb{Q}$. Their common marked fundamental group does not identify the domains, coefficient theories, or transfer maps.

Even such an isomorphism would not prove that μ survives. The inclusion $j : X \rightarrow \mathcal{Z}_{\text{st}}$ can identify distinct free homotopy classes. A class formerly represented by μ can then belong to the contractible sector in the larger domain without changing total symplectic homology. An isomorphism of the total graded vector spaces is not an injection of the ambient fundamental groups.

Ambient local systems make the missing condition explicit. A representation ρ of Q_X extends across the two handles exactly when

$$\rho([\mu, x]) = \rho([\mu, y]) = 1.$$

To detect the central element it must also satisfy $\rho(\mu) \neq 1$. Constructing it therefore still requires a representation of the full quotient Q_X/C_μ with the specified nonidentity image. Rank-one ambient local systems are trivial because $H_1(X) = 0$. Likewise, based-loop coefficients attached to a Legendrian stop concern that stop's loop space [Ekh24, Section 4.5]; they do not by themselves identify the invariant with loops in the filling.

20. THE REMAINING QUESTIONS AND THEIR LOGICAL RELATIONS

The established constructions can be organized as follows:

$$\begin{aligned} \Pi &\hookrightarrow Q_c \hookrightarrow Q_2 \hookrightarrow K \twoheadrightarrow H_\beta \twoheadrightarrow G, \\ Q_X &\twoheadrightarrow \widehat{G} \twoheadrightarrow G, \quad \pi_1(\mathcal{P}) = \pi_1(\mathcal{Z}) = \widehat{G}. \end{aligned}$$

The first three inclusions are proved before the final filling. The kernel of $K \rightarrow H_\beta$ is free with the module calculated in Proposition 11.1; the remaining kernel is unknown. The second line describes centralization of the section meridian, followed by killing its central image. These two lines isolate different possible ways to decide the same complete group.

The homomorphism $h : K \rightarrow \mathbb{Z}$ taking A to 1 and Q_2 to 0 is well defined. Since $h(\lambda) = 1$, the final attaching word is nontrivial in K . The complete group is obtained only after it is imposed.

Question 20.1. In the group K of Theorem 1.3, is

$$J \in \langle\langle \lambda \rangle\rangle_K?$$

Equivalently, do there exist finitely many $h_i \in K$ and $\varepsilon_i \in \{1, -1\}$ with

$$(55) \quad J = \prod_{i=1}^n h_i \lambda^{\varepsilon_i} h_i^{-1}?$$

An equality (55) would prove $G = 1$ by Proposition 9.2 and complete the conditional conclusions of Corollary 1.2. A proof of nonmembership would show $G \neq 1$. By Proposition 4.1, it would also show that this particular double is not simply connected and that this particular W is not homeomorphic to $\mathcal{B}$.

The stronger statement

$$(56) \quad \langle\langle \lambda \rangle\rangle_K \cap \Pi = \{1\}$$

would prove nonmembership, but it is not established. The surface injection (34) precedes the final quotient and supplies no such conclusion by itself. In fact, the whole vertex group Q_2 cannot inject under the actual map $K \rightarrow G$. The element

$$\kappa_* = a_* x a_*^{-1} r^{-1}$$

dies there, because $\lambda = 1$ sets $A = a_*$ and the A edge relation gives $Ax A^{-1} = r$. On the other hand, $H_1(Q_2) = \mathbb{Z}\langle r, s, B \rangle$: its relations kill x, y and impose no relation on these three classes. The class of κ_* is $-r$, so it is nontrivial in Q_2 . A theorem asserting injection of that entire HNN base cannot apply to this quotient. This obstruction does not settle the smaller surface injection question (56).

The intermediate free-kernel result likewise does not control the new relations after λ is imposed. One must either produce an equality (55) in the complete intermediate group or give an argument excluding every such equality. An explicit representation of all fifteen relations with nonidentity J would also prove nonmembership. Failure to find such a representation in any chosen collection does not prove membership.

Sufficient statements and unresolved hypotheses. For the central route, the following three statements are equivalent:

$$t \neq 1 \text{ in } \widehat{G} \iff \mu \notin C_\mu \iff \text{im}(\pi_2(V) \xrightarrow{\langle e, - \rangle} \mathbb{Z}) \neq \mathbb{Z}.$$

Each implies $G \neq 1$. Equivalently the retained orbit γ does not bound a mapped disk in $\mathcal{Z}$. None follows from its infinite order in the boundary, the index calculation, or exactness of the filling. The proposed capacity obstruction, exact-torus transfer classes, and conditional total-homology isomorphism in Sections 17–19 do not supply this implication either. Conversely $t = 1$ would leave the decision of G open. A proof about arbitrary spherical classes or arbitrary relations in (40) is required.

A group-algebra criterion for infinite order. There is also an elementary algebraic formulation of a stronger sufficient condition. Let $\mathbb{F} = \mathbb{F}_2(q)$, with q transcendental, and use the group algebra with finite group support. Since t is central, the quotient

$$(57) \quad B_q = \mathbb{F}[\widehat{G}]/(qt - 1)$$

is a unital $\mathbb{F}$ -algebra, possibly the zero algebra.

Proposition 20.2. *The algebra B_q is nonzero if and only if t has infinite order. It has no nonzero finite-dimensional unital module over $\mathbb{F}$.*

Proof. If t has finite order d , then

$$(qt - 1)(1 + qt + \cdots + (qt)^{d-1}) = q^d - 1.$$

The right side is a nonzero scalar, so $qt - 1$ is invertible and $B_q = 0$. If t has infinite order, decompose $\mathbb{F}[\widehat{G}]$ into the direct sum of its $\langle t \rangle$ -orbit subspaces. Each is a copy of $\mathbb{F}[z, z^{-1}]$, on which multiplication by $qt - 1$ is multiplication by $qz - 1$. Its cokernel is $\mathbb{F}$, by evaluation at

$z = q^{-1}$. Centrality of t identifies this cokernel with the quotient by the two-sided ideal in (57), so $B_q \neq 0$.

A unital module of positive finite dimension N would give a representation of $\widehat{G}$ with $t = q^{-1} \text{id}$. The determinant character is trivial because $\widehat{G}$ is perfect, whereas this matrix has determinant $q^{-N} \neq 1$. This contradiction proves the second assertion. $\square$

The algebra has a concrete presentation using the invertible generators A, x, y and the words $\mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S}$ of (24): impose

$$\mathcal{R} = q^{-1}, \quad \mathcal{T} = \mathcal{X} = \mathcal{Y} = \mathcal{S} = 1.$$

This is the central-extension presentation with scalar central character; it does not replace the surface relation of G . A nonzero module for this algebra would prove that t has infinite order, hence $G \neq 1$. No such module or independent nonvanishing argument is supplied here. This criterion does not detect finite nontrivial order of t .

Finite group support is essential. In the larger ring $\mathbb{F}_2[\widehat{G}((q))]$, which allows group support to grow with the power of q , the series $-\sum_{n \geq 0} q^n t^n$ inverts $qt - 1$ for every t . That completion destroys the criterion; it is different from $\mathbb{F}_2((q))[\widehat{G}]$, whose elements retain finite group support.

The full group remains undecided. For a triviality argument, any one of the collapse relations in Proposition 12.2, proved from the existing relations rather than added as a hypothesis, would suffice. The cyclic-centralizer assertion (67) is another sufficient statement, but its centralizer is taken after imposing the surface relation. The known cyclic centralizer in the earlier trefoil amalgam does not supply it. Likewise a new geometric decomposition would have to prove its gluing-face injections, avoiding the two specific obstructions in Section 12.1.

These formulations are useful because they name the missing assertions and retain the complete construction. They do not provide a theorem whose verified hypotheses already settle one of them. The four unit fillings remain geometrically eligible, but the known reflection changes a section index and supplies no equivalence between the two opposite sign choices with indices $(0, 0)$. No alternative has acquired a proved advantage for the final group calculation.

The present results establish marked geometric and algebraic structure, and exclude several specific shortcuts. They do not indicate which answer the final normal-closure question has. In particular neither the failure of those methods nor unsuccessful representation searches is evidence that G is trivial or nontrivial. The exotic-manifold conclusions remain conditional, and the fixed construction remains an open problem.

APPENDIX A. FREE IDENTITIES FOR THE PRESENTATION CHANGE

We give the products used in Proposition 9.1, so that the equivalence with every geometric relation can be checked without a group-search algorithm. All identities in this appendix are free word identities after the definitions (23) are substituted. Put $g = \ell^{-1}x^{-1}$. Then

$$(58) \quad [M, (rx)^{-1}] = g\mathcal{T}g^{-1}.$$

For the beta peripheral commutator put

$$b_* = sz^{-1}s^{-1}, \quad u_* = b_*r^{-1}A, \quad v_* = sz^{-1}r^{-1}A.$$

One has

$$(59) \quad [N, E] = (u_* \mathcal{T}^{-1} u_*^{-1})(u_* \mathcal{R}^{-1} u_*^{-1}) \cdot ((u_* J A) \mathcal{R} (u_* J A)^{-1})(v_* \mathcal{T} v_*^{-1}).$$

If $w_* = s^{-1} E^{-1}$, then

$$(60) \quad [B, r] = w_* [N, E]^{-1} w_*^{-1}.$$

These products account for both the $[N, E]$ and Br rows. The alpha meridian word $AMA^{-1}K_0^{-1}$ is freely trivial.

The exact identity $K_0 = (xA)^{-1} \mathcal{T}^{-1}$ gives (26). Finally, after the defining elimination $B = s^{-1} N s$, put $D = B K_0^{-1}$. Then

$$(61) \quad \begin{aligned} \mathcal{S}_0 &= D N^{-1}, & F_\beta &= \mathcal{S}_0^{-1}, \\ B(K_0^{-1} N K_0) B^{-1} N^{-1} &= \mathcal{S}_0 (N \mathcal{S}_0^{-1} N^{-1}). \end{aligned}$$

This accounts for the beta meridian transport as well as the original beta filling. The rows used to define r, s, M, N vanish freely; the replacement defining B was reversible by (25). The surface, Ar , Bx , and By rows are exactly the first four retained relators in (24); the fifth is recovered from (26). Hence the argument works in both directions.

The two free equalities in (31) provide the further reversible replacement used to obtain $K/\langle\langle\lambda\rangle\rangle_K$.

Verification of the displayed algebra. The word identities can be verified by substituting the printed definitions and cancelling adjacent inverse letters. An accompanying Python program¹ performs precisely these checks, reconstructs the fifteen original rows, and checks the integral exponent calculations. It also checks the two cotangent normal matrices and the two rank-three homology triples. These finite verifications concern identities and integer linear algebra. The geometric inclusions, HNN injectivity, and classification arguments are the proofs and cited theorems in the text. No finite-quotient computation is used to decide G . The companion `verify_expanded.py` checks the nineteen-factor identity, the lift normalization, the alternative trefoil dictionary and labelled action, the Whitehead cut table, and the displayed exterior-algebra identities. It imports only arithmetic routines from the accompanying checker and uses no historical research files.

APPENDIX B. THE IDENTITY USED IN THE CENTRAL EXTENSION

To make Theorem 13.1 checkable without a separate data archive, we print its entire free identity. In this appendix only, encode the generator A by the lower-case letter **a**, and encode inverses by upper-case letters. Thus **A** means A^{-1} , whereas **X, Y** mean x^{-1}, y^{-1} . The five relators in this alphabet are

$\mathcal{R}$	xyXYaxyXYA
$\mathcal{T}$	aaxAAX
$\mathcal{X}$	AYaayAxaYAAyay
$\mathcal{Y}$	AYaayxyaYAAyaXY
$\mathcal{S}$	AYaayAxyaaYA.

¹The accompanying `verify.py` uses only the Python standard library; run `python3 -B verify.py` from the directory containing the ancillary files.

For each row of the following table take $v_i r_i^{\varepsilon_i} v_i^{-1}$ in its displayed order. Their product freely reduces to the empty word.

i	ε_i	r_i	v_i
1	1	$\mathcal{X}$	1
2	1	$\mathcal{Y}$	Y
3	-1	$\mathcal{X}$	xAYaayAXaYAAya
4	-1	$\mathcal{Y}$	xyAYaayAYXYAAya
5	-1	$\mathcal{T}$	xyXYaxAAX
6	-1	$\mathcal{R}$	xyXYaxAAXYa
7	1	$\mathcal{R}$	xyXYaxAAYX
8	1	$\mathcal{T}$	xyXYaxAAYX
9	1	$\mathcal{T}$	xyXYaxAAYaayA
10	1	$\mathcal{S}$	xyXYaxA
11	-1	$\mathcal{T}$	xyXYaxyAAYaayXYAAyaaxAAYX
12	-1	$\mathcal{R}$	xyXYaxyAAYaayXYAAyaaxAAYX
13	1	$\mathcal{R}$	xyXYaxyAAYaayXYAAyaaxAAXYa
14	1	$\mathcal{T}$	xyXYaxyAAYaayXYAAyaaxAAX
15	-1	$\mathcal{S}$	xyXYaxyXAAYaaxAYAAya
16	-1	$\mathcal{T}$	xyXYaxyXAAYaaxAA
17	1	$\mathcal{R}$	1
18	1	$\mathcal{T}$	AYaayA
19	-1	$\mathcal{R}$	AYaayAxa

The signed counts are $(0, 1, 0, 0, 0)$ in the order $\mathcal{R}, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{S}$. This is the identity used to force the $\mathcal{T}$ value to vanish in Theorem 13.1.

APPENDIX C. A TREFOIL AMALGAM DESCRIPTION

A different exact presentation exposes a trefoil subgroup before two remaining relations are imposed. We give the dictionary and the subgroup calculation because neither the trefoil center nor the infinitude of this intermediate group decides the complete quotient. Throughout this appendix the filling is still $W(+1, -1; 0, 0)$.

C.1. Two generators and the two remaining equations. On free generators A, s define

$$\begin{aligned}
 (62) \quad & y = A^{-1}sA, & z = A^2, & N = [s, z^{-1}], & B = s^{-1}Ns, \\
 & k = [s, B^{-1}], & r = A^{-1}k^{-1}, & x = A^{-1}rA, & M = rkr^{-1}, \\
 & R_0 = [x, y][r, s], & T_0 = [A^2, x], & X_0 = BxB^{-1}y, \\
 & Y_0 = ByB^{-1}(BMB^{-1}yx)^{-1}.
 \end{aligned}$$

Then

$$(63) \quad G = \langle A, s \mid R_0, T_0, X_0, Y_0 \rangle.$$

To verify reversibility, the As row solves for N , and the beta filling with the Bs row solves for B and k . The alpha filling then gives $r = A^{-1}k^{-1}$, while the clean Ax row gives x . The still-required Ar row is $T_0 = 1$. The peripheral and meridian redundancies are precisely the free products of Appendix A; retain R_0, T_0 when using them. The two remaining beta transports are X_0, Y_0 . Reversing these eliminations recovers every original row.

Put $A = u$, $s = ub$, and introduce $a = u^{-1}bu$, $c = ubu^{-1}$, $d = u^2bu^{-2}$. There is a free identity, with $e = u^{-3}bu^3$,

$$BT_0B^{-1}X_0 = ca^{-2}ea.$$

Conjugating its right side by u^2 shows that the single equation $BT_0B^{-1}X_0 = 1$ defines

$$\mathcal{H} = F(a, b, c, d) \rtimes_{\Phi} \langle u \rangle, \quad \Phi(a, b, c, d) = (b, c, d, c^{-1}a^{-1}c^2).$$

Its inverse is $(a, b, c, d) \mapsto (b^2d^{-1}b^{-1}, a, b, c)$. Retaining $T_0 = 1$ as well gives $\mathcal{P}_- = \mathcal{H}/\langle\langle T_0 \rangle\rangle$. In these coordinates $A = u$, $s = cu$, $y = bu$. If $f = a^{-1}cac^{-1}a^{-1}$, the relation $T_0 = 1$ is equivalent, in $\mathcal{H}$, to

$$\sigma_0 = \Phi(f)b^{-1}f^{-1}b = 1, \quad \text{or equivalently } [ua, f] = 1.$$

For a direct word comparison, σ_0 is conjugate to T_0 by $b^{-1}a^{-1}cu$; substitution in the free-by-cyclic normal form verifies this equality. Thus the transport group

$$\mathcal{P}_- := \langle A, s \mid T_0, X_0 \rangle = \mathcal{H}/\langle\langle \sigma_0 \rangle\rangle$$

retains exactly those two equations.

The remaining words are fully specified by

$$\begin{aligned} k_f &= ac^{-1}bdb^{-1}, \quad J_0 = k_f^{-1}a\Phi(k_f)b^{-1}, \\ R_f &= J_0\Phi(J_0), \quad \eta = a^{-1}cac^{-1}bdb^{-1}c^{-1}acbd^{-1}b^{-1}ca^{-2}cd^{-1}b^{-1}. \end{aligned}$$

Indeed $x_0 = k_f^{-1}u^{-1} = T_0x$ in $\mathcal{H}$, so J_0 is the original J after T_0 is imposed. The original Y_0 has normal form $\eta^{-1}u$. Consequently

$$(64) \quad G = \mathcal{P}_- / \langle\langle R_f, \eta^{-1}u \rangle\rangle = \langle a, b, c, d \mid \sigma_0, R_f, \eta i \eta^{-1} = \Phi(i) \ (i = a, b, c, d) \rangle.$$

Eliminating u retains all four action equations; it does not turn this into a quotient with only σ_0, R_f .

C.2. An amalgamated product with injective edge maps. Let

$$\mathcal{K} = \langle a, c, z \mid zaz^{-1} = c, zcz^{-1} = c^{-1}a^{-1}c^2 \rangle, \quad g = afa^{-1}.$$

This is a free-by-cyclic group on $F(a, c)$, with $z = u^2$ in $\mathcal{H}$. Put $v = cz$ and $h = av$. Reversible substitution gives

$$\mathcal{K} = \langle h, v \mid hvh = vhw, \quad a = hv^{-1}, \quad c = vhw^{-2}, \quad z = v^2h^{-1} \rangle,$$

the trefoil group. The following explicit action proves that $\mathcal{D}_0 = \langle z, f, g \rangle$ is free on those generators.

In the modular group $\langle \alpha, \beta \mid \alpha^2 = \beta^3 = 1 \rangle$ set $h = \beta^{-1}\alpha$, $v = \alpha^{-1}\beta^2$. These satisfy the braid relation and therefore give a homomorphism from $\mathcal{K}$. Let $L = F(e_1, e_2, e_3)$. On $L \times \{0, \dots, 29\}$ define right actions by

$$(w, i)\alpha = (w\ell_\alpha(i), p(i)), \quad (w, i)\beta = (w\ell_\beta(i), q(i)).$$

Here are the complete permutation arrays, split at state 15:

$p(0), \dots, p(14)$	1, 0, 17, 4, 3, 8, 7, 6, 5, 10, 9, 27, 13, 12, 28
$p(15), \dots, p(29)$	16, 15, 2, 24, 20, 19, 29, 23, 22, 18, 26, 25, 11, 14, 21
$q(0), \dots, q(14)$	25, 2, 3, 1, 5, 6, 4, 0, 9, 16, 12, 10, 11, 15, 13
$q(15), \dots, q(29)$	14, 8, 19, 17, 18, 22, 20, 21, 24, 26, 7, 23, 28, 29, 27

All labels are one except

$$\ell_\alpha(12) = e_1^{-1}, \quad \ell_\alpha(13) = e_1, \quad \ell_\alpha(22) = e_2, \quad \ell_\alpha(23) = e_2^{-1}, \quad \ell_\beta(4) = e_3, \quad \ell_\beta(6) = e_3^{-1}.$$

The permutations and orbit labels satisfy $\alpha^2 = \beta^3 = 1$. The words z, f, g return state zero to itself with labels $e_3, e_3e_1^{-1}, e_2e_3$, respectively. These form a free basis of L , with inverse expressions $e_1 = f^{-1}z$, $e_2 = gz^{-1}$, $e_3 = z$. Any relation among z, f, g would therefore be a relation among a free basis. This proves the claimed subgroup injection; faithfulness of the whole modular action is unnecessary.

Now the map

$$\mathcal{D}_0 \longrightarrow F(u, F_0), \quad z \mapsto u^2, \quad f \mapsto F_0, \quad g \mapsto u^{-1}F_0u$$

is injective: its images form a Schreier basis of the subgroup of even u parity. The resulting amalgam is

$$(65) \quad \mathcal{P}_- \cong \mathcal{K} *_{\mathcal{D}_0} F(u, F_0).$$

Indeed identifying $F_0 = f$ leaves precisely $u^2 = z$ and $u^{-1}fu = afa^{-1}$, the latter being $[ua, f] = 1$. Both directions are presentation eliminations, and both edge maps are injective by the preceding free-basis calculation. Thus $\mathcal{K}$ embeds in $\mathcal{P}_-$.

This also supplies a complete membership test in the edge group. Apply a word of $\mathcal{K}$ to state zero. If it moves that state, it is outside $\mathcal{D}_0$. If it returns, convert its label to the unique candidate word on z, f, g by the inverse expressions above, and compare that candidate with the original word using the $F(a, c) \rtimes \langle z \rangle$ normal form. Equality is necessary and sufficient. In the other factor membership is even u parity. Thus the amalgam normal form is effective without a bounded search.

The height kernel of $\mathcal{P}_- \rightarrow \mathbb{Z}$ is generated by a, b, c, d and is nontrivial, perfect, and torsion-free. For perfectness, four translates of its defining relation give the exponent vectors

$$(-1, 1, 0, 0), \quad (0, 0, -1, 1), \quad (0, -1, 1, 0), \quad (-1, 0, 1, -1),$$

which span $\mathbb{Z}^4$. Nontriviality follows from the embedded $F(a, c)$; torsion-freeness follows from the injective amalgam of torsion-free groups. In particular this intermediate group is not locally indicable. These conclusions precede the two relations in (64).

C.3. Failure of height-kernel injection and simplicity. The whole height kernel cannot survive the remaining height-one relation, even though its free factors embed before that relation. To see this precisely, let $E_0 = F(a, c)$, $O_0 = F(b, d)$, let $j : E_0 \rightarrow O_0$ send (a, c) to (b, d) , and set $S = \Phi^2|_{E_0}$. The subgroup

$$U_0 = \mathcal{D}_0 \cap E_0 = F(f_n, g_n : n \in \mathbb{Z}), \quad f_n = S^n(f), \quad g_n = S^n(g),$$

has the displayed free basis by Schreier rewriting of $F(z, f, g) \rightarrow \mathbb{Z}$. Define its automorphism $\vartheta(f_n) = g_n$, $\vartheta(g_n) = f_{n-1}$. The kernel $N_0 = \ker(\mathcal{P}_- \rightarrow \mathbb{Z})$ is exactly

$$(66) \quad N_0 = E_0 *_{U_0} O_0,$$

with edge maps inclusion and $j\vartheta$. Indeed $fj(g)^{-1} = b\sigma_0^{-1}b^{-1}$ and its Φ -image is $j(f)S(g)^{-1}$. Their even and odd translates impose exactly $f_n = j(g_n)$ and $g_{n+1} = j(f_n)$ for all n . This proves the presentation and both edge injections for the entire infinite family. Both edge images are proper, since they have infinite free rank.

The factors E_0, O_0 are not conjugate in N_0 : they fix vertices of different types in its Bass-Serre tree, and properness of the edge subgroups makes each fixed vertex unique. If N_0 injected into $\mathcal{P}_- / \langle\langle w^{-1}u \rangle\rangle$ for any $w \in N_0$, the equation $u = w$ would force Φ to be

conjugation by w on N_0 . But $\Phi(E_0) = O_0$, contradicting that nonconjugacy. Thus no such height-one filling preserves the whole kernel. In the actual filling,

$$\kappa_a = \eta a \eta^{-1} b^{-1}$$

is an explicit killed element of infinite order. Its fourteen alternating syllables in (66) are all outside the edge groups: the six syllables of η , after relabeling the odd factor by j^{-1} , end at states 3, 16, 18, 21, 15, 2 in the printed modular action, and a ends at state 3. Thus the word is cyclically reduced. It dies because $u = \eta$ and $uau^{-1} = b$.

Nor can failure of injection be promoted to collapse by assuming that N_0 is simple. In fact N_0 is not simple. Here is the specific intersection calculation supporting that conclusion. Put $v_0 = c^{-1}a^2c^{-1} \in E_0$. Project $\mathcal{K}$ to the modular group as above. The based product of the folded subgroup graphs for $\mathcal{D}_0$ and $v_0\mathcal{D}_0v_0^{-1}$ has ten vertices, which may be labeled $q_0, \dots, q_9$, and exactly the following edges:

$$\begin{array}{l|l} \alpha \text{ pairs} & (q_0, q_1), (q_2, q_4), (q_5, q_7) \\ \beta \text{ cycles} & (q_1, q_3, q_2), (q_4, q_6, q_5), (q_7, q_9, q_8). \end{array}$$

Here is a direct way to reproduce the graph. First reduce the modular words for z, f, g and their v_0 conjugates. For each of the two triples, attach the three based loops, completing every alpha edge to a two-cycle and every beta edge to a three-cycle. Whenever two edges with the same label leave a vertex, identify their endpoints; repeat until each label defines a partial permutation. Completing a finite-factor cycle uses only $\alpha^2 = \beta^3 = 1$, and folding preserves the subgroup represented by closed paths. Finally form the product of the two resulting labelled graphs and keep the component of the pair of basepoints. It is the ten-vertex graph above, up to relabelling.² For an explicit identification, number the vertices of the first folded graph by $0, \dots, 26$ and of the second by $0, \dots, 36$. Their complete partial permutations are the following cycles; an omitted vertex has no edge of that label. Both basepoints are 0.

First graph, α	(0 1), (2 4), (5 7), (6 9), (10 12), (13 15), (16 11), (3 18), (19 21), (22 24), (25 20)
First graph, β	(2 1 3), (5 4 6), (7 0 8), (9 10 11), (12 13 14), (15 16 17), (18 19 20), (21 22 23), (24 25 26)
Second graph, α	(0 1), (2 4), (5 7), (8 10), (11 13), (14 16), (17 19), (20 22), (23 25), (26 21), (24 28), (29 31), (32 34), (35 30), (18 12)
Second graph, β	(2 1 3), (5 4 6), (8 7 9), (11 10 12), (13 14 15), (16 17 18), (19 20 21), (23 22 24), (25 26 27), (28 29 30), (31 32 33), (34 35 36)

The product-component dictionary is $q_i = (i, i)$ for $0 \leq i \leq 8$ and $q_9 = (0, 9)$, with basepoint q_0 . Thus its ten vertices and all their edges can be read directly from these two tables.

Every closed path in that component has trivial label: it consists of three beta triangles connected by alpha edges, with one remaining alpha edge to the basepoint. The projected intersection is therefore trivial. The labelled action already proves injectivity of the projection on $\mathcal{D}_0$, so

$$\mathcal{D}_0 \cap v_0\mathcal{D}_0v_0^{-1} = 1, \quad U_0 \cap v_0U_0v_0^{-1} = 1.$$

²The functions `modular` and `fold` in the accompanying `verify_expanded.py` perform these steps from the printed words and check the rooted labelled graph isomorphism.

The amalgam criterion of [MO15, Corollary 2.2] now makes both $\mathcal{P}_-$ and N_0 acylindrically hyperbolic: the edge inclusions are proper and an edge group has finite intersection with one of its conjugates. The virtually cyclic alternative is excluded by the embedded $F(a, c)$. Here acylindrical hyperbolicity has its usual meaning of admitting a non-elementary acylindrical action on a hyperbolic space; it is not an assertion about this particular tree action. Such groups have uncountably many normal subgroups [MO15, Theorem 3.7(f)], so N_0 is not simple. None of these properties is asserted for the prescribed quotient G .

C.4. What the alternative description does not imply. Normal forms give, for every integer j ,

$$C_{\mathcal{P}_-}(u^{2j+1}) = \langle u \rangle.$$

An odd power lies outside the normal index-two edge subgroup in the free factor $F(u, F_0)$. Since that edge subgroup is normal, the odd power lies in no conjugate of it inside this factor. It therefore fixes exactly one vertex of the Bass–Serre tree. Its centralizer preserves that vertex and lies in the free factor, where it is the indicated cyclic group. This is not the centralizer needed after imposing the surface relation.

To state that different question, put $\mathcal{Q} = \langle A, s \mid R_0, T_0, X_0 \rangle$ and

$$\varepsilon = (yx)^{-1}B(M^{-1}y)B^{-1}.$$

Then $G = \mathcal{Q}/\langle\langle \varepsilon \rangle\rangle$ and $[\varepsilon, y] = 1$ in $\mathcal{Q}$. Indeed the corrected beta transports, applied to (36), give $[y^{-1}, yx\varepsilon] = [x, y]$; the free identity

$$[y^{-1}, yx\varepsilon][x, y]^{-1} = x[\varepsilon, y]x^{-1}$$

proves the commutation. The abelianization of $\mathcal{Q}$ is $\mathbb{Z}$, with both y and ε mapping to its generator. Therefore

$$(67) \quad C_{\mathcal{Q}}(y) \text{ cyclic} \implies G = 1.$$

If that centralizer were generated by v , the primitive height of y would force $y = v^{\pm 1}$, and the height of ε would force $\varepsilon = y$. Killing it then kills all generators by (62) and X_0 . The cyclic-centralizer hypothesis is unproved. Neither (65) nor the centralizer of odd powers of u establishes it.

APPENDIX D. A RESTRICTION ON HANDLEBODY ATTACHING CURVES

The five conjugacy classes in (24) cannot be represented simultaneously by disjoint simple closed curves on the boundary of a genus-three handlebody with its specified free fundamental group. This statement allows arbitrary free basis changes. It concerns this tuple of relators, not every possible presentation of G .

Apply the free automorphism $A \mapsto A$, $x \mapsto Ax$, $y \mapsto yA^{-1}$ and cyclically reduce. In the alphabet of Appendix B, the resulting words are

$$\mathbf{xyAXYaaxyAXY}, \quad \mathbf{aaxAAX}, \quad \mathbf{YaayAxaaYAAyyA}, \quad \mathbf{YaayxyaYAAyXY}, \quad \mathbf{YaayAxyaaYA}.$$

Their total length is 56. None is a proper power, as one checks from its cyclic letters, and their distinct lengths distinguish their maximal cyclic-subgroup classes. Define the Whitehead graph by an edge from w_i to w_{i+1}^{-1} for every cyclic adjacency. For a signed multiplier v , the change in length for its Whitehead cut S is $|\partial S| - \deg(v)$, where $v \in S$ and $v^{-1} \notin S$. The following table prints every difference. Subsets of the other four letters are ordered by increasing binary mask in the signed-letter order (Y, X, A, a, x, y) , omitting v, v^{-1} , with the first remaining letter corresponding to the least significant bit.

v	Sixteen cut differences
Y	0, 8, 15, 17, 15, 19, 10, 8, 2, 10, 9, 11, 15, 19, 2, 0
X	0, 17, 21, 26, 23, 28, 24, 17, 11, 24, 24, 25, 18, 19, 11, 0
A	0, 7, 4, 9, 2, 1, 6, 3, 11, 14, 7, 8, 11, 6, 7, 0
a	0, 7, 6, 11, 8, 7, 14, 11, 3, 6, 1, 2, 9, 4, 7, 0
x	0, 11, 19, 18, 25, 24, 24, 11, 17, 24, 28, 23, 26, 21, 17, 0
y	0, 2, 19, 15, 11, 9, 10, 2, 8, 10, 19, 15, 17, 15, 8, 0

Whitehead minimality follows because all differences are nonnegative. The graph contains all nine edges of $K_{3,3}$ with parts $\{Y, X, x\}$ and $\{A, a, y\}$. A length-minimal Whitehead graph of a geometric multiword must be planar, by Zieschang’s theorem [Zie65]; see [Cas16, Section 2E1 and the discussion following Proposition 5.2]. This proves the obstruction. Further changes of relators, as opposed to free changes of generators, are outside its scope.

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