

ANNULAR SCATTERING IN QUARTIC OPEN FJRW THEORY

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ABSTRACT. For the open FJRW theory of Gross, Kelly and Tessler attached to the quartic Fermat polynomial in two variables, we prove that, with its two monomials matched to toric divisors of $\mathbb{P}(1, 1, 2)$, the smallest cover of the tropical base of Carl, Pumperla and Siebert on which the affine monodromy preserves the lattice spanned by the exponents of the base and of the first-descendent wall-crossing generators of twists $(1, 0)$ and $(0, 1)$ is the double cover branched at two of its three singular points, an integral-affine annulus. Its core holonomy is hyperbolic, and lifting its transitions requires roots of the slab functions that generate an infinite Kummer extension under transport. The annulus carries a wall structure, consistent at its two joints in a sense weaker than that of Gross and Siebert, whose wall-crossing agrees with that of Gross, Kelly and Tessler between the extremal systems of invariants for a single marking but not for two; the difference does not change periods, and its removal from the annular structure is left open. The homotopies of boundary conditions realizing this wall-crossing ray by ray can be chosen compatibly for all markings.

1. INTRODUCTION

1.1. The question. The Gross–Siebert program constructs mirrors by placing wall-crossing automorphisms on an integral-affine base with singularities and reading off broken lines and theta functions [GHK15, GHKK18, GS22]. Its input is geometric: a base, its singular locus, a polyhedral decomposition, and a polarization. This paper asks whether the wall-crossing of an open Landau–Ginzburg theory supplies such a base, and answers the question for one model.

For $x^4 + y^4$ the answer is partial. Under the matching of x^4 and y^4 with the rays of the coordinate divisors $\{X = 0\}$ and $\{Y = 0\}$ of $\mathbb{P}(1, 1, 2)$, the exponents of the wall-crossing generators attached to one internal marking, together with the integral exponents of the base of Carl, Pumperla and Siebert, determine an integral-affine annulus, a double cover of that base branched at two of its three singular points, and the annulus carries a wall structure that is consistent at its two joints, in a sense weaker than that of Gross and Siebert. The wall-crossing of this structure agrees with the transition of Gross, Kelly and Tessler between their two extremal boundary conditions (defined below) for one internal marking of twist $(1, 0)$ or $(0, 1)$ and descendent degree one, and differs from it for one such internal marking of each of these twists, by a factor that does not change periods (Theorems A and B and Proposition 7.4).

Gross, Kelly, and Tessler (GKT) define genus-zero open FJRW invariants for $(W, G) = (x^r + y^s, \mu_r \times \mu_s)$ as relative Euler numbers on moduli spaces of W -spin disks [GKT26a]. The invariants depend on a choice of canonical boundary conditions, and changes of boundary conditions act through a formal wall-crossing group closely related to the tropical vertex group. This action has the algebra of a scattering diagram but no integral-affine realization: no affine manifold, singular locus, or polyhedral decomposition carries its walls. Gross, Kelly, and Tessler

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list a tropical version of open FJRW theory among the open problems of their survey [GKT26b, §6], and Maher observes that it is not obvious how to incorporate Landau–Ginzburg mirror symmetry into the Gross–Siebert program, since a Landau–Ginzburg model has no target space and hence no SYZ fibration [Mah24, Rem. 5.1.1].

In algebraic terms, the theory for $W = x^4 + y^4$ reads as follows (Section 5.1 recalls the definitions). The internal markings of a disk carry distinct *labels*, each with a twist, and we refer to an internal marking by its label. Each internal marking also carries a descendent degree $d \geq 0$, and a label i contributes the nilpotent deformation variables $u_{i,d}$, $d \geq 0$, any two of which have product zero; we write $t_i = u_{i,1}$, so that $t_i^2 = 0$, and say that a label has descendent degree one when only its variable t_i is used. Boundary markings are of three kinds, of twists $(2, 0)$, $(0, 2)$ and $(2, 2)$. A choice of canonical boundary conditions chooses the multisections of the descendent Witten bundles whose weighted zero counts are the open invariants. The invariants appear, with signs, as the coefficients of the *potential* $\sum_{\Gamma} \pm \nu_{\Gamma} x^{k_1} y^{k_2} u_{\Gamma}$, a deformation of $x^4 + y^4$. Its monomials correspond to the *balanced* disk graphs Γ , those with one boundary marking of the third kind on whose moduli space the rank of the descendent Witten bundle equals the dimension; ν_{Γ} is the invariant of Γ , k_1 and k_2 are its numbers of boundary markings of the first two kinds, and u_{Γ} is the product of the variables $u_{i,d}$ of its labels i and their descendent degrees d ; the two balanced graphs without labels give $x^4 + y^4$. The wall-crossing group is generated, with coefficients in these variables, by the Hamiltonian vector fields X_k of the monomials $x^{k_1+1} y^{k_2+1}$ for $dx \wedge dy$, one for each *critical* graph, which has $k_1 + 1$ and $k_2 + 1$ boundary markings of the first two kinds, none of the third, and a descendent Witten bundle of rank one more than the dimension. These fields are $(xy)^{-1}$ times the generators $x^{m_1} y^{m_2} \partial_n$, $\langle m, n \rangle = 0$, with $\partial_n = n_1 x \partial_x + n_2 y \partial_y$, of the Lie algebra of the tropical vertex group of [GPS10] (Section 3.1).

The companion paper [Wue26b] factors the GKT wall-crossing into factors ordered by slope. The *boundary ray* of the generator X_k is the ray spanned by the numbers $k + (1, 1)$ of boundary markings of the first two kinds on its critical graph. This is a ray in the plane of marking counts, not in any affine base. GKT characterize the vectors of invariants that boundary conditions can produce by explicit equations, the chamber conditions; their solutions, the *chamber indices*, are exactly the vectors of open invariants of transverse canonical boundary conditions (Section 5.1). For a nonempty set of labels and a descendent vector on it, the coefficients of the potential at the balanced graphs of these data form their *balanced string*, indexed by the exponents of the corresponding monomials; these exponents form an arithmetic progression with step $(4, -4)$. The two *extremal* chamber indices are those for which, on every balanced string, only the coefficient with the least, respectively greatest, first exponent survives; we call their potentials the *extremal GKT potentials* $W^{\min}$ and $W^{\max}$, and canonical boundary conditions realizing them *extremal boundary conditions*. The *GKT transition* between the two extremal chamber indices is the element of the wall-crossing group that carries $W^{\min}$ to $W^{\max}$ (Section 5.2). Modulo any finite set of deformation monomials closed under taking divisors, it factors uniquely into factors ordered by the slope of their boundary rays [Wue26b, Thm. 1.1 and Prop. 3.1]. The factorization supplies an ordered family of wall-crossing operations, but no surface on which their rays lie and no affine monodromy.

We construct such a surface, and a wall structure on it that is consistent at its two joints, for the parabolic quartic $W = x^4 + y^4$, the singularity X_9 (also denoted $\tilde{E}_7$). Its central charge $\hat{c} = 2 - \frac{2}{r} - \frac{2}{s}$ equals one; by [Wue26a] the primary invariants are independent of the boundary conditions exactly when $\hat{c} < 1$. The suspension of the marginal pencil of X_9 is an anticanonical elliptic curve in $\mathbb{P}(1, 1, 2)$, and the toric-boundary degeneration of that pair gives the disk-shaped intersection complex of the Landau–Ginzburg construction of Carl–Pumperla–Siebert [CPS24],

the *CPS complex*. It is the natural candidate base: the anticanonical polygon of $\mathbb{P}(1, 1, 2)$, subdivided by the three segments from its interior lattice point, the *central vertex*, to its vertices, each segment carrying one singular point of the affine structure (Section 2.3).

Independently of the annulus, Appendix E normalizes the GKT primary potential of $x^4 + y^4$. For two labels of twist $(2, 2)$ it gives open invariants whose sum is $-\frac{1}{4}$, in agreement with the period computation of [Wue26b, Rem. 4.3]; this is the sum considered in [GKT26b, Ex. 5.5], where the value $-\frac{1}{8}$ is printed (Theorem E.3).

1.2. Notation. Appendix F lists the principal symbols of the paper and compares the lattice conventions with those of toric geometry and of [GS11, CPS24]. Let $M \simeq \mathbb{Z}^2$ be the lattice of the fan of $\mathbb{P}(1, 1, 2)$. In toric geometry this lattice is called N ; we write M for it because it carries the exponents of all Laurent monomials, slab functions and wall functions below, as M does in [GPS10]. The polygon of the CPS complex lies in $M^\vee \otimes \mathbb{R}$, with $M^\vee = \text{Hom}(M, \mathbb{Z})$. The discrete Legendre transform of the CPS complex is its *dual intersection complex* in the sense of [CPS24, §7.1]; its bounded cell is the polar polygon in $M \otimes \mathbb{R}$, and its tangent lattice is M (Section 2.3). Let $\mathbb{Z}_{x,y}^2$ be the lattice of bidegrees (a, b) of monomials $x^a y^b$, and put

$$(1) \quad F = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}, \quad \widetilde{M} = F(\mathbb{Z}_{x,y}^2), \quad \widetilde{M}/M \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

The *exponent map* F sends $x^a y^b$ to the character $z^{F(a,b)}$, whose exponent has coordinates (a, b) in the basis $F(1, 0), F(0, 1)$ of $\widetilde{M}$, its *F-coordinates*. It is the unique linear map that matches x^4 and y^4 with the rays of the coordinate divisors $\{X = 0\}$ and $\{Y = 0\}$ of $\mathbb{P}(1, 1, 2)$ (Section 2.2). The CPS complex has three singular points A, B, C (Section 2.3). The slab function attached to B is $1 + a_\rho z^u + z^{2u}$, with z^u a primitive character and a free coefficient a_ρ , the *slab parameter*; at the *nodal values* $a_\rho = \pm 2$ it is a square, and the associated anticanonical curve acquires a node (Section 2).

For $q = (q_1, q_2) \in \mathbb{Z}_{x,y}^2$, the derivation

$$X_q^\kappa = x^{q_1-1} y^{q_2-1} (q_2 x \partial_x - q_1 y \partial_y)$$

is the Hamiltonian vector field of the function $x^{q_1} y^{q_2}$ for $dx \wedge dy$; the superscript records the shift of the exponent by $\kappa = (1, 1)$, the exponent of xy , and F carries X_q^κ to a monomial derivation of the character ring of $\widetilde{M}$ (Proposition 2.2). The *cofactor* $L = -\frac{1}{2} F^{-T}$ of (8), a linear map from bidegrees to $M^\vee \otimes \mathbb{R}$ that intertwines the exchange of x and y on bidegrees with its action on the plane of the polygon, gives the direction Lq in that plane along which the wall of X_q^κ is placed (Section 2.2). We write $X_{a,b} = X_{\kappa+(a,b)}^\kappa$ and $E_{a,b} = t_1^a t_2^b X_{a,b}$, so that $A = -\frac{1}{2} E_{1,0}$ and $B = -\frac{1}{2} E_{0,1}$. The sans-serif letters A and B always denote these two Hamiltonian vector fields; outside the GKT notation of Section 5 and Appendices B and E, the roman letters A, B, C , also with decorations such as $\widetilde{A}$ or B_0 and as indices, refer to singular points of the affine structure, apart from the ring A of the root-stack convention of Appendix A, the logarithms C_τ of ray factors and the corrections C_i of Appendix D. The *singleton parameters* t_1, t_2 are the variables t_i of a single label of twist $(1, 0)$, respectively $(0, 1)$, and descendent degree one. The *cuts* are six arcs, from the four singular points of the annulus of Theorem A to its boundary, that carry the transitions between charts (Section 2.4). Their complement in the interior of the annulus consists of two simply connected *sheets* S_0 and S_1 , which contain the two lifts O_0 and O_1 of the central vertex, respectively. Over a simply connected region of the annulus away from its singular points, a *frame* is a trivialization of the local system $\widetilde{M}$, that is, an identification of the algebra of characters over the region with $\mathbb{k}[\widetilde{M}]$; the frame of S_0 is the one induced by the chart pulled back from the CPS complex, and the other frames are transported from it across

the cuts (Section 2.3). The monodromy of a local system of lattices along a loop is also called its holonomy, and we use the two words interchangeably; the *affine holonomy* of a loop is the affine map by which the developing map changes along it. A *Jacobi bracket* on a line bundle is a Lie bracket on its local sections that is a first-order bidifferential operator, a local Lie algebra with one-dimensional fibers in the sense of Kirillov [Kir76, Lic78]; the notion is unrelated to the Jacobian algebra $\text{Jac}(W)$ of a potential.

1.3. Main results. The generators of the GKT critical Lie algebra of $x^4 + y^4$ attached to a single label of twist $(1, 0)$, respectively $(0, 1)$, and descendent degree one are, in the notation of Section 1.2,

$$A = -\frac{1}{2}t_1 X_{(2,1)}^\kappa, \quad B = -\frac{1}{2}t_2 X_{(1,2)}^\kappa,$$

the logarithms of the two singleton factors of the GKT transition between the extremal chamber indices (Section 5.3). Their exponents, with standard coordinates $F(2, 1) = (\frac{1}{4}, -\frac{1}{2})$ and $F(1, 2) = (-\frac{1}{4}, -1)$ in $M \otimes \mathbb{Q}$, are fractional: they do not lie in M , and together with M they span the overlattice $\widetilde{M}$ of index eight.

Theorem A uses the terms *wall structure* and *consistency* in a weaker sense than in the Gross–Siebert program. The walls of $\mathfrak{D}$, the wall structure of Theorem A(4), are not tangent to the exponents of their functions, as they are in [GPS10, GHS22]: the exponents are tangent vectors of the dual intersection complex, and the supports lie in the plane of the polygon. Tangency transported by the discrete Legendre transform would place each wall along the annihilator of its exponent, a rule incompatible with the exchange of x and y ; the compatible linear rules are the gradient maps of exchange-invariant quadratic functions, and the supports are placed by one of them, the cofactor L , chosen by convention (Section 2.2). Consistency is the local condition around the two joints O_0 and O_1 , the lifts of the central vertex. It is weaker than the consistency of a wall structure in the sense of Gross and Siebert [GS11], which is required at every joint, every codimension-two cell of the decomposition cut out by the walls and slabs that does not lie in the boundary of the base, including the interior endpoints of walls [GS11, Defs. 2.27–2.28]: here the diagonal walls end at the singular points B_0 and B_1 with no condition there, and, as in [GS11], nothing is imposed along the core loop.

Theorem A (Annular base and wall structure). (1) *The smallest cover of the CPS complex on which the affine monodromy preserves $\widetilde{M}$ is the double cover branched at two of its three singular points. It is an integral-affine annulus $\overline{A}$ with integral tangent lattice $\widetilde{M}^\vee$, so that exponents lie in the dual lattice $\widetilde{M}$, with two focus–focus points B_0, B_1 , and two ramification points $\tilde{A}, \tilde{C}$ of the cover at which the structure is the pullback of a focus–focus germ under $z \mapsto z^2$; the pullback of the polarization of the discrete Legendre transform of the CPS complex becomes integral on it after multiplication by four (Theorem 2.3). Its holonomy K along the core loop γ_{core} , which separates $\{\tilde{A}, B_0\}$ from $\{B_1, \tilde{C}\}$, is hyperbolic of trace 18, and its monodromies around the two boundary circles are conjugate to $\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -16 \\ 0 & 1 \end{pmatrix}$.*

(2) *Lifting the slab transitions requires roots of the slab functions, of degrees 2, 4, 4, 2 at $\tilde{A}, B_0, B_1, \tilde{C}$ (the degree at B_0, B_1 drops to two at the nodal values). The roots of the slab functions transported along all paths generate an infinite Kummer extension: no finite Kummer extension containing the initial roots is stable under transport, and the system of all transported roots is not represented by a finite root stack. The functions transported along any finite set of paths lie in a single finite Kummer extension (Theorems 2.5 and 2.7).*

- (3) *The lifted slab maps, which multiply each character z^m , $m \in \widetilde{M}$, by an integral power of a root of the slab function as in part (2) and do not change its exponent, rescale the volume form $dx \wedge dy$ by the inverse of the slab function at $\widetilde{A}$ and $\widetilde{C}$, by the inverse of a square root of it at B_0 , and by a square root of it at B_1 , and the transitions carried by the cuts do not preserve $dx \wedge dy$, so its Poisson bivector cannot serve in every chart (Proposition 3.2). The volume form glues as a form with values in a line bundle $\mathcal{L}_\kappa$ over the covers obtained by adjoining the roots of part (2), and its Poisson bracket, transported along paths from the frame of the sheet S_0 , is a Jacobi bracket on $\mathcal{L}_\kappa$ whose bivector part, with respect to the transported local generators of $\mathcal{L}_\kappa$, is the bivector of $dx \wedge dy$ in the frame transported to every vertex of the path tree, the tree of reduced sequences of crossings of the cuts that start in S_0 (Proposition 3.3).*
- (4) *Place e^A and e^B on the two lines through a lift O_0 of the central vertex in the directions $L(2, 1)$ and $L(1, 2)$. On the ray from O_0 in direction Lq , place the inverse of the factor on the ray of q in the ordered factorization of the group commutator $e^{Ae^{-B}}e^{-A}e^B$, and transport the result to the other lift O_1 . This gives a wall structure $\mathfrak{D}$ on the complement $\mathcal{A}^\circ$ of the singular points that, at every finite order, is locally finite, has properly embedded arcs as supports, and is consistent at O_0 and O_1 , wall structures and consistency being taken in the weaker sense of Definition 3.4 and the paragraph after it (Proposition 3.6 and Theorem 3.7).*
- (5) *The holonomy K acts hyperbolically on $\widetilde{M}$, with no nonzero finite orbit. Consequently the local system $\widetilde{M}$ on $\mathcal{A}^\circ$ has no nonzero global section, and neither does its pullback to any connected finite cover of $\mathcal{A}^\circ$ (Theorem 3.9).*

In part (4), consistency at the joints depends on the placement only through the cyclic order of the supports about O_0 , which is the reverse of the cyclic order of their exponents in F -coordinates (L reverses orientation), an exponent q being replaced by $-q$ on the negative halves of the two lines; since F also reverses orientation, this is the same cyclic order as that of the characters $F(q)$ in the standard coordinates of $M \otimes \mathbb{R}$ (Section 2.2).

Part (1) rests on the monodromies of the CPS complex about its three singular points. In the basis $F(1, 0), F(0, 1)$ of $\widetilde{M}$, the monodromy about B is an integral matrix, whereas those about A and C have half-integral entries and therefore do not preserve $\widetilde{M}$ (95). Let χ be the character of the fundamental group of the disk punctured at A, B, C with values in $\mathbb{Z}/2$ that sends the loops about A and C to 1 and the loop about B to 0. The monodromies of a basis of its kernel are integral in this basis (Appendix A), so the loops whose monodromy preserves $\widetilde{M}$ form a proper subgroup containing the kernel, which has index two; the two subgroups therefore coincide, and the smallest cover is the double cover classified by χ , the *parity cover*. It is branched at A and C , the point B has two unramified lifts B_0, B_1 , and the boundary loop, on which χ takes the value $1 + 0 + 1 = 0$, has two lifts. By Riemann–Hurwitz the cover has Euler characteristic $2 \cdot 1 - 2 = 0$, so it is an annulus with two boundary circles. With respect to $\widetilde{M}$ the monodromies about B_0 and B_1 are primitive unipotent, and these points are primitive focus–focus singularities, whereas a small loop about $\widetilde{A}$ or $\widetilde{C}$ develops onto a curve winding twice around the developed singular point (Theorem 2.3).

Part (2) rests on one computation. Transport about $\widetilde{A}$ fixes the invariant character x_A of $\widetilde{A}$ and multiplies the invariant character x_B of B_0 by the nonconstant function $(1 + x_A^{-1})^4$, so the transported slab functions $f_B((1 + x_A^{-1})^{4n}x_B)$, $n \in \mathbb{Z}$, with $f_B(x) = 1 + a_\rho x + x^2$, are pairwise coprime, each with a simple prime factor that divides no other (after extracting the square at the nodal values), and their fourth roots generate an infinite Kummer extension (Section 2.5).

In part (3), a divergence computation shows that conjugation by a lifted slab map gives each singleton field a nonzero divergence for $dx \wedge dy$, so that it is no longer Hamiltonian (Proposition 3.2); the bracket is instead transported along the path tree of Section 2.5, the tree of reduced sequences of crossings of the cuts that start in the sheet S_0 . Every vertex of a tree is reached from its root by a unique route, so the transport is unambiguous, and at each crossing the change of local generator of $\mathcal{L}_\kappa$ compensates the factor by which the transition rescales $dx \wedge dy$, so that the bivector part remains that of $dx \wedge dy$ (Proposition 3.3). In part (4), beyond the Kontsevich–Soibelman factorization at a point, the content of $\mathfrak{D}$ lies in the annular cover that carries it, the cyclic order of the cofactor directions, and the fact that the walls avoid the cuts (Proposition 3.6); consistency at O_1 follows from consistency at O_0 : the deck involution of the cover carries the supports through O_0 , with their crossing order and coorientations, onto those through O_1 , and the factors there are the images of the factors at O_0 under one Jacobi isomorphism (Theorem 3.7).

Part (5) rests on the trace. In the basis $F(1, 0), F(0, 1)$ of $\widetilde{M}$, the holonomy along the core loop γ_{core} of Section 2.3 is

$$(2) \quad K = \begin{pmatrix} -1 & 2 \\ -10 & 19 \end{pmatrix}, \quad \text{tr } K = 18.$$

Its eigenvalues $9 \pm 4\sqrt{5}$ are not roots of unity, so no power K^r with $r \neq 0$ fixes a nonzero element of $\widetilde{M}$, whereas a global section of $\widetilde{M}$ over a connected finite cover of $\mathcal{A}^\circ$ would be fixed by such a power (Section 3.4).

As recalled above, each GKT generator X_k is the Hamiltonian field of $x^{k_1+1}y^{k_2+1}$ for $dx \wedge dy$; in the notation of Section 1.2 it is $X_{k+\kappa}^\kappa$, indexed by the shift $k + \kappa$ of k , so the GKT critical Lie algebra of every finite set of labels embeds, without change of coefficients, in the Lie algebra $\mathfrak{g}_\kappa$ of Hamiltonian derivations for the Jacobi bracket (Theorem 5.1 and Proposition 5.2). The comparison of $\mathfrak{D}$ with GKT wall-crossing is then as follows.

Theorem B (Comparison with GKT wall-crossing). (1) *The wall-crossing transformations of $\mathfrak{D}$ are not those of the GKT transition. For two labels of twists $(1, 0)$ and $(0, 1)$ and descendent degree one, the first mixed ray $\mathbb{R}_{\geq 0}(1, 1)$, the boundary ray of the generator $X_{1,1}$ attached to both labels, has coefficient $-3/4$ in $\mathfrak{D}$ and $1/4$ in the slope-ordered factorization of the GKT transition (Proposition 5.3), and the path-ordered product of $\mathfrak{D}$ along a path passing the joint carries $W^{\min}$ to $\exp(-t_1 t_2 X_{1,1}) W^{\max}$ rather than to $W^{\max}$ (Proposition 7.1).*

(2) *Fix a finite set of labels of twists $(1, 0)$ and $(0, 1)$ and descendent degree one, let J be a subset consisting of p_J labels of twist $(1, 0)$ and q_J of twist $(0, 1)$, and put $t_J = \prod_{i \in J} t_i$. After the base change sending t_1 and t_2 to the sums of the variables t_i of the labels of twist $(1, 0)$ and $(0, 1)$, respectively, the t_J -component of the logarithms of the wall factors of $\mathfrak{D}$ is*

$$S_J = p_J! q_J! a_{p_J, q_J} t_J X_{p_J, q_J},$$

where $a_{p,q}$ is the coefficient of $E_{p,q}$ in the factorization of $e^A e^B$ into ray factors in which the factor of least slope acts first ($a_{p,q} = -c_{p,q}$ for $p, q \geq 1$, with $c_{p,q}$ in Table 1). It agrees with the t_J -component of the logarithms of the factors of the GKT transition between the extremal chamber indices when $|J| = 1$, and not when J consists of one label of each twist, where it is $-\frac{3}{4} t_J X_{1,1}$ and that of the GKT transition is $\frac{1}{4} t_J X_{1,1}$ (Proposition 5.5).

(3) *There exist an increasing sequence of finite assembling-closed systems of labels, divisor-closed sets of deformation monomials and graph data that exhausts all graph data;*

extremal boundary conditions compatible with these systems; and a homotopy of canonical multisections between them that is smooth on each graph datum, with the following properties. Here a graph datum is a graph together with descendent degrees of its labels, a set of monomials is divisor-closed if it contains every divisor of each of its elements, and a system is assembling-closed if it satisfies the conditions of Definition B.1, among them closure under the assembling rule of GKT described after the theorem. The countably many ray factors occur in pairwise disjoint open time intervals, ordered by increasing slope. On each graph datum the homotopy is nonconstant on only finitely many of these intervals and is constant near the ends of each of them. On each of the systems, modulo the monomials outside it, only finitely many rays carry a nontrivial factor of the GKT transition between the extremal chamber indices, and the potential at the right end of the interval of each ray is the image of the potential at its left end under the factor of the GKT transition on that ray. The restrictions to the systems are compatible, and the boundary conditions and the homotopy are their inverse limits. Thus the homotopy realizes the GKT transition factor by factor in increasing slope (Theorem 5.7 and Appendix B).

The two coefficients in part (1) have different origins. The coefficient $-\frac{3}{4}$ is forced by consistency at the joint: modulo t_1^2 and t_2^2 the only other walls through O_0 lie on the two lines carrying e^A and e^B , so the wall of the first mixed ray, the diagonal segment from O_0 to B_0 , must carry $e^{-B}e^Ae^{B-A} = \exp([A, B])$, the inverse of the group commutator in Theorem A(4), and $[A, B] = -\frac{3}{4}t_1t_2X_{1,1}$. The coefficient $\frac{1}{4}$ is forced instead by GKT's chamber conditions, which fix the coefficients of $W^{\min}$ and $W^{\max}$, the mixed ones included. The mixed factor of the GKT transition must remove the term $2t_1t_2xy^5$ that its singleton factors leave in the image of $W^{\min}$, since $W^{\max}$ has no such term; as $X_{1,1}(x^4 + y^4) = 8x^5y - 8xy^5$, this factor is $\exp(\frac{1}{4}t_1t_2X_{1,1})$ (Proposition 5.3). In part (2), $p_J!q_J!$ is the coefficient of t_J in the image of $t_1^{p_J}t_2^{q_J}$ under the base change, because $t_i^2 = 0$ for every label i . On the GKT side, the critical generators of a support act on its balanced monomials through the weighted incidence matrix of a path graph, which has full column rank and whose one left-kernel relation is the chamber condition of that support; hence the two extremal chamber indices determine the logarithms of the factors of the GKT transition uniquely, support by support (Proposition 5.5).

Part (3) is a statement about GKT theory itself and does not involve the annulus. GKT realize a transition between two chamber indices by a homotopy of canonical multisections, with time parameter in $[0, 1]$, which on the critical graphs of a set J of labels vanishes only at one time $\mathfrak{t}(J)$, at isolated points; at that time the potential changes by the exponential of the critical generators whose coefficients are products of one variable of each label in J , weighted, with signs, by these zero counts (Section 5.1). Part (3) belongs to the comparison because every factor of the GKT transition compared in parts (1) and (2) is the total change of the potential across the time interval of its ray, for one such compatible homotopy restricted to a finite system (Section 5), and it answers the compatibility question left open in [Wue26b]. The finite systems of Definition B.1 consist of finitely many labels, deformation monomials and graph data, and are closed under the following rule of GKT: along each boundary stratum of the moduli space of a graph, the multisection is assembled from the multisections on the connected components of its base, essentially the pieces into which the disks of that stratum break at their boundary nodes (Section 5.1), so a system that contains a graph datum contains the data of these components. The proof in Appendix B extends canonical multisections and homotopies relative to the data already chosen, and three features of the construction make these extensions compatible. First, with every deformation monomial that carries a balanced graph, a system contains all balanced and critical graphs of that monomial; since the chamber

conditions couple the invariants of all balanced graphs of one monomial, and these invariants are adjusted through the critical graphs of the same monomial, prescribing the invariants of a new monomial then never alters an old multisection. Second, the times assigned to new sets of labels are chosen distinct from the old times, and away from the times, a set of measure zero, at which an old homotopy on a graph whose descendent Witten bundle has rank equal to the dimension and which has at most one boundary marking of the third kind, restricted to that time, fails to be transverse to zero in the interior of its moduli space; hence no old homotopy has to be changed. Third, the countably many ray factors receive pairwise disjoint time intervals, ordered by slope, through one embedding of their slope order into the ordered set of complementary open intervals of a Cantor set, fixed for all systems. Each system then restricts to the previous one, and the homotopy of part (3) is the inverse limit of these restrictions.

Along one path, transport through $\mathfrak{D}$ and transport by the factors of the GKT transition differ by one automorphism of the algebra of functions in the chart at the end of the path, the residual transport of the path; it relates products of functions transported along that path but not along different paths. Appendix D gives a degree-weighted correction, a first step towards multiplying functions carried along different paths, coherent up to an explicit associator, which is nontrivial for residual transports in general position (Proposition D.4).

The difference in Theorem B(1) is invisible to periods. For two labels of twists $(1, 0)$ and $(0, 1)$ and descendent degree one, for the potentials of Section 7 with the singleton terms of $W^{\max}$ and mixed terms $t_1 t_2 (a x y^5 + b x^5 y)$, the mixed period class is $(6 - \frac{a+b}{2})[xy dx \wedge dy]$. The GKT chamber condition, which is a period condition, therefore fixes $a + b = 12$, and the flows of the GKT generator $X_{1,1}$ move (a, b) along the line $a + b = 12$ without changing any period (Proposition 7.4). Within that orbit $W^{\max}$ is singled out by a normal form: on each balanced string only the coefficient with the greatest first exponent survives. This *maximal normal form* carries the transport of $W^{\min}$ by $\mathfrak{D}$ to $W^{\max}$ for every finite set of such labels (Proposition 7.6). The same happens for two labels of the same twist, where the GKT transition has a factor on a ray carrying no wall of $\mathfrak{D}$ (Proposition 7.5).

Problem C. Construct, from the annular structure of Theorem A, an operation at a boundary circle of the annulus that realizes the maximal normal form, and so carries the transport of $W^{\min}$ to the transport of $W^{\max}$ (Problem 7.7).

1.4. Further results. *Two labels of twist $(2, 2)$.* The normalization of the GKT primary potential in Appendix E gives, for two labels of twist $(2, 2)$ and descendent degree zero, the open invariants $-\frac{1}{8}$ at the balanced graphs with $(k_1, k_2) = (4, 0)$ and $(0, 4)$, the *balanced profiles* of these labels. These are the invariants of every chamber index that is *symmetric* in the sense of GKT, that is, invariant under the permutations of the labels that preserve their twists, and whose primary potential is the unique one invariant under the exchange of x and y , a symmetry of a different kind (Lemma E.1 and Theorem E.3). The sum $-\frac{1}{4}$ of the two invariants agrees with the period computation of [Wue26b, Rem. 4.3]; it is the sum considered in [GKT26b, Ex. 5.5], and it differs from the value $-\frac{1}{8}$ printed there.

Three labels. The discrepancy of Theorem B(1) recurs for three labels, where Theorem B(2) gives the wall side. Let J consist of three labels of descendent degree one, two of twist $(1, 0)$ and one of twist $(0, 1)$, or conversely. The only critical exponent k of J is $(2, 1)$, respectively $(1, 2)$. The t_J -component of the logarithms of the wall factors is $-\frac{3}{4}t_J X_k$, which is S_J of Theorem B(2) with $a_{2,1} = a_{1,2} = -\frac{3}{8}$, whereas that of the factors of the GKT transition is $-\frac{1}{2}t_J X_k$; the two differ by $-\frac{1}{4}t_J X_k$ (Section 5.4).

Obstructions. Given the dictionary F , the annulus is forced, and the simpler candidates fail. Exactly one of the three affine monodromies of the CPS complex preserves $\widetilde{M}$ (by (95) and

Theorem 2.3), so $\widetilde{M}$ is not preserved by the affine monodromy of the CPS complex and is not the character lattice of its integral-affine structure. Theorem A(2), (3) and (5) record the other obstructions: for each value of the slab parameter a_ρ no finite Kummer extension containing the initial roots is stable under transport (Theorems 2.5 and 2.7); the Lie algebra of Hamiltonian derivations of the structure sheaf containing the two singleton fields is not preserved by the lifted slab maps (Proposition 3.2); and the local system $\widetilde{M}$ on $\mathcal{A}^\circ$ has no nonzero global section, even after pullback to a connected finite cover, so that exponents in one fixed copy of $\widetilde{M}$ cannot serve as labels compatible with parallel transport (Theorem 3.9 and the discussion after it). Being an annulus, the base is not homeomorphic to the intersection complex of a distinguished toric degeneration of del Pezzo surfaces in the sense of Carl–Pumperla–Siebert, which is a disk (Proposition 4.5).

Boundary. The walls of $\mathfrak{D}$ other than the two diagonal segments leave the annulus transversely through its boundary circles. Outward integral-affine cylinders, of widths 4 and 16 (the integers w in the normal forms $\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}$ of Theorem A(1)), continue the annulus beyond its boundary circles and separate the exiting walls (Propositions 4.2 and 4.3). A logarithmic compactification is not constructed for the finite levels, the finite root stacks of finite subtrees of the path tree, base-changed to a truncated coefficient ring (Section 2.4): charts identified along dense opens would force the holonomy around a boundary circle, a shear, to fix every character, and the coefficient rings contain no degeneration parameter to absorb it (Remark 4.4). These root stacks and their Kummer covers are singular in general, and are dominated by log smooth Deligne–Mumford stacks with simple normal crossings boundary (Appendix C).

Primary Frobenius structure. Transport through the annular charts preserves the primary Frobenius structure of the GKT open mirror theorem [GKT26a, Thm. 0.3, Cor. 4.17, Cor. 5.14]: the transported derivatives of the primary potential form a basis of the transported Jacobian algebra, with product, unit and residue pairing independent of the path (Proposition 6.3). This is transport of structure. What is specific to the quartic is the normalization of the primary potential in the chart where the transport starts, which Appendix E determines: the primary potential has a unique representative invariant under exchanging x and y (Lemma E.1), and on the marginal line it is the flat structure of Noumi and Yamada [NY98], with primitive form normalized to $dx \wedge dy$ (Theorem E.3). The open invariants for two labels of twist (2, 2) stated at the beginning of this subsection are read off from this normalization.

1.5. Method and organization. The proof of Theorem A has three steps. The discrete Legendre transform and the exponents of the two singleton fields determine the enlarged lattice, the parity cover, and the pathwise Kummer rings (Section 2). A divergence computation shows that the lifted slab maps do not preserve the Hamiltonian derivations of $dx \wedge dy$, which leads to the Jacobi bracket on the line bundle $\mathcal{L}_\kappa$, and the ordered factorization of the commutator of the two singleton factors, placed along cofactor directions, gives the wall structure $\mathfrak{D}$ (Section 3). The boundary circles are treated in Section 4. For Theorem B, the GKT notions used are recalled in Section 5.1; the κ -shift embeds the GKT critical Lie algebra, and comparing the increasing-slope factorizations ray by ray gives the residual factors, the quotients on each ray of the wall factor by the factor of the GKT transition (Section 5). The compatible homotopies are built by relative extension of canonical multisections, one graph and descendent vector at a time in a well-founded order, along the finite systems of Definition B.1, using the multisection extension argument of [BCT24] in its W -spin form [GKT26a, Lem. 5.15] (Appendix B). Section 6 transports the primary Frobenius structure, and Section 7 compares transport through $\mathfrak{D}$ with the GKT transition along a path for two labels of descendent degree one, and states the maximal normalization problem. Appendix A records the affine and logarithmic calculations,

Appendix C the regular presentation, Appendix D products of functions transported along different paths, and Appendix E the normalization of the GKT primary potential of $x^4 + y^4$. The proofs of Theorems A and B do not use Section 6 or Appendices C, D and E, and of Section 4 they use only the labelled truncations of Definition 4.1 and the base change (52).

The construction follows the separation between affine and wall data of [GHK15, GHKK18, GS22] but produces no toric degeneration of the quartic pair, and it does not construct open FJRW invariants from the annular data. Its enumerative inputs are the two singleton Hamiltonian vector fields and, for Theorem B, Proposition 6.3 and Section 7, the GKT critical Lie algebra, the GKT homotopies, the GKT primary mirror theorem, and the extremal GKT potentials of the truncations considered.

Conventions. Throughout, $\mathbb{k}$ is an algebraically closed field of characteristic zero. Formal-algebraic constructions are made over $\mathbb{k}$; oscillatory integrals and Saito theory [Sai83] are understood after base change to $\mathbb{C}$. The enumerative coefficient rings are defined over $\mathbb{Q}$ and are base-changed to $\mathbb{k}$ when used in the root-stack construction.

The sources of this paper and its two companions [Wue26a, Wue26b], with code reproducing the finite computations used here (Table 1, the matrices of Appendix A, and the identities of Sections 5 to 7), are available at <https://github.com/bwuebben/open-fjrw-scattering>.

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2. THE ANNULAR AFFINE MANIFOLD AND ITS COEFFICIENT SYSTEM

2.1. The quartic degeneration. Let $M = \mathbb{Z}^2$ be the lattice of the fan of $\mathbb{P}(1, 1, 2)$, with primitive ray generators

$$v_X = (1, 0), \quad v_Y = (-1, -2), \quad v_Z = (0, 1).$$

This is the lattice called N in toric geometry; Appendix F compares the two conventions. The torus of $\mathbb{P}(1, 1, 2)$ has character lattice $M^\vee = \text{Hom}(M, \mathbb{Z})$, and we write (s_1, s_2) for the standard coordinates of the plane $M^\vee \otimes \mathbb{R}$, dual to those of M . The anticanonical polygon $\Delta \subset M^\vee \otimes \mathbb{R}$, with vertices given in the coordinates (s_1, s_2) , and its polar $\Delta^\vee \subset M \otimes \mathbb{R}$ are

$$(3) \quad \Delta = \text{conv}\{(-1, -1), (-1, 1), (3, -1)\}, \quad \Delta^\vee = \text{conv}\{v_X, v_Z, v_Y\}.$$

The edge $[v_Y, v_X]$ has lattice length two and midpoint $m_* = (0, -1) \in M$. This is the unique non-basic edge of $\Delta^\vee$.

Consider the marginal quartic pencil

$$(4) \quad E_s = \{X^4 + Y^4 + Z^2 + sX^2Y^2 = 0\} \subset \mathbb{P}(1, 1, 2)$$

and the toric-boundary degeneration

$$(5) \quad XYZ + \eta(X^4 + Y^4 + Z^2 + sX^2Y^2) = 0.$$

For $\eta \neq 0$, division by η and the substitution $Z' = Z + XY/(2\eta)$ give the same pencil with effective parameter $s - 1/(4\eta^2)$. At $\eta = 0$ the fiber is the toric boundary.

Proposition 2.1. *The family (5) is a relative anticanonical section of the coherent Mumford degeneration associated with the star subdivision of Δ at the origin. Its logarithmic smoothing charts are semistable on the canonical stack. At the coarse A_1 point the smoothing is Kummer of index two.*

Proof. The lattice points of Δ index the anticanonical Cox monomials by

$$m \mapsto X^{\langle m, v_X \rangle + 1} Y^{\langle m, v_Y \rangle + 1} Z^{\langle m, v_Z \rangle + 1}.$$

The five terms in (5) correspond to the origin and the four boundary points indexing X^4, Y^4, Z^2, X^2Y^2 . Assigning height zero to the origin and height one to the boundary gives the star subdivision, hence the stated coherent degeneration.

On $Y \neq 0$, with $x = X/Y$ and $z = Z/Y^2$, the point $x = z = 0$ is a node of the toric boundary at a smooth point of $\mathbb{P}(1, 1, 2)$, and near it the family has equation

$$xz + \eta(z^2 + x^4 + sx^2 + 1) = 0.$$

The parenthesized term is a unit at the node, so an étale change of coordinates gives $xz = \eta$. At $[0 : 0 : 1]$, pass to the canonical stack chart with residual involution $(x, y) \mapsto (-x, -y)$. The equation is again $xy = \eta$ up to a unit. On the coarse chart set $\xi_0 = x^2$, $\xi_1 = xy$, $\xi_2 = y^2$; since $\xi_0\xi_2 = \xi_1^2$, eliminating ξ_1 gives $\xi_0\xi_2 = \eta^2$ up to a unit. $\square$

Let ρ be the edge $[v_Y, v_X]$ of $\Delta^\vee$, the slab of lattice length two. It is an edge of the dual intersection complex, the discrete Legendre transform of the CPS complex (Section 1.2), in $M \otimes \mathbb{R}$; the transform matches it with the edge of the star subdivision of Δ from the origin to the vertex $(-1, 1)$ (Section 2.2), which carries the singular point B of Section 2.3. Let u denote the primitive character along ρ . The slab function of ρ may be written $f_\rho = 1 + a_\rho z^u + z^{2u}$, with a free coefficient a_ρ , the *slab parameter*. The associated curve is compared with (4) by

$$(6) \quad s^2 = \frac{4a_\rho^2}{a_\rho^2 - 4}, \quad a_\rho^2 = \frac{4s^2}{s^2 - 4}.$$

Indeed, write $U = z^{v_X}$ and $V = z^{v_Z}$ for the coordinates of the torus $\text{Spec } \mathbb{k}[M]$, in which the Landau–Ginzburg Laurent polynomial on the side of the CPS complex is $U + V + a_\rho V^{-1} + U^{-1}V^{-2}$ (Section 2.2). Eliminating U from $U + V + a_\rho V^{-1} + U^{-1}V^{-2} = 0$ gives $Y^2 = (V^2 + a_\rho)^2 - 4$, and rescaling by a square root of $a_\rho^2 - 4$ gives the quartic. Equation (6) is a comparison of the anticanonical curves. At the two values $a_\rho = \pm 2$ the slab function $f_\rho = (1 \pm z^u)^2$ is a square, the curve $Y^2 = (V^2 + a_\rho)^2 - 4$ acquires a node, and (6) gives $s = \infty$. We call these values the *nodal values* of the slab parameter.

2.2. Exponent lattices and the placement of walls. Let $\mathbb{Z}_{x,y}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be the lattice of bidegrees of Laurent monomials in the variables of $W = x^4 + y^4$. We write its elements as pairs (a, b) and read (a, b) as the exponent of $x^a y^b$. Bidegrees count boundary markings: for the GKT potentials and generators recalled in Section 5.1, the bidegree of the monomial attached to a disk graph, a monomial $x^{k_1} y^{k_2}$ of a potential or the Hamiltonian function $x^{k_1+1} y^{k_2+1}$ of a critical generator X_k , is the pair of numbers of boundary tails of twists $(2, 0)$ and $(0, 2)$ on the graph. The coordinate divisors give

$$(7) \quad \iota : \mathbb{Z}_{x,y}^2 \longrightarrow M, \quad \iota = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}, \quad [M : \iota(\mathbb{Z}_{x,y}^2)] = 2,$$

so that $\iota(e_1) = v_X$, $\iota(e_2) = v_Y$ and $\iota(e_1 + e_2) = 2m_*$. The support map for walls is the cofactor

$$(8) \quad L = \det(\iota) \iota^{-T} = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}.$$

For a pair (a, b) , whether a bidegree, an element of M in standard coordinates or a vector in the coordinates (s_1, s_2) , write $(a, b)^\perp = (b, -a)$. Then, for $q \in \mathbb{Z}_{x,y}^2$,

$$(9) \quad (Lq)^\perp = \iota(q^\perp), \quad L^T \iota = \det(\iota) \text{Id},$$

where $(Lq)^\perp$ is formed in the coordinates (s_1, s_2) and $\iota(q^\perp)$ in the standard coordinates of M , which are dual to them. By the second identity, $\iota(q^\perp) \in M$ annihilates Lq . Section 3.3 coorients the wall of direction Lq by this normal.

The Landau–Ginzburg variables enter through the linear map

$$(10) \quad F = \frac{1}{4}\iota, \quad \widetilde{M} = F(\mathbb{Z}_{x,y}^2) \supset M, \quad [\widetilde{M} : M] = 8,$$

and the dictionary $x^a y^b \mapsto z^{F(a,b)}$. The map F is the unique linear map sending the exponents $(4, 0)$ and $(0, 4)$ of the two monomials of W to the rays v_X and v_Y of the coordinate divisors $\{X = 0\}$ and $\{Y = 0\}$; the factor $\frac{1}{4}$ is the reciprocal of the degree of W , that is, the common weight of x and y . The dictionary sends x^4 , y^4 , $x^2 y^2$ and xy to z^{v_X} , z^{v_Y} , z^{m_*} and $z^{F(1,1)}$. Thus $\widetilde{M}$ is, under the dictionary, the lattice of exponents of Laurent monomials in x and y , while the fan lattice M is the character lattice of the dual torus $\text{Spec } \mathbb{k}[M]$, on which the Laurent polynomial $U + V + a_\rho V^{-1} + U^{-1} V^{-2}$ above and the slab functions live. As for the Landau–Ginzburg mirror of a toric variety, whose monomials have the rays of the fan as exponents, the exponents of slab functions, wall functions and derivations are tangent vectors of the dual intersection complex, in M and $\widetilde{M}$, while (5) is written on the torus of $\mathbb{P}(1, 1, 2)$, with character lattice $M^\vee$. The character of $x^a y^b$ lies in M exactly when b is even and $a \equiv b \pmod{4}$. In particular $4\widetilde{M} = \iota(\mathbb{Z}_{x,y}^2)$ has index two in M .

The dictionary. The Landau–Ginzburg variables x, y are the Cox coordinates X, Y of $\mathbb{P}(1, 1, 2)$, of weight one, and the Cox coordinate Z , of weight two, is the suspension variable: $W(X, Y) + Z^2$ is the quartic (4) at $s = 0$, and the marginal monomial $x^2 y^2$ is the deformation direction $X^2 Y^2$ of the pencil. The dictionary therefore matches the pure powers x^4 and y^4 of the two Landau–Ginzburg variables with the rays v_X and v_Y of the divisors $\{X = 0\}$ and $\{Y = 0\}$ of the corresponding Cox coordinates. The third ray v_Z , that of the divisor $\{Z = 0\}$ of the suspension variable, corresponds to no monomial of W . The extension from x^4, y^4 to all monomials is linear because the dictionary is to be multiplicative: it must send products of monomials to products of characters, so that it extends to a homomorphism from Laurent polynomials in x, y to $\mathbb{k}[\widetilde{M}]$ and carries derivations to derivations (Proposition 2.2). A multiplicative assignment of characters to monomials is a homomorphism of exponent lattices, and it is determined by its values on $(4, 0)$ and $(0, 4)$, which span $\mathbb{Z}_{x,y}^2 \otimes \mathbb{Q}$. On the side of the CPS complex, the characters of M are the exponents of the slab functions and of the Laurent polynomial $U + V + a_\rho V^{-1} + U^{-1} V^{-2}$ above. In the coordinates $U = z^{v_X}$ and $V = z^{v_Z}$ of the torus $\text{Spec } \mathbb{k}[M]$, its monomials U , $U^{-1} V^{-2}$ and V have the three rays v_X , v_Y and v_Z of the fan as exponents, and V^{-1} has the midpoint m_* of the non-basic edge. The dictionary sends x^4 , y^4 and $x^2 y^2$ to U , $U^{-1} V^{-2}$ and V^{-1} . The coefficient of V^{-1} is the slab parameter a_ρ , which is related to the marginal parameter s by (6), and the remaining monomial V has as exponent the ray v_Z of the divisor of the suspension variable.

Coordinates. The map F is an isomorphism of $\mathbb{Z}_{x,y}^2$ onto $\widetilde{M}$, and we give $\widetilde{M}$ the coordinates it transports. The F -coordinates of $m \in \widetilde{M}$ are the coordinates of $F^{-1}(m) \in \mathbb{Z}_{x,y}^2$, that is, the coordinates of m in the basis $F(1, 0), F(0, 1)$; the character of $x^a y^b$ has F -coordinates (a, b) , and we write $z^{(a,b)}$ for $z^{F(a,b)}$. Dually, the F -coordinates of $n \in M^\vee \otimes \mathbb{Q}$, in particular of $n \in \widetilde{M}^\vee$, are the coordinates of $F^T n$, that is, the coordinates of n in the dual basis; the element with F -coordinates (c, d) is $F^{-T}(c, d)$. Throughout the paper, exponents of characters of $\widetilde{M}$, matrices acting on $\widetilde{M}$, and determinants of pairs of elements of $\widetilde{M}$ are written in F -coordinates. In particular $\kappa = (1, 1)$ is the exponent of xy , and with $z_1 = z^{(1,0)}$ and $z_2 = z^{(0,1)}$, the characters of x and y , the form $z^\kappa d \log z_1 \wedge d \log z_2$ of Section 3 corresponds to $dx \wedge dy$. Standard coordinates

of $M \otimes \mathbb{Q}$ are used only for elements of M and for values such as $F(2, 1) = (\frac{1}{4}, -\frac{1}{2})$. In passages that use them, an element of $\widetilde{M}$ with F -coordinates (a, b) is written $F(a, b)$, never as a bare pair. Pairs describing points and directions in the plane $M^\vee \otimes \mathbb{R}$ of Δ are in the coordinates (s_1, s_2) of (3) and are marked as such, as are F -coordinates of elements of $M^\vee \otimes \mathbb{Q}$.

Let $q = (q_1, q_2) \in \mathbb{Z}_{x,y}^2$, so that $q^\perp = (q_2, -q_1)$, and denote by ∂_v the invariant logarithmic derivation in direction $v \in \mathbb{Z}^2$, so that $\partial_v(x^a y^b) = \langle v, (a, b) \rangle x^a y^b$. The derivation

$$X_q^\kappa = x^{q_1-1} y^{q_2-1} \partial_{q^\perp}$$

is the Hamiltonian vector field of $x^{q_1} y^{q_2}$ for $dx \wedge dy$, in the sense that $(dx \wedge dy)(X_q^\kappa, \cdot) = d(x^{q_1} y^{q_2})$; the superscript records the shift of the exponent by $\kappa = (1, 1)$, the exponent of xy . Put

$$(11) \quad \tilde{n}_q = F^{-T} q^\perp \in \widetilde{M}^\vee,$$

where F^{-T} is computed in the standard coordinates of $M^\vee \otimes \mathbb{Q} = \text{Hom}(M, \mathbb{Q})$; thus $\tilde{n}_q$ is the element with F -coordinates $q^\perp$. Consequently, in F -coordinates, where ∂_v denotes the derivation of $\mathbb{k}[\widetilde{M}]$ in the direction $F^{-T}v$, the transported derivation (12) below is $z^{q-\kappa} \partial_{q^\perp}$. This is the field X_q^κ of (31) and of the introduction, and we use the same notation for both.

Proposition 2.2. *Under the dictionary $x^a y^b \mapsto z^{F(a,b)}$, every logarithmic Hamiltonian derivation X_q^κ becomes an integral monomial derivation of $\mathbb{C}[\widetilde{M}]$. More precisely,*

$$(12) \quad F_* X_q^\kappa(z^m) = \langle \tilde{n}_q, m \rangle z^{m+F(q-\kappa)}, \quad m \in \widetilde{M}.$$

The derivation annihilates the wall character $z^{F(q)}$ and shifts exponents by $F(q - \kappa)$.

Proof. For $q' \in \mathbb{Z}_{x,y}^2$,

$$\langle F^{-T} q^\perp, F q' \rangle = \langle q^\perp, q' \rangle.$$

This proves integrality and (12). Since $\langle \tilde{n}_q, F(q) \rangle = \langle q^\perp, q \rangle = 0$, the derivation annihilates $z^{F(q)}$. $\square$

Wall directions. In the scattering diagrams of [GPS10, GHS22] exponents and wall supports lie in one real plane, and a wall is tangent to the exponent of its function. Here the two lie on opposite sides of the discrete Legendre transform (Section 2.3): the character $F(q)$ of the Hamiltonian function $x^{q_1} y^{q_2}$ of X_q^κ lies in $\widetilde{M} \subset M \otimes \mathbb{Q}$, the tangent space of the dual intersection complex, whereas cuts and wall supports are drawn in the plane $M^\vee \otimes \mathbb{R}$ of Δ , through its central vertex O , the origin. A support rule must therefore carry exponents across the transform. The transform matches each cell with a dual cell whose tangent space is the annihilator of its own: the segment from O to a vertex p of Δ (a *star edge*) is matched with the edge $\Delta^\vee \cap \{\langle p, \cdot \rangle = -1\}$ of the polar polygon, which is parallel to the annihilator of p . The singular points A, B, C of Section 2.3 are the midpoints of the star edges, B on the edge to $(-1, 1)$, and each star edge is annihilated by the invariant character $\bar{u}_V$ of the monodromy about its singular point V (Appendix A). If a support through O is regarded as an edge of a refinement of the star subdivision, the same rule matches a wall tangent to $F(q)$ on the dual intersection complex with a support along the annihilator of $F(q)$, that is, along $\tilde{n}_q = F^{-T} q^\perp$, the direction of the derivation X_q^κ . We do not use this rule, because it does not respect the exchange of x and y . The exchange $\sigma(a, b) = (b, a)$ acts on M by $S = F\sigma F^{-1}$, which fixes v_Z and exchanges v_X and v_Y , and on the plane of Δ by S^T , an involution of Δ that fixes the star edge through B pointwise and exchanges A and C . Since $(\sigma q)^\perp = -\sigma(q^\perp)$, the rule above satisfies $\tilde{n}_{\sigma q} = -S^T \tilde{n}_q$: the involution carries the support of X_q^κ to the opposite of the support

of $X_{\sigma q}^\kappa$. No choice of orientations repairs this: σ fixes $(1, 1)$, while S^T reverses both rays of the annihilator of $F(1, 1)$, the line $s_2 = 0$, which moreover misses A , B and C .

We place the wall of X_q^κ along the cofactor direction $Lq \in M^\vee \otimes \mathbb{R}$ instead. By (8), $L = -\frac{1}{2}F^{-T}$, so that

$$(13) \quad \langle Lq, F(q') \rangle = -\frac{1}{2} q \cdot q', \quad q, q' \in \mathbb{Z}_{x,y}^2,$$

with $q \cdot q'$ the standard scalar product of bidegrees, and $L\sigma = S^T L$. Equivalently,

$$Lq = -8(u^T)^{-1}F(q) = d\psi(F(q)), \quad \psi(F(a, b)) = -\frac{1}{4}(a^2 + b^2),$$

where $d\psi : M \otimes \mathbb{R} \rightarrow M^\vee \otimes \mathbb{R}$ is the gradient map, or Legendre transform, of the quadratic function ψ on $M \otimes \mathbb{R}$. Thus $d\psi$ carries the ray of each exponent to the support of its wall. Here ψ is an auxiliary quadratic function, unrelated to the polarization φ ; in F -coordinates on $\widetilde{M}$ and the dual F -coordinates on $\widetilde{M}^\vee$ the rule is $q \mapsto -\frac{1}{2}q$. What singles out rules of this kind is the exchange: a linear map $P : \mathbb{Z}_{x,y}^2 \otimes \mathbb{R} \rightarrow M^\vee \otimes \mathbb{R}$ satisfies $P\sigma = S^T P$ exactly when $P = F^{-T}Q$ for a matrix Q commuting with σ . Such a Q is symmetric, so P is the composite of F with the gradient map of the σ -invariant quadratic function $F(v) \mapsto \frac{1}{2}v^T Q v$, and P sends $(1, 1)$ into the line through O and B . The cofactor is the case $Q = -\frac{1}{2}\text{Id}$. Every negative definite Q gives supports in the cyclic order of the characters $F(q)$, with the diagonal support running from O along the star edge to B , and for all Q except the multiples of $-\begin{pmatrix} 7 & -5 \\ -5 & 7 \end{pmatrix}$, whose chords pass through A and C , the other supports avoid the singular points; among these the choice is a convention. The map $d\psi$ is symmetric and negative definite, of determinant 16 for the standard orientation of M and the dual orientation of $M^\vee$, so the supports occur about O in the same cyclic order as the characters $F(q)$ about the origin. Since F reverses orientation ($\det F = -\frac{1}{8}$), this is the reverse of the cyclic order of the exponents q in F -coordinates, the form in which it is stated after Theorem A; equivalently, L reverses orientation. This cyclic order is the only property of the placement used in the proof of consistency at the joints (Theorem 3.7): a path passing above O from left to right meets the supports in order of increasing slope of q , and its path-ordered product is the ordered factorization (42).

The pairs $(2, 1)$ and $(1, 2)$ are the exponents of x^2y and xy^2 in $\mathbb{Z}_{x,y}^2$. Their characters $F(2, 1) = (\frac{1}{4}, -\frac{1}{2})$ and $F(1, 2) = (-\frac{1}{4}, -1)$ do not lie in M , and the corresponding wall supports have the cofactor directions $L(2, 1) = (-4, 3)$ and $L(1, 2) = (-2, 3)$ in the coordinates (s_1, s_2) .

The lattice M is generated by $F(4, 0) = v_X$ and $F(-2, -2) = v_Z$. The matrix of their F -coordinates $(4, 0)$ and $(-2, -2)$ has determinant -8 and entries with greatest common divisor two, so the quotient is

$$(14) \quad \widetilde{M}/M \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

We call $\widetilde{M}/M$ the *fractional-exponent group*: it records an exponent in $\widetilde{M}$ modulo the integral exponents M . The classes $\alpha = [F(1, 0)]$ and $\beta = [F(0, 1)]$, by which the singleton fields $X_{(2,1)}^\kappa$ and $X_{(1,2)}^\kappa$ shift exponents, have order four, whereas $\alpha + \beta$ has order two. The total degree modulo four,

$$(15) \quad \deg_4 : \widetilde{M}/M \longrightarrow \mathbb{Z}/4, \quad \deg_4([F(a, b)]) = a + b \pmod{4},$$

is a homomorphism with kernel generated by $\alpha - \beta$.

2.3. The annular double cover. We keep the two sides of the discrete Legendre transform apart. The polygon Δ lies in $M^\vee \otimes \mathbb{R}$. With its star subdivision at the origin O , the *central vertex*, and the singular points A, B, C at the midpoints of the three star edges, it carries the integral-affine structure with straight boundary of Appendix A, whose tangent lattice is $M^\vee$.

Cuts, wall supports, flares and developed pictures are drawn in Δ . The dual lattice M is the tangent lattice of the dual intersection complex. Exponents of slab functions, wall functions and derivations lie in $\widetilde{M} \subset M \otimes \mathbb{Q}$, and the polarization φ is a piecewise-linear function on the dual intersection complex whose kinks are integral vectors of $M^\vee$. As topological manifolds we identify the dual intersection complex with the interior of Δ by the correspondence between cells and dual cells of the discrete Legendre transform [GS06, §1.4] (for a polygon with boundary, [CPS24, §7.1]), under which the singular points correspond to A, B, C ; in particular φ is a function on the interior of Δ , and it pulls back along the covers of Δ constructed below. Along a loop, the monodromy of M is the inverse transpose of the monodromy of $M^\vee$. The *width* of a unipotent monodromy $H \neq \text{Id}$ is the positive integer w with $H - \text{Id} = \pm w u n^T$ for primitive vectors u and n , so that H is conjugate in $\text{GL}_2(\mathbb{Z})$ to $\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}$; H is *primitive* when $w = 1$. Widths, traces and primitivity of a monodromy do not change under inverse transposition, so statements about them refer to either lattice. For the monodromy about a singular point with width $w > 1$, [CPS24, §8.1] call w the order of the singular point. The monodromies about A, B and C have widths 1, 2 and 1 (Appendix A); B is the singular point on the star edge matched with the slab ρ . On the double cover constructed below, $\widetilde{M}$ and its dual $\widetilde{M}^\vee = F^{-T}(\mathbb{Z}^2) \subset M^\vee \otimes \mathbb{Q}$ take the places of M and $M^\vee$; we call the pulled-back structure with these lattices the *$\widetilde{M}$ -affine structure*. Over a simply connected region of the double cover, away from its singular points, a *frame* is a trivialization of the local system $\widetilde{M}$, that is, an identification of it with the fixed lattice $\widetilde{M} = F(\mathbb{Z}_{x,y}^2)$, equivalently of the algebra of characters over the region with $\mathbb{k}[\widetilde{M}]$; exponents, monodromy matrices and F -coordinates are read in a frame. A chart pulled back from Δ identifies $\widetilde{M} \otimes \mathbb{Q}$ over the region with $M \otimes \mathbb{Q}$. On the sheet S_0 this identification carries $\widetilde{M}$ onto $F(\mathbb{Z}_{x,y}^2)$, and it is the frame we fix there; the frames used below are transported from S_0 across the cuts (Section 2.4). On the sheet S_1 the pulled-back chart carries $\widetilde{M}$ onto $T_A F(\mathbb{Z}_{x,y}^2) \neq F(\mathbb{Z}_{x,y}^2)$, with the monodromy T_A of (16) below, and induces no frame.

For the loops $\gamma_A, \gamma_B, \gamma_C$ oriented as in Appendix A, the monodromies of M about the three singular points are, in standard coordinates,

$$(16) \quad T_A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad T_B = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \quad T_C = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}.$$

Only T_B preserves $\widetilde{M}$. Indeed, conjugating the other two by F gives half-integral matrices; see Appendix A. Let $\Delta_{\text{aff}}^\circ$ be the affine base disk with the three singular points A, B, C removed, and define

$$(17) \quad \chi(\gamma_A) = \chi(\gamma_C) = 1, \quad \chi(\gamma_B) = 0 \quad \text{in } \mathbb{Z}/2.$$

We call $\chi(\gamma)$ the *parity* of a loop γ , $\ker \chi$ the *parity subgroup*, and the double cover classified by χ the *parity cover*.

Theorem 2.3 (Annular parity cover). *The connected double cover of $\Delta_{\text{aff}}^\circ$ classified by χ extends over the compact disk as a cover $p: \overline{\mathcal{A}} \rightarrow \Delta$ ramified at A and C , and $\overline{\mathcal{A}}$ is an annulus. The point B has two unramified lifts B_0, B_1 , which are primitive focus-focus singularities of the $\widetilde{M}$ -affine structure. The ramification points $\tilde{A}, \tilde{C}$ over A, C are singular points of a different kind: near each of them the structure is, as a real affine structure, the pullback of the focus-focus germ at A , respectively C , under the branched double cover $z \mapsto z^2$; the lattice is $\widetilde{M}^\vee$, which the monodromy of a single loop about A or C does not preserve. Their linear monodromies on $\widetilde{M}$ are primitive unipotent, but a small loop about $\tilde{A}$ or $\tilde{C}$, based on its invariant line, develops onto a closed curve winding twice around the developed singular point, whereas at a focus-focus*

point it winds once. The pullback polarization becomes integral after multiplication by four, and $4p^*\varphi$ is the minimal integral multiple.

Proof. Cut the disk along an arc from A to C disjoint from B , and glue two copies crosswise along it. We take for this arc the *branch segment*, the straight segment from A to C in the coordinates (s_1, s_2) . The local model at A and at C is $z \mapsto z^2$. Since the boundary loop has parity $1+0+1=0$, it has two lifts. Writing χ_{top} for the Euler characteristic, Riemann–Hurwitz gives $\chi_{\text{top}}(\bar{\mathcal{A}}) = 2\chi_{\text{top}}(\Delta) - 2 = 0$, since the disk Δ has Euler characteristic one; a connected oriented surface with Euler characteristic zero and two boundary components is an annulus.

Appendix A computes the monodromy of the parity subgroup on $\widetilde{M}$. With the base point in the sheet S_0 of Section 2.4, the monodromies about the four singular points are, in F -coordinates,

$$(18) \quad \begin{aligned} H_A &= \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}, & H_{B,0} &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \\ H_{B,1} &= \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}, & H_C &= \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}. \end{aligned}$$

Their rank-one parts factor as

$$(19) \quad \begin{aligned} H_A - \text{Id} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}(-1, 3), & H_{B,0} - \text{Id} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1, 1), \\ H_{B,1} - \text{Id} &= \begin{pmatrix} 5 \\ 3 \end{pmatrix}(-3, 5), & H_C - \text{Id} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}(-3, 1), \end{aligned}$$

and in each factorization both vectors are primitive. Hence every H_V is primitive unipotent on $\widetilde{M}$. The points B_0, B_1 are unramified lifts of B , so their germs are copies of the germ at B , a singular point of the disk with unipotent monodromy of width two on M ; with respect to $\widetilde{M}$ the monodromies $H_{B,0}, H_{B,1}$ are primitive. A small loop about $\tilde{A}$ maps to the loop γ_A traversed twice. Develop it from a point of the invariant line through $\tilde{A}$. The image is the developed image of γ_A followed by its image under the affine monodromy about A , which fixes the invariant line pointwise and preserves orientation; each is a closed curve winding once around the developed image of A , so the lift winds twice. The same argument applies at $\tilde{C}$. Winding numbers of developed loops are invariant under affine isomorphism of germs, so the germs at $\tilde{A}, \tilde{C}$ are not focus–focus germs. The preimage of the star edge through A consists of two arcs crossing at $\tilde{A}$, so $\tilde{A}$ and $\tilde{C}$ are four-valent vertices of the lifted decomposition. In the polarized complexes of [GS06, CPS24] the discriminant locus is disjoint from the vertices [CPS24, §1], so the lifted decomposition is not of that kind at $\tilde{A}$ and $\tilde{C}$; the last assertion of the theorem concerns only the integrality of $4p^*\varphi$.

The polarization φ is that of the discrete Legendre transform [GS06, §1.4]: its kink across an edge of the decomposition of the dual intersection complex is the lattice vector of the dual edge of the star subdivision. The kinks across the three bounded edges are therefore, up to sign, the vertices p_A, p_B, p_C of Δ , since the star edges from O to these vertices have lattice length one. Their F -coordinates have denominator exactly four (Appendix A). The kinks across the unbounded edges are the edge vectors of Δ , whose F -coordinates are integral in the frame of S_0 ; they are not the primitive edge directions of the normalization of [CPS24, §7.1], in which the slope changes by one along the unbounded edges. The frame of S_1 differs from the identification given by the chart of S_1 pulled back from Δ by the monodromy about A , which acts on F -coordinates of $\widetilde{M}^\vee$ by $(F^{-1}T_A F)^{-T}$; in the frame of S_1 the bounded kinks again have denominator exactly four, and the unbounded kinks have F -coordinates $(-\frac{1}{2}, \frac{1}{2})$,

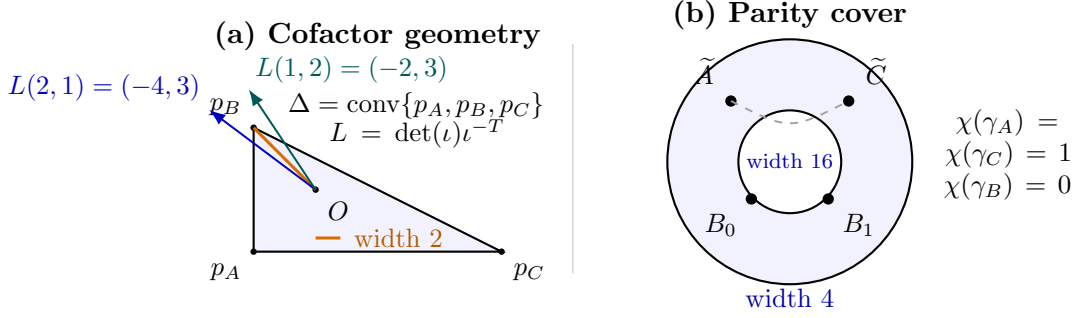


FIGURE 1. The quartic polygon and its annular parity cover. (a) The polygon Δ with vertices p_A, p_B, p_C , in the coordinates (s_1, s_2) ; the highlighted star edge from O to p_B contains the singular point B , whose monodromy has width two. The first two singleton supports are the cofactor directions $L(2, 1)$ and $L(1, 2)$. (b) Two copies of the disk, cut along the branch segment from A to C , are glued crosswise; the dashed arc is one of the two lifts of this segment. The boundary circles have widths 4 and 16. The cover is branched over A and C , whose preimages are the ramification points $\tilde{A}, \tilde{C}$, and B has two unramified lifts B_0, B_1 .

$(\frac{5}{2}, -\frac{9}{2})$ and $(-2, 4)$. Multiplication by four therefore makes $p^*\varphi$ integral on both sheets, and no smaller positive multiple does. $\square$

The two boundary circles of $\bar{\mathcal{A}}$ are the lifts of the boundary loop $\gamma_A\gamma_B\gamma_C$ of Δ , whose monodromy on M is, in standard coordinates, $T_AT_BT_C = \begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix}$, of width eight. Their monodromies U_0, U_1 on $\widetilde{M}$ are computed in Appendix A and have normal forms

$$(20) \quad \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -16 \\ 0 & 1 \end{pmatrix}.$$

Their widths are therefore 4 and 16.

Let $\mathcal{A}$ be the interior of $\bar{\mathcal{A}}$, and let $\mathcal{A}^\circ$ be the complement in $\mathcal{A}$ of the four singular points $\tilde{A}, B_0, B_1, \tilde{C}$. Loops in $\mathcal{A}^\circ$ that are freely homotopic in $\bar{\mathcal{A}}$ to a core circle of the annulus are not all freely homotopic in $\mathcal{A}^\circ$; they differ by loops about the singular points, and their holonomies need not have the same trace. We fix the *core loop* $\gamma_{\text{core}} = \gamma_C\gamma_A^{-1}$, based in the sheet S_0 of Section 2.4. Write $\gamma_{\tilde{A}} = \gamma_A^2$, $\gamma_{B_0} = \gamma_B$, $\gamma_{B_1} = \gamma_A\gamma_B\gamma_A^{-1}$ and $\gamma_{\tilde{C}} = \gamma_C^2$ for the loops about the four singular points. In the fundamental group the two boundary loops factor as

$$\gamma_A\gamma_B\gamma_C = \gamma_{B_1}\gamma_{\text{core}}^{-1}\gamma_{\tilde{C}}, \quad \gamma_A^2\gamma_B\gamma_C\gamma_A^{-1} = \gamma_{\tilde{A}}\gamma_{B_0}\gamma_{\text{core}},$$

so γ_{core} is the core loop separating $\{\tilde{A}, B_0\}$ from $\{B_1, \tilde{C}\}$. Its linear holonomy is the hyperbolic matrix K of (2), of trace 18, and $U_0 = H_{B,1}K^{-1}H_C$, $U_1 = H_A H_{B,0}K$. Other loops of this kind have other traces: for instance $\gamma_A\gamma_C$ has holonomy $K^{-1}H_C$, of trace -14 , which exchanges α and β .

The annulus $\bar{\mathcal{A}}$ with its $\widetilde{M}$ -affine structure is thus determined by the lattice $\widetilde{M}$ spanned by M and the exponents of the two singleton fields. Section 4 compares it with the affine bases of distinguished toric degenerations in the sense of [CPS24].

2.4. Local Kummer charts. In outline, the coefficient system constructed in this subsection and the next is one inverse system of Kummer covers of the character torus $\text{Spec } \mathbb{k}[\widetilde{M}]$, indexed

by the finite rooted subtrees T of the path tree, the universal covering tree of the graph whose vertices are the two sheets and whose edges are the cuts: the cover of T adjoins roots of the slab functions transported along the paths of T , it is finite étale away from the zeros and poles of these functions, and $\pi_1(\mathcal{A}^\circ)$ acts on the colimit of their Kummer rings through transport. The chart algebras, Kummer rings, finite root stacks and the annular coefficient system defined below are parts of this one system.

Each matrix in (18) exchanges α and β in $\widetilde{M}/M$, while the core matrix K acts trivially. The monodromy of the local system with fiber $\widetilde{M}/M$ therefore acts through the exchange of α and β , and $\deg_4$ of (15) is the projection onto its largest quotient on which the monodromy acts trivially. The local system becomes constant on the connected double cover

$$p_{\text{fr}} : \mathcal{A}_{\text{fr}}^\circ \longrightarrow \mathcal{A}^\circ$$

defined by the kernel of the monodromy action on $\widetilde{M}/M$. We call p_{fr} the *fractional-exponent cover*: it is the smallest connected cover on which the fractional-exponent group forms a constant local system. Its deck involution δ_{deck} exchanges the classes α and β . The lifted slab maps require, in addition, roots of the slab functions; these are the Kummer roots constructed below.

Placement of the transitions. For $V \in \{A, B, C\}$ let c_V be the outer half of the star edge through V , the segment from V to the vertex p_V of Δ . The complement of $c_A \cup c_B \cup c_C$ in the interior of Δ is simply connected and contains O , the inner half-edges and the branch segment from A to C . Its preimage in $\overline{\mathcal{A}}$ therefore consists of two *sheets* S_0 and S_1 , each mapped homeomorphically onto it; we write $O_0 \in S_0$ and $O_1 \in S_1$ for the two lifts of the central vertex O . The preimage of c_B consists of one arc from each B_i to the boundary, and the preimages of c_A and c_C consist of two arcs from $\widetilde{A}$ and two from $\widetilde{C}$, one to each boundary circle. We call these six arcs the *cuts*. All transitions at the singular points, namely the affine transitions, the lifted slab maps below and the line transitions of Section 3, are carried by the cuts, which run from the singular points outward to the boundary. The sheets carry no transition; in particular none is carried by the inner half-edges, by the preimage of the branch segment, or near O_0 and O_1 . By Proposition 3.6, no wall support of Section 3 meets an arc carrying a transition.

We give S_0 the frame of its chart pulled back from Δ (Section 2.3), and S_1 the continuation of that frame across the arc over c_A that γ_A crosses from S_0 , so that this arc carries no transition. We place the lifted slab map at each singular point on a single arc from that point to the boundary: at B_0 and B_1 on the arcs over c_B , at $\widetilde{A}$ on the arc over c_A that γ_A crosses from S_1 , and at $\widetilde{C}$ on the arc over c_C that γ_C crosses from S_1 . The arc over c_C that γ_C crosses from S_0 carries only the change of affine chart along γ_{core} . As in Appendix A, a product gh of loops traverses g first. Thus $\gamma_{\text{core}} = \gamma_C \gamma_A^{-1}$ crosses only the arc carrying the change of chart and the arc carrying no transition, and the transport along it has linear part K and no slab factor. The dual graph of this decomposition is the Schreier graph of the parity subgroup for the transversal $\{1, \gamma_A\}$, and its five edges other than the transversal edge correspond to the generators (96).

Slabs and cuts. This placement is a convention on where transitions are drawn. In the structures of [GS11] the transition at a singular point is carried by the slab through it, which carries a slab function at each of its points; under the identification of Section 2.3 the slab through V , the cell dual to the star edge through V , crosses that edge at V . Here the lifted slab map at V is carried instead by a single cut, which runs along the outer half c_V of the star edge, and the inner half-edges carry no transition. On the star edge through B the two halves have different roles: the arcs over the outer half c_B carry the lifted slab maps at B_0 and B_1 , and over the inner half runs the diagonal wall of Section 3.3, from O_i to B_i in the sheet S_i . A

small loop about B_i therefore crosses the arc of the slab map at B_i and the diagonal wall once each (Section 3.3).

Chart algebras. Every vertex of the path tree $\mathcal{T}$ of Section 2.5, a reduced edge path from S_0 to a sheet, carries the frame obtained from that of S_0 by transport along the path; the frames of S_0 and S_1 fixed above are those at the empty path and at the path that crosses the arc over c_A carrying no transition. The *chart algebra* of a sheet, or of a point or a region of $\mathcal{A}^\circ$ read in the frame at a vertex, is $\mathbb{k}[\widetilde{M}]$ in that frame over the Kummer ring of a finite level. A *finite level* is given by a finite rooted subtree $T \subset \mathcal{T}$ and a truncated coefficient ring, such as $R/\mathfrak{m}^N$ in (29) or the ring $R_{\mathcal{W},N}$ of a labelled truncation (Definition 4.1); geometrically it is the finite root stack of T (Section 2.5) base-changed to that ring, written $\mathfrak{X}_{T,\mathcal{W},N}$ in Section 4. Its Kummer ring is the Kummer ring R_T^{root} of (27) tensored with the coefficient ring; when T contains the vertex, $\mathbb{k}[\widetilde{M}]$ in the frame of the vertex maps into R_T^{root} by pull-back along the path of the vertex (Section 4.5).

For $V \in \{\widetilde{A}, B_0, B_1, \widetilde{C}\}$ let $u_V \in \widetilde{M}$ be the primitive vector spanning the invariant line of H_V , and let $x_V = z^{2\epsilon_V u_V}$ with the signs ϵ_V below. The exponent $2\epsilon_V u_V$ is the invariant character $\bar{u}_V \in M$ of the disk monodromy T_V about the image of V , that is, one of the characters $\bar{u}_A, \bar{u}_B, \bar{u}_C$ of T_A, T_B, T_C listed in Appendix A, transported linearly along γ_A for B_1 . The pulled-back slab functions are

$$f_{\widetilde{A}} = 1 + x_{\widetilde{A}}, \quad f_{B_i} = 1 + a_\rho x_{B_i} + x_{B_i}^2, \quad f_{\widetilde{C}} = 1 + x_{\widetilde{C}},$$

of degree $e_V = 1, 2, 2, 1$ in x_V . Let $N_V \in \widetilde{M}^\vee$ be the primitive conormal of the invariant line, with the sign fixed by

$$(21) \quad H_V - \text{Id} = -\frac{e_V}{d_V} (2\epsilon_V u_V) N_V^T = -\epsilon_V u_V N_V^T,$$

where d_V is the root degree below. The data, with u_V and N_V given by their F -coordinates, are

$$(22) \quad \begin{array}{c|cccc} V & \widetilde{A} & B_0 & B_1 & \widetilde{C} \\ \hline F^{-1}u_V & (3, 1) & (1, -1) & (5, 3) & (1, 3) \\ \epsilon_V & 1 & 1 & -1 & -1 \\ d_V & 2 & 4 & 4 & 2 \\ F^T N_V & (1, -3) & (-1, -1) & (-3, 5) & (-3, 1). \end{array}$$

On the complement of the slab divisor put

$$(23) \quad R_V^{\text{root}} = \mathbb{k}[\widetilde{M}][f_V^{-1}, g_V]/(g_V^{d_V} - f_V), \quad \tilde{\theta}_V(z^m) = g_V^{\langle N_V, m \rangle} z^m.$$

The ring R_V^{root} is the Kummer ring of f_V . Its spectrum is the restriction, over the complement of the slab divisor, of the Kummer cover $\text{Spec } \mathbb{k}[\widetilde{M}][g_V]/(g_V^{d_V} - f_V)$, whose quotient by μ_{d_V} , acting by $g_V \mapsto \zeta g_V$ and trivially on characters, is the d_V th root stack of the slab divisor (Appendix A). Over the complement of the divisor the Kummer cover is a μ_{d_V} -torsor, the root stack is the complement itself, and g_V is not a function on it. Since $\langle N_V, u_V \rangle = 0$, the map $\tilde{\theta}_V$ fixes f_V and g_V and is an automorphism of R_V^{root} . It commutes with μ_{d_V} on z^m exactly when d_V divides $\langle N_V, m \rangle$. This holds for $m \in M$, where $\langle N_V, m \rangle \in 4\mathbb{Z}$ and $\tilde{\theta}_V$ multiplies z^m by an integral power of f_V , and it fails for some $m \in \widetilde{M}$ because N_V is primitive. The lifted slab maps, the line transitions of Section 3 and every transport built from them therefore act on Kummer rings and Kummer covers, and not on root stacks.

The lifted slab map at V is placed on its arc so that the transport along the loop γ_V of Section 2.3, based in S_0 , is $z^m \mapsto g_V^{\langle N_V, m \rangle} z^{H_V m}$. By (21), $z^{H_V m} = z^{m - \epsilon_V \langle N_V, m \rangle u_V}$, and

$g_V^{d_V} x_V^{-e_V} = f_V x_V^{-e_V}$ tends to one where x_V is large; in this sense the transport is trivial to leading order there, which fixes the sign of N_V .

Proposition 2.4. *The formula (23) defines an automorphism of the Kummer ring R_V^{root} extending the transition of the disk: on characters $m \in M$, the transports about $\tilde{A}$, B_0 and $\tilde{C}$ based in S_0 are the disk transports about A twice, B once and C twice, and the transport about B_1 is the disk transport about B written in the frame of S_1 . For $a_\rho \neq \pm 2$ the root degrees 2, 4, 4, 2 are minimal.*

Proof. The root g_V is a unit on the localized chart and $\langle N_V, m \rangle$ is integral. Moreover $N_V = 4n_V$, where $n_V \in M^\vee$ is the primitive conormal of the disk at the image of V , transported linearly along γ_A for B_1 ; the arc that γ_A crosses from S_0 carries no transition. The disk transport about that singular point is $z^m \mapsto f_V^{\langle n_V, m \rangle} z^{T_V m}$, where $T_V - \text{Id} = -e_V \bar{u}_V n_V^T$ by (94); at B_1 , the slab function, the invariant character, the conormal and the monodromy are those of B transported linearly along γ_A , that is, written in the frame of S_1 . The disk transport fixes f_V and n_V . At $\tilde{A}$ and $\tilde{C}$ the loop traverses the disk loop twice, so on $m \in M$ the slab factor is $f_V^{2\langle n_V, m \rangle} = g_V^{\langle N_V, m \rangle}$; at B_0 and B_1 it traverses it once, giving $f_V^{\langle n_V, m \rangle} = g_V^{\langle N_V, m \rangle}$. Since N_V is primitive, $\langle N_V, m \rangle = 1$ for some $m \in \tilde{M}$, so the slab factor of that character is a d_V th root of f_V . For $a_\rho \neq \pm 2$ the Laurent polynomial f_V vanishes to order one along each component of its zero divisor, so its class in $\mathbb{k}(\tilde{M})^\times / \mathbb{k}(\tilde{M})^{\times d_V}$ has order d_V , and no proper divisor of d_V suffices. $\square$

The placement above thus defines the lifted coefficient system. On characters in M it agrees with the pullback of the disk system germwise at each singular point (Proposition 2.4), but it is not that pullback. In the pulled-back system each of the two arcs over c_A , and each of the two over c_C , carries the disk transition, whose slab factor on a single crossing is $f_V^{\langle n_V, m \rangle}$ with $\langle n_V, m \rangle \in \frac{1}{4}\mathbb{Z}$ for $m \in \tilde{M}$. On characters in M , the pulled-back disk transport along $\gamma_A \gamma_B \gamma_A^{-1}$ is the transport about B_1 conjugated by $z^m \mapsto f_A^{\langle n_A, m \rangle} z^m$, and the pulled-back transport along γ_{core} has slab factors, powers of f_C and of the transport of f_A , whereas the transport of the lifted system along γ_{core} is the monomial map with linear part K .

At the two nodal values $a_\rho = \pm 2$ the degree at B_0 and B_1 drops to two. If $f_B(a_\rho, z) = 1 + a_\rho z + z^2$, then

$$(24) \quad \begin{aligned} a_\rho = 2: \quad & g^4 - f_B = (g^2 - (1+z))(g^2 + (1+z)), \\ a_\rho = -2: \quad & g^4 - f_B = (g^2 - (1-z))(g^2 + (1-z)). \end{aligned}$$

Away from the reduced zero divisor these are two connected quadratic torsors; their closures meet over the divisor. We retain this fourth-root stack rather than replacing it by one connected square-root cover.

2.5. An infinite Kummer extension. The failure of finite Kummer closure already occurs on the overlap of the $\tilde{A}$ and B_0 charts. Write

$$x_A = x_{\tilde{A}} = z^{2u_{\tilde{A}}}, \quad x_B = x_{B_0} = z^{2u_{B_0}}, \quad f_A = 1 + x_A, \quad f_B(x) = 1 + a_\rho x + x^2,$$

and let $\mathcal{K} = \mathbb{k}(x_A, x_B)$. The lifted slab map $\tilde{\theta}_{\tilde{A}}$ of (23) multiplies each character by a power of the root $g_{\tilde{A}}$ and does not change its exponent. Since $\langle N_{\tilde{A}}, 2u_{B_0} \rangle = 8$ and $d_{\tilde{A}} = 2$, the lifted slab map $\tilde{\theta}_{\tilde{A}}$ fixes f_A and sends

$$(25) \quad x_B \mapsto f_A^4 x_B.$$

The transport about $\tilde{A}$ also applies the linear monodromy, and $H_A(2u_{B_0}) = 2u_{B_0} - 4 \cdot 2u_{\tilde{A}}$; it therefore fixes x_A and sends x_B to $x_A^{-4} f_A^4 x_B = (1 + x_A^{-1})^4 x_B$. For $a_\rho \neq \pm 2$ write $f_B(x) = (1 + \lambda x)(1 + \lambda^{-1}x)$ with $\lambda + \lambda^{-1} = a_\rho$ and $\lambda^2 \neq 1$.

Theorem 2.5 (Infinite Kummer orbit). *Let $h \in \mathbb{k}(x_A)$ be nonconstant, and let ω_h be the automorphism of $\mathcal{K}$ fixing x_A with $\omega_h(x_B) = hx_B$. The lifted slab map $\tilde{\theta}_{\tilde{A}}$, with $h = f_A^4$ by (25), and the transport about $\tilde{A}$, with $h = (1 + x_A^{-1})^4$, are of this form.*

- (1) *If $a_\rho \neq \pm 2$, closure of a fourth root of $f_B(x_B)$ under all powers of ω_h requires the fourth roots g_n of*

$$(26) \quad g_n^4 = f_B(h^n x_B) = (1 + \lambda h^n x_B)(1 + \lambda^{-1} h^n x_B), \quad n \in \mathbb{Z},$$

and the classes of these radicands are linearly independent over $\mathbb{Z}/4$ in $\mathcal{K}^\times / \mathcal{K}^{\times 4}$.

- (2) *If $a_\rho = \pm 2$, then $f_B(h^n x_B) = (1 \pm h^n x_B)^2$, and the classes of $1 \pm h^n x_B$, $n \in \mathbb{Z}$, are linearly independent over $\mathbb{Z}/2$ in $\mathcal{K}^\times / \mathcal{K}^{\times 2}$.*
- (3) *In either case no finite extension of $\mathcal{K}$ containing a fourth root of $f_B(x_B)$ admits an automorphism extending ω_h . In particular no finite algebraic Kummer extension containing the initial $\tilde{A}$ - and B_0 -roots is stable under the lifted slab map $\tilde{\theta}_{\tilde{A}}$ or under transport about $\tilde{A}$.*

Proof. Suppose $a_\rho \neq \pm 2$. In the principal ideal domain $\mathbb{k}(x_A)[x_B]$,

$$f_B(h^n x_B) = h^{2n} (x_B + \lambda^{-1} h^{-n}) (x_B + \lambda h^{-n}).$$

The elements $\lambda^{\pm 1} h^{-n}$, $n \in \mathbb{Z}$, of $\mathbb{k}(x_A)$ are pairwise distinct: an equality $\lambda^\epsilon h^{-n} = \lambda^{\epsilon'} h^{-m}$ makes h^{m-n} constant, hence $m = n$ because h is nonconstant, and then $\lambda^\epsilon = \lambda^{\epsilon'}$ forces $\epsilon = \epsilon'$ because $\lambda^2 \neq 1$. Thus $\pi_n = x_B + \lambda^{-1} h^{-n}$ is a prime dividing the n th radicand exactly once and no other radicand. If a finite product $\prod_n f_B(h^n x_B)^{k_n}$ is a fourth power in $\mathcal{K}$, its π_n -adic valuation k_n is divisible by four. This proves (1). For (2), the same argument applies to the primes $x_B \pm h^{-n}$ of $1 \pm h^n x_B$.

For (3), let $\mathcal{K}' \supset \mathcal{K}$ be finite, let $\tilde{\omega}$ be an automorphism of $\mathcal{K}'$ extending ω_h , and let $g_0 \in \mathcal{K}'$ satisfy $g_0^4 = f_B(x_B)$. Then $\tilde{\omega}^n(g_0)^4 = f_B(h^n x_B)$ for every $n \in \mathbb{Z}$. If $a_\rho \neq \pm 2$, the field $\mathbb{k}$ contains the fourth roots of unity, so by Kummer theory and (1) the subfield generated by the $\tilde{\omega}^n(g_0)$ with $|n| \leq r$ has degree 4^{2r+1} over $\mathcal{K}$ for every r , which is impossible. If $a_\rho = \pm 2$, then $\tilde{\omega}^n(g_0)^2 = \pm(1 \pm h^n x_B)$, and since -1 is a square in $\mathbb{k}$, $\mathcal{K}'$ contains square roots of all $1 \pm h^n x_B$; by (2) the same degree argument applies with 2^{2r+1} . $\square$

The path tree. Let $\mathfrak{C}$ be the graph whose vertices are the two sheets S_0, S_1 and whose edges are the six cuts, each joining the sheets on its two sides. The sheets are simply connected and the cuts are contractible, so the fundamental group of $\mathfrak{C}$ based at S_0 is $\Pi = \pi_1(\mathcal{A}^\circ)$, a free group of rank five. The *path tree* $\mathcal{T}$ is the universal covering tree of the graph $\mathfrak{C}$. Its vertices are the reduced edge paths in $\mathfrak{C}$ beginning at S_0 , equivalently the homotopy classes of paths in $\mathcal{A}^\circ$ from a base point $b_0 \in S_0$ to b_0 or to the other lift $b_1 \in S_1$ of its image in Δ ; its root is the empty path at S_0 .

Let $\mathbb{T} = \text{Spec } \mathbb{k}[\tilde{M}]$ be the torus of the sheet S_0 , and let $T \subset \mathcal{T}$ be a finite rooted subtree. We call an edge of T a *Kummer edge* if it crosses, in either direction, an arc carrying a lifted slab map; edges crossing the other two arcs adjoin no root. Number the Kummer edges $e = 1, \dots, \ell$, the edge e crossing the arc of the singular point V_e . It has the root degree $d_e = d_{V_e}$ and the *transported radicand* ϱ_e , the transport of f_{V_e} to S_0 along the path of T preceding e . The vectors u_V lie in the sublattice of $\tilde{M}$ of elements whose F -coordinates (m_1, m_2) satisfy $m_1 \equiv m_2 \pmod{2}$, which the matrices (96) preserve and on which every N_V is even. A lifted slab map

therefore multiplies a transported character $z^{2\epsilon_{v'u_{v'}}$ by a power of the root it uses with exponent divisible by four, that is, by an integral power of the corresponding transported radicand. By induction along the path, every ϱ_e is a rational function on $\mathbb{T}$. Let $U_T \subset \mathbb{T}$ be the complement of the zeros and poles of $\varrho_1, \dots, \varrho_\ell$, and put $G_T = \prod_e \mu_{d_e}$. The *Kummer ring* of T is

$$(27) \quad R_T^{\text{root}} = \mathcal{O}(U_T)[r_1, \dots, r_\ell] / (r_e^{d_e} - \varrho_e)_e,$$

on which $\zeta \in G_T$ acts by $r_e \mapsto \zeta_e r_e$. It is finite étale over $\mathcal{O}(U_T)$, and $\text{Spec } R_T^{\text{root}}$ is a G_T -torsor over U_T . The lifted slab map at the Kummer edge e uses the root r_e , the transport of g_{V_e} , so transport along the paths of T is defined on R_T^{root} . Let D_e be the divisor of zeros of ϱ_e on $\mathbb{T}$. The *finite root stack* of T is

$$\mathfrak{X}_T = \sqrt[d_1]{(\mathbb{T}, D_1)} \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} \sqrt[d_\ell]{(\mathbb{T}, D_\ell)},$$

in the convention (101). Over U_T it is U_T itself, the quotient of $\text{Spec } R_T^{\text{root}}$ by G_T . Section 4 extends $\text{Spec } R_T^{\text{root}}$ to a Kummer cover P_T of $\mathbb{T}$ with $\mathfrak{X}_T = [P_T/G_T]$; we call $\mathfrak{X}_T$ with its atlas P_T the *finite root-stack presentation* of T . Transport acts on R_T^{root} , that is, on P_T over U_T , and not on $\mathfrak{X}_T$, because the lifted slab maps do not commute with G_T . By Appendix A, the stabilizer of $\mathfrak{X}_T$ at a geometric point over $z \in \mathbb{T}$ is the product of the factors μ_{d_e} with $z \in D_e$. If $T \subset T'$, localization followed by adjunction of the new roots maps R_T^{root} injectively to $R_{T'}^{\text{root}}$, and forgetting the new roots gives $\mathfrak{X}_{T'} \rightarrow \mathfrak{X}_T$. The paths γ_A^n followed by a crossing of the B_0 -arc give the radicands $f_B(h^n x_B)$, $h = (1 + x_A^{-1})^4$, of Theorem 2.5.

Definition 2.6. Let $R = \mathbb{k}[t_1, t_2]$ be the deformation coefficient algebra, where t_1, t_2 are the singleton parameters, and let $\mathfrak{m} = (t_1, t_2)$ be the singleton ideal. The *pathwise Kummer ring* is $R_{\text{path}}^{\text{root}} = \varinjlim_{T \subset \mathcal{T}} R_T^{\text{root}}$, the colimit over the finite rooted subtrees $T \subset \mathcal{T}$. The pathwise root object and its adic completion are the formal systems

$$(28) \quad \mathfrak{X}_{\text{path}} = \varinjlim_{T \subset \mathcal{T}} \mathfrak{X}_T, \quad \widehat{\mathfrak{X}}_{\text{path}} = \varinjlim_N \varinjlim_{T \subset \mathcal{T}} (\mathfrak{X}_T \times \text{Spec } R/\mathfrak{m}^N).$$

The group Π acts on $\mathcal{T}$ by deck transformations, a loop acting by concatenation, and through transport on $R_{\text{path}}^{\text{root}}$, the Kummer root of an edge going to the Kummer root of its translate. The *annular coefficient system* is $\widehat{\mathfrak{X}}_{\text{path}}$ together with this action of Π . A structure over it, such as a module, an algebra, a bracket or a wall structure, is a Π -equivariant compatible system of such structures over the rings

$$(29) \quad R_T^{\text{root}} \otimes_{\mathbb{k}} R/\mathfrak{m}^N, \quad T \subset \mathcal{T} \text{ a finite rooted subtree, } N \geq 1.$$

The coefficient ring at every vertex of $\mathcal{T}$ is $R_{\text{path}}^{\text{root}} \otimes_{\mathbb{k}} R$, transport along the path of the vertex identifying it with the ring at the empty path. A *finite transport diagram modulo $\mathfrak{m}^N$* is a finite diagram of finitely presented modules or algebras over the coefficient rings of finitely many vertices of $\mathcal{T}$, reduced modulo $\mathfrak{m}^N$, together with finitely many homomorphisms, transports along edges of $\mathcal{T}$, and identities among their composites.

The limit over T is formed along the root-forgetting maps, the colimit of Kummer rings along the injections $R_T^{\text{root}} \rightarrow R_{T'}^{\text{root}}$, and the colimit over N along the closed immersions $\text{Spec } R/\mathfrak{m}^N \rightarrow \text{Spec } R/\mathfrak{m}^{N+1}$. The limits are formal systems: by Theorem 2.7, $\mathfrak{X}_{\text{path}}$ is not represented by a formal scheme or by an algebraic stack whose diagonal is locally of finite type. The group Π acts on the Kummer rings, preserving the root degrees, but not on the root stacks $\mathfrak{X}_T$, whose root groups G_T do not commute with the lifted slab maps. For this reason no quotient by Π is formed: structures over the annular coefficient system are defined through the Kummer rings (29) and the action of Π on them. This inverse system is of the same formal

kind as the infinite root stacks of [TV18, Def. 3.3 and Prop. 3.5]; the additional feature here is that the roots are indexed by reduced transport paths and permuted by the deck group Π . Wall structures over this object and their consistency are the subject of Section 3.

Theorem 2.7 (Definability over finite subtrees; non-representability). *Every finite transport diagram modulo $\mathfrak{m}^N$ is defined over $R_T^{\text{root}} \otimes_{\mathbb{k}} R/\mathfrak{m}^N$ for some finite rooted subtree T , that is, over U_T on the Kummer cover P_T of the finite root-stack presentation of T . Any two such subtrees have a common finite refinement. On the other hand, if all transported roots and their automorphisms are retained, the pathwise root object $\mathfrak{X}_{\text{path}}$ is not represented by a formal scheme, a finite root stack, or an algebraic stack whose diagonal is locally of finite type.*

Proof. A finite transport diagram involves finitely many vertices of $\mathcal{T}$ and, modulo $\mathfrak{m}^N$, finitely many deformation bidegrees. The rooted union of the corresponding paths is a finite subtree. The ring $R_{\text{path}}^{\text{root}}$ is the colimit of the Kummer rings of the finite subtrees (Appendix A), so by finite presentation the finitely many generators, relations, arrows and identities are defined over one of them after a further finite enlargement. Rooted union gives a common refinement.

For the negative assertion, let $h = (1 + x_A^{-1})^4$, let λ be a root of $\lambda^2 - a_\rho \lambda + 1$ (so $\lambda = \pm 1$ when $a_\rho = \pm 2$), and let w be a geometric point of $\mathbb{T}$ with $x_A = -\frac{1}{2}$ and $x_B = -\lambda^{-1}$. There $f_A = \frac{1}{2}$ and $h = 1$, so every radicand $f_B(h^n x_B)$ in (26) vanishes. For a finite set $S \subset \mathbb{Z}$, let T be the rooted union of the paths γ_A^n followed by a crossing of the B_0 -arc, $n \in S$. Since the transport about $\tilde{A}$ fixes f_A , every root adjoined along γ_A^n has the radicand f_A , which does not vanish at w . The root adjoined at the final crossing has the radicand $\varrho_n = f_B(h^n x_B)$, which is regular at w , because x_A and f_A are units there, and vanishes at w ; hence $w \in D_n$. The stabilizer of $\mathfrak{X}_T$ at the point over w is therefore $\prod_{n \in S} \mu_4$. The automorphism group of a compatible point of $\mathfrak{X}_{\text{path}}$ over w is the inverse limit of the stabilizers over all finite subtrees, and it contains

$$\prod_{n \in \mathbb{Z}} \mu_4$$

as a closed subgroup. This group scheme is not of finite type. A formal scheme has trivial stabilizers, a finite root stack has finite-type diagonal, and an algebraic stack with diagonal locally of finite type cannot have such a geometric stabilizer. $\square$

Theorem 2.7 is the only property of the inverse system used later: every product, period and logarithmic chart below is constructed over the Kummer ring of a finite level, and the pro-object records the compatibility of these constructions.

3. SCATTERING FOR THE JACOBI BRACKET; HYPERBOLIC CORE HOLONOMY

3.1. A line bundle for the volume form, and its Jacobi bracket. Local formulas are written in the fixed frame of the sheet $S_0 \ni O_0$ of Section 2.4, which is the frame at the root of the path tree $\mathcal{T}$ of Section 2; the frames at the other vertices of $\mathcal{T}$ are reached by transport along the edges of the graph $\mathfrak{C}$. As in Section 2.2, elements of $\tilde{M}$, matrices acting on $\tilde{M}$ and determinants of pairs of its elements are written in F -coordinates, so that $z_1 = z^{(1,0)}$ and $z_2 = z^{(0,1)}$ are the characters of x and y under the dictionary $x^a y^b \mapsto z^{F(a,b)}$. Put

$$(30) \quad \kappa = (1, 1), \quad \Omega_0 = d \log z_1 \wedge d \log z_2, \quad \Omega_\kappa = z^\kappa \Omega_0,$$

so that κ is the exponent $F(1, 1)$ of xy and Ω_κ corresponds to $dx \wedge dy$. For $q = (q_1, q_2)$ put $q^\perp = (q_2, -q_1)$, and denote the invariant logarithmic derivation in direction v by ∂_v , so that $\partial_v z^m = \langle v, m \rangle z^m$.

Proposition 3.1 (Shifted Hamiltonian bracket). *The inverse bivector $\pi_\kappa = z^{-\kappa}\pi_0$ of Ω_κ has Hamiltonian vector fields*

$$(31) \quad X_q^\kappa = z^{q-\kappa}\partial_{q^\perp},$$

and

$$(32) \quad [X_q^\kappa, X_r^\kappa] = -\det(q, r)X_{q+r-\kappa}^\kappa.$$

Thus, after writing $q = k + \kappa$ and $r = \ell + \kappa$, the output is indexed by $k + \ell$.

Proof. The Hamiltonian field of z^q for $z^{-\kappa}\pi_0$ is (31). The bracket of two monomial derivations has direction

$$\langle q^\perp, r - \kappa \rangle r^\perp - \langle r^\perp, q - \kappa \rangle q^\perp = -\det(q, r)(q + r - \kappa)^\perp,$$

which gives (32). $\square$

The fields X_q^κ , $q \in \widetilde{M}$, span over $\mathbb{k}$ the Lie algebra $\mathfrak{g}_\kappa$ of π_κ -Hamiltonian derivations of $\mathbb{k}[\widetilde{M}]$, with the bracket (32). We call $\mathfrak{g}_\kappa$ the *Hamiltonian wall algebra*: the wall automorphisms constructed below are exponentials of elements of $\mathfrak{g}_\kappa$ with coefficients in the ideal generated by the deformation parameters, completed with respect to that ideal; they are written in the frame of S_0 and transported to the other frames by Proposition 3.3. For $\kappa = 0$ the field $X_q^0 = z^q\partial_{q^\perp}$ is the Hamiltonian vector field of z^q for Ω_0 , and $X_0^0 = 0$. Thus $\mathfrak{g}_0$ is spanned by the fields $z^q\partial_{q^\perp}$ with $q \neq 0$, and for an Artinian local $\mathbb{k}$ -algebra R with maximal ideal $\mathfrak{m}_R$ the Lie algebra $\mathfrak{g}_0 \otimes_{\mathbb{k}} \mathfrak{m}_R$ is the Lie algebra $\mathfrak{h}_R$ of the tropical vertex group of Gross–Pandharipande–Siebert [GPS10, Sect. 1.1], with $\widetilde{M}$ in place of the lattice denoted M there and $\mathbb{k}$ in place of $\mathbb{C}$. As in Section 1.2 we write $X_{a,b} = X_{\kappa+(a,b)}^\kappa$.

The lifted slab map $\widetilde{\theta}_V$ of (23) is symplectic for Ω_0 , but

$$(33) \quad \widetilde{\theta}_V^* \Omega_\kappa = g_V^{\langle N_V, \kappa \rangle} \Omega_\kappa, \quad (\langle N_V, \kappa \rangle)_V = (-2, -2, 2, -2).$$

The transition carried by a cut is not the bare slab map: by Section 2.4 it has the form $\phi(z^m) = c_\phi(m)z^{H_\phi m}$, where $H_\phi \in \mathrm{SL}_2(\mathbb{Z})$ is its linear part and c_ϕ is a multiplicative character with values in the units generated by the roots; $c_\phi = 1$ on the arc carrying only the change of chart, and ϕ is the identity on the arc carrying no transition. Such a transition preserves Ω_0 , as the lifted slab maps do and $\det H_\phi = 1$, so $\phi^* \Omega_\kappa = c_\phi(\kappa)z^{H_\phi \kappa - \kappa} \Omega_\kappa$, and the transition law of Ω_κ requires a line bundle. Choose a local generator e of the line, a nowhere-vanishing local section, on each side of the cut and identify the generators across it by

$$(34) \quad \phi^* e' = c_\phi(\kappa)^{-1} z^{\kappa - H_\phi \kappa} e.$$

The products $\Omega_\kappa \otimes e$ then glue to a form with values in the resulting line bundle $\mathcal{L}_\kappa$, which we call the *volume line*. The frame at every vertex of the path tree $\mathcal{T}$ thus comes with a local generator e of $\mathcal{L}_\kappa$, the one in which the glued form is $\Omega_\kappa \otimes e$; brackets on sections of $\mathcal{L}_\kappa$ at a vertex are written with respect to it. Along the loop γ_V of Section 2.3, based in S_0 , the transport is $z^m \mapsto g_V^{\langle N_V, m \rangle} z^{H_V m}$ (Section 2.4), and $H_V \kappa - \kappa = -\epsilon_V \langle N_V, \kappa \rangle u_V$ by (21). Since $e_V/d_V = \frac{1}{2}$ at every singular point, the composite of the identifications (34) along γ_V is multiplication by

$$g_V^{-\langle N_V, \kappa \rangle} x_V^{\langle N_V, \kappa \rangle/2} = (f_V x_V^{-e_V})^{-\langle N_V, \kappa \rangle/d_V},$$

which is the normalized slab function $f_V x_V^{-e_V}$ at $\widetilde{A}$ and $\widetilde{C}$, a square root of it at B_0 , and the inverse of a square root of it at B_1 ; its square tends to one where x_V is large, as in Section 2.4. Along γ_{core} the transport is $z^m \mapsto z^{Km}$, and the composite is $z^{\kappa - K\kappa} = z^{(0, -8)}$. For a general path the composite is (48) below. Like the lifted slab maps (Section 2.4), the line $\mathcal{L}_\kappa$ and all

transports below are defined on the Kummer covers and not on the root stacks, namely on the spectra of the rings (23) and, over a finite rooted subtree $T \subset \mathcal{T}$, on $\text{Spec } R_T^{\text{root}}$ of (27).

Proposition 3.2 (Hamiltonian derivations do not glue). *The Lie algebra of π_κ -Hamiltonian derivations of the structure sheaf, which contains the two singleton fields $X_{(2,1)}^\kappa$ and $X_{(1,2)}^\kappa$, is not preserved under conjugation by any lifted slab transition. Moreover, the brackets given by π_κ in the individual charts do not glue to a bracket on the structure sheaf.*

Proof. For a derivation δ let $\text{div } \delta$ be defined by $L_\delta \Omega_\kappa = (\text{div } \delta) \Omega_\kappa$. Every π_κ -Hamiltonian derivation X_h has $\text{div } X_h = 0$, since $\iota_{X_h} \Omega_\kappa = \pm dh$. If θ is an automorphism with $\theta^* \Omega_\kappa = G \Omega_\kappa$, then $L_{\theta^{-1} \delta \theta} = \theta^{-1} L_\delta \theta$ on forms gives

$$\text{div}(\theta^{-1} \delta \theta) = \theta^{-1} \left(\text{div } \delta + \frac{\delta(G)}{G} \right).$$

Let u_V , ϵ_V and $x_V = z^{2\epsilon_V u_V}$ be as in (22), so that $\langle N_V, u_V \rangle = 0$ and the pulled-back slab function f_V is a nonconstant polynomial in x_V . Then $\tilde{\theta}_V$ fixes f_V and g_V . Take $\theta = \tilde{\theta}_V$, $G = g_V^{\langle N_V, \kappa \rangle}$, and $\delta = X_q^\kappa$ with $q \in \{(2, 1), (1, 2)\}$. Since $g_V^{d_V} = f_V$,

$$\text{div}(\tilde{\theta}_V^{-1} X_q^\kappa \tilde{\theta}_V) = \frac{\langle N_V, \kappa \rangle}{d_V} \tilde{\theta}_V^{-1} \left(\frac{X_q^\kappa(f_V)}{f_V} \right), \quad X_q^\kappa(f_V) = 2\epsilon_V \langle q^\perp, u_V \rangle z^{q-\kappa} x_V f'_V(x_V),$$

where f'_V is the derivative of f_V as a polynomial in x_V . The pairings $\langle q^\perp, u_V \rangle$ are

	$\tilde{A}$	B_0	B_1	$\tilde{C}$
$q = (2, 1)$	1	3	-1	-5
$q = (1, 2)$	5	3	7	-1,

all nonzero, and $\langle N_V, \kappa \rangle = \pm 2$. Hence every conjugate of a singleton field by a lifted slab transition has nonzero divergence and is not π_κ -Hamiltonian. The argument uses only that f_V is a nonconstant polynomial in x_V , so it applies at every value of the slab parameter, including $a_\rho = 0$, which corresponds by (6) to the Fermat member $s = 0$ of the pencil (4). If the brackets given by π_κ in the individual charts glued to a bracket on the structure sheaf, the transport along every loop based in S_0 would be a Poisson automorphism for the nondegenerate bivector π_κ , and hence would preserve Ω_κ . The transport along γ_V multiplies Ω_κ by $(f_V x_V^{-e_V})^{\langle N_V, \kappa \rangle / d_V}$, the inverse of the multiplier computed after (34), which is not constant. $\square$

By Proposition 3.2 the bivector π_κ cannot serve as the bracket in every chart. Transporting the Poisson bracket of π_κ from the frame of S_0 along the path tree $\mathcal{T}$ gives a bracket on functions over $\mathcal{T}$, whose bivector in the frame beyond a cut carrying a transition differs from π_κ by a nonconstant factor. Proposition 3.3 instead transports it as a bracket on the sections of $\mathcal{L}_\kappa$, whose bivector part is π_κ in the frame at every vertex of $\mathcal{T}$. Recall that a *Jacobi bracket*, or Jacobi structure, on a line bundle $\mathcal{L}$ is a Lie bracket on local sections of $\mathcal{L}$ that is a first-order bidifferential operator [Kir76, Lic78, VW20] (Section 1.2). In terms of a local generator e of $\mathcal{L}$ it is given by a bivector π and a vector field E through

$$(35) \quad \{fe, he\} = (\pi(df, dh) + fE(h) - hE(f))e.$$

For $E = 0$ the coefficient of e is the Poisson bracket $\pi(df, dh)$.

Proposition 3.3 (Jacobi transport). *The line $\mathcal{L}_\kappa$ over the path tree $\mathcal{T}$ carries a Jacobi bracket, determined by the choice of the frame of S_0 at the root of $\mathcal{T}$, whose expression in that frame is the Poisson bracket of π_κ . In the frame at every vertex of $\mathcal{T}$ its bivector part is π_κ . Every transition carried by a cut, with the identification (34), is a Jacobi isomorphism. The construction is compatible with finite root restriction and equivariant under the deck action of Π .*

Proof. In terms of a local generator e , write a Jacobi bracket as in (35). Under $e' = ae$ the local pair transforms by

$$(36) \quad \pi' = a\pi, \quad E' = aE + \pi^\sharp(da).$$

Start at the root of $\mathcal{T}$, in the frame of S_0 , with $(\pi, E) = (\pi_\kappa, 0)$ and transport the pair across each edge using the transition ϕ of the cut, (34) and (36). Every vertex of the path tree has a unique route from the root, so the construction is unambiguous and compatible with restriction to finite subtrees. Change of generator and pullback preserve the Jacobi identity. The deck action transports the same data, giving deck equivariance. If $\pi = \pi_\kappa$ before a cut, then (34) is $\phi^*e' = G^{-1}e$ with $\phi^*\Omega_\kappa = G\Omega_\kappa$, so by (36) the bivector part with respect to the generator ϕ^*e' is $G^{-1}\pi_\kappa$, which is the pullback under ϕ of the bivector π_κ of the chart beyond the cut. By induction along the route from the root, the bivector part is π_κ in the frame at every vertex of $\mathcal{T}$. $\square$

The bracket depends on the choice of base frame. The cut at B_0 is crossed by γ_B from S_0 and carries $z^m \mapsto g_{B_0}^{\langle N_{B_0}, m \rangle} z^{H_{B_0, 0}m}$, so by (34), with $\langle N_{B_0}, \kappa \rangle = -2$ and $\epsilon_{B_0} = 1$, its multiplier is $a = g_{B_0}^2 x_{B_0}^{-1}$. It is nonconstant and π_κ is nondegenerate, so by (36) with $E = 0$ the pair transported across that cut has a vector field which, expressed in the chart of S_0 , is $\pi_\kappa^\sharp(da) \neq 0$. Starting instead from $(\pi_\kappa, 0)$ in the frame beyond that cut therefore gives a different Jacobi bracket.

The Jacobi bracket of $\mathcal{L}_\kappa$ can equivalently be described by its Poissonization, a homogeneous Poisson structure on $\mathcal{L}_\kappa^\vee \setminus 0$ [VW20]. We use the line-bundle formulation because it displays the multiplier in (34) explicitly.

Definition 3.4 (Jacobi walls and consistency). Fix a finite rooted subtree $T \subset \mathcal{T}$ and work on the Kummer covers over T modulo $\mathfrak{m}^N$, where $\mathfrak{m} = (t_1, t_2)$ is the singleton ideal of Definition 2.6. A *Jacobi wall* is a pair $(\mathfrak{d}, \Theta_\mathfrak{d})$, where $\mathfrak{d}$, the *support* of the wall, is a cooriented rational affine segment, ray, or full chord (a segment both of whose ends lie on the boundary) in the smooth locus $\mathcal{A}^\circ$, and $\Theta_\mathfrak{d}$ is a locally constant unipotent Jacobi automorphism along $\mathfrak{d}$, congruent to the identity modulo $\mathfrak{m}$. The automorphism is written in the frame at a vertex of $\mathcal{T}$, with its local generator of $\mathcal{L}_\kappa$, and is transported between charts by Proposition 3.3; the charts and the transitions between them are placed as in Section 2.4. Unlike [GPS10, GHK15, GS22], we do not require the exponents of $\Theta_\mathfrak{d}$ to be tangent to $\mathfrak{d}$; the placement rule is that of Section 2.2.

Let γ be a path transverse to the walls. A crossing is *positive* if the velocity of γ pairs positively with the coorientation of the wall; a positive crossing contributes $\Theta_\mathfrak{d}$ and a negative crossing contributes $\Theta_\mathfrak{d}^{-1}$. The *path-ordered product* Θ_γ is the composite of these contributions, each transported to the frame at the initial point of γ along γ , with the first crossing acting first; in a product of operators the rightmost factor acts first. A *joint* of a locally finite collection of Jacobi walls is a point of $\mathcal{A}^\circ$ at which two supports of different directions meet or at which a support ends, and the collection is *consistent at a joint* if the path-ordered product around a sufficiently small positively oriented loop about that joint is the identity. The *regions* of the collection are the connected components of the complement in $\mathcal{A}^\circ$ of the supports of those walls whose automorphisms are not the identity; for the wall structure $\mathfrak{D}$ of Theorem 3.7 we speak of the *regions of $\mathfrak{D}$* .

A *wall structure* at a finite level is a locally finite collection of Jacobi walls, and all the notions above are notions at a finite level. A wall structure over $\mathcal{T}$ means a compatible system of wall structures at finite levels as T and N vary, and its path-ordered products are the compatible systems of the path-ordered products at finite levels.

3.2. Scattering of the two singleton Hamiltonian vector fields. With the singleton parameters t_1, t_2 , put

$$(37) \quad E_{a,b} = t_1^a t_2^b X_{a,b} = t_1^a t_2^b X_{\kappa+(a,b)}^\kappa, \quad (a,b) \in \mathbb{N}^2 \setminus \{(0,0)\}.$$

The *shifted exponent* of $E_{a,b}$ is $\kappa + (a,b) = (a+1, b+1)$. The ray $\mathbb{R}_{\geq 0}(a+1, b+1)$ is the *boundary ray* of the GKT generator $X_{a,b}$ (Section 1), and we call it the boundary ray of $E_{a,b}$; its slope is $(b+1)/(a+1)$. By (32),

$$(38) \quad [E_{a,b}, E_{c,d}] = -\det(\kappa + (a,b), \kappa + (c,d)) E_{a+c, b+d}.$$

The two singleton Hamiltonian vector fields, with GKT's normalization, are

$$(39) \quad A = -\frac{1}{2}E_{1,0}, \quad B = -\frac{1}{2}E_{0,1}.$$

The sans-serif letters distinguish these two fields from the singular points A and B . Since the shifted exponents of A and B are $(2,1)$ and $(1,2)$,

$$(40) \quad [A, B] = -\frac{3}{4}E_{1,1}.$$

Let $\widehat{\mathfrak{g}}_{\text{sing}}$ be the product of the lines spanned by the $E_{a,b}$, completed in the total degree $a+b$; it is a Lie subalgebra of the $\mathfrak{m}$ -adic completion of $\mathfrak{g}_\kappa \otimes_{\mathbb{K}} \mathbb{K}[t_1, t_2]$. For a ray $\mathfrak{r} \subset \mathbb{R}_{\geq 0}^2$ we let $\widehat{\mathfrak{g}}_{\mathfrak{r}}$ be the closed span of the $E_{a,b}$ whose shifted exponent lies on $\mathfrak{r}$. By (38) each $\widehat{\mathfrak{g}}_{\mathfrak{r}}$ is abelian, and we call the abelian group $\exp(\widehat{\mathfrak{g}}_{\mathfrak{r}})$ the *ray group* of $\mathfrak{r}$. Rays are ordered by their slopes, and $\mathfrak{r}_{1,1}, \mathfrak{r}_{1,2}$ denote the rays of $(2,1)$ and $(1,2)$, the *incoming rays*.

Proposition 3.5 (Kontsevich–Soibelman factorization of the singleton commutator). *Let operators act on functions, the rightmost factor acting first.*

(1) *There are unique elements $C_{\mathfrak{r}} \in \widehat{\mathfrak{g}}_{\mathfrak{r}}$ such that*

$$(41) \quad e^A e^{-B} e^{-A} e^B = \overrightarrow{\prod}_{\mathfrak{r}} \exp(C_{\mathfrak{r}}),$$

the product taken over rays with slopes increasing from left to right. Modulo $\mathfrak{m}^N$ only finitely many $C_{\mathfrak{r}}$ are nonzero, and the factorization is compatible with reduction of N .

(2) *Write $C_{\mathfrak{r}} = \sum c_{a,b} E_{a,b}$. Then $c_{a,b} = 0$ unless $a, b \geq 1$ and $(a+1, b+1)$ lies in the cone spanned by $(2,1)$ and $(1,2)$. Moreover $c_{b,a} = c_{a,b}$, $c_{1,1} = \frac{3}{4}$, and the coefficients of total degree at most 10 are those of Table 1.*

(3) *Equivalently,*

$$(42) \quad e^A e^B = e^{B - C_{\mathfrak{r}_{1,2}}} \left(\overleftarrow{\prod}_{\frac{1}{2} < \text{slope}(\mathfrak{r}) < 2} e^{-C_{\mathfrak{r}}} \right) e^{A - C_{\mathfrak{r}_{2,1}}},$$

where the middle product has slopes decreasing from left to right. This is the unique factorization of $e^A e^B$ into ray groups in which the factor of least slope acts first.

Proof. The bracket (38) is additive in (a,b) . Hence the elements of total degree at least d form an ideal $\widehat{\mathfrak{g}}^{\geq d}$, the quotient $\widehat{\mathfrak{g}}^{\geq d}/\widehat{\mathfrak{g}}^{\geq d+1}$ is central in $\widehat{\mathfrak{g}}_{\text{sing}}/\widehat{\mathfrak{g}}^{\geq d+1}$, and modulo $\mathfrak{m}^N$ only degrees below N survive. Suppose that a product P of ray factors satisfies $e^A e^{-B} e^{-A} e^B P^{-1} \in \exp(\widehat{\mathfrak{g}}^{\geq d})$, the first four factors forming the left side of (41). The degree- d part of the logarithm of this element decomposes uniquely into ray components; since this part is central modulo degree $d+1$, adding each component to the logarithm of the corresponding factor removes the discrepancy in degree d , and no other choice does. Induction on d gives existence and uniqueness in (1). This is the argument of the Kontsevich–Soibelman lemma [GPS10, Thm. 1.4]; it uses only the

TABLE 1. The coefficients $c_{a,b}$ of Proposition 3.5 with $a \geq b$ and $a + b \leq 10$. The entries are obtained by solving the degree-by-degree recursion in the proof of Proposition 3.5 through total degree ten. The coefficients with $a < b$ follow from $c_{b,a} = c_{a,b}$; every other coefficient of total degree at most 10 vanishes.

$a + b$	coefficients
2	$c_{1,1} = \frac{3}{4}$
3	$c_{2,1} = \frac{3}{8}$
4	$c_{3,1} = \frac{1}{16}, c_{2,2} = \frac{3}{8}$
5	$c_{3,2} = \frac{3}{8}$
6	$c_{4,2} = \frac{3}{32}, c_{3,3} = \frac{5}{16}$
7	$c_{5,2} = \frac{3}{128}, c_{4,3} = \frac{7}{16}$
8	$c_{5,3} = \frac{3}{16}, c_{4,4} = \frac{21}{64}$
9	$c_{6,3} = \frac{7}{128}, c_{5,4} = \frac{9}{16}$
10	$c_{7,3} = \frac{13}{1024}, c_{6,4} = \frac{19}{64}, c_{5,5} = \frac{63}{160}$

grading and not convexity, which fails here: the shifted exponent $q + r - \kappa$ of a bracket need not lie between q and r .

For the support statement, let $\mathcal{C}$ be the cone spanned by $(2, 1)$ and $(1, 2)$. If $q, r \in \mathcal{C} \cap \mathbb{Z}^2$ are not parallel, then

$$2(q_2 + r_2 - 1) - (q_1 + r_1 - 1) = (2q_2 - q_1) + (2r_2 - r_1) - 1 \geq 0,$$

because the two nonnegative integers $2q_2 - q_1$ and $2r_2 - r_1$ vanish together only when q and r both lie on the ray of $(2, 1)$; the other inequality is symmetric. Parallel shifted exponents have vanishing bracket. Hence the closed span of the $E_{a,b}$ with shifted exponent in $\mathcal{C}$ is a Lie subalgebra. It contains A and B , and the inductive construction stays in it. Its only generators with $a = 0$ or $b = 0$ are $E_{1,0}$ and $E_{0,1}$, of degree one, and the left side of (41) is the identity modulo degree two. This proves the support statement. Modulo degree three the left side equals $\exp(-[A, B]) = \exp(\frac{3}{4}E_{1,1})$ by (40).

For the symmetry, let $\tilde{\sigma}$ be the linear map with $\tilde{\sigma}(E_{a,b}) = -E_{b,a}$; it is induced by the exchange σ of the coordinates (Section 2.2) together with the exchange of t_1 and t_2 , because $(\sigma q)^\perp = -\sigma(q^\perp)$. The map $\tilde{\sigma}$ preserves (38), since exchanging the coordinates of both vectors changes the sign of the determinant. It sends A to $-B$ and B to $-A$, hence the left side of (41) to $e^{-B}e^Ae^Be^{-A}$, which is its inverse, and it reverses the order of slopes. Applying $\tilde{\sigma}$ to (41), and inverting (41), gives two factorizations of this inverse with slopes decreasing from left to right. By uniqueness, $\tilde{\sigma}(C_{\mathbf{r}}) = -C_{\mathbf{r}'}$, where $\mathbf{r}'$ is the reflection of $\mathbf{r}$; comparing coefficients gives $c_{b,a} = c_{a,b}$.

For (3), invert (41), multiply by e^B on the left and by e^A on the right, and use that $C_{\mathbf{r}_{1,2}}$ commutes with B and $C_{\mathbf{r}_{2,1}}$ with A . The factorization with the least slope acting first is unique by the same induction. The entries of Table 1 are computed by solving the degree-by-degree recursion in the proof of (1) through total degree ten. $\square$

All coefficients in Table 1 are positive, and every bidegree of total degree at most 10 allowed by (2) occurs. The factors $\exp(C_{\mathbf{r}_{2,1}})$ and $\exp(C_{\mathbf{r}_{1,2}})$ on the incoming rays are nontrivial from total degree four on: their first coefficients are $c_{3,1} = c_{1,3} = \frac{1}{16}$.

3.3. Wall supports and consistency at the joints. We work in the chart of Δ drawn in Figure 2(a), with affine coordinates (s_1, s_2) in which $\Delta = \text{conv}\{(-1, -1), (-1, 1), (3, -1)\}$ and the central vertex O of the star subdivision is the origin. For $V \in \{A, B, C\}$ let p_V be the vertex of Δ on the star edge through V , so that $p_A = (-1, -1)$, $p_B = (-1, 1)$, $p_C = (3, -1)$. In the normalization of Appendix A, V is the midpoint of the edge from O to p_V , so that $A = (-\frac{1}{2}, -\frac{1}{2})$, $B = (-\frac{1}{2}, \frac{1}{2})$ and $C = (\frac{3}{2}, -\frac{1}{2})$. Points and directions in this chart are given by their coordinates (s_1, s_2) , and linear forms by their coefficients with respect to s_1, s_2 ; shifted exponents, which are elements of $\widehat{M}$, are written in F -coordinates.

Transitions and frames. The transitions at the singular points, including the line transitions (34), are placed as in Section 2.4, on the cuts over the outer half-edges c_V from V to p_V , and each sheet $S_i \ni O_i$ carries one frame. By Proposition 3.6 below, no support meets an arc carrying a transition and the supports through O_i lie in S_i ; hence the factor on every support through O_i is written in the frame of S_i along the whole support. The supports through O_1 are the deck images of those through O_0 , so the placement rule below holds for them in the chart of S_1 pulled back from Δ ; the frame of S_1 differs from the identification given by that chart by the affine monodromy about A . Their cyclic order about O_1 and their factors are those at O_0 , which is all that consistency at O_1 uses.

Supports. The singleton shifted exponents $(2, 1)$ and $(1, 2)$ have the support directions, in the chart (s_1, s_2) ,

$$(43) \quad L(2, 1) = (-4, 3), \quad L(1, 2) = (-2, 3).$$

Each is placed on the full chord through O in its direction. For $q \in \{(2, 1), (1, 2)\}$ the *positive* and *negative half-chords* of Lq are the parts of this chord in $\mathbb{R}_{\geq 0}Lq$ and in $\mathbb{R}_{\leq 0}Lq$. Full chords are forced: if a degree-one wall ended at an interior point, a small loop around that endpoint would cross the wall once, while every correction begins in degree two. If $q = c_1(2, 1) + c_2(1, 2)$ with $c_1, c_2 \geq 0$, then, in the chart (s_1, s_2) ,

$$(44) \quad Lq = (-4c_1 - 2c_2, 3c_1 + 3c_2).$$

For $c_1 > c_2$ the ray $\mathbb{R}_{\geq 0}Lq$ leaves Δ through the facet $s_1 = -1$, and for $c_2 > c_1$ through the facet $s_1 + 2s_2 = 1$; these are the two facets meeting at p_B . For $c_1 = c_2$ it runs along the inner half-edge from O towards p_B and reaches B . For each ray $\mathfrak{r}$ with $\frac{1}{2} < \text{slope}(\mathfrak{r}) < 2$, the *outgoing wall* of $\mathfrak{r}$ is the ray $\mathbb{R}_{\geq 0}Lq$, $q \in \mathfrak{r}$; for the *first mixed ray* $\mathfrak{r}_{1,1}$, the ray of $(1, 1)$, it is the segment from O to B , the *diagonal wall*. The factors of the incoming rays $\mathfrak{r}_{2,1}$ and $\mathfrak{r}_{1,2}$ are placed on the positive half-chords of $L(2, 1)$ and $L(1, 2)$.

Placement. The wall of the Hamiltonian vector field with shifted exponent q is placed along Lq , the image of its exponent under the gradient map of an auxiliary quadratic function, rather than along the annihilator of the exponent, which is what the orthogonality of dual cells in the discrete Legendre transform would suggest. Section 2.2 explains the choice: among linear rules, the exchange of x and y restricts the placement to a two-parameter family, of which the cofactor is one member. Consistency at the joints uses only the cyclic order of these directions about O . Since L reverses orientation ($\det L = -2$ in these coordinates; F reverses orientation and $d\psi$ preserves it) and maps the cone of shifted exponents into the half-plane $s_2 > 0$, a path passing above O from left to right meets the supports in increasing slope of their boundary rays.

Coorientations. We coorient every wall so that crossing it in the direction of increasing slope of the boundary rays is the positive crossing. Thus the chord and the outgoing wall in the direction Lq , $q = c_1(2, 1) + c_2(1, 2)$, are cooriented, each chord along its whole length, by the linear form

$$(3c_1 + 3c_2)s_1 + (4c_1 + 2c_2)s_2,$$

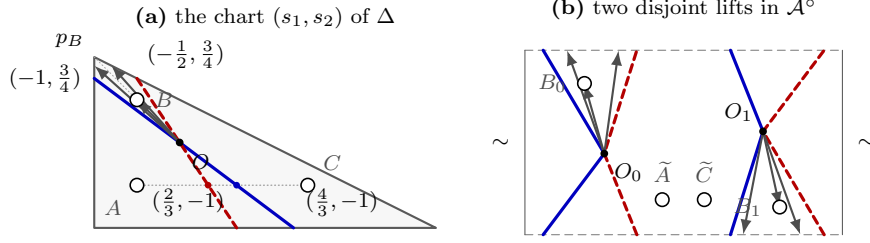


FIGURE 2. Wall supports along the cofactor directions. The two full chords carry the singleton factors. The off-diagonal outgoing walls leave through the boundary; the diagonal support ends at a lift of B . The dotted segment from B to p_B is the outer half-edge carrying the transition at B ; those at A and C run to p_A and p_C and are not drawn. Crosswise gluing along the branch segment produces two copies of the joint without adding intersections. Panel (b) is schematic: the deck involution exchanges the two boundary circles, so the outgoing walls through O_1 end on the circle on which the negative half-chords through O_0 end, and conversely.

which vanishes on Lq . We call its coefficient vector $(3c_1 + 3c_2, 4c_1 + 2c_2)$ the *normal* of the wall; by (9) it equals $(Lq)^\perp = \iota(q^\perp)$, the image under ι of the direction $q^\perp$ of the derivation X_q^κ . In the chart (s_1, s_2) , the normals of the two chords are $(3, 4)$ and $(3, 2)$, and that of the diagonal wall is proportional to $(1, 1)$. Every normal has positive first coordinate, so a path moving in the direction of increasing s_1 crosses every wall through O positively. With C_τ the elements of Proposition 3.5, whose coefficients $c_{a,b}$ are listed in Table 1, the walls carry the following factors.

- (1) The negative half-chords of $L(2, 1)$ and $L(1, 2)$ carry e^A and e^B .
- (2) The positive half-chords carry $e^A \exp(-C_{\tau_{2,1}}) = e^{A-C_{\tau_{2,1}}}$ and $e^B \exp(-C_{\tau_{1,2}}) = e^{B-C_{\tau_{1,2}}}$.
- (3) The outgoing wall of τ carries $\exp(-C_\tau)$. In particular the diagonal wall, the outgoing wall of the first mixed ray $\tau_{1,1}$, carries $\exp(-C_{\tau_{1,1}})$, which is $\exp(-\frac{3}{4}E_{1,1})$ modulo $\mathfrak{m}^4$; it is the first outgoing wall, the only one whose factor is nontrivial modulo $\mathfrak{m}^3$.

These are the coorientations and factors of the table in Section 7.

Proposition 3.6. *Modulo $\mathfrak{m}^N$, the supports placed above along the cofactor directions, lifted to $\mathcal{A}^\circ$, form a locally finite collection of properly embedded arcs in $\mathcal{A}^\circ$. Its only joints are the two lifts O_0, O_1 of O . No support meets an arc carrying a transition, the supports through O_i lie in the sheet S_i , and the diagonal supports end at B_0 and B_1 . The lifts of the supports to the universal cover of $\mathcal{A}^\circ$ meet only at lifts of O_0 and O_1 , even where their images under the developing map intersect in an affine plane.*

Proof. Modulo $\mathfrak{m}^N$ there are finitely many deformation bidegrees. Since the factors $C_{\tau_{2,1}}, C_{\tau_{1,2}}$ on the incoming rays (from total degree four) are placed on the positive half-chords, the compact polygon contains a finite embedded chord-and-ray skeleton whose members meet only at O . In the chart (s_1, s_2) , the positive half-chords leave Δ at $(-1, \frac{3}{4})$ and $(-\frac{1}{2}, \frac{3}{4})$. The negative half-chords cross the branch segment $s_2 = -\frac{1}{2}$ at $(\frac{2}{3}, -\frac{1}{2})$ and $(\frac{1}{3}, -\frac{1}{2})$, away from O , and leave Δ at $(\frac{4}{3}, -1)$ and $(\frac{2}{3}, -1)$. The off-diagonal rays lie in $s_2 > 0$ and leave Δ through the two facets at p_B . The segments c_A and c_C lie in $s_2 < 0$, with $s_1 \leq -\frac{1}{2}$ and $s_1 \geq \frac{3}{2}$ respectively, and c_B lies on the line $s_1 + s_2 = 0$ beyond B , which the other supports meet only at O . Hence no support meets $c_A \cup c_B \cup c_C$, and the diagonal support ends at B . Crosswise gluing lifts each chord

smoothly and produces two copies of the skeleton, one in each sheet, without creating another intersection; the arcs carrying the transitions lie over $c_A \cup c_B \cup c_C$, so no support meets them. Deleting the endpoints on the boundary and at B_0, B_1 makes each support relatively closed. A compact subset of $\mathcal{A}^\circ$ stays a positive distance from these ends and therefore meets finitely many supports. Finally, each support is an embedded arc in $\mathcal{A}^\circ$, so its lifts to the universal cover are pairwise disjoint, and lifts of two different supports can meet only over a point at which the supports meet, that is, over O_0 or O_1 . $\square$

Place the factors listed under *Coorientations* on the supports through O_0 , in the frame of S_0 , and transport them by Proposition 3.3: at a vertex of the path tree over the sheet S_i , the supports through O_i carry the factors transported along the path of that vertex.

Theorem 3.7 (Consistency at the joints). *The transported factors on the supports of Proposition 3.6 define a deck-equivariant wall structure $\mathfrak{D}$ on the Kummer covers over the path tree $\mathcal{T}$. It is consistent at O_0 and O_1 . For every N , the path-ordered product of a compact transverse path modulo $\mathfrak{m}^N$ is defined on the Kummer covers over one finite rooted subtree $T \subset \mathcal{T}$ and depends only on the homotopy class of the path in $\mathcal{A}^\circ$ relative to its endpoints.*

Proof. By Proposition 3.6 and the placement of the transitions, a small disk about O_0 meets only the chords and outgoing walls through O_0 and lies in one chart and one frame. Let γ_+ and γ_- run from $(s_1, s_2) = (-\varepsilon, 0)$ to $(s_1, s_2) = (\varepsilon, 0)$ along $s_2 = \varepsilon/10$ and $s_2 = -\varepsilon/10$, joined to their endpoints by vertical segments; these are the paths of Section 7. By (44) a support of direction Lq meets $s_2 = \varepsilon/10$ at $s_1 = -\frac{\varepsilon}{10} \frac{4c_1+2c_2}{3c_1+3c_2}$, which increases with the slope of q . Hence γ_+ meets the positive $(2, 1)$ -half-chord, then the outgoing walls in increasing slope of their boundary rays, then the positive $(1, 2)$ -half-chord. The path γ_- meets only the two negative half-chords, first that of $(1, 2)$ and then that of $(2, 1)$. Both paths move in the direction of increasing s_1 where they meet a wall, so they cross every wall positively. The vertical segments meet no support. With the factors listed under *Coorientations*, and the first crossing acting first,

$$\Theta_{\gamma_-} = e^A e^B, \quad \Theta_{\gamma_+} = e^{B-C_{\tau_{1,2}}} \left(\overleftarrow{\prod_{\frac{1}{2} < \text{slope}(\tau) < 2}} e^{-C_\tau} \right) e^{A-C_{\tau_{2,1}}}.$$

Since $C_{\tau_{1,2}}$ commutes with B and $C_{\tau_{2,1}}$ with A ,

$$\Theta_{\gamma_+} = e^B \left(\overrightarrow{\prod_{\tau} \exp(C_\tau)} \right)^{-1} e^A = e^B (e^{-B} e^A e^B e^{-A}) e^A = e^A e^B = \Theta_{\gamma_-}$$

by (41); equivalently, $\Theta_{\gamma_+} = \Theta_{\gamma_-}$ is (42). The loop formed by γ_+ and the reverse of γ_- encircles O_0 once, so the path-ordered product around a small loop about O_0 is the identity modulo $\mathfrak{m}^N$ for every N .

The deck involution of the parity cover p preserves orientation and is the identity in the charts pulled back from Δ . It carries O_0 and the supports through O_0 onto O_1 and the supports through O_1 , together with their crossing order and coorientations. The factors on the supports through O_1 are the transports of those through O_0 by one Jacobi isomorphism, so the product around O_1 is the transport of the identity. The same argument applies at every vertex of the path tree. Transport along the path tree is by Jacobi isomorphisms (Proposition 3.3), which gives deck equivariance.

A compact path crosses finitely many walls modulo $\mathfrak{m}^N$. Theorem 2.7 places all of its root and wall transitions over one finite rooted subtree. A homotopy in $\mathcal{A}^\circ$ relative to the endpoints can be decomposed into moves across a single wall, which do not change Θ_γ because the factors are locally constant along the walls; moves across O_0 or O_1 , which do not change it by consistency;

and moves across an arc carrying a transition, which do not change it because transport of frames is invariant under homotopy in $\mathcal{A}^\circ$ and no wall meets such an arc. This gives homotopy invariance. $\square$

The consistency of $\mathfrak{D}$ is the local Kontsevich–Soibelman condition for walls in one chart at the two joints, together with homotopy invariance in $\mathcal{A}^\circ$; nothing is imposed around a singular point or around the core, where the prescribed affine, root and line holonomy is not trivialized. The diagonal walls end at B_0 and B_1 : the terms $c_{a,a}E_{a,a}$ of $C_{\tau_{1,1}}$ have exponents (a, a) , which the monodromy at B_0 does not fix, since $H_{B,0}(1, 1) = (3, -1)$, and likewise at B_1 after transport. A small loop about B_i crosses the arc carrying the transition at B_i and the diagonal wall once each, so its transport is the prescribed transition at B_i composed with the diagonal wall factor $\exp(-C_{\tau_{1,1}})$ or its inverse.

Definition 3.8 (Admissible paths). An *admissible path* $\mathbf{p}$ is a triple consisting of a reduced edge path in the graph $\mathfrak{E}$ beginning at the sheet S_0 (a vertex of the path tree $\mathcal{T}$), a finite rooted subtree $T \subset \mathcal{T}$ and a deformation order N , such that all transition maps along the path, modulo $\mathfrak{m}^N$, are defined over $R_T^{\text{root}} \otimes_{\mathbb{k}} R/\mathfrak{m}^N$, that is, over U_T on the Kummer cover P_T of the finite root-stack presentation of T (Theorem 2.7).

3.4. Hyperbolic holonomy and finite covers. The matrix K in (2) satisfies

$$(45) \quad \chi_K(\theta) = \det(\theta \text{Id} - K) = \theta^2 - 18\theta + 1, \quad \theta_{\pm} = 9 \pm 4\sqrt{5}.$$

For $q_1 = (2, 1)$ and $q_2 = (1, 2)$, put $q_{i,n} = K^n q_i$. Then

$$(46) \quad q_{i,n+2} = 18q_{i,n+1} - q_{i,n}, \quad \det(q_{1,n}, q_{2,n}) = 3.$$

If a path γ has linear holonomy H_γ , the transport of the coefficient system along γ , by slab transitions and chart changes and without wall factors, has in a developed chart the form

$$(47) \quad \Psi_\gamma(z^m) = c_\gamma(m) z^{H_\gamma m},$$

where c_γ is a multiplicative character with values in the group of units generated by the transported roots. Each slab transition preserves Ω_0 , and so does each chart change, whose linear part has determinant one. Hence $\Psi_\gamma(\Omega_\kappa) = c_\gamma(\kappa) z^{H_\gamma \kappa - \kappa} \Omega_\kappa$, and the compatible transport of $\mathcal{L}_\kappa$, the composite of the identifications (34) along γ , has multiplier

$$(48) \quad c_\gamma(\kappa)^{-1} z^{\kappa - H_\gamma \kappa}.$$

Thus the annular holonomy is a semidirect action on exponents, roots, and the volume line.

Theorem 3.9 (Hyperbolic core holonomy). *Let $\gamma_{\text{core}} = \gamma_C \gamma_A^{-1}$ be the core loop of Section 2.3, which separates $\{\tilde{A}, B_0\}$ from $\{B_1, \tilde{C}\}$ and has linear holonomy K on $\tilde{M}$.*

- (1) *K acts trivially on the fractional-exponent group $\tilde{M}/M$ and hyperbolically on $\tilde{M}$; its eigenvalues (45) are not roots of unity.*
- (2) *No nonzero $m \in \tilde{M}$ has finite K -orbit. Consequently the local system $\tilde{M}$ on $\mathcal{A}^\circ$ has no nonzero global section, and neither does its pullback to any connected finite cover of $\mathcal{A}^\circ$.*
- (3) *Let $\Psi = \Psi_{\gamma_{\text{core}}}$. For every nonzero $m \in \tilde{M}$ the elements $\Psi^n(z^m)$, $n \in \mathbb{Z}$, are units with pairwise distinct exponents $K^n m$; hence the Ψ -orbit of z^m is infinite, and the sum of its elements does not converge $\mathfrak{m}$ -adically.*

Proof. Appendix A shows that K acts trivially on $\tilde{M}/M$ and has characteristic polynomial χ_K . If $K^r m = m$ with $r \neq 0$, then $m = 0$, because $\det(K^r - \text{Id}) = 2 - \theta_+^r - \theta_-^r$ is nonzero. A connected cover of finite degree d contains a closed lift of γ_{core}^r for some $1 \leq r \leq d$, and a global section of the pulled-back local system is fixed by K^r . This proves (2). For (3), the transport

along γ_{core} carries no slab factor (Section 2.4), so $c_{\gamma_{\text{core}}} = 1$ in (47) and $\Psi^n(z^m) = z^{K^n m}$ is a unit, hence not in $\mathfrak{m}$. By (2) the exponents $K^n m$ are pairwise distinct for $m \neq 0$, so the orbit is infinite. An infinite sum converges $\mathfrak{m}$ -adically only if all but finitely many of its terms lie in $\mathfrak{m}$. $\square$

Exponents therefore cannot serve as global labels: a labelling of sections by one fixed copy of $\widetilde{M}$, compatible with parallel transport of exponents, would be a trivialization of the local system $\widetilde{M}$ over $\mathcal{A}^\circ$, and by Theorem 3.9(2) there is none, even after passing to a finite cover. Nor can single-valued elements be obtained from monomials by averaging over the core: by Theorem 3.9(3) the Ψ -orbit of every nonconstant monomial is infinite, and its sum diverges. For these reasons we do not define broken lines or theta functions on $\mathcal{A}^\circ$.

Two finite covers retain the finite grading data. The first is the fractional-exponent cover p_{fr} of Section 2.4, of degree two, with deck involution δ_{deck} . For the second, put

$$(49) \quad 4\widetilde{M} \subset M \subset \widetilde{M}, \quad Q = \widetilde{M}/4\widetilde{M} \simeq (\mathbb{Z}/4)^2;$$

here $4\widetilde{M}$ has index two in M . We call Q the *twist group*: the class of $F(a, b)$ in Q is the pair (a, b) modulo 4, so that under the dictionary Q is the group of GKT twists (a, b) modulo 4 (Section 5.1). The monodromy about each singular point acts on Q by the order-four matrix

$$(50) \quad P_{\text{res}} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \pmod{4},$$

the holonomy K along γ_{core} acts by $K \equiv P_{\text{res}}^2$, and the boundary holonomies act trivially. The *twist cover* is the connected cover of $\mathcal{A}^\circ$ on which the local system Q becomes constant, that is, the cover defined by the kernel of the monodromy action on Q . It has degree four, the order of P_{res} , and since $4\widetilde{M} \subset M$ it factors through p_{fr} . The transported Kummer roots of the annular coefficient system (Definition 2.6), the volume line and the wall structure pull back to both covers. The hyperbolic exponent holonomy persists on both: the first closed lift of γ_{core} to the twist cover acts by K^2 .

Since the ground field has characteristic zero, the idempotents

$$e_+ = \frac{1 + \delta_{\text{deck}}}{2}, \quad e_- = \frac{1 - \delta_{\text{deck}}}{2}$$

split the pushforward along p_{fr} of a deck-linearized object on the fractional-exponent cover into its δ_{deck} -invariant and δ_{deck} -anti-invariant summands; Remark 6.4 applies this splitting to the transported Jacobian algebras.

4. THE ANNULAR ENDS AND THE FINITE ROOT-STACK PRESENTATIONS

Except for the two diagonal segments, which end at B_0 and B_1 , the walls of the wall structure $\mathfrak{D}$ of Theorem 3.7 reach the boundary circles of $\overline{\mathcal{A}}$ transversely. On the affine side each boundary circle has an outward integral-affine continuation, a flare, on which the continued supports separate (Propositions 4.2 and 4.3), and Remark 4.4 explains why the natural gluing of toric charts over the flares does not close the ends. The finite root-stack presentations of Section 2 are quotients of Kummer covers of a torus by finite groups of roots of unity; the lifted transitions are defined on these covers away from the root divisors, and not on the quotients (Section 4.5). The covers are singular in general, and both the presentations and their Kummer covers are dominated by log smooth objects with regular closed fiber; these two facts are proved in Appendix C. Throughout, $\partial_i \overline{\mathcal{A}}$ ($i = 0, 1$) denotes the boundary circle whose holonomy has the normal form of width w_i in (20), so that $(w_0, w_1) = (4, 16)$.

4.1. Truncated coefficient rings.

Definition 4.1. A *labelled truncation* $\mathcal{W} = (I_{\mathcal{W}}, \tau, \mathcal{P})$ consists of a finite set $I_{\mathcal{W}}$ of labels, each of descendent degree one; a map $\tau : I_{\mathcal{W}} \rightarrow \{\tau_1, \tau_2\}$, where $\tau_1 = (1, 0)$ and $\tau_2 = (0, 1)$, assigning to each label its twist; and a finite set $\mathcal{P}$ of monomials in variables t_i , $i \in I_{\mathcal{W}}$, which contains 1 and is closed under taking divisors. For $N \geq 1$, the *truncated coefficient ring* of $\mathcal{W}$ at order N is

$$(51) \quad R_{\mathcal{W}, N} = \mathbb{k}[t_i : i \in I_{\mathcal{W}}] / (\mathfrak{m}_{\mathcal{W}}^N + \langle m : m \notin \mathcal{P} \rangle), \quad \mathfrak{m}_{\mathcal{W}} = (t_i : i \in I_{\mathcal{W}}).$$

Since $\mathcal{P}$ is closed under divisors, the monomials in $\mathcal{P}$ of degree less than N form a basis of $R_{\mathcal{W}, N}$, which is therefore a finite local Artin $\mathbb{k}$ -algebra with maximal ideal $\mathfrak{m}_{\mathcal{W}}$ and residue field $\mathbb{k}$. The labels enter the wall structure $\mathfrak{D}$ through the $\mathbb{k}$ -algebra homomorphism

$$(52) \quad v_{\mathcal{W}} : \mathbb{k}[t_1, t_2] / \mathfrak{m}^N \longrightarrow R_{\mathcal{W}, N}, \quad t_j \longmapsto \sum_{\tau(i)=\tau_j} t_i \quad (j = 1, 2),$$

where $\mathfrak{m} = (t_1, t_2)$ is the singleton ideal. It is well defined because $v_{\mathcal{W}}(\mathfrak{m}) \subset \mathfrak{m}_{\mathcal{W}}$, so that $v_{\mathcal{W}}(\mathfrak{m}^N) \subset \mathfrak{m}_{\mathcal{W}}^N$. The walls of $\mathfrak{D}$, which are defined modulo $\mathfrak{m}^N$, are base-changed to $R_{\mathcal{W}, N}$ along $v_{\mathcal{W}}$. Let $\mathcal{W}' = (I_{\mathcal{W}'}, \tau', \mathcal{P}')$ contain $\mathcal{W}$, in the sense that $I_{\mathcal{W}} \subset I_{\mathcal{W}'}$, $\tau'|_{I_{\mathcal{W}}} = \tau$ and $\mathcal{P} \subset \mathcal{P}'$, and let $N \leq N'$. The $\mathbb{k}$ -algebra surjection $R_{\mathcal{W}', N'} \rightarrow R_{\mathcal{W}, N}$ that fixes t_i for $i \in I_{\mathcal{W}}$ and sends t_i to 0 for $i \notin I_{\mathcal{W}}$ is well defined, because a monomial outside $\mathcal{P}'$ either contains such a t_i or lies outside $\mathcal{P}$. It is compatible with $v_{\mathcal{W}'}$, $v_{\mathcal{W}}$ and the reduction $\mathbb{k}[t_1, t_2] / \mathfrak{m}^{N'} \rightarrow \mathbb{k}[t_1, t_2] / \mathfrak{m}^N$. For a finite rooted subtree $T \subset \mathcal{T}$, the finite level (Section 2.4) of T with coefficient ring $R_{\mathcal{W}, N}$ is

$$\mathfrak{X}_{T, \mathcal{W}, N} = \mathfrak{X}_T \times_{\text{Spec } \mathbb{k}} \text{Spec } R_{\mathcal{W}, N}.$$

It is the base change along $v_{\mathcal{W}}$ of the finite root-stack presentation $\mathfrak{X}_T \times_{\text{Spec } \mathbb{k}} \text{Spec } \mathbb{k}[t_1, t_2] / \mathfrak{m}^N$ of Theorem 2.7. Its Kummer cover, on which the lifted transitions are defined, is described in Section 4.5.

4.2. Exit points on the boundary circles. Fix $i \in \{0, 1\}$. By Appendix A, a collar of $\partial_i \overline{\mathcal{A}}$ develops, in affine coordinates (x, y) whose x -axis is the invariant line of the boundary holonomy, onto a strip $\{0 \leq y < \epsilon\}$ with $\partial_i \overline{\mathcal{A}}$ on the line $y = 0$. We write these tangent–normal coordinates in sans-serif type to distinguish them from the Landau–Ginzburg variables x, y . The affine holonomy of the boundary loop is

$$(53) \quad \gamma_i(x, y) = (x + \ell_i - w_i y, y), \quad \ell_i = w_i h_i,$$

with (ℓ_i, w_i, h_i) as in (99). Its linear part is the normal form in (20), and it fixes the common apex $(x, y) = (\phi_i, h_i)$ of the developed facets, which is the developed image of both joints O_0 and O_1 . A wall support meeting $\partial_i \overline{\mathcal{A}}$ develops to a segment of a ray from the apex. In the outward normal coordinate $r = -y$ it has the equation

$$(54) \quad x = \phi_i - \xi(r + h_i),$$

and its slope ξ lies in one of the bounded intervals of (100). In particular every support meeting $\partial_i \overline{\mathcal{A}}$ meets it transversely, in one point, which we call the *exit point* of the support.

The consistency criterion of Gross, Hacking and Siebert at a boundary joint [GHS22, Prop. 3.13] does not apply at an exit point. It is formulated for walls whose functions have exponents tangent to the wall [GHS22, Def. 2.11], through the convexity condition at a boundary joint [GHS22, Def. 3.12], defined by the slopes [GHS22, (3.7)], whereas here the wall supports are transverse to their exponents.

A boundary circle cannot be removed by an affine double either. A double of $\overline{\mathcal{A}}$ along $\partial_i \overline{\mathcal{A}}$ would carry an integral-affine involution fixing $\partial_i \overline{\mathcal{A}}$ pointwise. The involution maps the

boundary loop to itself, so its linear part along the circle commutes with the linear holonomy of the circle. In the tangent–normal basis of (53) these are

$$U = \begin{pmatrix} 1 & -w_i \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & c \\ 0 & -1 \end{pmatrix},$$

and $SU - US = \begin{pmatrix} 0 & -2w_i \\ 0 & 0 \end{pmatrix}$ is nonzero, since $w_i \in \{4, 16\}$.

4.3. Affine continuation beyond the boundary circles.

Proposition 4.2 (Outward affine flares). *Let $i \in \{0, 1\}$. In the coordinates (x, r) of (54), in which the tangent lattice $\widetilde{M}^\vee$ is $2\mathbb{Z}^2$ (Appendix A), let $\overline{\mathcal{F}}_i$ be the quotient of $\{(x, r) \in \mathbb{R}^2 : r \geq 0\}$ by the transformation*

$$(55) \quad (x, r) \mapsto (x + \ell_i + w_i r, r).$$

- (1) *The group $\mathbb{Z}$ generated by (55) acts freely and properly discontinuously, and $\overline{\mathcal{F}}_i$ is an integral-affine cylinder with tangent lattice $\widetilde{M}^\vee$. For $r_0 \geq 0$ the circle $\{r = r_0\}$ has affine length $\frac{1}{2}(\ell_i + w_i r_0)$, and r descends to a proper integral affine function $\overline{\mathcal{F}}_i \rightarrow [0, \infty)$.*
- (2) *Identify $\{r = 0\}$ with $\partial_i \overline{\mathcal{A}}$ by $y = -r$. The union $\overline{\mathcal{A}} \cup \overline{\mathcal{F}}_i$ is an integral-affine surface in which $\partial_i \overline{\mathcal{A}}$ is an interior circle. A neighborhood of this circle contains no singular point and is affinely isomorphic to the quotient of a strip $\{-\epsilon < y < \epsilon\}$ by (53).*
- (3) *Modulo $\mathfrak{m}^N$, the straight continuation into $\overline{\mathcal{F}}_i$ of each wall support meeting $\partial_i \overline{\mathcal{A}}$ is a properly embedded ray along which r increases strictly.*

Proof. The n th power of (55) translates x by $n(\ell_i + w_i r)$, and $\ell_i + w_i r \geq \ell_i > 0$ for $r \geq 0$. Hence the action is free, and a compact set meets only finitely many of its translates. The linear part $\begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}$ lies in $\mathrm{SL}_2(\mathbb{Z})$, so it preserves $2\mathbb{Z}^2$, and the translation $(\ell_i, 0)$ lies in $2\mathbb{Z}^2$, since $\ell_i \in \{4, 8\}$ by (99); so the quotient is integral affine with tangent lattice $\widetilde{M}^\vee$. The transformation preserves r and translates the line $\{r = r_0\}$ by $(\ell_i + w_i r_0, 0)$, which is $\frac{1}{2}(\ell_i + w_i r_0)$ times the primitive vector $(2, 0)$ of $2\mathbb{Z}^2$ in this direction; this gives the affine length. The function r is invariant and integral affine, and $r^{-1}([0, r_0])$ is the image of the compact set $[0, \ell_i + w_i r_0] \times [0, r_0]$, which proves (1).

In the coordinate $y = -r$, (55) becomes $(x, y) \mapsto (x + \ell_i - w_i y, y)$, which is (53). The charts of the collar and of $\overline{\mathcal{F}}_i$ therefore glue to the quotient of a strip about $\{y = 0\}$ by (53), which acts freely there because $\ell_i - w_i y = w_i(h_i - y) > 0$ for $y < h_i$. This proves (2).

By (54) the continuation of a support is the image of $r \mapsto (\phi_i - \xi(r + h_i), r)$, $r \geq 0$. Since (55) preserves r , the image meets each circle $\{r = r_0\}$ in exactly one point. The map is therefore injective, and it is proper because its composition with the proper function r is the identity of $[0, \infty)$. An injective proper immersion is a closed embedding, which proves (3). $\square$

Write

$$\overline{\mathcal{A}}^{\mathrm{fl}} = \overline{\mathcal{F}}_0 \cup \overline{\mathcal{A}} \cup \overline{\mathcal{F}}_1.$$

By Proposition 4.2(2) it is an integral-affine surface without boundary, homeomorphic to an open cylinder, whose singular points are the four singular points of $\overline{\mathcal{A}}$; the functions r on $\overline{\mathcal{F}}_0$ and $\overline{\mathcal{F}}_1$ are proper on its two ends. Figure 3 shows a fundamental domain of a flare.

Proposition 4.3 (Separation of the continued supports). *Let $i \in \{0, 1\}$ and $N \geq 1$. The continuations in $\overline{\mathcal{F}}_i$ of two wall supports meeting $\partial_i \overline{\mathcal{A}}$ modulo $\mathfrak{m}^N$ are either disjoint or equal. There is a support-free ray $x = \phi_i - \xi_*(r + h_i)$, $r \geq 0$, in $\overline{\mathcal{F}}_i$, that is, a ray disjoint from all continued supports.*

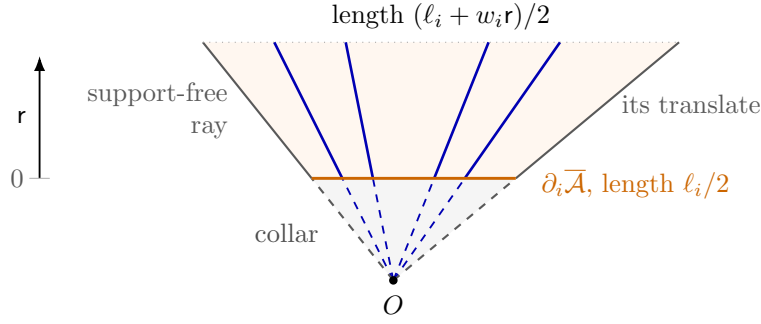


FIGURE 3. A fundamental domain of the flare $\overline{\mathcal{F}}_i$ in the coordinates (x, r) of Proposition 4.2. Its slanted sides are identified by (55), so the circle at height r has affine length $\frac{1}{2}(\ell_i + w_i r)$. Below the boundary circle $\partial_i \overline{\mathcal{A}}$ lies the collar, in which the developed supports are rays from the apex O , the common developed image of the joints O_0 and O_1 . Their continuations are disjoint rays along which r increases, and the left side is a support-free ray (Proposition 4.3). The picture is schematic.

Proof. Since $\ell_i = w_i h_i$, the k th power of (55) maps the line (54) of slope ξ to the line of slope $\xi - kw_i$ through the same point $(x, r) = (\phi_i, -h_i)$. Two such lines meet at a point with $r + h_i > 0$ only if their slopes coincide, and then they are equal. Hence the continuations of two supports of slopes ξ, ξ' meet in $\overline{\mathcal{F}}_i$ if and only if $\xi - \xi' \in w_i \mathbb{Z}$, and then they are equal. By (100), each family of slopes lies in an interval of width smaller than w_i , and the slope of a support of direction Lq exceeds that of a support of direction $-Lq$ by an amount strictly between 0 and w_i . Hence $\xi - \xi' \in w_i \mathbb{Z}$ only if $\xi = \xi'$. Modulo $\mathfrak{m}^N$ there are only finitely many supports, all passing through the two joints (Proposition 3.6 and its proof), so a slope ξ_* with $\xi_* - \xi \notin w_i \mathbb{Z}$ for all their slopes ξ exists, and the corresponding ray is disjoint from every continued support. $\square$

Remark 4.4 (Closing the ends). The flares are integral-affine objects. The natural way to close the ends algebraically places a toric chart, with the Kummer roots of the finite level, over each of the regions into which the continued supports and a support-free ray cut the compact part $\{r \leq r_0\}$ of $\overline{\mathcal{F}}_i$, for some $r_0 > 0$. It transports these charts through the wall automorphisms met on the way from a base region and identifies consecutive charts along dense toric opens. Composing each chart with the inverse of its transport identifies all of them with a single chart over $R_{\mathcal{W}, N}$, every identification across a support with the identity, and the identification across the support-free ray with the holonomy of the coefficient system around $\partial_i \overline{\mathcal{A}}$: the transport (47) along a loop parallel to $\partial_i \overline{\mathcal{A}}$, together with the wall factors met by that loop. Since the identifications are along dense opens, their composite around $\partial_i \overline{\mathcal{A}}$ is defined on a dense open subset of a chart. If the charts are glued to an algebraic space or a formal scheme, the cocycle condition forces this composite to be the identity there. If they are glued as étale charts of a Deligne–Mumford stack, in the way that the Kummer covers P_T of Section 4.5 are charts of the root stacks $\mathfrak{X}_T$, the cocycle condition forces the composite, over the dense open on which the stabilizers are trivial, to be an automorphism of the chart over the stack; such an automorphism fixes the characters z^m , which are functions on the coarse space. In either case the composite fixes every character on a dense open. By (47), modulo $\mathfrak{m}_{\mathcal{W}}$ it sends z^m to $c_\gamma(m)z^{U^m}$, where γ is the loop, U is the linear holonomy of $\partial_i \overline{\mathcal{A}}$, of width w_i , and $c_\gamma(m)$ is a product of transported roots. The gluing therefore requires $c_\gamma(m) = z^{m-U^m}$ for every $m \in \widetilde{M}$, and this fails. The requirement is unchanged under conjugation by transport, so we may base γ in S_0 . With the

conventions of Section 2.4, γ is then $\gamma_A \gamma_B \gamma_C$, with $U = U_0$, for $i = 0$, and $\gamma_A^2 \gamma_B \gamma_C \gamma_A^{-1}$, with $U = U_1$, for $i = 1$; it crosses the arcs carrying the lifted slab maps at B_1 and $\tilde{C}$, respectively at $\tilde{A}$ and B_0 . For $m = u_{\tilde{C}}$ if $i = 0$, and $m = K^{-1}u_{B_0}$ if $i = 1$, the root factor of the second of these slab maps is trivial, and

$$c_\gamma(u_{\tilde{C}}) = f_{B_1}^{-1}, \quad c_\gamma(K^{-1}u_{B_0}) = f_{\tilde{A}}^2,$$

with the slab functions of Section 2.4. These have nonzero divisor on the torus, whereas z^{m-Um} is a unit, so the natural construction does not close the ends. In Mumford's degenerations, of which the Tate curve is the simplest case, and in the constructions of [GHK15, GS22], a nontrivial holonomy around a circle is realized over a degeneration parameter: the special fibers of the charts are then reducible, and charts that are not adjacent need not meet. The rings $R_{\mathcal{W},N}$ contain no such parameter. A logarithmic compactification of the finite levels, and in particular a closure of the flares by boundary divisors flat over $R_{\mathcal{W},N}$, remains open.

4.4. Comparison with the Carl–Pumperla–Siebert class.

Proposition 4.5 (Not homeomorphic to a Carl–Pumperla–Siebert complex). *Let $(\tilde{X} \rightarrow \tilde{S}, \tilde{D})$ be a distinguished toric degeneration of del Pezzo surfaces in the sense of [CPS24, Def. 7.1]; the generic fiber is allowed to be a Gorenstein surface. Then neither $\overline{\mathcal{A}}$ nor any open cylinder, such as $\overline{\mathcal{A}}^\Pi$, is homeomorphic to the intersection complex of $(\tilde{X} \rightarrow \tilde{S}, \tilde{D})$. If the degeneration has simple singularities in the sense of [GS06, Def. 1.60], neither is homeomorphic to its dual intersection complex.*

Proof. The intersection complex is homeomorphic to a closed disk [CPS24, Thm. 7.5 and Thm. 8.1]. Under the simplicity assumption the dual intersection complex is homeomorphic to $\mathbb{R}^2$ [CPS24, Prop. 7.13]. Both have vanishing first cohomology, whereas an annulus and an open cylinder have $H^1(-, \mathbb{k}) \simeq \mathbb{k}$. $\square$

The disk-shaped complex of the quartic pair in Section 2 is of the kind treated in [CPS24, §8.1]: the monodromy T_B has width two, so the singular point B , of order two in the terminology of [CPS24, §8.1], is not simple. The first assertion of the proposition applies to it, and the second does not.

4.5. Kummer covers of the finite root-stack presentations. Let $\mathbb{T} = \text{Spec } \mathbb{k}[\widetilde{M}]$ be the character torus in the frame of the sheet S_0 of Section 2.4, which is the frame at the root of the path tree $\mathcal{T}$, and let $T \subset \mathcal{T}$ be a finite rooted subtree. As in Section 2, a *Kummer edge* of T is an edge e that crosses an arc carrying a lifted slab map, say the arc at V ; it has the root degree $d_e = d_V \in \{2, 4\}$ and the transported radicand $\varrho_e \in \mathbb{k}(\widetilde{M})^\times$, the slab function f_V pulled back to the frame of S_0 along the path of T that precedes e . We write $e' \prec e$ if the Kummer edge e' lies on that path, and r_e for the root adjoined at e , so that $r_e^{d_e} = \varrho_e$. In F -coordinates put

$$\widetilde{M}_u = \{m \in \widetilde{M} : m_1 + m_2 \equiv 0 \pmod{4}\}.$$

Lemma 4.6 (Transported radicands). (1) *The lattice $\widetilde{M}_u$ has index four in $\widetilde{M}$ and is spanned by the four vectors u_V of (22). It is preserved by the matrices (96) and their inverses, and every conormal N_V of (22) takes values in $4\mathbb{Z}$ on it.*
 (2) *For every Kummer edge e of T , at the arc of V ,*

$$\varrho_e = f_V(y_e), \quad y_e = z^{m_e} \prod_{e' \prec e} \varrho_{e'}^{k_{ee'}}, \quad m_e \in 2\widetilde{M}_u, \quad k_{ee'} \in \frac{8}{d_{e'}}\mathbb{Z}.$$

- (3) *There are Laurent polynomials $\widehat{\varrho}_e, H_e \in \mathbb{k}[\widetilde{M}]$ without common prime factor such that $\varrho_e = \widehat{\varrho}_e H_e^{-d_e}$ and every prime factor of H_e divides $\widehat{\varrho}_{e'}$ for some $e' \prec e$. In particular $\operatorname{div} \widehat{\varrho}_e$ is the divisor of zeros of ϱ_e on $\mathbb{T}$.*

Proof. (1) The lattice $\widetilde{M}_u$ is the kernel of $m \mapsto m_1 + m_2$ modulo four, so it has index four. It contains the vectors u_V of (22), and $u_{B_0} = (1, -1)$ and $u_{\widetilde{A}} - u_{B_0} = (2, 2)$ already span a sublattice of index four. Each matrix G in (96) satisfies $(1, 1)G \equiv (1, 1) \pmod{4}$, hence so does G^{-1} , and $G^{\pm 1}$ preserves $\widetilde{M}_u$. The conormals $N_{\widetilde{A}}, N_{B_0}, N_{B_1}, N_{\widetilde{C}}$ pair with $(1, -1)$ to 4, 0, -8, -4 and with $(2, 2)$ to -4, -4, 4, -4.

(2) Pull-back along the path that precedes e composes the transitions across the arcs met by the path (Section 2.4). On a character $z^{m'}$ such a transition replaces m' by its image under one of the matrices (96), their inverses or the identity; if the arc carries the lifted slab map at $V_{e'}$ and is crossed at the Kummer edge e' , it also multiplies by $r_{e'}^{\pm \langle N_{V_{e'}}, m'' \rangle}$, where m'' is m' or its image under the linear part of the transition. By induction along the path, the pull-back of z^m is $z^{Gm} \prod_{e'} r_{e'}^{j_{e'}}$, where G is a product of these matrices and their inverses and each $j_{e'}$ has the form $\pm \langle N_{V_{e'}}, G'm \rangle$ for such a product G' . In the frame of the sheet from which the arc of V is crossed, f_V is a polynomial in a character whose exponent is the image of $2\epsilon_V u_V$ under such a product, so it lies in $2\widetilde{M}_u$ by (1). By (1) again every $j_{e'}$ lies in $8\mathbb{Z}$, and $r_{e'}^{j_{e'}} = \varrho_{e'}^{j_{e'}/d_{e'}}$.

(3) Let $Z \subset \mathbb{T}$ be a prime divisor and v_Z its valuation. By (2), $v_Z(y_e) = \sum_{e'} k_{ee'} v_Z(\varrho_{e'})$ is even, since $8/d_{e'} \in \{2, 4\}$. The polynomial f_V is monic of degree $d_V/2$ with constant term one. Hence $v_Z(\varrho_e) = 0$ if $v_Z(y_e) > 0$; $v_Z(\varrho_e) = \frac{d_V}{2} v_Z(y_e)$, which is divisible by d_V because $v_Z(y_e)$ is even, if $v_Z(y_e) < 0$; and $v_Z(\varrho_e) \geq 0$ if $v_Z(y_e) = 0$. Thus every pole of ϱ_e has multiplicity divisible by d_e and lies on a prime divisor along which some $\varrho_{e'}$, $e' \prec e$, has a zero or a pole, and hence, by induction along the path, along which some $\widehat{\varrho}_{e''}$, $e'' \prec e$, vanishes. Since $\mathbb{k}[\widetilde{M}]$ is factorial, we may take for H_e the product of $h_Z^{-v_Z(\varrho_e)/d_e}$ over the poles $Z = \{h_Z = 0\}$ of ϱ_e , and $\widehat{\varrho}_e = \varrho_e H_e^{d_e}$, which has no poles on $\mathbb{T}$. $\square$

The argument uses only that f_V is monic of degree $d_V/2$ with constant term one, so it applies at the nodal values $a_\rho = \pm 2$ as well; there the B -radicands are squares by (24). We fix such $\widehat{\varrho}_e$ for every Kummer edge e of $\mathcal{T}$; they are unique up to d_e th powers of units of $\mathbb{k}[\widetilde{M}]$. By Lemma 4.6(3), the divisor D_e of zeros of ϱ_e (Section 2) is the effective Cartier divisor $\operatorname{div} \widehat{\varrho}_e$. Let $D_T = \bigcup_e D_e$ with its reduced structure; since every pole of ϱ_e lies on some $D_{e'}$, the complement U_T of the zeros and poles of the transported radicands of T is $\mathbb{T} \setminus D_T$. A divisor D_e may share components with a divisor $D_{e'}$, $e' \prec e$: along $\gamma_{\widetilde{A}}^2$ both crossings of the arc at $\widetilde{A}$ have the radicand $f_{\widetilde{A}}$. Put

$$P_T = \operatorname{Spec} \mathbb{k}[\widetilde{M}][g_e] / (g_e^{d_e} - \widehat{\varrho}_e), \quad G_T = \prod_e \mu_{d_e},$$

over the Kummer edges e of T , with $\zeta \in G_T$ acting by $g_e \mapsto \zeta_e g_e$. The scheme P_T is finite and flat over $\mathbb{T}$. With the convention (101), the quotient $\mathfrak{X}_T = [P_T/G_T]$ is the fiber product over $\mathbb{T}$ of the root stacks $\sqrt[d_e]{(\mathbb{T}, D_e)}$, and an iterated root stack along pulled-back divisors is this fiber product [Cad07, Def. 2.6]. Thus $P_T \rightarrow \mathfrak{X}_T$ is a G_T -torsor, and the finite levels are $\mathfrak{X}_{T, \mathcal{W}, N} = [P_{T, \mathcal{W}, N}/G_T]$ with $P_{T, \mathcal{W}, N} = P_T \times_{\operatorname{Spec} \mathbb{k}} \operatorname{Spec} R_{\mathcal{W}, N}$. We call P_T the *Kummer cover* of $\mathfrak{X}_T$.

Let $P_T^\circ \subset P_T$ be the preimage of $\mathbb{T} \setminus D_T$. It is a finite étale G_T -torsor over $\mathbb{T} \setminus D_T$, over which $\mathfrak{X}_T \rightarrow \mathbb{T}$ is an isomorphism. By Lemma 4.6(3), H_e is a unit on $\mathbb{T} \setminus D_T$, and

$$r_e = g_e H_e^{-1} \in \mathcal{O}(P_T^\circ), \quad r_e^{d_e} = \varrho_e,$$

so that over $\mathbb{T} \setminus D_T$ the radicands $\widehat{\varrho}_e$ and the transported radicands ϱ_e define the same Kummer cover. Thus P_T° is the spectrum of the Kummer ring R_T^{root} of Section 2, on which Sections 2 and 3 define the lifted transitions and $\mathcal{L}_\kappa$, and P_T extends it over $\mathbb{T}$. For every vertex v of T , pull-back along the path of T that ends at v , with the root at each Kummer edge e of the path taken to be r_e , is therefore a homomorphism of $\mathbb{k}$ -algebras from $\mathbb{k}[\widetilde{M}]$, in the frame at v , to $\mathcal{O}(P_T^\circ)$. The local generators of $\mathcal{L}_\kappa$ at the vertices of T , related by (34), differ by units of $\mathcal{O}(P_T^\circ)$. A wall automorphism, base-changed along $v_{\mathcal{W}}$, is the exponential of a derivation of the character ring in the frame at a vertex v of T , tensored with $R_{\mathcal{W},N}$, with coefficients in $\mathfrak{m}_{\mathcal{W}}$; the derivation extends uniquely along the étale morphism $P_T^\circ \rightarrow \text{Spec } \mathbb{k}[\widetilde{M}]$ given by pull-back along the path to v , and so does the automorphism. In this sense the lifted transitions along T , the volume line $\mathcal{L}_\kappa$ with its Jacobi bracket, and the wall automorphisms are defined on P_T° , after base change along $v_{\mathcal{W}}$ on $P_T^\circ \times_{\text{Spec } \mathbb{k}} \text{Spec } R_{\mathcal{W},N}$. The lifted transitions are not G_T -equivariant, and they do not descend to $\mathfrak{X}_T$: G_T fixes the characters and multiplies r_e by ζ_e , whereas the lifted slab map at a Kummer edge e multiplies z^m by $r_e^{\langle N, m \rangle}$ for a primitive conormal N , and $\langle N, m \rangle = 1$ for some m . Over $\mathbb{T} \setminus D_T$ the functions on $\mathfrak{X}_T$ are the functions on $\mathbb{T} \setminus D_T$, and the roots are not among them.

The Kummer covers, and with them the presentations, are not regular in general. If T contains the three Kummer edges at which the transported B_0 -radicands $f_B((1 + x_A^{-1})^{4n} x_B)$, $n = 0, 1, 2$, of Theorem 2.5 are adjoined, then at the points over a specific point of $\mathbb{T}$, at which the root divisors of these three radicands meet pairwise transversely, P_T is neither regular nor log regular, for the trivial log structure or for the log structure of its roots, and $\mathfrak{X}_T$ is not regular (Proposition C.1). Regularity is recovered by a proper modification (Theorem C.2). For every finite rooted subtree T there is a proper morphism $p_T : \mathcal{Y}_T \rightarrow \mathfrak{X}_T$, an isomorphism over $\mathbb{T} \setminus D_T$, from a smooth separated Deligne–Mumford stack $\mathcal{Y}_T$ with a divisor $\mathcal{E}_T$ that is a simple normal crossings divisor on an étale atlas; $\mathcal{Y}_T$ is the stack of fourth roots of the boundary components of the minimal log resolution of $(\mathbb{T}, D_T)$. Its fiber product with P_T is proper over P_T and isomorphic to it over P_T° , so the lifted transitions, the Jacobi bracket and the wall automorphisms pull back to it unchanged. After base change to $R_{\mathcal{W},N}$ both stacks are flat and log smooth, with log regular closed fibers, and the construction is compatible with enlarging T and with the surjections $R_{\mathcal{W}',N'} \rightarrow R_{\mathcal{W},N}$ between truncated coefficient rings. Appendix C proves both results.

5. COMPARISON WITH GROSS–KELLY–TESSLER WALL-CROSSING

The wall structure $\mathfrak{D}$ of Theorem 3.7 is determined by the two singleton factors and by consistency at its joints. The wall-crossing group of Gross, Kelly and Tessler has a different origin: its elements are the transitions between the systems of open invariants defined by different canonical multisections. After the κ -shift both act through the Hamiltonian wall algebra $\mathfrak{g}_\kappa$ of Section 3, the Lie algebra of Ω_κ -Hamiltonian derivations spanned by the fields X_q^κ of (31) with the bracket (32), but their coefficients need not agree. We compare them one ray at a time.

5.1. Recollections from Gross–Kelly–Tessler. We recall the notions of [GKT26a] used in this section and in Appendix B, for $W = x^4 + y^4$, thus $r = s = 4$ in the notation of [GKT26a].

Labels and coefficients. Internal marked points carry labels from a fixed countable set, distinct internal markings of a disk carrying distinct labels, and we refer to an internal marking by its label. Each label i has a twist $\tau(i) = (a_i, b_i) \in \{0, 1, 2, 3\}^2$, and every twist is carried by infinitely many labels [GKT26a, §0.1]. All constructions are bounded by a finite set I of labels. A descendent vector on $J \subseteq I$ is $\mathbf{d} = (d_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$, and for $J \neq \emptyset$ we write $r_J, s_J \in \{0, 1, 2, 3\}$

for the residues of $\sum_{i \in J} a_i$ and $\sum_{i \in J} b_i$ modulo 4. The coefficient ring is

$$(56) \quad A_I = \mathbb{Q}[\{u_{i,d} : i \in I, d \geq 0\}] / \langle u_{i,d} u_{i,d'} : i \in I \rangle$$

[GKT26a, Def. 4.6], in which two variables belonging to the same label have zero product, and $u_{J,\mathbf{d}} = \prod_{i \in J} u_{i,d_i}$. These are GKT's coefficient variables; they are unrelated to the invariant vectors u_V of Section 2.4, which do not occur in this section.

Graphs and bundles. The graphs are the graded open $(4,4)$ -graphs Γ , the dual graphs of genus-zero graded W -spin disks, with internal labels $I(\Gamma) \subseteq I$ and unlabelled boundary tails [GKT26a, Defs. 1.41 and 3.21]. A smooth connected graph has one vertex and, besides its internal tails, $k_1(\Gamma)$ boundary r -tails, of twist $(2,0)$, $k_2(\Gamma)$ boundary s -tails, of twist $(0,2)$, and $k_{12}(\Gamma)$ *fully twisted* boundary tails, of twist $(2,2)$ [GKT26a, §2.4 and Notation 2.27]; the names of the first two kinds refer to the exponents of $x^r + y^s$. Each of the two spin structures carries a *grading*, an orientation of a real line bundle over the boundary of the disk away from its special points, subject to the conditions of [GKT26a, Def. 1.29], and the *alternation* of a boundary marking or half-edge for a grading is 1 if this orientation does not extend across it and 0 if it does [GKT26a, Defs. 1.21, 1.23 and 1.29]. The moduli space $\overline{\mathcal{M}}_\Gamma^W$ of such disks carries the descendent Witten bundle $E_\Gamma(\mathbf{d})$, the direct sum of the Witten bundle and, for each $i \in I(\Gamma)$, of d_i copies of the cotangent line at the i th internal marking [GKT26a, Def. 2.16].

Balanced and critical graphs. A smooth connected graph with one fully twisted tail is *balanced* for $\mathbf{d}$ if its moduli space is nonempty and $\text{rank } E_\Gamma(\mathbf{d}) = \dim \overline{\mathcal{M}}_\Gamma^W$ [GKT26a, Def. 2.32]. A smooth connected graph without fully twisted tails that is relevant, that is, not irrelevant in the sense of [GKT26a, Def. 1.57], is *critical* for $\mathbf{d}$ if the rank exceeds the dimension by one [GKT26a, Def. 2.33(1)]. Write $\text{Balanced}(J, \mathbf{d})$ for the set of balanced graphs with internal labels J . Such a graph Γ is determined by its numbers $(k_1(\Gamma), k_2(\Gamma))$ of r -tails and s -tails, and these pairs, for $\Gamma \in \text{Balanced}(J, \mathbf{d})$, are the pairs (k_1, k_2) with $k_1 \equiv r_J$, $k_2 \equiv s_J$ modulo 4 and

$$k_1 + k_2 = 4 + \sum_{i \in J} (a_i + b_i + 4(d_i - 1))$$

[GKT26a, Prop. 2.31 and Notation 2.40]. A critical graph with $k_1 + 1$ r -tails and $k_2 + 1$ s -tails has *critical exponent* (k_1, k_2) ; the critical exponents of $(J, \mathbf{d})$ satisfy the same congruences and $k_1 + k_2 = \sum_{i \in J} (a_i + b_i + 4(d_i - 1))$ [GKT26a, Prop. 2.39 and Notation 2.40].

Boundary strata. Let $\mathcal{PM}_\Gamma$ be the complement in $\overline{\mathcal{M}}_\Gamma^W$ of the strata of boundary graphs with a positive boundary half-edge, one with positive twist and alternation zero for one of the gradings [GKT26a, Def. 1.53], the space written $\overline{\mathcal{PM}}_\Gamma$ in [GKT26a, Notation 3.1]. Its boundary is the union of the strata $\mathcal{PM}_\Lambda$ of two kinds of boundary graphs Λ [GKT26a, Def. 3.2]. The *exchange boundary graphs* $\Lambda \in \partial^{\text{exchange}} \Gamma$ have a vertex which, after detaching one boundary edge, carries exactly three boundary half-edges with the twists of [GKT26a, Def. 1.54]; two such strata that differ by reversing the cyclic order at that vertex carry matching boundary conditions, and their contributions to zero counts cancel ([GKT26a], discussion after Def. 1.56, and [GKT26a, Lem. 5.3]). The *zero boundary graphs* $\Lambda \in \partial^0 \Gamma$ are the remaining boundary graphs with a boundary edge or a contracted boundary half-edge [GKT26a, Def. 1.56]. For Λ of either kind, the *base* $B\Lambda$ is obtained by detaching the boundary edges at which, for each grading, one half-edge has twist and alternation zero, and forgetting the boundary tails of twist and alternation zero; the induced map is $F_\Lambda : \mathcal{M}_\Lambda^W \rightarrow \mathcal{M}_{B\Lambda}^W$ [GKT26a, Def. 3.5 and Notation 3.6], and the Witten bundle over $\mathcal{PM}_\Lambda$ is pulled back along it [GKT26a, Obs. 3.11].

Canonical multisections. A multisection of $E_\Gamma(\mathbf{d})$ is *strongly positive* if, near the removed strata and along the strata of irrelevant graphs, its Witten components evaluate positively, with respect to the gradings, at the boundary points and on the boundary intervals specified in [GKT26a, Defs. 3.17 and 3.18]; this keeps zeros from leaving through these strata. A *family of*

canonical multisections bounded by I is a collection $\mathfrak{s} = (s_{\Gamma, \mathbf{d}})$ of strongly positive multisections $s_{\Gamma, \mathbf{d}}$ of $E_{\Gamma}(\mathbf{d})$ over $\mathcal{PM}_{\Gamma}$, one for every graph Γ with $I(\Gamma) \subseteq I$ and every descendent vector $\mathbf{d}$, that satisfy the assembling relation

$$(57) \quad s_{\Gamma, \mathbf{d}}|_{\mathcal{PM}_{\Lambda}} = F_{\Lambda}^* \left(\bigoplus_{\Xi \in \text{Conn}(B\Lambda)} s_{\Xi, \mathbf{d}|_{I(\Xi)}} \right)$$

for every zero or exchange boundary graph Λ of Γ ; here $\bigoplus$ is the external sum over the connected components Ξ of the base [GKT26a, Defs. 3.20, 3.27 and 3.28].

Invariants and chamber indices. For a transverse family $\mathfrak{s}$ and $\Gamma \in \text{Balanced}(J, \mathbf{d})$, the invariant $\nu_{\Gamma, \mathbf{d}}^{\mathfrak{s}} \in \mathbb{Q}$ is the weighted count of zeros of $s_{\Gamma, \mathbf{d}}$ [GKT26a, (3.20) and Def. 3.32]. The vector $\nu^{\mathfrak{s}}$ lies in $\mathbb{Q}\text{-Bal}(I) = \prod_{J, \mathbf{d}} \mathbb{Q}^{\text{Balanced}(J, \mathbf{d})}$ [GKT26a, (0.3)]; for ν in this space and $\Gamma \in \text{Balanced}(J, \mathbf{d})$ with $b = (k_1(\Gamma), k_2(\Gamma))$ we write $\nu_{J, b}$ for the coordinate of ν at Γ . With the quartic Gamma weight

$$(58) \quad w_J(u, v) = \frac{\Gamma((u+1)/4)\Gamma((v+1)/4)}{\Gamma((r_J+1)/4)\Gamma((s_J+1)/4)},$$

which is nonzero for $u, v \geq 0$, put

$$A(J, \mathbf{d}, \nu) = \sum_{\pi \in \text{Part}(J)} \sum_{(b_C)_{C \in \pi}} w_J \left(\sum_{C \in \pi} b_C \right) \prod_{C \in \pi} \nu_{C, b_C},$$

where $\text{Part}(J)$ is the set of set partitions of J and b_C runs over the pairs $(k_1(\Gamma), k_2(\Gamma))$ of the graphs $\Gamma \in \text{Balanced}(C, \mathbf{d}|_C)$ [GKT26a, Notation 3.42]. A *chamber index* bounded by I is an element ν such that ν equals 1 on balanced graphs with one internal marking and $\mathbf{d} = 0$ and -1 on balanced graphs without internal markings; $A(\{i\}, (d), \nu) = (-1)^d$; and $A(J, \mathbf{d}, \nu) = 0$ whenever $|J| \geq 2$ and $4 + \sum_{i \in J} (a_i + b_i + 4(d_i - 1)) \geq r_J + s_J$ [GKT26a, Def. 3.47]. The chamber indices are exactly the vectors $\nu^{\mathfrak{s}}$ of transverse canonical families [GKT26a, Cor. 5.14]. The potential of ν is

$$W^{\nu} = \sum_{J \subseteq I, \mathbf{d} \in \mathbb{Z}_{\geq 0}^J} \sum_{\Gamma \in \text{Balanced}(J, \mathbf{d})} (-1)^{|J|-1} \nu_{\Gamma, \mathbf{d}} x^{k_1(\Gamma)} y^{k_2(\Gamma)} u_{J, \mathbf{d}} \in A_I[[x, y]]$$

[GKT26a, Def. 4.6].

The wall-crossing group. The GKT critical Lie algebra [GKT26a, Def. 4.23] is spanned over $\mathbb{Q}$ by the elements $u_{J, \mathbf{d}} X_k$ with $\emptyset \neq J \subseteq I$ and

$$(59) \quad k \equiv \sum_{i \in J} \tau(i) \pmod{4}, \quad k_1 + k_2 = \sum_{i \in J} (a_i + b_i + 4(d_i - 1)),$$

that is, with k a critical exponent of $(J, \mathbf{d})$, where

$$(60) \quad X_k = x^{k_1} y^{k_2} ((k_2 + 1)x \partial_x - (k_1 + 1)y \partial_y).$$

The definition in [GKT26a] excludes $k = (0, 0)$. We adjoin the degree-zero generator $X_{0,0} = x \partial_x - y \partial_y$ whenever $k = (0, 0)$ satisfies (59). The exponential group of the GKT critical Lie algebra acts on $A_I[[x, y]]$ [GKT26a, Def. 4.19 and (4.18)], and on chamber indices by $g(W^{\nu}) = W^{g(\nu)}$ with trivial stabilizers (Step 2 of the proof of [GKT26a, Thm. 4.27]). GKT state that this action is faithful and transitive [GKT26a, Thm. 4.27 and Cor. 5.14]. Transitivity requires the degree-zero generator as soon as a marginal critical graph occurs, that is, a critical graph of critical exponent $(0, 0)$. For two labels of twist $(2, 2)$ and descendent degree zero, $(0, 0)$ is the only critical exponent, and the chamber indices form a one-parameter family on which the group without $X_{0,0}$ acts trivially [Wue26b, Rem. 2.3]. The group enlarged by $X_{0,0}$ acts freely and transitively [Wue26b, Lem. 2.5]. A *canonical homotopy* is a multisection over $[0, 1] \times \mathcal{PM}_{\Gamma}$ that is strongly positive at every time and satisfies the assembling relation in the

form of [GKT26a, Def. 3.36]; it is *simple* if on every boundary stratum it is assembled from one homotopy for each component of the base, independent of the stratum. We write $t \in [0, 1]$ for the time parameter of a homotopy. Between two transverse families, [GKT26a, Lem. 3.40] provides a family of simple canonical homotopies and a distinct time $t(J) \in (0, 1)$ for every nonempty $J \subseteq I$ such that, on a critical graph with internal labels J , the homotopy vanishes only at time $t(J)$, at isolated points. The proof of [GKT26a, Thm. 5.9] shows that at time $t(J)$ the potential changes by the exponential of the critical generators with support J , weighted by $(-1)^{|J|+k_2}$ times the zero counts of the homotopy on the corresponding critical graphs.

5.2. Labelled critical Lie algebras. Let $Q = \widetilde{M}/4\widetilde{M} \simeq (\mathbb{Z}/4)^2$ be the twist group of (49), the group of GKT twists modulo 4. An object $\mathcal{I} = (I, \tau)$ consists of a finite set of labels and a twist map $\tau : I \rightarrow Q$; morphisms are twist-preserving bijections. We write $A_{\mathcal{I}}$ for the ring (56) of I , and $\tau(i) = (a_i, b_i)$ with representatives in $\{0, 1, 2, 3\}^2$.

Theorem 5.1 (Labelled critical Lie algebra embedding). *For every $\mathcal{I}$, the assignment*

$$(61) \quad j_{\mathcal{I}} : u_{J,\mathbf{d}} X_k \longmapsto u_{J,\mathbf{d}} X_{k+\kappa}^{\kappa}$$

is a natural coefficient-preserving injective Lie map from the GKT critical Lie algebra, enlarged by the degree-zero generator, to the Hamiltonian wall algebra $\mathfrak{g}_{\kappa} \otimes A_{\mathcal{I}}$ in the frame of the sheet S_0 . It sends $X_{0,0}$ to X_{κ}^{κ} . Jacobi transport along paths in the twist cover, on which the Q -grading is constant, carries this embedding to the frame of every other region of $\mathfrak{D}$, compatibly with composition of paths. The construction commutes with relabelling, with restriction to a subset of the labels, and with reduction modulo powers of the ideal generated by the $u_{i,d}$.

Proof. Specializing the bracket formula [GKT26a, (4.17)] to two variables gives

$$[X_k, X_{\ell}] = -\det(k + \kappa, \ell + \kappa) X_{k+\ell}.$$

Equation (32) gives the same formula after applying (61). If two coefficient monomials share a label, their product vanishes in (56); if their supports are disjoint, both conditions in (59) add, as in [GKT26a, Prop. 4.25]. Hence the assignment is a Lie map, and it is injective on the displayed monomial basis. The bracket with the degree-zero generator is preserved because

$$[X_{0,0}, X_{a,b}] = (a - b) X_{a,b} = -\det(\kappa, (a, b) + \kappa) X_{a,b}.$$

Jacobi path transport is a Lie isomorphism by Proposition 3.3, and it is compatible with composition of paths. On the twist cover the Q -grading is constant; affine transport moves the exponent of the Hamiltonian function and the κ -shift together. The remaining naturality statements are immediate from the labelled formula. $\square$

Transport along the twist cover is not induced by a change of the twists of the labels. The monodromy about each singular point acts on Q by the matrix P_{res} of (50), which permutes the singleton twists cyclically,

$$(1, 0) \mapsto (2, 3) \mapsto (3, 2) \mapsto (0, 1) \mapsto (1, 0).$$

Replacing the twist $\tau(i)$ of every label by $P_{\text{res}}\tau(i)$, with the labels and descendent degrees unchanged, does not induce an isomorphism of critical Lie algebras that maps the $u_{J,\mathbf{d}}$ -component to the $u_{J,\mathbf{d}}$ -component. Indeed, for two labels of twist $(1, 0)$ and descendent degree one, the conditions (59) have the single solution $k = (2, 0)$, whereas for two labels of twist $(2, 3)$ they have the three solutions $(0, 10)$, $(4, 6)$ and $(8, 2)$. Jacobi transport instead moves the exponent and the local generator of the volume line together, and in the target chart it may produce Laurent expressions whose coefficients involve roots of the slab functions.

Under the dictionary $x^a y^b \leftrightarrow z^{(a,b)}$ of Section 3, the field $X_{k+\kappa}^\kappa$ is the vector field X_k of (60); accordingly $X_{a,b}$ denotes both, as in the notation $X_{a,b} = X_{\kappa+(a,b)}^\kappa$ of Section 1. For a ray $\mathfrak{r}$, the *ray group* of $\mathfrak{r}$ with coefficients in $A_{\mathcal{I}}$ is the exponential of the span of the elements uX_q^κ with $q \in \mathfrak{r}$ and u in the ideal of $A_{\mathcal{I}}$ generated by the $u_{i,d}$; it is abelian by (32).

In a finite nilpotent quotient, exponential integrates (61) to an injective group homomorphism. Let ν and ν' be chamber indices bounded by I . The family of simple canonical homotopies of [GKT26a, Lem. 3.40] between transverse families realizing them changes the potential at the successive times $\mathfrak{t}(J)$ by the exponentials recalled in Subsection 5.1 [GKT26a, Thm. 5.9, with Lem. 3.40]; their product g satisfies $g(W^\nu) = W^{\nu'}$, and we call g a *GKT transition* from ν to ν' . The slope-ordered factorization of [Wue26b, Prop. 3.1] writes it uniquely as a product of factors, each supported on the generators X_k whose boundary rays $\mathbb{R}_{\geq 0}(k + \kappa)$ coincide, ordered by increasing slope of that ray; we call the factor on a ray the *factor of the GKT transition* on that ray. The *extremal chamber indices* $\nu^{\min}$ and $\nu^{\max}$ are the unique chamber indices bounded by I such that, for every $(J, \mathbf{d})$ with $J \neq \emptyset$, the only coordinate at the balanced graphs of $(J, \mathbf{d})$ that may be nonzero is the one of least, respectively greatest, k_1 [Wue26b, Thm. 1.1]. Their potentials $W^{\min} = W^{\nu^{\min}}$ and $W^{\max} = W^{\nu^{\max}}$ are the *extremal GKT potentials*.

Proposition 5.2 (Finite GKT transitions). *In every finite nilpotent quotient, every GKT transition maps to an automorphism of the chart algebra of S_0 in the group generated by the ray groups with coefficients in $A_{\mathcal{I}}$, and the factors of its slope factorization map factor by factor to the ray groups of the boundary rays $\mathbb{R}_{\geq 0}(k + \kappa)$. The GKT transition from ν to ν' , and hence its image, depends only on ν and ν' , not on the chosen canonical homotopy. The construction is natural under labelled restriction and path transport.*

Proof. Exponentiation and Baker–Campbell–Hausdorff are finite in the chosen quotient and are functorial for the map of Theorem 5.1. Step 2 of the proof of [GKT26a, Thm. 4.27] shows that the stabilizers of the action of their group on chamber indices are trivial. Because a transition may use the degree-zero summand excluded in [GKT26a, Def. 4.23], we use the enlarged-torsor statement [Wue26b, Lem. 2.5]: the action of the enlarged group, which contains the degree-zero generator, is free and transitive. Its freeness makes the element carrying W^ν to $W^{\nu'}$ unique. The slope assertion follows because k maps to the shifted exponent $k + \kappa$, which spans the boundary ray of X_k . $\square$

The finite homotopies realizing GKT transitions can be chosen compatibly as the labelled system grows (Theorem 5.7, proved in Appendix B). Every factor of the GKT transition used below is therefore the total jump, across its time interval, of a finite homotopy of canonical multisections.

5.3. Two labels of twists $(1, 0)$ and $(0, 1)$. Take two distinct labels with

$$t_1 = u_{i,1}, \quad \tau(i) = (1, 0), \quad t_2 = u_{j,1}, \quad \tau(j) = (0, 1),$$

and work modulo (t_1^2, t_2^2) . By [GKT26a, Def. 4.6], the potential of a chamber index ν has coefficient $(-1)^{|J|-1} \nu_{\Gamma, \mathbf{d}}$ on the monomial of a balanced graph Γ with internal labels J . The extremal GKT potentials are

$$(62) \quad \begin{aligned} W^{\min} &= x^4 + y^4 - 4t_1xy^4 - 2t_2y^5 + 12t_1t_2xy^5, \\ W^{\max} &= x^4 + y^4 - 2t_1x^5 - 4t_2x^4y + 12t_1t_2x^5y. \end{aligned}$$

The GKT transition from $\nu^{\min}$ to $\nu^{\max}$, written in increasing slope with the rightmost factor acting first, is

$$(63) \quad \exp\left(-\frac{1}{2}t_2X_{0,1}\right) \exp\left(\frac{1}{4}t_1t_2X_{1,1}\right) \exp\left(-\frac{1}{2}t_1X_{1,0}\right).$$

Write $\Sigma_1 = \exp(-\frac{1}{2}t_1X_{1,0}) = e^A$ and $\Sigma_2 = \exp(-\frac{1}{2}t_2X_{0,1}) = e^B$ for its two singleton factors, with A and B the fields (39), not the singular points A and B, and for $c \in \mathbb{Q}$ put

$$(64) \quad Z_c = \exp(ct_1t_2X_{1,1}).$$

Each Z_c lies in the ray group of the first mixed ray, is central in the group generated by Σ_1 , Σ_2 and the $Z_{c'}$, and $Z_c Z_{c'} = Z_{c+c'}$.

With the coorientations of Section 3.3, a path that passes the joint O_0 from left to right on the side of the outgoing walls, such as the path γ_+ of Proposition 7.1, crosses the positive half-chords and the outgoing walls in increasing slope of their boundary rays and crosses each of them positively; this is the order in which the GKT transition applies its factors. The path γ_- of Proposition 7.1 crosses only the two negative half-chords, also positively. Modulo (t_1^2, t_2^2) the factors $\exp(-C_r)$ of Section 3.3 are the identity except on the first mixed ray, so both halves of the two chords carry Σ_1 and Σ_2 , and the diagonal wall from O_0 to B_0 , the wall of the first mixed ray, carries $Z_{-3/4}$, the image of $\exp(-\frac{3}{4}E_{1,1})$. The factor of the GKT transition and the residual factor on the first mixed ray are

$$(65) \quad G_{\text{mix}} = Z_{1/4}, \quad R_{\text{mix}}^{\text{res}} = Z_{-1}.$$

Proposition 5.3 (Factorization on the first mixed ray). *Modulo (t_1^2, t_2^2) , the two chords of $\mathfrak{D}$ carry the GKT singleton factors Σ_1 and Σ_2 , and the diagonal wall carries $Z_{-3/4}$. The elements $Z_{-3/4}$, G_{mix} and $R_{\text{mix}}^{\text{res}}$ lie in the ray group of the first mixed ray, and*

$$(66) \quad Z_{-3/4} = G_{\text{mix}} R_{\text{mix}}^{\text{res}}, \quad -\frac{3}{4} = \frac{1}{4} - 1.$$

Here G_{mix} is the mixed factor of the GKT transition (63), which is unique by Proposition 5.2; hence $R_{\text{mix}}^{\text{res}}$ is determined by the wall. If the factor carried by the diagonal wall is replaced by G_{mix} , the product around a small loop about the joint becomes $(R_{\text{mix}}^{\text{res}})^{\pm 1}$ instead of the identity, the sign depending on the orientation of the loop.

Proof. By [GKT26a, Notation 2.40], the balanced graphs of $(\{i\}, (1))$ have $(k_1, k_2) = (1, 4)$ and $(5, 0)$, those of $(\{j\}, (1))$ have $(0, 5)$ and $(4, 1)$, and those of $(J, (1, 1))$, $J = \{i, j\}$, have $(1, 5)$ and $(5, 1)$; the only critical exponent of J is $(1, 1)$. The chamber index $\nu^{\min}$ can be nonzero only at the pairs $(1, 4)$, $(0, 5)$ and $(1, 5)$ of least k_1 . By (58), the single-label chamber conditions $A(\{i\}, (1), \nu^{\min}) = A(\{j\}, (1), \nu^{\min}) = -1$ read $\frac{1}{4}\nu_{\{i\}, (1, 4)}^{\min} = -1$ and $\frac{1}{2}\nu_{\{j\}, (0, 5)}^{\min} = -1$. The set J has the two partitions $\{J\}$ and $\{\{i\}, \{j\}\}$, and $(1, 4) + (0, 5) = (1, 9)$; since $r_J = s_J = 1$, the weights are $w_J(1, 5) = \Gamma(\frac{1}{2})\Gamma(\frac{3}{2})/\Gamma(\frac{1}{2})^2 = \frac{1}{2}$ and $w_J(1, 9) = \Gamma(\frac{1}{2})\Gamma(\frac{5}{2})/\Gamma(\frac{1}{2})^2 = \frac{3}{4}$. The chamber condition $A(J, (1, 1), \nu^{\min}) = 0$ [GKT26a, Def. 3.47(3) and Notation 3.42] is therefore

$$w_J(1, 5)\nu_{J, (1, 5)}^{\min} + w_J(1, 9)\nu_{\{i\}, (1, 4)}^{\min}\nu_{\{j\}, (0, 5)}^{\min} = \frac{1}{2}\nu_{J, (1, 5)}^{\min} + \frac{3}{4} \cdot (-4) \cdot (-2) = 0,$$

so $\nu_{J, (1, 5)}^{\min} = -12$. The same computation for $\nu^{\max}$, with the pairs $(5, 0)$, $(4, 1)$, $(5, 1)$ and $(9, 1)$ in place of $(1, 4)$, $(0, 5)$, $(1, 5)$ and $(1, 9)$, gives the single-label values -2 and -4 and $\nu_{J, (5, 1)}^{\max} = -12$.

By [GKT26a, Def. 4.6] these invariants, with the signs $(-1)^{|J|-1}$, are the coefficients in (62); in particular the mixed coefficient is $(-1)^{|J|-1}\nu = 12$.

The factor Σ_1 carries $W^{\min}$ to $x^4 + y^4 - 2t_1x^5 - 2t_2y^5 + 2t_1t_2xy^5$, because $X_{1,0}(x^4 + y^4) = 4x^5 - 8xy^4$ and $X_{1,0}(y^5) = -10xy^5$. The factor Σ_2 does not change the coefficient of $t_1t_2xy^5$, and

$$X_{1,1}(x^4 + y^4) = 8x^5y - 8xy^5.$$

Since $W^{\max}$ has no $t_1t_2xy^5$ term, the mixed GKT generator has coefficient $2/8 = 1/4$, proving (63).

Let Θ_{mix} be the factor carried by the diagonal wall. The path γ_+ crosses the Σ_1 -chord, the diagonal wall and the Σ_2 -chord positively and in this order, and γ_- crosses the negative half-chords of Σ_2 and of Σ_1 positively and in this order. Consistency at the joint (Theorem 3.7) therefore requires $\Sigma_2\Theta_{\text{mix}}\Sigma_1 = \Sigma_1\Sigma_2$. Equation (40) gives $\Sigma_2\Sigma_1 = Z_{3/4}\Sigma_1\Sigma_2$, and the factors on the first mixed ray are central, so $\Theta_{\text{mix}} = Z_{-3/4}$. Subtraction of logarithms on this ray gives $-\frac{3}{4} - \frac{1}{4} = -1$, which is (66). Finally, replacing $Z_{-3/4}$ by $G_{\text{mix}} = Z_{-3/4}(R_{\text{mix}}^{\text{res}})^{-1}$ multiplies the product along γ_+ by the central element $(R_{\text{mix}}^{\text{res}})^{-1}$, so the two loop products are $(R_{\text{mix}}^{\text{res}})^{-1}$ and $R_{\text{mix}}^{\text{res}}$. $\square$

Proposition 7.1(3) is the corresponding comparison along a path, with correction $(R_{\text{mix}}^{\text{res}})^{-1}$. The factor of the GKT transition is not the image of the wall factor under any homomorphism fixing the singleton factors, since such a homomorphism fixes their commutator $Z_{-3/4} = \Sigma_2^{-1}\Sigma_1\Sigma_2\Sigma_1^{-1}$. Nor does an abelian quotient separate the two factors: a character of a group containing Σ_1 , Σ_2 and G_{mix} is trivial on the commutator $Z_{-3/4}$ and therefore takes inverse values on G_{mix} and $R_{\text{mix}}^{\text{res}}$.

5.4. Factorization ray by ray in all orders. Let $\mathcal{I}$ be a finite set of distinct labels of descendent degree one, each of twist $\tau_1 = (1, 0)$ or $\tau_2 = (0, 1)$. We write $t_i = u_{i,1} \in A_{\mathcal{I}}$, so that $t_i^2 = 0$, and base-change the walls of $\mathfrak{D}$ along

$$t_1 \mapsto \bar{t}_1 = \sum_{\tau(i)=\tau_1} t_i, \quad t_2 \mapsto \bar{t}_2 = \sum_{\tau(i)=\tau_2} t_i,$$

as in (52). This is the base change for the labelled truncation $(\mathcal{I}, \tau, \mathcal{P}_{\mathcal{I}})$ of Definition 4.1, with $\mathcal{P}_{\mathcal{I}}$ the set of square-free monomials in the t_i ; we call it the *labelled truncation* of $\mathcal{I}$. For $N > |\mathcal{I}|$ its coefficient ring is $\mathbb{k}[t_i : i \in \mathcal{I}]/(t_i^2 : i \in \mathcal{I})$, the base change to $\mathbb{k}$ of the subring of $A_{\mathcal{I}}$ generated by the t_i . For $\emptyset \neq J \subseteq \mathcal{I}$ put

$$p_J = |J_1|, \quad q_J = |J_2|, \quad t_J = \prod_{i \in J} t_i,$$

where J_1 and J_2 are the labels in J of twist τ_1 and τ_2 , and write $p_J = r_J + 4\ell_J$, $q_J = s_J + 4m_J$ with $0 \leq r_J, s_J \leq 3$ as in Subsection 5.1. By [GKT26a, Notation 2.40 and (2.30)–(2.31)], the pairs $(k_1(\Gamma), k_2(\Gamma))$ of the balanced graphs Γ of J , which we call the *balanced profiles* of J , and the critical exponents of J , are

$$(67) \quad b_{J,h} = (r_J + 4h, s_J + 4(\ell_J + m_J + 1 - h)), \quad 0 \leq h \leq \ell_J + m_J + 1,$$

$$(68) \quad k_{J,h} = (r_J + 4h, s_J + 4(\ell_J + m_J - h)), \quad 0 \leq h \leq \ell_J + m_J.$$

Write

$$b_J^{\min} = b_{J,0}, \quad b_J^{\max} = b_{J,\ell_J+m_J+1}.$$

Among the balanced graphs of support t_J , the extremal chamber index ν^ϵ , $\epsilon \in \{\min, \max\}$, is nonzero at most at the graph with $(k_1, k_2) = b_J^\epsilon$; we write ν_J^ϵ for its coordinate there and call it the *extremal invariant* of J . For two labels of twists τ_1 and τ_2 the extremal GKT potentials are (62).

In the labelled truncation of $\mathcal{I}$ every label has descendent degree one, so the inequality in the third chamber condition holds for every J , and the chamber conditions read $A(J, \mathbf{d}, \nu) = -1$ for $|J| = 1$ and $A(J, \mathbf{d}, \nu) = 0$ for $|J| \geq 2$. Since ν^ϵ has on each block C of a partition only the coordinate at b_C^ϵ , the formula for A in Subsection 5.1 turns these conditions into the *extremal recursion*

$$(69) \quad w_J(b_J^\epsilon)\nu_J^\epsilon + \sum_{\substack{\pi \in \text{Part}(J) \\ |\pi| \geq 2}} w_J\left(\sum_{C \in \pi} b_C^\epsilon\right) \prod_{C \in \pi} \nu_C^\epsilon = \begin{cases} -1, & |J| = 1, \\ 0, & |J| \geq 2. \end{cases}$$

The coefficient $w_J(b_J^\epsilon)$ of ν_J^ϵ is nonzero, so the recursion determines the extremal invariants by induction on $|J|$.

On the scattering side, the path γ_+ crosses positively the positive half-chord of least slope, which carries $e^{A-C_{\tau_2,1}}$, then the outgoing walls in increasing slope of their boundary rays, the wall of τ carrying e^{-C_τ} , and then the positive half-chord of greatest slope, which carries $e^{B-C_{\tau_1,2}}$. By Theorem 3.7 its path-ordered product equals that of γ_- , which crosses the negative half-chords, carrying e^B and e^A ; this is (42). Grouping the factors by boundary ray gives, for every N ,

$$(70) \quad e^A e^B = \prod_{\tau}^{\leftarrow} \exp\left(\sum_{\kappa+(p,q) \in \tau} a_{p,q} E_{p,q}\right) \pmod{\mathfrak{m}^N},$$

where $\prod^{\leftarrow}$ places the factor of least slope on the right, so that it acts first. By Proposition 3.5(3) this factorization is unique, and $a_{p,q}$ is the coefficient of $E_{p,q}$ in the logarithm of the factor carried by the wall of $\mathfrak{D}$ through O_0 whose factor lies in the ray group of the ray of $\kappa + (p, q)$; on a chord ray this is the positive half-chord. Thus $a_{1,0} = a_{0,1} = -\frac{1}{2}$ are the chord factors, $a_{p,0} = a_{0,q} = 0$ for $p, q \geq 2$, and $a_{p,q} = -c_{p,q}$ for $p, q \geq 1$, where $c_{p,q}$ are the coefficients of Proposition 3.5 and Table 1. In particular $a_{p,q} = a_{q,p}$, and the first values are

$$a_{1,1} = -\frac{3}{4}, \quad a_{2,1} = a_{1,2} = -\frac{3}{8}, \quad a_{3,1} = a_{1,3} = -\frac{1}{16}, \quad a_{2,2} = -\frac{3}{8}.$$

Definition 5.4 (Transition, wall and residual logarithms). In the labelled truncation of $\mathcal{I}$, let τ be a boundary ray.

- (1) The *transition logarithm* G_τ is the logarithm of the factor on τ of the GKT transition from $\nu^{\min}$ to $\nu^{\max}$.
- (2) The *wall logarithm* S_τ is the logarithm of the base change of the factor on τ in (70), that is, of the factor carried by the wall of $\mathfrak{D}$ on τ (on a chord ray, by its positive half) in the labelled truncation; $S_\tau = 0$ if no wall carries a factor in the ray group of τ .
- (3) The *residual logarithm* is

$$(71) \quad R_\tau^{\text{res}} = S_\tau - G_\tau,$$

and $\exp(R_\tau^{\text{res}})$ is the *residual factor* on τ .

For $J \subseteq \mathcal{I}$ we write G_J , S_J and R_J^{res} for the sums over τ of the t_J -components of G_τ , S_τ and R_τ^{res} . By (68),

$$(72) \quad G_J = t_J \sum_h g_{J,h} X_{\kappa_{J,h}}$$

for rational coefficients $g_{J,h}$.

The generators on one boundary ray commute, so (71) exponentiates to $\exp(S_\tau) = \exp(G_\tau) \exp(R_\tau^{\text{res}})$. The product of the factors $\exp(G_\tau)$, with least slope acting first, is the GKT

transition, which carries $W^{\nu^{\min}}$ to $W^{\nu^{\max}}$, and the product of the factors $\exp(S_{\tau})$ is the base change of (70), the path-ordered product across the joint in the labelled truncation. Hence

$$(73) \quad \overleftarrow{\prod}_{\tau} \exp(S_{\tau}) = \overleftarrow{\prod}_{\tau} (\exp(G_{\tau}) \exp(R_{\tau}^{\text{res}})),$$

in which each factor on the left is replaced, in its own position, by two commuting factors in the same ray group. On a ray that carries no wall, $S_{\tau} = 0$ and $R_{\tau}^{\text{res}} = -G_{\tau}$, so the right side of (73) contains the pair $\exp(G_{\tau}) \exp(-G_{\tau})$; the factorization adds no wall to $\mathfrak{D}$.

Proposition 5.5 (Factorization ray by ray in all orders). *In the labelled truncation of every finite set $\mathcal{I}$ of labels of twists τ_1 and τ_2 and descendent degree one:*

- (1) *the extremal invariants determine the coefficients $g_{J,h}$ in (72) uniquely, support by support in increasing order of $|J|$;*
- (2) *the t_J -component S_J of the wall logarithms is*

$$(74) \quad S_J = p_J! q_J! a_{p_J, q_J} t_J X_{p_J, q_J};$$

- (3) *the residual logarithms vanish on singleton supports: $G_J = S_J = -\frac{1}{2} t_i X_{\tau(i)}$ and $R_J^{\text{res}} = 0$ for $J = \{i\}$;*
- (4) *the factorization (73) is natural under relabelling, deletion of labels, reduction of deformation order, and Jacobi transport along admissible paths (Section 3.3).*

Proof. The GKT dimension formula gives (67)–(68). The critical generators act on the balanced monomials of one support as the weighted incidence matrix of the path graph with vertices $b_{J,0}, \dots, b_{J, \ell_J + m_J + 1}$. Indeed,

$$(75) \quad X_{u,v}(x^4 + y^4) = 4(v+1)x^{u+4}y^v - 4(u+1)x^u y^{v+4}.$$

For $(u, v) = k_{J,h}$ the two monomials in (75) are precisely the adjacent vertices $b_{J,h+1}$ and $b_{J,h}$. The resulting path incidence matrix has full column rank, and its one left-kernel relation is the chamber equation, which the difference of two chamber indices satisfies, because in the labelled truncation the chamber conditions [GKT26a, Def. 3.47] prescribe $A(J, \mathbf{d}, \cdot)$ at every support. Hence the coefficients $g_{J,h}$ exist and are unique, support by support in increasing order of $|J|$. For $J = \{i\}$ this gives $G_J = -\frac{1}{2} t_i X_{\tau(i)}$, as in (63).

The base change carries $e^A e^B$ to the path-ordered product in the labelled truncation and each ray factor of (70) to a labelled ray factor. Since $t_i^2 = 0$, the coefficient of t_J in $\bar{t}_1^{p_J} \bar{t}_2^{q_J}$ is $p_J! q_J!$, and no other monomial $\bar{t}_1^{p_J} \bar{t}_2^{q_J}$ contains t_J ; this gives (74). For $J = \{i\}$ it gives $S_J = -\frac{1}{2} t_i X_{\tau(i)} = G_J$, so the singleton residuals vanish. Both constructions are triangular in proper subsets, proving naturality under restriction; the other naturality statements are immediate from the constructions. $\square$

For the mixed pair $J = \{i, j\}$, the logarithms G_J , S_J and R_J^{res} are the logarithms of the factors G_{mix} , $Z_{-3/4}$ and $R_{\text{mix}}^{\text{res}}$ of Proposition 5.3. The residual logarithm is nonzero at the next order as well. If J consists of two labels of twist τ_1 and one of twist τ_2 , or of one of twist τ_1 and two of twist τ_2 , the only critical exponent is $k = (2, 1)$, respectively $(1, 2)$, and in both cases G_J , S_J and R_J^{res} have coefficients $-\frac{1}{2}$, $-\frac{3}{4}$ and $-\frac{1}{4}$ on $t_J X_k$. For two labels i, i' of twist τ_1 , the GKT transition contains $\exp(\frac{1}{2} t_i t_{i'} X_{2,0})$ on the boundary ray $\mathbb{R}_{\geq 0}(3, 1)$, which lies outside the cone spanned by $(2, 1)$ and $(1, 2)$ and carries no wall; there $R_{\tau}^{\text{res}} = -G_{\tau}$.

The comparison of Proposition 5.3 concerns only the coefficient of $t_1 t_2 X_{1,1}$ on the first mixed ray $\mathbb{R}_{\geq 0} \kappa$. This ray also contains the image X_{κ}^{κ} of the degree-zero generator. If labels of twist $(2, 2)$ and descendent degree zero are admitted, the GKT transition has on this ray the degree-zero component $\exp(f X_{0,0})$, where $f = \frac{1}{32} t_{2,2}^2 + O(t_{2,2}^4)$. Here $t_{2,2}$ is the variable of

GKT's symmetric potential [GKT26a, Def. 4.12] for these labels: the ring map of [GKT26a, Lem. 4.14] sends $t_{2,2}^m/m!$ to the sum of the monomials $u_{J,0}$ over the m -element sets J of them [GKT26a, (4.12)]. It is the variable t of Lemma E.1. This component survives on the *primary slice*, where all descendent variables vanish. Every wall of $\mathfrak{D}$ is congruent to the identity modulo $\mathfrak{m} = (t_1, t_2)$ (Definition 3.4), so $\mathfrak{D}$ has no wall on the primary slice, and the residual component there is $\exp(-fX_{0,0})$.

Remark 5.6 (Products along different paths). The comparison extends from wall factors to transport along paths. Along a path transverse to the walls, the transport of functions through $\mathfrak{D}$ and the transport by the factors of the GKT transition, obtained from it by replacing every wall factor with the factor of the GKT transition on its ray, differ by one automorphism of the target chart algebra, the *residual transport* of the path. Products of functions carried along the same path are therefore related by this automorphism. For functions carried along different paths the residual transports differ in general, already for the paths γ_+ and γ_- of Proposition 7.1, which have the same endpoints (see the example after Definition D.1). Appendix D gives a degree-weighted correction of the products, coherent up to an explicit associator (Lemma D.2 and Propositions D.4 and D.5).

5.5. Compatible homotopies of canonical multisections. A *graph datum* is a graded open $(4, 4)$ -graph Γ together with a descendent vector $\mathbf{d}$ on its internal labels; its support is the monomial $u_{I(\Gamma), \mathbf{d}}$. A *finite assembling-closed system* $\mathbb{D} = (I, \mathcal{P}, \mathcal{E})$ records a finite twist-labelled set I , a finite set $\mathcal{P}$ of labelled monomials closed under divisors, and a finite set $\mathcal{E}$ of graph data whose supports lie in $\mathcal{P}$. The graph data are closed under the assembling relation (57): with each graph datum they contain the data of its connected components and of the connected components of the bases of its boundary graphs, whose multisections prescribe its multisection on the boundary. Moreover, for every support in the monomial set that carries a balanced graph, they contain all graph data of that support whose graph is balanced or critical; the monomials and the graph data are invariant under twist-preserving permutations of the labels. The precise definition is Definition B.1, and Appendix B proves the following theorem by relative extension arguments.

Theorem 5.7 (Compatible GKT homotopies). *There are finite assembling-closed systems $\mathbb{D}_n = (I_n, \mathcal{P}_n, \mathcal{E}_n)$, $n \geq 1$, with $\mathbb{D}_1 \subset \mathbb{D}_2 \subset \dots$, exhausting all graph data, compatible canonical families realizing the two extremal chamber indices, and a homotopy of canonical multisections between them whose multisection on each graph datum $(\Gamma, \mathbf{d})$ is smooth on $[0, 1] \times \mathcal{PM}_\Gamma$, such that:*

- (1) *the countably many ray factors occur in pairwise disjoint open time intervals, ordered by increasing slope;*
- (2) *the multisection on each graph datum is nonconstant on only finitely many such intervals and is constant near the ends of each of them;*
- (3) *for every n , modulo the monomials of A_{I_n} outside $\mathcal{P}_n$, the potential at the right end of the interval of a ray is the image of the potential at its left end under the factor on that ray of the slope factorization [Wue26b, Thm. 1.1 and Prop. 3.1] of the GKT transition from $\nu^{\min}$ to $\nu^{\max}$, and only finitely many of these factors are nontrivial; and*
- (4) *the restrictions to the $\mathbb{D}_n$ are compatible under the restriction maps of Definition B.1, and the families and the homotopy are the inverse limit of these restrictions over the inverse system $\mathbb{D}_1 \subset \mathbb{D}_2 \subset \dots$.*

On one interval of (1) and one finite system, the homotopy is a simple canonical homotopy in the normal form of [GKT26a, Lem. 3.40] for its own assignment of times to label subsets

(Theorem B.5). Thus every factor of the GKT transition in Proposition 5.5 is the total jump, across its time interval, of a finite restriction of one compatible homotopy; inside that interval the potential jumps at the times $t(J)$ assigned to the label subsets J .

6. TRANSPORT OF THE PRIMARY FROBENIUS STRUCTURE

Section 5 compares wall-crossing groups. A comparison of Frobenius structures requires further data: a primitive form, oscillatory periods, a residue pairing, and a Kodaira–Spencer basis. This section supplies these data on the primary slice, where every descendent variable vanishes, starting from the primary potential of [GKT26a] with the primitive form $dx \wedge dy$ in the sense of [GKT26a, Def. 4.3], recalled below. The annular geometry enters only through the transition maps between charts; these are algebra isomorphisms onto their images, and they preserve the Frobenius structure by transport of structure (Proposition 6.3). The structure at the start of the transport is normalized in Appendix E, independently of the annulus: the primary potential has a unique representative $W_{\text{prim}}^{\text{GKT}}$ invariant under the exchange of the coordinates, for which $dx \wedge dy$ is a primitive form with the nine primary variables as flat coordinates (Lemma E.1 and the discussion after it). On the marginal line $W_{\text{prim}}^{\text{GKT}}$ is the flat structure of Noumi and Yamada with primitive form normalized to $dx \wedge dy$, and its coefficients give the first open invariants of every symmetric chamber index with primary potential $W_{\text{prim}}^{\text{GKT}}$ (Theorem E.3).

In the *source chart*, given by the fixed frame of the sheet S_0 (Sections 2 and 3), we write $x = z_1$ and $y = z_2$. Under the dictionary of Section 3 these are the coordinates of the GKT potentials in [GKT26a] and in Sections 5 and 7, not the Cox coordinates of (4). Then $dx \wedge dy = \Omega_\kappa$ by (30), and the Hamiltonian fields $X_{a,b} = X_{\kappa+(a,b)}^\kappa$ of Proposition 3.1, of divergence zero for $dx \wedge dy$, are

$$(76) \quad X_{a,b} = x^a y^b ((b+1)x\partial_x - (a+1)y\partial_y).$$

The symmetric deformation variables of [GKT26a, Def. 4.10] are $t_{a,b,d}$, $0 \leq a, b \leq 2$, $d \geq 0$, where (a, b) is a twist and d is the descendent degree. The *primary variables* are those with $d = 0$, and we write $t_{a,b} = t_{a,b,0}$. The variable $t_{a,b}$ is the parameter dual to the monomial $x^a y^b$; the nine monomials $x^a y^b$, $0 \leq a, b \leq 2$, form a basis of the Jacobian ring $\text{Jac}(x^4 + y^4)$ and correspond to the nine narrow elements $(e^{2\pi i(a+1)/4}, e^{2\pi i(b+1)/4})$ of $\mu_4 \times \mu_4$, those whose fixed locus in $\mathbb{C}^2$ is the origin. Put

$$(77) \quad A_{\text{prim}} = \mathbb{Q}[[t_{a,b} : 0 \leq a, b \leq 2]], \quad \mathfrak{m}_{\text{prim}} = (t_{a,b}).$$

A statement over this complete ring means a compatible system of statements over the quotients $\mathbb{Q}[t_{a,b}]/(t_{a,b}^{n_{a,b}+1})$. For the source potential the compatibility is part of Lemma E.1. The exchange of the coordinates and the degree-zero field, the case $(a, b) = (0, 0)$ of (76), are

$$(78) \quad \sigma(x, y, t_{a,b}) = (y, x, t_{b,a}), \quad X_{0,0} = x\partial_x - y\partial_y.$$

The periods are taken over the good basis of [GKT26a, §4.1]: the product cycles $\Xi_{a,b}$, $0 \leq a, b \leq 2$, in the relative homology $H_2(\mathbb{C}^2, \text{Re}(x^4 + y^4)/\hbar \ll 0; \mathbb{C})$, dual to the monomial basis in the sense that

$$\int_{\Xi_{a,b}} x^c y^d e^{(x^4 + y^4)/\hbar} dx \wedge dy = \delta_{ac} \delta_{bd}, \quad 0 \leq c, d \leq 2$$

[GKT26a, (4.3)]. The same integrals for arbitrary $c, d \geq 0$, the *moments*, follow by integration by parts [GKT26a, Lem. 4.2]. For a deformation W of $x^4 + y^4$ over a complete local ring, the period of $(W, dx \wedge dy)$ over $\Xi_{a,b}$ is the formal series in $\hbar$ and $\hbar^{-1}$ obtained by expanding

$\int_{\Xi_{a,b}} e^{W/\hbar} dx \wedge dy$ in the moments. Following [GKT26a, Def. 4.3], $dx \wedge dy$ is a *primitive form* for W if, for every $0 \leq a, b \leq 2$, the period over $\Xi_{a,b}$ contains no positive power of $\hbar$ and has constant term $\delta_{(a,b),(0,0)}$; the coefficients of $\hbar^{-1}$ in the nine periods are then the *flat coordinates*. Thus $dx \wedge dy$ is a primitive form with flat coordinates $t_{a,b}$ exactly when

$$\int_{\Xi_{a,b}} e^{W/\hbar} dx \wedge dy = \delta_{(a,b),(0,0)} + \hbar^{-1} t_{a,b} + O(\hbar^{-2}), \quad 0 \leq a, b \leq 2,$$

which is the form in which [GKT26a, Cor. 4.17] states it. GKT formulate the definition for a form $f dx \wedge dy$ and present it as a simplified version of Saito's theory of primitive forms; only $f = 1$ occurs here.

6.1. The Frobenius algebra in the source chart. At the Fermat point the primary state space is

$$(79) \quad H_{4,4} = \mathbb{Q}[x, y]/(x^3, y^3), \quad e_{a,b} = [x^a y^b], \quad 0 \leq a, b \leq 2.$$

Normalize the Grothendieck residue trace by

$$(80) \quad \epsilon_0(f) = 16 \operatorname{Res}_0 \frac{f dx \wedge dy}{(4x^3)(4y^3)}.$$

Proposition 6.1 (Residue pairing at the Fermat point). *The pairing $\eta_0(f, g) = \epsilon_0(fg)$ on $H_{4,4}$ is nondegenerate and*

$$(81) \quad \eta_0(e_{a,b}, e_{c,d}) = \delta_{a+c,2} \delta_{b+d,2}.$$

Equivalently, ϵ_0 is the coefficient of the socle monomial $e_{2,2}$, and $e_{a,b} e_{2-a,2-b} = e_{2,2}$ for $0 \leq a, b \leq 2$.

Proof. The residue in (80) extracts the coefficient of $x^2 y^2$. Since $e_{a,b} e_{c,d} = e_{a+c,b+d}$ when $a+c \leq 2$ and $b+d \leq 2$ and vanishes otherwise, this proves (81), the socle identity, and nondegeneracy. $\square$

Over the primary base, the Jacobian algebra of the reflection-fixed primary potential and its normalized residue trace are

$$(82) \quad \mathcal{R}_0 = A_{\text{prim}}[x, y]/(\partial_x W_{\text{prim}}^{\text{GKT}}, \partial_y W_{\text{prim}}^{\text{GKT}}), \quad \epsilon(h) = 16 \operatorname{Res}_0 \frac{h dx \wedge dy}{\partial_x W_{\text{prim}}^{\text{GKT}} \partial_y W_{\text{prim}}^{\text{GKT}}},$$

with residue pairing $\eta(f, g) = \epsilon(fg)$ and Kodaira–Spencer classes $\Phi_{a,b}^0 = [\partial W_{\text{prim}}^{\text{GKT}} / \partial t_{a,b}]$, $0 \leq a, b \leq 2$.

Lemma 6.2 (Source Frobenius algebra). *The classes $\Phi_{a,b}^0$ form an A_{prim} -basis of $\mathcal{R}_0$, with $\Phi_{a,b}^0 = x^a y^b + O(\mathfrak{m}_{\text{prim}})$ and $\Phi_{0,0}^0 = [1]$. The pairing η is nondegenerate and makes $\mathcal{R}_0$ a Frobenius algebra. Under the Kodaira–Spencer map $\partial/\partial t_{a,b} \mapsto \Phi_{a,b}^0$, the multiplication of $\mathcal{R}_0$ and η are the product and the flat metric of the primary Saito–Givental Frobenius manifold of $W_{\text{prim}}^{\text{GKT}}$; in particular η is constant in the flat coordinates $t_{a,b}$. By [GKT26a, Thm. 0.3 and Rem. 0.4(2)], this Frobenius manifold is the genus-zero primary closed FJRW Frobenius manifold of $(x^4 + y^4, \mu_4^2)$.*

Proof. Modulo quadratic terms in the $t_{a,b}$, $W_{\text{prim}}^{\text{GKT}} \equiv x^4 + y^4 + \sum t_{a,b} x^a y^b$ [GKT26a, Def. 3.47(1) and (4.11)], which gives the congruence for $\Phi_{a,b}^0$. Over each quotient A_I^{prim} of A_{prim} (Appendix E.1), where $\mathfrak{m}_{\text{prim}}$ is nilpotent, the nine monomials $x^a y^b$ span $\mathcal{R}_0$ by Nakayama's lemma, since they span $\mathcal{R}_0/\mathfrak{m}_{\text{prim}} \mathcal{R}_0 = H_{4,4}$. The functional ϵ vanishes on the Jacobian ideal, and the Gram matrix of η on these monomials reduces modulo $\mathfrak{m}_{\text{prim}}$ to the invertible matrix

of Proposition 6.1, so its determinant is a unit. Hence $\mathcal{R}_0$ is free with basis the $x^a y^b$ and η is nondegenerate; the $\Phi_{a,b}^0$ form a basis by Nakayama's lemma, $\Phi_{0,0}^0 = [1]$ by (108), and $\eta(fg, h) = \epsilon(fgh) = \eta(f, gh)$. The flat coordinates $t_{a,b}$ of $(W_{\text{prim}}^{\text{GKT}}, dx \wedge dy)$ (the discussion after Lemma E.1) agree with those of Saito–Givental theory in the good-basis description of [HLSW22, §2.2] [GKT26a, Rems. 0.4(2) and 4.5], so the Kodaira–Spencer map identifies the product and flat metric of Saito's Frobenius manifold [Sai83] with the multiplication of $\mathcal{R}_0$ and η . Every symmetric chamber index is enumerative [GKT26a, Prop. 3.46 and Cor. 5.14], so [GKT26a, Thm. 0.3 and Rem. 0.4(2)] applies. $\square$

6.2. Transport through the annular charts. Let R be an Artinian local $\mathbb{Q}$ -algebra of labelled deformation coefficients, base-changed to $\mathbb{k}$ where it meets the chart algebras (Section 1). For $W \in R[x, y]$ the Brieskorn module is

$$(83) \quad \text{GM}_{\hbar}(W) = H^2\left(\Omega_{R[x,y]/R}^{\bullet}[[\hbar]], \hbar d + dW \wedge\right).$$

For an admissible path $\mathfrak{p}$ (Definition 3.8), let $P_{\mathfrak{p}}$ be the composite of the transition maps along it: the slab maps (23) with the affine holonomy they carry, and the wall automorphisms. These are injective homomorphisms of R -algebras between chart algebras, so $U_{\mathfrak{p}} = P_{\mathfrak{p}}(x)$ and $V_{\mathfrak{p}} = P_{\mathfrak{p}}(y)$ are algebraically independent over R , and $P_{\mathfrak{p}}$ is an isomorphism onto the image algebra

$$(84) \quad P_{\mathfrak{p}}(R[x, y]) = R[U_{\mathfrak{p}}, V_{\mathfrak{p}}]$$

inside the target chart algebra. Brieskorn modules, Jacobian algebras, residue pairings and good-basis periods (the moment functionals of [GKT26a, Lem. 4.2]) of $P_{\mathfrak{p}}W$ are formed over this image algebra by the same formulas in $U_{\mathfrak{p}}, V_{\mathfrak{p}}$, not over the target chart algebra, a Laurent algebra over which the Jacobian quotient of a transported primary potential vanishes because its $U_{\mathfrak{p}}$ -derivative is a unit. The periods are formal functionals, since the torus of the target chart does not contain the critical point $U_{\mathfrak{p}} = V_{\mathfrak{p}} = 0$. By (34), and since the wall factors are transports of automorphisms that preserve Ω_{κ} in the frames of S_0 and S_1 , every step of $P_{\mathfrak{p}}$ multiplies Ω_{κ} by a unit; hence $dU_{\mathfrak{p}} \wedge dV_{\mathfrak{p}}$ is the form Ω'_{κ} of the target chart times the unit recorded by the line $\mathcal{L}_{\kappa}$.

On the primary slice put $W_{\mathfrak{p}} = P_{\mathfrak{p}}(W_{\text{prim}}^{\text{GKT}})$,

$$(85) \quad \mathcal{R}_{\mathfrak{p}} = A_{\text{prim}}[U_{\mathfrak{p}}, V_{\mathfrak{p}}]/(\partial_{U_{\mathfrak{p}}} W_{\mathfrak{p}}, \partial_{V_{\mathfrak{p}}} W_{\mathfrak{p}}),$$

let $\eta_{\mathfrak{p}}$ be the residue pairing of $(W_{\mathfrak{p}}, dU_{\mathfrak{p}} \wedge dV_{\mathfrak{p}})$, and let

$$(86) \quad \Phi_{a,b}^{\mathfrak{p}} = \left[\frac{\partial W_{\mathfrak{p}}}{\partial t_{a,b}} \right] \in \mathcal{R}_{\mathfrak{p}}, \quad 0 \leq a, b \leq 2,$$

be the transported Kodaira–Spencer classes; for the trivial path these are the $\Phi_{a,b}^0$ of Lemma 6.2.

Proposition 6.3 (Transported primary Frobenius structure). *Let $\mathfrak{p}$ be an admissible path.*

- (1) *For $W \in R[x, y]$, $P_{\mathfrak{p}}$ induces an isomorphism of twisted de Rham complexes over $R[[\hbar]]$, and hence*

$$(87) \quad \text{GM}_{\hbar}(W) \xrightarrow{\sim} \text{GM}_{\hbar}(P_{\mathfrak{p}}W),$$

carrying $[dx \wedge dy]$ to $[dU_{\mathfrak{p}} \wedge dV_{\mathfrak{p}}]$. Modulo $\hbar$ it identifies the Jacobian algebras and the residue pairings of $(W, dx \wedge dy)$ and $(P_{\mathfrak{p}}W, dU_{\mathfrak{p}} \wedge dV_{\mathfrak{p}})$, and it preserves every good-basis period.

- (2) On the primary slice, $P_{\mathfrak{p}}$ is the base change to A_{prim} of an injective homomorphism of chart algebras over $\mathbb{k}$, independent of the $t_{a,b}$. For the transported periods, $dU_{\mathfrak{p}} \wedge dV_{\mathfrak{p}}$ is a primitive form for $W_{\mathfrak{p}}$ in the sense of [GKT26a, Def. 4.3], and the nine $t_{a,b}$ are simultaneous flat coordinates.
- (3) The classes (86) form an A_{prim} -basis of $\mathcal{R}_{\mathfrak{p}}$ with

$$(88) \quad \Phi_{a,b}^{\mathfrak{p}} = P_{\mathfrak{p}}(\Phi_{a,b}^0), \quad \Phi_{a,b}^0 = x^a y^b + O(\mathfrak{m}_{\text{prim}}),$$

and their multiplication, the unit $\Phi_{0,0}^{\mathfrak{p}} = [1]$ and the residue pairing $\eta_{\mathfrak{p}}$ have coefficient matrices independent of $\mathfrak{p}$, namely the structure constants of the primary Saito–Givental Frobenius manifold of $W_{\text{prim}}^{\text{GKT}}$, which [GKT26a, Thm. 0.3 and Rem. 0.4(2)] identifies with the genus-zero primary closed FJRW Frobenius manifold of $(x^4 + y^4, \mu_4^2)$.

These maps compose under concatenation of paths and are compatible with reduction of R , in particular to the quotients A_I^{prim} , and with refinement of the root-stack presentation. For a second admissible path $\mathfrak{p}'$, the isomorphism $P_{\mathfrak{p}'} P_{\mathfrak{p}}^{-1} : R[U_{\mathfrak{p}}, V_{\mathfrak{p}}] \rightarrow R[U_{\mathfrak{p}'}, V_{\mathfrak{p}'}]$ has the properties in (1) and carries $\Phi_{a,b}^{\mathfrak{p}}$ to $\Phi_{a,b}^{\mathfrak{p}'}$.

Proof. Each part is transport of structure along $P_{\mathfrak{p}}$. (1) Since $P_{\mathfrak{p}}$ carries dx, dy to $dU_{\mathfrak{p}}, dV_{\mathfrak{p}}$, it satisfies $(\hbar d + d(P_{\mathfrak{p}}W) \wedge) P_{\mathfrak{p}} = P_{\mathfrak{p}}(\hbar d + dW \wedge)$ on forms, so pullback is a chain isomorphism; modulo $\hbar$ it is an isomorphism of Koszul complexes, whose top cohomology is the Jacobian algebra times the volume form. The residue and the moment functionals are given by the same formulas in both coordinate systems, so they are preserved. Composition, reduction and refinement follow from functoriality of the transition maps, and the argument applies equally to $P_{\mathfrak{p}'} P_{\mathfrak{p}}^{-1}$.

(2) Every wall automorphism is the identity modulo $\mathfrak{m} = (t_1, t_2)$ (Definition 3.4), an ideal generated by descendent variables, so on the primary slice $P_{\mathfrak{p}}$ is a composite of slab and holonomy maps, which involve no deformation parameters. The conditions of [GKT26a, Def. 4.3] concern only the good-basis periods; they hold for $(W_{\text{prim}}^{\text{GKT}}, dx \wedge dy)$ by the discussion after Lemma E.1 and are preserved by (1).

(3) By the chain rule and (2), $P_{\mathfrak{p}}$ induces an isomorphism of A_{prim} -algebras $\mathcal{R}_0 \rightarrow \mathcal{R}_{\mathfrak{p}}$ that commutes with $\partial/\partial t_{a,b}$, which gives the first identity in (88), carries $[1]$ to $[1]$, and by (1) carries η to $\eta_{\mathfrak{p}}$. Hence the multiplication, the unit and the pairing have the same coefficient matrices in the bases $\Phi_{a,b}^{\mathfrak{p}}$ as for the trivial path, $P_{\mathfrak{p}'} P_{\mathfrak{p}}^{-1}$ carries $\Phi_{a,b}^{\mathfrak{p}}$ to $\Phi_{a,b}^{\mathfrak{p}'}$, and the remaining assertions follow from Lemma 6.2. $\square$

Wall factors, and the residual factors by which they differ from the factors of the GKT transition (Section 5), are flows of the fields (76) with nilpotent coefficients, hence volume-preserving right equivalences of $(W, dx \wedge dy)$. By Proposition 6.3(1), the identities between algebra automorphisms of Appendix D therefore become composition identities of Brieskorn chain maps. Since $P_{\mathfrak{p}}$ carries Jacobian ideals to Jacobian ideals, a coefficient of the Jacobian algebra on transported source sections, such as $1/32$ in (117), does not depend on the admissible path, although $U_{\mathfrak{p}}$ and $V_{\mathfrak{p}}$ do, through the hyperbolic holonomy.

Remark 6.4 (Deck transformations and core holonomy). The deck transformations of the fractional-exponent cover p_{fr} and of the twist cover, and the core holonomy, relate admissible paths $\mathfrak{p}, \mathfrak{p}'$ ending over the same region of $\mathfrak{D}$. By Proposition 6.3, $\phi = P_{\mathfrak{p}'} P_{\mathfrak{p}}^{-1} : \mathcal{R}_{\mathfrak{p}} \rightarrow \mathcal{R}_{\mathfrak{p}'}$ is A_{prim} -linear and carries transported bases to transported bases, so the structure constants are invariant under these maps. If the two sheets of p_{fr} over a region are identified by such a ϕ , the involution $(r, r') \mapsto (\phi^{-1} r', \phi r)$ splits $\mathcal{R}_{\mathfrak{p}} \oplus \mathcal{R}_{\mathfrak{p}'}$ into its (± 1) -eigensummands, free of rank nine

over A_{prim} and orthogonal for the residue pairing; the $(+1)$ -eigensummand $\{(r, \phi r)\}$ is a unital subalgebra isomorphic to $\mathcal{R}_{\text{p}}$.

7. TWO LABELS: PATH-ORDERED PRODUCTS AND THE MAXIMAL NORMAL FORM

Proposition 5.3 compares the GKT transition with the wall structure $\mathfrak{D}$ one ray at a time. This section compares the two along a path, first for two labels of twists $(1, 0)$ and $(0, 1)$, where $\mathfrak{D}$ has a diagonal wall, and then for two labels of the same twist $(1, 0)$, where it has none. We keep the two labels i, j , the parameters $t_1 = u_{i,1}$ and $t_2 = u_{j,1}$, and the operators Σ_1, Σ_2 and Z_c of Section 5.3, and work over $R_2 = \mathbb{Q}[t_1, t_2]/(t_1^2, t_2^2)$. Operators act on polynomials in x, y over R_2 through the vector fields (60), and the rightmost factor of a product acts first. The proof of Proposition 5.3 gives the commutation relation

$$(89) \quad \Sigma_2 \Sigma_1 = Z_{3/4} \Sigma_1 \Sigma_2.$$

Of the two balanced monomials $t_1 t_2 x y^5$ and $t_1 t_2 x^5 y$ of the support $J = \{i, j\}$, the extremal GKT potentials $W^{\min}$ and $W^{\max}$ of (62) keep only the one of least, respectively greatest, x -degree; its coefficient $12 = (-1)^{|J|-1} \nu$ is determined by the GKT invariant $\nu = -12$ of J . Direct differentiation gives

$$(90) \quad \begin{aligned} X_{1,0}(x^4 + y^4) &= 4x^5 - 8xy^4, & X_{1,0}(y^5) &= -10xy^5, \\ X_{0,1}(x^4 + y^4) &= 8x^4y - 4y^5, & X_{0,1}(x^5) &= 10x^5y, \\ X_{1,1}(x^4 + y^4) &= 8x^5y - 8xy^5. \end{aligned}$$

7.1. Path-ordered products near the joint. Let (s_1, s_2) be affine coordinates on the chart of Δ drawn in Figure 2(a), with the joint O at the origin. By (43) the two singleton chords are the lines $3s_1 + 4s_2 = 0$ and $3s_1 + 2s_2 = 0$. By (44) with $c_1 = c_2$, the diagonal wall is the segment from O to B on the line $s_1 + s_2 = 0$, $s_2 \geq 0$. Modulo (t_1^2, t_2^2) these are the only walls through O with a nontrivial factor. With the coorientations of Section 3, the three walls carry the following factors:

Support	Positive normal	Attached factor
$3s_1 + 4s_2 = 0$	$(3, 4)$	Σ_1
$3s_1 + 2s_2 = 0$	$(3, 2)$	Σ_2
$s_1 + s_2 = 0$, from O to B	$(1, 1)$	$Z_{-3/4}$

A path crossing a wall in the direction of its positive normal applies the attached factor, and otherwise its inverse (Definition 3.4). Choose $\varepsilon > 0$ so small that the disk of radius 2ε about O meets no wall whose factor is nontrivial modulo (t_1^2, t_2^2) other than these three and contains no singular point; this is possible by Proposition 3.6. The transitions at the singular points are carried by cuts running from the singular points outward (Section 3), so they do not meet this disk, and the regions of $\mathfrak{D}$ adjacent to O share one frame. Let γ_+ and γ_- run from $o_- = (-\varepsilon, 0)$ to $o_+ = (\varepsilon, 0)$ along the horizontal segments $s_2 = \varepsilon/10$ and $s_2 = -\varepsilon/10$, respectively, joined to o_- and o_+ by vertical segments, and write $\Theta_{\gamma_{\pm}}$ for the path-ordered products. Along γ_+ the walls are met in the order of increasing slope of their boundary rays, so o_- lies in the region preceding all three walls in that order, and we place $W^{\min}$ there.

Proposition 7.1 (Two-label comparison). (1) *The path-ordered products*

$$\Theta_{\gamma_+} = \Sigma_2 Z_{-3/4} \Sigma_1, \quad \Theta_{\gamma_-} = \Sigma_1 \Sigma_2$$

are equal.

(2) Both carry $W^{\min}$ to

$$(91) \quad W^{\text{tr}} = x^4 + y^4 - 2t_1x^5 - 4t_2x^4y + t_1t_2(8xy^5 + 4x^5y).$$

(3) The GKT transition (63) satisfies

$$\Sigma_2 Z_{1/4} \Sigma_1 = Z_1 \Theta_{\gamma_+}, \quad Z_1 W^{\text{tr}} = W^{\max}.$$

Thus the GKT transition and the path-ordered product of $\mathfrak{D}$ differ by $Z_1 = \exp(t_1 t_2 X_{1,1})$.

Proof. On $s_2 = \varepsilon/10$ the three supports are met at $s_1 = -2\varepsilon/15$, $-\varepsilon/10$ and $-\varepsilon/15$, in the order Σ_1 -chord, diagonal wall, Σ_2 -chord. The direction of travel $(1, 0)$ has positive pairing with $(3, 4)$, $(1, 1)$ and $(3, 2)$, so $\Theta_{\gamma_+} = \Sigma_2 Z_{-3/4} \Sigma_1$. On $s_2 = -\varepsilon/10$ the chords are met at $s_1 = \varepsilon/15$ and $2\varepsilon/15$, both positively, and the diagonal wall is not met. The vertical segments lie on $s_1 = \pm\varepsilon$ with $|s_2| \leq \varepsilon/10$, where no support passes: the chords meet these lines at $|s_2| \geq 3\varepsilon/4$ and the diagonal wall at $s_2 = \varepsilon$. Hence $\Theta_{\gamma_-} = \Sigma_1 \Sigma_2$.

Since $Z_{-3/4}$ is central, (89) gives $\Theta_{\gamma_+} = Z_{-3/4} \Sigma_2 \Sigma_1 = \Sigma_1 \Sigma_2 = \Theta_{\gamma_-}$, which proves (1).

For (2), use (90) and $t_1^2 = t_2^2 = 0$. First,

$$\Sigma_1 W^{\min} = x^4 + y^4 - 2t_1x^5 - 2t_2y^5 + 2t_1t_2xy^5.$$

The flow Z_c changes only mixed terms, and $Z_c V = V + 8ct_1t_2(x^5y - xy^5)$ for every potential V of the form $x^4 + y^4 + O(t_1, t_2)$. Hence

$$Z_{-3/4} \Sigma_1 W^{\min} = x^4 + y^4 - 2t_1x^5 - 2t_2y^5 + t_1t_2(8xy^5 - 6x^5y).$$

Applying Σ_2 adds $-4t_2x^4y + 2t_2y^5$ from $x^4 + y^4$ and $10t_1t_2x^5y$ from $-2t_1x^5$, which gives (91). Part (1) gives the same result along γ_- .

For (3), since Z_c is central, $\Sigma_2 Z_{1/4} \Sigma_1 = Z_{1/4+3/4} \Sigma_2 Z_{-3/4} \Sigma_1 = Z_1 \Theta_{\gamma_+}$. Finally $Z_1 W^{\text{tr}}$ has mixed terms $t_1t_2((8-8)xy^5 + (4+8)x^5y)$, which is $W^{\max}$ by (62). $\square$

Remark 7.2 (Relation with the factorization ray by ray). Proposition 5.3 splits the factor of the diagonal wall into the factor of the GKT transition on the first mixed ray and a residual factor, and Proposition 7.1(3) is the corresponding statement for the whole path: removing the residual turns the path-ordered product of $\mathfrak{D}$ into the GKT transition. By Proposition 7.1(2), $\mathfrak{D}$ itself does not carry $W^{\min}$ to $W^{\max}$.

7.2. The maximal normal form. For $a, b \in \mathbb{Q}$ put

$$V(a, b) = x^4 + y^4 - 2t_1x^5 - 4t_2x^4y + t_1t_2(axy^5 + bx^5y),$$

so that $W^{\text{tr}} = V(8, 4)$ and $W^{\max} = V(0, 12)$. By the proof of Proposition 7.1, $Z_c V(a, b) = V(a - 8c, b + 8c)$. Each orbit of the flows Z_c in this family therefore contains exactly one potential without an xy^5 term, and we define

$$(92) \quad \mathcal{N}^{\max} V(a, b) = Z_{a/8} V(a, b) = V(0, a + b).$$

On the family $V(a, b)$ this operation is the maximal normal form of Proposition 7.6 for the labels i, j ; that normal form keeps, for each support, only the balanced coefficient of greatest x -degree. The operation $\mathcal{N}^{\max}$ is idempotent and constant on the orbits of the flows Z_c . Deleting the xy^5 term instead would give $V(0, b)$, which differs from $\mathcal{N}^{\max} V(a, b)$ whenever $a \neq 0$.

Proposition 7.3 (The normal form carries the mixed coefficient). *Let $W_m^{\min}$ be $W^{\min}$ with the coefficient 12 of $t_1t_2xy^5$ replaced by $m \in \mathbb{Q}$. Then*

$$\Theta_{\gamma_+} W_m^{\min} = V(m - 4, 4), \quad \mathcal{N}^{\max} \Theta_{\gamma_+} W_m^{\min} = V(0, m).$$

In particular $\mathcal{N}^{\max} \Theta_{\gamma_+} W^{\min} = W^{\max}$.

Proof. The operators Σ_1 , Σ_2 and Z_c fix the term $m t_1 t_2 x y^5$, and their other contributions to the coefficient of $t_1 t_2 x y^5$ do not involve m . The computation in the proof of Proposition 7.1 with 12 replaced by m therefore gives $V(m-4, 4)$, and (92) gives $V(0, m)$. $\square$

The normal form transfers the mixed coefficient of $W^{\min}$ to $W^{\max}$; it does not compute it. The coefficient is fixed by periods, which in turn do not select a representative in an orbit of the flows Z_c . Let $W_0 = x^4 + y^4$ and $\Omega = dx \wedge dy$, and let $\nabla_0 = \hbar d + dW_0 \wedge$ act on polynomial differential forms over $R_2[\hbar, \hbar^{-1}]$; this is the operator of (83) over that ring. Call a one-form *relative* if its pullbacks to both coordinate axes vanish, and call a two-form *relatively exact* if it equals $\nabla_0 \beta$ for a relative one-form β .

Proposition 7.4 (Periods of the two-label family). (1) *The $t_1 t_2$ -part of $e^{(V(a,b)-W_0)/\hbar} \Omega$ is ∇_0 -cohomologous to $(6 - \frac{a+b}{2}) xy \Omega$.*
 (2) *The one-form $\beta = t_1 t_2 d(x^2 y^2)$ is relative, and $\nabla_0 \beta = (W^{\max} - W^{\text{tr}}) \Omega$; in particular $(W^{\max} - W^{\text{tr}}) \Omega$ is relatively exact.*
 (3) *Let $\hbar \in \mathbb{C}^\times$ and let $\mathcal{S} \subset \mathbb{C}^2$ be a properly embedded oriented real surface whose boundary lies in the union of the coordinate axes, on which $\text{Re}(W_0/\hbar) \leq -c|z|^4 + c'$ for some constants $c > 0$ and c' , and whose area in the ball of radius ρ grows at most polynomially in ρ . Then*

$$\int_{\mathcal{S}} e^{W^{\max}/\hbar} \Omega = \int_{\mathcal{S}} e^{W^{\text{tr}}/\hbar} \Omega \pmod{t_1^2, t_2^2}.$$

Proof. For a polynomial f , $\nabla_0(f dy) = (4x^3 f + \hbar \partial_x f) \Omega$ and $\nabla_0(f dx) = -(4y^3 f + \hbar \partial_y f) \Omega$. Taking $f = x^2 y$, xy^2 and $x^6 y$ gives the relations $x^5 y \Omega \sim -\frac{\hbar}{2} xy \Omega \sim xy^5 \Omega$ and $x^9 y \Omega \sim \frac{3}{4} \hbar^2 xy \Omega$. Modulo (t_1^2, t_2^2) the $t_1 t_2$ -part of $e^{(V(a,b)-W_0)/\hbar}$ is $\hbar^{-1}(a xy^5 + b x^5 y) + \hbar^{-2} 8x^9 y$, which proves (1).

The function $x^2 y^2$ vanishes on both axes, so its differential pulls back to zero there. Since $d\beta = 0$,

$$\nabla_0 \beta = dW_0 \wedge \beta = t_1 t_2 (4x^3 dx + 4y^3 dy) \wedge (2xy^2 dx + 2x^2 y dy) = t_1 t_2 (8x^5 y - 8xy^5) \Omega,$$

which is $(W^{\max} - W^{\text{tr}}) \Omega$ by (91) and (62). Put $\delta = W^{\max} - W^{\text{tr}} \in t_1 t_2 \mathbb{Q}[x, y]$. Because $W^{\text{tr}} - W_0$ lies in (t_1, t_2) and $t_1 t_2(t_1, t_2) = 0$,

$$e^{W^{\max}/\hbar} \Omega - e^{W^{\text{tr}}/\hbar} \Omega = \hbar^{-1} e^{W_0/\hbar} \delta \Omega = \hbar^{-1} e^{W_0/\hbar} \nabla_0 \beta = d(e^{W_0/\hbar} \beta).$$

Under the stated growth conditions both integrals converge absolutely. By the coarea formula there is a sequence $\rho_n \rightarrow \infty$ of regular values of $|z|$ on $\mathcal{S}$ and on $\partial \mathcal{S}$ for which the length of $\mathcal{S} \cap \{|z| = \rho_n\}$ grows at most polynomially in ρ_n . By Stokes' theorem the integral of $d(e^{W_0/\hbar} \beta)$ over $\mathcal{S} \cap \{|z| \leq \rho_n\}$ equals the integral of $e^{W_0/\hbar} \beta$ over $\partial \mathcal{S} \cap \{|z| \leq \rho_n\}$, which vanishes because β is relative, plus an integral over $\mathcal{S} \cap \{|z| = \rho_n\}$, which tends to zero because $e^{W_0/\hbar}$ decays faster than the polynomial form β and the length grow. $\square$

The class in Proposition 7.4(1) vanishes exactly when $a + b = 12$. This is the GKT chamber condition for the support $\{i, j\}$, which is a period condition [GKT26a, Def. 3.47(3) and Thm. 4.9]. The periods of $V(a, b)$ depend only on $a + b$, and potentials in one orbit of the flows Z_c have the same periods. Selecting the extremal representative within an orbit therefore requires a normal-form rule; for these two labels, the rule producing $W^{\max}$ is $\mathcal{N}^{\max}$.

7.3. Two labels of the same twist. The same phenomenon occurs without a diagonal wall. Take two labels with parameters t, t' , both of twist $(1, 0)$ and descendent degree one, and work modulo (t^2, t'^2) . Their singleton walls lie on the same chord and commute, so modulo (t^2, t'^2)

the wall structure $\mathfrak{D}$ has no further wall, and every path crossing the chord in the direction of increasing slope has path-ordered product

$$\Theta' = \exp\left(-\frac{1}{2}(t+t')X_{1,0}\right).$$

The support of the two labels has balanced profiles $(2, 4)$ and $(6, 0)$ and critical exponent $(2, 0)$. Each label has the twist and descendent degree of the label i , so the singleton terms are those of t_1 in (62), and the extremal GKT potentials are

$$x^4 + y^4 - 4(t+t')xy^4 + ptt'x^2y^4 \quad \text{and} \quad x^4 + y^4 - 2(t+t')x^5 + qtt'x^6$$

for some $p, q \in \mathbb{Q}$. As in Proposition 7.4(1), the GKT chamber condition for the support of the two labels is the vanishing of the class of the tt' -part of $e^{(W-W_0)/\hbar} \Omega$ for each extremal potential W . Write $\sim$ for ∇_0 -cohomology. The formulas for $\nabla_0(f dx)$ with $f = x^2y$, x^2y^5 and for $\nabla_0(f dy)$ with $f = x^3$, x^7 in the proof of Proposition 7.4 give

$$x^2y^4 \Omega \sim -\frac{\hbar}{4} x^2 \Omega, \quad x^2y^8 \Omega \sim \frac{5\hbar^2}{16} x^2 \Omega, \quad x^6 \Omega \sim -\frac{3\hbar}{4} x^2 \Omega, \quad x^{10} \Omega \sim \frac{21\hbar^2}{16} x^2 \Omega.$$

Since $(t+t')^2 = 2tt'$ modulo (t^2, t'^2) , the tt' -parts for the two potentials are $(\hbar^{-1}p x^2y^4 + 16\hbar^{-2}x^2y^8) \Omega$ and $(\hbar^{-1}q x^6 + 4\hbar^{-2}x^{10}) \Omega$, which are ∇_0 -cohomologous to $(5 - \frac{p}{4}) x^2 \Omega$ and $(\frac{21}{4} - \frac{3q}{4}) x^2 \Omega$. The chamber conditions therefore give $p = 20$ and $q = 7$, and the extremal potentials are

$$W'^{\min} = x^4 + y^4 - 4(t+t')xy^4 + 20tt'x^2y^4, \quad W'^{\max} = x^4 + y^4 - 2(t+t')x^5 + 7tt'x^6.$$

Proposition 7.5 (Same twist). *The wall structure $\mathfrak{D}$ carries $W'^{\min}$ to*

$$\Theta' W'^{\min} = x^4 + y^4 - 2(t+t')x^5 + tt'(6x^2y^4 + 5x^6),$$

and the GKT transition is $\Theta' \exp(\frac{1}{2}tt'X_{2,0})$. Its extra factor lies on the ray of shifted exponent $(3, 1)$, which lies outside the cone spanned by $(2, 1)$ and $(1, 2)$ and carries no wall of $\mathfrak{D}$. The flows $\exp(ctt'X_{2,0})$ change the coefficients (p, q) of $tt'x^2y^4$ and $tt'x^6$ to $(p - 12c, q + 4c)$, and the normal form without an x^2y^4 term carries $\Theta' W'^{\min}$ to $W'^{\max}$. If the coefficient 20 is replaced by n , the transported potential has coefficients $(n - 14, 5)$ and its normal form has coefficient $(n + 1)/3$ on $tt'x^6$.

Proof. Modulo (t^2, t'^2) we have $(t+t')^2 = 2tt'$ and $\Theta' = 1 - \frac{1}{2}(t+t')X_{1,0} + \frac{1}{4}tt'X_{1,0}^2$. With $X_{1,0}(xy^4) = -7x^2y^4$ and $X_{1,0}(x^5) = 5x^6$, the three terms of Θ' applied to $W'^{\min}$ give the displayed potential: the coefficient of $tt'x^2y^4$ is $20 - 28 + 14 = 6$ and that of $tt'x^6$ is 5; with n in place of 20 the first becomes $n - 14$. Since $X_{2,0}(x^4 + y^4) = 4x^6 - 12x^2y^4$, the flow $\exp(ctt'X_{2,0})$ changes (p, q) to $(p - 12c, q + 4c)$, and $c = \frac{1}{2}$ carries $(6, 5)$ to $(0, 7)$; in general $c = p/12$ gives $(0, q + p/3)$, which is $(0, (n + 1)/3)$ for $(p, q) = (n - 14, 5)$. The flow of $tt'X_{2,0}$ commutes with Θ' modulo (t^2, t'^2) . The shifted exponent $(3, 1) = c_1(2, 1) + c_2(1, 2)$ has $c_2 = -\frac{1}{3} < 0$. $\square$

The path γ_+ meets the ray $L(3, 1)$ before the $(2, 1)$ -chord. In the comparison of Section 5 this ray has wall factor 1 and residual equal to the inverse of the factor of the GKT transition.

7.4. Finitely many labels.

Proposition 7.6 (Maximal normal form). *Let $\mathcal{W}$ be a labelled truncation (Definition 4.1) whose monomial set consists of square-free monomials, so that $R_{\mathcal{W}, N}$ is a quotient of $A_{\mathcal{I}} \otimes_{\mathbb{Q}} \mathbb{k}$, with $A_{\mathcal{I}}$ the labelled ring (56), under $u_{i,1} \mapsto t_i$ and $u_{i,d} \mapsto 0$ for $d \neq 1$. For a support J , the balanced string of J consists of the coefficients of the monomials $t_J x^{k_1} y^{k_2}$ with (k_1, k_2) one of the balanced profiles $b_{J,h}$ of (67), which form an arithmetic progression with step $(4, -4)$. Let $W^{\min}$ and $W^{\max}$ denote the images over $R_{\mathcal{W}, N}$ of the extremal GKT potentials of the labels of $\mathcal{W}$.*

- (1) Let $V = x^4 + y^4 + \dots$ be a potential over $R_{\mathcal{W},N}$ supported on balanced monomials. The orbit of V under the group of the GKT critical Lie algebra of $\mathcal{W}$ contains exactly one potential whose coefficients on the balanced string of each support J vanish except at $b_J^{\max}$. We call it the maximal normal form $\mathcal{N}^{\max}V$ of V .
- (2) On the family $V(a, b)$, $\mathcal{N}^{\max}$ is the operation (92); for the two labels of twist $(1, 0)$ of Proposition 7.5 it is the normal form used there.
- (3) The path-ordered product of $\mathfrak{D}$ along a path in a disk about the joint O_0 contained in the sheet S_0 , such as γ_+ , carries $W^{\min}$ to the potential of a chamber index, and $\mathcal{N}^{\max}$ carries this potential to $W^{\max}$.

Proof. (1) By (75), the generator $X_{k,J,h}$ carries $x^4 + y^4$ to a combination of the monomials of the adjacent profiles $b_{J,h+1}$ and $b_{J,h}$ with nonzero coefficients. Modulo the contributions of proper divisors of t_J , the generators of support J therefore act on the balanced string of J through a bidiagonal incidence matrix with nonzero entries; its image has codimension one and meets the line of $b_J^{\max}$ only in zero, so it is a complement of that line. Hence the normal form exists and is unique; it is constructed support by support, by induction on $|J|$, as in [Wue26b, Thm. 1.1].

(2) The balanced profiles $b_J^{\max}$ of the supports $\{i\}$, $\{j\}$ and $\{i, j\}$ are $(5, 0)$, $(4, 1)$ and $(5, 1)$. The potential $V(0, a + b) = Z_{a/8}V(a, b)$ lies in the orbit of $V(a, b)$, and its coefficients on the balanced strings vanish except at these profiles, so it is $\mathcal{N}^{\max}V(a, b)$ by (1). For the two labels of Proposition 7.5 the profiles $b_J^{\max}$ of the singletons and of the pair are $(5, 0)$ and $(6, 0)$. The normal form used there is obtained by a flow $\exp(ctt'X_{2,0})$ of a critical generator, keeps the singleton terms $-2(t + t')x^5$ and has no $tt'x^2y^4$ term, so it is $\mathcal{N}^{\max}$ as well.

(3) Such a disk meets only supports through O_0 , since the supports through O_1 lie in S_1 (Proposition 3.6). The factors on the supports through O_0 are exponentials of elements of the GKT critical Lie algebra of $\mathcal{W}$, so the path-ordered product along such a path lies in its group and carries $W^{\min}$ to the potential of a chamber index [GKT26a, Thm. 4.27]. The maximal normal form of this potential lies in the same orbit, so it is again the potential of a chamber index, and its only nonzero balanced coefficients of nonempty support are those at the profiles $b_J^{\max}$. The only such chamber index is $\nu^{\max}$ [Wue26b, Thm. 1.1], [GKT26a, Cor. 5.14], so $\mathcal{N}^{\max}$ carries the transported $W^{\min}$ to $W^{\max}$. $\square$

Propositions 7.1, 7.3 and 7.5 make this correction explicit for two labels of twists $(1, 0)$ and $(0, 1)$ and for two labels of twist $(1, 0)$.

Problem 7.7 (Maximal normalization). Let γ be a path from o_- to a region of $\mathfrak{D}$ adjacent to a boundary circle of the annulus. Construct, from the annular structure of Theorem A, an operation $\mathcal{E}_\gamma$ on potentials in the chart at the end of γ such that, with $\mathcal{E}_\gamma$ read in the frame at o_- by transport along γ , the operation $\Theta_\gamma^{-1}\mathcal{E}_\gamma\Theta_\gamma$ agrees with $\mathcal{N}^{\max}$ on every truncation of Proposition 7.6. For an intrinsic enumerative construction, the construction should moreover single out the coefficients of $W^{\min}$, for instance the value $m = 12$ in Proposition 7.3, without using the GKT chamber conditions of the supports involved.

The comparisons above are local and are made in one frame at O : the regions containing o_- and o_+ are adjacent to the joint, not to the boundary circles, and the computations see neither the core holonomy K nor the transitions at the singular points, so a solution must be global. A comparison theorem could also take the GKT canonical systems as input, provided it identifies the boundary evaluation with the normal form by a geometric argument. Theta functions defined by broken lines on the annulus, and a logarithmic compactification of the finite levels (Section 4), are further open problems. The comparison between annular scattering and GKT open FJRW theory proved here, Theorem B, concerns wall factors and finite homotopies; Proposition 6.3 transports the primary Frobenius structure through the annular charts.

APPENDIX A. AFFINE AND LOGARITHMIC CALCULATIONS

This appendix collects the basis conventions and local algebra used in the main proofs. It records the lattice data, the widths of the monodromies and the separation estimate in explicit coordinates. The conventions for the two sides of the discrete Legendre transform are those of Section 2.3: affine charts and developments are in the plane $M^\vee \otimes \mathbb{R}$ of Δ , and monodromies of characters act on the fan lattice M . Elements of M and of $M^\vee$ are written in standard coordinates, which on $M^\vee \otimes \mathbb{R}$ are the coordinates (s_1, s_2) of (3); elements of $\widetilde{M}$ and $\widetilde{M}^\vee$ are written in F -coordinates (Section 2.2).

A.1. Straight-boundary charts. At a vertex p_V of Δ , let ℓ_V, r_V be the primitive boundary tangents toward its two neighbors, and let $-p_V$ be the inward radial vector. Both $(\ell_V, -p_V)$ and $(r_V, -p_V)$ are unimodular. Define the left and right straight-boundary charts by

$$\mathcal{B}_{V,L}\ell_V = (-1, 0), \quad \mathcal{B}_{V,L}(-p_V) = (0, 1), \quad \mathcal{B}_{V,R}r_V = (1, 0), \quad \mathcal{B}_{V,R}(-p_V) = (0, 1).$$

The resulting matrices in the standard basis of $M^\vee$ are

$$(93) \quad \begin{array}{c|cc} V & \mathcal{B}_{V,L} & \mathcal{B}_{V,R} \\ \hline A & \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\ B & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \\ C & \begin{pmatrix} -1 & -3 \\ -1 & -2 \end{pmatrix} & \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix} \end{array}.$$

The matrix $\tilde{T}_V = \mathcal{B}_{V,R}^{-1}\mathcal{B}_{V,L}$ is the monodromy of the tangent lattice $M^\vee$ of Δ about V ; it fixes p_V . Along a loop the dual lattice M has the inverse transpose monodromy. We orient the loop γ_V so that the monodromy of M along it is $T_V = \tilde{T}_V^T$, that is, oppositely to the loop along which the tangent monodromy is $\tilde{T}_V$. Direct multiplication yields precisely the matrices (16). Their invariant primitive characters are, in standard coordinates of M , $\bar{u}_A = (1, -1)$, $\bar{u}_B = (1, 1)$, $\bar{u}_C = (1, 3)$, and

$$(94) \quad \begin{aligned} T_A - \text{Id} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} (1, 1) = -\bar{u}_A n_A^T, \\ T_B - \text{Id} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (2, -2) = -2 \bar{u}_B n_B^T, \\ T_C - \text{Id} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} (3, -1) = -\bar{u}_C n_C^T, \end{aligned}$$

with the primitive conormals $n_A = (1, 1)$, $n_B = (-1, 1)$, $n_C = (-3, 1)$ in $M^\vee$, in the coordinates (s_1, s_2) . The conormals of (22) are $N_{\tilde{A}} = 4n_A$, $N_{B_0} = 4n_B$ and $N_{\tilde{C}} = 4n_C$: the F -coordinates of $4n_A$, $4n_B$, $4n_C$ are $(1, -3)$, $(-1, -1)$, $(-3, 1)$. Thus the monodromies of the disk about A , B and C have widths 1, 2 and 1.

In standard coordinates the exponent map and its inverse are

$$F = \begin{pmatrix} 1/4 & -1/4 \\ 0 & -1/2 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 4 & -2 \\ 0 & -2 \end{pmatrix}.$$

Conjugating by F gives the disk monodromies in F -coordinates:

$$(95) \quad \begin{aligned} F^{-1}T_A F &= \begin{pmatrix} -1/2 & 9/2 \\ -1/2 & 5/2 \end{pmatrix}, \\ F^{-1}T_B F &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \\ F^{-1}T_C F &= \begin{pmatrix} -1/2 & 1/2 \\ -9/2 & 5/2 \end{pmatrix}. \end{aligned}$$

The loops whose monodromy preserves $\widetilde{M}$ form a subgroup that contains the parity subgroup, by (96) below, and does not contain γ_A . Since the parity subgroup has index two, it is exactly the stabilizer of $\widetilde{M}$; in particular the monodromies about A and C do not preserve $\widetilde{M}$.

For a loop γ in the parity subgroup write H_γ for the monodromy of $\widetilde{M}$ along γ , with the convention $H_{gh} = H_g H_h$, where a product gh traverses g first. For the Schreier transversal $\{1, \gamma_A\}$, the Reidemeister–Schreier basis of the parity subgroup is

$$\gamma_B, \quad \gamma_C \gamma_A^{-1}, \quad \gamma_A^2, \quad \gamma_A \gamma_B \gamma_A^{-1}, \quad \gamma_A \gamma_C.$$

The matrices of these loops, in F -coordinates, are

$$(96) \quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ -10 & 19 \end{pmatrix}, \quad \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -14 & 25 \\ -9 & 16 \end{pmatrix}, \quad \begin{pmatrix} -20 & 11 \\ -11 & 6 \end{pmatrix}.$$

All are integral. The first is $H_{B,0}$; the second is the core matrix $K = H_{\gamma_{\text{core}}}$ of the loop $\gamma_{\text{core}} = \gamma_C \gamma_A^{-1}$; the third and fourth are H_A and $H_{B,1}$; the fifth is $H_{\gamma_A \gamma_C} = K^{-1} H_C$, of trace -14 . Squaring the loop γ_C gives $H_C = H_{\gamma_C^2} = H_{\gamma_C \gamma_A^{-1}} H_{\gamma_A \gamma_C}$, the final local matrix in (18). The primitive factorizations (19) follow by subtracting the identity.

The kinks of φ across the bounded edges of the decomposition of the dual intersection complex are, up to sign, the vertices of Δ . Equivalently, they are the primitive facet conormals of $\Delta^\vee$, which in the coordinates (s_1, s_2) are $-p_A = (1, 1)$, $-p_B = (1, -1)$ and $-p_C = (-3, 1)$, each taking the value 1 on its facet. Their F -coordinates are

$$(97) \quad F^T(1, 1) = (\tfrac{1}{4}, -\tfrac{3}{4}), \quad F^T(1, -1) = (\tfrac{1}{4}, \tfrac{1}{4}), \quad F^T(-3, 1) = (-\tfrac{3}{4}, \tfrac{1}{4}).$$

For the polarization of the discrete Legendre transform (Section 2.3), the kinks across the unbounded edges are the edge vectors of Δ , which are $(0, 2)$, $(4, -2)$ and $(-4, 0)$ in the coordinates (s_1, s_2) and have the integral F -coordinates $(0, -1)$, $(1, 0)$ and $(-1, 1)$ in the chart of S_0 ; in the frame of S_1 two of them are half-integral (proof of Theorem 2.3). Thus $p^* \varphi$ is quarter-integral and $4p^* \varphi$ is the minimal integral multiple.

A.2. Core and boundary holonomy. The core matrix satisfies

$$K - \text{Id} = \begin{pmatrix} -2 & 2 \\ -10 & 18 \end{pmatrix}.$$

The lattice M is generated by $g_1 = F(4, 0)$ and $g_2 = F(-2, -2)$. The columns of $K - \text{Id}$ are the F -coordinates of $(K - \text{Id})F(1, 0) = 2g_1 + 5g_2$ and $(K - \text{Id})F(0, 1) = -4g_1 - 9g_2$, so K acts trivially on $\widetilde{M}/M$. It has trace 18 and determinant one, which proves the spectral statements in Section 3.

The two boundary circles are the lifts of the boundary loop $\gamma_A \gamma_B \gamma_C$, whose monodromy on M is, in standard coordinates, $T_A T_B T_C = \begin{pmatrix} 1 & 0 \\ -8 & 1 \end{pmatrix}$. Their monodromies on $\widetilde{M}$ are, in

F -coordinates,

$$(98) \quad U_0 = H_{\gamma_A \gamma_B \gamma_C} = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}, \quad U_1 = H_{\gamma_A^2 \gamma_B \gamma_C \gamma_A^{-1}} = \begin{pmatrix} 33 & -64 \\ 16 & -31 \end{pmatrix},$$

and $U_0 = H_{B,1} K^{-1} H_C$, $U_1 = H_A H_{B,0} K$. They factor as

$$U_0 - \text{Id} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1, -1), \quad U_1 - \text{Id} = 16 \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1, -2).$$

In the invariant tangent–normal bases given by the columns of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ for } U_0, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ for } U_1,$$

the matrices of U_0 and U_1 are

$$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -16 \\ 0 & 1 \end{pmatrix}.$$

This gives the two global widths used in Section 4.

On the side of Δ , the invariant line of $\tilde{T}_V$ is the line through O and p_V , which contains V . In coordinates centered at O the affine transition at each singular point is therefore linear, and so is the affine holonomy of every loop. In the coordinates (s_1, s_2) , as in Section 3.3, the branch segment joins $A = (-\frac{1}{2}, -\frac{1}{2})$ to $C = (\frac{3}{2}, -\frac{1}{2})$ and crosses the negative half-chords at $(\frac{2}{3}, -\frac{1}{2})$ and $(\frac{1}{3}, -\frac{1}{2})$. Its interior lies in the complement of $c_A \cup c_B \cup c_C$, so its lifts lie in the sheets S_0, S_1 of Section 2.4.

A.3. Boundary development and exit slopes. In the chart of the sheet S_0 one has $\widetilde{M}^\vee = F^{-T}(\mathbb{Z}^2) = 2L(\mathbb{Z}_{x,y}^2)$. The development below is written in the plane of Δ , in the invariant tangent–normal coordinates (x, y) of (53), normalized so that $\widetilde{M}^\vee$ is $2\mathbb{Z}^2$, equivalently so that the lattice $L(\mathbb{Z}_{x,y}^2)$ of cofactor directions is $\mathbb{Z}^2$. The development data of the two boundary circles are

$$(99) \quad \begin{array}{c|c|c|c} i & \ell_i & w_i & (\phi_i, h_i) \\ \hline 0 & 4 & 4 & (7/2, 1) \\ 1 & 8 & 16 & (15/2, 1/2). \end{array}$$

Here w_i is the width of the boundary monodromy, $\ell_i = w_i h_i$ is the length of the boundary circle in these coordinates, and (ϕ_i, h_i) is the common apex of the developed facets, in the coordinates (x, y) . For $i = 0$ the data are obtained as follows. Starting from the facet $s_2 = -1$ in the chart of S_0 , develop the facet $s_1 + 2s_2 = 1$ through the vertex p_C and then the facet $s_1 = -1$ through p_B , by the transitions between the straight-boundary charts (93) at these vertices, which fix the vertex and O . The three facets develop onto the line $s_2 = -1$, and the vertex $p_A = (-1, -1)$ returns at $(7, -1)$. In the coordinates $x = (7 - s_1)/2$ and $y = s_2 + 1$, in which $\widetilde{M}^\vee$ is $2\mathbb{Z}^2$, the developed image of O lies at $(\frac{7}{2}, 1)$ and the boundary circle has length 4. For $\partial_1 \overline{A}$ one starts from the facet $s_2 = -1$ in the chart of S_1 pulled back from Δ , in which $\widetilde{M}^\vee$ is $2\mathbb{Z} \oplus 4\mathbb{Z}$ in the coordinates (s_1, s_2) , develops in the same way, and uses $x = 7 - s_1 + \frac{1}{2}(s_2 + 1)$ and $y = \frac{1}{2}(s_2 + 1)$, in which $\widetilde{M}^\vee$ is $2\mathbb{Z}^2$; the developed image of O lies at $(\frac{15}{2}, \frac{1}{2})$, and the boundary circle has length 8. Affine lengths measured in $\widetilde{M}^\vee$ are half the coordinate lengths; the widths and the slopes below do not depend on this normalization. In the coordinate $r = -y$, a support of slope ξ lies on the line $x = \phi_i - \xi(r + h_i)$ of (54). On each line $\{r = r_0\}$ the holonomy (53) is the translation by $w_i(r_0 + h_i)$, so its k th power maps the line of slope ξ to the line of slope $\xi - kw_i$.

By Proposition 3.6 no support meets an arc carrying a transition, so each support has direction Lq , or $-Lq$ for a negative half-chord, in the chart of its sheet pulled back from Δ . The supports through O_1 are the deck images of those through O_0 ; the frame of S_1 of Section 2.4 differs from the identification given by the chart of S_1 pulled back from Δ by the affine monodromy about A , and in the frame of S_1 their directions are the images of $\pm Lq$ under this monodromy. The development below uses the pulled-back charts. Write $q = c_1(2, 1) + c_2(1, 2)$ with $c_1, c_2 \geq 0$, as in (44). A support of direction $-Lq$ leaves Δ through the facet $s_2 = -1$, and one of direction Lq through the facet $s_1 = -1$ if $c_1 \geq c_2$ and through $s_1 + 2s_2 = 1$ if $c_1 \leq c_2$. Developing the three facets along $\partial_0 \bar{A}$ into the coordinates above gives the slopes

$$\xi_- = \frac{2c_1 + c_2}{3(c_1 + c_2)}, \quad \xi_+ = \begin{cases} \frac{3(7c_1 + 3c_2)}{4(2c_1 + c_2)}, & c_1 \geq c_2, \\ \frac{11c_1 + 19c_2}{4(c_1 + 2c_2)}, & c_1 \leq c_2, \end{cases}$$

for the directions $-Lq$ and Lq ; along $\partial_1 \bar{A}$ the slopes are $4\xi_{\pm} + 1$. Each slope is read off from the developed point at which the support meets its facet. For instance, a support of direction $-Lq$ meets $s_2 = -1$ at $s_1 = (4c_1 + 2c_2)/(3(c_1 + c_2))$, and $x = \phi_0 - \xi_- h_0$ there gives ξ_- ; for the direction Lq the meeting point on $s_1 + 2s_2 = 1$ is developed through p_C , and that on $s_1 = -1$ through p_C and p_B , as above. All of these slopes are positive numbers. Each of the functions above is decreasing in $c_2/c_1 \in [0, \infty]$, and the two expressions for ξ_+ agree at $c_1 = c_2$. The slopes of the supports of direction $-Lq$, respectively Lq , therefore lie in the closed intervals bounded by their values at $c_2 = 0$ and at $c_1 = 0$:

i	direction $-Lq$	direction Lq
0	$[1/3, 2/3]$	$[19/8, 21/8]$
1	$[7/3, 11/3]$	$[21/2, 23/2]$

(100)

For $i = 0$, the slope of a support of direction Lq exceeds that of a support of direction $-Lq$ by an amount in $(0, 4)$, and for $i = 1$ by an amount in $(0, 16)$; each interval has width smaller than w_i . Thus the slopes of two supports never differ by a nonzero multiple of w_i , and no nonzero power of the holonomy (53) maps a support onto a support. These are the inequalities used in the proof of Proposition 4.3.

A.4. Finite root algebra. For an effective Cartier divisor $D_f = (f)$ on $X = \text{Spec } A$, we call $\text{Spec } A[g]/(g^d - f)$ the *Kummer cover* of (X, f) , with μ_d acting by $g \mapsto \zeta g$ and trivially on A . Our root-stack convention is

$$(101) \quad \sqrt[d]{(X, D_f)} = \left[\text{Spec } A[g]/(g^d - f) / \mu_d \right].$$

Over $X \setminus D_f$ the Kummer cover is the μ_d -torsor $\text{Spec } A[f^{-1}][g]/(g^d - f)$, finite étale because d is invertible in $\mathbb{k}$, and the root stack is $X \setminus D_f$ itself: its functions there are the invariants $A[f^{-1}]$, and g , of weight one, is not among them. Along D_f the quotient retains the root automorphism group, and $g = 0$ defines the root divisor. If A is a character algebra and N an integral conormal vanishing on the exponents of f , the slab map $z^m \mapsto g^{\langle N, m \rangle} z^m$ is an automorphism of the Kummer ring $A[f^{-1}][g]/(g^d - f)$ fixing f and g . It commutes with the action of μ_d on z^m exactly when d divides $\langle N, m \rangle$. It therefore acts on the Kummer cover over $X \setminus D_f$, and it descends to the root stack only if d divides $\langle N, m \rangle$ for every character m . For finitely many divisors $D_e = (f_e)$, with degrees d_e , the fiber product over X of the root stacks $\sqrt[d_e]{(X, D_e)}$ is the quotient of the Kummer cover $\text{Spec } A[g_e]/(g_e^{d_e} - f_e)$ by $\prod_e \mu_{d_e}$, and the stabilizer of a point of this cover is the product of the factors μ_{d_e} whose coordinate g_e vanishes

at the point, that is, of the factors with $x \in D_e$ for the image x of the point in X . Section 4 applies this to the finite root stacks $\mathfrak{X}_T$ of Section 2.

Let $\mathfrak{J}$ be the filtered poset of finite rooted subtrees of the path tree $\mathcal{T}$. For $T \subset T'$ in $\mathfrak{J}$, the ring $R_{T'}^{\text{root}}$ of (27) is free over $\mathcal{O}(U_{T'})$ on the monomials in the roots with exponents smaller than their degrees, and these contain the monomials of T ; hence $R_T^{\text{root}} \rightarrow R_{T'}^{\text{root}}$ is injective. The colimit over $\mathfrak{J}$ is the pathwise Kummer ring $R_{\text{path}}^{\text{root}}$. A finitely presented module, algebra, morphism, or finite collection of identities among composites over $R_{\text{path}}^{\text{root}} \otimes_{\mathbb{k}} R/\mathfrak{m}^N$ uses finitely many generators, relations, and matrix entries. All therefore occur over $R_T^{\text{root}} \otimes_{\mathbb{k}} R/\mathfrak{m}^N$ for one $T \in \mathfrak{J}$, after replacing the chosen subtrees by their rooted union. This proves the finite-presentation assertion in Theorem 2.7.

APPENDIX B. COMPATIBLE COMPLETION OF GKT HOMOTOPIES

This appendix proves Theorem 5.7, the compatibility statement in Part (3) of Theorem B. The labels and monomials of a labelled truncation (Definition 4.1) do not by themselves define a restriction map on canonical families: by the assembling relation (57), the multisection on one graph datum is prescribed on the boundary of its moduli space by the multisections on the connected components of the bases of its boundary graphs. We therefore restrict along finite systems of graph data closed under this relation, the finite assembling-closed systems of Definition B.1.

We use the notions of [GKT26a] recalled in Subsection 5.1. Fix a countable set I_∞ of labels with a twist map, containing infinitely many labels of each twist. Every construction below uses only a finite subset. The relative extension argument is the multisection argument of [BCT24, Lem. 6.5] in its W -spin form [GKT26a, Lem. 5.15]; these statements, the homotopy lemma [GKT26a, Lem. 5.17], and the part of the proof of [GKT26a, Thm. 5.13] that we use are recalled in Subsection B.2. The lemma of GKT that provides simple canonical homotopies with assigned times, [GKT26a, Lem. 3.40], is recalled below together with the parts of its proof that we use. From [Wue26b] we use only the extremal chamber indices [Wue26b, Thm. 1.1], the freeness of the enlarged action [Wue26b, Lem. 2.5], and the slope factorization [Wue26b, Prop. 3.1].

B.1. Finite assembling-closed systems. For a graded open $(4, 4)$ -graph Γ with finite internal label set and descendent vector $\mathbf{d}$, write

$$(102) \quad \delta = (\Gamma, \mathbf{d}), \quad E_\delta = E_\Gamma(\mathbf{d}) \longrightarrow \mathcal{PM}_\Gamma, \quad \text{supp}(\delta) = u_{I(\Gamma), \mathbf{d}}.$$

We call δ a *graph datum*. A family of canonical multisections satisfies the assembling relation (57) on every zero or exchange boundary stratum Λ of Γ . The *assembling constituents* of δ are the connected component data of $B\Gamma$ and of all $B\Lambda$ occurring there, with restricted descendent vectors; by (57), their multisections prescribe the multisection of δ on the boundary. The *assembling closure* of a set of graph data consists of these data and their iterated assembling constituents.

Definition B.1. A *finite assembling-closed system* is a triple $\mathbb{D} = (I, \mathcal{P}, \mathcal{E})$ such that:

- (1) $I \subset I_\infty$ is finite and twist labelled; $\mathcal{P}$ is a finite divisor-closed set of labelled monomials, containing 1 and invariant under twist-preserving permutations of I ;
- (2) $\mathcal{E}$ is a finite set of graph data containing every iterated assembling constituent and connected component datum of each of its elements;
- (3) $\text{supp}(\delta) \in \mathcal{P}$ for every $\delta \in \mathcal{E}$;
- (4) if a support $u \in \mathcal{P}$ carries a balanced graph, then $\mathcal{E}$ contains every graph datum of support u whose graph is balanced or critical, together with its assembling closure;

(5) $\mathcal{E}$ is invariant under the twist-preserving permutation group of I .

An inclusion $\mathbb{D} \subset \mathbb{D}'$ means inclusion of the label sets, of the monomial sets and of the sets of graph data. Restriction is therefore the forgetful map.

In Theorems B.4 and B.5 below, the smaller system may also be the *empty system*, with no labels, monomials or graph data; then there are no old data, and the two theorems are [GKT26a, Thm. 5.13] and the construction of [GKT26a, Lem. 3.40], with the time collars and the corner modification described below.

Lemma B.2 (Finite closure). *The assembling closure of a finite set of graph data, together with all balanced and critical graph data of finitely many supports, is finite. Moreover, if δ' lies in the assembling closure of δ , then*

$$(103) \quad \text{supp}(\delta') \mid \text{supp}(\delta).$$

Proof. An assembling constituent uses a subset of the internal labels and the restricted descendent vector, and iteration proves (103). Order connected graphs lexicographically by

$$(|I(\Gamma)|, n(\Gamma), -k_{12}(\Gamma)), \quad n = 4k_1 + 4k_2 + 8k_{12}.$$

Every proper connected base component precedes the original graph. There is no infinite descending chain, and every graph has finitely many boundary types. Thus the finitely branching tree of iterated assembling constituents is finite. A support has finitely many balanced and critical graph data [GKT26a, Notation 2.40], and permutation orbits are finite. $\square$

Let $\mathcal{C}(\mathbb{D})$ be the collections of transverse, strongly positive, automorphism-invariant canonical multisections on $\mathcal{E}$ satisfying (57). Independent choices are made only on graph data whose graph is smooth and connected; the other data are the pullbacks and external sums prescribed by GKT. A chamber index, or a family of canonical multisections, is *symmetric* in the sense of [GKT26a, Notation 3.42 and Def. 3.43] if it is invariant under the twist-preserving permutations of the labels, which act on families through their natural lifts to the moduli spaces.

We write $t \in [0, 1]$ for the time of a homotopy. For $r = s = 4$, the lemma of GKT on simple canonical homotopies reads as follows; for a graph Γ and a descendent vector $\mathbf{d}$ we abbreviate $\text{rk } \Gamma = \text{rank } E_\Gamma(\mathbf{d})$ and $\dim \Gamma = \dim \mathcal{PM}_\Gamma$.

Theorem ([GKT26a, Lem. 3.40]). Fix a set I of labels, and let $\mathfrak{s}^0$ and $\mathfrak{s}^1$ be families of transverse canonical multisections bounded by I . Then there are a family H of simple canonical homotopies from $\mathfrak{s}^0$ to $\mathfrak{s}^1$, bounded by I , and pairwise distinct times $t(A) \in (0, 1)$, one for each $A \subseteq I$, with the following properties for every smooth graph Γ with only open vertices and $I(\Gamma) \subseteq I$ and every descendent vector $\mathbf{d}$.

- (1) The homotopy H^Γ on Γ is nowhere zero if (a) Γ is irrelevant; or (b) Γ is connected and $\text{rk } \Gamma \geq \dim \Gamma + 2$; or (c) Γ contains two connected components Ξ_1, Ξ_2 with $I(\Xi_1) \neq I(\Xi_2)$ such that $\text{rk } \Xi_j = \dim \Xi_j + 1$ and $k_{12}(\Xi_j) \leq 2$ for $j = 1, 2$.
- (2) If Γ is connected and either (a) $\text{rk } \Gamma = \dim \Gamma + 1$ and $k_{12}(\Gamma) \leq 2$, or (b) $\text{rk } \Gamma = \dim \Gamma$ and $k_{12}(\Gamma) \leq 1$, then H^Γ is transverse to zero on $[0, 1] \times \mathcal{PM}_\Gamma$ and on $[0, 1] \times \mathcal{PM}_\Lambda$ for every $\Lambda \in \partial^0 \Gamma \cup \partial^{\text{exchange}} \Gamma$.
- (3) Let Γ be connected. (a) If $\text{rk } \Gamma = \dim \Gamma + 1$ and $k_{12}(\Gamma) \leq 2$, then H^Γ is nowhere zero on $([0, 1] \setminus \{t(I(\Gamma))\}) \times \mathcal{PM}_\Gamma$. (b) If $\text{rk } \Gamma = \dim \Gamma$ and $k_{12}(\Gamma) \leq 1$, then for every $A \subseteq I$ the restriction of H^Γ to $\{t(A)\} \times (\mathcal{PM}_\Gamma \setminus \partial \mathcal{PM}_\Gamma)$ is transverse to zero.

The proof constructs H^Γ for smooth connected graphs by induction on $\dim \Gamma$; on disconnected graphs the homotopy is determined by simplicity, and (1)(c) follows from (3)(a). The boundary values are, on a stratum Λ with a contracted boundary half-edge, the linear homotopy

$(1 - \mathfrak{t})s_\Lambda^0 + \mathfrak{t}s_\Lambda^1$ between the components s_Λ^0 and s_Λ^1 of $\mathfrak{s}^0$ and $\mathfrak{s}^1$, and on the other strata the assembled homotopy $F_\Lambda^*(\boxplus_\Xi H^\Xi)$ over the connected components Ξ of $B\Lambda$. On irrelevant boundary strata the boundary values are nowhere zero, inductively by (1)(a). For a relevant connected graph Γ the proof distinguishes three cases.

- (a) If $\text{rk } \Gamma \geq \dim \Gamma + 2$, the boundary values are nowhere zero, and a transverse extension is nowhere zero.
- (b) If $\text{rk } \Gamma = \dim \Gamma + 1$ and $k_{12}(\Gamma) \leq 2$, a dimension count shows that the base of every relevant boundary stratum has a component Ξ with $\text{rk } \Xi \geq \dim \Xi + 2$, or two components Ξ with $\text{rk } \Xi = \dim \Xi + 1$, distinct label sets and $k_{12}(\Xi) \leq 2$, so that the boundary values are nowhere zero by (1)(b) and (1)(c) for the base. A transverse extension has finitely many zeros; after a small perturbation no two of them lie over the same point of $\mathcal{PM}_\Gamma$, and a diffeomorphism of $[0, 1] \times \mathcal{PM}_\Gamma$ that commutes with the projection to $\mathcal{PM}_\Gamma$ and is the identity on $[0, 1] \times \partial\mathcal{PM}_\Gamma$ and near the positive boundary moves them to the time $\mathfrak{t}(I(\Gamma))$.
- (c) If $\text{rk } \Gamma = \dim \Gamma$ and $k_{12}(\Gamma) \leq 1$, then on every relevant boundary stratum either the boundary values are nowhere zero or their transversality follows from (1)–(3) for the components of the base. For a base with a component Ξ_1 with $\text{rk } \Xi_1 = \dim \Xi_1$ and a component Ξ_2 with $\text{rk } \Xi_2 = \dim \Xi_2 + 1$, it uses the transversality of the slice of H^{Ξ_1} at the time $\mathfrak{t}(I(\Xi_2))$, the only time at which H^{Ξ_2} vanishes. Near each boundary zero the data are extended so that their restriction to the interior slice through it is transverse; the data are then made transverse on the slices at the times $\mathfrak{t}(A)$ and extended transversally.

On the remaining relevant connected graphs the homotopy is extended arbitrarily.

For homotopies, fix $\varepsilon \in (0, \frac{1}{2})$ and a nondecreasing smooth collar function $c : [0, 1] \rightarrow [0, 1]$ with $c = 0$ on $[0, \varepsilon]$ and $c = 1$ on $[1 - \varepsilon, 1]$, and fix distinct times $\mathfrak{t}(J) \in (\varepsilon, 1 - \varepsilon)$ for the subsets $J \subseteq I$. We call $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$ the *time collars*; the homotopies below are constant in time on them, and the assigned times lie outside them. Let $\mathcal{H}(\mathbb{D}; \mathfrak{s}^0, \mathfrak{s}^1; (\mathfrak{t}(J))_{J \subseteq I})$ denote the canonical homotopies on $\mathcal{E}$ from $\mathfrak{s}^0$ to $\mathfrak{s}^1$ that are simple in the sense of [GKT26a, Def. 3.36] and satisfy, in addition, the normal-form conditions of [GKT26a, Lem. 3.40] for the times $\mathfrak{t}(J)$, under the hypotheses stated there. Irrelevant components, and connected components whose rank exceeds the dimension by at least two, are nowhere zero. A connected component Γ whose rank exceeds the dimension by one and with $k_{12}(\Gamma) \leq 2$ is transverse on $[0, 1] \times \mathcal{PM}_\Gamma$ and on every stratum $[0, 1] \times \mathcal{PM}_\Lambda$ with $\Lambda \in \partial^0 \Gamma \cup \partial^{\text{exchange}} \Gamma$, and vanishes only over $\mathfrak{t}(I(\Gamma))$. A connected component Γ whose rank equals the dimension and with $k_{12}(\Gamma) \leq 1$ is transverse on the same strata and on the slices $\{\mathfrak{t}(J)\} \times (\mathcal{PM}_\Gamma \setminus \partial\mathcal{PM}_\Gamma)$ at the assigned times. Since the times are distinct, the external sum of two components of the first kind with different internal label sets is nowhere zero, which is [GKT26a, Lem. 3.40(1)(c)]. No condition is imposed on the other relevant components. The critical graphs ($k_{12} = 0$) and the balanced graphs ($k_{12} = 1$), which are the graphs entering the wall-crossing formulas, are covered by these conditions.

Collars are part of the data. A *collar* of a multisection or a homotopy along a closed subset of its domain, namely a graph boundary stratum, the positive boundary, or one of the ends $\mathfrak{t} = 0$ and $\mathfrak{t} = 1$, is a fixed germ of it along that subset. Along the ends, a homotopy from $\mathfrak{s}^0$ to $\mathfrak{s}^1$ equals $\mathfrak{s}^0$, respectively $\mathfrak{s}^1$, on the time collars. Along a graph boundary stratum the germ is not required to be constant in the normal direction: near an isolated boundary zero of a component whose rank equals the dimension, it is the extension into the interior used in the proof of [GKT26a, Lem. 3.40(3)(b)], for which the interior slice through that zero is transverse. A germ constant in the normal coordinate would make this slice contain a segment of zeros.

B.2. Relative extension. We first recall the extension and modification results on which the constructions below rest. In them, $\mathcal{PM}$ has the meaning of Subsection 5.1; it is the space written $\overline{\mathcal{PM}}$ in [GKT26a] and [BCT24].

Theorem ([GKT26a, Lem. 5.15]). Let Γ be a smooth connected open graded W -spin graph, and let ζ be a multisection of $E_\Gamma(\mathbf{d})$ defined over $\partial\mathcal{PM}_\Gamma$. Suppose that, for every $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$,

$$\zeta|_{\mathcal{PM}_\Lambda} = F_\Lambda^* \zeta^{B\Lambda},$$

where $\zeta^{B\Lambda}$ is a transverse multisection of $E_{B\Lambda}(\mathbf{d})$ over $\mathcal{PM}_{B\Lambda}$ that is strongly positive with respect to $B\Lambda$ and $\text{Aut}(B\Gamma)$ -invariant. Then ζ extends to a transverse, strongly positive, $\text{Aut}(\Gamma)$ -invariant multisection of $E_\Gamma(\mathbf{d})$.

The statement also holds when $\partial\mathcal{PM}_\Gamma$ is empty [GKT26a, Rem. 5.16], and its proof is that of the following r -spin statement of Buryak, Clader and Tessler, whose first two items we use; the Witten bundle there is that of graded r -spin disks.

Theorem ([BCT24, Lem. 6.5(1)–(2)]). Let Γ be a smooth, connected, open, graded r -spin dual graph, and let ζ be a multisection of the Witten bundle over $\bigcup_{\Lambda \in \partial\Gamma \setminus \partial^+\Gamma} \mathcal{M}_\Lambda^{1/r}$.

- (1) Suppose that, for every $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$, $\zeta|_{\mathcal{PM}_\Lambda^{1/r}} = F_\Lambda^* \zeta^{B\Lambda}$ for a multisection $\zeta^{B\Lambda}$ of the Witten bundle over $\mathcal{PM}_{B\Lambda}^{1/r}$ that is positive with respect to $B\Lambda$, $\text{Aut}(B\Gamma)$ -invariant and transverse. Then ζ extends to a positive, transverse, $\text{Aut}(\Gamma)$ -invariant multisection σ of the Witten bundle over $\mathcal{PM}_\Gamma^{1/r}$.
- (2) Suppose, in addition, that for every $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$ such that $B\Lambda$ has at least one vertex without boundary tails, the restriction of ζ to $\mathcal{M}_\Lambda^{1/r}$ is strongly positive in the sense of [BCT24, Def. 6.1]. Then σ can be chosen to be strongly positive on $\mathcal{PM}_\Gamma^{1/r}$.

The next lemma of GKT prescribes the zero count of a homotopy; we write its time as $\mathfrak{t}$.

Theorem ([GKT26a, Lem. 5.17]). Let $E \rightarrow Y$ be an oriented orbifold bundle over an oriented orbifold Y with corners, such that the generic isotropy of $E \rightarrow Y$ is trivial and $\text{rank}_{\mathbb{R}} E = \dim_{\mathbb{R}} Y + 1$. Let s be a nowhere vanishing multisection of E , and let $c \in \mathbb{Q}$. Then there are a multisection s' of E that agrees with s outside a compact subset of the interior of Y , and a transverse homotopy $H : [0, 1] \times Y \rightarrow E$ with $H(0, -) = s$, $H(1, -) = s'$ and $H(\mathfrak{t}, y) = s(y)$ for every $\mathfrak{t}$ and every $y \in Y$ outside that compact subset, such that the rational signed count of zeros of H is c .

Lemma B.3 (Equivariant relative multisection extension). *Let $\delta = (\Gamma, \mathbf{d})$ be a graph datum with Γ smooth and connected, and let Y be either $\mathcal{PM}_\Gamma$ or $[0, 1] \times \mathcal{PM}_\Gamma$. Suppose compatible invariant data are prescribed on a neighborhood of a collared invariant closed set $K_0 \subset Y$, where K_0 contains every collar on which data are prescribed: the time collars, the collars along the graph boundary strata and along the positive boundary, and those around the data already constructed on a smaller system and around the slices at the assigned times. If the prescribed data are strongly positive and already satisfy the required transversality or nonvanishing conditions on K_0 , they extend to an invariant global multisection that coincides with them near K_0 and satisfies the same open conditions. A nowhere-vanishing critical section may additionally be modified in the smooth interior through a transverse homotopy with prescribed rational signed zero count.*

Proof. The argument of [BCT24, Lem. 6.5], in the W -spin form [GKT26a, Lem. 5.15], extends assembling data prescribed on the boundary strata $\Lambda \in \partial\Gamma \setminus \partial^+\Gamma$ to a strongly positive

multisection near these strata and near the positive boundary $\partial^+\Gamma$; we include the time collars and the collars around the slices at the assigned times in the prescribed data. On the compact complement choose finitely many compactly supported local multisections whose branches span every fiber. Relative parametric transversality, applied to the finitely many graph strata and slices, gives a generic linear combination with the required properties. The perturbation vanishes near K_0 , hence leaves the data on all prescribed collars unchanged and preserves strong positivity.

Perform the construction on the *injectively labelled cover* of Y , the corresponding space of disks whose k boundary markings are numbered bijectively by $\{1, \dots, k\}$, as in the labelled moduli spaces of [GKT26a, Def. 2.2], multiplied by $[0, 1]$ if $Y = [0, 1] \times \mathcal{PM}_\Gamma$; it maps onto Y with finite fibers by forgetting this numbering. Then take the normalized union of all finite automorphism translates. On K_0 every translate is the same invariant prescribed multisection, so the result coincides with it there. The generic smooth interior of the injectively labelled cover has trivial isotropy. On this locus, [GKT26a, Lem. 5.17] supplies a compactly supported critical modification whose transverse homotopy has any prescribed rational signed zero count. Relative transversality preserves that count and leaves the prescribed collars unchanged. $\square$

Completion to all graph data. Every $\mathfrak{s} \in \mathcal{C}(\mathbb{D})$ extends, without change on $\mathcal{E}$, to transverse, strongly positive, invariant canonical multisections on all graph data with labels in I . Indeed, process the missing data with smooth connected graphs in the order of Lemma B.2. Assembling from data already constructed prescribes boundary values, which Lemma B.3 extends; the other data are the forced pullbacks and external sums. No invariant at a balanced graph datum is prescribed in this step.

The realization of prescribed invariants at balanced graph data, that is, of prescribed coordinates of a chamber index, follows the proof of the realization theorem of GKT.

Theorem ([GKT26a, Thm. 5.13]). Every chamber index bounded by I is realized as the vector $\nu^\mathfrak{s}$ of open invariants of some canonical family $\mathfrak{s}$ of multisections bounded by I .

The proof is an induction on the number of internal labels. For a set J of labels and a descendent vector $\mathbf{d}$ on J , let $\Gamma_{J,0}, \dots, \Gamma_{J,N}$ be the balanced graphs of $(J, \mathbf{d})$, with $k_1(\Gamma_{J,p}) = r_J + 4p$, and let $\Lambda_{J,1}, \dots, \Lambda_{J,N}$ be its critical graphs, with $k_1(\Lambda_{J,p}) = r_J + 4(p-1) + 1$ [GKT26a, Notation 2.40]. The inductive step rests on the following two assertions, stated here for $r = s = 4$.

Theorem ([GKT26a, proof of Thm. 5.13, (5.50) and (5.52)–(5.54)]). Let ν be a chamber index bounded by I , let $J \subseteq I$ and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^J$, and let $\mathfrak{s}$ be a transverse canonical family bounded by J such that $\nu_{\Gamma, \mathbf{d}'}^\mathfrak{s} = \nu_{\Gamma, \mathbf{d}'}$ for every balanced graph datum $(\Gamma, \mathbf{d}')$ whose support is a proper divisor of $u_{J, \mathbf{d}}$. Then, with the weight (58),

$$\sum_{\Gamma \in \text{Balanced}(J, \mathbf{d})} w_J(k_1(\Gamma), k_2(\Gamma)) (\nu_{\Gamma, \mathbf{d}}^\mathfrak{s} - \nu_{\Gamma, \mathbf{d}}) = 0.$$

For $1 \leq p \leq N$, replace the multisection of $\Lambda_{J,p}$ by one that agrees with it near the boundary of $\overline{\mathcal{M}}_{\Lambda_{J,p}}^W$ and is joined to it by a transverse homotopy, constant in time near that boundary, with rational zero count $h_{p, \mathbf{d}}$, and redefine by [GKT26a, Lem. 5.15] the multisections of the graphs that have a zero or exchange boundary graph Λ with $\Lambda_{J,p}$ a connected component of BA . Then the invariants of balanced graphs with fewer than $|J|$ labels do not change, and the

invariant $o'_{J,p,\mathbf{d}}$ of $\Gamma_{J,p}$ becomes

$$\begin{aligned} o'_{J,0,\mathbf{d}} &= \nu_{\Gamma_{J,0},\mathbf{d}}^{\mathfrak{s}} + (-1)^{k_2(\Lambda_{J,1})} 4k_1(\Lambda_{J,1}) h_{1,\mathbf{d}}, \\ o'_{J,p,\mathbf{d}} &= \nu_{\Gamma_{J,p},\mathbf{d}}^{\mathfrak{s}} + (-1)^{k_2(\Lambda_{J,p})+1} 4k_2(\Lambda_{J,p}) h_{p,\mathbf{d}} \\ &\quad + (-1)^{k_2(\Lambda_{J,p+1})} 4k_1(\Lambda_{J,p+1}) h_{p+1,\mathbf{d}} \quad (0 < p < N), \\ o'_{J,N,\mathbf{d}} &= \nu_{\Gamma_{J,N},\mathbf{d}}^{\mathfrak{s}} + (-1)^{k_2(\Lambda_{J,N})+1} 4k_2(\Lambda_{J,N}) h_{N,\mathbf{d}}. \end{aligned}$$

In [GKT26a] the hypothesis is stated for all balanced graphs with fewer than $|J|$ labels; the proof of (5.50) uses it only at proper divisors of $u_{J,\mathbf{d}}$, since apart from the coordinates at the balanced graphs of $(J, \mathbf{d})$ these are the only coordinates on which $A(J, \mathbf{d}, \cdot)$ depends. The first term of each formula is written there as the sum of $\nu_{\Gamma_{J,p},\mathbf{d}}$ and the difference $\nu_{\Gamma_{J,p},\mathbf{d}}^{\mathfrak{s}} - \nu_{\Gamma_{J,p},\mathbf{d}}$. The zero counts $h_{p,\mathbf{d}}$ can be prescribed by [GKT26a, Lem. 5.17].

Theorem B.4 (Relative lifting of canonical families). *Let $\mathbb{D} \subset \mathbb{D}'$ and let $\mathfrak{s} \in \mathcal{C}(\mathbb{D})$, where $\mathbb{D} = (I, \mathcal{P}, \mathcal{E})$ and $\mathbb{D}' = (I', \mathcal{P}', \mathcal{E}')$. Prescribe a value at every balanced graph datum in $\mathcal{E}' \setminus \mathcal{E}$, assuming that these values and the invariants of $\mathfrak{s}$ at the balanced graph data in $\mathcal{E}$ are the restriction of a chamber index. Then there exists $\mathfrak{s}' \in \mathcal{C}(\mathbb{D}')$ realizing those values and restricting to $\mathfrak{s}$. If the input and target index are symmetric, the extension may be chosen symmetric.*

Proof. Construct the new data with smooth connected graphs in the order of Lemma B.2. Since $\mathcal{E}'$ is assembling-closed, the associativity of assembling prescribes compatible boundary values, which Lemma B.3 extends. Define the other new data by their prescribed pullbacks and assemblies.

Process the supports of the new balanced graph data by increasing number of labels, following the proof of [GKT26a, Thm. 5.13]. For one support $u = u_{J,\mathbf{d}}$, temporarily extend the partial family by the completion to all graph data above. Proper-divisor coordinates have already been fixed, so the chamber equations reduce to the one weighted linear relation displayed above among the coordinates at the balanced graph data of the present support. The critical sections have rank one greater than the dimension and are nowhere zero. On the injectively labelled smooth-interior cover, [GKT26a, Lem. 5.17] supplies compactly supported modifications with prescribed rational signed zero counts. The formulas [GKT26a, (5.52)–(5.54)] recalled above are triangular in these counts: the coefficient $4k_1(\Lambda_{J,p+1})$ of $h_{p+1,\mathbf{d}}$ in $o'_{J,p,\mathbf{d}}$ is nonzero, because $k_1(\Lambda_{J,p+1}) = r_J + 4p + 1$. Choose $h_{1,\mathbf{d}}, \dots, h_{N,\mathbf{d}}$ successively so that the invariants of $\Gamma_{J,0}, \dots, \Gamma_{J,N-1}$ take their prescribed values; the last balanced coordinate is then forced by the weighted chamber relation [GKT26a, (5.50)], whose weights are nonzero. Reapply the relative extension lemma and reassemble the affected data.

A change made for a support u affects only data whose support is a multiple of u . Suppose such a datum δ were old. Its support lies in the old monomial set by condition (3) of Definition B.1, so u lies there by divisor closure, and condition (4) places all balanced and critical graph data of support u in the old system. Then u is not the support of a new balanced graph datum, a contradiction. Hence the original family is unchanged. For symmetry, make choices orbitwise and take normalized finite orbit-unions; on old orbits these equal the old invariant section. $\square$

Theorem B.5 (Relative simple-homotopy lifting). *Let $\mathbb{D} \subset \mathbb{D}'$, let $\mathfrak{s}^0, \mathfrak{s}^1 \in \mathcal{C}(\mathbb{D}')$, and suppose*

$$H \in \mathcal{H}(\mathbb{D}; \mathfrak{s}^0|_{\mathbb{D}}, \mathfrak{s}^1|_{\mathbb{D}}; (\mathfrak{t}(J))_{J \subseteq I}).$$

Then the assignment $J \mapsto \mathfrak{t}(J)$ extends to an injective assignment on the subsets of I' , and H extends to an element of $\mathcal{H}(\mathbb{D}'; \mathfrak{s}^0, \mathfrak{s}^1; (\mathfrak{t}(J))_{J \subseteq I'})$ which agrees with the old homotopies, equals

$\mathfrak{s}^0$ and $\mathfrak{s}^1$ on the time collars, and keeps the collars of the old homotopies. In particular, all old subset times are retained.

Proof. For an old component whose rank equals the dimension and with $k_{12}(\Gamma) \leq 1$, the times at which a slice $\{\mathfrak{t}\} \times (\mathcal{PM}_\Gamma \setminus \partial\mathcal{PM}_\Gamma)$ is not transverse form a measure-zero set by Sard's theorem on the finitely many zero-set strata. Case (c) in the proof of [GKT26a, Lem. 3.40] examines the relevant boundary strata of such a component and shows that on each of them the assembled data either vanish nowhere or vanish only at the time $\mathfrak{t}(A)$ of a base component whose rank exceeds its dimension by one, A being its label set. Hence the boundary values of such a component vanish only at times $\mathfrak{t}(A)$ with $\emptyset \neq A \subseteq I(\Gamma)$. Choose the new subset times distinctly, away from the old values and these exceptional sets, and outside the time collars. No old homotopy must be changed.

Induct on $\dim \mathcal{PM}_\Gamma$, choosing only new data with smooth connected graphs. The values at $\mathfrak{t} = 0$ and $\mathfrak{t} = 1$ are prescribed. On a boundary stratum Λ without a contracted boundary half-edge use the assembly $F_\Lambda^*(\boxplus_\Xi H^\Xi)$ of the homotopies H^Ξ of the connected components Ξ of $B\Lambda$, as in (57); on disconnected faces take external sums. These prescriptions agree at corners by associativity of assembling. On a boundary stratum Λ with a contracted boundary half-edge, the proof of [GKT26a, Lem. 3.40] uses the linear homotopy between the components of $\mathfrak{s}^0$ and $\mathfrak{s}^1$; at a corner with a stratum of the preceding kind it need not agree with the assembled homotopy, which need not be linear in time. We use instead a homotopy on Λ that agrees with the assembled homotopies near the corners, is positive at the contracted boundary half-edges in the sense of [GKT26a, Def. 3.18(1)], and equals the components of $\mathfrak{s}^0$ and $\mathfrak{s}^1$ on the time collars, for instance the combination, by a partition of unity, of a positive extension of the corner values near the corners with $(1 - c(\mathfrak{t}))s_{\Lambda, \mathfrak{d}}^0 + c(\mathfrak{t})s_{\Lambda, \mathfrak{d}}^1$; it is positive because this positivity is preserved by convex combinations.

Apply Lemma B.3 with all old data and boundary data prescribed. For a component Γ whose rank exceeds the dimension by one and with $k_{12}(\Gamma) \leq 2$, the prescribed data are nowhere zero on $[0, 1] \times \partial\mathcal{PM}_\Gamma$: on irrelevant strata by the condition on irrelevant components and positivity, and on relevant strata by the argument of case (b) in the proof of [GKT26a, Lem. 3.40], a dimension count showing that the base of every relevant boundary stratum has a component whose rank exceeds its dimension by at least two, or two components whose ranks exceed their dimensions by one and whose label sets are distinct. That argument uses only the conditions above for the base components, all of smaller dimension, together with [GKT26a, Lem. 3.39] and the injectivity of $J \mapsto \mathfrak{t}(J)$, and old base components satisfy these conditions for their old times, which are retained. We then make the homotopy transverse. It has finitely many zeros, and after a small perturbation no two of them have the same image under the projection to $\mathcal{PM}_\Gamma$. A diffeomorphism of $[0, 1] \times \mathcal{PM}_\Gamma$ that commutes with this projection and is the identity near the boundary of $[0, 1] \times \mathcal{PM}_\Gamma$ and near the positive boundary then moves all zeros to the assigned time $\mathfrak{t}(I(\Gamma))$, as in the proof of [GKT26a, Lem. 3.40]. For a component whose rank equals the dimension and with $k_{12}(\Gamma) \leq 1$, the prescribed data are transverse on the boundary strata, by the argument of case (c) there, which derives transversality on each relevant boundary stratum from the conditions above on the components of its base. For base components Ξ_1 of this kind and Ξ_2 of the preceding kind with $A = I(\Xi_2)$, that argument uses the transversality of the interior slice of H^{Ξ_1} at $\mathfrak{t}(A)$, which holds for old Ξ_1 by the choice of the new times, and the absence of boundary zeros of H^{Ξ_1} at $\mathfrak{t}(A)$, which holds because these zeros lie at times $\mathfrak{t}(A')$ with $A' \subseteq I(\Xi_1)$, hence $A' \neq A$. The prescribed data vanish only at isolated points of codimension-one strata, at times $\mathfrak{t}(A)$ with $\emptyset \neq A \subseteq I(\Gamma)$; near each of them extend the data into the interior so that the restriction to the interior slice is transverse, as in the proof of [GKT26a, Lem. 3.40(3)(b)]. Then make the slices at the assigned times transverse, prescribe

the data on collars around them, and perturb on the complement. Irrelevant components and components whose rank exceeds the dimension by at least two are made nowhere zero relative to the prescribed data, by positivity and by transversality respectively. On the other relevant components the homotopy is extended arbitrarily relative to the prescribed data, as at the end of the proof of [GKT26a, Lem. 3.40]. Automorphism orbit-unions preserve the assigned time because it depends only on the invariant label set. The homotopies on the remaining data are then the forced pullbacks and assemblies. $\square$

The inductive step of this proof, applied to the missing data with smooth connected graphs in the order of Lemma B.2 after the canonical families at $t = 0$ and $t = 1$ have been completed, gives the corresponding completion of homotopies to all graph data.

B.3. The compatible union. The proof of Theorem 5.7 assembles the finite constructions above along an increasing sequence of systems. Restriction is taken along assembling-closed systems because, by the assembling relation (57), a canonical family is determined on the boundary of each moduli space by its multisections on other graph data; on such systems, restriction to a smaller system is the forgetful map. A system contains all balanced and critical graph data of each of its supports that carries a balanced graph, because the chamber conditions couple the coordinates at all balanced graphs of a support through one weighted relation, these coordinates are adjusted through the critical graphs of the same support, and this completeness ensures that realizing the prescribed invariants at the balanced graphs of a new support never alters an old datum (proof of Theorem B.4). The rays that carry a nontrivial factor of the GKT transition on some system $\mathbb{D}_n$ form a countable set, whose slope order need not be the order of a sequence; the complementary open intervals of a Cantor set in $[0, 1]$, ordered by position, form a countable dense linear order without endpoints, into which every countable linear order embeds. One such embedding, fixed for all systems, gives each of these rays its own time interval, the intervals being pairwise disjoint and ordered by slope. Passing from $\mathbb{D}_n$ to $\mathbb{D}_{n+1}$ one carries along the canonical families that realize the extremal and the intermediate chamber indices, and the homotopies between consecutive families on their intervals; the data on $\mathbb{D}_{n+1}$ restrict to those on $\mathbb{D}_n$, and the families and the homotopy of Theorem 5.7 are the inverse limit of these data.

The proof below uses the companion paper [Wue26b] only for three facts in every finite divisor-closed quotient: the two extremal chamber indices are unique [Wue26b, Thm. 1.1], the enlarged transition group acts freely [Wue26b, Lem. 2.5], and the transition between the extremal indices has a unique increasing-slope factorization [Wue26b, Thm. 1.1 and Prop. 3.1]. The finite homotopies themselves are constructed by Theorems B.4 and B.5.

Proof of Theorem 5.7. Choose increasing finite label sets I_n exhausting I_∞ , finite divisor-closed sets of labelled monomials in the labels of I_n whose union is the set of all labelled monomials, and an enumeration of all graph data. Let $\mathcal{P}_n$ be the smallest divisor-closed set invariant under the twist-preserving permutations of I_n that contains $\mathcal{P}_{n-1}$, the n th of these monomial sets, and the supports of the first n graph data with labels in I_n . Let $\mathcal{E}_n$ consist of $\mathcal{E}_{n-1}$, these graph data, and all balanced and critical graph data of every support in $\mathcal{P}_n$ that carries a balanced graph, closed under taking assembling constituents and under the twist-preserving permutations of I_n . By Lemma B.2, $\mathcal{E}_n$ is finite and every support of a datum in $\mathcal{E}_n$ lies in $\mathcal{P}_n$. This gives the systems $\mathbb{D}_n = (I_n, \mathcal{P}_n, \mathcal{E}_n)$.

The union over all n of the finite sets of nonzero ray factors on the systems $\mathbb{D}_n$ is countable. Embed its slope order into the ordered set of complementary open intervals of a Cantor set in $[0, 1]$; each ray factor thereby receives an open time interval. Inductively apply Theorem B.4 to all intermediate canonical families and Theorem B.5 to the elementary homotopies, each of

which realizes one ray factor on its interval, always relative to the previous system (for $n = 1$ relative to the empty system, so that there are no old data).

Fix $\delta \in \mathbb{D}_{n_0}$. Only the finitely many ray factors on the system $\mathbb{D}_{n_0}$ can be nonconstant on it: a factor introduced on a later system is extended relative to the constant homotopy on the old system, which lies in $\mathcal{H}$ for any distinct times: its components of rank exceeding the dimension are nowhere zero, and those of rank equal to the dimension have nowhere-zero boundary values and transverse slices. This finiteness proves Part (2). On the interval of such a factor use its elementary homotopy; elsewhere use the constant value reached after the finitely many such intervals to the left. Constant collars make these pieces join smoothly. Every assembling identity involves finitely many graph data and hence holds on some system $\mathbb{D}_n$.

The completion to all graph data permits application of the argument of [GKT26a, Thm. 5.13] on each system $\mathbb{D}_n$. Uniqueness of the factorization, support by support, identifies the jump with the required ray factor in every finite quotient. Since the transition maps are forgetful maps, the compatible restrictions form an inverse limit in sets. $\square$

Enlargement of the labelled system leaves the realization on the old system unchanged.

APPENDIX C. REGULARITY OF THE FINITE ROOT-STACK PRESENTATIONS

This appendix proves the two statements on regularity summarized at the end of Section 4.5, whose notation we keep. Thus $\mathbb{T} = \operatorname{Spec} \mathbb{k}[\widetilde{M}]$ is the character torus in the frame of the sheet S_0 , and $T \subset \mathcal{T}$ is a finite rooted subtree. At a Kummer edge e of T we have the root degree $d_e \in \{2, 4\}$, the transported radicand ϱ_e , the Laurent polynomial $\widehat{\varrho}_e$ of Lemma 4.6 and the divisor $D_e = \operatorname{div} \widehat{\varrho}_e$ of zeros of ϱ_e ; moreover $D_T = \bigcup_e D_e$, the scheme P_T is the Kummer cover, with group $G_T = \prod_e \mu_{d_e}$, so that $\mathfrak{X}_T = [P_T/G_T]$, and P_T° is the preimage of $\mathbb{T} \setminus D_T$ in P_T .

C.1. Non-regularity where three root divisors meet.

Proposition C.1 (Non-regularity where three root divisors meet). *Write $f_B(x) = 1 + a_\rho x + x^2 = (1 + \lambda x)(1 + \lambda^{-1}x)$ for the B_0 -slab function, with $\lambda \neq \pm 1$, and let*

$$\varrho_n = f_B(h^n x_B), \quad h = (1 + x_A^{-1})^4, \quad n \in \mathbb{Z},$$

be the transported B_0 -radicands of Theorem 2.5, adjoined at the crossings e_n of the B_0 -arc that follow the paths γ_A^n . Let $T \subset \mathcal{T}$ contain e_0, e_1, e_2 , and let $w \in \mathbb{T}$ be a point with $(x_A, x_B)(w) = (-\frac{1}{2}, -\lambda^{-1})$. Then the Kummer cover P_T is not regular at any point $\hat{w}$ over w . At $\hat{w}$ it is not log regular, either with the trivial log structure or with the log structure given by the chart from the free commutative monoid on the Kummer edges of T that sends each Kummer edge e to g_e . In particular $\mathfrak{X}_T$ is not regular.

Proof. The characters $x_A = z^{2u_{\bar{A}}}$ and $x_B = z^{2u_{B_0}}$ have linearly independent exponents, so $(x_A, x_B) : \mathbb{T} \rightarrow \mathbb{G}_m^2$ is étale, and $x_A + \frac{1}{2}$, $x_B + \lambda^{-1}$ form a regular system of parameters of $\mathcal{O}_{\mathbb{T}, w}$; we compute differentials in these coordinates. Put $\varrho_n^\pm = 1 + \lambda^{\pm 1} h^n x_B$, so that $\varrho_n = \varrho_n^+ \varrho_n^-$. Then $h(w) = 1$ and $f_A(w) = \frac{1}{2} \neq 0$, and for every n

$$\varrho_n^+(w) = 0, \quad \varrho_n^-(w) = 1 - \lambda^{-2} \neq 0, \quad \nabla \varrho_n^+(w) = (-16n, \lambda).$$

Near w each ϱ_n is a unit times ϱ_n^+ . For $n = 0, 1, 2$ the gradients $(0, \lambda), (-16, \lambda), (-32, \lambda)$ are pairwise independent, so the three root divisors are smooth and pairwise transverse at w .

For $n \geq 0$ the radicand ϱ_n is a Laurent polynomial, so we may take $\widehat{\varrho}_{e_n} = \varrho_n$ in Lemma 4.6; the fixed $\widehat{\varrho}_{e_n}$ differs from ϱ_n by the fourth power of a unit v , which changes P_T by the isomorphism $g \mapsto vg$. Write g_0, g_1, g_2 for the roots at e_0, e_1, e_2 , of degree four, and $g_3, \dots, g_k$ for the roots at the other Kummer edges of T , with d_j and $\widehat{\varrho}_j$ accordingly. Then $P_T = \operatorname{Spec} \mathcal{O}_{\mathbb{T}}[g_0, \dots, g_k]/(g_j^{d_j} - \widehat{\varrho}_j)$

is finite and flat over the surface $\mathbb{T}$, hence of dimension two at every point, and g_0, g_1, g_2 vanish at every point $\hat{w}$ over w . The Jacobian of the defining equations at $\hat{w}$ has rank at most $2 + (k - 2) = k$, because the rows of $g_j^4 - \varrho_j$, $j \leq 2$, vanish in the g -columns. The tangent space at $\hat{w}$ therefore has dimension at least $(k + 3) - k = 3 > 2$, and P_T is not regular at $\hat{w}$. With the trivial log structure, log regularity at $\hat{w}$ is regularity [Kat94, Def. 2.1], so it fails. With the log structure of the roots, the characteristic monoid at $\hat{w}$ is free on the Kummer edges whose roots vanish at $\hat{w}$, so its rank is at least three, and the ideal $\mathfrak{a}_{\hat{w}} \subset \mathcal{O}_{\hat{w}}$ generated by the images of the nonunits of the log structure contains g_0, g_1, g_2 . The quotient $\mathcal{O}_{\hat{w}}/\mathfrak{a}_{\hat{w}}$ is finite over $\mathcal{O}_{\mathbb{T},w}/(\varrho_0^+, \varrho_1^+) = \mathbb{k}$, because ϱ_0^+ and ϱ_1^+ vanish transversely at w , and so has dimension zero. Kato's condition $\dim \mathcal{O}_{\hat{w}} = \dim(\mathcal{O}_{\hat{w}}/\mathfrak{a}_{\hat{w}}) + \text{rank}$ [Kat94, Def. 2.1] would read $2 = 0 + \text{rank}$ with a rank of at least three, so it fails. Since $P_T \rightarrow \mathfrak{X}_T$ is étale and surjective, $\mathfrak{X}_T$ is not regular. A tree of this kind exists: by Theorem 2.5 the radicands $\varrho_0, \varrho_1, \varrho_2$ occur along the finitely many paths $\gamma_{\tilde{A}}^n$, $n = 0, 1, 2$, followed by a crossing of the B_0 -arc, and their rooted union is a finite subtree. $\square$

At the nodal values $a_\rho = \pm 2$ the same point, with $\lambda = \pm 1$, gives the same conclusion. There $\varrho_n = (\varrho_n^+)^2$ by (24), and both the fourth roots of ϱ_n and the square roots of ϱ_n^+ , whose gradients at w are again $(-16n, \lambda)$, give Kummer covers whose tangent space at the points over w has dimension at least three. The argument for log regularity applies verbatim: the three roots at e_0, e_1, e_2 vanish at these points, so the rank is at least three, while the quotient by the ideal of the log structure has dimension zero.

C.2. A regular dominating presentation. Regularity is recovered by a proper modification of the finite presentations, together with the modification of their Kummer covers that it induces; the latter carries the lifted transitions over the preimage of $\mathbb{T} \setminus D_T$ (Section 4.5). A reduced curve on a smooth surface is a *simple normal crossings divisor* if its components are smooth, meet pairwise transversely, and no three pass through a point. The modification is the stack of fourth roots of the boundary components of the minimal log resolution of $(\mathbb{T}, D_T)$; fourth roots suffice because every root degree d_e divides 4.

Theorem C.2 (Regular dominating presentation). *For every finite rooted subtree $T \subset \mathcal{T}$ there are a smooth separated Deligne–Mumford stack $\mathcal{Y}_T$ over $\mathbb{k}$, a divisor $\mathcal{E}_T \subset \mathcal{Y}_T$ that is a simple normal crossings divisor on an étale atlas, and a proper morphism $p_T : \mathcal{Y}_T \rightarrow \mathfrak{X}_T$. Put $\tilde{\mathcal{Y}}_T = \mathcal{Y}_T \times_{\mathfrak{X}_T} P_T$, let $\tilde{p}_T : \tilde{\mathcal{Y}}_T \rightarrow P_T$ be the projection, and let $\tilde{\mathcal{E}}_T$ be the preimage of $\mathcal{E}_T$. Then the following hold.*

- (1) *p_T is an isomorphism over $\mathbb{T} \setminus D_T$. The projection $\tilde{\mathcal{Y}}_T \rightarrow \mathcal{Y}_T$ is a G_T -torsor, so $\tilde{\mathcal{Y}}_T$ is a smooth separated Deligne–Mumford stack and $\mathcal{Y}_T = [\tilde{\mathcal{Y}}_T/G_T]$. The morphism $\tilde{p}_T$ is proper and G_T -equivariant, and it is an isomorphism over P_T° . Consequently every function, section of $\mathcal{L}_\kappa$, bracket or automorphism defined over $P_T^\circ \times_{\text{Spec } \mathbb{k}} \text{Spec } R_{\mathcal{W},N}$ pulls back unchanged to the preimage of $\mathbb{T} \setminus D_T$ in $\tilde{\mathcal{Y}}_T \times_{\text{Spec } \mathbb{k}} \text{Spec } R_{\mathcal{W},N}$; this applies in particular to the lifted transitions along T , to the Jacobi bracket of Proposition 3.3, and to the wall automorphisms base-changed along $v_{\mathcal{W}}$ of (52).*
- (2) *For every labelled truncation $\mathcal{W}$ and every $N \geq 1$, the base changes of $\mathcal{Y}_T$ and of $\tilde{\mathcal{Y}}_T$ to $\text{Spec } R_{\mathcal{W},N}$, with the log structures of $\mathcal{E}_T$ and of $\tilde{\mathcal{E}}_T$, are flat and log smooth over $\text{Spec } R_{\mathcal{W},N}$ with the trivial log structure, and they have smooth atlases over $R_{\mathcal{W},N}$. Their closed fibers $(\mathcal{Y}_T, \mathcal{E}_T)$ and $(\tilde{\mathcal{Y}}_T, \tilde{\mathcal{E}}_T)$ are log regular.*
- (3) *If $T \subset T'$, there is a proper morphism $q_{T':T} : \mathcal{Y}_{T'} \rightarrow \mathcal{Y}_T$ with $p_T \circ q_{T':T} \simeq \pi_{T':T} \circ p_{T'}$, where $\pi_{T':T} : \mathfrak{X}_{T'} \rightarrow \mathfrak{X}_T$ forgets the roots along the divisors indexed by $T' \setminus T$. It lifts to*

a proper morphism $\tilde{q}_{T',T} : \tilde{\mathcal{Y}}_{T'} \rightarrow \tilde{\mathcal{Y}}_T$ over the projection $P_{T'} \rightarrow P_T$ that forgets the same roots. All constructions commute with the base changes $R_{\mathcal{W}',N'} \rightarrow R_{\mathcal{W},N}$.

Proof. Embedded resolution. Let $\pi_T : \hat{\mathbb{T}}_T \rightarrow \mathbb{T}$ be the minimal log resolution of $(\mathbb{T}, D_T)$, obtained by repeatedly blowing up the finitely many points at which the reduced total transform of D_T is not a simple normal crossings divisor, and let $E_T = E_1 \cup \cdots \cup E_{n_T}$ be the reduced total transform of D_T in $\hat{\mathbb{T}}_T$. The process stops, and every composition of point blowups with centers over $\mathbb{T}$ under which the reduced total transform of D_T is a simple normal crossings divisor factors through π_T . Indeed, on a smooth projective compactification of $\mathbb{T}$ such compositions exist by embedded resolution of curves on surfaces [Har77, Thm. V.3.9], followed if necessary by the blowups of the points where three components meet. If such a composition factors through a stage of the process, it is not an isomorphism over any neighborhood of a center of the next step, since the reduced total transform of D_T is not a simple normal crossings divisor there, so its inverse is not defined at that center; by [Har77, Prop. V.5.3], applied after blowing up the same centers in a smooth projective compactification of $\mathbb{T}$, it factors through the blowup of each such center, hence through the next stage. As each stage adds an exceptional curve of the composition, the process stops. All centers lie over D_T , so π_T is an isomorphism over $\mathbb{T} \setminus D_T$.

Fourth roots of the boundary. Let $\mathcal{Y}_T$ be the fiber product over $\hat{\mathbb{T}}_T$ of the root stacks $\sqrt[4]{(\hat{\mathbb{T}}_T, E_j)}$, and let $\mathcal{E}_j$ be the reduced preimage of E_j , so that E_j pulls back to $4\mathcal{E}_j$. Where $E_{j_1}, \dots, E_{j_l}$ meet, with local equations $x_1, \dots, x_l$ forming part of a regular system of parameters, $\mathcal{Y}_T$ has the étale chart $U = \operatorname{Spec} \mathcal{O}[y_1, \dots, y_l] / (y_1^4 - x_1, \dots, y_l^4 - x_l)$ with its action of μ_4^l [Cad07, Ex. 3.1 and Def. 2.6]; it is smooth over $\mathbb{k}$, and $\mathcal{E}_T = \bigcup_j \mathcal{E}_j$ is $y_1 \cdots y_l = 0$ on it. Since μ_4 is étale in characteristic zero, $\mathcal{Y}_T$ is a smooth separated Deligne–Mumford stack and $\mathcal{E}_T$ is a simple normal crossings divisor on an étale atlas.

The morphism p_T . Write $\pi_T^* D_e = \sum_j a_{ej} E_j$ with $a_{ej} \geq 0$. On $\mathcal{Y}_T$ this divisor is d_e times $\sum_j (4a_{ej}/d_e) \mathcal{E}_j$, which is integral because d_e divides 4; the line bundle of the latter, its canonical section and the canonical isomorphism of its d_e th power with $\mathcal{O}(\pi_T^* D_e)$ define a morphism $\mathcal{Y}_T \rightarrow \sqrt[4]{(\mathbb{T}, D_e)}$ over π_T [Cad07, Defs. 2.1, 2.3]. Together these give p_T , which is proper because $\mathcal{Y}_T \rightarrow \mathbb{T}$ is proper and $\mathfrak{X}_T \rightarrow \mathbb{T}$ is separated. Locally $\pi_T^* \hat{\varrho}_e = v_e \prod_j x_j^{a_{ej}}$ with v_e a unit, so after the étale extension $\eta_e^{d_e} = v_e$ a d_e th root of $\pi_T^* \hat{\varrho}_e$ is a unit times a monomial in the boundary coordinates:

$$(104) \quad g_e = \eta_e \prod_j y_j^{4a_{ej}/d_e}, \quad g_e^{d_e} = \pi_T^* \hat{\varrho}_e.$$

The Kummer cover. As a base change of the G_T -torsor $P_T \rightarrow \mathfrak{X}_T$, with G_T finite and étale, $\tilde{\mathcal{Y}}_T \rightarrow \mathcal{Y}_T$ is a finite étale G_T -torsor. Hence $\mathcal{Y}_T = [\tilde{\mathcal{Y}}_T/G_T]$, the stack $\tilde{\mathcal{Y}}_T$ is smooth, separated and Deligne–Mumford, and $\tilde{p}_T$, the base change of p_T , is proper and G_T -equivariant. Over a chart U as above, $\tilde{U} = U \times_{\mathcal{Y}_T} \tilde{\mathcal{Y}}_T$ is a scheme, finite and étale over U and hence smooth over $\mathbb{k}$, and $\tilde{\mathcal{E}}_T$ is $y_1 \cdots y_l = 0$ on it. On $\tilde{U}$ the quotient $g_e \prod_j y_j^{-4a_{ej}/d_e}$ of the root g_e by the monomial is a rational function whose d_e th power is the unit v_e , hence a unit because $\tilde{U}$ is normal, and (104) holds on $\tilde{U}$.

Proof of (1) and (2). No divisor that is blown up or rooted meets $\mathbb{T} \setminus D_T$, so p_T is an isomorphism over $\mathbb{T} \setminus D_T$ and $\tilde{p}_T$ is one over P_T° ; this proves (1). The chart $\mathbb{N}^l \rightarrow \mathcal{O}_U$ that sends the j th basis vector to y_j induces smooth morphisms from U and from $\tilde{U}$, which is étale over U , to $\operatorname{Spec} \mathbb{k}[\mathbb{N}^l]$, so both are log smooth over $\mathbb{k}$ by Kato’s criterion [Kat89, Thm. 3.5]; log smoothness, flatness and smooth atlases persist after base change to $R_{\mathcal{W},N}$. At a point on exactly l components, the characteristic monoid is $\mathbb{N}^l$, the ideal generated by its nonunits is

$(y_1, \dots, y_l)$, and the quotient by this ideal is regular of dimension $2 - l$; this is Kato's condition for log regularity [Kat94, Def. 2.1], which proves (2).

Proof of (3). Since $D_T \subset D_{T'}$, the reduced total transform of D_T in $\widehat{\mathbb{T}}_{T'}$ is a union of components of $E_{T'}$, hence a simple normal crossings divisor. By minimality $\pi_{T'} = \pi_T \circ b_{T'T}$ for a morphism $b_{T'T} : \widehat{\mathbb{T}}_{T'} \rightarrow \widehat{\mathbb{T}}_T$, and since $b_{T'T}^{-1}(E_T) \subset E_{T'}$, each E_j pulls back to $\sum_k c_{jk} E'_k$ with $c_{jk} \geq 0$, which is $4 \sum_k c_{jk} \mathcal{E}'_k$ on $\mathcal{Y}_{T'}$. As for p_T , this defines $q_{T'T}$ over $b_{T'T}$, proper because $\mathcal{Y}_{T'} \rightarrow \widehat{\mathbb{T}}_T$ is proper and $\mathcal{Y}_T \rightarrow \widehat{\mathbb{T}}_T$ is separated. For each Kummer edge e of T , both $p_T \circ q_{T'T}$ and $\pi_{T'T} \circ p_{T'}$ correspond, on the factor ${}^d\sqrt{(\mathbb{T}, D_e)}$, to the line bundle $\mathcal{O}(\sum_k (4a'_{ek}/d_e) \mathcal{E}'_k)$ with its canonical section and isomorphism, where a'_{ek} is the multiplicity of $\pi_{T'}^* D_e$ along E'_k ; hence they are 2-isomorphic. The projection $P_{T'} \rightarrow P_T$ is equivariant for $G_{T'} \rightarrow G_T$ and lies over $\pi_{T'T}$; with $q_{T'T}$ and this 2-isomorphism it induces $\widetilde{q}_{T'T}$, which is proper because $\widetilde{\mathcal{Y}}_{T'} \rightarrow \mathbb{T}$ is proper and $\mathcal{Y}_T \rightarrow \mathbb{T}$ is separated. Every object is a base change from $\mathbb{k}$, which gives the last assertion. $\square$

APPENDIX D. PRODUCTS OF FUNCTIONS TRANSPORTED ALONG DIFFERENT PATHS

Section 5 compares $\mathfrak{D}$ with GKT wall-crossing one ray at a time. Multiplicative structures also involve functions that reach one region along different paths: in the theta-function constructions of [GHK15, GHKK18, GHS22], structure constants come from pairs of broken lines that end at a common point after travelling along different paths. We do not define broken lines on $\mathcal{A}^\circ$ (Section 3.4), but the same question arises for admissible paths. Along one path, transport through $\mathfrak{D}$ and transport by the factors of the GKT transition differ by one automorphism of the target chart algebra, the *residual transport* of the path (Definition D.1); being an algebra automorphism, it relates the products of functions carried along that path. Along different paths the residual transports differ, and the product of the transports through $\mathfrak{D}$ need not be the image of the product of the transports by the factors of the GKT transition under any automorphism (see the example after Definition D.1). As a first step towards multiplying functions carried along different paths, we correct each factor by a degree-weighted interpolation of the residual transports, so that the corrected product is such an image, and we compute the associator between parenthesizations.

Definition D.1 (Transports along a path). Fix a labelled truncation (Section 5.4), a deformation order, and a region of $\mathfrak{D}$ (Definition 3.4), the *target region*, and work in the corresponding finite nilpotent quotient. Let γ be a path transverse to the walls that ends in the target region and realizes an admissible path $\mathfrak{p}$ (Definition 3.8).

- (1) The *transport through $\mathfrak{D}$* along γ is $P_\gamma = P_{\mathfrak{p}}$ (Section 6.2), the composite, in the order met, of the slab transitions and of the contributions $\Theta_{\mathfrak{d}}^{\pm 1}$ of the crossed walls (Definition 3.4), from the chart algebra at the start of γ to that of the target region.
- (2) The *transport P_γ^{GKT} by the factors of the GKT transition* is obtained from P_γ by replacing, for every crossed wall $\mathfrak{d}$ whose factor lies in the ray group of a boundary ray $\mathfrak{r}$, the contribution $\Theta_{\mathfrak{d}}^{\pm 1}$ by $\exp(\pm G_{\mathfrak{r}})$, with $G_{\mathfrak{r}}$ the logarithm of the factor of the GKT transition on $\mathfrak{r}$ (Definition 5.4), transported along γ to the frame of $\mathfrak{d}$; at a chord, $\mathfrak{r}$ is the chord ray, on either half of the chord. The slab transitions are not changed. Factors of the GKT transition on rays that carry no wall of $\mathfrak{D}$ are not inserted, so P_γ^{GKT} is not in general a GKT transition.
- (3) The *residual transport* of γ is the automorphism $\mathcal{Q}_\gamma = P_\gamma(P_\gamma^{\text{GKT}})^{-1}$ of the chart algebra of the target region, congruent to the identity modulo the ideal generated by the t_i .

For the two labels of twists $(1, 0)$ and $(0, 1)$ of Proposition 5.3, the paths $\gamma_\pm$ of Proposition 7.1 lie in one chart and have the same endpoints. Both have transport $\Sigma_1 \Sigma_2$ through $\mathfrak{D}$, but by

(63) and Proposition 7.1(3) their transports $P_{\gamma_+}^{\text{GKT}}$ and $P_{\gamma_-}^{\text{GKT}}$ are $\Sigma_2 Z_{1/4} \Sigma_1 = Z_1 \Sigma_1 \Sigma_2$ and $\Sigma_1 \Sigma_2$, with Z_c as in (64). Thus P_{γ}^{GKT} depends on the path and not only on its endpoints, and the residual transports are Z_{-1} and the identity. For f_1, f_2 in the chart algebra at the initial point, the product of the transports through $\mathfrak{D}$ is $Z_{-1}(a)b$, where $a = Z_1 \Sigma_1 \Sigma_2 f_1 = P_{\gamma_+}^{\text{GKT}} f_1$ and $b = \Sigma_1 \Sigma_2 f_2 = P_{\gamma_-}^{\text{GKT}} f_2$ run through the whole algebra. Since $Z_{-1} \neq \text{id}$, this product is not the image of ab under any automorphism independent of f_1, f_2 , by Proposition D.5.

D.1. Corrected products. Let $\gamma_1, \dots, \gamma_n$ be paths as in Definition D.1, let f_i be a function in the chart algebra at the initial point of γ_i , and write $\mathcal{Q}_i = \mathcal{Q}_{\gamma_i}$, so that $P_{\gamma_i} f_i = \mathcal{Q}_i P_{\gamma_i}^{\text{GKT}} f_i$. Rational powers of automorphisms congruent to the identity are the finite polynomials $g^c = \exp(c \log g)$, $c \in \mathbb{Q}$, and for such g, h we put $g \#_c h = g(g^{-1}h)^c$, which runs from g at $c = 0$ to h at $c = 1$. If g and h commute, then $\log(g \#_c h) = (1 - c) \log g + c \log h$. For $n = 2$ the midpoint of the residual transports is $H = \mathcal{Q}_1 \#_{1/2} \mathcal{Q}_2$, and the corrections are $C_i = H \mathcal{Q}_i^{-1}$.

Lemma D.2 (Midpoint correction). *Let g, h be as above and $c \in \mathbb{Q}$.*

- (1) $g \#_c h = h \#_{1-c} g$; in particular H is symmetric in $\mathcal{Q}_1, \mathcal{Q}_2$.
- (2) For $n = 2$,

$$(105) \quad C_i P_{\gamma_i} = H P_{\gamma_i}^{\text{GKT}}, \quad (C_1 P_{\gamma_1} f_1)(C_2 P_{\gamma_2} f_2) = H((P_{\gamma_1}^{\text{GKT}} f_1)(P_{\gamma_2}^{\text{GKT}} f_2)).$$

- (3) *These constructions are natural under finite restriction. If all paths are continued by a common path from the target region, with transport Ψ through $\mathfrak{D}$ and transport Ψ' by the factors of the GKT transition, then $\mathcal{Q}_i$ is replaced by $\Psi \mathcal{Q}_i \Psi'^{-1}$, and $(\Psi g \Psi'^{-1}) \#_c (\Psi h \Psi'^{-1}) = \Psi(g \#_c h) \Psi'^{-1}$; in particular H is replaced by $\Psi H \Psi'^{-1}$.*

Proof. (1) With $\phi = g^{-1}h$, $h(h^{-1}g)^{1-c} = g\phi(\phi^{-1})^{1-c} = g\phi^c$. (2) The first identity is $\mathcal{Q}_i = P_{\gamma_i}(P_{\gamma_i}^{\text{GKT}})^{-1}$, and the second follows because H is an algebra automorphism. (3) Finite restriction commutes with composition, log and exp. After continuation, P_{γ_i} and $P_{\gamma_i}^{\text{GKT}}$ become ΨP_{γ_i} and $\Psi' P_{\gamma_i}^{\text{GKT}}$, so $\mathcal{Q}_i$ becomes $\Psi \mathcal{Q}_i \Psi'^{-1}$; moreover $(\Psi g \Psi'^{-1})^{-1} \Psi h \Psi'^{-1} = \Psi' g^{-1} h \Psi'^{-1}$, and rational powers commute with conjugation, since log and exp do. $\square$

For γ_+ and γ_- above, $H = Z_{-1/2}$, and the corrected product $(Z_{1/2} \Sigma_1 \Sigma_2 f_1)(Z_{-1/2} \Sigma_1 \Sigma_2 f_2)$ is the image under $Z_{-1/2}$ of the product $(Z_1 \Sigma_1 \Sigma_2 f_1)(\Sigma_1 \Sigma_2 f_2)$ of the transports $P_{\gamma_+}^{\text{GKT}} f_1$ and $P_{\gamma_-}^{\text{GKT}} f_2$.

Definition D.3 (Interpolants and corrected products). Let each input i carry a positive rational degree r_i . A *product tree* T on the inputs $1, \dots, n$ is a planar rooted binary tree whose leaves are $1, \dots, n$ in this order, that is, a parenthesization of a product of n factors; its degree $|\mathsf{T}|$ is the sum of the degrees of its leaves. If T is the leaf i , its *interpolant* is $H_{\mathsf{T}} = \mathcal{Q}_i$ and its *corrected product* is $\mathsf{P}_{\mathsf{T}} = P_{\gamma_i} f_i$. If the root of T has left and right subtrees $\mathsf{T}_1, \mathsf{T}_2$, then

$$H_{\mathsf{T}} = H_{\mathsf{T}_1} \#_{|\mathsf{T}_2|/|\mathsf{T}|} H_{\mathsf{T}_2}, \quad \mathsf{P}_{\mathsf{T}} = (H_{\mathsf{T}} H_{\mathsf{T}_1}^{-1} \mathsf{P}_{\mathsf{T}_1})(H_{\mathsf{T}} H_{\mathsf{T}_2}^{-1} \mathsf{P}_{\mathsf{T}_2}).$$

For two leaves of equal degree, $H_{\mathsf{T}} = H$ and the corrections $H_{\mathsf{T}} H_{\mathsf{T}_j}^{-1}$ are the C_i . If the $\mathcal{Q}_i$ commute, induction on T gives $\log H_{\mathsf{T}} = \sum_i r_i \log \mathcal{Q}_i / \sum_i r_i$ for every product tree T , whereas iterated midpoints, which ignore the degrees, give three commuting residual transports the weights $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, depending on the parenthesization.

Proposition D.4 (Degree-weighted coherence). *For every product tree T , $\mathsf{P}_{\mathsf{T}} = H_{\mathsf{T}} \prod_i P_{\gamma_i}^{\text{GKT}} f_i$. For product trees $\mathsf{T}, \mathsf{T}', \mathsf{T}''$ on the same inputs, the tree transition $\mathcal{V}_{\mathsf{T} \leftarrow \mathsf{T}'} = H_{\mathsf{T}} H_{\mathsf{T}'}^{-1}$ carries $\mathsf{P}_{\mathsf{T}'}$ to P_{T} , and $\mathcal{V}_{\mathsf{T} \leftarrow \mathsf{T}'} \mathcal{V}_{\mathsf{T}' \leftarrow \mathsf{T}''} = \mathcal{V}_{\mathsf{T} \leftarrow \mathsf{T}''}$. In particular the composite of the tree transitions around any cycle of trees is the identity; for four inputs this includes the pentagon of the associahedron.*

Let $\mathcal{U}$ be the quotient of the free associative algebra $\mathbb{Q}\langle x, y, z \rangle$ by the span of the words containing some letter twice, let $\mathcal{Q}_1 = e^x$, $\mathcal{Q}_2 = e^y$, $\mathcal{Q}_3 = e^z$ in $\mathcal{U}$ have equal degrees, and let $\mathsf{T}_\ell = ((1, 2), 3)$ and $\mathsf{T}_r = (1, (2, 3))$. Then

$$(106) \quad \log \mathcal{V}_{\mathsf{T}_\ell \leftarrow \mathsf{T}_r} = \log(H_{\mathsf{T}_\ell} H_{\mathsf{T}_r}^{-1}) = \frac{1}{72} [[x, z], y].$$

In particular $\mathcal{V}_{\mathsf{T}_\ell \leftarrow \mathsf{T}_r}$ is not the identity in $\mathcal{U}$, so for residual transports in general position the two corrected triple products differ.

Proof. For a leaf, $\mathsf{P}_\mathsf{T} = \mathcal{Q}_i P_{\gamma_i}^{\text{GKT}} f_i$. By induction, the two corrected subtree products at the root are H_T applied to the products of the transports $P_{\gamma_i}^{\text{GKT}} f_i$ over the two subtrees; since H_T is an algebra automorphism, this gives P_T , and the statements on tree transitions follow by cancellation. For $g = e^U$, $h = e^V$, the Baker–Campbell–Hausdorff formula gives, modulo brackets of four or more factors U, V ,

$$\log(g \#_c h) = (1 - c)U + cV - \frac{1}{12}c(1 - c)([U, [U, V]] + [V, [V, U]]).$$

In $\mathcal{U}$ every word of length at least four contains a repeated letter and vanishes, and so do the cubic terms for $(U, V) = (x, y)$ and (y, z) . Hence $\log H_{(1,2)} = \frac{1}{2}(x + y)$ and $\log H_{(2,3)} = \frac{1}{2}(y + z)$. With $(U, V, c) = (\frac{1}{2}(x + y), z, \frac{1}{3})$ and $(x, \frac{1}{2}(y + z), \frac{2}{3})$, for which $c(1 - c)/12 = 1/54$,

$$\begin{aligned} \log H_{\mathsf{T}_\ell} &= \frac{1}{3}(x + y + z) - \frac{1}{216}([x, [y, z]] + [y, [x, z]]), \\ \log H_{\mathsf{T}_r} &= \frac{1}{3}(x + y + z) - \frac{1}{216}([y, [z, x]] + [z, [y, x]]). \end{aligned}$$

Their linear parts agree and they have no quadratic part, so their brackets vanish in $\mathcal{U}$, and $\log(H_{\mathsf{T}_\ell} H_{\mathsf{T}_r}^{-1})$ is their difference. Since $-[y, [z, x]] = [y, [x, z]]$ and, by the Jacobi identity, $[x, [y, z]] - [z, [y, x]] = [y, [x, z]]$, this difference is $-\frac{3}{216}[y, [x, z]] = \frac{1}{72}[[x, z], y]$, a nonzero element of $\mathcal{U}$. $\square$

The associator is computed here in the universal algebra $\mathcal{U}$. The residual transports of the paths $\gamma_\pm$ computed after Definition D.1 commute, and three residual transports of paths in $\mathfrak{D}$ with a nontrivial associator are not exhibited here.

D.2. Change of paths. Let $\gamma = (\gamma_i)$ and $\gamma' = (\gamma'_i)$ be tuples of paths as above such that γ_i and γ'_i have the same endpoints for every i , and write $H_{\gamma, \mathsf{T}}$ for the interpolant of γ and the product tree T . The *leg maps* $\mathcal{G}_i^{\gamma' \leftarrow \gamma} = P_{\gamma'_i}^{\text{GKT}} (P_{\gamma_i}^{\text{GKT}})^{-1}$ and the *transitions* $\mathcal{V}_{(\gamma', \mathsf{T}') \leftarrow (\gamma, \mathsf{T})} = H_{\gamma', \mathsf{T}'} H_{\gamma, \mathsf{T}}^{-1}$ are automorphisms of the chart algebra of the target region.

Proposition D.5 (Change of admissible paths). *For a product tree T and $a_i = P_{\gamma_i}^{\text{GKT}} f_i$, the corrected product of γ' and T is*

$$(107) \quad H_{\gamma', \mathsf{T}} \prod_i P_{\gamma'_i}^{\text{GKT}} f_i = \mathcal{V}_{(\gamma', \mathsf{T}) \leftarrow (\gamma, \mathsf{T})} H_{\gamma, \mathsf{T}} \prod_i \mathcal{G}_i^{\gamma' \leftarrow \gamma} a_i.$$

The leg maps satisfy $\mathcal{G}_i^{\gamma'' \leftarrow \gamma'} \mathcal{G}_i^{\gamma' \leftarrow \gamma} = \mathcal{G}_i^{\gamma'' \leftarrow \gamma}$, and every composite of transitions from (γ, T) to (γ', T') , in particular of tree transitions ($\gamma' = \gamma$) and of transitions changing only the paths ($\mathsf{T}' = \mathsf{T}$), equals $\mathcal{V}_{(\gamma', \mathsf{T}') \leftarrow (\gamma, \mathsf{T})}$. If $\phi \neq \text{id}$ is an automorphism of the target algebra, no automorphism carries ab to $\phi(a)b$ for all a, b ; hence, if one leg map is not the identity and another is, the leg maps cannot be absorbed into one automorphism of the target algebra.

Proof. The identities $P_{\gamma'_i}^{\text{GKT}} f_i = \mathcal{G}_i^{\gamma' \leftarrow \gamma} a_i$ and $\mathcal{V}_{(\gamma', \mathsf{T}) \leftarrow (\gamma, \mathsf{T})} H_{\gamma, \mathsf{T}} = H_{\gamma', \mathsf{T}}$ give (107), whose left side is the corrected product by Proposition D.4. The leg maps and the transitions are ratios, so their composites telescope. Finally, an automorphism carrying ab to $\phi(a)b$ for all a, b equals ϕ , by $b = 1$, and the identity, by $a = 1$. $\square$

APPENDIX E. NORMALIZATION OF THE GKT PRIMARY POTENTIAL OF $x^4 + y^4$

This appendix normalizes the source of the transport in Section 6; nothing in it involves the annulus. Appendix E.1 shows that, for a set of labels stable under the exchange of the coordinates, the primary parts of the GKT potentials of the symmetric chamber indices form a single orbit under the flows of the degree-zero field $X_{0,0}$, that this orbit contains exactly one potential invariant under the exchange, and that these potentials define the reflection-fixed primary potential $W_{\text{prim}}^{\text{GKT}}$ used in Section 6. Appendix E.2 computes its restriction to the marginal line, identifies it with the flat structure of Noumi and Yamada, and determines the first open invariants of the symmetric chamber indices with primary potential $W_{\text{prim}}^{\text{GKT}}$. We use the notation of Section 6: the source-chart coordinates x, y , the fields (76), the primary variables $t_{a,b}$ and the ring A_{prim} of (77), the good-basis cycles $\Xi_{a,b}$, the primitive-form conditions of [GKT26a, Def. 4.3], and the involution σ and the field $X_{0,0}$ of (78). The GKT notions are those recalled in Section 5.1.

E.1. The reflection-fixed primary potential. Let I be a finite set of labels, each carrying a twist (a, b) with $0 \leq a, b \leq 2$, and let $n_{a,b}$ be the number of labels of twist (a, b) . Put $A_I^{\text{prim}} = \mathbb{Q}[t_{a,b}]/(t_{a,b}^{n_{a,b}+1})$. For a symmetric chamber index ν bounded by I [GKT26a, Def. 3.47], let W_{prim}^ν be the reduction of the potential $W^{\nu, \text{sym}}$ of [GKT26a, Def. 4.12] modulo the descendent variables $t_{a,b,d}$, $d > 0$; its coefficients lie in A_I^{prim} . In [GKT26a] *symmetric* means invariance under twist-preserving permutations of the labels [GKT26a, Notation 3.42]; the exchange of the two coordinates is the separate involution σ of (78). Let $\text{Pot}(I)$ be the set of the potentials W_{prim}^ν . We call I σ -stable if $n_{a,b} = n_{b,a}$ for all a, b ; then σ acts on $A_I^{\text{prim}}[x, y]$. The σ -stable label sets are cofinal, and $\varprojlim A_I^{\text{prim}} = A_{\text{prim}}$.

Lemma E.1 (Reflection-fixed representative). *Let I be σ -stable and put $t = t_{2,2}$.*

- (1) *Every element of $\text{Pot}(I)$ is a polynomial of degree at most four in x, y . The automorphisms $\exp(fX_{0,0})$, with f in the image of $t^2\mathbb{Q}[t^2]$ in A_I^{prim} , act freely and transitively on $\text{Pot}(I)$.*
- (2) *The involution σ maps $\text{Pot}(I)$ to itself, and $\sigma \exp(fX_{0,0}) \sigma^{-1} = \exp(-fX_{0,0})$.*
- (3) *$\text{Pot}(I)$ contains exactly one σ -fixed potential W_I . If $I \subset I'$ are σ -stable, $W_{I'}$ reduces to W_I .*

Consequently the W_I define a polynomial $W_{\text{prim}}^{\text{GKT}} \in A_{\text{prim}}[x, y]$ of degree at most four with $\sigma(W_{\text{prim}}^{\text{GKT}}) = W_{\text{prim}}^{\text{GKT}}$, whose reduction to every σ -stable I is the primary part of a symmetric chamber index.

Proof. (1) By [GKT26a, (0.6)], $W^{\nu, \text{sym}}$ is homogeneous of degree 16 for the Euler field in which x, y have weight 4 and $t_{a,b,0}$ has weight $16 - 4a - 4b \geq 0$. Hence every monomial of W_{prim}^ν has degree at most four in x, y .

Let ν, ν' be symmetric chamber indices bounded by I , and let $W^\nu, W^{\nu'}$ be the labelled potentials of [GKT26a, Def. 4.6]. By [GKT26a, Cor. 4.15], $\psi_I(W^{\nu, \text{sym}}) = W^\nu$, where ψ_I is the injective ring homomorphism of [GKT26a, Lem. 4.14]. By [Wue26b, Lem. 2.5], together with [GKT26a, Cor. 5.14], the group $\tilde{G}$ generated by the wall-crossing group of [GKT26a, Def. 4.23] and the bidegree-(0,0) summands of [Wue26b, Rem. 2.3] acts freely and transitively on the labelled potentials of the chamber indices bounded by I . Definition 4.23 of [GKT26a] excludes bidegree (0,0); without these summands the group would act trivially on primary parts, as the next paragraph shows. Let $g \in \tilde{G}$ satisfy $gW^\nu = W^{\nu'}$. A twist-preserving permutation π of I fixes both potentials, so $\pi(g) = g$ by freeness.

Modulo the descendent variables, a generator of $\tilde{G}$ has the form

$$u_J X_{k_1, k_2}, \quad 4k_1 + 4k_2 = m(J, \mathbf{0}) - 16 = \sum_{j \in J} (4a_j + 4b_j - 16),$$

where X_{k_1, k_2} is the field (76), u_J is the product of the variables of a nonempty set J of primary labels, $k_1, k_2 \geq 0$, and $k_1 \equiv r_J$, $k_2 \equiv s_J$ modulo 4 [GKT26a, Def. 4.23 and Notation 2.30]. Here $m(J, \mathbf{d}) = 16 + \sum_{j \in J} (4a_j + 4b_j + 16(d_j - 1))$ is the number of [GKT26a, Notation 2.30] for $r = s = 4$, so these are the conditions (59) for $\mathbf{d} = \mathbf{0}$: the pair (k_1, k_2) is a critical exponent of $(J, \mathbf{0})$. Since $a_j, b_j \leq 2$, the sum in the display is at most zero, with equality only if every twist in J is $(2, 2)$. Hence $(k_1, k_2) = (0, 0)$, every twist in J is $(2, 2)$, and $r_J = 0$ forces $|J|$ to be even. Such J carry exactly one critical graph, of critical exponent $(0, 0)$ [GKT26a, (2.31)], and the corresponding generator, of bidegree $(0, 0)$, is the summand $u_J X_{0,0}$ adjoined in [Wue26b, Rem. 2.3]. These generators commute, so modulo the descendent variables

$$g = \exp\left(\sum_J c_J u_J X_{0,0}\right), \quad c_J \in \mathbb{Q}.$$

Invariance under π gives $c_J = c_{|J|}$, and [GKT26a, (4.12)] gives $\sum_{|J|=m} u_J = \psi_I(t^m/m!)$. The map ψ_I commutes with $X_{0,0}$ and, by its monomial description [GKT26a, (4.12)], remains injective modulo the descendent variables. Hence $\exp(fX_{0,0})W_{\text{prim}}^\nu = W_{\text{prim}}^{\nu'}$ with $f = \sum_k c_{2k} t^{2k}/(2k)!$.

Conversely, for $f \in t^2\mathbb{Q}[t^2]$ the element $\psi_I(f)X_{0,0}$ lies in the Lie algebra of $\tilde{G}$, so $\exp(\psi_I(f)X_{0,0})W^\nu = W^{\nu''}$ for a chamber index ν'' bounded by I . A labelled potential determines its chamber index, because the balanced graphs over a given label set and descendent vector have distinct balanced profiles. Since $\exp(\psi_I(f)X_{0,0})$ and W^ν are fixed by every twist-preserving permutation, ν'' is symmetric, and its primary part is $\exp(fX_{0,0})W_{\text{prim}}^\nu$. This proves transitivity.

For freeness, let f have lowest term $ct^{2k} \neq 0$ in A_I^{prim} . Since $W \equiv x^4 + y^4$ modulo $\mathfrak{m}_{\text{prim}}$ and $X_{0,0}x^4 = 4x^4$, the coefficient of the monomial $t^{2k}x^4$ in $\exp(fX_{0,0})W - W$ is $4c$.

(2) Write $\Gamma_{0,k_1,k_2,1,J}$, as in [GKT26a, Notation 2.27], for the smooth genus-zero graph with k_1 r -tails, k_2 s -tails, one fully twisted boundary tail and internal labels J (Section 5.1). Choose a bijection ς of I taking each label of twist (a, b) to a label of twist (b, a) . For ν symmetric define ν^σ by $\nu_{\Gamma_{0,k_1,k_2,1,J}, \mathbf{d}}^\sigma = \nu_{\Gamma_{0,k_2,k_1,1,\varsigma(J)}, \mathbf{d} \circ \varsigma^{-1}}$. Since $r = s = 4$, the balancing conditions [GKT26a, Prop. 2.31], the numbers $m(J, \mathbf{d})$, and the Gamma weights of [GKT26a, Notation 3.42] are invariant under exchanging the two entries of every twist and of every balanced profile. Hence $A(\varsigma(J), \mathbf{d} \circ \varsigma^{-1}, \nu^\sigma) = A(J, \mathbf{d}, \nu)$, the conditions of [GKT26a, Def. 3.47] for ν^σ follow from those for ν , and ν^σ is symmetric. Formula [GKT26a, (4.11)] gives $W^{\nu^\sigma, \text{sym}} = \sigma(W^{\nu, \text{sym}})$. The conjugation formula follows from $\sigma(t) = t$ and $\sigma X_{0,0} \sigma^{-1} = -X_{0,0}$.

(3) Fix $W \in \text{Pot}(I)$ and write $\sigma W = \exp(gX_{0,0})W$, using (1) and (2). For h in the image of $t^2\mathbb{Q}[t^2]$,

$$\sigma(\exp(hX_{0,0})W) = \exp(-hX_{0,0})\sigma W = \exp((g - h)X_{0,0})W,$$

which equals $\exp(hX_{0,0})W$ if and only if $g = 2h$, by freeness. Thus $W_I = \exp(\frac{1}{2}gX_{0,0})W$ is the unique σ -fixed element. For $I \subset I'$, restricting a chamber index to the graphs with labels in I preserves the conditions of [GKT26a, Def. 3.47], which for a subset of I involve only such graphs. Restriction reduces W_{prim}^ν along $A_{I'}^{\text{prim}} \rightarrow A_I^{\text{prim}}$ and commutes with σ , so the image of $W_{I'}$ is a σ -fixed element of $\text{Pot}(I)$, namely W_I . $\square$

We call $W_{\text{prim}}^{\text{GKT}}$ the *reflection-fixed primary potential*, and a symmetric chamber index ν , bounded by a σ -stable label set I , with $W_{\text{prim}}^\nu = W_I$ a *reflection-fixed chamber index*. Lemma E.1 fixes only the primary part, so such indices are not unique; their primary invariants, the only ones quoted below, are determined by $W_{\text{prim}}^{\text{GKT}}$. The variable $t_{0,0}$ has Euler weight 16, so it occurs only in monomials $t_{0,0}t_{2,2}^k$ of degree zero in x, y ; write their sum as $c(t_{2,2})t_{0,0}$. The $\Xi_{0,0}$ period of $W_{\text{prim}}^{\text{GKT}}$ is $\exp(c(t_{2,2})t_{0,0}/\hbar)$ times the $\Xi_{0,0}$ period of its restriction to $t_{0,0} = 0$, and by [GKT26a, Cor. 4.17] the latter is $1 + O(\hbar^{-1})$. The part of the $\hbar^{-1}$ coefficient that is linear in $t_{0,0}$ is therefore $c(t_{2,2})t_{0,0}$, and by the same corollary it equals $t_{0,0}$. Hence

$$(108) \quad W_{\text{prim}}^{\text{GKT}} = t_{0,0} + (\text{terms free of } t_{0,0}).$$

On the marginal line $t_{a,b} = 0$, $(a, b) \neq (2, 2)$, the monomials of $W_{\text{prim}}^{\text{GKT}}$ have degree four in x, y , since $t_{2,2}$ has Euler weight zero. The congruences $k_1 \equiv r_J$, $k_2 \equiv s_J$ modulo 4 of [GKT26a, Prop. 2.31(a)] allow only $t^{2k}x^4$, $t^{2k}y^4$ and $t^{2k+1}x^2y^2$, and σ -invariance equates the coefficients of x^4 and y^4 . By [GKT26a, Cor. 4.17], applied for every σ -stable label set I , $dx \wedge dy$ is a primitive form for $W_{\text{prim}}^{\text{GKT}}$ in the sense of [GKT26a, Def. 4.3], with the nine $t_{a,b}$ as flat coordinates. By [GKT26a, Rem. 4.5] these flat coordinates agree with those of [HLSW22, §2.2], which come from Saito's theory of primitive forms [Sai83].

Remark E.2 (Reflection). The involution σ fixes $W_{\text{prim}}^{\text{GKT}}$ and pulls $dx \wedge dy$ back to $-dx \wedge dy$. Differentiating $\sigma(W_{\text{prim}}^{\text{GKT}}) = W_{\text{prim}}^{\text{GKT}}$ gives $\sigma(\Phi_{b,a}^0) = \Phi_{a,b}^0$ for the Kodaira–Spencer classes of Lemma 6.2. The sign of $\sigma^*(dx \wedge dy)$ is compensated by the exchange of $\partial_x W_{\text{prim}}^{\text{GKT}}$ and $\partial_y W_{\text{prim}}^{\text{GKT}}$, in which the Grothendieck residue symbol is alternating; hence the functional ϵ of (82) satisfies $\epsilon(\sigma h) = \sigma(\epsilon(h))$, where σ acts on coefficients by $t_{a,b} \mapsto t_{b,a}$. Thus σ is an automorphism of the source Frobenius manifold of Lemma 6.2 covering a nontrivial involution of the base. It is not A_{prim} -linear; at the Fermat point its invariants in $H_{4,4}$ form a subspace of dimension six.

E.2. The flat coordinate on the marginal line. Set $t = t_{2,2}$. The marginal quartics are

$$(109) \quad W_{\mathbf{a}, \mathbf{b}, \mathbf{a}'} = \mathbf{a}x^4 + \mathbf{b}x^2y^2 + \mathbf{a}'y^4, \quad \Omega = dx \wedge dy.$$

For a unit $q \in \mathbb{Q}[[t]]$ the coordinate change $(x, y) \mapsto (qx, q^{-1}y)$ preserves Ω and acts by

$$(110) \quad (\mathbf{a}, \mathbf{b}, \mathbf{a}') \mapsto (q^4\mathbf{a}, \mathbf{b}, q^{-4}\mathbf{a}').$$

It is the monomial rescaling $z^m \mapsto q^{m_1-m_2}z^m$, a single-chart instance of the zero-cochain action of [GHS22, Prop. 5.2]; for $q = \exp(f)$ it is the flow $\exp(fX_{0,0})$ of Lemma E.1. Its invariants are $\mathbf{b}$ and $\mathbf{a}\mathbf{a}'$. An orbit with $\mathbf{a}(0) = \mathbf{a}'(0) = 1$ has a unique representative with $\mathbf{a} = \mathbf{a}'$, namely $\mathbf{a} = \mathbf{a}' = \sqrt{\mathbf{a}\mathbf{a}'}$ with the square root equal to 1 at $t = 0$.

For the good-basis cycles $\Xi_{0,0}$ and $\Xi_{2,2}$ define the period series

$$(111) \quad \varpi_0(\mathbf{a}, \mathbf{b}) = \sum_{i,j,\ell \geq 0} \frac{(-1)^{i+j} \binom{1}{4}_{i+\ell} \binom{1}{4}_{j+\ell}}{i! j! (2\ell)!} (\mathbf{a} - 1)^{i+j} \mathbf{b}^{2\ell},$$

$$(112) \quad \varpi_2(\mathbf{a}, \mathbf{b}) = \sum_{i,j,\ell \geq 0} \frac{(-1)^{i+j} \binom{3}{4}_{i+\ell} \binom{3}{4}_{j+\ell}}{i! j! (2\ell+1)!} (\mathbf{a} - 1)^{i+j} \mathbf{b}^{2\ell+1}.$$

By the proof of Theorem E.3, ϖ_0 and $\varpi_2/\hbar$ are the periods of $(W_{\mathbf{a}, \mathbf{b}, \mathbf{a}}, \Omega)$ over $\Xi_{0,0}$ and $\Xi_{2,2}$, and the other seven periods vanish.

Theorem E.3 (Periods and flat coordinate on the marginal line). *There is a unique pair $\mathbf{a}(t), \mathbf{b}(t) \in \mathbb{Q}[[t]]$, with $\mathbf{a}(0) = 1$ and $\mathbf{b}(t) = t + O(t^2)$, such that*

$$(113) \quad \varpi_0(\mathbf{a}(t), \mathbf{b}(t)) = 1, \quad \varpi_2(\mathbf{a}(t), \mathbf{b}(t)) = t.$$

The quartic

$$W_t^{\text{flat}} = \mathfrak{a}(t)(x^4 + y^4) + \mathfrak{b}(t)x^2y^2$$

is the restriction of $W_{\text{prim}}^{\text{GKT}}$ to the marginal line, and

$$(114) \quad \mathfrak{a}(t) = 1 + \frac{t^2}{16} + \frac{t^4}{768} + \frac{t^6}{61440} + O(t^8),$$

$$(115) \quad \mathfrak{b}(t) = t + \frac{t^5}{1280} + O(t^9).$$

For two labels of twist $(2, 2)$, the open invariants of every reflection-fixed chamber index at the balanced profiles $(4, 0)$ and $(0, 4)$ are

$$(116) \quad \nu_{(4,0)} = \nu_{(0,4)} = -\frac{1}{8},$$

and for three such labels its open invariant vanishes. The first correction visible in the Jacobian algebra of W_t^{flat} is the term $t^3/32$ in the relations

$$(117) \quad x^3 = \left(-\frac{t}{2} + \frac{t^3}{32}\right)xy^2, \quad y^3 = \left(-\frac{t}{2} + \frac{t^3}{32}\right)x^2y \pmod{t^4}.$$

Proof. By [GKT26a, Lem. 4.2 and (4.3)] the quartic good-basis moments are

$$P_0(4k) = (-\hbar)^k \left(\frac{1}{4}\right)_k, \quad P_2(4k+2) = (-\hbar)^k \left(\frac{3}{4}\right)_k,$$

and vanish in the complementary congruence classes. Expanding

$$\exp\left(\frac{(\mathfrak{a}-1)(x^4 + y^4) + \mathfrak{b}x^2y^2}{\hbar}\right)$$

against $\Xi_{0,0}$ and $\Xi_{2,2}$ gives precisely ϖ_0 and $\varpi_2/\hbar$: an even number of insertions of x^2y^2 contributes to the first period and an odd number to the second, and every power of $\hbar$ cancels except the displayed one. The seven remaining periods vanish by the phase. Hence $dx \wedge dy$ is a primitive form for a marginal quartic with $\mathfrak{a} = \mathfrak{a}'$, with t as flat coordinate, if and only if (113) holds.

At $(\mathfrak{a}, \mathfrak{b}) = (1, 0)$ the Jacobian matrix of (ϖ_0, ϖ_2) with respect to $(\mathfrak{a}, \mathfrak{b})$ is $\text{diag}(-1/2, 1)$. The formal inverse function theorem gives existence and uniqueness; coefficient extraction gives (114)–(115). By the discussion after Lemma E.1, the marginal restriction of $W_{\text{prim}}^{\text{GKT}}$ has the form $\mathfrak{a}(x^4 + y^4) + \mathfrak{b}x^2y^2$ with $\mathfrak{a}(0) = 1$ and $\mathfrak{b} = t + O(t^2)$, and it satisfies (113); by uniqueness it is W_t^{flat} .

At order t^2 ,

$$\varpi_0 = 1 - \frac{1}{2}(\mathfrak{a}-1) + \frac{\mathfrak{b}^2}{32} + O(t^3),$$

so $\mathfrak{a}-1 = t^2/16 + O(t^3)$. By the sign and the order-two automorphism factor in [GKT26a, (4.11)], the coefficient of t^2x^4 , respectively t^2y^4 , is $-1/2$ times the corresponding invariant for two labels. Hence (116). In particular $\nu_{(4,0)} + \nu_{(0,4)} = -\frac{1}{4}$, in agreement with the period computation of [Wue26b, Rem. 4.3]; this is the sum considered in [GKT26b, Ex. 5.5], and it differs from the value $-\frac{1}{8}$ printed there. The coefficient of t^3 in the second period equation gives the vanishing of the invariant for three labels. Finally the Jacobian relation is $x^3 = -\mathfrak{b}(2\mathfrak{a})^{-1}xy^2$, and expansion of $-\mathfrak{b}/(2\mathfrak{a})$ gives (117). $\square$

In the coordinates $\tilde{x} = \mathbf{a}(t)^{1/4}x$, $\tilde{y} = \mathbf{a}(t)^{1/4}y$ the pair $(W_t^{\text{flat}}, dx \wedge dy)$ becomes $(\tilde{x}^4 + \tilde{y}^4 + s\tilde{x}^2\tilde{y}^2, \mathbf{a}(t)^{-1/2}d\tilde{x} \wedge d\tilde{y})$ with $s = \mathbf{b}(t)/\mathbf{a}(t)$. The same substitution in the moments gives $\varpi_0(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{-1/2}\varpi_0(1, s)$ and $\varpi_2(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{-1/2}\varpi_2(1, s)$, while

$$\varpi_0(1, s) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \frac{s^2}{4}\right), \quad \varpi_2(1, s) = s {}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; \frac{s^2}{4}\right).$$

Hence (113) is equivalent to

$$(118) \quad \mathbf{a}(t)^{1/2} = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \frac{s^2}{4}\right), \quad t = \frac{s {}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; \frac{s^2}{4}\right)}{{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \frac{s^2}{4}\right)}.$$

The second relation is the flat coordinate of the simple elliptic singularity $x^4 + y^4$ along its marginal deformation computed by Noumi and Yamada [NY98], as quoted in [Str12, (7), case $\tilde{E}_7$]. The statements of [Mah24, Props. 4.2.6–4.2.8] for elliptic singularities of Fermat type (the primitive form as the reciprocal of a Gauss hypergeometric function times the volume form after Morse stabilization, the flat coordinate, and $s(t)$ as a Schwarz triangle function) specialize here to ${}_2F_1(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; z)$, $z = s^2/4$. By the first relation in (118), the resulting primitive form $d\tilde{x} \wedge d\tilde{y}/{}_2F_1(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \frac{s^2}{4})$ is $dx \wedge dy$, the normalization of [GKT26a] used in Theorem E.3. In this normalization the coefficients of the potential are determined by the open invariants of a reflection-fixed chamber index through [GKT26a, (4.11)], and the series (114)–(115) and the invariants (116) refer to it.

The residue pairing detects this normalization at quadratic order. For $W_c = (1 + ct^2)(x^4 + y^4) + tx^2y^2$, the normalized residue of the socle monomial is

$$(119) \quad 16 \operatorname{Res}_0 \frac{x^2y^2 dx \wedge dy}{\partial_x W_c \partial_y W_c} = \frac{1}{(1 + ct^2)^2 - t^2/4}.$$

Since $x^4 \equiv -\frac{t}{2(1+ct^2)}x^2y^2$ in the Jacobian algebra, the residue pairing η of $(W_c, dx \wedge dy)$ satisfies

$$(120) \quad \eta(\partial_t W_c, 1) = \frac{1 - 2ct^2/(1 + ct^2)}{(1 + ct^2)^2 - t^2/4} = 1 + \left(\frac{1}{4} - 4c\right)t^2 + O(t^4).$$

When $dx \wedge dy$ is a primitive form and t a flat coordinate, this entry of the flat metric is constant, equal to its value 1 at $t = 0$. Its constancy forces $c = 1/16$ at order t^2 , whereas the product first sees c at order t^3 : in (117) the relevant coefficient is $-t/(2(1 + ct^2)) = -t/2 + (c/2)t^3 + O(t^5)$.

Off the marginal line the flat coordinates are simultaneous, and on the first two-parameter slice this already fixes a term that the critical locus does not see. Set $v = t_{1,1}$ and $t = t_{2,2}$. Through total degree three, the restriction of $W_{\text{prim}}^{\text{GKT}}$ to the slice on which the other seven $t_{a,b}$ vanish is

$$(121) \quad W_{v,t} = \left(1 + \frac{t^2}{16}\right)(x^4 + y^4) + tx^2y^2 + \left(v - \frac{vt^2}{16}\right)xy - \frac{v^2t}{32} + O((v, t)^4).$$

The two new open invariants of every reflection-fixed chamber index are

$$\nu_{vt^2; (1,1)} = -\frac{1}{8}, \quad \nu_{v^2t; (0,0)} = -\frac{1}{16}.$$

They are the solutions of the chamber conditions [GKT26a, Def. 3.47(3)] for the supports J consisting of one label of twist (1, 1) and two of twist (2, 2), respectively two of twist (1, 1) and one of twist (2, 2); both supports satisfy the inequality in that condition. These supports have the single balanced profiles (1, 1) and (0, 0), and no two-element support containing a label of

twist $(1, 1)$ has a balanced profile. In the expression $A(J, \mathbf{0}, \nu)$ of Section 5.1, with the weights (58), the two conditions therefore read

$$\nu_{vt^2; (1,1)} + \frac{1}{2}(\nu_{(4,0)} + \nu_{(0,4)}) + \frac{1}{4} = 0, \quad \nu_{v^2t; (0,0)} + \frac{1}{16} = 0,$$

where the middle term comes from the partition that separates the label of twist $(1, 1)$, the terms $\frac{1}{4}$ and $\frac{1}{16}$ come from the partitions into singletons, and $\nu_{(4,0)} = \nu_{(0,4)} = -\frac{1}{8}$ by (116). The order-two automorphism factors in [GKT26a, (4.11)], from the two labels of equal twist, divide the two invariants by two, giving the last two corrections in (121). The periods confirm these values. For the $\Xi_{1,1}$ period, the direct correction $-\frac{vt^2}{16}xy$, the cross term of vxy with $\frac{t^2}{16}(x^4 + y^4)$, and the cubic term $vxy(tx^2y^2)^2$ contribute

$$-\frac{vt^2}{16\hbar}, \quad -\frac{vt^2}{16\hbar}, \quad \frac{vt^2}{8\hbar};$$

for $\Xi_{0,0}$ the two contributions are $-v^2t/(32\hbar)$ and $v^2t/(32\hbar)$. Both sums vanish, as the flatness of v and t requires. Thus the constant term $-v^2t/32$ is fixed by the oscillatory integral although it is invisible to the critical locus.

APPENDIX F. NOTATION

The principal symbols, with the place where each is defined. Local symbols used in a single proof are not listed. The letter T with a singular point as index, as in T_A, T_B, T_C , denotes a monodromy of the disk; otherwise T , also as an index, denotes a finite rooted subtree of the path tree $\mathcal{T}$. The sans-serif letters A, B denote the singleton Hamiltonian vector fields, and the roman letters A, B, C singular points; exceptions are GKT's rings $A_I, A_{\mathcal{I}}, A_I^{\text{prim}}$ and functional $A(J, \mathbf{d}, \nu)$, the ring A of the root-stack convention of Appendix A, the label subsets A and graph bases BA of Appendix B, the base B of [GS11, CPS24], the logarithms C_{τ} of ray factors (Section 3) and the corrections C_i of Appendix D.

Lattice conventions. The fan lattice is called M here because it carries the exponents. The following dictionary compares the conventions with those of toric geometry and of [GS11, CPS24]; in the latter, walls lie on a base B and are tangent to their exponents.

This paper	Toric geometry; [GS11, CPS24]
M , the lattice of the fan of $\mathbb{P}(1, 1, 2)$	N , the lattice of the fan
exponents of slab functions, wall functions and derivations lie in M , and in $\widetilde{M}$ on the annulus	exponents are integral tangent vectors of the base B that carries the walls
the polygon Δ , the cuts and the wall supports lie in $M^{\vee} \otimes \mathbb{R}$	$M_{\mathbb{R}}$; in [CPS24, §7.1] the polytope is the intersection complex $\check{B}$, the Legendre dual of B

Symbol	Meaning	Defined in
<i>Lattices and exponents</i>		
$M, M^{\vee}$	fan lattice of $\mathbb{P}(1, 1, 2)$ (the lattice N of toric geometry), and its dual, the plane of Δ	Section 2
$\mathbb{Z}_{x,y}^2$	bidegrees (a, b) of monomials $x^a y^b$	(7)
$\iota, F = \frac{1}{4}\iota$	coordinate-divisor map; exponent map $x^a y^b \mapsto z^{F(a,b)}$	(7), (10)
$\widetilde{M}, \widetilde{M}^{\vee}$	$F(\mathbb{Z}_{x,y}^2) \supset M$, index eight; its dual	(10); Section 2.3

Symbol	Meaning	Defined in
$\widetilde{M}/M$	fractional-exponent group $\mathbb{Z}/2 \oplus \mathbb{Z}/4$	(14)
L	cofactor $\det(\iota)\iota^{-T} = -\frac{1}{2}F^{-T}$, the support map	(8)
ψ	auxiliary quadratic function with $Lq = d\psi(F(q))$	Section 2.2
σ, S	exchange of x and y ; its action $F\sigma F^{-1}$ on M	Section 2.2
κ, X_q^κ	the exponent $(1, 1)$ of xy ; Hamiltonian field of $x^{q_1}y^{q_2}$	Section 2.2
$\widetilde{n}_q$	$F^{-T}q^\perp$, the direction of X_q^κ	(11)
$X_{a,b}, E_{a,b}$	$X_{\kappa+(a,b)}^\kappa$ and $t_1^a t_2^b X_{a,b}$	Section 1.2
<i>The base</i>		
$\Delta, \Delta^\vee, O$	anticanonical polygon, its polar, central vertex	(3); Section 2.3
φ	polarization	Section 2.3
dual intersection complex	discrete Legendre transform of the CPS complex, with tangent lattice M	Section 1.2
A, B, C, p_V	singular points; vertex of Δ on the star edge through V	Section 2.3
width	the integer w with a unipotent monodromy conjugate to $\begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}$; for $w > 1$ at a singular point, its order in [CPS24, §8.1]	Section 2.3
c_V	outer half-edge of Δ from V to p_V ; its preimage arcs in $\overline{\mathcal{A}}$ are the cuts	Section 2.4
T_A, T_B, T_C	monodromies of M about the singular points	(16)
a_ρ	slab parameter; nodal values $a_\rho = \pm 2$	Section 2
$\overline{\mathcal{A}}, \mathcal{A}^\circ$	the annulus; complement of its four singular points	Theorem 2.3
$B_0, B_1, \widetilde{A}, \widetilde{C}$	focus-focus points; ramification points	Theorem 2.3
γ_{core}, K	core loop; holonomy along it	Section 2.3, (2)
S_0, S_1, O_0, O_1	sheets; lifts of O	Section 2.4
$p_{\text{fr}}, \delta_{\text{deck}}$	fractional-exponent cover; its deck involution	Section 2.4
Q	twist group $\widetilde{M}/4\widetilde{M}$; the twist cover trivializes it	Section 3.4
<i>Coefficients</i>		
H_V	monodromy of $\widetilde{M}$ about a singular point V of $\overline{\mathcal{A}}$, in F -coordinates	(18)
u_V, N_V, d_V, g_V, f_V	invariant character, conormal, root degree, root and slab function at $V \in \{\widetilde{A}, B_0, B_1, \widetilde{C}\}$	(22)
$\widetilde{\theta}_V$	lifted slab map	Section 2.4
frame	trivialization of the local system $\widetilde{M}$ over a region, an identification with $F(\mathbb{Z}_{x,y}^2)$; on S_0 the pulled-back chart induces one	Section 2.3
chart algebra	$\mathbb{k}[\widetilde{M}]$ in the frame of a region, over a Kummer ring	Section 2.4
$\Pi, \mathcal{T}, T$	$\pi_1(\mathcal{A}^\circ)$; path tree; finite rooted subtree	Section 2.5
e, ϱ_e, D_e	Kummer edge; its radicand; its root divisor	Section 2.5

Symbol	Meaning	Defined in
$\widehat{\varrho}_e$	Laurent polynomial with $D_e = \operatorname{div} \widehat{\varrho}_e$ and $\varrho_e \widehat{\varrho}_e^{-1}$ a d_e th power	Lemma 4.6
D_T	union of the root divisors of T	Section 4.5
R_T^{root}	Kummer ring of T	(27)
$R_{\text{path}}^{\text{root}}, \mathfrak{X}_{\text{path}}$	pathwise Kummer ring; pathwise root object	Definition 2.6
$\mathbb{T}, P_T, G_T, \mathfrak{X}_T$	character torus; Kummer cover; its group; root stack $[P_T/G_T]$	Sections 2.5, 4.5
finite level, $\mathfrak{X}_{T, \mathcal{W}, N}$	a finite rooted subtree T with a truncated coefficient ring; the root stack $\mathfrak{X}_T$ base-changed to that ring	Sections 2.4, 4
<i>Brackets and walls</i>		
$\Omega_\kappa, \pi_\kappa, \mathcal{L}_\kappa$	volume form, bivector, volume line	Section 3.1
$\mathfrak{g}_\kappa$	Hamiltonian wall algebra	Section 3.1
$A, B; t_1, t_2; \mathfrak{m}$	singleton Hamiltonian vector fields; singleton parameters; singleton ideal	Section 1.2; Definition 2.6
$C_{\mathfrak{r}}, c_{p,q}$	logarithms of the ray factors of the singleton commutator; their coefficients	Proposition 3.5
$a_{p,q}$	coefficients in the ordered factorization of $e^A e^B$	(70)
Θ_γ	path-ordered product along γ	Definition 3.4
$\gamma_\pm, o_\pm$	paths passing the joint above and below O ; their endpoints	Theorem 3.7, Section 7
$\mathfrak{D}$	the wall structure on $\mathcal{A}^\circ$	Theorem 3.7
$\mathfrak{r}_{1,1}$	first mixed ray, the ray of $(1, 1)$; its wall is the diagonal wall	Section 3.3
$\mathfrak{p}, P_{\mathfrak{p}}$	admissible path; its transition map	Definition 3.8, Section 6.2
<i>GKT data and the comparison</i>		
k_1, k_2, k_{12}	numbers of r -tails, s -tails and fully twisted tails	Section 5.1
ν, W^ν	chamber index; its potential	Section 5.1
X_k	GKT critical generator, $X_k = X_{k+\kappa}^\kappa$	Section 5.1
boundary ray	the ray $\mathbb{R}_{\geq 0}(k+\kappa)$ of X_k , spanned by its shifted exponent	Sections 1, 3.2
$\mathcal{I}, A_{\mathcal{I}}, u_{i,d}$	labels with twists; labelled ring; its variables	(56)
$\tau; \tau_1, \tau_2$	twist map of the labels; the twists $(1, 0)$ and $(0, 1)$	Section 5.1; Definition 4.1
$b_{J,h}, k_{J,h}$	balanced profiles and critical exponents of J	(67)
$\nu^{\min}, \nu^{\max}, W^{\min}, W^{\max}$	extremal chamber indices; extremal GKT potentials	Section 5.2, (62)
Σ_1, Σ_2, Z_c	singleton factors; flows $\exp(ct_1 t_2 X_{1,1})$	Section 5.3
$G_{\text{mix}}, R_{\text{mix}}^{\text{res}}$	mixed factor of the GKT transition; mixed residual factor	Section 5.3
$G_{\mathfrak{r}}, S_{\mathfrak{r}}, R_{\mathfrak{r}}^{\text{res}}$	transition, wall and residual logarithms	Definition 5.4
$\mathcal{W}, I_{\mathcal{W}}, \mathcal{P}, R_{\mathcal{W}, N}, v_{\mathcal{W}}$	labelled truncation, its labels, its monomial set, its ring, base change	Definition 4.1

Symbol	Meaning	Defined in
$\mathbb{D} = (I, \mathcal{P}, \mathcal{E})$	finite assembling-closed system: labels, monomial set, graph data	Definition B.1
$\mathfrak{t}, \mathfrak{t}(J)$	time of a homotopy; time assigned to a set J of labels	Section 5.1
$t_{a,b}$	primary variables	Section 6
$V(a, b), \mathcal{N}^{\max}$	two-label family; maximal normal form	Section 7.2

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