

THE CENTRAL-CHARGE THRESHOLD FOR WALL-CROSSING IN TWO-VARIABLE OPEN FJRW THEORY

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ABSTRACT. For the genus-zero open FJRW theory of Gross, Kelly and Tessler attached to a two-variable Fermat polynomial, we prove, as they predicted, that the primary invariants do not depend on the choice of symmetric canonical boundary conditions exactly for the simple singularities, of types A , D_4 , E_6 and E_8 , that is, if and only if the central charge is less than one. The proof bounds the critical graphs, which carry the jumps, by the central charge and passes to the invariants through the wall-crossing group. Whenever a critical graph has one boundary marking of each type, this group acts simply transitively only after the vector field of the anti-diagonal torus action is adjoined. Beyond the simple range, dependence first appears with two interior insertions when both exponents are at least four and with three otherwise, while one insertion of descendent degree one produces wall-crossing for every pair. At central charge one every primary wall generator is a multiple of the torus vector field. A B-model analogue, for open Saito potentials, is due to Maher.

1. INTRODUCTION

Background. For a nondegenerate quasi-homogeneous polynomial W and a group G of diagonal symmetries, FJRW theory [FJR13] assigns to the Landau–Ginzburg model (W, G) a cohomological field theory: a genus-zero A -model built from moduli of W -spin curves and the Euler class of a Witten bundle. In their monograph [GKT22] (whose numbering and page references we take from arXiv version 3), Gross, Kelly and Tessler construct the *open* analogue for the two-variable Fermat model

$$W = x^r + y^s, \quad G = \mu_r \times \mu_s, \quad r, s \geq 2,$$

replacing closed W -spin curves by W -spin disks. The resulting open FJRW invariants are integrals of multisections of a descendent Witten bundle over moduli spaces that are real orbifolds with corners, and their central output is an open mirror theorem: a generating function of the open invariants is a versal deformation of W whose oscillatory period integrals recover the closed extended FJRW invariants, with the deformation parameters already flat [GKT22, Theorem 0.3].

A structural feature distinguishes the open theory from its closed counterpart and from the open r -spin theory of Buryak–Clader–Tessler [BCT24, BCT22]: the invariants are *not* uniquely determined. Boundary conditions for the Witten bundle over the corners of moduli space are constrained, but not determined, by the positivity requirements of [GKT22]; the residual freedom makes the open invariants jump as the boundary data cross certain walls. Gross, Kelly and Tessler organize these jumps into a pro-nilpotent Landau–Ginzburg wall-crossing group $G^{r,s}$ acting on the set of possible systems of invariants, and state that this action is faithful and transitive [GKT22, Theorems 0.7 and 4.27], so that the systems of open invariants

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form a single torsor, while the period integrals, hence the closed theory, are constant along it. As explained below, transitivity requires one further generator whenever a critical graph with one boundary marking of each type occurs.

The source of every wall is a *critical graph*: a stable W -spin graph Λ whose Witten bundle has rank one above the dimension of its moduli space, $\text{rank } E_\Lambda(\mathbf{d}) = \dim \overline{\mathcal{M}}_\Lambda^W + 1$ [GKT22, Definition 2.33]. Such a graph indexes a one-parameter family of codimension-one boundary strata through which zeros of the multisection can escape. The wall-crossing Lie algebra $\mathfrak{g}^{r,s}$ of [GKT22, Definition 4.23] is a direct sum of one-dimensional pieces indexed by critical graphs of nonzero bidegree (defined below); the excluded bidegree $(0,0)$ is spanned by $x\partial_x - y\partial_y$, and we return to it below.

The prediction. Whether critical graphs exist depends on (r,s) and on whether one allows descendents. In the descendent-free (*primary*) theory, [GKT22, Remark 0.9] predicts that wall-crossing is absent exactly for the simple singularities, and asserts this for an explicit list of pairs. As stated, that remark contains a discrepancy between its list and its naming, which we record and correct below. The prediction itself, that the primary open FJRW invariants are independent of the boundary conditions if and only if W is a simple singularity, is left open in [GKT22, Remark 0.9].

Results. We prove the prediction as a threshold on the central charge. Recall that the *central charge* of $W = x^r + y^s$, whose variables carry weights $q_x = 1/r$ and $q_y = 1/s$, is

$$(1) \quad \hat{c}_W = (1 - 2q_x) + (1 - 2q_y) = 2 - \frac{2}{r} - \frac{2}{s}.$$

Fix a finite set I of labeled interior markings, each label i carrying a twist (a_i, b_i) with $0 \leq a_i \leq r - 2$ and $0 \leq b_i \leq s - 2$. A family of GKT canonical boundary conditions is called *symmetric* if it is invariant under permutations of labels with the same twist.

Theorem 1.1 (Central-charge criterion). *Let $W = x^r + y^s$ with $r, s \geq 2$ and central charge $\hat{c}_W$ as in (1). The following are equivalent.*

- (i) $\hat{c}_W < 1$.
- (ii) $\frac{1}{r} + \frac{1}{s} > \frac{1}{2}$.
- (iii) W is a simple singularity: up to interchange of x and y ,

$$(r, s) \in \{(n, 2) : n \geq 2\} \cup \{(3, 3), (3, 4), (3, 5)\},$$

of type A_{n-1} , D_4 , E_6 , E_8 respectively.

- (iv) For every finite labeled marking set I as above, the genus-zero primary open FJRW invariants of $(x^r + y^s, \mu_r \times \mu_s)$ are independent of the choice of symmetric canonical boundary conditions.

When these fail, that is when $\hat{c}_W \geq 1$, there is a finite marking set I for which the primary invariants depend on the boundary conditions.

The equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is elementary and classical [Arn72, Sai74]; the content is (iv). The threshold $\hat{c}_W = 1$ is exactly the boundary between the simple singularities ($\hat{c}_W < 1$) and the parabolic (simple elliptic, $\hat{c}_W = 1$) and hyperbolic ($\hat{c}_W > 1$) regimes; here *hyperbolic* refers to the triangle group $(2, r, s)$, not to Arnold's hyperbolic singularities $T_{p,q,r}$, which are not quasi-homogeneous. Thus the presence of primary wall-crossing detects, through open enumerative geometry, the classical ADE threshold.

Beyond the dichotomy, the same count determines *when* wall-crossing first appears. For an interior insertion the relevant twists lie in the range $0 \leq a \leq r - 2$, $0 \leq b \leq s - 2$; write n for the number of interior insertions.

Theorem 1.2 (Least number of insertions). *Let $n_*(r, s)$ be the least number of primary interior insertions admitting a critical graph. Then*

$$n_*(r, s) = \begin{cases} \infty, & \frac{1}{r} + \frac{1}{s} > \frac{1}{2} \quad (\text{simple}), \\ 2, & \min(r, s) \geq 4, \\ 3, & \min(r, s) = 3 \text{ and } \max(r, s) \geq 6. \end{cases}$$

The case $n_* = 2$ is first realized by $W = x^4 + y^4$ with two interior insertions of twist $(2, 2)$, the example of [GKT26, Examples 4.9 and 5.5].

By the *first wall* of (r, s) we mean the set of critical graphs over interior multisets of the minimal size $n_*(r, s)$. By Proposition 4.1 below, $n_*(r, s)$ is also the least size of a marking set over which the primary invariants depend on the boundary conditions.

Finally, descendents change the picture entirely: they generate walls in every case, including the simple singularities, as [GKT22, Remark 0.9] anticipates. The first such wall is universal.

Proposition 1.3 (First descendent wall). *For every (r, s) with $r, s \geq 2$, the data of a single interior insertion of descendent degree one admit exactly one critical graph. Hence the open FJRW invariants with descendents exhibit wall-crossing for every two-variable Fermat model, simple or not.*

Thus the descendent-free theory is independent of the boundary conditions exactly on the ADE locus, while with a single descendent the invariants depend on them in every case.

The torus vector field. The generator attached to a critical graph with $k_1 + 1$, respectively $k_2 + 1$, boundary markings of the two types is a multiple of $x^{k_1}y^{k_2}((k_2 + 1)x\partial_x - (k_1 + 1)y\partial_y)$ with a coefficient in the deformation parameters; we call (k_1, k_2) its *bidegree*. GKT's Lie algebra $\mathfrak{g}^{r,s}$ contains the generators of nonzero bidegree but excludes the bidegree $(0, 0)$ [GKT22, Def. 4.23], whose generator is a multiple of the *torus vector field* $x\partial_x - y\partial_y$, the generator of the torus action $(x, y) \mapsto (\lambda x, \lambda^{-1}y)$, which preserves xy and $dx \wedge dy$. The critical graphs of bidegree $(0, 0)$, those with one boundary marking of each type, are exactly those of Euler weight zero (Section 2); we call them *weight-zero critical graphs*. They occur for every (r, s) : a single interior insertion of twist $(0, 0)$ with descendent degree one carries one. At central charge one the first wall consists of weight-zero critical graphs whose insertions carry the maximal twist $(r - 2, s - 2)$, and $x^{r-2}y^{s-2}$ is the marginal deformation of W . The description of the algebra in [GKT22, §0.4 and Remark 4.24] includes the torus vector field, the jump argument of [GKT22, Theorem 5.9] uses it, and the realization argument of [GKT22, Theorem 5.13] realizes the corresponding motion of coefficients; the exclusion lies only in the explicit definition.

Proposition 1.4 (Enlargement of the Gross–Kelly–Tessler action). *Let $\tilde{G}^{r,s}$ be the group generated by the wall-crossing group of [GKT22] and the exponentials of the torus vector field multiplied by rational multiples of the deformation monomials $u_{J,\mathbf{d}}$ whose data $(J, \mathbf{d})$ admit a weight-zero critical graph; equivalently $\tilde{G}^{r,s} = \exp(\tilde{\mathfrak{g}}^{r,s})$ of (9). It preserves the set of systems of symmetric canonical open FJRW invariants with marking set I and acts on it freely and transitively.*

The wall-crossing group of [GKT22] alone does not act transitively as soon as I carries a weight-zero critical graph; the smallest case is a single insertion of twist $(0, 0)$ with descendent degree one (Example 3.2). The proof of preservation also extends the period argument of [GKT22] to all residues (Remark 3.1). In the parabolic regime the torus vector field generates every primary wall: every primary critical graph has weight zero, because equality in the bound of the Method paragraph below forces maximal twists and vanishing residues (Proposition 7.1).

The wall generators. Gross, Kelly and Tessler write the wall-crossing Lie algebra in terms of the vector fields above [GKT22, eqn. (4.19)] and observe that they preserve $dx \wedge dy$ [GKT22, p. 89]; conversely, every derivation preserving $dx \wedge dy$ and the ideal (xy) is a combination of these fields [GKT22, Remark 4.24], and Maher’s algebra [Mah24, (5.2.1) and Proposition 5.2.1] consists of the same fields. Each of them is the Hamiltonian vector field of the monomial $x^{k_1+1}y^{k_2+1}$ for $dx \wedge dy$, and the structure constants of the bracket are the determinants of the numbers of boundary markings, as in the tropical vertex Lie algebra of the logarithmic form $d \log x \wedge d \log y$; the two Lie algebras are nevertheless not isomorphic as graded Lie algebras (Section 6).

Together, Theorems 1.1 and 1.2 and Propositions 1.3 and 1.4 describe when walls exist and how the wall-crossing group acts. The slope-ordered factorization of the wall-crossing group, with rays indexed by the numbers of boundary markings $(k_1 + 1, k_2 + 1)$ of critical graphs, is constructed in the companion paper [Wue26].

Method. Nothing here rebuilds the machinery of [GKT22]. GKT’s count [GKT22, Notation 2.40] says that a critical graph with interior data $(J, \mathbf{d})$ exists if and only if $N(J, \mathbf{d}) = \ell_1 + \ell_2 - |J| + 1 + \sum_i d_i \geq 1$, where ℓ_1 and ℓ_2 are the carries in the twist sums $\sum a_i$ and $\sum b_i$. Since Neveu–Schwarz twists are at most $(r - 2, s - 2)$, the single inequality of Lemma 4.2 gives $\ell_1 + \ell_2 \leq |J| \hat{c}_W$, so $N(J, \mathbf{0}) \leq 0$ when $\hat{c}_W < 1$. When $\hat{c}_W \geq 1$, n copies of the maximal twist give $N = 1$ for suitable n ; since N is nondecreasing in each twist, the least such n is n_* . A single insertion of descendent degree d has $N = d$, which gives Proposition 1.3. The step from graphs to invariants is the reduction to critical graphs (Proposition 4.1): the generator of a primary critical graph changes a primary coefficient of the potential; if there is none, every generator carries a descendent variable, and transitivity forces all systems to agree. For Proposition 1.4, preservation holds because every generator is divergence-free and tangent to the axes, so it preserves the integral of $e^{W/h}$ over the positive quadrant, whose coefficients are, up to fixed nonzero factors, GKT’s polynomials $\mathcal{A}(J, \mathbf{d}, \nu)$; this replaces the period formula of [GKT22, Theorem 4.9], which is stated only for residues at most $r - 2$ and $s - 2$. Freeness follows by looking at a divisibility-minimal deformation monomial in the support of a nonzero element v of the enlarged Lie algebra, for $g = \exp v$. Transitivity is proved by induction over divisibility: at each monomial the difference of two systems lies in the kernel of one nonzero linear functional on the coefficients of the balanced graphs, and the vectors by which the critical graphs move these coefficients span that kernel. All numerical claims are proved in §§4–7; finite exact checks are described in §8.

Relation to prior work. The uniqueness of the primitive form for simple singularities is classical [Sai83, NY98], and is the closed-mirror shadow of the statement that these are the potentials without wall-crossing. On the open side, Basalaev–Buryak constructed open Saito potentials for the A - and D -series by solving the open WDVV equations [BB21], and Maher computed open Saito potentials for the Fermat form $x^3 + y^3$ of D_4 and for the chain-type E_7 [Mah24, Examples 4.2.3 and 4.2.4]; the present criterion explains, uniformly,

why the primary open FJRW theory is unambiguous on the whole ADE locus. The wall-crossing group of [GKT22] is drawn, “in spirit,” by analogy with the tropical vertex group of Kontsevich–Soibelman [KS06] and its role in Fano mirror symmetry [GPS10, Gro10] ([GKT22, Remark 0.8]); separately, a tropical version of open FJRW theory is raised as an open problem in [GKT26, §6], there through tropical moduli. The classification of critical graphs organized here is the starting point for the slope-ordered factorization of the wall-crossing group in [Wue26].

Maher proves a B-model analogue of the threshold: for invertible polynomials, an axiomatically defined primary wall-crossing group of open Saito potentials is trivial exactly for the simple singularities [Mah24, Proposition 1.0.5 and Corollary 5.2.6]. His wall-crossing algebra [Mah24, (5.2.1)] includes the torus vector field, and for simple elliptic singularities of rank two or three he shows that the primary wall-crossing elements are exponentials of combinations of torus fields (in rank two, of $x\partial_x - y\partial_y$) with coefficients powers of the marginal parameter satisfying an integrality condition [Mah24, Propositions 1.0.5 and 5.2.10]; Proposition 7.1 below is the corresponding statement in the open FJRW theory of [GKT22]. The A-model statement (i) $\Leftrightarrow$ (iv) of Theorem 1.1, the first-wall count of Theorem 1.2 and the descendent wall are, to our knowledge, not in [Mah24]; we give an elementary proof in the Fermat case.

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2. THE COUNT OF CRITICAL GRAPHS

We recall only what is needed and refer to [GKT22] for all definitions. Fix $W = x^r + y^s$. A genus-zero smooth (r, s) -disk carries interior marked points with twists (a_i, b_i) , $0 \leq a_i \leq r-1$, $0 \leq b_i \leq s-1$, and a descendent vector $\mathbf{d} = (d_i) \in \mathbb{Z}_{\geq 0}^J$. The interior insertions contributing to the primary potential lie in the Neveu–Schwarz range $0 \leq a_i \leq r-2$, $0 \leq b_i \leq s-2$ [GKT22, eqn. (0.5)].

Given a set J of interior markings with twists $(a_i, b_i)_{i \in J}$ and descendents $\mathbf{d} = (d_i)_{i \in J}$, define, following [GKT22, Notation 2.30],

$$(2) \quad \sum_{i \in J} a_i = r(J) + \ell_1 r, \quad \sum_{i \in J} b_i = s(J) + \ell_2 s,$$

where $r(J) \in \{0, \dots, r-1\}$ and $s(J) \in \{0, \dots, s-1\}$ are the residues, so that

$$(3) \quad \ell_1 = \left\lfloor \frac{1}{r} \sum_{i \in J} a_i \right\rfloor, \quad \ell_2 = \left\lfloor \frac{1}{s} \sum_{i \in J} b_i \right\rfloor.$$

The classification of balanced and critical graphs with interior markings exactly J is [GKT22, Notation 2.40]: setting

$$(4) \quad N(J, \mathbf{d}) = \ell_1 + \ell_2 - |J| + 1 + \sum_{i \in J} d_i,$$

one has $|\text{Balanced}(J, \mathbf{d})| = \max(N(J, \mathbf{d}) + 1, 0)$ and

$$(5) \quad |\text{Crit}(J, \mathbf{d})| = \max(N(J, \mathbf{d}), 0).$$

Here the balanced graphs are the disk types whose invariants occur as coefficients of the open potential; the critical graphs index the possible codimension-one jumps between those coefficients. In particular a critical graph with interior markings J and descendent vector $\mathbf{d}$ exists if and only if $N(J, \mathbf{d}) \geq 1$. (Throughout, $J \neq \emptyset$. For $J = \emptyset$, the balanced equation

has precisely the solutions $(k_1, k_2) = (r, 0), (0, s)$, the two boundary-only graphs appearing in [GKT22, Example 3.4]; the critical equation forces $(0, 0)$, whose two boundary markings are unstable. Thus (5) is not asserted in the boundary-only case.)

Weights and the enlarged algebra. For a nonempty subset $J \subseteq I$ and descendent degrees $\mathbf{d} = (d_i)_{i \in J}$, set

$$(6) \quad r(J) \equiv \sum_{i \in J} a_i \pmod{r}, \quad 0 \leq r(J) < r, \quad s(J) \equiv \sum_{i \in J} b_i \pmod{s}, \quad 0 \leq s(J) < s,$$

$$(7) \quad m(J, \mathbf{d}) = rs + \sum_{i \in J} (sa_i + rb_i + rs(d_i - 1)), \quad w(J, \mathbf{d}) = m(J, \mathbf{d}) - rs.$$

Here $m(J, \mathbf{d})$ is the quantity of [GKT22, Notation 2.30]. The generator attached to a critical graph with $k_1 + 1$, respectively $k_2 + 1$, boundary markings of the two types is a deformation-monomial multiple of $x^{k_1}y^{k_2}((k_2 + 1)x\partial_x - (k_1 + 1)y\partial_y)$, whose weight under the Euler vector field $sx\partial_x + ry\partial_y$ of W is $sk_1 + rk_2$, and the critical condition [GKT22, Proposition 2.39] reads

$$(8) \quad sk_1 + rk_2 = w(J, \mathbf{d}).$$

We therefore call w the *Euler weight*. Write $t_{a,b,d}$ for the deformation variable associated with an insertion $\tau_d^{(a,b)}$ and $u_{J,\mathbf{d}} = \prod_{i \in J} t_{a_i,b_i,d_i}$. The coefficient ring $A_{I,\text{sym}}$ is the symmetric GKT deformation ring in which the total multiplicity of each twist is bounded by the number of labels of that twist in I ; in particular every positive-degree element is nilpotent. For a balanced graph, k_1 and k_2 denote its numbers of boundary markings of the two types; for a critical graph with $k_1 + 1$ and $k_2 + 1$ boundary markings, (k_1, k_2) is its bidegree, as in the introduction. Since $k_1, k_2 \geq 0$, (8) shows that a critical graph has Euler weight zero exactly when its bidegree is $(0, 0)$, that is, when it is a weight-zero critical graph in the sense of the introduction.

The *enlarged algebra* and group of Proposition 1.4 over $A_{I,\text{sym}}$ are

$$(9) \quad \begin{aligned} \tilde{\mathfrak{g}}_{A_{I,\text{sym}}}^{r,s} &= \mathfrak{g}_{A_{I,\text{sym}}}^{r,s} \oplus \sum_{\substack{\emptyset \neq J \subseteq I, \mathbf{d} \geq 0 \\ w(J, \mathbf{d})=0, r(J)=s(J)=0}} \mathbb{Q} u_{J,\mathbf{d}} (x\partial_x - y\partial_y), \\ \tilde{G}_{A_{I,\text{sym}}}^{r,s} &= \exp(\tilde{\mathfrak{g}}_{A_{I,\text{sym}}}^{r,s}). \end{aligned}$$

The exponential is finite over $A_{I,\text{sym}}$. This notation keeps GKT's original positive-weight group distinct from the enlargement used here.

3. THE TORUS VECTOR FIELD AND THE ENLARGED ACTION

We recall the objects of [GKT22] on which the enlarged action acts. A system of open invariants bounded by I is a vector $\nu = (\nu_{\Gamma,\mathbf{d}})$ indexed by the balanced graphs Γ with internal labels in I and their descendent vectors $\mathbf{d}$ [GKT22, (0.3)]. For $J \subseteq I$ and $\mathbf{d}$, the chamber-index polynomial of [GKT22, Notation 3.42], which we call the chamber-index polynomial, is

$$(10) \quad \mathcal{A}(J, \mathbf{d}, \nu) = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{J_1 \sqcup \dots \sqcup J_h = J} \sum_{\Theta_1, \dots, \Theta_h} \frac{\Gamma\left(\frac{1+\sum_i k_1(\Theta_i)}{r}\right) \Gamma\left(\frac{1+\sum_i k_2(\Theta_i)}{s}\right)}{\Gamma\left(\frac{1+r(J)}{r}\right) \Gamma\left(\frac{1+s(J)}{s}\right)} \prod_{i=1}^h \nu_{\Theta_i, \mathbf{d}|_{J_i}},$$

the sum over ordered decompositions of J into nonempty blocks and over balanced graphs Θ_i with internal labels J_i , where $k_1(\Theta_i)$ and $k_2(\Theta_i)$ are the numbers of boundary markings of the two types and Γ is the Gamma function. A *chamber index* is a system with $\nu_{\Theta, \mathbf{0}} = 1$ when the balanced graph Θ has one internal marking, $\nu_{\Theta, \mathbf{0}} = -1$ when it has none, $\mathcal{A}(J, \mathbf{d}, \nu) = (-1)^d$

when $J = \{j\}$ and $\mathbf{d} = (d)$, and $\mathcal{A}(J, \mathbf{d}, \nu) = 0$ when $|J| \geq 2$ and $m(J, \mathbf{d}) \geq sr(J) + r s(J)$ [GKT22, Definition 3.47]. By [GKT22, Corollary 5.14], the chamber indices are exactly the systems realized by canonical boundary conditions; we call them *realizable*.

Remark 3.1 (The torus vector field in [GKT22]). Besides the descendent singleton of twist $(0, 0)$, examples of weight-zero critical graphs are [GKT22, Example 2.38] and [GKT26, Example 4.9]. With the Lie algebra of [GKT22, Definition 4.23], which omits the bidegree $(0, 0)$, the transitivity in [GKT22, Theorems 0.7 and 4.27] fails as soon as I carries such a graph: by the freeness in Proposition 1.4, the only element of $\tilde{G}^{r,s}$ carrying a realizable system ν to $\exp(cu(x\partial_x - y\partial_y))\nu$, with $c \neq 0$ and u the deformation monomial of the graph, is that element itself, which does not lie in $G^{r,s}$; Example 3.2 is the smallest case. On the family of balanced graphs of such a u , the element written in [GKT22, proof of Theorem 4.27, Step 3] is a multiple of the torus vector field. The jump automorphisms in the proof of [GKT22, Theorem 5.9], over the labeled ring of [GKT22, Definition 4.6], lie in the analogous enlargement of GKT's group over that ring, because the jump sets $Q_0(A, \mathbf{d})$ there impose only the congruences of [GKT22, Definition 4.23] and (8), which admit $(k_1, k_2) = (0, 0)$ when $r(A) = s(A) = 0$. The equality of the chamber indices with the realizable systems [GKT22, Corollary 5.14], proved through [GKT22, Theorem 5.13], does not depend on transitivity: at [GKT22, (5.50)] the two required values of $\mathcal{A}(J, \mathbf{d}, -)$ agree because both are prescribed by [GKT22, Definition 3.47]. The preservation step of [GKT22, proof of Theorem 4.27, Step 1] uses the period formula of [GKT22, Theorem 4.9], which is stated for residues at most $r - 2$ and $s - 2$; the Laplace integral in the proof below covers the residues $r - 1$ and $s - 1$ as well.

Example 3.2 (Failure of transitivity without the torus vector field). Let $I = \{1\}$ consist of one label of twist $(0, 0)$ and consider descendent degree one, so $u = t_{0,0,1}$, $J = \{1\}$, $r(J) = s(J) = 0$ and $N(J, (1)) = 1$. The balanced graphs have boundary markings $(0, s)$ and $(r, 0)$, and the unique critical graph has bidegree $(0, 0)$. By (10), the chamber-index condition $\mathcal{A}(J, (1), \nu) = -1$ reads

$$\frac{\Gamma(\frac{1}{r})\Gamma(\frac{1+s}{s})}{\Gamma(\frac{1}{r})\Gamma(\frac{1}{s})} \nu_{(0,s)} + \frac{\Gamma(\frac{1+r}{r})\Gamma(\frac{1}{s})}{\Gamma(\frac{1}{r})\Gamma(\frac{1}{s})} \nu_{(r,0)} = \frac{\nu_{(0,s)}}{s} + \frac{\nu_{(r,0)}}{r} = -1,$$

a single linear condition on two coefficients. Every solution extends to a chamber index, since for $|I| = 1$ the conditions for different descendent degrees involve disjoint coordinates, and is therefore realizable [GKT22, Corollary 5.14]. A generator of $\mathfrak{g}^{r,s}$ changes the u -coefficients of a potential only if its coefficient divides u , that is, equals u ; the generators with coefficient u are attached to the critical graphs of $(J, (1))$, and the only one has bidegree $(0, 0)$, which [GKT22, Definition 4.23] excludes. Hence no element of $G^{r,s}$ relates two realizable systems with different values of $\nu_{(r,0)}$. The element $\exp(cu(x\partial_x - y\partial_y))$ of the enlarged group moves the pair $(\nu_{(0,s)}, \nu_{(r,0)})$ along the solution line, by a multiple of $(-s, r)$.

Proof of Proposition 1.4. Closure and finiteness. The enlarged algebra is closed under the bracket: for a weight-zero coefficient u and a generator $u'X_{k_1,k_2}$, $[u(x\partial_x - y\partial_y), u'X_{k_1,k_2}] = (k_1 - k_2)uu'X_{k_1,k_2}$, and uu' satisfies the residue and weight conditions of u' because those of u vanish; brackets of positive-weight generators are as in [GKT22, Prop. 4.25]. Every generator has a coefficient in the ideal generated by the deformation variables, which is nilpotent in $A_{I,\text{sym}}$, so the exponential is finite. The argument of Step 1 in the proof of [GKT22, Thm. 4.27] applies verbatim to the weight-zero generators and shows that $e^v W^\nu$ is again the potential of a symmetric system of balanced coefficients.

Preservation. Take $\hbar < 0$ and expand, coefficientwise, the positive-quadrant Laplace integral

$$\mathcal{L}_\hbar(W) = \int_0^\infty \int_0^\infty e^{W/\hbar} dx dy.$$

Its monomial moments are

$$\int_0^\infty \int_0^\infty x^\alpha y^\beta e^{(x^r+y^s)/\hbar} dx dy = \frac{1}{rs} (-\hbar)^{\frac{\alpha+1}{r} + \frac{\beta+1}{s}} \Gamma\left(\frac{\alpha+1}{r}\right) \Gamma\left(\frac{\beta+1}{s}\right).$$

Write $W = x^r + y^s + V$ and expand $e^{V/\hbar} = \sum_h V^h / (h! \hbar^h)$. The $u_{J,\mathbf{d}}$ -coefficient of V^h is a sum over ordered decompositions $J = J_1 \sqcup \dots \sqcup J_h$ of products of coefficients of V , each attached to a balanced graph with internal labels J_i . Since $\sum_i (s k_1(\Theta_i) + r k_2(\Theta_i)) = m(J, \mathbf{d}) + (h-1)rs$, all these terms carry the same power of $-\hbar$; the sign $(-1)^h$ from $\hbar^{-h}$ and the signs $(-1)^{|J_i|-1}$ of the coefficients of the potential combine to $(-1)^{|J|}$, and the moment of the product monomial is the Gamma product in (10). Hence the $u_{J,\mathbf{d}}$ -coefficient of $\mathcal{L}_\hbar(W^\nu)$ is a nonzero multiple, depending only on $(J, \mathbf{d})$, of $\mathcal{A}(J, \mathbf{d}, \nu)$, for every pair of residues $r(J), s(J)$. This is the termwise calculation in the proof of [GKT22, Theorem 4.9], with the positive quadrant in place of the good-basis cycles, carried out over the labeled ring of [GKT22, Definition 4.6]; the injective map ψ_I of [GKT22, Lemma 4.14] sends $W^{\nu, \text{sym}}$ to W^ν [GKT22, Corollary 4.15] and is compatible with the action, so over $A_{I, \text{sym}}$ the coefficient acquires the factor $1/|\text{Aut}(A)|$ of [GKT22, (4.11)]. Every enlarged generator is divergence-free (Lemma 6.1) and tangent to both coordinate axes, since in (13) the coefficient of ∂_x vanishes on $x = 0$ and that of ∂_y on $y = 0$. Integration by parts therefore gives $\mathcal{L}_\hbar(e^\nu W) = \mathcal{L}_\hbar(W)$. Thus all the $\mathcal{A}$ -conditions are preserved, including the residue classes $r-1$ and $s-1$, which the good-basis periods of [GKT22, §4.1] do not detect. The normalizations of Definition 3.47 are also preserved: they prescribe the coefficients of balanced graphs with at most one internal marking and no descendants, and a generator changes the coefficient of a deformation monomial only if the deformation monomial of some critical graph divides it; a primary critical graph has at least two interior markings, and a critical graph with one interior marking carries a descendent. Since the chamber indices are exactly the realizable systems, this proves preservation.

The wall vectors. Fix a deformation monomial $u = u_{J,\mathbf{d}}$ and write $N = N(J, \mathbf{d})$. The balanced disk types over $(J, \mathbf{d})$ are indexed by $0 \leq p \leq N$ and have boundary degrees

$$(r(J) + pr, s(J) + (N-p)s).$$

Let $V_u = \bigoplus_{p=0}^N \mathbb{Q}e_p$ be their coefficient space, where e_p changes only the coefficient of the p th balanced type. For $1 \leq p \leq N$, the critical graph between the $(p-1)$ st and p th balanced types acts on V_u , through $x^r + y^s$, by

$$v_p = \alpha_p e_p - \beta_p e_{p-1}, \quad \alpha_p = r(s(J) + (N-p)s + 1), \quad \beta_p = s(r(J) + (p-1)r + 1).$$

Both coefficients are positive, and reading successively the $e_N, e_{N-1}, \dots, e_1$ coordinates shows that the v_p are independent. If u carries a weight-zero critical graph, the weight equation forces $N = 1$ and $r(J) = s(J) = 0$; its only wall vector is $v_1 = re_1 - se_0$, the vector of the torus vector field adjoined in (9).

Freeness. Let $g = \exp(v)$ with $v \neq 0$, and let u be minimal for divisibility among the deformation monomials in the support of v . The change of the u -coefficients of any potential under g is $v_u(x^r + y^s)$, where v_u is the u -component of v : every other contribution carries a product of a support monomial with a further deformation monomial, which by minimality is not u . The summands of v_u act by the independent vectors v_p , so $v_u(x^r + y^s) \neq 0$ and g moves every potential. The same argument applies modulo the monomials outside any finite divisor-closed set of deformation monomials.

Transitivity. Let S be a finite set of deformation monomials that contains every divisor of each of its elements, and pass to the quotient that retains exactly the coefficients indexed by S . Order S by a linear extension of proper divisibility, and suppose two realizable systems agree on every proper divisor of $u \in S$. They satisfy the same prescribed value of $\mathcal{A}(J, \mathbf{d}, -)$: by Definition 3.47(2) of [GKT22] if $|J| = 1$, and by its condition (3) if $|J| \geq 2$, which applies because the existence of a balanced type forces $m(J, \mathbf{d}) \geq sr(J) + rs(J)$. Every term of (10) with two or more blocks already agrees, so their difference in V_u lies in the kernel of the remaining one-block term, a nonzero linear functional $L_u : V_u \rightarrow \mathbb{Q}$ whose coefficients are the Gamma factors computed above; thus $\dim \ker L_u = N$. The Laplace invariance gives $L_u(v_p) = 0$, so the v_p form a basis of $\ker L_u$, and a unique combination of the critical-graph generators with coefficient u removes the difference in V_u . Acting by that combination can alter only coefficients indexed by strict multiples of u , so it does not disturb earlier steps. Induction proves transitivity in the quotient defined by S . By freeness, the element so constructed is the unique one in the quotient group carrying the first system to the second, so the elements obtained for $S \subset S'$ are compatible. For each bound on the degree $k_1 + k_2$ of the generators X_{k_1, k_2} only finitely many deformation monomials occur, because the weight equation (8) bounds the descendent degrees. The compatible elements therefore define an element of the group (9), which, as in [GKT22, Def. 4.19], is the inverse limit of its truncations by the degree $k_1 + k_2$; it carries one system to the other, which proves transitivity over $A_{I, \text{sym}}$. $\square$

4. PROOF OF THE CENTRAL-CHARGE CRITERION

Both directions of Theorem 1.1 rest on the following reduction from invariants to graph counts.

Proposition 4.1 (Reduction to critical graphs). *Fix a finite marking set I and work over the coefficient ring $A_{I, \text{sym}}$ of §2.*

- (a) *If no critical graph with descendent vector $\mathbf{0}$ has interior markings contained in I , then $\tilde{G}_{A_{I, \text{sym}}}^{r, s}$ fixes every primary open FJRW invariant; the primary invariants are independent of the boundary conditions.*
- (b) *If some critical graph with descendent vector $\mathbf{d}$ has interior markings $J \subseteq I$, then there are two systems of symmetric canonical open FJRW invariants over I that differ in an invariant with interior markings J and descendent vector $\mathbf{d}$; if $\mathbf{d} = \mathbf{0}$, this is a primary invariant.*

Proof. By [GKT22, Definition 4.23], the original wall-crossing Lie algebra $\mathfrak{g}_{A_{I, \text{sym}}}^{r, s}$ is spanned by the summands corresponding to critical graphs with $(k_1, k_2) \neq (0, 0)$. Equation (9) adds the omitted $(0, 0)$ summands. After passage to symmetric coefficients, critical graphs with the same twist–descendent multiset determine the same summand. For a critical graph with interior markings J , descendents $\mathbf{d}$, and bidegree (k_1, k_2) , its generator is

$$(11) \quad v_\Lambda = u_{J, \mathbf{d}} x^{k_1} y^{k_2} ((k_2 + 1)x\partial_x - (k_1 + 1)y\partial_y), \quad u_{J, \mathbf{d}} = \prod_{i \in J} t_{a_i, b_i, d_i}.$$

Here (k_1, k_2) is the bidegree used in [GKT22, Definition 4.23]: the critical graph has $k_1 + 1$ and $k_2 + 1$ boundary marked points of the two types [GKT22, eqn. (2.31), Proposition 2.39], and the bidegree satisfies the critical condition $sk_1 + rk_2 = w(J, \mathbf{d})$ of (8). Positive weight gives a summand of $\mathfrak{g}^{r, s}$, while $(k_1, k_2) = (0, 0)$ gives the adjoined torus vector field. GKT’s group $G_{A_{I, \text{sym}}}^{r, s} = \exp(\mathfrak{g}_{A_{I, \text{sym}}}^{r, s})$ acts on the symmetric chamber indices of [GKT22, Definition 3.47]

by $g(W^\nu) = W^{g(\nu)}$ [GKT22, Theorem 4.27] (the action itself is reproved in Proposition 1.4); Proposition 1.4 extends the action transitively to $\tilde{G}_{A_I, \text{sym}}^{r,s}$.

For (a): if there is no $\mathbf{0}$ -critical graph, every generator (11) has some $d_i > 0$, hence $u_{J, \mathbf{d}}$ is divisible by a variable $t_{a,b,d}$ with $d > 0$. Acting on a potential $W^\nu = x^r + y^s + \sum_{0 \leq a \leq r-2, 0 \leq b \leq s-2} t_{a,b,0} x^a y^b + \dots$ [GKT22, eqn. (0.5)], such a generator raises the total degree in the positive-descendent variables and therefore leaves unchanged the coefficients of the purely primary monomials $u_{J', \mathbf{0}} x^\alpha y^\beta$. Since each primary open invariant $\nu_{\Gamma, \mathbf{0}}$, the coefficient attached to a graph Γ in [GKT22, eqns. (0.4)–(0.5)], is, up to the fixed nonzero sign and automorphism factor in the symmetric potential, such a coefficient, it is fixed by every generator, hence by $\tilde{G}_{A_I, \text{sym}}^{r,s}$. Transitivity of the enlarged action then makes these primary invariants equal in every system.

For (b): a critical graph Λ with data $(J, \mathbf{d})$ furnishes a nonzero generator v_Λ with coefficient $u_{J, \mathbf{d}}$, a monomial in primary variables only when $\mathbf{d} = \mathbf{0}$. Fix any system ν of symmetric canonical invariants, which exists by [GKT22, Theorem 0.1]. The coefficient of the deformation monomial $u_{J, \mathbf{d}}$ in $\exp(v_\Lambda)(W^\nu) - W^\nu$ is exactly $v_\Lambda(x^r + y^s)$, namely

$$u_{J, \mathbf{d}} (r(k_2 + 1) x^{k_1+r} y^{k_2} - s(k_1 + 1) x^{k_1} y^{k_2+s}),$$

which is nonzero. Proposition 1.4 ensures that $\exp(v_\Lambda) \cdot \nu$ is another system of symmetric canonical invariants over I . The two systems therefore differ in an invariant with interior markings J and descendent vector $\mathbf{d}$. $\square$

The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are immediate: (1) gives $\hat{c}_W < 1 \Leftrightarrow \frac{2}{r} + \frac{2}{s} > 1$, and $\frac{1}{r} + \frac{1}{s} > \frac{1}{2}$ characterizes the two-variable Fermat simple singularities, whose only types are $A_{n-1} = (n, 2)$, $D_4 = (3, 3)$, $E_6 = (3, 4)$ and $E_8 = (3, 5)$ (there is no Fermat E_7 , as $E_7 = x^3 + xy^3$ is not of the form $x^r + y^s$; likewise $D_n = x^{n-1} + xy^2$ is Fermat only for $n = 4$, where $x^3 + y^3 \cong D_4$). It remains to prove (i) $\Leftrightarrow$ (iv), for which, by Proposition 4.1, it suffices to decide the existence of a primary ($\mathbf{d} = \mathbf{0}$) critical graph. The whole matter rests on one inequality.

Lemma 4.2. *Let J be a set of $n = |J|$ interior markings with Neveu–Schwarz twists, $0 \leq a_i \leq r - 2$ and $0 \leq b_i \leq s - 2$. Then, with ℓ_1, ℓ_2 as in (3),*

$$\ell_1 + \ell_2 \leq n \hat{c}_W.$$

Consequently the primary count (4) obeys $N(J, \mathbf{0}) \leq 1 - n(1 - \hat{c}_W)$.

Proof. Dropping the floor functions and using the twist bounds,

$$\ell_1 + \ell_2 = \left\lfloor \frac{1}{r} \sum a_i \right\rfloor + \left\lfloor \frac{1}{s} \sum b_i \right\rfloor \leq \frac{\sum a_i}{r} + \frac{\sum b_i}{s} \leq \frac{n(r-2)}{r} + \frac{n(s-2)}{s} = n \left(2 - \frac{2}{r} - \frac{2}{s} \right) = n \hat{c}_W.$$

The second claim is $N(J, \mathbf{0}) = \ell_1 + \ell_2 - n + 1 \leq n \hat{c}_W - n + 1$. $\square$

Proof of (i) $\Rightarrow$ (iv). Assume $\hat{c}_W < 1$. For any J with $n = |J| \geq 1$ interior markings, Lemma 4.2 gives $\ell_1 + \ell_2 \leq n \hat{c}_W < n$; as $\ell_1 + \ell_2$ is an integer strictly below the integer n , we get $\ell_1 + \ell_2 \leq n - 1$, i.e. $N(J, \mathbf{0}) \leq 0$. By (5) there is no primary critical graph, for any J and any finite I . Proposition 4.1(a) then says the primary invariants are independent of the boundary conditions for every I . (The case $n = 0$ is vacuous: there is no interior marking to twist, and $N = 1$ corresponds to the two boundary-only balanced graphs of [GKT22, Example 3.4], which are not critical.) $\square$

Proof of $\neg(i) \Rightarrow \neg(iv)$. Assume $\hat{c}_W \geq 1$, i.e. $\frac{2}{r} + \frac{2}{s} \leq 1$; in particular $r, s \geq 3$ and not both equal to 3, so $r, s \geq 3$ with $\max(r, s) \geq 4$, and we may take all interior twists maximal,

$(a_i, b_i) = (r - 2, s - 2)$. With n such markings,

$$(12) \quad N = \left\lfloor \frac{n(r-2)}{r} \right\rfloor + \left\lfloor \frac{n(s-2)}{s} \right\rfloor - n + 1 = \left(n - \left\lceil \frac{2n}{r} \right\rceil \right) + \left(n - \left\lceil \frac{2n}{s} \right\rceil \right) - n + 1 = n - \left\lceil \frac{2n}{r} \right\rceil - \left\lceil \frac{2n}{s} \right\rceil + 1,$$

using $\lfloor n(r-2)/r \rfloor = \lfloor n - 2n/r \rfloor = n - \lceil 2n/r \rceil$. We exhibit n with $N \geq 1$, i.e. $\lceil 2n/r \rceil + \lceil 2n/s \rceil \leq n$.

If $\frac{2}{r} + \frac{2}{s} < 1$ (the hyperbolic regime $\hat{c}_W > 1$), then $\lceil 2n/r \rceil + \lceil 2n/s \rceil \leq \frac{2n}{r} + \frac{2n}{s} + 2 = n(\frac{2}{r} + \frac{2}{s}) + 2$, which is $\leq n$ once $n(1 - \frac{2}{r} - \frac{2}{s}) \geq 2$; such n exists since $1 - \frac{2}{r} - \frac{2}{s} > 0$.

If $\frac{2}{r} + \frac{2}{s} = 1$ (the parabolic regime $\hat{c}_W = 1$: the pairs $(4, 4)$, $(3, 6)$, $(6, 3)$), choose n with $r \mid 2n$ and $s \mid 2n$, e.g. $n = \text{lcm}(r, s)/2$, an integer for all three parabolic pairs; then $\lceil 2n/r \rceil + \lceil 2n/s \rceil = \frac{2n}{r} + \frac{2n}{s} = n$, so $N = 1$.

In either case a primary critical graph exists once I contains n interior labels of maximal twist, which is possible because each twist is realized infinitely often in the marking data [GKT22, §2.4]. By Proposition 4.1(b) the primary invariants over such an I depend on the boundary conditions. This proves $\neg(\text{iv})$, and with it Theorem 1.1. $\square$

Remark 4.3 (On Remark 0.9 of [GKT22]). [GKT22, Remark 0.9] (arXiv versions 2 and 3) states that the pairs $(r, s) \in \{(n, 2), (2, n), (3, 3), (3, 4), (4, 3)\}$ give simple singularities “ A_{n-1} , E_6 , and E_8 ”. Two adjustments reconcile the list with the naming. First, $(3, 3)$ appears in the list but D_4 is not named; $x^3 + y^3$ is the D_4 singularity (three distinct lines, Milnor number 4). Second, E_8 is named but its pairs $(3, 5)$, $(5, 3)$ are absent from the list; these are $x^3 + y^5$, Milnor number 8. The list, completed and matched to its names, is

$$\{(n, 2), (2, n) : n \geq 2\} \cup \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\},$$

of types A_{n-1}, D_4, E_6, E_8 , exactly as in Theorem 1.1(iii). The theorem shows this list is not merely a set of examples but the complete answer to the prediction.

5. THE FIRST WALL AND THE DESCENDENT WALL

Proof of Theorem 1.2. The simple case $n_* = \infty$ is Theorem 1.1. Assume $\hat{c}_W \geq 1$ and use (12), which gives the maximal N over size- n primary multisets, since N is nondecreasing in the twists and maximal at $(r - 2, s - 2)$. Thus $n_*(r, s)$ is the least n with $\lceil 2n/r \rceil + \lceil 2n/s \rceil \leq n$.

$\min(r, s) \geq 4$. For $n = 1$, $\lceil 2/r \rceil + \lceil 2/s \rceil = 1 + 1 = 2 > 1$ (each $2/r, 2/s \leq \frac{1}{2}$). For $n = 2$ and $r, s \geq 4$, $\lceil 4/r \rceil + \lceil 4/s \rceil = 1 + 1 = 2 = n$, so $N = 1$ and $n_* = 2$.

$\min(r, s) = 3$, say $r = 3$, with $\max = s \geq 6$. Here $a_i \leq 1$, so $\ell_1 = \lfloor n/3 \rfloor$. For $n = 2$, $\lceil 4/3 \rceil + \lceil 4/s \rceil = 2 + 1 = 3 > 2$. For $n = 3$, $\lceil 6/3 \rceil + \lceil 6/s \rceil = 2 + 1 = 3 = n$ (using $s \geq 6$), so $N = 1$ and $n_* = 3$. For $3 \leq s \leq 5$ the pair is simple and $n_* = \infty$, already covered.

Finally $\min(r, s) = 2$ forces the A -type simple case, $n_* = \infty$. This exhausts all (r, s) . For $x^4 + y^4$, the witnessing multiset is two insertions of twist $(r - 2, s - 2) = (2, 2)$, giving $N = 1$: the example of [GKT26, Examples 4.9 and 5.5]. $\square$

Proof of Proposition 1.3. Take $|J| = 1$ with any twist (a, b) in the Neveu–Schwarz range. Then $\ell_1 = \ell_2 = 0$ and, by (4), $N(J, \mathbf{d}) = 0 + 0 - 1 + 1 + d = d$, where d is the descendent degree of the single insertion. Hence $d = 1$ gives $N = 1$: exactly one critical graph, for every (r, s) . Proposition 4.1(b) then gives two systems of invariants that differ in a descendent invariant. Hence the descendent theory wall-crosses in every case. When $(a, b) = (0, 0)$, and in particular for $(r, s) = (2, 2)$, where this is the only twist, the critical graph has weight zero and its generator is a multiple of the torus vector field, so the conclusion rests on Proposition 1.4. $\square$

Remark 5.1. The trichotomy of Theorem 1.2 shows the $x^4 + y^4$ example of [GKT26] is not isolated: two interior insertions already force a wall for every (r, s) with $r, s \geq 4$, and three

suffice whenever $\min(r, s) = 3 \leq \frac{1}{2} \max(r, s)$. Combined with Proposition 1.3, the qualitative picture is complete.

6. THE WALL GENERATORS ARE HAMILTONIAN

Throughout this section $\omega = dx \wedge dy$, and for a polynomial H the Hamiltonian vector field is $X_H := (\partial_y H) \partial_x - (\partial_x H) \partial_y$, so that $\iota_{X_H} \omega = dH$. For $k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$ put

(13)

$$X_k = X_{k_1, k_2} = x^{k_1} y^{k_2} ((k_2 + 1)x \partial_x - (k_1 + 1)y \partial_y) = (k_2 + 1)x^{k_1+1} y^{k_2} \partial_x - (k_1 + 1)x^{k_1} y^{k_2+1} \partial_y,$$

and $H_k = x^{k_1+1} y^{k_2+1}$; for a critical graph Λ of bidegree k write $H_\Lambda = H_k$. The generator of a critical graph of bidegree k is a multiple of X_k , and the exponent $k + (1, 1)$ of H_k records the numbers of its boundary markings.

Lemma 6.1. *For every k , $X_k = X_{H_k}$. In particular $\operatorname{div} X_k = 0$, so X_k preserves ω , and X_k is tangent to both coordinate axes.*

Proof. $\partial_y H_k = (k_2 + 1)x^{k_1+1} y^{k_2}$ and $\partial_x H_k = (k_1 + 1)x^{k_1} y^{k_2+1}$, which gives (13). A Hamiltonian vector field is divergence-free, which recovers the volume-preservation of [GKT22, p. 89]. The coefficient of ∂_x vanishes on $x = 0$ and that of ∂_y on $y = 0$. $\square$

Proposition 6.2 (Hamiltonian wall generators). *For $k, l \in \mathbb{Z}_{\geq 0}^2$,*

$$(14) \quad [X_k, X_l] = ((k_2 + 1)(l_1 + 1) - (k_1 + 1)(l_2 + 1)) X_{k+l},$$

equivalently $\{H_k, H_l\} = ((k_2 + 1)(l_1 + 1) - (k_1 + 1)(l_2 + 1)) H_{k+l}$ for the Poisson bracket $\{H, K\} := X_H(K)$ of ω . Thus $H_k \mapsto X_k$ is an isomorphism of graded Lie algebras from the Poisson algebra of ω on the span of the monomials $x^p y^q$, $p, q \geq 1$, onto its image. After extension of scalars to $A_{I, \text{sym}}$ it maps the span of the products $u_{J, \mathbf{d}} H_\Lambda$, over critical graphs Λ of positive weight with data $(J, \mathbf{d})$, onto $\mathfrak{g}_{A_{I, \text{sym}}}^{r, s}$, and, with the products $u_{J, \mathbf{d}} xy$ of the weight-zero critical graphs adjoined, onto $\widetilde{\mathfrak{g}}_{A_{I, \text{sym}}}^{r, s}$. The (s, r) -weighted degree of H_Λ is $w(J, \mathbf{d}) + r + s$.

Proof. Write $X_k = f \partial_x + g \partial_y$ and $X_l = F \partial_x + G \partial_y$ as in (13). The ∂_x -coefficient of $[X_k, X_l]$ is

$$X_k F - X_l f = (k_2 + l_2 + 1)((k_2 + 1)(l_1 + 1) - (k_1 + 1)(l_2 + 1)) x^{k_1+l_1+1} y^{k_2+l_2},$$

the stated multiple of the ∂_x -coefficient of X_{k+l} , and the ∂_y -coefficients match in the same way. The Poisson form follows from Lemma 6.1, since $X_k(H_l)$ is the stated multiple of H_{k+l} . The generators of $\mathfrak{g}^{r, s}$ and of its enlargement are the products $u_{J, \mathbf{d}} X_k$ over the critical graphs of bidegree k [GKT22, Definition 4.23], (9). For the grading, the critical condition (8) gives $sk_1 + rk_2 = w(J, \mathbf{d})$; add $r + s$. $\square$

The coefficient in (14) is, up to sign, the determinant of the boundary exponents $k + (1, 1)$ and $l + (1, 1)$, as in the tropical vertex Lie algebra of Kontsevich–Soibelman [KS06, Gro10].

Remark 6.3 (Relation to the tropical vertex algebra). The tropical vertex Lie algebra is the Hamiltonian algebra of the *logarithmic* form $d \log x \wedge d \log y$, for which $\{x^u, x^v\} = \langle u, v \rangle x^{u+v}$ with $\langle u, v \rangle$ the determinant, so that Hamiltonians add exponents. Proposition 6.2 uses instead the *affine-plane* form $dx \wedge dy = xy (d \log x \wedge d \log y)$, under which $\{x^u, x^v\} = \langle u, v \rangle x^{u+v-(1,1)}$, and the Hamiltonian vector fields of the two forms are related by $X_H^{\text{aff}} = (xy)^{-1} X_H^{\text{tor}}$. The structure constants agree, but the exponent of a bracket differs by $(1, 1)$, and the two Lie algebras are not isomorphic as graded Lie algebras. For instance, in $\mathfrak{g}^{4,6}$ the generators $t_{2,3,0} t_{2,4,0} X_{0,1}$ and $t_{2,4,0}^2 X_{0,2}$ of Table 1 have nonzero bracket once I carries one label of twist $(2, 3)$ and three of twist $(2, 4)$, whereas a graded map sends $X_{0,1}$ and $X_{0,2}$ to tropical vertex

elements with parallel exponents, which commute. The analogy that [GKT22, Remarks 0.8 and 4.20] draw “in spirit” with the Kontsevich–Soibelman group is therefore an analogy of structure constants. The companion paper [Wue26] groups the corresponding automorphisms by the rays spanned by (k_1+1, k_2+1) , orders those ray factors by slope, and uses their products to transport the open potentials. No result of this paper depends on that construction. The tropical open FJRW theory called for in [GKT26, §6] is posed there through tropical moduli, a different route.

7. THE FIRST WALL AT THE PARABOLIC BOUNDARY

The threshold $\hat{c}_W = 1$ is not only where wall-crossing begins across the family; the walls it produces have a uniform shape. Recall that the first wall of (r, s) is the set of critical graphs over interior multisets of the minimal size $n_*(r, s)$. Combining the count (5) with the Hamiltonian dictionary of §6 determines the first wall of any given (r, s) by a finite enumeration, recorded in Table 1 for the parabolic and the smallest hyperbolic pairs; at the parabolic boundary it collapses to a single wall.

Proposition 7.1 (Parabolic first wall). *For any interior multiset J with $N(J, \mathbf{0}) = 1$, the bidegree of its unique critical graph is $(k_1, k_2) = (r(J), s(J))$, of Euler weight $sr(J) + rs(J)$. For the three parabolic pairs, $(r, s) \in \{(4, 4), (3, 6), (6, 3)\}$ with $\hat{c}_W = 1$, the first wall consists of exactly one critical graph, its multiset forced to be the maximal-twist one: two copies of $(2, 2)$ for $(4, 4)$, three copies of $(1, 4)$ for $(3, 6)$, three copies of $(4, 1)$ for $(6, 3)$. Its residues vanish, so its bidegree is $(0, 0)$, its weight is zero, and its generator is a multiple of the torus vector field $x\partial_x - y\partial_y$ of (9), with Hamiltonian xy . More generally, for the parabolic pairs every primary critical graph has weight zero, so positive-weight walls occur exactly in the hyperbolic regime.*

Proof. If $N(J, \mathbf{0}) = 1$ there is a unique critical graph and, by [GKT22, Notation 2.40, eqn. (2.31)] read in the indexing of (11), its bidegree is $(k_1, k_2) = (r(J), s(J))$; the weight statement is the critical condition (8). For each parabolic pair the minimal critical multiset is uniquely the maximal-twist one: at $n = n_*$ the inequality $\ell_1 + \ell_2 \geq n$ is attained only with equality and only at $(a_i, b_i) = (r - 2, s - 2)$ (for $(4, 4)$, $\ell_1, \ell_2 \leq 1$ force $a_i = b_i = 2$; for $(3, 6)$, $\ell_1 \leq 1, \ell_2 \leq 2$ force $a_i = 1, b_i = 4$). There the residues vanish, so $(k_1, k_2) = (0, 0)$: the torus vector field, with Hamiltonian $H_{0,0} = xy$ (Lemma 6.1). For the last statement, Lemma 4.2 gives $N(J, \mathbf{0}) \leq 1$ in the parabolic regime, and $N(J, \mathbf{0}) = 1$ forces equality there, hence no loss in the floors and maximal twists $(r - 2, s - 2)$; then $r \mid \sum a_i$ and $s \mid \sum b_i$, so $r(J) = s(J) = 0$ and the bidegree is $(0, 0)$. In the hyperbolic regime, n copies of the maximal twist with $N \geq 1$ exist by the proof of Theorem 1.1, and by (7) every critical graph over them has Euler weight $n(rs - 2r - 2s) > 0$. $\square$

For the hyperbolic pairs of Table 1, the enumeration of the interior multisets of size n_* with $N \geq 1$ shows that the first wall comprises several critical graphs, some of positive weight and some of weight zero.

For $(4, 4)$ and $(3, 6)$ the minimal multisets match the integrality condition on the exponent of the marginal parameter in Maher’s description of the elliptic primary wall-crossing group [Mah24, Proposition 5.2.10 and Example 5.2.13], obtained there for open Saito potentials.

The parabolic pairs are the two-variable Fermat *simple elliptic* singularities $(x^4 + y^4 = \tilde{E}_7, x^3 + y^6 = \tilde{E}_8)$. Along their marginal deformation Saito’s primitive form is $dx \wedge dy$ divided by a period of an elliptic curve; in the normalization of [GKT22, Theorem 0.3] the form $dx \wedge dy$ is kept and the period enters through the flat coordinates. The elliptic wall-crossing group of

(r, s)	$\hat{c}_W$	regime	n_*	graphs	weight zero	wall Hamiltonians
$(4, 4)$	1	parabolic	2	1	1	— (xy only)
$(3, 6)$	1	parabolic	3	1	1	— (xy only)
$(6, 3)$	1	parabolic	3	1	1	— (xy only)
$(4, 5)$	11/10	hyperbolic	2	2	1	xy^2
$(3, 7)$	22/21	hyperbolic	3	2	1	xy^2
$(4, 6)$	7/6	hyperbolic	2	4	2	xy^2, xy^3
$(5, 5)$	6/5	hyperbolic	2	5	2	xy^2, x^2y, x^2y^2

TABLE 1. The first wall of $x^r + y^s$: the number of critical graphs at $n = n_*(r, s)$, how many have weight zero (bidegree $(0, 0)$, the torus vector field with Hamiltonian xy), and the Hamiltonians $x^{k_1+1}y^{k_2+1}$ of the positive-weight wall generators. At the parabolic boundary the first wall has weight zero; positive-weight walls appear only in the hyperbolic regime. Computed exactly; see §8.

[Mah24, §5.2.2] is generated by the torus vector field, and the explicit jumping of the $x^4 + y^4$ invariants in [GKT26, Example 4.9] is its action.

The first hyperbolic Fermat pair with $r = s$ is $x^5 + y^5$, with boundary monomials xy^2, x^2y, x^2y^2 in Table 1; its wall functions and slope-ordered boundary-ray factorization are computed in [Wue26].

8. FINITE EXACT CHECKS

The proofs above are uniform in r and s ; the following finite calculations serve only as independent reproducibility checks. Substitution in (4) for every $2 \leq r, s \leq 40$ reproduces the ADE list and the trichotomy of Theorem 1.2. The absence assertion outside this finite range is supplied by Lemma 4.2, not by enumeration. The same calculation reproduces the first-wall data printed in Table 1, including the weight-zero/positive-weight distinction, and the bidegrees of the single-descendent critical graphs of Proposition 1.3. Finally, direct polynomial differentiation verifies Lemma 6.1 for $0 \leq k_1, k_2 \leq 4$ and identity (14), componentwise, for symbolic k and l .¹

CONCLUDING REMARKS AND OPEN DIRECTIONS

The companion paper [Wue26] continues the Hamiltonian identification by grouping the automorphisms on rays of boundary degree and factoring them in slope order. Two further directions remain open for the threshold problem itself.

- (1) *Explicit ADE potentials.* On the wall-crossing-free locus the primary potential is unambiguous, and by [GKT22, Theorem 0.3] it is the unfolding of W in flat coordinates for the primitive form $dx \wedge dy$. The Fermat D_4 case is computed in [Mah24, Example 4.2.3]; writing out the E_6 and E_8 cases from the classical flat coordinates [Sai83, NY98] and relating the open invariants to Coxeter and root-system data remain to be done.
- (2) *The parabolic boundary.* Proposition 7.1 pins the first wall of the parabolic pairs to the torus vector field. What remains is the analytic side: these are the simple elliptic singularities $\tilde{E}_7, \tilde{E}_8$, whose flat structure along the marginal deformation is

¹The exact-arithmetic files are `src/census.py` and `src/gkt.algebra.py` in the repository cited in [Wue26].

governed by modular forms rather than polynomials [Mah24, §4.2.2]. Maher studies the action of the torus vector field on the modular potential on the B-model side [Mah24, Example 5.2.12]; identifying the corresponding action on the open FJRW invariants, beyond the $x^4 + y^4$ jumping of [GKT26, Example 4.9], is the first transcendental case of the theory.

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