

SLOPE-ORDERED FACTORIZATION OF WALL-CROSSING IN TWO-VARIABLE OPEN FJRW THEORY

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ABSTRACT. For the genus-zero open FJRW invariants of Gross, Kelly and Tessler attached to a two-variable Fermat polynomial, we prove that the wall-crossing transformation between the two extremal systems of invariants factors uniquely, in every finite truncation, into automorphisms supported on rays in the quadrant of boundary-marking numbers, ordered by slope. In these two systems, within each set of invariants mixed by wall-crossing, all vanish except the one with the fewest, respectively most, boundary markings of the first type. Each critical graph contributes a generator on the ray of its boundary-marking numbers, the only ray whose monomials that generator fixes. The proof combines the Kontsevich–Soibelman ordered-product lemma, run with a filtration in place of a grading, with the simply transitive action of the wall-crossing group enlarged by a torus direction. The factorization is realized ray by ray by concatenated homotopies of boundary conditions, though in general not by a single homotopy from the extension lemma of Gross, Kelly and Tessler. Finally, in each truncation the potentials between consecutive rays are transported from one set of initial coefficients, analogous to theta-function coefficients, which can be nonzero on terms with several internal markings.

1. INTRODUCTION

1.1. The theory. Gross, Kelly and Tessler [GKT22] (henceforth GKT) defined genus-zero open FJRW invariants

$$\nu_{\Gamma, \mathbf{d}} = \left\langle \prod_i \tau_{d_i}^{(a_i, b_i)} \sigma_1^{\beta_1} \sigma_2^{\beta_2} \sigma_{12} \right\rangle^{\mathfrak{s}, o}$$

for the Landau–Ginzburg model $(W = x^r + y^s, \mu_r \times \mu_s)$: relative Euler numbers of descendent Witten bundles over moduli of W -spin disks, which are real orbifolds with corners. Here $\tau_d^{(a, b)}$ is an internal marking of twist (a, b) , $0 \leq a \leq r-2$, $0 \leq b \leq s-2$, with descendent exponent d , and $\sigma_1, \sigma_2, \sigma_{12}$ are boundary markings of twists $(r-2, 0)$, $(0, s-2)$ and $(r-2, s-2)$, the last being the *fully twisted* one. The invariants depend on a family $\mathfrak{s}$ of canonical multisections (boundary conditions), unlike the open r -spin intersection numbers of Buryak, Clader and Tessler, whose construction GKT generalize and which do not depend on this choice [BCT24, Thm. 3.17].

GKT record the combinatorial type of a disk by a *graph* Γ . The graphs that carry invariants are the *balanced* ones: they have one σ_{12} -point, and the descendent Witten bundle has rank equal to the dimension of their moduli space, so that its relative Euler number is a number once boundary conditions are fixed [GKT22, Def. 2.32]. A *critical* graph has no σ_{12} -point and rank one more than its dimension [GKT22, Def. 2.33], so that a homotopy of boundary conditions transverse to the zero section vanishes at finitely many points of $[0, 1]$ times its moduli space.

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The invariants are collected in the potential [GKT22, (4.11)]

$$W^\nu = \sum_{J, \mathbf{d}} \sum_{\Gamma} \frac{(-1)^{|J|-1}}{|\text{Aut}(J, \mathbf{d})|} \nu_{\Gamma, \mathbf{d}} u_{J, \mathbf{d}} x^{\beta_1(\Gamma)} y^{\beta_2(\Gamma)} = x^r + y^s + \cdots,$$

the sums over multisets J of internal twists (a_j, b_j) with descendent vectors $\mathbf{d}$ and over the balanced graphs Γ with these internal markings. Here $u_{J, \mathbf{d}} = \prod_j t_{a_j, b_j, d_j}$ is a monomial in deformation parameters $t_{a, b, d}$ dual to the insertions $\tau_d^{(a, b)}$, and $\beta(\Gamma) = (\beta_1(\Gamma), \beta_2(\Gamma))$ is the *boundary degree* of Γ , its numbers of boundary markings of types σ_1 and σ_2 ; the empty multiset contributes $x^r + y^s$, by the normalization $\nu = -1$ for balanced graphs without internal markings, and over the labeled ring of [GKT22, Def. 4.6] the same formula holds without the symmetry factor. Up to the sign and symmetry factor, which are constant along each balanced string (see the notation in §1.2), the coefficient of $u_{J, \mathbf{d}} x^{\beta_1} y^{\beta_2}$ is thus the invariant of the balanced graph of boundary degree β over $(J, \mathbf{d})$.

On the boundary of a balanced moduli space a homotopy of boundary conditions can vanish only on the strata where a critical graph and a balanced graph meet at a boundary node, and the invariants change only through these zeros. GKT assemble the change at each time at which such zeros occur into an automorphism of the potential: the exponential of the wall terms of the critical graphs, the Hamiltonian vector fields attached to them (see the notation in §1.2), multiplied, up to explicit signs, by their zero counts. The total change is the time-ordered product of these exponentials [GKT22, proofs of Thms. 5.4 and 5.9]; this jump formula is restated in §3.

Three of GKT's results organize the dependence on boundary conditions completely. First, the systems of invariants arising from canonical families form a set $\Omega_{r, s}(I)$, for I a finite set of internal marking labels as in [GKT22], characterized by explicit polynomial conditions, the *chamber-index* conditions of [GKT22, Def. 3.47, Cor. 5.14]: normalizations for at most one internal marking; the condition $\mathcal{A}(J, \mathbf{d}, \nu) = (-1)^{d_1}$ when J is a single marking with descendent exponent d_1 ; and the condition $\mathcal{A}(J, \mathbf{d}, \nu) = 0$ when $|J| \geq 2$ and $N(J, \mathbf{d}) \geq 0$, that is, when balanced graphs over $(J, \mathbf{d})$ exist (the integer $N(J, \mathbf{d})$ is defined in Definition 2.1, and §2 shows that this is the inequality of [GKT22, Def. 3.47]). Here $\mathcal{A}$ is a polynomial in the invariants with Gamma-function coefficients, recalled in (5). A system satisfying these conditions is a *chamber index* in GKT's terminology; the word does not refer to a chamber of the wall complement in the quadrant used below. Since these are exactly the systems in $\Omega_{r, s}(I)$, we call them *realizable*.

Second, GKT state that their wall-crossing group acts faithfully and transitively on these systems [GKT22, Thm. 4.27 and Cor. 5.14]; transitivity requires the marginal torus summands adjoined in §1.2 ([Wue26a, Prop. 1.4 and Rem. 3.1], Proposition 2.6). Third, the potential W^ν of any system has oscillatory integrals over the good basis, the cycles $\Xi_{(a, b)}$, $0 \leq a \leq r-2$, $0 \leq b \leq s-2$, with $\int_{\Xi_{(a, b)}} x^{a'} y^{b'} e^{(x^r + y^s)/h} dx \wedge dy = \delta_{aa'} \delta_{bb'}$ for (a', b') in the same range [GKT22, (4.3)], equal to the closed extended FJRW generating function [GKT22, Thm. 0.3], an open mirror theorem building on the closed Landau–Ginzburg mirror symmetry of [LLSS17, HLSW22].

GKT remark [GKT22, Rem. 0.8] that their wall-crossing group is close in spirit to the tropical vertex group of Kontsevich–Soibelman [KS06], as used for Fano mirror symmetry in [GPS10, Gro10] and in the log Calabi–Yau setting [GHK15], and their survey raises the problem of a tropical formulation of the open theory [GKT26]. Such a formulation is not immediate. A Landau–Ginzburg model carries no target space and hence no SYZ fibration, and Maher notes that without an SYZ fibration the Gross–Siebert machinery does not apply directly to these models [Mah24, Rem. 5.1.1].

This paper organizes the wall-crossing group of GKT by rays in the quadrant of boundary degrees. It proves that in each finite truncation each element of the group factors uniquely as a slope-ordered product of factors supported on single rays, the *ray factors*, and computes the first factors; the construction is algebraic and does not presuppose a degenerating target. Here the *boundary ray* of a wall term is the ray spanned by the boundary degree of its critical graph, that is, by its numbers of boundary marked points; it is not one of the two edge rays of the quadrant, which we call the *edges* of the quadrant. The paper then shows that in each finite truncation the factorization of the *transition element* $g^{\min \rightarrow \max}$, the wall-crossing transformation between the two extremal systems, in which the coefficients on every wall-bearing string (§1.2) vanish except at one end, is realized by homotopies of boundary conditions; the ray factors, but not the chosen homotopies, are compatible in the coefficientwise limit.

The paper also shows that the system of open invariants at every slope cut, a direction in the quadrant taken on one side of any wall it meets (§1.2), is obtained by transport through the ray factors from initial coefficients, constants attached to the monomials of the potential that do not depend on the cut. The quadrant organizes wall-crossing from one of its edges to the other; it does not by itself carry the singular integral-affine structure of a Gross–Siebert base. For $x^4 + y^4$ the companion paper [Wue26b] constructs a singular integral-affine annulus with a wall structure, consistent at its two joints, whose walls for a single first-descendent marking of twist $(1, 0)$ or $(0, 1)$ are ray factors of $g^{\min \rightarrow \max}$ (Remark 6.26).

1.2. Results. The organizing fact is a dictionary between the wall-crossing algebra of [GKT22] and the tropical vertex. The wall term of a critical graph is the monomial of its internal markings times the Hamiltonian vector field, for the area form $dx \wedge dy$, of the boundary monomial $x^{\#\sigma_1} y^{\#\sigma_2}$ of the critical graph. Since $dx \wedge dy = xy d \log x \wedge d \log y$, this vector field is $(xy)^{-1}$ times the tropical-vertex vector field whose exponent is the boundary degree $(\#\sigma_1, \#\sigma_2)$ (Proposition 2.2). The shift by $(1, 1)$ between this exponent and the Lie bidegree, the exponent shift that the wall term effects on monomials, is the removal of the single fully twisted boundary point σ_{12} , which balanced graphs carry and critical graphs do not. Consequently the wall term of a critical graph lies on its boundary ray, and all results below are organized by these rays.

In order of importance, the main new results are the following.

- (a) The wall term of a critical graph fixes exactly the monomials whose exponents are proportional to its boundary degree, as a wall of a scattering diagram fixes the monomials along it; its boundary ray is therefore the unique ray whose monomials it fixes, and the walls are placed there. Along these rays the transition element $g^{\min \rightarrow \max}$ between the two extremal systems factors without choices (Theorem 1.1). By contrast, the jump formula of [GKT22, proof of Thm. 5.9] orders the total jump of a normal-form homotopy by homotopy time, with one factor for each subset of internal markings and factors that mix boundary rays; the times and their order are auxiliary choices of the extension lemma (Remark 3.3).
- (b) In each finite coefficient quotient this factorization is realized by concatenated homotopies of boundary conditions, one ray at a time, whereas a single homotopy in the normal form of the GKT extension lemma makes all critical graphs with the same internal markings jump at once and so cannot in general separate their rays (Theorem 1.2).
- (c) The invariants at every slope cut are transported through the ray factors from initial coefficients (Theorem 1.3); nonzero initial coefficients can occur beyond those of x^r , y^s and the monomials with one internal marking, so the open theory is not in general generated from these by wall-crossing; for $x^5 + y^5$ it is strictly richer (Observation 6.11).

Theorems 1.1, 1.2 and 1.3 below state (a)–(c) in this order; they rest on the ordered factorization of Proposition 3.1, discussed below. Under *Further results*, Theorem 1.4 makes explicit on each string how the conditions of GKT determine the invariants, and Theorem 1.5 classifies the wall terms by weight.

Two further results are recalled from the companion note [Wue26a], where they are proved over the symmetric coefficient ring. First, the transitivity asserted in [GKT22, Thm. 4.27] requires adjoining to the Lie algebra of [GKT22, Def. 4.23] the torus direction $x\partial_x - y\partial_y$ of the wall terms of critical graphs with one boundary marking of each type, and the enlarged group acts freely and transitively [Wue26a, Prop. 1.4 and Rem. 3.1]; the proof extends the invariance of the polynomials $\mathcal{A}$, which GKT derive from a period formula stated for residues at most $(r-2, s-2)$, to all residues through a Laplace integral. Here the torsor property is proved over the labeled ring of [GKT22, Def. 4.6] as well (Lemma 2.5), as Theorem 1.2 requires. The proof uses the all-residue invariance (Lemma 2.4), whose Laplace computation [Wue26a, proof of Prop. 1.4] already carries out over that ring. Proposition 2.6 adds an explicit primary instance of the failure, for $r = s = 4$ and two labels of twist $(2, 2)$; [Wue26a, Ex. 3.2] gives a descendent one. Second, Theorem 1.5, including the threshold (iii), is also in the companion note [Wue26a, §2, Lem. 4.2 and Prop. 7.1].

Notation. The statements below use the following notation; further conventions are in §2.1. The Landau–Ginzburg wall-crossing Lie algebra $\mathfrak{g}^{r,s}$ of [GKT22, Def. 4.23] is spanned by the *wall terms* $u_{J,\mathbf{d}}X_k$, $k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$, where $u_{J,\mathbf{d}}$ is the deformation monomial of §1.1 (we write $t_{a,b} = t_{a,b,0}$) and

$$X_k = X_{k_1,k_2} = x^{k_1}y^{k_2}((k_2+1)x\partial_x - (k_1+1)y\partial_y)$$

is the Hamiltonian vector field of $x^{k_1+1}y^{k_2+1}$ for the area form $dx \wedge dy$, subject to the dimension constraint

$$(1) \quad sk_1 + rk_2 = w(J, \mathbf{d}), \quad w(J, \mathbf{d}) = \sum_j (sa_j + rb_j + rs(d_j - 1)),$$

where $w(J, \mathbf{d})$ is the *Euler weight* of $u_{J,\mathbf{d}}$ (the integer $m(J, \mathbf{d})$ of [GKT22, Not. 2.30] is $w(J, \mathbf{d}) + rs$), and to the congruences $k_1 \equiv r(J) \pmod{r}$, $k_2 \equiv s(J) \pmod{s}$, where $\sum_j a_j = r(J) + r\ell_1$ and $\sum_j b_j = s(J) + s\ell_2$ with $0 \leq r(J) < r$ and $0 \leq s(J) < s$. We call k the *Lie bidegree*, or *bidegree*, of X_k . A critical graph of boundary degree $\beta = (\beta_1, \beta_2)$ carries wall terms of Lie bidegree $k = \beta - (1, 1)$ (Proposition 2.2; for a marginal critical graph, of boundary degree $(1, 1)$, this is the torus term adjoined below), and its *boundary ray* is $\mathbb{R}_{\geq 0}\beta = \mathbb{R}_{\geq 0}(k_1 + 1, k_2 + 1)$. An element of the Lie algebra is *supported on* a boundary ray ρ if it is a combination of wall terms whose boundary ray is ρ , and an automorphism is supported on ρ if it is the exponential of such an element. The *slope* of β , and of its ray, is $\beta_2/\beta_1 \in [0, \infty]$. A *slope-ordered product* of automorphisms supported on rays composes them in the order of the slopes of their rays; in increasing order the factor of least slope acts first, so that for factors $\exp(\eta_i)$ on rays ρ_i with $\text{slope}(\rho_1) < \dots < \text{slope}(\rho_L)$ the product is $\exp(\eta_L) \circ \dots \circ \exp(\eta_1)$. *Primary* means that all descendent exponents d_j vanish, and the *deformation ideal* $\mathfrak{m}$ is the ideal generated by the parameters $t_{a,b,d}$. For a primary monomial, $-w(J, \mathbf{0})/rs$ is the degree of $u_{J,\mathbf{0}}$ for which $x^r + y^s + \sum t_{a,b}x^a y^b$ is quasi-homogeneous of degree one, with $\deg x = 1/r$ and $\deg y = 1/s$; thus a single parameter $t_{a,b}$ has $w > 0$, $w = 0$ or $w < 0$ exactly when the deformation $x^a y^b$ is irrelevant, marginal or relevant.

For fixed $(J, \mathbf{d})$ let $N = N(J, \mathbf{d}) = \ell_1 + \ell_2 - |J| + 1 + \sum_j d_j$ (Definition 2.1). There are $N + 1$ balanced graphs when $N \geq 0$ and N critical graphs when $N \geq 1$ [GKT22, Not. 2.40]. The $N + 1$ coefficients of the balanced graphs in the potential, the *balanced coefficients*, form the

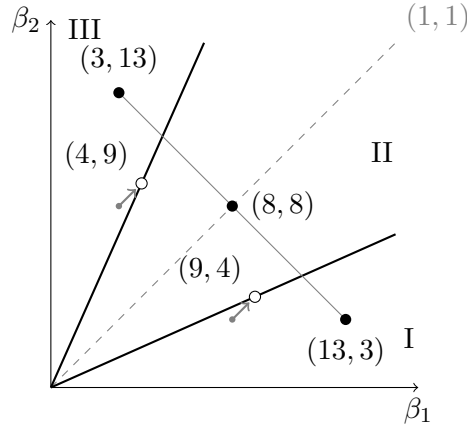


FIGURE 1. The balanced string of $t_{3,3,2}$ for $x^5 + y^5$ (Example 6.12). Filled dots: the balanced boundary degrees; open dots: the boundary degrees of the two critical graphs, on their boundary rays (solid), obtained from the Lie bidegrees $(3, 8)$ and $(8, 3)$ by the $(1, 1)$ shift (arrows). Balanced and critical degrees interlace in slope. In the three sectors I, II, III cut out by the two rays (after reduction to the divisors of $t_{3,3,2}$), the coefficients of $(3, 13)$, $(8, 8)$, $(13, 3)$ are $(\frac{25}{36}, 0, 0)$, $(\frac{25}{36}, -\frac{25}{16}, \frac{25}{36})$ and $(0, 0, \frac{25}{36})$. Sector I, along the β_1 -axis, carries the minimal system $\nu^{\min}$ of Theorem 1.1, whose surviving coefficient is the one of least x -degree, at the far end of the string.

$(J, \mathbf{d})$ -string, also called the *balanced string* of $u_{J, \mathbf{d}}$; their boundary degrees form an arithmetic progression with step $(r, -s)$. The wall term of each of the N intervening critical graphs changes $x^r + y^s$ by a *wall vector* supported on two adjacent coefficients of the string (6) (Figure 1). The string and its monomial are *wall-bearing* if $N \geq 1$, and *marginal* if $w(J, \mathbf{d}) = 0$. The *enlarged algebra* $\tilde{\mathfrak{g}}^{r,s}$ adjoins to $\mathfrak{g}^{r,s}$ the wall terms $u_{J, \mathbf{d}}(x\partial_x - y\partial_y)$ of bidegree $(0, 0)$ of the critical graphs on marginal strings with $J \neq \emptyset$; their critical graphs have boundary degree $(1, 1)$. The group $\tilde{G}^{r,s} = \exp(\tilde{\mathfrak{g}}^{r,s})$ acts freely and transitively on the set $\Omega_{r,s}(I)$ of realizable systems, the *torsor action* (Lemma 2.5; [Vue26a, Prop. 1.4]), so any two realizable systems ν, ν' are related by a unique element of $\tilde{G}^{r,s}$, the *transition element* from ν to ν' . All statements are for the enlarged algebra and its group. A finite set S of deformation monomials is *divisor-closed* if every monomial dividing an element of S belongs to S , and *reduction to S* means quotienting out all other deformation monomials.

Main theorems. Every element of the enlarged group factors along the boundary rays (Proposition 3.1): in the primary algebra, modulo every power of $\mathfrak{m}$, it is uniquely a slope-ordered product of exponentials of elements supported on single boundary rays, and with descendents the same holds on every finite divisor-closed coefficient set. This is the Kontsevich–Soibelman ordered-product lemma [KS06], [GPS10, §0.3 and Thm. 1.4], here over the boundary rays, whose directions are the Lie bidegrees shifted by $(1, 1)$ and therefore do not form a Lie grading. The induction still works because each ray subalgebra is abelian, the bracket being proportional to the determinant of the directions of the two boundary rays, and the filtration by the number of markings replaces the grading. The main theorem applies this to one canonical element.

Theorem 1.1 (Slope-ordered factorization between the extremal systems; §5).

- (i) *On every wall-bearing string, including a marginal string, the wall vectors of its N critical graphs are supported on the N pairs of adjacent balanced coefficients, one pair each, with nonzero entries at both; consequently there exist unique realizable systems $\nu^{\min}$ and $\nu^{\max}$ in which, on every such string, only the coefficient of least, respectively greatest, x -degree survives.*
- (ii) *In the primary algebra, modulo every power of the deformation ideal, the transition element $g^{\min \rightarrow \max}$ from $\nu^{\min}$ to $\nu^{\max}$ factors uniquely as a product, in increasing order of slope, of exponentials of elements supported on single boundary rays. These factorizations are compatible and define the completed primary factorization $\mathfrak{D}_0^{r,s}$, which involves no choices and is computable order by order.*
- (iii) *The rays are forced: the vector field of the wall term of a critical graph annihilates a monomial $x^{\alpha_1}y^{\alpha_2}$, $\alpha \in \mathbb{Z}_{\geq 0}^2$, if and only if α lies on the boundary ray of that graph.*

We call $\nu^{\min}$ and $\nu^{\max}$ the *minimal* and *maximal systems*, together the *extremal systems*: on each wall-bearing string the surviving coefficient is the one of least x -degree in $\nu^{\min}$ and the one of greatest x -degree in $\nu^{\max}$. For a finite labeled set I , $\nu_I^{\min}$ and $\nu_I^{\max}$ denote the restrictions of $\nu^{\min}$ and $\nu^{\max}$ to the labeled set I . In the full descendent algebra the proof gives compatible factorizations of $g^{\min \rightarrow \max}$ on all finite divisor-closed coefficient sets (Proposition 3.1); their coefficientwise limit, read on slope cuts (defined before Theorem 1.3), is the descendent extension $\mathfrak{D}_0^{r,s}$ of $\mathfrak{D}_0^{r,s}$. Part (iii) is the identity $X_k(x^{\alpha_1}y^{\alpha_2}) = ((k_2+1)\alpha_1 - (k_1+1)\alpha_2)x^{\alpha_1+k_1}y^{\alpha_2+k_2}$ (Proposition 6.2): the factors are placed on the boundary rays, and not on the rays of the Lie bidegrees, because these are the rays whose monomials they fix. In (i) the wall vector of the p th critical graph is supported on the coefficients with indices $p-1$ and p , $1 \leq p \leq N$ ($\lambda_p, \mu_p > 0$ in (6)), so the wall vectors form a triangular system, and the existence of $\nu^{\min}$ and $\nu^{\max}$ is the triangular elimination of [GKT22, proof of Thm. 4.27, Step 3], with the torus direction adjoined; the new statements in Theorem 1.1 are the factorization (ii) of $g^{\min \rightarrow \max}$ along the boundary rays, which involves no choices, and the placement (iii). We write θ_ρ for the factor of $\mathfrak{D}_0^{r,s}$ supported on the ray ρ and call it the *ray factor* of ρ ; it collects all critical-graph terms for which (k_1+1, k_2+1) lies on ρ (Proposition 2.2). For $x^5 + y^5$ the walls of $\mathfrak{D}_0^{5,5}$ begin

$$\begin{aligned} \theta_{(1,2)} &= \exp\left(\frac{1}{25} t_{2,3} t_{3,3} X_{0,1} + \cdots\right), & \theta_{(2,1)} &= \exp\left(\frac{1}{25} t_{3,2} t_{3,3} X_{1,0} + \cdots\right), \\ \theta_{(1,1)} &= \exp\left(\frac{1}{50} t_{3,3}^2 X_{1,1} + \frac{1}{25} (t_{2,2} t_{3,3} + t_{2,3} t_{3,2}) X_{0,0} + \cdots\right), \end{aligned}$$

where the subscripts are the primitive vectors on the rays. The ray factors are computed by solving $\nu^{\max} - \nu^{\min}$ string by string, order by order, against the wall vectors (Example 5.1).

In each finite truncation the factorization of an element g is a finite set of rays carrying automorphisms, in the shape of a scattering diagram, whose path-ordered product from one edge of the quadrant to the other is g . Apart from the factors on the $(1,1)$ -ray, its automorphisms are the rational-power wall-crossings of Proposition 2.2, which multiply x and y by rational powers of binomials $1 + c x^{k_1} y^{k_2}$ [GKT22, Lem. 4.21], not elements of the tropical vertex group of [GPS10]. For the transition element $g^{\min \rightarrow \max}$ this product is not the identity unless every ray factor is trivial. Accordingly the diagram $\mathfrak{D}_0^{r,s}$ is not the consistent completion of a diagram of incoming walls: it is the Kontsevich–Soibelman factorization of one prescribed element, $g^{\min \rightarrow \max}$. In the graded setting of Kontsevich–Soibelman and Bridgeland [KS06, Bri17], ordered factorization over the rays of the grading identifies each group element with the consistent scattering diagram whose path-ordered product between two opposite chambers is that element; $\mathfrak{D}_0^{r,s}$ is the analogue of that diagram for $g^{\min \rightarrow \max}$, read along a path from one edge of the quadrant to the other, but the boundary rays do not come from a grading of $\tilde{\mathfrak{g}}^{r,s}$ (Proposition 2.2), so the standard bijection does not apply verbatim, and Proposition 3.1 supplies the factorization instead.

The worked case $x^5 + y^5$ (§§4–5) produces, beyond the ray factors above, the invariant relations and values

$$(2) \quad \langle (\tau_0^{(3,3)})^2 \sigma_1 \sigma_2^6 \sigma_{12} \rangle + \langle (\tau_0^{(3,3)})^2 \sigma_1^6 \sigma_2 \sigma_{12} \rangle = -\frac{2}{5},$$

$$(3) \quad \langle (\tau_0^{(3,3)})^3 \sigma_1^4 \sigma_2^4 \sigma_{12} \rangle = \frac{1}{5}, \quad \langle \tau_0^{(1,2)} \tau_0^{(2,1)} \tau_0^{(2,2)} \sigma_{12} \rangle = -\frac{1}{25},$$

the first a wall relation (only the sum is invariant; the difference changes across the $(1, 1)$ -ray, which carries the first symmetric positive-weight primary wall term $t_{3,3}^2 X_{1,1}$), the second and third absolute: they live on strings with no wall terms, the second on a broad string, where $r(J) = r - 1$ or $s(J) = s - 1$. For the third, $J = \{(1, 2), (2, 1), (2, 2)\}$ has $N = 0$, every term of $\mathcal{A}(J, \mathbf{0}, \nu)$ with two blocks vanishes because each pair of these twists has $N = -1$, and the term with three blocks, the product of the three one-marking normalizations, has Gamma coefficient $\Gamma(\frac{6}{5})^2 / \Gamma(\frac{1}{5})^2 = \frac{1}{25}$; so the condition $\mathcal{A} = 0$ gives $-\frac{1}{25}$. The *two-insertion relation*, satisfied by every realizable system, takes the uniform form

$$(4) \quad \frac{\nu_{(r,0)}}{r} + \frac{\nu_{(0,s)}}{s} = -\frac{1}{rs} \quad (\text{e.g. } \langle \sigma_1^r \sigma_{12} \cdots \rangle + \langle \sigma_2^r \sigma_{12} \cdots \rangle = -\frac{1}{r} \text{ at } r = s),$$

for every two-insertion string $\{(a_1, b_1), (a_2, b_2)\}$, $|J| = 2$, with $a_1 + a_2 = r$, $b_1 + b_2 = s$ (Remark 4.3). Here ν_β denotes the invariant of boundary degree $\beta = (\beta_1, \beta_2)$, with boundary insertions $\sigma_1^{\beta_1} \sigma_2^{\beta_2} \sigma_{12}$, and the suppressed insertions are the two interior markings (a_1, b_1) and (a_2, b_2) .

Theorem 1.2 concerns homotopies of boundary conditions. A *homotopy of boundary conditions* is a family of simple canonical homotopies [GKT22, Defs. 3.36–3.37], a homotopy of families of canonical multisections compatible on boundary strata. A *normal-form homotopy* is one constructed by the extension lemma [GKT22, Lem. 3.40] (Lemma 3.4): each subset A of the internal labels has one time T_A , and every zero of the homotopy of a critical graph with internal marking set A occurs at T_A (§3). For a normal-form homotopy, or a concatenation of such, the *jumps* are the automorphisms by which its potential changes at these zeros, given by the jump formula of [GKT22, proof of Thm. 5.9] (§3), and their time-ordered product is its *total jump*. Freeness of the action gives path independence: for such a homotopy with equal endpoints the total jump is the identity (Remark 3.2). This is consistency in the space of boundary conditions, distinct from consistency of the ray diagram in the quadrant in the sense of [GPS10]. In Theorem 1.2(i) the labeled set I carries enough labels that no monomial of S lies in the ideal $\text{Ideal}(I)$ of [GKT22, Def. 4.10], on whose quotient symmetrization is injective [GKT22, Lem. 4.14]; part (ii) concerns labeled systems and needs only labels of twist $(3, 3)$.

Theorem 1.2 (Realization by homotopies in a finite coefficient quotient; §6.3).

- (i) *Let S be a finite divisor-closed set of deformation monomials containing 1, and choose a finite labeled set I with at least*

$$n_{a,b}(S) := \max_{u \in S} \sum_{d \geq 0} \text{ord}_{t_{a,b,d}}(u)$$

labels of every twist (a, b) occurring in S . The reduction to S of the factorization $\mathfrak{D}^{r,s}$ (Theorem 1.1 and the paragraph after it), the quotient factorization, has finitely many rays $\rho_1, \dots, \rho_L$, listed in increasing slope. There is a homotopy of boundary conditions, obtained by concatenating normal-form homotopies, from a family of canonical multisections realizing $\nu_I^{\min}$ to one realizing $\nu_I^{\max}$; every time slice is a family of canonical multisections [GKT22, Def. 3.22]. After reduction to S every nonzero jump is supported on a single boundary ray, the rays carrying nonzero reduced jumps occur in increasing

slope as the homotopy time increases, and the total reduced jump on ρ_i is the ray factor θ_{ρ_i} of Theorem 1.1.

- (ii) A single normal-form homotopy does not suffice in general: for $x^5 + y^5$ modulo $\mathfrak{m}^6$, with I carrying at least five labels of twist $(3, 3)$, no normal-form homotopy between families of canonical multisections realizing $\nu_I^{\min}$ and $\nu_I^{\max}$ has all of its jumps supported on single rays.

The proof of (i) (§6.3) treats one ray at a time. The intermediate systems, obtained from $\nu_I^{\min}$ by applying the factors $\theta_{\rho_1}, \theta_{\rho_2}, \dots$ in turn, are realized by canonical families by [GKT22, Cor. 5.14], and consecutive ones are joined by a single normal-form homotopy of [GKT22, Lem. 3.40]. By the jump formula of [GKT22, proof of Thm. 5.9] (§3) the total jump of such a homotopy is a time-ordered product of factors, one for each subset of internal labels and each with exact label support on that subset, and freeness of the action forces this total to equal the labeled lift of θ_{ρ_i} , its image under symmetrization in the labeled coefficient ring of [GKT22, Def. 4.6]. A factorization with exact label supports is unique, and this uniqueness, together with the abelianness of the subalgebra of each ray, forces every internal jump of that homotopy onto the prescribed ray; the concatenation may reuse a marking subset in later homotopies. A final normal-form homotopy joining the result to a realization of $\nu_I^{\max}$ is trivial in the quotient. The label count $n_{a,b}(S)$ keeps the monomials of S out of $\text{Ideal}(I)$ [GKT22, Def. 4.10], on whose quotient symmetrization is injective [GKT22, Lem. 4.14], so the labeled statement descends to the symmetric one. The increasing order of slopes is the convention fixed in Theorem 1.1: by Proposition 3.1, $g^{\min \rightarrow \max}$ also factors uniquely in decreasing order of slopes, and the same construction realizes that factorization (Remark 6.23).

Part (ii) is Corollary 6.21, an instance of a general obstruction: a concatenation is needed whenever a string carries two wall terms with nonzero reduced coefficients that receive no contribution from lower-order terms, as for $t_{3,3}^5$ in $x^5 + y^5$ or for any descendent singleton $t_{a,b,2}$: a single normal-form homotopy of [GKT22, Lem. 3.40] cannot separate them, since all wall terms of a string jump at one time (Proposition 6.19). The exact first order at which two primary wall terms share a string is computed for every hyperbolic pair, that is, every pair with $rs > 2r + 2s$ (Proposition 6.17, Remark 6.22). Realizing each boundary-ray factor by its own normal-form homotopy avoids the obstruction. On the punctured quadrant the product of ray factors along a path transverse to the walls depends only on the chambers of its angular endpoints (Remark 6.25).

The last main result is algebraic and concerns transport through the *boundary-degree quadrant* C_∂ , the closed first quadrant of the boundary-degree plane, with *edges* the σ_1 -axis and the σ_2 -axis. A *slope cut* is a direction in C_∂ , taken on one side of the wall when a wall lies on that direction (Definition 6.1); it plays the role of the endpoint of a path in the path-ordered products of [GPS10]. The σ_1 -*edge cut* and the σ_2 -*edge cut* are the cuts at the two edges. The *transport* $\Psi(q)$ to a slope cut q is the slope-ordered product of the ray factors from the σ_1 -edge cut up to q , that is, the path-ordered product, in the sense of [GPS10], along a path from the σ_1 -edge to q , taken coefficientwise (Definition 6.1). The *cut potential* at q is $W(q) = \Psi(q)(W^{\nu^{\min}})$. Here m denotes a monomial $u_{J,\mathbf{d}}x^{\beta_1}y^{\beta_2}$ of the potential. The *anchor cut* q_m of m is the slope cut through the direction of its exponent, on the side toward the $(1, 1)$ -direction (Definition 6.4), and the coefficient of m at q_m is its *anchored value*.

Theorem 1.3 (The transport representation; §6). *There are unique constants init_m , one for each monomial m of the potential, such that every cut potential is their coefficientwise transport,*

$$W(q) = \sum_m \Psi(q) \Psi(q_m)^{-1} (\text{init}_m m),$$

the sum over the monomials m of the potential and q_m the anchor cut of m . On every monomial $u_{J,\mathbf{d}}$ the difference between the coefficient vector of $W(q)$ and the transported contributions of the constants init_m of the proper divisor monomials is independent of the cut q , and it equals the vector of the constants init_m on the string of $u_{J,\mathbf{d}}$.

We call init_m the *initial coefficient* of m ; the initial coefficients form the *initial system*, and by the last statement they are determined recursively in the divisibility order of deformation monomials.

The proof of Theorem 1.3 is an induction in the divisibility order of deformation monomials: on the string of a monomial u , the change of the potential across a wall and the change of the transported lower initial coefficients are the same universal expression in lower-order data, so their difference does not depend on the cut. This part is formal. What gives the representation content is that the initial system is not in general generated by wall-crossing (Observation 6.11): for $x^5 + y^5$, transport through the ray factors from $x^r + y^s$ and the monomials with one internal marking does not produce the potential, because nonzero initial coefficients occur beyond these, for instance the invariant $\frac{1}{5}$ of (3), with three internal markings, on a string without wall terms, which no ray action can produce. In the smooth Fano case of [Gro10, CPS24], by contrast, the potential is generated by broken lines from its initial monomials. A coefficient whose initial coefficient is zero is given at every cut by transport of lower initial coefficients alone. Two consequences, stated and proved after Theorem 6.8, compute an initial coefficient directly when no bending from another initial coefficient reaches the relevant cut. Theorem 1.3 underlies the tropical description of the open theory in §6: the cut potentials are computed by bend chains from the initial coefficients, where a *bend chain* is a term of the expansion of Lemma 6.9, a sequence of wall terms applied to a transported monomial along a monotone path in the slope line. Bend chains play the role of the broken lines of [Gro10, CPS24], but there is no affine manifold and no asymptotic direction. In the language of theta functions [GHK15, GHKK18], written ϑ as there (the letter θ_ρ for ray factors follows [GPS10]), init_m plays the role of the coefficient of ϑ_m in an expansion $\sum_m \text{init}_m \vartheta_m$ of the potential, which is therefore not the analogue of $\vartheta_{(r,0)} + \vartheta_{(0,s)}$, and the anchor cut replaces the asymptotic end of a broken line with initial monomial z^m . The initial coefficients do not depend on the side of the anchor cut, since the factor on the ray of m fixes m (Proposition 6.2); the anchored values do, and the $(1,1)$ -side is the one for which Proposition 6.5 controls the foreign bending at the anchor cut. The triangular inversion of Proposition 6.10 expresses each initial coefficient as an alternating sum of anchored values multiplied by sums of bend chains.

Further results. The *good-basis period functional* of a string is the linear functional given by the coefficient, on that string, of the oscillatory periods of [GKT22, Thm. 0.3] (Lemma 4.1). We call the $(J, \mathbf{d})$ -string *narrow* if $r(J) \leq r - 2$ and $s(J) \leq s - 2$, and *broad* if $r(J) = r - 1$ or $s(J) = s - 1$: as for marked points in [GKT22, §1.1], its residue twist $(r(J), s(J))$ is then Neveu–Schwarz for both bundles, respectively Ramond for one of them. The good-basis periods vanish on the broad strings, which are therefore invisible to them.

Theorem 1.4 (Effective determination; §4). *(i) On each narrow $(J, \mathbf{d})$ -string the wall vectors span exactly the kernel of the good-basis period functional. (ii) On each $(J, \mathbf{d})$ -string the chamber-index conditions of [GKT22] amount to one linear condition, with the Gamma coefficients of (5) and with right-hand side determined by strictly lower order. (iii) Together with the characterization of $\Omega_{r,s}(I)$ as the set of solutions of the chamber-index conditions [GKT22, Cor. 5.14], this determines all open FJRW invariants of $x^r + y^s$, up to the torsor action, from the normalizations alone. (iv) On every string that carries a balanced coefficient the prescribed value of $\mathcal{A}$ is 0 or, for a single marking with descendent exponent d_1 , $(-1)^{d_1}$; the closed extended*

invariants occur only as the values of $\mathcal{A}$ for $(J, \mathbf{d})$ with $N(J, \mathbf{d}) < 0$, which have no balanced coefficients [GKT22, Cor. 5.11(3)].

Theorem 1.4 says that the boundary-ray action, together with the period functional, determines the open theory up to that action. That the chamber-index conditions determine the invariants up to the action follows from the transitivity of [GKT22, Thm. 4.27] together with [GKT22, Cor. 5.14], once the Lie algebra is enlarged (Lemma 2.5). What is added is the explicit identification of the wall vectors on each string. On a narrow string they span exactly the kernel of the good-basis period functional, which also follows from Steps 1 and 3 of [GKT22, proof of Thm. 4.27]. On a broad string, where the good-basis periods vanish and [GKT22, Thm. 4.9] does not apply, the Laplace functional of Lemma 2.4 replaces the period formula. Lemma 2.4 shows, by integration by parts, that every wall term preserves the Laplace integral of $e^{W/h}$ over the positive quadrant, whose coefficients are fixed nonzero multiples of the polynomials $\mathcal{A}$ for all residues, the broad ones included. Besides this lemma, the proof of Theorem 1.4 has two ingredients. Each wall vector is annihilated by the period functional, because the period moments obey the recurrence of Lemma 4.1, which is the form that $\Gamma(z+1) = z\Gamma(z)$ takes on the good-basis cycles. The N wall vectors of a string with $N+1$ coefficients are independent and lie in the kernel of the linear part of $\mathcal{A}$, a nonzero functional on these coefficients, so they span that kernel, which has dimension N .

The last theorem, which is also in [Wue26a], is expressed through the central charge $\hat{c}_W = 2 - \frac{2}{r} - \frac{2}{s}$. We call the pair (r, s) *hyperbolic* if $\hat{c}_W > 1$, *parabolic* if $\hat{c}_W = 1$, and *simple* if $\hat{c}_W < 1$. As in [Wue26a], *hyperbolic* refers to the sign of $rs - 2r - 2s = rs(\hat{c}_W - 1)$, the sign that makes the triangle group $(2, r, s)$ hyperbolic, and not to Arnold's hyperbolic singularities $T_{p,q,r}$. The *threshold* is the value $\hat{c}_W = 1$. The *wall terms over a monomial* $u_{J,\mathbf{d}}$ are the wall terms $u_{J,\mathbf{d}}X_k$ of $\tilde{\mathfrak{g}}^{r,s}$ with coefficient $u_{J,\mathbf{d}}$, including the torus term $u_{J,\mathbf{d}}X_{0,0}$, $X_{0,0} = x\partial_x - y\partial_y$, when it is present; each is the wall term of a critical graph over $(J, \mathbf{d})$ (Proposition 2.2).

Theorem 1.5 (Weight trichotomy and the threshold; §2). *In the enlarged algebra $\tilde{\mathfrak{g}}^{r,s}$, described in §1.2 and displayed in §2.1, the following hold.*

- (i) *For a nonempty marking multiset J , wall terms over the monomial $u_{J,\mathbf{d}}$ exist if and only if $N(J, \mathbf{d}) \geq 1$, which forces $w(J, \mathbf{d}) \geq 0$.*
- (ii) *The wall terms of nonzero bidegree are exactly those of positive weight; the others are the weight-zero (torus) terms. At $w = 0$ the constraint (1) admits only the marginal torus bidegree $(0, 0)$, and at $w < 0$ it admits nothing. Consequently a monomial of nonpositive weight carries no wall term of nonzero bidegree of its own.*
- (iii) *The central charge $\hat{c}_W$ governs a trichotomy: positive-weight $d = 0$ wall-crossing exists if and only if $\hat{c}_W > 1$; at $\hat{c}_W = 1$ primary wall-crossing exists but is carried entirely by the marginal torus direction; for $\hat{c}_W < 1$ there is no primary wall-crossing at all.*

Positive weight of the monomial is necessary but not sufficient for a positive-weight wall term: at $r = s = 5$ the monomial $t_{3,3}$ has $w = 5$ but $N = 0$, hence no wall term. The qualifier *of its own* is needed, since higher-order transport from divisor monomials may reach the coefficients of a monomial of nonpositive weight. The threshold is a weight count: a primary insertion contributes at most $rs(\hat{c}_W - 1)$ to the Euler weight, with equality at the maximal twist $(r - 2, s - 2)$. Conversely, when $\hat{c}_W > 1$, the monomial of sufficiently many insertions of maximal twist has $N \geq 1$ and positive weight, and so carries positive-weight wall terms.

1.3. Comparison with the literature. Our computations bear on three points in [GKT22, GKT26], recorded here and, with the supporting derivations, in Remark 2.3, Proposition 2.6,

Remark 4.3 and Lemmas 2.4 and 2.5. Theorem, equation and section numbers for [GKT22] refer to arXiv:2203.02435v3, which is the text quoted throughout.

First, as the companion note shows [Wue26a, Rem. 3.1 and Ex. 3.2], the marginal torus summands adjoined in §2.1 are needed for the transitivity of [GKT22, Thm. 4.27]: with the Lie algebra of [GKT22, Def. 4.23], the action is not transitive, for instance, for $r = s = 4$ and two labels of twist $(2, 2)$ (Proposition 2.6), and Remark 2.3 locates where the definitions and proofs of [GKT22] treat this direction. On the primary part of the parabolic cases the enlarged group is the elliptic wall-crossing group of Maher’s thesis [Mah24, §5.2.2]: the primary wall-crossing group that Maher determines from open Saito potentials for simple elliptic singularities, generated by the products of $x\partial_x - y\partial_y$ with those positive powers of the marginal deformation parameter allowed by [Mah24, (5.2.8)] (see [Mah24, Cor. 5.2.7, Prop. 5.2.10]).

Second, there is an apparent discrepancy in a worked example of the survey. Imposing the open mirror theorem [GKT22, Thm. 0.3] directly on the oscillatory expansion of the potential, which is the mechanism of [GKT22, Thm. 0.2], gives the two-insertion values (4); at $r = s = 4$ this yields $-\frac{1}{4}$ for the sum in [GKT26, Ex. 5.5], where the value $-\frac{1}{8}$ is stated. Remark 4.3 traces this apparent factor-of-two discrepancy to the count of decompositions into two blocks in the quadratic part of $\mathcal{A}$, in the numerical evaluation of [GKT26, Ex. 5.5], and confirms the count independently through [GKT22, Cor. 5.11(3)]; the open mirror theorem itself is not affected.

Third, two points concern the proof of [GKT22, Thm. 4.27]. Its preservation step [GKT22, proof of Thm. 4.27, Step 1] uses the period formula of [GKT22, Thm. 4.9], which is stated for residues at most $r - 2$ and $s - 2$; Lemma 2.4 establishes the preservation also for the broad residues $r(J) = r - 1$ or $s(J) = s - 1$, as does the proof of [Wue26a, Prop. 1.4]. And the argument of [GKT22, Thm. 4.27, Step 2] establishes freeness of the action, which is stated there as faithfulness; freeness is the form used here (Lemma 2.5).

1.4. Method. The argument combines the grading and torsor action of the GKT wall-crossing group with ordered factorization in a strictly convex cone, the Kontsevich–Soibelman ordered-product lemma over the boundary rays (Proposition 3.1 and the discussion before Theorem 1.1). The period relations determine the transverse coordinate on each string, with the Laplace functional of Lemma 2.4 on the broad strings. The GKT extension lemma realizes each prescribed finite sequence of group factors by canonical homotopies. Theorem 1.5 is proved here (§2) and also in [Wue26a], which proves in addition the central-charge threshold for the dependence of the primary invariants on boundary conditions, and the first-wall count, the least number of primary interior insertions that admits a critical graph. The exact calculations for $x^5 + y^5$ in Section 7 are consistency checks and enter no proof; the programs are available at <https://github.com/bwuebben/open-fjrw-scattering>.

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2. THE WALL-CROSSING ALGEBRA OF CRITICAL GRAPHS

2.1. Setting and notation. This subsection completes the notation of §1.2. In a deformation monomial $u_{J,\mathbf{d}} = \prod_j t_{a_j,b_j,d_j}$ the descendent index of $t_{a,b,d}$ is suppressed when $d = 0$. For a multiset J of internal twists with descendents $\mathbf{d}$, write

$$\sum_{j \in J} a_j = r(J) + r\ell_1, \quad 0 \leq r(J) < r, \quad \sum_{j \in J} b_j = s(J) + s\ell_2, \quad 0 \leq s(J) < s.$$

A wall term $u_{J,\mathbf{d}} X_k$ of $\mathfrak{g}^{r,s}$ is subject to the congruences $k_1 \equiv r(J) \pmod{r}$, $k_2 \equiv s(J) \pmod{s}$ and to (1). The integer $N(J, \mathbf{d}) = \ell_1 + \ell_2 - |J| + 1 + \sum_{j \in J} d_j$ of Definition 2.1 counts the balanced and the critical graphs over $(J, \mathbf{d})$; below the respective thresholds $N \geq 0$ and $N \geq 1$ those sets are empty. The points of the $(J, \mathbf{d})$ -string are $(r(J) + pr, s(J) + (N - p)s)$, $0 \leq p \leq N$.

Including the wall terms of bidegree $(0, 0)$ that occur when $w = 0$, the enlarged algebra is

$$\tilde{\mathfrak{g}}^{r,s} = \mathfrak{g}^{r,s} \oplus \bigoplus_{\substack{\emptyset \neq J, w(J, \mathbf{d})=0, N(J, \mathbf{d}) \geq 1 \\ r(J)=s(J)=0}} \mathbb{Q} u_{J, \mathbf{d}} (x \partial_x - y \partial_y), \quad \tilde{G}^{r,s} = \exp(\tilde{\mathfrak{g}}^{r,s}).$$

Remark 2.3 proves that this is a Lie algebra; its marginal torus summands are omitted from [GKT22, Def. 4.23]. Lemma 2.5 proves the torsor property of §1.2.

As in GKT, a *family* is indexed by graphs, and a *homotopy* is a one-parameter deformation, of a single multisection or of a family. A homotopy of boundary conditions $(\mathfrak{s}_T)_{T \in [0,1]}$ is thus a homotopy of families of canonical multisections, compatible on boundary strata. The extension lemma [GKT22, Lem. 3.40], recalled as Lemma 3.4, constructs such a homotopy between any two families of canonical multisections that are transverse to the zero section; its jumps are given by the jump formula of [GKT22, proof of Thm. 5.9], restated in §3. Because the descendent walls are dense in C_∂ (Proposition 6.14), slope cuts replace the chambers of a finite set of walls (Definition 6.1).

2.2. Critical graphs and the enlarged algebra. We recall the objects of [GKT22, §2.4, §4.3] in the notation used throughout, with the internal and boundary markings of §1.1 and with $r(J)$, $s(J)$ as in §2.1 and $w(J, \mathbf{d})$ as in (1).

Definition 2.1 ([GKT22, Not. 2.40]). Write $\sum_j a_j = r(J) + \ell_1 r$, $\sum_j b_j = s(J) + \ell_2 s$ and $N = N(J, \mathbf{d}) = \ell_1 + \ell_2 - |J| + 1 + \sum_j d_j$. The *balanced* graphs over $(J, \mathbf{d})$ are $\Gamma_{J,p}$, $0 \leq p \leq N$, with boundary degrees $\beta = (r(J) + pr, s(J) + (N - p)s)$ and one σ_{12} -point; the *critical* graphs are $\Lambda_{J,p}$, $1 \leq p \leq N$, with boundary degrees $(r(J) + (p - 1)r + 1, s(J) + (N - p)s + 1)$ and no σ_{12} -point. We refer to the set of balanced coefficients over a fixed $(J, \mathbf{d})$ as its *balanced string*. The balanced range is empty for $N < 0$, and the critical range is empty for $N < 1$. By [GKT22, Prop. 2.31], the boundary degrees of the string are the $\beta \in \mathbb{Z}_{\geq 0}^2$ with $s\beta_1 + r\beta_2 = w(J, \mathbf{d}) + rs$, $\beta_1 \equiv r(J) \pmod{r}$ and $\beta_2 \equiv s(J) \pmod{s}$, so a string lies on a segment parallel to $(r, -s)$ in the boundary-degree plane.

A system of open invariants bounded by I is a vector $\nu = (\nu_{\Gamma, \mathbf{d}})$ indexed by the balanced graphs Γ with internal labels in I and their descendent vectors $\mathbf{d}$ [GKT22, (0.3)]. For $J \subseteq I$ and $\mathbf{d}$, the chamber-index polynomial of [GKT22, Notation 3.42] is

$$(5) \quad \mathcal{A}(J, \mathbf{d}, \nu) = \sum_{h=1}^{|J|} \frac{1}{h!} \sum_{J_1 \sqcup \dots \sqcup J_h = J} \sum_{\Theta_1, \dots, \Theta_h} \frac{\Gamma(\frac{1+\sum_i \beta_1(\Theta_i)}{r}) \Gamma(\frac{1+\sum_i \beta_2(\Theta_i)}{s})}{\Gamma(\frac{1+r(J)}{r}) \Gamma(\frac{1+s(J)}{s})} \prod_{i=1}^h \nu_{\Theta_i, \mathbf{d}|_{J_i}},$$

the sum over ordered decompositions of J into nonempty blocks and over balanced graphs Θ_i (so written to keep Γ for the Gamma function) with internal labels J_i , where $\beta(\Theta_i) = (\beta_1(\Theta_i), \beta_2(\Theta_i))$ is the boundary degree of Θ_i , its numbers of boundary markings of types σ_1 and σ_2 . A *chamber index* is a system with $\nu_{\Gamma, \mathbf{0}} = 1$ when Γ has one internal marking, $\nu_{\Gamma, \mathbf{0}} = -1$ when it has none, $\mathcal{A}(J, \mathbf{d}, \nu) = (-1)^d$ when $J = \{j\}$ and $\mathbf{d} = (d)$, and $\mathcal{A}(J, \mathbf{d}, \nu) = 0$ when $|J| \geq 2$ and $N(J, \mathbf{d}) \geq 0$ [GKT22, Definition 3.47]; we write $\Upsilon_{r,s}(I)$ for the set of chamber indices bounded by I . By [GKT22, Corollary 5.14], $\Omega_{r,s}(I) = \Upsilon_{r,s}(I)$: the chamber indices are exactly the systems realized by canonical boundary conditions, the *realizable* systems of §1.1.

The inequality of [GKT22, Def. 3.47] in the last condition reads $N(J, \mathbf{d}) \geq 0$, because

$$w(J, \mathbf{d}) + rs = sr(J) + rs(J) + rsN(J, \mathbf{d}).$$

Proposition 2.2 (The $(1, 1)$ shift). *The Lie bidegree of the critical graph $\Lambda_{J,p}$, in the sense of (1), is its boundary degree minus $(1, 1)$ ([GKT22, Prop. 2.39]); equivalently, the vector field of the wall term attached to Λ is the Hamiltonian vector field, for the area form $dx \wedge dy$, of the boundary monomial $x^{\# \sigma_1} y^{\# \sigma_2}$ of the critical graph. With $g_\alpha = z^\alpha (\alpha_2 x \partial_x - \alpha_1 y \partial_y)$ the corresponding vector field of the tropical vertex (Hamiltonian for $d \log x \wedge d \log y$), one has $X_{k_1, k_2} = (xy)^{-1} g_{(k_1+1, k_2+1)}$, and the wall-crossings are the rational-power maps of [GKT22, Lem. 4.21]. The shift is the removal of the σ_{12} -insertion. The algebra is $\mathbb{Z}^2$ -graded by (k_1, k_2) ; the ray assignment uses $(k_1, k_2) + (1, 1)$, which is not additive under the bracket.*

Proof. The first statement is [GKT22, Prop. 2.39(b)] read against Definition 2.1. Since $dx \wedge dy = xy d \log x \wedge d \log y$, the Hamiltonian vector field of a function for $dx \wedge dy$ is $(xy)^{-1}$ times its Hamiltonian vector field for $d \log x \wedge d \log y$; for $x^{k_1+1} y^{k_2+1}$ this is the dictionary $X_k = (xy)^{-1} g_{k+(1,1)}$, and the explicit wall maps are [GKT22, Lem. 4.21]. The grading follows from $[X_k, X_{k'}] \in \mathbb{Q} X_{k+k'}$ (§3). $\square$

Proof of Theorem 1.5. The constraint (1) reads $sk_1 + rk_2 = w(J, \mathbf{d})$ with $k_1, k_2 \geq 0$: for $w < 0$ there are no solutions, for $w = 0$ only $(0, 0)$, and for $w > 0$ the N solutions of Definition 2.1. A twist (a, b) contributes $sa + rb - rs \leq s(r-2) + r(s-2) - rs = rs(\hat{c}_W - 1)$ to the weight, with equality at the maximal twist; descendants contribute $rs d_j > 0$. Hence positive-weight $d = 0$ monomials exist if and only if $\hat{c}_W > 1$, and for $\hat{c}_W = 1$ every $d = 0$ monomial has $w \leq 0$ with equality exactly on products of marginal twists; whenever such an equality monomial is wall-bearing, its unique direction is the torus one. For the existence of wall terms (not merely of positive-weight monomials) when $\hat{c}_W > 1$: take n interior markings of maximal twist $(r-2, s-2)$; then $N = n - \lfloor 2n/r \rfloor - \lfloor 2n/s \rfloor + 1 \geq 1$ once $n(1 - \frac{2}{r} - \frac{2}{s}) \geq 2$, while the weight is $nrs(\hat{c}_W - 1) > 0$, so the resulting critical graphs carry wall terms; this is the maximal-twist construction of [Wue26a]. At $\hat{c}_W = 1$, those analogous multisets for which $N \geq 1$ are critical and marginal, realizing the torus wall-crossing; for $\hat{c}_W < 1$ and any primary J with $|J| = n$, $\ell_1 + \ell_2 \leq n\hat{c}_W < n$ forces $N \leq 0$: no primary critical graphs at all. The qualifier in the theorem is necessary: acting on a lower divisor coefficient multiplies deformation monomials, so nonlinear transport can contribute to a nonpositive-weight target even though no wall term is supported there. $\square$

Remark 2.3 (The marginal torus direction). At a wall-bearing marginal monomial ($w = 0$ and $N \geq 1$; necessarily $r(J) = s(J) = 0$ and $N = 1$) the critical graph exists (for $x^4 + y^4$ it is precisely the disk of [GKT26, Fig. 9, Ex. 4.9]), but its Lie bidegree is $(0, 0)$, the direction $x\partial_x - y\partial_y$, which is not included in the explicit description [GKT22, (4.16), (4.19)], although the intrinsic description of the group in [GKT22, §0.4 and Rem. 4.24] admits it. With coefficients in the ideal generated by the deformation variables, the exponential of this direction converges adically. The proofs of [GKT22, Thm. 5.9] and of [GKT22, Thm. 5.13] (which gives [GKT22, Cor. 5.14]) realize the torus motion. In the proof of [GKT22, Thm. 5.9], the jump sets $Q_0(A, \mathbf{d})$ of the jump formula (§3) impose only the congruences and (1), which admit the bidegree $(0, 0)$ when $r(A) = s(A) = 0$. In the proof of [GKT22, Thm. 5.13], the realizability step modifies the multisection of a critical graph Λ by a homotopy [GKT22, Lem. 5.17]. This moves the two adjacent balanced coefficients by $h_\Lambda \cdot ((-1)^{\beta_2(\Lambda)} s \beta_1(\Lambda), (-1)^{\beta_2(\Lambda)+1} r \beta_2(\Lambda))$, where $\beta(\Lambda)$ is the boundary degree of Λ and $h_\Lambda \in \mathbb{Q}$ is the signed number of zeros of that homotopy, the number $h_{p, \mathbf{d}}$ of [GKT22, (5.52)–(5.54)]. On a marginal string this vector is exactly the antisymmetric torus motion. We therefore work with the enlarged algebra $\tilde{\mathfrak{g}}^{r,s}$ of §2.1, whose new summands

commute with one another and preserve the bidegree of every wall term. This is a Lie algebra: with $X_{0,0} = x\partial_x - y\partial_y$, $[uX_{0,0}, vX_k] = (k_1 - k_2)uvX_k$ and $[uX_{0,0}, vX_{0,0}] = 0$; since a marginal coefficient u has weight and residues zero, uv satisfies the same wall-term constraints as v . In the primary algebra nothing is added when $\hat{c}_W < 1$; for the parabolic pairs the primary group is purely torus, matching the elliptic wall-crossing group of Maher [Mah24, §5.2.2]; for $\hat{c}_W > 1$ the torus summands sit over the marginal submonomials. Without the torus summands transitivity fails (Proposition 2.6). The next two lemmas supply the torsor properties over the enlargement, which over the symmetric ring are [Wue26a, Prop. 1.4].

Lemma 2.4 (All-residue Γ -functionals). *For every nonempty labeled marking set J and descendent vector $\mathbf{d}$, the polynomial $\mathcal{A}(J, \mathbf{d}, \nu)$ is preserved by every wall term of $\tilde{\mathfrak{g}}^{r,s}$, including when $r(J) = r - 1$ or $s(J) = s - 1$. (The preservation step of [GKT22, proof of Thm. 4.27, Step 1] uses the period formula of [GKT22, Thm. 4.9], which is stated for residues at most $r - 2$ and $s - 2$.)*

Proof. Take $\hbar < 0$ and expand, coefficientwise in the deformation variables, the positive-quadrant Laplace integral

$$\mathcal{L}_\hbar(W) = \int_0^\infty \int_0^\infty e^{W/\hbar} dx dy.$$

The monomial moments are

$$\int_0^\infty x^q e^{x^r/\hbar} dx = \frac{1}{r} (-\hbar)^{(q+1)/r} \Gamma\left(\frac{q+1}{r}\right),$$

and similarly for y . Therefore the coefficient of $u_{J,\mathbf{d}}$ in $\mathcal{L}_\hbar(W^\nu)$ is a fixed nonzero power of $-\hbar$, multiplied by fixed nonzero Gamma factors depending only on $r(J), s(J)$, times $\mathcal{A}(J, \mathbf{d}, \nu)$. This is the termwise calculation in the proof of [GKT22, Thm. 4.9], before the good-basis Kronecker factors are imposed. Unlike those good-basis periods, the displayed Laplace functional is nonzero also for residues $r - 1$ and $s - 1$.

Let $\eta = uX_k$ be any wall term of the enlarged algebra. The vector field X_k is divergence-free and tangent to both coordinate axes. Hence, coefficientwise,

$$\left. \frac{d}{dT} \right|_{T=0} \mathcal{L}_\hbar(e^{T\eta}W) = \int_{[0,\infty)^2} \eta(e^{W/\hbar}) dx dy = 0 :$$

integration by parts has no contribution on the axes, and the exponential decay removes the boundary at infinity. The same identity after replacing W by $e^{T\eta}W$ shows that the full exponential action preserves $\mathcal{L}_\hbar$, and comparison of deformation coefficients proves the claim. Equivalently, the calculation is the Gamma recurrence $\Gamma(z+1) = z\Gamma(z)$ on every residue class. $\square$

Lemma 2.5 (The enlarged torsor). *The action of $\tilde{G}^{r,s}$ on potentials over the labeled ring A_I of [GKT22, Def. 4.6] preserves $\Omega_{r,s}(I)$ and is free and transitive on it, so that $\Omega_{r,s}(I)$ is a torsor in the strict sense, and the jump automorphisms of [GKT22, Thm. 5.9] lie in $\tilde{G}^{r,s}$. The same holds for the symmetric systems over the symmetric ring $A_{I,\text{sym}}$.*

Proof. Membership of the jumps: the jump sets $Q_0(A, \mathbf{d})$ of the jump formula (§3) impose only the congruences and (1), which admit the bidegree $(0,0)$ when $r(A) = s(A) = 0$, so the jump elements are sums of positive-weight and weight-zero (torus) wall terms.

Preservation: the coefficient of every wall term has positive deformation degree, so its exponential converges in that filtration. Lemma 2.4 preserves all the $\mathcal{A}$ -conditions of [GKT22, Def. 3.47]. It also preserves the normalization there: a primary critical monomial has at least

two markings, while a descendent singleton does not alter the $\mathbf{d} = \mathbf{0}$ normalization. Thus the enlarged group preserves $\Upsilon_{r,s}(I)$.

Transitivity: first work modulo a finite divisor-closed set of deformation monomials and order those monomials by any linear extension of proper divisibility. Suppose two realizable systems have already been made equal on every proper divisor of $u = u_{J,\mathbf{d}}$. On the string of u , all $h \geq 2$ terms of $\mathcal{A}(J, \mathbf{d}, -)$ involve proper divisors and hence already agree. The difference of the two coefficient vectors therefore lies in the kernel of the nonzero $h = 1$ Gamma functional. If the string has $N + 1$ balanced coefficients, that kernel has dimension N . In the basis $\varepsilon_0, \dots, \varepsilon_N$ of the coefficients of the string, the N adjacent wall vectors (computed in §4)

$$v_p = \lambda_p \varepsilon_p - \mu_p \varepsilon_{p-1}, \quad \lambda_p, \mu_p > 0, \quad 1 \leq p \leq N,$$

are independent and lie in the kernel by Lemma 2.4; hence they span it. On a wall-bearing marginal string $N = 1$, and the sole vector is the adjoined torus vector $r\varepsilon_1 - s\varepsilon_0 = X_{0,0}(W_0)$. A unique combination of wall terms with coefficient u therefore removes the difference. Its action on every nonconstant term of the potential carries a strict multiple of u , so it changes no coefficient on a proper divisor of u and acts on the string of u only through W_0 . Induction proves transitivity in every finite quotient. By the freeness proved below, whose argument applies verbatim modulo the monomials outside a finite divisor-closed set, the element constructed in each quotient is unique, so the elements for nested quotients are compatible; since only finitely many deformation monomials occur below each bound on the degree of the wall terms, they define an element of the completed group.

The equality $\Omega_{r,s}(I) = \Upsilon_{r,s}(I)$ remains GKT's realization theorem [GKT22, Thm. 5.13 and Cor. 5.14]. Its inclusion $\Upsilon_{r,s}(I) \subseteq \Omega_{r,s}(I)$ does not require the omitted marginal transitivity: at equation (5.50) of [GKT22], the two required $\mathcal{A}$ -values agree directly because both are the prescribed value of [GKT22, Def. 3.47]. The enlarged action therefore preserves and is transitive on the realizable systems.

Freeness: let $g = \exp(\eta)$ with $\eta \neq 0$, and let u be minimal in the divisibility order among the monomials in the support of η (labeled monomials over A_I , symmetric monomials over $A_{I,\text{sym}}$; the argument is the same). The change of the string of u in any potential is exactly $\eta_u(W_0)$, $W_0 = x^r + y^s$: contributions of other wall terms, or of higher exponential terms, carry monomials properly containing a support monomial, which cannot equal u by minimality. And $\eta_u(W_0) \neq 0$. Indeed, on the string belonging to u , the N positive-weight wall terms act by the adjacent vectors $\lambda_p \varepsilon_p - \mu_p \varepsilon_{p-1}$ with $\lambda_p, \mu_p > 0$. These vectors are linearly independent: reading a relation successively in the $\varepsilon_N, \varepsilon_{N-1}, \dots, \varepsilon_1$ coordinates kills all its coefficients. On a marginal string there is instead the single nonzero torus vector $r\varepsilon_1 - s\varepsilon_0$. Thus no nonzero linear combination of the wall terms in η_u can cancel. (Individually, a positive-weight wall term contributes $r(k_2+1)x^{k_1+r}y^{k_2} - s(k_1+1)x^{k_1}y^{k_2+s}$, while a torus term contributes $rx^r - sy^s$.) So g moves every potential, which is freeness and not merely faithfulness. The distinction matters downstream: uniqueness of the transition element, used in Remark 3.2, needs trivial stabilizers, and faithfulness alone does not supply them for a nonabelian group. The argument of [GKT22, Thm. 4.27, Step 2] likewise establishes freeness, though it is stated there as faithfulness. All uses below are through this lemma. $\square$

Proposition 2.6 (Transitivity requires the torus summands). *Let $r = s = 4$, and let $I = \{i, j\}$ consist of two labels of twist $(2, 2)$, with primary label variables u_i, u_j . The chamber-index conditions leave a one-parameter set of values for the coefficients of $u_i u_j$ in the potential, the set exhibited in [GKT26, Ex. 4.9], and no generator of the Lie algebra $\mathfrak{g}^{4,4}$ of [GKT22, Def. 4.23] changes these coefficients. Consequently $\exp(\mathfrak{g}^{4,4})$ does not act transitively on $\Omega_{r,s}(I)$.*

Proof. For $J = \{i, j\}$ and $\mathbf{d} = \mathbf{0}$ one has $r(J) = s(J) = 0$, $\ell_1 = \ell_2 = 1$, $N = 1$ and $w = 0$, with balanced boundary degrees $(0, 4)$ and $(4, 0)$. The chamber-index condition on this string is the single condition $\mathcal{A}(J, \mathbf{0}, \nu) = 0$, whose linear part has two nonzero Gamma coefficients, so it leaves a one-parameter set of values for the two coefficients. Since $u_i u_j$ divides no other monomial of the labeled ring, no other condition involves these coefficients, so every value in this set occurs in a chamber index, hence in $\Omega_{r,s}(I)$ by [GKT22, Cor. 5.14]. A generator multiplies the deformation monomials of the terms on which it acts by its own coefficient, so it can change these coefficients only if its coefficient is u_i , u_j or $u_i u_j$. The singletons have $N = 0$ and carry no generator, and $u_i u_j$ has weight zero, so by (1) its only admissible bidegree is $(0, 0)$, which $\mathfrak{g}^{4,4}$ does not contain. Hence every element of $\exp(\mathfrak{g}^{4,4})$ fixes these coefficients, and two realizable systems with different values in this set lie in different orbits. $\square$

3. SLOPE-ORDERED BOUNDARY-RAY FACTORIZATION

Throughout, a *normal-form homotopy* is a homotopy of boundary conditions supplied by [GKT22, Lem. 3.40], recalled as Lemma 3.4 at the end of this section: for each subset A of the internal labels there is one time $T_A \in (0, 1)$, all times distinct, and every jump of a critical graph with internal marking set A occurs at T_A . The times are auxiliary choices (Remark 6.18). We restate the jump formula of [GKT22, proof of Thm. 5.9]. The potential W^{sr} of such a homotopy is constant in the homotopy time T except at the times T_A , and at T_A it changes by

$$W \mapsto \exp\left(\sum_{\mathbf{d}} \sum_{\Lambda} (-1)^{|A|+\beta_2(\Lambda)-1} \Delta(\Lambda) u_{A,\mathbf{d}} X_{\Lambda}\right)(W) \quad [\text{GKT22, proof of Thm. 5.9}],$$

the sums over the descendent vectors $\mathbf{d}$ on A and the critical graphs Λ with internal marking set A and descendent vector $\mathbf{d}$. Here $X_{\Lambda} = X_{k(\Lambda)}$, where $k(\Lambda) = \beta(\Lambda) - (1, 1)$ is the Lie bidegree of Λ (Proposition 2.2), so that the exponent $\beta_2(\Lambda) - 1$ in the sign is its second component, written $k_2(0)$ in [GKT22, Not. 5.7]; $u_{A,\mathbf{d}} = \prod_{j \in A} u_{j,d_j}$ is the monomial of the internal markings in the labeled ring of [GKT22, Def. 4.6], so that $u_{A,\mathbf{d}} X_{\Lambda}$ is the wall term of Λ ; and $\Delta(\Lambda)$ is the signed count of zeros of the homotopy of Λ on $[0, 1] \times \mathcal{PM}_{\Lambda}$ [GKT22, (5.33)], with $\mathcal{PM}_{\Lambda}$ GKT's partial compactification of the moduli space of Λ . These automorphisms are the *jumps* of the homotopy. (In [GKT22] the sum over Λ runs over the set $Q_0(A, \mathbf{d})$ of [GKT22, Not. 5.7], whose conditions are the congruences and (1) for $k(\Lambda)$.)

Proposition 3.1 (Ordered factorization). *Over the symmetric coefficient ring and over the labeled ring A_I of [GKT22, Def. 4.6], the following holds. In the primary algebra, modulo every power of the deformation ideal, every element of the enlarged wall-crossing group $\tilde{G}^{r,s}$ factors uniquely as a slope-ordered product of exponentials supported on the boundary rays $\mathbb{R}_{\geq 0}(k_1+1, k_2+1)$; the marginal torus terms belong to the $(1, 1)$ -ray. The compatible factorizations define its completed boundary-ray factorization. In the full descendent algebra the same factorization is taken on finite divisor-closed coefficient sets and passed to the coefficientwise limit.*

Proof. Let $\mathfrak{m}$ be the deformation ideal and first work in the primary quotient

$$F^p \tilde{\mathfrak{g}}_0^{r,s} := \text{span}\{u_{J,\mathbf{0}} X : |J| \geq p\}, \quad L_{<K} := \tilde{\mathfrak{g}}_0^{r,s} / F^K.$$

It is a finite-dimensional nilpotent Lie algebra and $[F^p, F^q] \subseteq F^{p+q}$ (here and below $F^{\bullet}$ also denotes the induced filtration on the quotient). For a rational boundary ray ρ let L_{ρ} be the span of the wall terms $u X_{k_1, k_2}$ for which $(k_1+1, k_2+1) \in \rho$, placing a marginal wall term $u(x\partial_x - y\partial_y)$ in $\rho = \mathbb{R}_{\geq 0}(1, 1)$; by Proposition 6.2 these are the rays whose monomials the wall terms fix. Then $L_{<K} = \bigoplus_{\rho} L_{\rho}$, with only finitely many summands.

Each L_ρ is abelian. Indeed,

$$[X_{k_1, k_2}, X_{k'_1, k'_2}] = \det((k'_1+1, k'_2+1), (k_1+1, k_2+1)) X_{k_1+k'_1, k_2+k'_2},$$

so the bracket vanishes when the boundary-ray directions (k_1+1, k_2+1) and (k'_1+1, k'_2+1) are parallel; on the $(1, 1)$ -ray the marginal vector field also commutes with every $X_{q, q}$. The deformation coefficient of a marginal wall term has positive $\mathfrak{m}$ -degree, so it belongs to the same nilpotent filtration.

Fix any slope order. The multiplication map

$$\prod_{\rho} \exp(L_\rho) \xrightarrow{\quad} \exp(L_{<K})$$

is bijective. Modulo F^2 this is the direct-sum decomposition above. Suppose the factors have been found through F^n . The logarithm of the residual in F^n/F^{n+1} decomposes uniquely by ρ ; insert these components into the corresponding factors. Commuting a degree- n correction past a previously chosen positive-degree term changes the product only in F^{n+1} . This proves existence by induction, as in the Kontsevich–Soibelman lemma [KS06], [GPS10, Thm. 1.4]. At the first filtration degree where two ordered factorizations differ, the same direct-sum decomposition forces all differences to vanish, proving uniqueness. The construction is compatible as K varies.

Over the labeled ring A_I the same argument applies with the filtration by the number $|A|$ of labels of the label monomials u_A in place of $|J|$: the products of label monomials are again label monomials or zero, so $[F^p, F^q] \subseteq F^{p+q}$, and the bracket formula above does not involve the coefficients.

For descendents, perform the same construction after quotienting to any finite divisor-closed set of deformation monomials. Its complement is upward closed, so this is a Lie-algebra quotient, and the compatible finite constructions give the asserted coefficientwise factorization. $\square$

Remark 3.2 (Path independence). For any homotopy of boundary conditions supplied by [GKT22, Lem. 3.40], or concatenation of such homotopies, with equal endpoints, the time-ordered product of its jumps is the identity; for such homotopies with arbitrary endpoints, the total jump depends only on the endpoints. Indeed, let $(\mathfrak{s}_T)_{T \in [0, 1]}$ be a homotopy of boundary conditions supplied by [GKT22, Lem. 3.40], with $\mathfrak{s}_1 = \mathfrak{s}_0$, jump times $T_1 < \dots < T_M$, the times of the marking subsets $A_1, \dots, A_M$, and jump elements $g_i \in \tilde{G}^{r, s}$ (Lemma 2.5) given by the jump formula above ([GKT22, proof of Thm. 5.9, eqns. (5.33)–(5.37)]). By that proof the potential $W^{\mathfrak{s}_T}$ is constant between jumps and transforms by g_i at T_i ; around the loop, $W^{\mathfrak{s}_0} = (g_M \circ \dots \circ g_1)(W^{\mathfrak{s}_0})$, and freeness (Lemma 2.5) gives $g_M \circ \dots \circ g_1 = \text{id}$. For two such paths with fixed endpoints, their total jumps g, g' satisfy $g'g^{-1}(W^{\mathfrak{s}_1}) = W^{\mathfrak{s}_1}$, hence $g' = g$. For a concatenation of such homotopies the same argument applies: its potential is constant between the finitely many jump times of its pieces and transforms by an element of $\tilde{G}^{r, s}$ at each of them.

Remark 3.3. The proof of [GKT22, Thm. 5.9] produces the total jump in a homotopy-time-ordered factorization, one factor per subset of internal markings, mixing boundary rays; the proof of [GKT22, Lem. 3.40] shows the jump times and their order are auxiliary choices. The boundary-ray factorization is a reorganization, involving no choices, by the boundary rays, whose directions are the Lie bidegrees shifted by $(1, 1)$ as forced by Proposition 2.2, and not by the rays of the Lie bidegrees k , which are additive under the bracket; this shift is what makes the boundary-ray factorization $\mathfrak{D}^{r, s}$ of Theorem 1.1 geometrically meaningful.

For later reference we recall the parts of the extension lemma of [GKT22] that are used below. For a graph Γ , let $\mathcal{PM}_\Gamma$ be as above, $E_\Gamma(\mathbf{d})$ the descendent Witten bundle and $\#\sigma_{12}(\Gamma)$ the

number of σ_{12} -points (written $k_{12}(\Gamma)$ in [GKT22]). A critical graph is connected and relevant [GKT22, Def. 1.57], has $\#\sigma_{12} = 0$ and $\text{rank } E_\Gamma(\mathbf{d}) = \dim \mathcal{PM}_\Gamma + 1$, and a connected balanced graph has $\#\sigma_{12} = 1$ and rank equal to dimension [GKT22, Defs. 2.32–2.33].

Lemma 3.4 ([GKT22, Lem. 3.40]). *Let $\mathfrak{s}_0$ and $\mathfrak{s}_1$ be families of canonical multisections bounded by I that are transverse to the zero section. There are a family $H^* = (H^\Gamma)_\Gamma$ of simple canonical homotopies bounded by I from $\mathfrak{s}_0$ to $\mathfrak{s}_1$ and distinct times $T_A \in (0, 1)$, one for each $A \subseteq I$, such that the following hold for every smooth graded W -spin graph Γ with only open vertices and $I(\Gamma) \subseteq I$ and every descendent vector $\mathbf{d}$.*

- (1) H^Γ vanishes nowhere if Γ is irrelevant in the sense of [GKT22, Def. 1.57], if Γ is connected with $\text{rank } E_\Gamma(\mathbf{d}) \geq \dim \mathcal{PM}_\Gamma + 2$, or if two connected components of Γ have distinct internal marking sets and each has rank one more than its dimension and at most two σ_{12} -points.
- (2) If Γ is connected and either (a) $\text{rank } E_\Gamma(\mathbf{d}) = \dim \mathcal{PM}_\Gamma + 1$ and $\#\sigma_{12}(\Gamma) \leq 2$, or (b) $\text{rank } E_\Gamma(\mathbf{d}) = \dim \mathcal{PM}_\Gamma$ and $\#\sigma_{12}(\Gamma) \leq 1$, then the restriction of H^Γ to every stratum of $[0, 1] \times \mathcal{PM}_\Gamma$ is transverse to the zero section.
- (3) Let Γ be connected. (a) In case (2)(a), H^Γ has no zeros on $([0, 1] \setminus \{T_{I(\Gamma)}\}) \times \mathcal{PM}_\Gamma$. (b) In case (2)(b), for every $A \subseteq I$ the restriction of H^Γ to $\{T_A\} \times (\mathcal{PM}_\Gamma \setminus \partial \mathcal{PM}_\Gamma)$ is transverse to the zero section.

By (3)(a), every zero of the homotopy of a critical graph with internal marking set A occurs at the time T_A , which is the normal form used in this section; by (3)(b), the slices of the homotopies of balanced graphs at the times T_A are transverse on $\mathcal{PM}_\Gamma \setminus \partial \mathcal{PM}_\Gamma$.

4. PERIOD RELATIONS ON EACH BALANCED STRING

Two structural facts from [GKT22] organize this section. The open topological recursion factorizes through a cohomological-field-theory node splitting: in [GKT22, Lem. 6.4] the distinguished insertion splits off a closed extended invariant and continues as an open piece carrying the socle-dual twist $(a, b) \mapsto (r-2-a, s-2-b)$ of the Jacobi ring $\mathbb{C}[x, y]/(x^{r-1}, y^{s-1})$: the node is the residue pairing. And the Gamma coefficients of the polynomials $\mathcal{A}$ are periods of $x^r + y^s$: $\int_0^\infty e^{-x^r} x^{\beta_1} dx = \frac{1}{r} \Gamma(\frac{1+\beta_1}{r})$, so the Gamma coefficient attached to a boundary degree β is, up to a normalization depending only on $(r(J), s(J))$, the integral of $x^{\beta_1} y^{\beta_2} e^{-x^r - y^s}$ against Saito's primitive form $dx \wedge dy$ over the positive quadrant. The good-basis period moments themselves are explicit:

Lemma 4.1 (Periods). *For the good-basis cycles Ξ_a of x^r ($\int_{\Xi_a} x^{a'} e^{x^r/h} = \delta_{aa'}$, $0 \leq a, a' \leq r-2$), the moments $P_a(p) = \int_{\Xi_a} x^p e^{x^r/h}$ satisfy $P_a(p) = \delta_{ap}$ for $p \leq r-2$, $P_a(r-1) = 0$, and $P_a(p) = -\frac{h(p-r+1)}{r} P_a(p-r)$.*

Proof. Total derivatives integrate to zero on Ξ_a : $0 = \int_{\Xi_a} \frac{d}{dx} (x^q e^{x^r/h}) = q P_a(q-1) + \frac{r}{h} P_a(q+r-1)$. $\square$

Proof of Theorem 1.4. The case $N = 0$. If $N = 0$, there is no wall vector and the nonzero Gamma coefficient of the $h = 1$ part of $\mathcal{A}$ determines the sole coefficient modulo lower monomials, so the claim is immediate.

Wall-bearing strings. Fix now a wall-bearing string $(J, \mathbf{d})$, of positive or marginal weight, with coefficients ν_p , $0 \leq p \leq N$, as in Definition 2.1. The wall term of $\Lambda_{J,p}$ acts on $W_0 = x^r + y^s$ by

$$(6) \quad u_{J,\mathbf{d}} X_{k_1,k_2}(W_0) = u_{J,\mathbf{d}}(r(k_2+1) x^{k_1+r} y^{k_2} - s(k_1+1) x^{k_1} y^{k_2+s}),$$

which lands on the monomials of two adjacent balanced coefficients of the same string: in the basis $\varepsilon_0, \dots, \varepsilon_N$ of coefficients, it is the wall vector $v_p = \lambda_p \varepsilon_p - \mu_p \varepsilon_{p-1}$ with $\lambda_p = r(k_2+1) = r \beta_2(\Lambda_{J,p}) > 0$ and $\mu_p = s(k_1+1) = s \beta_1(\Lambda_{J,p}) > 0$, where $\beta(\Lambda_{J,p}) = k + (1, 1)$ is the boundary degree of $\Lambda_{J,p}$. (These are exactly the geometric jump coefficients of [GKT22, (5.52)–(5.54)].) The v_p are triangular, hence independent, and each is annihilated by the period functional: by Lemma 4.1, $P_a(k_1+r)P_b(k_2) = -\frac{h(k_1+1)}{r}P_a(k_1)P_b(k_2)$ cancels against the second term of (6).

Narrow strings. On a narrow string the period functional is nonzero on each ε_p , so its kernel has dimension N and coincides with $\text{span}(v_p)$. By [GKT22, Thm. 4.9] the coefficient of $u_{J,\mathbf{d}}$ in $\int_{\Xi_{(r(J),s(J))}} e^{W^\nu/h} dx \wedge dy$ equals $(-1)^{|J|}(-h)^{N(J,\mathbf{d})-1} \mathcal{A}(J, \mathbf{d}, \nu)$; the exponent is $-d(J, \mathbf{d}) - 2$ in the notation of [GKT22, Def. 4.8], where $d(J, \mathbf{d}) = -N(J, \mathbf{d}) - 1$ by the identity for $w(J, \mathbf{d}) + rs$ in §2.2. The $h = 1$ part of $\mathcal{A}$ (h the number of blocks in (5)) is the sum $\sum_p \omega_p \nu_p$ over the string, with ω_p the Gamma coefficients of the $h = 1$ part of (5), and its $h \geq 2$ parts involve only strictly smaller monomials. The chamber-index conditions [GKT22, Def. 3.47] prescribe the value of $\mathcal{A}$.

Broad strings. On broad strings ($r(J) = r - 1$ or $s(J) = s - 1$) the good-basis periods vanish identically on the string (Lemma 4.1), but the Gamma coefficients ω_p remain nonzero there (their arguments $(1 + \beta_1)/r$ and $(1 + \beta_2)/s$ are positive, and Γ does not vanish there), and $\text{span}(v_p)$ lies in the kernel of the functional $\sum_p \omega_p \nu_p$ modulo strictly smaller monomials, by Lemma 2.4; so the same one-linear-condition structure holds, imposed directly by the conditions, which by [GKT22, Cor. 5.14] characterize $\Omega_{r,s}(I)$ completely. Ordering monomials by degree and descendents determines every invariant, up to the span of the v_p (the torsor directions), proving the theorem. The prescribed values of $\mathcal{A}$ on strings with balanced coefficients are 0 or, for a single marking with descendent exponent d_1 , $(-1)^{d_1}$, because $d(J, \mathbf{d}) = -N(J, \mathbf{d}) - 1 < 0$ there [GKT22, Def. 4.8]; this proves the last statement. The open topological recursion [GKT22, Thm. 0.5] then relates these invariants to the closed extended invariants; it is not among the conditions. $\square$

Remark 4.2 (The first values over the twist $(3, 3)$). For $x^5 + y^5$ the conditions produce, order by order over the $t_{3,3}$ -monomials, the one-insertion normalization $\langle \tau_0^{(3,3)} \sigma_1^3 \sigma_2^3 \sigma_{12} \rangle = 1$; the first-wall relation (2) at $t_{3,3}^2$; the absolute value $\frac{1}{5}$ of (3) at $t_{3,3}^3$ (a string with $N = 0$, hence no wall terms, and $r(J) = 4 = r - 1$, hence broad and invisible to the good-basis periods, so the condition that determines it is not a period condition); and at $t_{3,3}^4$ a relation quadratic in the first-wall invariants, giving $\nu_{(2,7)}^{\min} = -\frac{12}{25}$ in the minimal system of Theorem 1.1. The descendent singleton at $d = 1$ gives $\langle \tau_1^{(3,3)} \sigma_1^8 \sigma_2^3 \sigma_{12} \rangle + \langle \tau_1^{(3,3)} \sigma_1^3 \sigma_2^8 \sigma_{12} \rangle = -\frac{5}{4}$: on its string ($N = 1$, balanced boundary degrees $(3, 8)$ and $(8, 3)$) the condition $\mathcal{A} = (-1)^1$ has only its $h = 1$ part, and both Gamma coefficients in (5) equal $\Gamma(\frac{9}{5})/\Gamma(\frac{4}{5}) = \frac{4}{5}$. The values at $t_{3,3}^3$ and $t_{3,3}^4$ solve the condition $\mathcal{A} = 0$ on the string of that monomial, with the lower values substituted into the terms with $h \geq 2$: at $t_{3,3}^3$ the term with $h = 1$ is the single unknown invariant, with Gamma coefficient 1, and at $t_{3,3}^4$ ($N = 1$, balanced boundary degrees $(2, 7)$ and $(7, 2)$) the minimal system sets the coefficient of $(7, 2)$ to zero and the condition determines that of $(2, 7)$.

Remark 4.3 (The two-insertion values, and [GKT26, Ex. 5.5]). Imposing [GKT22, Thm. 0.3] directly on the oscillatory expansion of the potential (the mechanism by which [GKT22, Thm. 0.2] is proved, so that no combinatorial convention enters) forces, at every two-insertion string $\{(a_1, b_1), (a_2, b_2)\}$ with $a_1 + a_2 = r$, $b_1 + b_2 = s$, $\mathbf{d} = 0$, the relation (4): the string supports the boundary degrees $(r, 0)$ and $(0, s)$ only, $N(J, \mathbf{0}) = 1$, so $d(J, \mathbf{0}) = -N(J, \mathbf{0}) - 1 < 0$ in the notation of [GKT22, Def. 4.8] and the mirror theorem admits no term there, and the linear and quadratic insertions contribute $\nu_{(r,0)}/r + \nu_{(0,s)}/s$ and $1/(rs)$ respectively (Lemma 4.1). At

$r = s = 4$ this gives $-\frac{1}{4}$ for the sum considered in [GKT26, Ex. 5.5], where $-\frac{1}{8}$ is stated; the discrepancy traces to the $h = 2$ partition count in the evaluation of $\mathcal{A}$ there, which [GKT22, Notation 3.42 and Thm. 4.9] fixes as the count of ordered partitions of the labeled marking set with $1/h!$. Notation 5.3 of [GKT26] writes the partition sum over the multiset of insertions; read as ordered decompositions of the multiset, the $h = 2$ term is counted once and gives $-\frac{1}{8}$, while read over ordered partitions of a labeled realization it is counted twice and gives $-\frac{1}{4}$. The same count is determined independently by [GKT22, Cor. 5.11(3)]: at $r = s = 5$, $\mathcal{A}(\{(1, 1), (1, 1)\}, 0)$ equals the closed three-point value $\langle (\tau_0^{(1,1)})^3 \rangle = 1$, which the labeled count reproduces and the halved count does not. (The corresponding statement in [GKT22, Cor. 0.6] carries an extra sign $(-1)^{d(J, \mathbf{d})-1}$, with $d(J, \mathbf{d}) = -N(J, \mathbf{d}) - 1$ as in [GKT22, Def. 4.8]. For $t_{1,1}^2$ one has $N = -1$, so the $t_{1,1}^2$ -coefficient of $\int_{\Xi_{2,2}} e^{W/\hbar}$ is $P_2(2)^2/(2\hbar^2) = 1/(2\hbar^2)$; by [GKT22, Thm. 0.3] this equals $\hbar^{-2} \langle (\tau_0^{(1,1)})^3 \rangle^{\text{ext}}/2$, which confirms the sign of [GKT22, Cor. 5.11(3)].) On the marginal two-insertion strings the two coefficients in (4) are moved (antisymmetrically) only by the torus direction of Remark 2.3; at $r = s$ the point of this one-parameter set of values with equal coefficients has $\nu_{(r,0)} = \nu_{(0,s)} = -\frac{1}{2r}$. The discrepancy concerns only the worked example; no theorem of [GKT22] or [GKT26] is affected.

5. EXTREMAL SYSTEMS AND THE BOUNDARY-RAY FACTORIZATION $\mathfrak{D}^{r,s}$

Proof of Theorem 1.1. Part (iii) is Proposition 6.2. For (i) and (ii), work in a primary quotient modulo F^K , or more generally on a finite divisor-closed coefficient set as in Proposition 3.1. For existence and uniqueness of $\nu^{\min}$, order its monomials by total degree. At each step, on each wall-bearing string the span of the wall vectors $v_p = \lambda_p \varepsilon_p - \mu_p \varepsilon_{p-1}$ ($\lambda_p, \mu_p > 0$) is complementary to $\mathbb{Q}\varepsilon_0$: a combination $\sum c_p v_p \in \mathbb{Q}\varepsilon_0$ forces, reading the $\varepsilon_N, \varepsilon_{N-1}, \dots$ components in turn, $c_N = \dots = c_1 = 0$. Hence there is a unique group correction at this order making all ν_p , $p \geq 1$, vanish, and it does not disturb lower orders; on marginal strings the torus direction plays the role of the single wall term. This triangular elimination is the argument of [GKT22, proof of Thm. 4.27, Step 3], with the torus direction adjoined. The resulting system is realizable because the group preserves $\Omega_{r,s}(I)$ (Lemma 2.5). Uniqueness follows from freeness (Lemma 2.5) plus the same triangularity: a nontrivial leading correction moves some $\nu_{p \geq 1}$. The construction for $\nu^{\max}$ is symmetric (complement of $\mathbb{Q}\varepsilon_N$). The factorization, in increasing order of slope, of the unique $g^{\min \rightarrow \max}$ (Proposition 3.1) is then determined without choices, and its ray factors are computed order by order from the extremal systems. Uniqueness makes these constructions compatible under restriction, giving the primary completion and the coefficientwise descendent extension. $\square$

Example 5.1 (The first walls of $\mathfrak{D}^{5,5}$). At order two in the deformation variables the three strings of $x^5 + y^5$ carrying positive-weight wall terms have balanced boundary degrees $(0, 6)$, $(5, 1)$; $(6, 0)$, $(1, 5)$; and $(1, 6)$, $(6, 1)$ respectively, with the invariant relations and extremal values below (§7). The two marginal strings $u = t_{2,2}t_{3,3}$ and $u = t_{2,3}t_{3,2}$, with balanced boundary degrees $(0, 5)$ and $(5, 0)$, carry the torus terms $\frac{1}{25}uX_{0,0}$ on the $(1, 1)$ -ray, by the two-insertion relation (4):

monomial	relation	extremal values
$t_{2,3}t_{3,3}$	$\frac{2}{5}\nu_{(0,6)} + \frac{1}{5}\nu_{(5,1)} = -\frac{2}{25}$	$\nu_{(0,6)}^{\min} = -\frac{1}{5}, \nu_{(5,1)}^{\max} = -\frac{2}{5}$
$t_{3,2}t_{3,3}$	the image under $x \leftrightarrow y$	by symmetry
$t_{3,3}^2$	$\frac{2}{5}(\nu_{(1,6)} + \nu_{(6,1)}) = -\frac{4}{25}$	$\nu_{(1,6)}^{\min} = \nu_{(6,1)}^{\max} = -\frac{2}{5}$

Each relation is the condition $\mathcal{A} = 0$ of (5). For $t_{2,3}t_{3,3}$ the Gamma coefficients of the $h = 1$ part are $\Gamma(\frac{7}{5})/\Gamma(\frac{2}{5}) = \frac{2}{5}$ and $\Gamma(\frac{6}{5})/\Gamma(\frac{1}{5}) = \frac{1}{5}$, and the $h = 2$ term, the product of the two one-marking normalizations, has Gamma coefficient $\Gamma(\frac{6}{5})\Gamma(\frac{7}{5})/(\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})) = \frac{2}{25}$; for $t_{3,3}^2$ the corresponding Gamma coefficients are $\frac{2}{5}$, $\frac{2}{5}$ and $(\frac{2}{5})^2$. In $\nu^{\min}$ only the coefficient of least x -degree survives, and in $\nu^{\max}$ only that of greatest x -degree. Solving $W^{\nu^{\max}} - W^{\nu^{\min}} = c \cdot u X_{k_1, k_2}(W_0)$ on each string is an overdetermined system (two monomial components, one coefficient); it is consistent because both systems satisfy the same linear relation, whose solutions differ by multiples of the wall vector, and it yields the ray factors of Theorem 1.1. The wall term $t_{3,3}^2 X_{1,1}$ is the first symmetric positive-weight primary wall term: only the sum (2) is invariant, the difference changes across the $(1, 1)$ -ray.

6. TRANSPORT OF THE POTENTIAL AND REALIZATION BY HOMOTOPIES

The results above define the boundary-ray factorization $\mathfrak{D}^{r,s}$ and show that its action, together with the period relations, determines the invariants. What they do not provide is an *affine base*: an intrinsic domain in which the walls are rays and the invariants are computed by broken lines, as for the Fano Landau–Ginzburg models of [Gro10, CPS24]. This section builds as much of that picture as is a theorem and isolates what is not.

The product $g^{\min \rightarrow \max}$ is the transport along a path from one edge of C_∂ to the other, whose two ends carry $\nu^{\min}$ and $\nu^{\max}$; it is not the holonomy of a loop (Remark 6.25).

The section is organized as follows. Section 6.1 places the walls; §6.2 proves the transport representation and its triangular inversion, and shows that the theory is not in general generated by wall-crossing from x^r , y^s and the monomials with one internal marking; §6.3 realizes every finite coefficient quotient by concatenated homotopies and proves the obstruction that makes the concatenation necessary; and §6.4 describes the punctured quadrant (Remark 6.25) and what it does not supply (Remark 6.26).

6.1. The quadrant and its rays.

Definition 6.1. The *boundary-degree lattice* is $M_\partial = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, where e_i records one boundary insertion of type σ_i . The *boundary-degree quadrant* is its effective cone $C_\partial = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$; its two *edges* are the rays of the two boundary types. A factor of $\mathfrak{D}^{r,s}$ is assigned to the boundary ray $\mathbb{R}_{\geq 0}(k_1+1, k_2+1)$ of its critical graph, whose direction is the Lie bidegree shifted by $(1, 1)$ (Proposition 2.2), and a chamber of a finite primary truncation is a connected component of the complement of these rays. For the completed descendent factorization, whose walls are dense (Proposition 6.14), a *slope cut* is a direction together with a choice of side when the direction itself carries a factor, that is, a point of $\mathbb{P}(C_\partial)$ doubled at each wall slope; for transport it matters only through the Dedekind cut it induces on the ordered set of wall slopes, which in a finite truncation is a chamber, or a wall ray approached from one side. The σ_1 -*edge cut* and the σ_2 -*edge cut* are the cuts at the edges of C_∂ along the σ_1 -axis and the σ_2 -axis, and the transport $\Psi(q)$ to a cut q is the slope-ordered product of the ray factors between the σ_1 -edge cut and q , defined on each finite divisor-closed coefficient set and then coefficientwise: on the coefficient of a fixed deformation monomial u , every factor is projected to the finite divisor-closed quotient consisting of the divisors of u , where only finitely many projected ray factors are nontrivial. Only the products $\Psi(q)\Psi(q')^{-1}$ enter below, so the choice of the σ_1 -edge cut as base point is a convention. For $\alpha, \alpha' \in \mathbb{Z}^2$ we write $\det(\alpha, \alpha') = \alpha_1\alpha'_2 - \alpha_2\alpha'_1$.

Thus the quadrant, its lattice, and its edges come from the open FJRW boundary data rather than from an auxiliary drawing. Only its projectivization $\mathbb{P}(C_\partial) \cong [0, \infty]$, the ordered slope space, is used by transport. The placement is not a convention but forced (Proposition 6.2),

and the same computation shows that the $(1,1)$ -shift of Proposition 2.2, the removal of the σ_{12} -insertion, is what makes bending along a transported monomial monotone in slope (Proposition 6.5).

Proposition 6.2 (Forced placement; = Theorem 1.1(iii)). *The vector field X_{k_1, k_2} annihilates a monomial $x^{\alpha_1} y^{\alpha_2}$, $\alpha \in \mathbb{Z}_{\geq 0}^2$, exactly when α lies on the ray of its boundary-monomial direction (k_1+1, k_2+1) . Thus the boundary ray of a critical graph is the unique ray whose monomials its wall term fixes, which is why the walls are placed there, and a monomial is fixed by the walls on the ray of its own exponent.*

Proof. $X_{k_1, k_2}(x^{\alpha_1} y^{\alpha_2}) = ((k_2+1)\alpha_1 - (k_1+1)\alpha_2)x^{\alpha_1+k_1} y^{\alpha_2+k_2}$, and the coefficient is the determinant of $\alpha = (\alpha_1, \alpha_2)$ against (k_1+1, k_2+1) . It vanishes precisely on the line through (k_1+1, k_2+1) , which meets $\mathbb{Z}_{\geq 0}^2$ only on the ray since $k_1 + 1, k_2 + 1 > 0$. $\square$

If the two edges of the quadrant were identified, their ordered crossing would define a loop with holonomy $g^{\min \rightarrow \max} \neq \text{id}$ (Example 5.1); in C_∂ the edges remain distinct (Remark 6.25).

Proposition 6.3 (Interlacing and extreme support). *On every string the balanced monomials and the critical graphs interlace in slope (writing bal_p for the balanced monomial of $\Gamma_{J,p}$), $\text{bal}_N < \Lambda_N < \dots < \Lambda_1 < \text{bal}_0$, and the extremal systems of Theorem 1.1 are the realizable systems at the two extreme chambers of every finite primary truncation, with the support of that theorem: near one edge only the monomial at the far end of each string survives.*

Proof. Immediate from the formulas of Definition 2.1: the critical graph Λ_p has boundary degree between those of bal_p and bal_{p-1} in slope; the support statement is the construction of the extremal systems read through the interlacing. $\square$

6.2. The transport representation.

Definition 6.4. The *anchor cut* q_m of a potential monomial m is the one-sided cut through its own direction, on the side toward the $(1,1)$ -direction. If m is off every wall in a finite truncation, this is represented by the chamber containing its direction, and if m lies on a wall, by the chamber adjacent to that wall on the side of the $(1,1)$ -direction; for m on the $(1,1)$ -direction either side gives the same coefficient, and the exponent $(0,0)$ is anchored at the σ_1 -edge cut. Its *anchored value* is its coefficient at q_m . For a candidate assignment of initial coefficients, $\widetilde{W}(q) := \sum_m \Psi(q) \Psi(q_m)^{-1} (\text{init}_m m)$ denotes the resulting transport potential; the *anchored-value assignment* sets each init_m equal to the anchored value of m .

Every bend along a transported monomial adds $\beta - (1,1)$ to the exponent, β the boundary degree of the wall term used, which lies on the crossed ray. As in the introduction, a *bend chain* is a term of the ordered bend expansion of Lemma 6.9 below: a sequence of wall terms applied to a transported monomial along a monotone path in the slope line. Throughout, initial monomial and target are taken off the crossed rays: an initial monomial on a bend ray has vanishing bend coefficient, and an on-ray target uses the one-sided anchor cut toward the $(1,1)$ -direction, the convention of Definition 6.4. Call a bending contribution to a coefficient *foreign* if it originates from an initial coefficient other than the coefficient's own.

Proposition 6.5 (Monotonicity of bends). *Along a bend chain with n_b bends, using wall terms of boundary degrees β_f , from an initial monomial of exponent α' to a monomial of exponent α ,*

$$n_b \det(\alpha, (1,1)) = \det(\alpha, \alpha') + \sum_f \det(\alpha, \beta_f).$$

Consequently no bending contribution originating from a different initial coefficient reaches a monomial's anchor cut from the far side of the monomial's own direction, the side away from the $(1, 1)$ -direction: at its anchor cut, a monomial below the $(1, 1)$ -direction receives foreign bending only along paths descending in slope, one above it only along paths ascending, and a monomial on the $(1, 1)$ -direction receives none.

Proof. Summing the per-bend exponent motion gives $\alpha + n_b(1, 1) = \alpha' + \sum_f \beta_f$; the displayed identity is obtained by applying $\det(\alpha, \cdot)$ to it. Transport is monotone, so every crossed ray lies between the directions of α' and α , a crossed ray on the direction of α contributing $\det(\alpha, \beta_f) = 0$; and $\det(\alpha, \alpha') \neq 0$, since otherwise $q_{m'} = q_m$ and the path is empty. For a path from below, the direction of α' is strictly shallower than α and no crossed ray is steeper, so the right side is negative, forcing $\det(\alpha, (1, 1)) < 0$, i.e. α strictly above the $(1, 1)$ -direction; from above it is positive, forcing α strictly below. A monomial on the $(1, 1)$ -direction, where $\det(\alpha, (1, 1))$ vanishes, excludes both. $\square$

Lemma 6.6 (Reduction). *For the anchored-value assignment, $\widetilde{W} = W$ at every cut if and only if, for every coefficient, the foreign bending contributions cancel at its anchor cut.*

Proof. Ray factors act additively, so $W - \widetilde{W}$ is itself transported by them; its coefficients on any divisibility-minimal nonzero monomial are cut-independent (all bending contributions come from proper divisors, where $W - \widetilde{W}$ vanishes), and evaluating each coefficient at its anchor cut gives zero exactly under the stated cancellation. $\square$

We call x^r and y^s , the two monomials of W_0 , the *axis monomials*. For a twist (a, b) , the *tower* of (a, b) is the set of singleton strings of $t_{a,b,d}$, $d \geq 1$.

Theorem 6.7 (Linear order and towers).

- (i) *On every string of every $(r, s, J, \mathbf{d})$, no bend of an axis monomial by a wall term over the monomial of the string reaches a balanced coefficient of the string at the anchor cut of that coefficient. Thus, at linear order in these wall terms, the foreign contributions of Lemma 6.6 vanish.*
- (ii) *On a singleton string (one internal marking, any descendent exponent) there are no other contributions, and the transport representation holds with the anchored-value assignment on every tower. The wall coefficients form the chain $c_{i+1} = c_i \lambda_i / \mu_{i+1}$, with c_1 fixed by the condition $\mathcal{A} = (-1)^d$ of the string, where λ_i, μ_i are the coefficients of the wall vectors $v_i = \lambda_i \varepsilon_i - \mu_i \varepsilon_{i-1}$ of §4.*

Proof. (i) By (6), $X_{\Lambda_i}(x^r)$ is a nonzero rational multiple of the monomial bal_i and $X_{\Lambda_i}(y^s)$ of bal_{i-1} . By the interlacing of Proposition 6.3, transport into the anchor cut of bal_p from below crosses exactly the rays of the Λ_i with $i \geq p+1$, whose bends of x^r land on $\text{bal}_i \neq \text{bal}_p$, and transport from above crosses exactly those with $i \leq p$, whose bends of y^s land on $\text{bal}_{i-1} \neq \text{bal}_p$. (ii) A singleton monomial has no proper divisor other than 1, so the only contributions are the linear ones of (i). $\square$

The interlacing of Proposition 6.3, as used in Theorem 6.7, is the analogue for this transport of the monotonicity of broken lines in [CPS24]. Bends of initial coefficients of divisors by the wall terms of other strings lie outside the theorem; Example 6.13 shows that they can arrive.

Two computed examples below (Examples 6.12 and 6.13) show what the initial coefficients are.

Theorem 6.8 (Transport representation; = Theorem 1.3). *With the anchor assignment of Definition 6.4, the difference between the potential's coefficient vector on a monomial and the*

transported contributions of the initial coefficients of its proper divisors is independent of the cut. Defining the initial system by these constants, every cut potential is the transport

$$W(q) = \sum_m \Psi(q) \Psi(q_m)^{-1} (\text{init}_m m)$$

of the initial coefficients, and the initial system is the unique one with this property.

Proof of Theorem 6.8 (= Theorem 1.3). Induct on the divisibility order of deformation monomials; the induction starts at $W_0 = x^r + y^s$, whose two coefficients are initial coefficients with value 1 in every cut. Fix a monomial $u = u_{J,\mathbf{d}}$ and assume the initial coefficients are defined, with the transport identity holding, on every proper divisor of u . Write $W|_{u'}$ for the coefficients of W on the string of a deformation monomial u' , so that $W|_1 = W_0$, and let $W_{<u}(q)$ denote the u -component of the sum of the transported lower initial coefficients.

Step 1: the change across a wall. Across any wall $\theta = \exp(\eta)$ the coefficients on the string of u change by

$$\theta(W)|_u - W|_u = \sum_{j \geq 1} \frac{1}{j!} \sum_{\substack{u', \eta_1, \dots, \eta_j \in \text{supp}(\eta) \\ u' u_{\eta_1} \dots u_{\eta_j} = u}} [\eta_1 \dots \eta_j (W|_{u'})]_u,$$

the sum over proper divisors u' of u , $u' = 1$ included, and over terms η_i of η with deformation coefficients u_{η_i} . This is a universal expression in η and the coefficients of W on proper divisors of u : any other source overshoots u , and the wall terms over u act on the string of u only through W_0 (their monomial is u itself), so the change is exactly linear there.

Step 2: cancellation, and chamber-independence of the residual. The change of $W_{<u}$ across the same wall is the same universal expression evaluated on the sum of transported lower initial coefficients, whose divisor coefficients equal W 's by the induction hypothesis and whose W_0 -coefficients are the same constants 1. Hence the two changes cancel in the difference and $V := W(q)|_u - W_{<u}(q)$ is independent of q .

Step 3: the initial coefficients on the string of u , and uniqueness. A u -term transports with unchanged u -component (every wall action on it overshoots u), so taking as initial coefficients init_m of the monomials m on the string of u the components of V , placed at the assigned anchor cuts, gives $\widetilde{W}|_u = V + W_{<u} = W|_u$ at every cut (all sums are coefficientwise finite: only finitely many initial coefficients of divisors and bends contribute to a fixed monomial), completing the induction. Uniqueness is immediate: V is determined by W and the lower initial coefficients. Anchor assignments through directions other than the monomial's own affect only the higher-order terms of the transports, hence the initial coefficients at higher orders, never the existence or uniqueness of the decomposition. The side of the anchor cut does not affect them, since the factor on the ray of m fixes m (Proposition 6.2); the $(1, 1)$ -side assignment is the one compatible with Proposition 6.5. $\square$

Consequences. (a) On a coefficient receiving no foreign bending at its anchor cut, the initial coefficient equals its anchored value. (b) An extreme coefficient of a wall-bearing string that receives no foreign bending at the cut where it vanishes in the corresponding extremal system, namely the σ_1 -edge cut for the coefficient of greatest x -degree and the σ_2 -edge cut for that of least x -degree, has initial coefficient zero.

Proof of (a) and (b). Evaluate the transport formula of Theorem 1.3 at a cut q . The term of the monomial m itself contributes init_m to the coefficient of m , because every wall term has a nonconstant deformation coefficient and therefore moves a term only into strictly larger deformation monomials; every other contribution is foreign. At $q = q_m$ this gives (a), as in Lemma 6.6. At the σ_1 -edge cut the cut potential is $W^{\nu^{\min}}$, and at the σ_2 -edge cut it is

$g^{\min \rightarrow \max}(W^{\nu^{\min}}) = W^{\nu^{\max}}$; there the coefficient of greatest, respectively least, x -degree of every wall-bearing string vanishes (Theorem 1.1; Proposition 6.3 identifies the extremal systems with the two extreme chambers), which gives (b). Proposition 6.5 restricts the direction from which foreign bending can arrive: none reaches the anchor cut of a monomial from beyond its own direction, and none reaches the anchor cut of a monomial on the $(1,1)$ -direction. $\square$

Lemma 6.9 (Ordered bend expansion). *Work in a finite divisor-closed coefficient quotient and write $z^\alpha = x^{\alpha_1}y^{\alpha_2}$. Suppose a transport path crosses, in chronological order, walls with automorphisms*

$$\Phi_j = \exp(\eta_j), \quad \eta_j = \sum_{f \in L_j} c_f u_f X_{k_f}, \quad 1 \leq j \leq M,$$

where for a negative crossing the c_f are the negatives of the coefficients of the ray factor. A bend chain γ is a choice of integers $n_j \geq 0$ and of ordered $f_{j,1}, \dots, f_{j,n_j} \in L_j$; list its bends $f_1, \dots, f_{|\gamma|}$ by increasing j , and for fixed j in the chosen order, and put $\alpha^h = \alpha^{h-1} + k_{f_h}$, starting from $\alpha^0 = \alpha$. Then, for a deformation monomial u' ,

$$(7) \quad (\Phi_M \circ \dots \circ \Phi_1)(u' z^\alpha) = \sum_{\gamma} \frac{\prod_h c_{f_h} \det(\alpha^{h-1}, k_{f_h} + (1,1))}{\prod_j n_j!} u' \prod_h u_{f_h} z^{\alpha^{|\gamma|}},$$

the finite sum over the bend chains γ with nonzero coefficient in the quotient.

Proof. Expand each exponential in powers of η_j and distribute each power over its summands. Successive applications use

$$X_k(z^\alpha) = ((k_2 + 1)\alpha_1 - (k_1 + 1)\alpha_2)z^{\alpha+k} = \det(\alpha, k + (1,1))z^{\alpha+k}.$$

The exponential supplies $1/n_j!$, the path supplies the displayed factor order, and nilpotence makes the sum finite. Wall terms on one boundary ray commute, so repeated bends may equivalently be grouped as multisets with the same factorials. The expansion is by chains: transport is linear in its initial monomial. Rooted Lie trees arise only after an additional BCH expansion of the ray factors, which depends on choices. $\square$

Write each monomial of the potential as $m = u z^\alpha$, where u is its deformation monomial and α a boundary degree on the string of u , and let q_m be its anchor cut. For a second monomial $m' = u' z^{\alpha'}$, define the strictly divisibility-triangular transport kernel

$$(8) \quad \mathcal{K}_{mm'} = [u z^\alpha] \Psi(q_m) \Psi(q_{m'})^{-1} (u' z^{\alpha'}) \quad \text{when } u' \mid u, \quad u' \neq u, \quad \mathcal{K}_{mm'} = 0 \quad \text{otherwise.}$$

Lemma 6.9 is an explicit formula for $\mathcal{K}_{mm'}$: retain the bend chains whose deformation coefficients multiply u' to u and whose exponent increments carry α' to α .

Proposition 6.10 (Triangular inversion). *On every finite divisor-closed coefficient set, let anc_m be the anchored value of the monomial m and init_m its initial coefficient. In vector form,*

$$(9) \quad \text{anc} = (1 + \mathcal{K}) \text{init}, \quad \text{init} = (1 + \mathcal{K})^{-1} \text{anc} = \sum_{n \geq 0} (-\mathcal{K})^n \text{anc}.$$

In particular,

$$(10) \quad \text{init}_m = \text{anc}_m + \sum_{n \geq 1} (-1)^n \sum_{m_0 < \dots < m_n = m} \mathcal{K}_{m_n m_{n-1}} \cdots \mathcal{K}_{m_1 m_0} \text{anc}_{m_0},$$

where $<$ includes strict divisibility of the deformation monomials and the intermediate boundary exponents are summed. Thus every initial coefficient is a finite alternating sum of anchored values multiplied by sums of ordered bend chains.

Proof. Evaluate Theorem 1.3 at the anchor cut of m . The initial coefficient of m contributes init_m , since a positive-degree wall term acting on it overshoots its deformation monomial. The initial coefficient of a monomial $m' < m$ contributes $\mathcal{K}_{mm'}\text{init}_{m'}$, and no other initial coefficient contributes. Hence $\text{anc}_m = \text{init}_m + \sum_{m' < m} \mathcal{K}_{mm'}\text{init}_{m'}$. The kernel strictly increases deformation monomials and is nilpotent in the finite quotient, so its geometric series inverse is finite and gives (9)–(10). $\square$

The following examples show that the initial coefficients contain information not generated by the ray factors themselves. Consider the part of the potential obtained by transport through the ray factors from x^r, y^s and the monomials with one internal marking, that is, the part generated from these by wall-crossing. In the language of theta functions [GHK15, GHKK18], the observation below says that for $x^5 + y^5$ the potential is not the analogue of $\vartheta_{(r,0)} + \vartheta_{(0,s)}$ plus the theta functions of the one-marking monomials: it requires further summands, with coefficients init_m . The observation rests on Example 6.12, which follows it.

Observation 6.11. Bend chains generate exactly the part of the potential reachable by transport from the initial coefficients, and the initial system is input that the ray factors do not produce. It need not be supported on x^r, y^s and the monomials with one internal marking: for $x^5 + y^5$ the invariant $\frac{1}{5}$ of (3), on a string without wall terms, which no ray action can produce, is a nonzero initial coefficient of a monomial with three internal markings (Example 6.12). Hence the open theory is not in general generated by wall-crossing from these monomials, and for $x^5 + y^5$ it is strictly richer; this has no analogue in the smooth Fano case of [Gro10, CPS24], where the potential is generated by broken lines from its initial monomials. With the anchor convention of Definition 6.4, nonzero initial coefficients also occur at coefficients of wall-bearing strings, for instance the value $-\frac{25}{16}$ of a middle balanced monomial in the same example.

Proof. By Theorem 6.8 and Lemma 6.9, every cut potential is the sum of the bend chains from the initial coefficients; by the uniqueness there, the part generated from $x^r + y^s$ and the monomials with one internal marking differs from the potential once an initial coefficient of a monomial with at least two internal markings is nonzero. In Example 6.12 the coefficient $\frac{1}{5}$, over $t_{3,3}^3$, receives no bending, so by Consequence (a) after Theorem 6.8 it is such an initial coefficient. The value $-\frac{25}{16}$, on the one-marking monomial $t_{3,3,2}$, is an initial coefficient by Theorem 6.7. $\square$

Example 6.12 (Initial coefficients not generated by wall-crossing). Two things this example shows, for $x^5 + y^5$: bending cannot generate everything, and, with the anchor convention of Definition 6.4, initial coefficients are not confined to strings without wall terms. For the first, the invariant $\langle (\tau_0^{(3,3)})^3 \sigma_1^4 \sigma_2^4 \sigma_{12} \rangle = \frac{1}{5}$ of (3) has no bending source (t -monomials multiply along a bend chain, and the unique candidate wall term fixes the initial monomial of $t_{3,3}$ exactly), so such coefficients are initial data. For the second, take the string of $t_{3,3,2}$ in the tower of the twist (3,3) (Figure 1). Here $N = 2$, the balanced boundary degrees are (3,13), (8,8) and (13,3), and the wall terms lie on the rays of (4,9) and (9,4), with wall vectors $v_1 = 45\varepsilon_1 - 20\varepsilon_0$ and $v_2 = 20\varepsilon_2 - 45\varepsilon_1$ in the basis $\varepsilon_0, \varepsilon_1, \varepsilon_2$ of coefficients, whose coordinates we write ν_0, ν_1, ν_2 (the coefficients ν_p of the proof of Theorem 1.4). The singleton condition $\mathcal{A} = (-1)^2$ has only its $h = 1$ part, with Gamma coefficients $\Gamma(\frac{4}{5})\Gamma(\frac{14}{5})/\Gamma(\frac{4}{5})^2 = \frac{36}{25}$ at the two ends and $\Gamma(\frac{9}{5})^2/\Gamma(\frac{4}{5})^2 = \frac{16}{25}$ in the middle:

$$\frac{36}{25} \nu_0 + \frac{16}{25} \nu_1 + \frac{36}{25} \nu_2 = 1.$$

In increasing slope the transport crosses the ray of (9,4) and then that of (4,9), so the coefficient vectors in the three chambers are $\nu^{\min} = \nu_0\varepsilon_0$, $\nu^{\min} + c_2v_2$ and $\nu^{\min} + c_2v_2 + c_1v_1 = \nu^{\max}$. The

$\mathcal{A}$ -condition for $\nu^{\min}$ and the vanishing of the ε_0 - and ε_1 -components of $\nu^{\max}$ form the linear system

$$\begin{pmatrix} \frac{36}{25} & 0 & 0 \\ 1 & -20 & 0 \\ 0 & 45 & -45 \end{pmatrix} \begin{pmatrix} \nu_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

whose solution is the extreme value $\nu_0 = \langle \tau_2^{(3,3)} \sigma_1^3 \sigma_2^{13} \sigma_{12} \rangle = \frac{25}{36}$ and equal wall coefficients $c_1 = c_2 = \frac{5}{144}$. The middle chamber contains the direction $(1, 1)$ of the middle balanced boundary degree $(8, 8)$, and there the coefficient vector is $\nu^{\min} + c_2 v_2 = (\frac{25}{36}, -\frac{25}{16}, \frac{25}{36})$. The middle coefficient $-\mu_2 c_2$ is nonzero because $c_2 = c_1 \lambda_1 / \mu_2 = c_1 = \nu_0 / \mu_1 \neq 0$, and by Theorem 6.7 it is the initial coefficient of the middle balanced monomial, $\langle \tau_2^{(3,3)} \sigma_1^8 \sigma_2^8 \sigma_{12} \rangle^{\text{mid}} = -\frac{25}{16}$, where the superscript denotes the value in the middle chamber. The $\mathcal{A}$ -condition holds in all three chambers, since both wall vectors lie in the kernel of its left-hand side.

Example 6.13 (An anchored value need not be an initial coefficient). Proposition 6.5 excludes foreign contributions only from beyond the target's own direction; contributions from the $(1, 1)$ -side may remain at the anchor cut. Composite strings (those of monomials with at least two internal markings) with descendents exhibit this behavior. For $x^5 + y^5$ we consider the composite strings of $J = \{(0, 0, 2), (1, 1, 0)\}$, whose wall term (the one over the composite monomial itself) lies on the $(1, 1)$ -ray, and of $J = \{(0, 0, 2), (2, 1, 0)\}$, the smallest example whose wall term lies off the $(1, 1)$ -ray, on the ray of $(3, 2)$. For each of them we solve the $\mathcal{A}$ -conditions on the strings of all divisors of the composite monomial and transport the resulting cut potentials across every ray carrying a wall term of these finitely many strings. With ν' the coefficients on the string of $t_{0,0,2}$, with balanced boundary degrees $(0, 10)$, $(5, 5)$, $(10, 0)$, and ν those on the composite strings, with balanced boundary degrees $(1, 6)$, $(6, 1)$ and $(2, 6)$, $(7, 1)$, the conditions read

$$\begin{aligned} t_{0,0,2} : \quad & \frac{6}{25} \nu'_{(0,10)} + \frac{1}{25} \nu'_{(5,5)} + \frac{6}{25} \nu'_{(10,0)} = 1, \\ t_{0,0,2} t_{1,1} : \quad & \frac{2}{5} \nu_{(1,6)} + \frac{2}{5} \nu_{(6,1)} + \frac{14}{25} \nu'_{(0,10)} + \frac{4}{25} \nu'_{(5,5)} + \frac{14}{25} \nu'_{(10,0)} = 0, \\ t_{0,0,2} t_{2,1} : \quad & \frac{2}{5} \nu_{(2,6)} + \frac{3}{5} \nu_{(7,1)} + \frac{14}{25} \nu'_{(0,10)} + \frac{6}{25} \nu'_{(5,5)} + \frac{24}{25} \nu'_{(10,0)} = 0; \end{aligned}$$

the first is the singleton condition $\mathcal{A} = (-1)^2$, and in the other two the last three terms are the parts with $h = 2$, which contain the one-marking normalization 1 of $t_{1,1}$, respectively $t_{2,1}$. Hence $\nu'^{\min} = (\frac{25}{6}, 0, 0)$ and $\nu'^{\max} = (0, 0, \frac{25}{6})$. With the wall vectors $v_1 = 30\varepsilon_1 - 5\varepsilon_0$ and $v_2 = 5\varepsilon_2 - 30\varepsilon_1$ on the rays of $(1, 6)$ and $(6, 1)$, $\nu'^{\max} - \nu'^{\min} = c_1 v_1 + c_2 v_2$ gives $c_1 = c_2 = \frac{5}{6}$, nonzero because $\nu'_{(0,10)}^{\min} \neq 0$, as in the proof of Proposition 6.14. The extremal values on the composite strings are $\nu_{(1,6)}^{\min} = \nu_{(6,1)}^{\max} = -\frac{35}{6}$ and $\nu_{(2,6)}^{\min} = -\frac{35}{6}$, $\nu_{(7,1)}^{\max} = -\frac{20}{3}$, and the potential carries them with the sign $(-1)^{|J|-1} = -1$. In the second case the potential coefficients of $(2, 6)$, $(7, 1)$, transported from $(\frac{35}{6}, 0)$ at the σ_1 -edge cut across the rays of $(6, 1)$, $(3, 2)$ and $(1, 6)$, with the term $c t_{0,0,2} t_{2,1} X_{2,1}$ on the ray of $(3, 2)$, become successively $(\frac{35}{6}, -\frac{10}{3})$, $(\frac{35}{6} - 15c, -\frac{10}{3} + 10c)$ and $(15 - 15c, -\frac{10}{3} + 10c)$; both components agree with the maximal values $(0, \frac{20}{3})$ exactly when $c = 1$. The same computation forces the coefficient 1 on the $(1, 1)$ -ray in the first case, and the $\mathcal{A}$ -conditions hold in every chamber (Section 7). In the second case the coefficient of $(7, 1)$ vanishes in $\nu^{\min}$, so its anchored value, read below the ray of $(6, 1)$, is 0. The monomial $t_{2,1} x^2 y$ has initial coefficient 1 and is anchored in the chamber between the rays of $(6, 1)$ and $(3, 2)$, so its transport to that anchor cut crosses the ray of $(6, 1)$ negatively, where

the logarithm of the ray factor is $\frac{5}{6} t_{0,0,2} X_{5,0}$. By (7) it gives the foreign contribution

$$-\frac{5}{6} \det((2, 1), (6, 1)) t_{0,0,2} t_{2,1} x^7 y = \frac{10}{3} t_{0,0,2} t_{2,1} x^7 y,$$

the only one at that cut. Hence the anchored value 0, the value in the minimal system, is not the initial coefficient, which is $0 - \frac{10}{3} = -\frac{10}{3}$.

6.3. Realization by concatenated homotopies. In this subsection and the next, a *leg* is one normal-form homotopy (§3) joining two consecutive intermediate canonical families, that is, canonical families realizing consecutive members of a sequence of realizable systems; the homotopies constructed below are concatenations of legs.

Proposition 6.14 (Descendent rays accumulate; primary rays are locally finite). *Fix $r, s \geq 2$ and an interior twist (a, b) . Let d range over $\mathbb{Z}_{\geq 1}$, so that $t_{a,b,d}$ runs over deformation parameters of total degree one. The rays of the associated critical graphs are exactly*

$$R(a, b) = \{(a + j_1 r + 1, b + j_2 s + 1) : j_1, j_2 \in \mathbb{Z}_{\geq 0}\}, \quad d = j_1 + j_2 + 1,$$

their coefficients in the ray factors of $\mathfrak{D}^{r,s}$ are all nonzero, and their slopes are dense in $(0, \infty)$. Hence the walls of $\mathfrak{D}^{r,s}$ accumulate in every open subcone of C_∂ already at total degree one in the deformation parameters, for every (r, s) , the simple singularities included; those carry no primary wall at all (Theorem 1.5). By contrast, write $\mathfrak{D}_0^{r,s}$ for the primary boundary-ray factorization, the factors with $\mathbf{d} = \mathbf{0}$, and $\mathfrak{m}$ for the ideal generated by the deformation parameters. Then $\mathfrak{D}_0^{r,s}$ has finitely many walls modulo $\mathfrak{m}^n$ for every n , and its chambers are open angular sectors.

Proof. For a singleton $J = \{(a, b)\}$ one has $\ell_1 = \ell_2 = 0$ and $N = d$, so by Definition 2.1 the critical graphs $\Lambda_{J,p}$, $1 \leq p \leq d$, have boundary degrees $(a + (p-1)r + 1, b + (d-p)s + 1)$; putting $j_1 = p-1$, $j_2 = d-p$ gives $j_1 + j_2 = d-1$ and the stated set, every element of which is attained by taking $d = j_1 + j_2 + 1$. For density, fix $\xi > 0$ and let j_1 be large enough that $\xi(a + j_1 r + 1) > 1 + b$; with $j_2 = \lceil (\xi(a + j_1 r + 1) - 1 - b)/s \rceil$ one has $j_2 s \in [\xi(a + j_1 r + 1) - 1 - b, \xi(a + j_1 r + 1) - 1 - b + s)$, whence

$$\left| \frac{b + j_2 s + 1}{a + j_1 r + 1} - \xi \right| < \frac{s}{a + j_1 r + 1} \xrightarrow{j_1 \rightarrow \infty} 0.$$

For the coefficients: on a singleton string the chamber-index condition is $\mathcal{A}(J, \mathbf{d}, \nu) = (-1)^d \neq 0$, whose $h = 1$ part is $\sum_p \omega_p \nu_p$ with all $\omega_p > 0$, so $\nu_N^{\max} \neq 0$; solving $W^{\nu^{\max}} - W^{\nu^{\min}} = \sum_p c_p v_p$ with $v_p = \lambda_p \varepsilon_p - \mu_p \varepsilon_{p-1}$, $\lambda_p, \mu_p > 0$, gives $c_N = \nu_N^{\max} / \lambda_N \neq 0$ and $c_p = c_{p+1} \mu_{p+1} / \lambda_p \neq 0$ for every p , and no cancellation between distinct rays is possible at total degree one, where the monomials are distinct. For the primary statement, modulo $\mathfrak{m}^n$ only monomials $u_{J,\mathbf{0}}$ with $|J| < n$ occur, and for each such $|J|$ the twists range over the finite range of narrow residues $0 \leq a \leq r-2$, $0 \leq b \leq s-2$, so only finitely many strings, hence finitely many walls, contribute; finitely many rays through the origin cut the quadrant into open angular sectors. $\square$

Theorem 6.15 (Realization by homotopies in a finite coefficient quotient; = Theorem 1.2(i)). *Let S be a finite divisor-closed set of deformation monomials containing 1, and choose a finite labeled set I with at least $n_{a,b}(S)$ labels of each twist occurring in S , with $n_{a,b}(S)$ as in Theorem 1.2. The quotient factorization, the reduction to S of the factorization $\mathfrak{D}^{r,s}$, has finitely many walls, with rays $\rho_1, \dots, \rho_L$ listed in increasing slope. There is a homotopy $(\mathfrak{s}_T)_{T \in [0,1]}$ of boundary conditions, that is, a family of simple canonical homotopies in the sense of [GKT22, Defs. 3.36–3.37], obtained by concatenating normal-form homotopies, from a family of canonical multisections realizing $\nu_I^{\min}$ to one realizing $\nu_I^{\max}$ (the restrictions of the minimal and maximal systems to I); every time slice is a family of canonical multisections [GKT22, Def. 3.22]. After reduction to S every nonzero jump is supported on a single boundary ray, the rays carrying*

nonzero reduced jumps occur in increasing slope as T increases, and the total reduced jump on ρ_i is the ray factor θ_{ρ_i} of Theorem 1.1 modulo that quotient.

Proof. Step 1: the labeled model, and symmetrization. Let $\mathfrak{q}_S$ be the monomial ideal spanned by monomials outside S ; it is an ideal because the complement of a divisor-closed set is upward closed. Write a bar for reduction modulo $\mathfrak{q}_S$. Let $A_I = \mathbb{Q}[u_{i,j} : i \in I, j \geq 0] / (u_{i,j} u_{i,j'})$ be the labeled coefficient ring of [GKT22, Def. 4.6], square-zero in each label; write $u_{A,\mathbf{d}} = \prod_{i \in A} u_{i,d_i}$ and $u_i = u_{i,0}$. The monomials $u_{J,\mathbf{d}}$ (indexed by a twist multiset, in the deformation parameters) and $u_{A,\mathbf{d}}$ (indexed by a label subset, in A_I) live in different rings. Symmetrization sends, for every descendent level,

$$t_{a,b,d} \mapsto \sum_{\substack{i \in I \\ \text{tw}(i) = (a,b)}} u_{i,d},$$

where $\text{tw}(i)$ denotes the twist of the label i . Let $\mathfrak{q}_{I,S}$ be the labeled monomial ideal whose monomials have symmetric type outside S . By [GKT22, Lem. 4.14] and the label bound, symmetrization induces an injection

$$\mathbb{Q}[t_{a,b,d}] / \mathfrak{q}_S \hookrightarrow A_I / \mathfrak{q}_{I,S}.$$

Indeed, a monomial in S maps to the nonempty sum over injective assignments of its occurrences to labels of the required twists; the label bound $n_{a,b}(S)$ makes the sum nonempty, and distinct symmetric monomials have disjoint labeled supports. The induced map on the enlarged Lie algebra preserves boundary rays, including the marginal $(1,1)$ -ray. The action after reduction is still free: the minimal-support argument of Lemma 2.5 detects every nonzero class supported in S on W_0 . Thus the $\mathfrak{q}_{I,S}$ -reductions of $\nu_I^{\min}, \nu_I^{\max}$ and of the labeled boundary-ray factorization identify with their universal symmetric counterparts. Below we suppress the labeled-lift symbol in the action of a symmetric factor.

Step 2: the ray factors are symmetric. Twist-preserving permutations fix the two symmetric extremal systems, hence by freeness fix $g^{\min \rightarrow \max}$; they also preserve every boundary-ray subspace. Uniqueness of the factorization in Proposition 3.1, whose filtered proof applies verbatim over the labeled coefficient ring, therefore makes each θ_{ρ_i} symmetric. Write the boundary-ray factorization, in the composition convention of §1.2, as

$$g^{\min \rightarrow \max} = \theta_{\rho_L} \circ \cdots \circ \theta_{\rho_1} \pmod{\mathfrak{q}_{I,S}}, \quad \text{slope}(\rho_1) < \cdots < \text{slope}(\rho_L).$$

Step 3: one leg per ray. For each quotient factor choose the symmetric full-group representative $\widehat{\theta}_{\rho_i} = \exp(\widehat{\eta}_{\rho_i})$ whose logarithm is the finite sum of its terms supported in S and has no terms outside S . Starting with a transverse canonical family realizing $\nu_I^{\min}$, that is, a family of canonical multisections [GKT22, Def. 3.22] transverse to the zero section, as required of the endpoints in Lemma 3.4, construct one leg for each factor. If the current system is ν , then $\widehat{\theta}_{\rho_i} \nu$ is again an element of $\Omega_{r,s}(I)$. The realizability statement $\Omega_{r,s}(I) = \Upsilon_{r,s}(I)$ of [GKT22, Cor. 5.14], read in the enlarged sense of Lemma 2.5 on the marginal ray, supplies a canonical endpoint family realizing it, and the constructive realization in [GKT22, Thm. 5.13, Lem. 5.15] may be chosen transverse. Only the systems, not the families, need to be symmetric. Join the two transverse endpoint families by a normal-form homotopy as in Lemma 3.4.

Step 4: every jump of the leg lies on one ray. We claim that every jump internal to this leg is supported on ρ_i . For the actual chronological order of its subset times, the jump formula (§3) writes the total labeled jump as

$$\prod_{\emptyset \neq A \subseteq I}^{\text{time}} \exp(\eta_A),$$

where η_A , the logarithm of the jump at T_A , has exact label support A (it is the sum over descendent vectors of terms $u_{A,\mathbf{d}}X$). In the labeled coefficient ring, $u_{A,\mathbf{d}}u_{B,\mathbf{e}} = 0$ when $A \cap B \neq \emptyset$. For any fixed order this exact-support factorization is unique. Indeed, a nonzero bracket of disjoint supports has support $A \cup B$; hence

$$\log \left(\prod_A \exp(\eta_A) \right) \Big|_A = \eta_A + P_A(\eta_B : B \subsetneq A),$$

and induction on $|A|$ recovers every η_A .

The total jump and the labeled lift of $\widehat{\theta}_{\rho_i}$ carry the same initial endpoint potential to the same final one. Freeness of the enlarged labeled action (Lemma 2.5) makes them equal. Now induct again on $|A|$. The exact- A component of the logarithm of the target is supported on ρ_i ; by induction all η_B for $B \subsetneq A$ are supported there as well. Every Lie word in P_A therefore vanishes because the boundary-ray subalgebra is abelian, as proved in Proposition 3.1. Thus η_A equals the target's exact- A component and is supported on ρ_i . This proves the claim, including for the torus terms of the enlarged algebra.

Step 5: the final leg. After the legs of Step 3, the resulting full system ν^* is congruent to $\nu_I^{\max}$ in the stated quotient. Let g_* be the transition element, before reduction, from ν^* to $\nu_I^{\max}$. Quotient freeness gives $\bar{g}_* = \text{id}$. Join the current endpoint family to a transverse canonical family realizing $\nu_I^{\max}$ by one further leg, chosen as above. Its total jump equals g_* by full freeness. Applying the exact-support induction above *after reduction* to its time-ordered factorization shows successively that every one of its internal jump factors has trivial class: this final leg is entirely zero modulo $\mathfrak{q}_{I,S}$.

Step 6: concatenation. Give all legs constant collars at their endpoints, reparametrize them to successive subintervals of $[0, 1]$, and concatenate. Canonicity [GKT22, Def. 3.22], strong positivity (the positivity conditions of [GKT22, Def. 3.18] at contracted boundary nodes and near the positive boundary strata) and simplicity [GKT22, Defs. 3.36–3.37] survive this operation: use identical collars and time reparametrizations simultaneously on every graph and component, which preserves the boundary-factorization identity [GKT22, (3.12)] and the component-independence defining simplicity; the conditions (1)–(3) of Lemma 3.4 hold in each leg, with leg-dependent times, and no transversality is asserted for the components on which that lemma imposes none; strong positivity is slicewise. The resulting homotopy has the exact endpoints and the asserted reduced order and total jumps. It is not one global normal-form homotopy: the same marking subset may jump in different ray legs. That distinction is precisely what evades Proposition 6.19.

□

Remark 6.16 (Finite versus completed realization). Theorem 6.15 is a theorem for each fixed finite quotient. The ray factors are algebraically compatible when the finite divisor-closed coefficient set grows, but the proof does not choose their endpoint families and homotopies compatibly. Such a choice would require relative lifting theorems for the spaces of canonical families and simple canonical homotopies. We therefore do not assert a single completed geometric homotopy for the dense descendent factorization. For $x^4 + y^4$ and finite systems of graph data closed under the assembling relation of [GKT22], compatible choices are constructed in [Wue26b, Theorem 5.7 and Appendix B]. Moreover, the final leg is zero in the stated quotient but may have jumps on several rays outside it.

A single normal-form homotopy already suffices below the first degree at which one string carries two wall terms.

Proposition 6.17 (Exact first two-wall order; realization by one normal-form homotopy). *For $n \geq 1$, the maximum of $N(J, \mathbf{0})$ over primary multisets with $|J| = n$ is*

$$N_{\max}(r, s; n) = n + 1 - \left\lceil \frac{2n}{r} \right\rceil - \left\lceil \frac{2n}{s} \right\rceil,$$

attained by $J = \{(r-2, s-2)^n\}$. Consequently the least primary degree admitting a string with two wall terms is

$$N_2(r, s) = \min \left\{ n \geq 3 : \left\lceil \frac{2n}{r} \right\rceil + \left\lceil \frac{2n}{s} \right\rceil \leq n - 1 \right\},$$

where the minimum of the empty set is ∞ .

- (i) $N_2 = \infty$ if and only if $\hat{c}_W \leq 1$, that is, if and only if W is simple or parabolic; when $\hat{c}_W > 1$, $N_2 \geq \lceil 1/(\hat{c}_W - 1) \rceil$.
- (ii) When $N_2 < \infty$, a single normal-form homotopy realizes the primary boundary-ray factorization modulo $\mathfrak{m}^{N_2}$. When $N_2 = \infty$, such a homotopy exists at every finite order, realizing the completed primary boundary-ray factorization as the inverse system of these finite-order statements; no compatibility of the homotopies themselves is asserted.

Proof. Write $\sum_j a_j = r(J) + r\ell_1$ and $\sum_j b_j = s(J) + s\ell_2$. Since $a_j \leq r-2$,

$$\ell_1 \leq \left\lfloor \frac{n(r-2)}{r} \right\rfloor = n - \left\lceil \frac{2n}{r} \right\rceil,$$

and similarly for ℓ_2 ; both bounds are attained simultaneously by the maximal twist. Substitution in $N = \ell_1 + \ell_2 - n + 1$ proves the formula for $N_{\max}$ and hence the displayed formula for N_2 . In particular $n = 1, 2$ never gives $N \geq 2$.

The coarser bound from the proof of Theorem 1.5 is $\ell_1 + \ell_2 \leq n\hat{c}_W$, hence

$$(11) \quad N(J, \mathbf{0}) \leq 1 + n(\hat{c}_W - 1),$$

which is at most 1 for every n when $\hat{c}_W \leq 1$, and, when $\hat{c}_W > 1$, forces $n \geq 1/(\hat{c}_W - 1)$ whenever $N \geq 2$; this is (i) apart from the converse. Conversely, if $\hat{c}_W > 1$, the exact maximal-twist count satisfies $N_{\max} > n(\hat{c}_W - 1) - 1$, which is at least 2 for all sufficiently large n , so $N_2 < \infty$.

For (ii), put $K = N_2$ when $N_2 < \infty$ and otherwise fix an arbitrary finite K . Work modulo $\mathfrak{q}_K = (\mathfrak{m}^K, \text{descendent parameters})$ and choose at least $K-1$ labels of every primary twist, so the symmetrization map used in Theorem 6.15 is injective in the degrees retained by the quotient. Then every retained string has $N \leq 1$, and a marking subset $A \subseteq I$ determines the single primary string $(A, \mathbf{0})$. Choose transverse symmetric families realizing $\nu_I^{\min}$ and $\nu_I^{\max}$ and take the one normal-form homotopy between them. Its jumps occur at the subset times T_A . By the jump formula of [GKT22, proof of Thm. 5.9] (§3), the reduction of the jump at T_A is a sum over the critical graphs with marking set A : it is zero if $|A| \geq K$ or no retained wall term exists, and otherwise is supported on the unique wall term of that string.

Order the times of all subsets whose retained primary string carries its unique wall term by the slope of that term's ray; place the remaining subset times arbitrarily. A resulting zero jump is harmless. This prescription is legitimate: [GKT22, Lem. 3.40] constrains the times only to be distinct in $(0, 1)$, and its construction takes them as given: the zeros are moved onto a prescribed $T_{I(\Gamma)}$ by a diffeomorphism preserving the moduli factor, and the finitely many non-transverse times are kept off the T_A by re-extending transversally at each prescribed time. Visible jumps sharing a ray commute by the abelianness proved in Proposition 3.1. Full freeness identifies the total jump with $g^{\min \rightarrow \max}$; after reduction, its grouping by ray is therefore the unique slope-ordered factorization $\prod_{\rho}^{\rightarrow} \theta_{\rho}$. Symmetry and injectivity descend this labeled statement to the universal primary quotient. Thus the total reduced jump on ρ_i is θ_{ρ_i} . $\square$

Remark 6.18. The ordering step above is the only place the freedom in the times of [GKT22, Lem. 3.40] is used; the times T_A there may be prescribed arbitrarily as distinct points of $(0, 1)$. What the ordering does *not* do is separate the wall terms of a single string: those all carry the one marking set J , so they share the one time T_J whatever order is chosen, which is Proposition 6.19.

At the first two-wall degree, one global normal-form homotopy provably does not suffice. This explains why the concatenation of legs in Theorem 6.15 is necessary.

Proposition 6.19 (One normal-form homotopy cannot separate a string's wall terms). *Let $(J, \mathbf{0})$ be a primary string with $N \geq 2$ and let H^* be a family of simple canonical homotopies with times $(T_A)_{A \subseteq I}$ as in [GKT22, Lem. 3.40]. Then every possible contribution from the string's N wall terms is concentrated at the single time T_J , with wall term p actually contributing if and only if $\Delta(\Lambda_{J,p}) \neq 0$. The primary part of the jump there, modulo the descendent parameters, is supported on one ray if and only if at most one of these zero counts is nonzero. In particular no single such normal-form homotopy gives the ray-by-ray realization of Theorem 6.15 once two of the string's wall terms actually contribute.*

Proof. The N critical graphs $\Lambda_{J,1}, \dots, \Lambda_{J,N}$ all carry the internal marking set J , so by Lemma 3.4(3)(a) their zeros occur at the one time T_J . By the jump formula of [GKT22, proof of Thm. 5.9] (§3), the logarithm of the jump there is

$$\sum_{p=1}^N (-1)^{|J|+\beta_2(\Lambda_{J,p})-1} \Delta(\Lambda_{J,p}) u_{J,\mathbf{0}} X_{\Lambda_{J,p}},$$

with $\beta(\Lambda_{J,p}) = k(\Lambda_{J,p}) + (1, 1)$ the boundary degree of $\Lambda_{J,p}$. Its boundary rays are pairwise distinct: along p the first boundary degree increases by r and the second decreases by s , both by at least 2 from values at least 1, so the slope strictly decreases; and for $N \geq 2$ none of them is the torus direction. The displayed wall terms lie in distinct bidegree summands, so no cancellation is possible. Its logarithm is therefore supported on one ray exactly when at most one coefficient is nonzero. $\square$

The concatenation of legs in Theorem 6.15 is not a single normal-form homotopy: it applies [GKT22, Lem. 3.40] separately to each boundary-ray factor and may therefore reuse the same marking subset at several leg times. Proposition 6.17 succeeds with a single leg because when $N \leq 1$ the at-most-one condition on $\Delta(\Lambda_{J,p})$ holds for free.

Example 6.20 (The first two-ray string of $x^5 + y^5$). For $x^5 + y^5$ the order $N_2 = 5$ of Proposition 6.17 is attained by $J = \{(3, 3)^5\}$, $\mathbf{d} = \mathbf{0}$: here the Euler weight is 25 and $N = 2$, with balanced boundary degrees $(0, 10), (5, 5), (10, 0)$ and critical graphs of boundary degrees $(1, 6)$ and $(6, 1)$: two wall terms on two *distinct* rays, the first such string for this Fermat pair. Their wall vectors are $v_1 = 30\varepsilon_1 - 5\varepsilon_0$ and $v_2 = 5\varepsilon_2 - 30\varepsilon_1$. Since $|J| \geq 2$ and $N \geq 0$, the chamber-index condition is $\mathcal{A}(J, \mathbf{0}, \nu) = 0$, which reads

$$\frac{6}{25} \nu_{(0,10)} + \frac{1}{25} \nu_{(5,5)} + \frac{6}{25} \nu_{(10,0)} = \frac{204}{15625} :$$

three coefficients against two torsor directions and one condition, as Theorem 1.4 requires. The right-hand side is minus the part of $\mathcal{A}$ with $h \geq 2$, evaluated on the lower-order values of the minimal system: $-\frac{2}{5}, \frac{1}{5}$ and $-\frac{12}{25}$ over $t_{3,3}^2, t_{3,3}^3$ and $t_{3,3}^4$, and the one-marking normalization 1. In the minimal system of Theorem 1.1 this gives

$$\langle (\tau_0^{(3,3)})^5 \sigma_2^{10} \sigma_{12} \rangle^{\min} = \frac{34}{625},$$

with $\langle (\tau_0^{(3,3)})^5 \sigma_1^{10} \sigma_{12} \rangle^{\max}$ the same by symmetry. This is the lowest primary degree for $x^5 + y^5$ at which the jump of a single normal-form homotopy would have to carry two rays, hence the first primary degree at which the realization of Theorem 6.15 differs from one global normal-form homotopy (Corollary 6.21); every other primary string computed in §5 and in §6 has $N \leq 1$, which is why the difficulty does not surface there.

The hypothesis of Proposition 6.19 is met here. Take I to carry at least five markings of twist $(3, 3)$. For each five-element set A of these labels the labeled string $(A, \mathbf{0})$ carries the same two wall terms, at the one time T_A , and the argument below applies to it verbatim. Writing $\nu^{\max} - \nu^{\min} = c_1 v_1 + c_2 v_2$ and reading the extreme rows gives $c_1 = c_2 = \frac{34}{3125}$, the middle row being consistent; more generally $c_1 = \nu_0/\mu_1$ and $c_{p+1} = \lambda_p c_p/\mu_{p+1}$, where ν_0 is the ε_0 -coefficient of $\nu^{\min}$ (here $\nu_{(0,10)}^{\min}$), so every c_p is nonzero as soon as ν_0 is, exactly as in the proof of Proposition 6.14. No lower-order term contributes to either coefficient: the wall terms over the powers of $t_{3,3}$ sit at $|J| = 2, 4, 5$ and no multiset drawn from the lower orders $|J| = 2, 4$ sums to 5, while a wall term acting on a lower coefficient of the potential contributes nothing here, its direction being parallel to that coefficient's and hence annihilating it (Proposition 6.2). So both zero counts are nonzero and both wall terms contribute to the jump at the one time.

The same computation shows the difficulty is not peculiar to the hyperbolic regime. A single insertion of maximal twist with two descendents has $N = 2$ for every (r, s) , with both coefficients again nonzero, so the obstruction appears in every model once descendents are admitted, the simple ones included, which carry no primary wall at all. What is special about $x^5 + y^5$ is that it is the first symmetric hyperbolic Fermat pair and this is its first primary occurrence.

Corollary 6.21 (= Theorem 1.2(ii)). *For $x^5 + y^5$ modulo $\mathfrak{m}^6$, and I carrying at least five labels of twist $(3, 3)$, no normal-form homotopy supplied by [GKT22, Lem. 3.40] between families realizing the minimal and maximal systems $\nu_I^{\min}$ and $\nu_I^{\max}$ has all of its jumps supported on single rays.*

Proof. By Example 6.20 both wall terms of the string $J = \{(3, 3)^5\}$ contribute, and by Proposition 6.19 they share the single time T_J , so the jump there meets two distinct rays. $\square$

Remark 6.22. For simple W the *primary* statement carries no walls at all: $N \leq 0$ throughout by Theorem 1.5, so $\mathfrak{D}_0^{r,s}$ is empty and the quadrant C_∂ is a single chamber. By Proposition 6.14, though, the descendent walls of the same model are dense in the same quadrant. The parabolic pairs are the first case with primary content, and there the unique ray is $(1, 1)$, carrying the marginal torus direction; the change it effects is the one exhibited in [GKT26, Ex. 4.9]. In the hyperbolic regime strings with $N \geq 2$ occur and the jump of a single normal-form homotopy can then carry several rays at once. The exact first degree is given by Proposition 6.17. Up to

the symmetry $(r, s) \leftrightarrow (s, r)$, its complete hyperbolic classification is

r	s	$N_2(r, s)$
3	7	21
3	8	12
3	$9 \leq s \leq 11$	9
3	$s \geq 12$	6
4	5	10
4	$6 \leq s \leq 7$	6
4	$s \geq 8$	4
5	$5 \leq s \leq 7$	5
5	$s \geq 8$	4
$r \geq 6$	$s \geq r$	3

Thus $N_2 \leq 21$ for hyperbolic integral Fermat exponents, with the maximum attained only at $(3, 7)$ up to symmetry. The coarser lower bound in Proposition 6.17 is useful for a fixed pair but does not produce an unbounded sequence approaching the threshold: the exponents are discrete.

Remark 6.23 (How the realization reads the quadrant). On every finite divisor-closed coefficient set, Theorem 6.15 realizes the successive slope cuts by intervals in $[0, 1]$, the two edges of the quadrant C_∂ by the two endpoints, and the realizable systems at the two extreme chambers by the extremal systems $\nu^{\min}$ and $\nu^{\max}$. In a finite primary quotient these cuts bound actual open chambers; in the descendent completion only the algebraic ray factors are asserted to be coefficientwise compatible. The marginal torus terms of Proposition 3.1 are carried by the $(1, 1)$ -ray, whose fixed monomials are exactly its own. The increasing order of slopes is a choice made in the construction: the same argument realizes the factorization of $g^{\min \rightarrow \max}$ in decreasing order of slopes.

6.4. The punctured boundary-degree quadrant. For a compact polygon P , a P -homotopy is a two-parameter family of multisections of the descendent Witten bundles over $P \times \mathcal{PM}_\Gamma$ satisfying the canonical boundary identities of [GKT22, Def. 3.22] on every parameter slice, with no transversality required.

Lemma 6.24 (Pullback fillers). *Let $H = (\mathfrak{s}_T)_{T \in [T^-, T^+]}$ be a canonical homotopy, let P be a compact polygon and let $\varphi : P \rightarrow [T^-, T^+]$ be smooth. Then $(p, x) \mapsto \mathfrak{s}_{\varphi(p)}(x)$ is a P -homotopy, strongly positive on every parameter slice, whose restriction to ∂P is the reparametrization of H by $\varphi|_{\partial P}$. If $\varphi_0, \varphi_1 : P \rightarrow [T^-, T^+]$ agree on ∂P , the pullbacks along the linear interpolation from φ_0 to φ_1 form a $(P \times [0, 1])$ -homotopy joining the two pullbacks relative to ∂P .*

Proof. The canonical boundary identities and strong positivity are slicewise conditions, and every slice of a pullback is a slice $\mathfrak{s}_T$ of H ; a convex combination of maps to an interval takes values in it and is constant on ∂P , where the maps agree. $\square$

Remark 6.25 (The punctured quadrant). Let S be finite and divisor-closed, and let $\mathfrak{D}_S$ be the union of the wall rays $\rho_1, \dots, \rho_L$ of the quotient factorization. The map $q \mapsto (q_1 + q_2, q_2 / (q_1 + q_2))$ identifies the punctured quadrant $C_\partial^\times = C_\partial \setminus \{0\}$ with $(0, \infty) \times [0, 1]$, and the walls with the half-lines $(0, \infty) \times \{\zeta_i\}$. Hence $C_\partial^\times$ is contractible, and the crossing word of a generic path, one with off-wall endpoints that is piecewise smooth and transverse to $\mathfrak{D}_S$, is that of its angular

projection. A homotopy of generic paths relative to the endpoints, through finitely many simple tangencies to $\mathfrak{D}_S$, inserts or deletes only adjacent inverse crossings, and the chamber-adjacency graph is a path. Assigning to a positive or negative crossing of ρ the reduction $\theta_{\rho,S}$ of θ_ρ to S or its inverse, the reduced path-ordered product therefore depends only on the endpoint chambers: it is trivial on closed generic paths and equals $g^{\min \rightarrow \max}$ on a path from the σ_1 -edge to the σ_2 -edge, radial paths act by the identity, and the product factors through the slope space $\mathbb{P}(C_\partial) \cong [0, \infty]$. These statements hold for any finitely many rays through the vertex of a closed strictly convex cone. What is specific to boundary conditions is that the cancellations are realized by homotopies. Assign to the i th chamber the intermediate family $\mathfrak{s}_i$ of Theorem 6.15 and to a crossing of ρ_i the ray leg $H_i : \mathfrak{s}_{i-1} \rightarrow \mathfrak{s}_i$, parametrized by $[T_i^-, T_i^+]$, or its reverse. A backtracking $H_i * H_i^{-1}$ is the reparametrization of H_i by a map $\partial P \rightarrow [T_i^-, T_i^+]$ for a polygon P ; any smooth extension $\varphi : P \rightarrow [T_i^-, T_i^+]$ gives a filler relative to the boundary by Lemma 6.24, and two fillers are joined by the pullback along the linear interpolation of their maps.

Remark 6.26 (Interpretation). The product $g^{\min \rightarrow \max}$ from one edge of C_∂ to the other is not the holonomy of a loop about the vertex: the two edges of C_∂ are not identified, and on the punctured quadrant the continuation along homotopies of boundary conditions sees only slope. In particular the marginal torus terms are not identified here with the holonomy of an integral-affine structure. In a tropical Landau–Ginzburg model in the sense of [CPS24], the input includes a singular integral-affine manifold, a polyhedral decomposition and a multivalued convex piecewise-linear function, and broken lines use that affine geometry; none of these data is recovered from the quadrant. Likewise the present results transport potentials and their decomposition into initial coefficients, but they do not construct the operation on pairs of broken lines that would identify theta-function multiplication constants with open invariants. Theta functions in the sense of [GHK15, GHKK18, GS22] might still span the Saito–Givental deformation of $x^r + y^s$ and recast the open mirror theorem [GKT22, Thm. 0.3] as intrinsic mirror symmetry; such a theory would need data not considered here, for example an energy valuation, a period normalization, or a logarithmic degeneration with affine gluing. The companion paper [Wue26b] constructs, for $x^4 + y^4$, a singular integral-affine annulus over the Carl–Pumperla–Siebert base, with four singular points, and on it a wall structure consistent at its two joints, whose singleton walls are the ray factors of $g^{\min \rightarrow \max}$ for a single first-descendent marking of twist $(1, 0)$ or $(0, 1)$, but whose first wall carrying one marking of each of these twists is not; it constructs no theta functions.

7. EXACT FINITE CALCULATIONS

The proofs of the structural theorems are independent of software. Exact calculations provide the following reproducibility checks; the repository <https://github.com/bwuebben/open-fjrw-scattering> contains the programs and a file-by-file map from these mathematical statements to their implementations.

Critical graphs and boundary rays. Direct enumeration from [GKT22, Not. 2.40] agrees with the closed formula for $N(J, \mathbf{d})$, the weight trichotomy, the values of $N_{\max}(r, s; n)$ and $N_2(r, s)$, and the two-ray example $J = \{(3, 3)^5\}$. The same calculation gives the ray set $R(a, b)$ of Proposition 6.14. Exact symbolic differentiation verifies bracket closure and the $(1, 1)$ exponent shift.

Periods, closed correlators, and recursion. Expansion of the oscillatory integral gives the two-insertion relation (4) and the Gamma functional of Lemma 4.1. The closed FJRW correlators of $x^5 + y^5$, which enter the open theory only as values of $\mathcal{A}$ for $(J, \mathbf{d})$ with $N(J, \mathbf{d}) < 0$ [GKT22, Cor. 5.11(3)], are computed as well. By Sebastiani–Thom [FJR13] the state space

and cohomological field theory factor as $A_4(x) \otimes A_4(y)$, so a closed correlator $\langle \prod_i \tau_{d_i}^{(a_i, b_i)} \rangle$ is the tensor correlator $\int_{\overline{\mathcal{M}}_{0,n}} c^{1/5}(\mathbf{a}) c^{1/5}(\mathbf{b}) \psi^{\mathbf{d}}$ of two Witten 5-spin classes. The single-variable A_4 correlators come from the Saito primitive form in flat coordinates, validated against the A_2 normalization $\langle (\tau_0^1)^4 \rangle = \frac{1}{3}$, WDVV and the $r = 2$ Witten–Kontsevich numbers [Kon92]; the tensor correlators in which one of the two classes has degree zero factorize, and those in which the smaller degree is one are obtained by reconstructing the class of degree one, a divisor on $\overline{\mathcal{M}}_{0,n}$, from its node-splitting restrictions and pairing it with the other class in the genus-zero tautological ring [Kee92]. An evaluation of Witten’s class through Chiodo’s formula [Chi08] independently reproduces these values. For instance $\langle (\tau_0^{(2,1)})^3 \tau_0^{(2,0)} \rangle_{\text{ext}} = \frac{2}{5}$ in the normalization of [GKT22, Thm. 0.3]. With these values, both sides of the open topological recursion [GKT22, Thm. 0.5], with the values of $\mathcal{A}$ given by [GKT22, Cor. 5.11], agree in the 72 instances with two internal markings of twists in $\{(1, 1), (2, 1), (1, 2), (2, 2), (1, 3), (3, 1), (2, 3), (3, 2)\}$ and in the 105 instances with three internal markings of twists in $\{(1, 1), (2, 1), (1, 2), (2, 2), (1, 3)\}$, all without descendants and including the extended sectors, in which a twist component equals -1 . This checks the closed correlators and involves no open invariant.

Ray factors and transport. Exact expansion through deformation degree four reproduces the displayed factors of Theorem 1.1 in increasing order of slopes, including the coefficient $-1/750$ of $t_{2,3}t_{3,3}^3X_{1,2}$ on the ray of $(2, 3)$, the coefficient $-1/1250$ of $t_{2,3}^2t_{3,3}^2X_{0,2}$ on the ray of $(1, 3)$, the vanishing coefficient of $t_{3,3}^4$ on the $(1, 1)$ ray, and the value $\nu_{(2,7)}^{\min} = -\frac{12}{25}$ of Remark 4.2. The transport identities reproduce (3), the initial coefficient $-25/16$ in Example 6.12, the foreign contribution $10/3$ in Example 6.13, and the bend-chain coefficients entering (10) in that example.

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