A GEOMETRIC PRODUCT FORMULA FOR UNIVERSAL CURVE COUNTING ON CALABI–YAU THREEFOLDS

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Abstract. We prove a product formula for the universal series of effective one-cycles on a smooth projective threefold X with trivial canonical bundle and $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$. The identity holds in Pardon's rational cycle algebra. Its genus-g factor is defined by the circle with opposite weights on two line bundles of degree $g-1$ whose tensor product is the canonical bundle of the curve. The exponents are the elementary-cluster integers of Ionel and Parker, with finite genus support in each integral curve class. We construct these integers as signed multiplicities in a decomposition of the cycles on X. In the algebra formed from almost-complex families, a deformation on the sum of two copies of a spin line identifies each antidiagonal class with the point class of the corresponding multiple of an elementary curve. Excision for disconnected cycles factors the image of the series of X at each energy bound. Stable maps of fixed genus distinguish the multiplicity-one classes, which suffices to lift the product to the complex cycle algebra.

1. Introduction

Let X be a smooth projective complex threefold with $K_X \simeq \mathcal{O}_X$ and $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$. For each effective class β, the characteristic function of the compact space of cycles representing β defines a class $U_{X,\beta}$ in the Chern-number-zero, virtual-dimension-zero part H of Pardon's rational algebra of semi-Fano one-cycles [Par25, §§2–3]. Write $\Gamma_X \subset H_2(X;\mathbb{Z})$ for the monoid of effective integral classes and put $U_{X,0} = 1$. We use the completion by degree against an ample integral divisor. A bounded degree contains only finitely many effective classes, by compactness of the Chow spaces and local constancy of the integral cycle class.

There are two geometric descriptions of the local curves which enter this problem. On a smooth curve C of genus g, take a line bundle L of degree $g-1$ and set $N = L \oplus (K_C \otimes L^{-1})$. The circle of weights $+1,-1$ on these two summands fixes the multiple zero sections $m[C]$. The associated equivariant Thom classes define elements $a_{g,m}$ of H. Write $A_g(q) = 1 + \sum_{m\geq 1} a_{g,m} q^m$. The second description, due to Ionel and Parker, uses an almost-complex structure for which every compact curve in a neighborhood of C is supported on C and the normal operator is injective on every cover. The contribution of such an elementary cluster is known numerically [IP18, §3]. We compare the two descriptions on the space of cycles.

Theorem 1.1. *For X as above there are unique integers $n_{\beta,g}(X)$ indexed by $\beta \in \Gamma_X \setminus \{0\}$ and $g \geq 0$, zero for all sufficiently large g at each fixed β, such that*

$$\sum_{\beta\in\Gamma_X} U_{X,\beta} Q^\beta = \prod_{\beta\in\Gamma_X\setminus\{0\}} \prod_{g\geq 0} A_g(Q^\beta)^{n_{\beta,g}(X)} \qquad \text{in } H[\![\Gamma_X]\!]. \tag{1}$$

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The integer $n_{\beta,g}(X)$ is the signed multiplicity of genus-g elementary clusters of core class β in a cluster decomposition at any energy bound containing β. It equals the coefficient of the genus-g elementary series at β in the connected Gromov–Witten expansion of Ionel and Parker.

The balanced factors and their operations were constructed in [Wue26, §§2–3]. Theorem 1.1 of that paper obtained (1) assuming a universal Gromov–Witten evaluation on analytic families. Here the integer exponents are determined by deformation of cycles, before taking the connected numerical invariant. The proof still uses compactness and gluing for stable maps of fixed genus. Integral homology is used throughout, including when multiplication by an integer identifies distinct classes.

The local comparison starts with a spin line L and the real rotations of the two copies of L in $L \oplus L$. These rotations preserve the Lee deformation of Ionel and Parker. On the integrable fiber the circle has opposite weights. On each positive fiber the only cycle of degree m is $m[C]$. A compactly supported class on the total cycle space restricts to the antidiagonal Thom class at one end and to the normalized point class at the other (§3). One can see the effect of the choice of circle already at the double of a genus-two curve. If $x_{g,m}$ denotes Pardon's scalar class, then

$$a_{2,2} - x_{2,2} = -2x_{4,1}.$$

In §8 we obtain the two negative terms separately from the two families of branched double covers of a generic balanced curve.

To pass to X, we use the cluster isotopy of Ionel–Parker and Doan–Ionel–Walpuski [IP18, §7], [DIW26, §3]. In a bounded energy range the disjoint connected supports can be separated in X. A cycle with components in several of the resulting open sets gives a product of compactly supported classes. At a fold two factors cancel up to factors of higher multiples of the core class. Iteration gives a finite signed product in an algebra of almost-complex families (§§2–4).

The difficulty in returning to the complex cycle algebra is that a relation among classes of curves can run over parameter spaces of arbitrary dimension. Only the multiplicity-one classes need to be distinguished. At the least genus in a putative relation the image-cycle map from stable maps singles out one embedded curve, regular with local degree one. Sections 5 and 6 show that its point orientation survives in compactly supported cohomology. A comparison of the first coefficient where two products differ then gives (1). The result concerns the balanced antidiagonal classes. The dependence of the other geometric local elements in [Par25, Remark 6.8] on their defining geometry is a separate question.

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2. Cycle classes in families

Let $Y \to B$ be a family of almost-complex six-manifolds. The base B is a finite simplicial complex. On each closed simplex the family is a smooth submersion with a smooth vertical almost-complex structure, in charts extending across the faces. The restrictions agree on common faces. Open and empty targets are allowed. Write $Z(Y/B)$ for the space of compact effective pseudoholomorphic one-cycles in the fibers, including the empty cycle. Its topology records the base parameter, the Hausdorff limit of the supports and weak convergence of their integration currents [DIW26, Definition 2.6]. Empty support is a separate open and

closed locus. A morphism of families is an inclusion of simplicial bases and a fiberwise almost-complex open embedding into the restricted target, smooth on every closed simplex.

For a locally compact Hausdorff space $\pi : Z \to B$ put

$$V(Z/B) = H_c^0(Z; \pi^* \omega_B),$$

where ω_B is the rational dualizing complex. Cohomology here and below is sheaf cohomology. In cellular form the summand over an oriented p-simplex σ consists of the compactly supported cochains of $Z|_\sigma$ shifted by p. The second differential takes signed restrictions to the faces. This follows by resolving ω_B by the constant sheaves on the closed simplices with the indicated shifts [Par25, Definitions 2.2 and 3.1, Lemma 3.4].

2.1. **Topology and operations.** Over a small base chart we can identify a neighborhood of any compact support with an open part of a product. Parallel transport on one closed simplex gives such a trivialization after shrinking about the compact set. A cutoff Hermitian form at the central parameter is uniformly positive on nearby complex lines, so convergence of currents bounds their masses in the chart. Consequently changes of trivialization preserve convergence. Requiring the supports to lie in a relatively compact neighborhood and imposing a closed bound on mass gives a compact containing set by [DIW26, Theorem 2.11]. For this application one may extend the structures by cutoff to the fixed central fiber on a slightly larger neighborhood. Strict inequalities give a neighborhood of the prescribed cycle. The finite gluing along closed face spaces gives a locally compact Hausdorff space over B.

On a complex analytic family the topology on each simplex agrees with Barlet's topology on compact cycles. Apply the support and volume criterion of [Bar99, §1] in the complexification of the simplex. Only the vertical $(1,1)$ part of a test two-form pairs with a complex curve. Differences of positive Hermitian forms suffice for these tests on a compact set. A positive form supported near a given point of the limiting curve detects lower continuity of supports. Conversely, compact cutoffs and uniform approximation on a set of bounded mass give Barlet's continuous volume tests.

Let S be the rational vector-space colimit of $V(Z(Y/B)/B)$ over these families, identifying subdivisions. A closed base inclusion includes the corresponding face summands. An open embedding of targets extends by zero on the open subset of cycles contained in the source. These two operations commute by restriction of extension by zero. The external product, with disjoint union of the two targets over the product base, defines a commutative multiplication on S. The empty target over a point gives the unit. These are the smooth counterparts of the operations of [Par25, §3.1]. Extension by zero from the open semi-Fano cycle locus, followed by passage to the underlying almost-complex family, thus gives a ring homomorphism

$$\Phi : H \longrightarrow S.$$

On the summand defining H the relative cohomological degree is zero.

If the base is a point, the characteristic function of a compact open subset of its cycle space gives a class of S. Such a class remembers the ambient almost-complex manifold. An insertion on a family over $B \times [0,1]$, with B a closed oriented manifold of dimension n, can also be represented by a compactly supported n-cocycle. Restrict it to each top simplex and sum with the product orientations. In the shifted complex this is a cochain of degree -1. Interior faces cancel in its differential, leaving the difference of the two endpoint insertions. We use this relation both for characteristic functions over an interval and for Thom classes over a finite Borel construction.

For the addition operation, retain all relative degrees in V and write S^* for the resulting graded group. Proper pullback by addition over the same base followed by pushforward

along the diagonal of bases takes values in $S^* \otimes S^*$, as in [Par25, Definitions 2.3 and 3.9]. Here the external product is an isomorphism for the bivariant groups over two independent bases. Indeed, for the compactly supported cohomology of two locally compact Hausdorff spaces use their one-point compactifications relative to the added points. On the compact product finite rectangular covers are cofinal. The nerve of a product cover maps to the product of the two nerves by taking marginal barycentric coordinates, with a section given by their products. Linear interpolation gives a homotopy on each product of simplices, also on the relative subspaces at infinity. Finite-polyhedron Künneth over $\mathbb{Q}$ therefore passes to relative Čech cohomology, hence to compactly supported sheaf cohomology. Filtering the two finite face complexes by dimension gives the bivariant assertion, since the product is an isomorphism on the first pages. Use product subdivisions containing the diagonal for the base pushforward. Additivity of support of effective cycles makes this construction commute with open extension, and restriction to faces commutes with both operations. The same argument for multiplication by d gives ρ_d.

3. The class of an elementary curve

For a smooth genus-g curve C and $L \in \mathrm{Pic}^{g-1}(C)$ write $L_+ = L$ and $L_- = K_C L^{-1}$. On a cycle of degree m in $\mathrm{Tot}(L_+ \oplus L_-)$, evaluate the elementary symmetric functions of the two fiber projections at $1 + \max(0, m(g-1))$ distinct points of C. The jth coefficients are holomorphic sections of $L_\pm^j$, including at ramification by taking the characteristic polynomial on the finite direct image from the normalization of each irreducible component and raising it to the multiplicity of that component. Their vanishing at the chosen points implies their identical vanishing for $1 \le j \le m$. Thus the common zero set is $m[C]$. In chosen frames the evaluations form an equivariant tuple F of nonzero weights $r_1, \dots, r_N$. On the associated bundle over $\mathbb{P}^N$ it defines a section of the pullback of $\bigoplus_j \mathcal{O}(r_j)$. The class

$$u_N = \big(\textstyle\prod_j r_j\big)^{-1} F^* \,\mathrm{Th}\big(\bigoplus_j \mathcal{O}(r_j)\big)$$

has compact support on the fixed section. Here Th is the relative Thom class of the zero section. If f is an additional test and $i : \mathbb{P}^N \hookrightarrow \mathbb{P}^{N+1}$ is a hyperplane inclusion, then $i_! u_N(F) = h\, u_{N+1}(F) = u_{N+1}(F, f)$, where $h = c_1(\mathcal{O}(1))$. The first equality is the hyperplane projection formula and the second is multiplication by the normalized Euler class of the appended line. Concatenation of two tuples therefore identifies their universal images [Wue26, §2]. This construction defines $a_{g,m}$. It applies also to continuous equivariant tests on an invariant neighborhood when their common zero set is the same section.

3.1. **Deformation of the spin model.** Fix $L^2 = K_C$ and a disk bundle U about the zero section in $L \oplus L$. Give the two summands the same Hermitian metric. On U use a Kähler form invariant under real rotations of the copies of L. In a spin frame $e^2 = dz$ the holomorphic two-form in the construction of [IP18, Proposition 3.2] is

$$\alpha = dz \wedge (w_1\, dw_1 + w_2\, dw_2).$$

It is the sum of the pullbacks of the tautological two-forms on the two line summands. The metric and α are rotation invariant. Hence the Lee almost-complex structures $J_{t\alpha}$ are invariant for $0 \le t \le T$ with $T > 0$ small. On the two eigenlines $w_1 \pm i w_2$ the rotations have weights $+1, -1$.

Proposition 3.1. *For $m > 0$, the image of $a_{g,m}$ in S equals the characteristic function of the single cycle $m[C]$ in the elementary germ.*

Proof. Let Z_m be the integrable degree-m cycle space in U, and let W_m be the corresponding space over the Lee interval. Each nonconstant compact component has positive degree over C by tameness and the retraction of U onto C. Image localization [IP18, Proposition 3.2] places every such component on C for $t > 0$. Thus W_m is the union of $\{0\} \times Z_m$ and $[0, T] \times \{m[C]\}$, glued at $(0, m[C])$. Both subsets are closed in the topology of parameter, support and current. Collapsing the interval gives a continuous proper equivariant retraction onto Z_m. Its inverse image of a compact set is that set together with the interval if the set contains $m[C]$.

Pass to the associated bundles for the Hopf circle bundle over $\mathbb{P}^N$. At $t = 0$ the unit-sphere reduction identifies this with the holomorphic Borel family restricted to the disk. The pullback of u_N under the proper retraction has compact support on $\mathbb{P}^N \times [0, T]$. At $t > 0$ it is the normalized Euler class $(\prod_j r_j)^{-1} \prod_j (r_j h) = h^N$, where $h = c_1(\mathcal{O}(1))$. Its integral is $+1$. By the interval relation of §2 the two endpoints give the same class. At the positive endpoint the Gysin map from a point of $\mathbb{P}^N$ identifies h^N with the unit at $m[C]$ in the corresponding target fiber. Excision at the fixed section permits any smaller disk. The retraction commutes with hyperplane inclusion, so stabilization respects the entire interval. □

The class in H itself is independent of the balanced splitting. In a relative Picard chart choose the marked evaluation points as disjoint local sections and trivialize the evaluation lines. The fiberwise degree argument gives the same compact section of zeros over a closed path segment. Its relative Thom class identifies the endpoints in the analytic diagram under the interval relation. The variety $\mathrm{Pic}^{g-1}(C)$ is connected. To vary C, take successive charts in connected Teichmüller space, with the relative degree-$(g-1)$ bundle $\mathcal{O}((g-1)p)$ for a local marked section p. The fixed-curve comparison identifies choices at common endpoints. Use piecewise real-analytic paths in the charts, so each segment has an analytic thickening. For genus zero use the constant sphere and for genus one use marked elliptic curves. This also joins different spin choices through balanced splittings.

3.2. **General elementary neighborhoods.** In an elementary neighborhood in the sense of [IP18, Definition 3.1], the core is the only irreducible curve, and its normal operator is the double of a real Cauchy–Riemann operator D_0 on a complex line. Every branched-cover pullback of this operator is injective. These conditions suffice to identify the point insertion of the given germ with the class in Proposition 3.1.

Let C be the core. Choose a tubular chart of its normal bundle N adapted to the complex splitting along C, and pull back the almost-complex structure J to that chart. On a smaller fixed disk use $J_s = \delta_s^* J$, $0 < s \leq 1$, where δ_s scales the normal fibers by the positive real number s. The family extends smoothly to the homogeneous almost-complex structure $J_{\mathfrak{d}}$ determined by the normal Cauchy–Riemann operator $\mathfrak{d}$. Indeed the only block in the pullback with a negative power of s is $s^{-1}B(x, sv)$ from tangent to normal. Since $B(x, 0) = 0$, its extension at zero is obtained by integrating $\partial_v B(x, \tau s v)[v]$ over $0 \leq \tau \leq 1$. Identification of the limit with $J_{\mathfrak{d}}$ is the normal rescaling formula. For $s > 0$ the elementary isolation condition gives only the cycles $m[C]$ in this disk. At zero an irreducible curve off the core would project to a branched cover f of C with a nonzero section in $\ker(f^*\mathfrak{d})$, contrary to super-rigidity. Thus for each $m > 0$ the singleton insertion continues across this ordinary interval [DW21, Lemmas 3.6 and 3.10 and Proposition 3.12] and [IP18, Definition 3.1].

By the elementary splitting, $\mathfrak{d} = D_0 \oplus D_0$ on $L \oplus L$, with D_0 a real Cauchy–Riemann operator on a complex line of degree $g - 1$. Identify its underlying smooth complex line bundle with that of a holomorphic spin line. On this bundle join D_0 to the spin Dolbeault operator by affine interpolation of the zero-order terms. The corresponding normal models $J_{D_t \oplus D_t}$ project holomorphically to C, induce the usual structure on each fiber and commute

with real fiber dilations and real rotations of the copies. These properties follow from the construction of $J_{\mathfrak{d}}$ from the operator. A lift over a holomorphic f is a pair of sections of f^*L satisfying f^*D_t. All models agree as almost-complex structures along C. Thus one fixed Kähler form ω for the spin model tames the whole path on a sufficiently small closed disk, by compactness of $C \times [0,1]$.

Fix $m > 0$ and let $r(Z)$ be the maximum fiber norm on the support of a cycle, using a rotation-invariant metric. For a radius $a > 0$ in the interior of the disk the locus T_m of all $r(Z) = a$, including the parameter t, is compact. Its supports lie in a fixed compact set, and the taming form has integral precisely $m \int_C \omega$ on each such cycle. Geometric compactness and continuity of the radius give the claim. There are no circle-fixed cycles in T_m. Each nonconstant irreducible component projects to a finite holomorphic cover, since a fiber contains no compact curve. An invariant cycle would therefore have a finite invariant set in every fiber. The connected rotation group fixes every element of such a set, and the origin is the only fixed vector [DIW26, Theorem 2.11].

On a compact Hausdorff circle space without fixed points there are finitely many continuous functions of nonzero weights having no common zero. At a point x its stabilizer is μ_k for some $k \geq 1$. The function $gx \mapsto g^k$ on its closed orbit extends continuously to the space by Tietze extension applied to the real and imaginary parts. Its weight-k Fourier projection agrees with the original function on the orbit. The nonvanishing loci so obtained cover the space, hence a finite set suffices. Apply this to T_m. For a test f there, define a test on the whole degree-m cycle family by

$$\widehat{f}(Z) = \begin{cases} r(Z)\, f\big(t, (\delta_{a/r(Z)})_* Z\big), & r(Z) > 0, \\ 0, & r(Z) = 0. \end{cases}$$

Homogeneity ensures that the rescaled cycle lies in T_m. Away from zero the formula is continuous by geometric convergence. At zero it is continuous because f is bounded. The tests retain their weights and vanish simultaneously precisely at $m[C]$ in each fiber. If T_m is empty the family already consists of the singleton in each fiber, and the empty tuple suffices.

Take Borel stage N equal to the number of tests and pull back their normalized relative Thom class. It has compact support on the distinguished section over $\mathbb{P}^N \times [0,1]$. At the superrigid end its restriction is h^N and gives the unit at $m[C]$. At the integrable spin end it gives $a_{g,m}$ because the tests are continuous circle-equivariant functions, and concatenation with the holomorphic evaluation tuple after hyperplane stabilization identifies their normalized Thom classes by the hyperplane formula.

The ordinary rescaling and this Borel deformation give the asserted identity for the original germ. Write $b_{g,m} = \Phi(a_{g,m})$ and $b_{g,0} = 1$. On the point representatives, pullback by addition counts the ordered pairs $(r[C], s[C])$ with $r + s = m$. Pullback by multiplication by d selects $(m/d)[C]$ when $d \mid m$ and is zero otherwise. Both pullbacks preserve compact supports on these families. For addition, the support and mass of either effective summand are bounded by those of the sum, so compactness follows from [DIW26, Theorem 2.11]. The same reasoning applies to multiplication by d. On these representatives the diagonal of the point base is a point. The summands are external products of degree-zero classes, giving

$$\Delta b_{g,m} = \sum_{r+s=m} b_{g,r} \otimes b_{g,s}, \qquad \rho_d b_{g,m} = \begin{cases} b_{g,m/d} & d \mid m, \\ 0 & d \nmid m. \end{cases}$$

4. Cluster decomposition at bounded energy

Choose a Kähler form ω in an integral ample class and fix a positive integer Λ. All series in this section are truncated above ω-degree Λ. Initially their labels are zero and all integral homology classes of positive ω-degree. Write $\mathcal{J}_{\mathrm{isol}}$ and $\mathcal{J}_*$ for the isolated and regular structures of [DIW26, Definition 3.13], respectively. At a structure in $\mathcal{J}_{\mathrm{isol}}$ the connected supports are embedded irreducible curves, disjoint from each other. At an energy bound, a cluster consists of a neighborhood $\mathcal{U}$ of such a support in the Hausdorff space of compact subsets, a structure J and an embedded core C. The boundary of $\mathcal{U}$ misses the bounded connected supports, their classes inside are positive multiples of $[C]$, and C is the unique curve of that class there [DIW26, Definition 3.9]. A connected support carries no multiplicity. The cycle insertions below retain it.

4.1. **Excision for disconnected cycles.** Fix $J \in \mathcal{J}_{\mathrm{isol}}$ and let $\mathscr{S}$ denote the space of nonempty connected supports at J of degree at most Λ. It is compact by [DIW26, Theorem 2.11 and Lemma 3.16]. Indeed connectedness is closed under Hausdorff convergence and every nonempty cycle has integral energy at least one. Fix a finite partition of $\mathscr{S}$ into relatively open and closed subsets $\mathscr{S}_i$. The union K_i of all curves in $\mathscr{S}_i$ is compact by the closed incidence relation in $\mathscr{S}_i \times X$. The sets K_i are pairwise disjoint since distinct irreducible curves are disjoint. Choose open neighborhoods V_i of K_i with disjoint closures. For every sufficiently close tame J' each bounded connected support is contained in some V_i. Otherwise a sequence of such supports violating the assertion would have a connected limit at J inside some K_i, contrary to Hausdorff convergence. For a prescribed finite partition by cluster neighborhoods with unoccupied boundaries, the same argument shows that the partition by the V_i continues to agree with it near J. Include the complement of the selected cluster in the initial partition.

For a union $\mathscr{U}$ of parts define $P(\mathscr{U}, J)$ to have constant term one. Its coefficient at $\beta \neq 0$ is the characteristic function of the cycles in class β all of whose connected components are supported in the corresponding open neighborhoods. Use the class in $H_2(X;\mathbb{Z})$ under inclusion for cycles in an open target. These locally constant functions have compact support. To see this over a short closed path, use compactness at fixed energy and the continuity of the integral homology label [DIW26, Theorem 2.11 and Remark 2.13]. Once the V_i are separated, restricting the integration current to one of them amounts to multiplying by a cutoff which is one on its closure and zero near the other closures. The nonempty supports converge there in the Hausdorff topology. Occupation of the several neighborhoods is locally constant. Thus the insertion for a union of parts extends over the path, and the interval relation preserves its class in S. These short transports may use smooth segments in charts of tame structures. Compactness and avoidance of the boundaries hold in a neighborhood of the given structure, so segments in a smaller convex coordinate neighborhood join the same endpoints even if the original isotopy path has only C^1 parameter dependence.

At either endpoint, open excision into $\coprod_i V_i$ decomposes each cycle uniquely as $\sum_i D_i$, including $D_i = 0$. In total class β the characteristic function is the sum of external products over all $\sum_i \beta_i = \beta$. There are finitely many relevant labels by compactness. Consequently

$$P(\mathscr{U}, J) = \prod_{i:\, \mathscr{S}_i \subset \mathscr{U}} P(\mathscr{S}_i, J). \tag{2}$$

Shrinking the V_i about the K_i gives the same insertions by open excision. The separation in the target is what identifies the right side with multiplication in S.

4.2. **Crossings of the wall.** For a core in class A consider the subgroup of the units of the truncated series algebra generated by the cluster factors with core class dA, $d \geq 2$, at isolated structures. Equivalence modulo this subgroup is an exact identity after multiplication by finitely many such factors and their inverses. Every factor has constant term one.

The subgroup is generated by corrections with unobstructed cores. Indeed, for a cluster around a possibly obstructed core in class dA, keep its outer neighborhood and its complement fixed as above. Choose a nearby structure in $\mathcal{J}_*$ in the neighborhood given by [DIW26, Propositions 3.14 and 3.21]. Transport its insertion along a short path. At the new endpoint the outer neighborhood partitions into finitely many clusters with unobstructed cores of classes kdA, $k \geq 1$, by Proposition 3.21(2) of that work. Include the complementary part and apply (2) to this partition. This expresses the entire correction as the product of the resulting regular factors, even if the former core disappears.

Consider first a transverse fold of class-A cores [DIW26, Proposition 3.33]. The whole path lies in $\mathcal{J}_{\mathrm{isol}}$, including the fold parameter, where the obstructed core is isolated. Spatial separation there extends the outer cluster insertion across the fold. At the two-core endpoint isolate the two degree-one branches and partition the remainder. At the other endpoint there are no class-A cores. Equation (2) at both ends gives

$$P_+P_- \prod_i H_i = \prod_j K_j,$$

where each H_i, K_j has core class a proper multiple of A. Thus the two distinguished factors are mutual inverses modulo those corrections. This product accounts also for cycles meeting both core neighborhoods.

Along a regular arc of cores in the isolated locus, the factor is constant modulo the same subgroup. Use the finite cover by outer neighborhoods in [DIW26, Proposition 3.30]. On an overlap choose a common smaller cluster around the core [DIW26, Proposition 3.19]. Each difference is a union of clusters in proper multiples, and (2) compares the factors. At a wall along a path fixing the core, use the two-parameter model $s = x(x - t)$ of [DIW26, Proposition 3.37]. Along $s = 0$ the two branches are $x = 0$ and $x = t$. At small $s > 0$ the root $(t + \sqrt{t^2 + 4s})/2$ joins the distinguished branch at $t = -\delta$ to the other branch at $t = \delta$. For $s < 0$ the two positive roots at the latter end meet at the transverse fold $t = 2\sqrt{-s}$. Use isolated auxiliary paths as in that proposition. The endpoint insertions persist under the small displacements in s by spatial separation at the two original endpoints. Regular transport on the first arc and the fold relation on the second therefore invert the distinguished factor at the fixed-core crossing.

4.3. **The elementary endpoint.** For an embedded core C with induced complex structure j, choose a spin line L and give $L \oplus L$ the Kähler form $\Omega = \pi^*(\omega|_C) + i\partial\bar{\partial}\|w\|^2$ on a small disk, using equal Hermitian metrics. The symplectic normal bundle of C has first Chern number $2g - 2$. Classification of complex rank-two bundles over a surface gives a unitary normal isomorphism with $L \oplus L$. Choose it as the derivative of a tubular embedding using the symplectic orthogonal to TC. The forms agree as bilinear forms along C. Linear interpolation on a smaller disk and the relative Moser argument, with primitive obtained by fiber contraction, identify the symplectic forms there. Extend the compatible integrable structure from a smaller tube by extending its metric and using polar decomposition with ω. Inside the tube introduce the Lee deformation with small amplitude $T\chi$, where χ is one on an inner disk U and supported in the region of agreement. The resulting ambient J_L

is tame and agrees with the elementary structure on U [IP18, Proposition 3.2 and proof of Theorem 7.1].

Take $C \subset V \Subset U$ a smaller disk bundle. At J_L no cycle contained in $\overline{V}$ meets ∂V, since every irreducible component there lies on C. Compactness at bounded energy excludes such boundary supports for all sufficiently close structures. Containment in V thus gives a compact open and closed insertion over any sufficiently short closed path. At J_L it equals the point insertion of Proposition 3.1. At a nearby generic structure fixing the unobstructed core, as in [DIW26, Proposition 3.54], it is the cluster insertion in a small Hausdorff ball about C. Indeed a sufficiently small ball implies containment in V, and otherwise a sequence of connected supports in $\overline{V}$ outside that ball would limit to C. This proves the comparison inside the original compact target.

Proposition 4.1. *Through degree Λ the image of the compact series in S is a finite product of elementary factors $\Phi(A_g(Q^\gamma))$ and their inverses. Each occurrence carries the genus and integral class of its core.*

Proof. Start with a generic structure and the finite partition of [DIW26, Proposition 3.18]. Follow the generic fixed-core path and its finite subdivision in the proof of [DIW26, Theorem 3.25] to the elementary endpoint just identified. Regular arcs transport the factor, and each crossing inverts it modulo the proper-multiple factors. The sign is $(-1)^w$, where w counts the fixed-core crossings relative to the elementary endpoint. Keep the corrections in the exact identities and replace each by its regular factors as above. For each recursive correction the positive integer $\lfloor \Lambda/\langle [\omega], A\rangle \rfloor$ strictly decreases. The branching and the depth are finite. Finally the insertion for the *entire* β-locus extends from the generic structure to the given projective structure over a tame path by compactness at bounded energy and continuity of the integral class. The endpoint relation in S completes the proof. □

5. Detection by maps of fixed genus

Write $x_{h,m}$ for the scalar classes of [Par25, Definition 4.7]. The map Φ of §2 will only have to distinguish scalar classes of multiplicity one.

Theorem 5.1. *The images $\Phi(x_{h,1})$, $h \geq 0$, are linearly independent over $\mathbb{Q}$.*

We first reduce an arbitrary linear relation to a statement about a single moduli space. Section 6 proves the required nonvanishing there.

5.1. **The image cycle.** Let $Y \to B$ be a family as in §2. Let M_k be the coarse moduli space of connected unmarked stable maps in the fibers, of arithmetic genus k, with nonconstant image and vertical Chern number zero. Write $z_k : M_k \to Z(Y/B)$ for the map taking the image cycle. A component is counted with its degree over the normalization of its image. Contracted components contribute zero.

Lemma 5.2. *The map z_k is continuous and proper.*

Proof. Gromov convergence gives convergence of the weighted currents and the Hausdorff supports. Contracted components have image in the nonconstant part of the image because the domain is connected. For properness, take a sequence above a compact subset of $Z(Y/B)$ and pass to a subsequence whose cycles converge. Work over one closed simplex, in a local target trivialization about the limiting support. Hausdorff convergence puts all the images in one relatively compact neighborhood. The masses of the cycles bound the energies of the maps, with multiplicity included. Indeed one can test against a cutoff Hermitian form of the

limiting parameter, uniformly positive on all nearby complex lines in this neighborhood. For the fixed parametrizing surface in Ivashkovich–Shevchishin's theorem take a closed oriented surface of genus k. Each connected nodal domain of arithmetic genus k is the quotient of such a surface by pinching disjoint circles, as one sees by smoothing its nodes. The boundary condition in their theorem is empty. The images stay in a common compact set, and the structures on its neighborhood converge smoothly after trivialization. Their theorem now gives a convergent subsequence of stable maps. On smooth parts away from the nodes elliptic regularity upgrades convergence to the smooth Gromov topology for these smoothly converging structures. If the limit map were constant, all nearby images would lie in a coordinate ball on which a fixed exact two-form tames the nearby structures. Its integral over a nonconstant closed curve is positive, whereas Stokes' theorem gives zero. Thus the limit is nonconstant and its image cycle is the prescribed limit. The empty cycle has a neighborhood disjoint from all nonempty cycles, and the integer Chern number is locally constant in the Gromov topology. These observations preserve the restrictions defining M_k. The compactness argument applies in metrizable local charts for the two topologies and over a finite union of simplices. For compactness see [IS99, Theorem 1.1], and for the image-cycle map see [DIW26, Definition 2.14]. □

5.2. **A regular map over the least genus.** Use a smooth curve C of genus h and a degree $h-1$ line L outside the theta divisor. For $h=0$ take $\mathcal{O}(-1)$ and for $h=1$ a nontrivial line of degree zero. Serre duality and Riemann–Roch give $H^0(C,N)=H^1(C,N)=0$ for $N=L\oplus K_C L^{-1}$. Every compact degree-one cycle in N is a section. Indeed it has one irreducible component mapping with degree one to C, no compact curve lies in an affine fiber, and the normalization gives the graph of a holomorphic section. Hence the core is isolated in that degree and the point insertion represents $x_{h,1}$, as in §3.

For a stable map over this insertion, exactly one normalized component maps with degree one to C and it maps isomorphically. Arithmetic genus equals the sum of the component genera plus the first Betti number of the dual graph. It is at least h. In the equality case the dual graph is a tree and every other vertex is a contracted rational component. Any nontrivial such tree has a contracted leaf, violating stability in the absence of markings. Thus $z_h^{-1}([C])$ is the single embedded map, with trivial map automorphisms in every genus. Its unparametrized deformation and obstruction spaces are $H^0(C,N)$ and $H^1(C,N)$, since the domain tangent complex cancels the tangential part of the target complex. The map is regular with complex orientation $+1$.

In this splitting the whole degree-one cycle locus is the singleton $[C]$. For both circles its unit represents the local class, by the empty equivariant tuple at Borel stage zero. In particular $a_{h,1}=x_{h,1}$. For $k<h$ the fiber of z_k over this cycle is empty.

Lemma 5.3. *Suppose a finite relation among $\Phi(x_{h,1})$ has least genus k with nonzero coefficient. There is a smooth family of almost-complex six-manifolds over an open subset of $\mathbb{C}^r$ for which the point orientation at one regular map, in a product neighborhood of a marked parameter, has zero image in $H_c^{2r}(M_k;\mathbb{Q})$.*

Proof. Represent the sum on a finite disjoint union of vertex targets, one per nonzero coefficient. In this comparison diagram the filtered-quotient argument of Pardon's Lemma 3.2 applies. Take disjoint unions for common targets. For parallel inclusions, attach a cylinder on the source base between their two images. Use the corresponding pulled-back larger target near each end and the source open target across the middle. Stacking three cylinders permits simplicial attachment even when the endpoint images overlap. The resulting target

is Hausdorff since near the middle slice only the common open target is present, with larger pieces on opposite sides. It is simplexwise smooth. The two inclusions induce equal maps after attachment by the constant-cylinder prism on compactly supported cochains and open extension. A subdivision likewise has a triangulated product concordance. Thus every finite vanishing relation is witnessed in one family X/B after these attachments. Distinct marked vertices of the initial disjoint-union base remain distinct under its inclusion [Par25, Lemmas 3.2–3.4].

At each marked vertex b attach a new closed interval to B by one endpoint only, and use the constant family with fiber X_b on that interval. Its other endpoint b' has an open neighborhood in the enlarged base on which the family is a product. The original base is a subcomplex, so its zero relation still vanishes after inclusion. The compact point insertion crosses the new edge by the interval relation for a constant family. Indeed the insertion extended constantly over the edge has the difference of its vertex insertions as its face boundary. Use distinct new endpoints for the different marks. Relabel the enlarged complex B and use these endpoints as the marked parameters below.

Embed a subdivision of B as a subcomplex in an oriented Euclidean space and take a compact PL regular neighborhood, with B in its interior W. After subdivision there is a PL neighborhood retraction onto B that fixes B. Pull the family back along this retraction. The inclusion of B takes the zero relation to zero by the bivariant pushforward. Equivalently, the dualizing complex of the neighborhood restricts in the interior to $\mathbb{Q}[n]$, and the closed-base pushforward has support on B in that interior. One can use the pushforward directly with target W, by properness of the closed inclusion, so no restriction of compact supports across an arbitrary open set is needed. Increase the ambient dimension if necessary. On W the relation is therefore zero in $H_c^n(Z(X_W/W);\mathbb{Q})$. This is the base thickening used in the proof of Pardon's Theorem 5.15. It requires only a PL retraction for the present families [Par25, proof of Theorem 5.15].

The closed-base pushforward for $i : B \hookrightarrow W$ uses the counit $i_*\omega_B = i_*i^!\omega_W \to \omega_W$. Pull back this counit to the cycle space and use proper base change for the closed inclusion of its restriction over B. This formula also gives functoriality for a vertex followed by i, and commutes with pulling back to the stable-map space by the proper map z_k.

Continuity of the retraction gives a Euclidean ball about each new endpoint mapping into its constant-family neighborhood. Over this ball the targets, cycle spaces and stable-map spaces are products. The singleton cycle is open in the original open target and remains open in the larger fiber by open embedding of the cycle spaces. It is compact there as well. The oriented point Gysin map in the ball is the external product of this singleton insertion and the orientation at the marked parameter. Pullback by z_k computes the two factors separately. At a higher-genus mark the inverse image of the singleton is empty by the genus bound above. At the least-genus mark it consists of the one regular embedded map, so over the ball it is precisely a copy of the ball. The pulled-back class is its point orientation extended by zero, with a nonzero coefficient. This proves the reduction.

It remains to smooth the dependence of the targets on the parameter. Let P denote the compact triangulated PL neighborhood just constructed, with interior W open in Euclidean space. Increase its dimension to an even integer $2r$. The family on P is simplexwise smooth after subdivision. Using the attached constant intervals, choose disjoint Euclidean balls $B_{4\delta}(b) \Subset W$ around the marked parameters on each of which the entire family is a product. Use a common $\delta > 0$ for these finitely many marks. The point orientation on the distinguished stable-map graph maps to zero in $H_c^{2r}(M_k)$ for a putative dependence. Subdivide to mesh smaller than $\delta/4$.

For every vertex v of the subdivision take its open star in P. Choose a partition of unity λ_v with closed supports contained in these stars, smooth on the interior in the Euclidean smooth structure. It can be obtained by restricting a smooth partition of unity from an open Euclidean cover near P. Define $F : P \to P$ by the barycentric combination of the vertices with weights $\lambda_v(y)$. At y, the vertices whose closed supports contain y belong to the least simplex containing y. Since the cover is finite, all other weights vanish in a whole neighborhood of y. Consequently F maps that neighborhood into a *single* simplex and its barycentric coordinates there are smooth in W. The straight-line homotopy from y to $F(y)$ stays in its simplex and moves points by at most the mesh. In particular it preserves the boundary of P.

Modify F to fix the marked balls. Choose a smooth function χ equal to zero on the balls of radius δ and one outside the balls of radius 2δ. Replace $F(y)$ by $(1-\chi(y))y + \chi(y)F(y)$, choosing $0 \le \chi \le 1$. This map remains in the least simplex containing y, moves points by less than $\delta/4$, and agrees with the old map near the boundary of P. Where it changes, its source and image lie in one of the balls of radius 3δ with constant target family. Outside the modifying balls the earlier factorization through one simplex applies. Write F for the modified map.

Set $W' = F^{-1}(W)$. The restriction $W' \to W$ is proper, as a base change of the continuous map of compact spaces $P \to P$, and W' lies in the Euclidean interior of P. Pull the target family back along this map. It is a smooth submersion with smooth vertical almost-complex structure on W'. On the modifying balls it is a product. Elsewhere it is the pullback of a smooth simplex model. On overlaps of simplex models the maps factor through the common face and the prescribed identifications are smooth. The product trivializations are the specified constant families on the attached intervals, so they agree with the simplex models there. For the stable-map spaces this is a Cartesian pullback, including their Gromov topologies, so their induced map is also proper.

For a marked b, the preimage of $B_{\delta/2}(b)$ equals that ball and F is its identity. Indeed any preimage lies within $3\delta/4$ of b by the displacement bound. Consequently the proper pullback of the point orientation on the distinguished graph is the very same point orientation extended by zero. It still vanishes in the global compactly supported group. □

6. The local virtual degree

The putative relation of Lemma 5.3 would kill a regular point orientation in the presence of other, possibly singular and noncompact, components of the moduli space. We use an implicit atlas on a neighborhood of the compact support of such a vanishing. First we construct its pairing with compactly supported cohomology. We then obtain the charts, including their orientation at nodal maps, from fixed-target gluing. In this section references to numbered axioms, deformation spaces and gluing charts are to [Par16].

6.1. **A functional on a finite union of regular footprints.** In an implicit atlas, X_I denotes the thickening belonging to a finite set I of indices, with obstruction space $E_I = \bigoplus_{\alpha\in I} E_\alpha$, section $s_I : X_I \to E_I$, and group $\Gamma_I = \prod_{\alpha\in I}\Gamma_\alpha$. The footprint is the image of the zero locus under the quotient map to the moduli space. The regular footprint uses the regular part of the thickening. The local orientation system is the virtual orientation system of the atlas.

Proposition 6.1. *Let M be locally compact Hausdorff. Suppose we have the chart, footprint and transversality data of an oriented topological implicit atlas of virtual dimension n on a*

neighborhood of a compact set. We use no boundary on these charts. Let O be an open set covered by regular footprints of a finite set of indices A, allowing multi-indices formed from A. All coefficients are rational. There is a functional

$$H_c^n(O) \longrightarrow \mathbb{Q}$$

for this choice of charts. Suppose a point $p \in O$ has an oriented manifold neighborhood of dimension n in the zero locus. For some index set $I \subset A$ assume its zero-coordinate lifts near p are regular, are freely permuted in fibers of size $|\Gamma_I|$, and that s_I on this regular thickening is a submersion there. Orient the neighborhood of p by the virtual orientation. The functional sends the point orientation class extended by zero to $+1$.

Proof. For $I \subset J \subset A$ take the deformation space $Y_{IJ} = X_{I,J,A}$ of Pardon's Definition 4.2.1. It is a manifold with corners of dimension $n + \dim E_A + |I|$. Write $e : Y_{IJ} \to E_A$ for its obstruction projection. On $e^{-1}(0)$ there is a continuous footprint map to the unthickened space. Delete the part of $e^{-1}(0)$ lying over the complement of O, an ordinary closed subset, and call the resulting open manifold Y_{IJ}^O. Write Z_{IJ} for its full zero locus and $\pi : Z_{IJ} \to O$ for the footprint map. These restrictions commute with the maps (4.2.1) for the one fixed set A. Away from $e = 0$ no map to O is required.

Let O^+ be the one-point compactification (with a disjoint point if O is compact). To a compact $K \subset O^+$ associate the complex of relative singular chains

$$F_{IJ}^\bullet(K) = C_{n+\dim E_A - \bullet}\big(Y_{IJ}^O, Y_{IJ}^O \setminus \pi^{-1}(K \cap O); o_{E_A}^\vee\big)^{\Gamma_A}.$$

Here $\pi^{-1}(K \cap O)$ is closed in Y_{IJ}^O, including when K contains infinity. Use the normalized transition maps of Definition 4.2.6 and take their finite homotopy colimit $F^\bullet$. This is the one-point compactification argument of Pardon's proof of Lemma A.6.4 applied to each of the deformation spaces.

We give the descent checks. The assignments on compact subsets are homotopy K-sheaves. The value at the empty set is zero, and the two-compact-set axiom is singular-chain excision for the open complements. For decreasing compact neighborhoods, a singular chain has compact image. Its intersection with Z_{IJ} is compact, and the image of that intersection in O^+ is compact. If the chain misses $\pi^{-1}(K \cap O)$, its footprint therefore misses a compact neighborhood of K as well. This proves continuity under shrinking neighborhoods, including at infinity. Finite group invariants are exact over $\mathbb{Q}$. Finite homotopy colimits preserve these conditions.

At each $x \in O$, the stalk calculation is the one in Lemma 4.3.1 and Proposition 4.3.3. Excision away from the deleted zero loci does not change the relative chains at x. The footprint diagrams give $o_{M,x}$ in degree zero and zero in every other degree. Coverage is required only over O. At infinity every relative chain complex is zero. For the uniform lower bound in the purity criterion, use Poincaré–Lefschetz duality with closed support in the finite-dimensional manifolds Y_{IJ}^O, as in Lemma A.6.4. The relative homology vanishes above their dimensions. There are finitely many pairs and finitely many nondegenerate strings in their indexing poset. Thus their shifted homotopy colimit still has a uniform lower bound, also on neighborhoods of infinity. The resulting degree-zero sheaf is $j_! o_O$ for $j : O \hookrightarrow O^+$, by adjunction and the stalk calculations. Proposition A.5.4 now identifies $H^*(F^\bullet(O^+))$ with $H_c^*(O; o_O)$ [Par16, §§4.2–4.3, Proposition A.5.4 and Lemma A.6.4].

At $K = O^+$ the complement in each pair is the complement of the *whole* zero set in Y_{IJ}^O. Projection to E_A therefore defines a map to the relative chains of $(E_A, E_A \setminus 0)$. In the convention where the transition $J \to J'$ is the ordinary pushforward divided by $|\Gamma_{J \setminus J'}|$, take

the projection on the IJ term divided by $|\Gamma_J|$. Indeed $|\Gamma_J| = |\Gamma_{J'}||\Gamma_{J\setminus J'}|$, and the obstruction projection is unchanged by the transition. This gives a chain map from the homotopy colimit by the augmentation on strings of length zero, with value zero on higher strings. In degree n, identify the resulting obstruction-space orientation class with 1. This gives the functional. Applying the projection directly to chains relative to a proper compact subset of a larger zero set would lack this map of pairs.

Normalization at the regular point. Let $V_I \subset O$ be the regular footprint for I, intersected with O. In the diagonal term $I = J$, take the slice $t_i = 1$ for $i \in I$. It is the open part of $E_{A\setminus I} \times X_I^{\mathrm{reg}}$ obtained by deleting zeros over the complement of O. Its obstruction projection is $(e', x) \mapsto (e', s_I(x))$. Its entire zero set is closed in this open manifold and has proper finite quotient onto V_I. Indeed it is the restriction of the finite group quotient of the zero-coordinate chart to its invariant open regular locus.

Here is the slice calculation directly from Definition 4.2.1. Denote the slice by D and its zero set by Z_D. Before removal of the zeros outside O the closed set over a compact $K \subset O$ in this slice is

$$\{0\} \times \big((s_I|_{X_I^{\mathrm{reg}}})^{-1}(0) \cap \psi_{\emptyset I}^{-1}(K)\big) \;\subset\; E_{A\setminus I} \times X_I^{\mathrm{reg}}.$$

The pair uses the complement of this entire set. This detail matters in using (4.3.14), whose displayed complement takes the product of $E_{A\setminus I}$ with the complement in the second factor. For example, take a point with trivial groups, $X_\emptyset = \{p\}$ regular, and one additional chart $X_\alpha = \mathbb{R}$ with $s_\alpha = \mathrm{id}$. For $I = J = \emptyset$, Definition 4.2.5 gives $H_1(\mathbb{R}, \mathbb{R}\setminus\{0\}) = \mathbb{Q}$. The pair $(\mathbb{R}, \varnothing)$ would give zero in degree one.

For any compact $K \subset O^+$ put $L = \pi^{-1}(K \cap O)$ in $Y = Y_{II}^O$. On the interior Y°, all $t_i > 0$ and (t, e', x) are product coordinates, with $e_i = s_i(x)/t_i$. In these coordinates the pair $(Y^\circ, Y^\circ \setminus L)$ is $(\mathbb{R}_{>0})^I$ times $(D, D\setminus(L\cap D))$. Retraction of the first factor to $(1, \dots, 1)$ proves the slice isomorphism on the interior. The inclusion of this interior pair into $(Y, Y\setminus L)$ induces an isomorphism as well. To see it, apply Poincaré–Lefschetz duality (A.6.6) to the closed set L in Y and to its intersection with Y° in the open manifold Y°. The coefficient sheaf on L in the first duality is the extension by zero of the orientation sheaf from $L \cap Y^\circ$. Inclusion of chains is the corresponding extension on compactly supported cohomology. This naturality follows from the relative-chain map in the proof of Lemma A.6.4, which is the orientation identification on interior stalks and has zero source at boundary stalks. Both dual groups are therefore the same. Tensoring with $o_{E_A}^\vee$ and taking finite-group invariants preserves the isomorphism. The argument also covers $K = O^+$, where L is the whole zero set.

Applying Lemma A.6.4 once more to Z_D in the slice, then taking finite-group invariants as in (4.3.7), identifies the diagonal term on O^+ with $H_c^*(V_I; o_O)$. The map of this term into the homotopy colimit induces the identity of orientation sheaves on V_I by Proposition 4.3.3. Its source sheaf is extension by zero from V_I. By adjunction this specifies the map of sheaves on all of O^+, so on cohomology it is precisely extension by zero $H_c^*(V_I; o_O) \to H_c^*(O; o_O)$.

Take a ball at p on which the zero-locus cover splits into $|\Gamma_I|$ sheets. Pullback of its point orientation to the cover gives coefficient one on each sheet. In the relative chains of the slice these classes are small oriented normal disks at the lifts. Submersivity lets us choose them with boundary disjoint from the entire zero set and obstruction projection of local degree one. In chains with obstruction orientation coefficients, first average one disk over $\Gamma_{A\setminus I}$, which fixes its lift and acts simultaneously on the complementary obstruction coordinates and their orientation coefficients. Its projected degree remains one. Sum its translates over Γ_I to obtain the invariant representative on the free sheets. The previous paragraph identifies

its image in the full complex with the point orientation extended by zero. Projection of the orbit sum and division by $|\Gamma_I|$ therefore give one. This also works when regularity of the unperturbed *fixed-domain* operator fails. It uses an ordinary submersion in one nonempty chart over the smooth ball. □

If the local orientation at p vanished in $H^n_c(M)$, it would already vanish with support in a compact set K_0 containing p. Choose finitely many regular footprints covering K_0 and an open O inside their union containing K_0. Excision for support in K_0 would make its image in $H^n_c(O)$ zero. The functional contradicts this.

6.2. **An implicit atlas for maps in the fibers.** Let $\pi : E \to W$ be the smooth family of Lemma 5.3, with W open in $\mathbb{C}^r$ and $n = 2r$. Choose a smooth horizontal distribution and give E the almost-complex structure given by the vertical structure and the lift of the standard structure on W. Every connected compact holomorphic map to E projects to a constant. Holomorphic functions on each normalized component are constant and agree at nodes. Thus unperturbed fixed-target maps are exactly the maps into the fibers, including their Gromov topology. Their Chern number in E equals their vertical Chern number since TW is trivial.

The fixed-target construction can be used separately in each homology class of maps to E. This class is locally constant in the labeled-domain topology: on nearby domains collapse the plumbing necks to the nodes, compare maps into a small tubular neighborhood of the limiting graph, and use short geodesics in E to homotope to the limiting map composed with the collapse. The collapse sends the fundamental cycle to the sum of the fundamental cycles of the limiting components. Equivalently one can integrate the corresponding integral currents in a compact neighborhood and use local constancy of the integral current homology class in the flat topology. A compact set of maps meets only finitely many such sectors. The test also restricts to nonconstant maps. A constant map has an image neighborhood in a coordinate ball with an exact taming form, which excludes a closed nonconstant holomorphic curve in that ball.

Use the indices of [Par16, §9.2] for the fixed target E with *vertical* perturbation sections. Use actual codimension-two incidence disks in E and the usual marked-domain orbifold charts. The positivity of rank of the differential on every unstable nonconstant component gives the stabilizing disks of Lemma 9.2.8. In a small target trivialization one can thicken a transverse disk in the fiber in all base directions and round its edge away from the marked intersections. For each finite index set I, take the fixed-target thickening with these vertical perturbations. Every solution, at arbitrary vertical obstruction coordinates, still projects holomorphically to W and lies in a single fiber. The spaces, group actions, footprint quotients, obstruction coordinates and forgetful maps are therefore the ordinary data of Definitions 9.2.1–9.2.4. The finite group need not act effectively on its marked-domain chart.

Choose indices on sufficiently small marked-domain charts. On each one we can also specify a complex Γ_α-representation P_α of dimension kr, where k is the fixed arithmetic genus, and equivariant perturbation forms whose projection induces an isomorphism

$$P_\alpha \;\simeq\; H^1(T, \mathcal{O}_T) \otimes \mathbb{C}^r$$

on each labeled domain T in the chart. Here is the construction. At the central stabilized curve use the right-hand space as the representation. Smooth $(0,1)$ forms supported in the smooth locus away from the marked points span its first factor by duality with dualizing differentials. Such a differential vanishing on the smooth locus is zero, including at every

branch of a node. Choose a complex linear right inverse and average over the finite group. The forms extend on disks in the smooth locus of the universal curve. Extend and average again, keeping the original equivariant values at the center. The classes still give a basis after shrinking the chart. Indeed the dual Hodge bundle is locally free for the proper flat family of connected nodal curves of fixed genus. In a Hodge frame the pairing with these disk forms is continuous up to the central fiber, by integration over a fixed compact set away from the nodes. The extension may be chosen smooth on the total smooth locus. This also handles normalization components on which the map is constant. Take the horizontal lifts of the resulting $\mathbb{C}^r$-valued forms on $\mathcal{C}_\alpha \times E$. They are allowed perturbations in Definition 9.2.1(vi) of [Par16]. For relative duality and base change see [Stacks, Tags 0E6N and 0E6S].

For example, take the rational nodal genus-one curve obtained by identifying 0 and ∞, with a stabilizing mark at 1. The involution $\tau(z) = 1/z$ acts by -1 on the dualizing differential dz/z, hence by -1 on $H^1(\mathcal{O}_T)$. Choose a form ϕ supported on a disk near 2 pairing nontrivially with dz/z. The form $(\phi - \tau^*\phi)/2$ is anti-invariant and has the same nonzero pairing. An invariant horizontal coordinate would miss this cokernel. Taking P_α with its actual stabilizer representation deals with this node without passing to an unlabeled quotient.

For each nonempty I select for the moment $\alpha \in I$. Replace the datum at α by the same datum with obstruction space $E_\alpha \oplus P_\alpha$ and the additional horizontal perturbations. Keep every other datum unchanged. This is an ordinary collection of Pardon's fixed-target indices. On any simultaneous thickening containing α the projected perturbed equation reads

$$\bar\partial w + \eta_\alpha(h) = 0, \qquad w = \pi u, \quad h \in P_\alpha.$$

Taking its Dolbeault class gives $h = 0$, and then w is constant. Conversely every solution of the original vertical thickening is a solution with $h = 0$. This is a homeomorphism of the two thickenings. All marks and domain labels are identical, the graph topologies agree, and the extra coordinate on one side is constantly zero. If the distinguished index is retained, the homeomorphism also intertwines the footprint maps on the locus where the new obstruction coordinates are zero.

The regularity test in Definition 9.2.5 is independent of which $\alpha \in I$ is chosen for this purpose. At a solution the horizontal block of the fixed-domain linearization is $(\xi, h) \mapsto \bar\partial\xi + \eta_\alpha(h)$. Its cokernel is zero, and its kernel is the constant horizontal displacement T_bW. Derivatives of the horizontal forms with respect to domain labels have coefficient $h = 0$. In the kernel of this block the remaining equation is exactly the vertical linearization with free base value b. Transverse intersections with incidence disks, used to define the labels in (9.2.3), move smoothly with the map on the fixed domain. Thus the full augmented operator is surjective if and only if the vertical operator with free base is surjective. Triviality of the automorphism group is the same on both sides.

Define regularity on nonempty thickenings by this intrinsic test, and set the regular locus of the empty chart to the empty set. For $I \subset J$ with I nonempty, augment *one index in I* on both sides of the inclusion. Under the two homeomorphisms just proved, fixed-target Proposition 9.2.6 gives the required openness and the topological submersion onto $E_{J\setminus I}$. The auxiliary space never occurs in this quotient. For I empty the submersion and regular-footprint freeness requirements have empty domains of verification. The footprint and quotient axioms still hold there without a regularity assumption. The dimension in the augmented model is

$$2r(1-k) + \dim_{\mathbb{R}} E_I + 2rk = n + \dim_{\mathbb{R}} E_I$$

for unmarked maps in the Chern-zero sector.

For coverage choose a stabilizing disk and finite-dimensional vertical perturbations spanning the fixed-domain *vertical* cokernel, as in the proof of Lemma 9.2.9. At the unperturbed map the label derivatives of these vertical perturbations have zero coefficients. Perturbations supported away from all special points suffice by that lemma's elliptic argument. Extend the sections locally in the vertical tangent bundle and use the induced finite group representation for equivariance. The choice of a labeled domain chart removes the residual automorphisms of the lifted object. This produces a nonempty regular footprint at every point. The construction uses compactness of the domain curves, and only finitely many indices are needed near a compact K_0. Pardon's compact-target assumption in the global GW application supplies compactness of the entire unperturbed space. Proposition 9.2.6 itself is a gluing calculation near a single compact map. Its stabilized fixed-target model in Appendix B.1 uses a smooth almost-complex target and transverse disks. All the maps in this local calculation stay near the fixed compact image. Local compactness of the unperturbed space also follows from the regular footprints themselves. Each is an open set in the Hausdorff moduli space and the finite-group quotient of the closed zero locus in a locally compact regular chart. These footprints cover.

6.3. **Orientation on overlaps.** At a smooth-domain regular point of X_I, retain the same domain labels and incidence disks. For $\alpha, \beta \in I$, write $H_T = H^1(T, \mathcal{O}_T) \otimes \mathbb{C}^r$ and denote the two maps to this quotient by $A_\alpha : P_\alpha \to H_T$ and $A_\beta : P_\beta \to H_T$. Identify P_α with P_β by the complex isomorphism $G = A_\beta^{-1} A_\alpha$. On the common linearized domain consider the horizontal perturbation forms

$$\eta_s = (1-s)\eta_\alpha + s\,\eta_\beta G, \qquad 0 \le s \le 1.$$

Every η_s projects to A_α. Thus $(\xi, h) \mapsto \bar\partial\xi + \eta_s(h)$ is onto for the whole path, and its kernel always has $h = 0$ and ξ constant. This also describes the horizontal equation with domain variations, since the projection of the original map is constant and the horizontal perturbation coefficient at the solution is zero. On this kernel the remaining linear equation, including the old perturbations and moving incidence marks, is the same for every s. Regularity therefore gives a path of surjective linearizations with the very same tangent kernel. The canonical determinant orientation varies continuously along that path in the fixed kernel. Identifying its two auxiliary factors by G preserves their real orientations. This proves equality of the two orientations of the regular thickening at the smooth point. Only a path of linear operators at that point is needed. The construction commutes with relabeling, since G is defined by the induced maps to the same horizontal cohomology group.

Each augmented chart already has a continuous orientation identification by Proposition 9.2.6(iii). Both live on the same topological regular locus by the homeomorphisms above. By [Par16, §B.12] the plumbing parameters can all be nonzero arbitrarily close to every nodal point of that locus. The two identifications therefore agree there as well, since their ratio is a locally constant sign. For $I \subset J$ with I nonempty, choose an index in I on both sides. On the regular footprint the forgetful map at zero extra coordinates is a finite covering of an open set in X_I^{reg}. Smooth-domain lifts are dense on this covering. On these lifts the ordinary determinant exact sequence for adding $E_{J\setminus I}$ gives the orientation transition for the vertical charts, since the same oriented P_α cancels from both sides. Continuity gives the transition at the nodal lifts. For a triple of nested nonempty index sets one may choose the same index in the smallest one throughout, so the transition maps compose. All extra complex factors have

even real dimension. This verifies orientation descent on the regular nodal overlaps used by the compact-support test [Par16, Proposition 9.2.6 and §§B.12–B.13].

Proof of Theorem 5.1. Use the regular point over the least genus in Lemma 5.3. Its normal bundle is acyclic, its map automorphism group is trivial, and the family is constant over a ball at its marked parameter. In any nonempty regular chart covering this point the map to the vertical obstruction coordinates is a submersion near its lift. Indeed the unperturbed linearization with domain variations is onto. On this smooth domain the implicit function theorem solves the perturbed equation with arbitrary prescribed small obstruction coordinates. Transverse intersections recover the stabilizing marks. The kernel is the tangent to the graph of the unperturbed solutions, oriented by the complex normal operator and the base. There are $|\Gamma_I|$ lifts. Proposition 6.1 therefore pairs the local point orientation to $+1$ on a neighborhood of every compact set containing it. The vanishing of Lemma 5.3 is impossible. □

7. Lifting the product to the complex cycle algebra

7.1. **Stabilization of the integers.** At cutoff Λ, let $N^{\Lambda}_{\gamma,h}$ be the sum of the signs of the elementary factors of genus h and core class γ in Proposition 4.1. These integers are zero outside a finite set of pairs. Suppose two decompositions, possibly one obtained by truncating a decomposition at a higher bound, give different integers. Cancel their common lower-degree factors and take a class β in the first degree of disagreement. Because every nonconstant term of a factor in class γ has degree at least that of γ, comparison of the coefficients of Q^{β} gives

$$\sum_{h}(N^{\Lambda}_{\beta,h}-\widetilde{N}_{\beta,h})\,\Phi(x_{h,1})=0.$$

Here we used $a_{h,1}=x_{h,1}$. Theorem 5.1 gives equality in every genus, a contradiction. The same induction proves that $N^{\Lambda}_{\beta,h}=0$ when β is ineffective for the original projective structure. The compact series has coefficient zero there, and sums of effective lower labels are effective.

For the chosen polarization denote the stabilized counts by $n_{\beta,h}$. Taking Λ equal to the degree of β proves finite genus support at β. Thus Proposition 4.1 has already given the product of Theorem 1.1 after application of Φ. We need only multiplicity-one independence to lift this identity.

7.2. **Primitive classes.** Let $P=\mathbb{Q}[X_{h,m}:h\geq 0,\ m\geq 1]$ and $X_{h,0}=1$. The map $p:P\to H$ taking $X_{h,m}$ to $x_{h,m}$ is surjective by the geometric generation theorem [Par25, Theorem 6.13]. The Chern numbers of the generators of virtual dimension zero are nonnegative, so only generators of Chern number zero can enter this summand. The scalar classes respect the coproduct Δ from addition of cycles and pullback ρ_d under multiplication by d. Their formulas are

$$\Delta X_{h,m}=\sum_{r+s=m}X_{h,r}\otimes X_{h,s},\qquad \rho_d X_{h,m}=\begin{cases}X_{h,m/d} & d\mid m,\\ 0 & d\nmid m.\end{cases}$$

For the scalar classes, apply the division formula and the diagonal calculation of [Par25, §4, in particular Lemma 4.13]. Each term with a nonzero shift has one tensor factor of negative virtual dimension and thus vanishes by [Par25, Theorem 6.13]. The same vanishing removes the shifted terms in the balanced calculation [Wue26, §3]. Thus both operation formulas use only the geometric generation and dimension statements on the complex side. The augmentation on both algebras is restriction to the empty cycle.

In the triangular coordinates $v_{h,m} = [q^m] \log(\sum_{j\geq 0} X_{h,j} q^j)$ the generators are primitive and $\rho_d v_{h,m} = v_{h,m/d}$ for $d \mid m$, zero otherwise. Suppose the kernel of p contains a nonzero polynomial f of smallest total polynomial degree in these coordinates. Its constant term is zero by augmentation. Its reduced coproduct lies in the tensor square of polynomials of strictly lower degree. The map p is injective on this vector space and remains so on its tensor square. Since the reduced coproduct maps to zero, f is primitive. Over $\mathbb{Q}$ an additive polynomial is linear, by differentiating $f(v+w) = f(v) + f(w)$ at $w = 0$. Applying ρ_j for the largest multiplicity j occurring in f gives a nonzero formal combination of the $X_{h,1}$ in the kernel. Their images even under Φp are independent by Theorem 5.1. Hence p is an isomorphism.

Lemma 7.1. *Let* $\mathrm{Prim}(H)$ *denote the primitive subspace for* Δ*. Then*

$$\mathrm{Prim}(H) \cap \bigcap_{d>1} \ker \rho_d = \bigoplus_{h\geq 0} \mathbb{Q} x_{h,1}.$$

Proof. In the coordinates $p(v_{h,m})$ a primitive is an additive polynomial, hence linear. If its largest multiplicity is $j > 1$, applying ρ_j leaves exactly the coefficients at multiplicity j, now on the linearly independent classes $x_{h,1}$. Conversely, each $x_{h,1}$ is primitive and is killed by every ρ_d with $d > 1$. □

7.3. **The first coefficient of disagreement.** Write F for either the compact series or the product with the stabilized exponents, formed in $H[\![\Gamma_X]\!]$. Each has $F_0 = 1$ and satisfies

$$\Delta F_\beta = \sum_{\gamma+\delta=\beta} F_\gamma \otimes F_\delta, \qquad \rho_d F_\beta = \sum_{\substack{\gamma\in\Gamma_X \\ d\gamma=\beta}} F_\gamma. \tag{3}$$

For the compact series take inverse images of the entire class locus under addition and multiplication. For the other series use the formulas on the balanced elements, multiply, and take inverses for negative exponents. The substitution $Q^\gamma \mapsto Q^{d\gamma}$ is multiplicative even when several exponents have the same image. Each effective root occurs separately in (3).

If R_β is a nonzero difference at a least ample degree, the terms with two positive summands cancel in its coproduct. It is primitive. For $d > 1$ all the roots on the right of (3) have smaller degree, so $\rho_d R_\beta = 0$. By Lemma 7.1, R_β is a linear combination of the $x_{h,1}$. The finite-energy identity says $\Phi(R_\beta) = 0$. Theorem 5.1 now forces $R_\beta = 0$. This proves the product formula. Uniqueness follows by comparing the first degree of disagreement between any two systems of exponents. A second polarization yields an identity in the same coefficients. For either polarization there are finitely many effective sublabels of any given β, and their possible decompositions are finite. Uniqueness therefore identifies the two systems.

7.4. **Identification with the elementary-cluster coefficients.** It remains to identify the integers already obtained with those of the connected Gromov–Witten expansion. Apply fixed-target connected counts to the *geometric decompositions* of §4. For a compact open insertion u let $\epsilon(u)$ be its value at the empty cycle, and let $D_k(u)$ be its localized positive-energy connected genus-k count. The same open part of the stable-map space computes this count on excision into a target neighborhood. For the compact symplectic targets and paths in that section the localized counts are additive on partitions and invariant under transport [DIW26, Lemma 3.8 and Proposition 3.10]. On disjoint targets a connected map lies in just one of the two, hence

$$D_k(uv) = D_k(u)\epsilon(v) + \epsilon(u)D_k(v).$$

Extend the equality coefficientwise with the integral class labels. Each of our normalized series has empty-cycle value one as a series. Thus connected counts add under their multiplication and change sign on inversion, which can equally be calculated in the truncated series algebra. This applies along the exact fold, subdivision and endpoint relations used to obtain each finite product.

For genus h the elementary connected contribution is

$$G_h(q,t) = \log\left(1 + \sum_{m\geq 1} q^m \sum_{\lambda\vdash m} \prod_{\square\in\lambda} \bigl(2\sin(\ell(\square)t/2)\bigr)^{2h-2}\right),$$

where $\ell(\square)$ is the hook length [IP18, Proposition 3.3]. Thus the signed occurrences of each factor give the coefficients of $G_h(Q^\beta, t)$ in the connected expansion of X. Uniqueness is [DIW26, Theorem 3.50]. At the first degree of disagreement, the least remaining genus is distinguished by $[q]G_h(q,t) = t^{2h-2} + O(t^{2h})$. This identifies the integers of (1) and completes the proof of Theorem 1.1.

8. A double curve of genus two

In genus two the first difference between the two circle actions can be calculated inside the complex cycle algebra. The identity also follows under numerical reconstruction in [Wue26, §6].

Proposition 8.1. *In H one has $a_{2,2} = x_{2,2} - 2x_{4,1}$.*

Proof. Let C be a smooth genus-two curve with hyperelliptic involution ι. Choose distinct non-Weierstrass points p, q in distinct orbits. Choose L with $L^2 = \mathcal{O}(p+q)$, possible by surjectivity of multiplication by two on the Jacobian. Put $M = K_C L^{-1}$. The divisors of chosen sections s_L, s_M of L^2, M^2 are $p+q, \iota p+\iota q$, respectively. Indeed $p+\iota p \sim q+\iota q \sim K_C$. Each section spans its space of global sections by Riemann–Roch, since neither divisor is canonical. A section of L would give a square section with a double zero, which is impossible. The same holds for M. Also $ML^{-1} = K_C(-p-q)$ is a nontrivial degree-zero line bundle. By the deformation of balanced pairs in §3 and the scalar construction [Par25, §4.3], both classes can be computed on this splitting.

In $Y = \mathrm{Tot}_C(L \oplus M)$, every irreducible compact curve projects finitely onto C. A degree-two cycle has zero trace in both fiber coordinates u, v, because $H^0(L) = H^0(M) = 0$. Its two fiber points, with multiplicity, thus have the form $\pm(u, v)$. The three quadratic invariants give global sections

$$u^2 = a s_L, \qquad v^2 = b s_M, \qquad uv = d \in H^0(K_C).$$

They extend over ramification by symmetric evaluation on the finite maps from the normalizations of the components. Since $d^2 = ab\, s_L s_M$ and the four zeros of $s_L s_M$ are simple, $ab = 0$ and $d = 0$. Conversely each point of the two coordinate axes gives a cycle. On the first axis it is $D_a = \{u^2 = a s_L,\ v = 0\}$, and there is an exchanged formula on the second. The origins are the same cycle $2[C]$. These parametrizations are continuous, including at zero, by the finite flat families of spectral divisors. Their inverse is continuous by fiber evaluation away from the zeros. This identifies the underlying cycle space equivariantly with the union of the two complex lines. For nonzero amplitude each cover is smooth and connected, with two simple branch points. Riemann–Hurwitz gives genus four. The two independent scalings of L, M act on a, b with characters $(2,0), (0,2)$.

The two Borel classes. Set $B = \mathbb{P}^1 \times \mathbb{P}^1$ and let h_1, h_2 be the positive hyperplane classes. Twist L, M by the hyperplane lines from the first and second factors, respectively. The relative degree-two cycle space W is the union of the total spaces $P_+ = \operatorname{Tot}\mathcal{O}(2,0)$ and $P_- = \operatorname{Tot}\mathcal{O}(0,2)$, glued along their zero sections B. Write U_+, U_- for the complex Thom classes of these two line bundles. Use the two base inclusions

$$i_s(z) = (z, z), \qquad i_a(z) = (z, \bar{z}),$$

orienting the source by its complex orientation in both cases. The first restricts the target family to the scalar Borel family. On the second, the second hyperplane line identifies real analytically and fiberwise complex linearly with $\mathcal{O}(-1)$ by the Hermitian metric. Thus its restriction is the antidiagonal Borel family. A real analytic base map and real analytic bundle identification suffice here. On real simplices they complexify to the family germs of [Par25, Definition 3.1]. The Gysin map of either embedding preserves the universal image. One can use a product triangulation subdivided along the diagonal, and transport it by conjugation of the second factor for the second graph. Both give the canonical concordance class of immersed real analytic triangulations of B. In the shifted compactly supported description the pushforward is the oriented normal Thom map, as in Pardon's diagonal computation [Par25, §3.1, Lemma 3.5 and proof of Lemma 4.13].

At Borel stage one the scalar cut is $(a+b)^*\mathrm{Th}/2$, where the sum is in $\mathcal{O}(2)$. For the antidiagonal cut use $a+j(b)$, where the Hermitian conjugate-linear identification $j : \mathcal{O}(-2) \to \mathcal{O}(2)$ reverses the real fiber orientation. Again divide by two. Each cut vanishes exactly on the common zero section. These choices represent the published normalizations. To check this even for the continuous test j, use the standard Hermitian metrics at every finite Borel stage. The tests restrict along hyperplanes, and each ordinary normalized Euler class is h. Append a defining tuple from [Wue26, §2] to the one-test tuple and append this test to the defining tuple. The two resulting compactly supported Thom products coincide at the common stage by naturality and the projection formula of the stabilization lemma. The same argument identifies the scalar test with that of [Par25, Definition 4.7].

The Poincaré duals of the two base graphs are $h_1 + h_2$ and $-h_1 + h_2$, respectively. For the second sign, integrate h_1 and h_2 on the graph to obtain $1, -1$. Naturality of the normal Thom map under restriction to either branch gives the following table in $H_c^4(P_\pm; \mathbb{Q})$.

	P_+	P_-
scalar	$(h_1+h_2)U_+/2$	$(h_1+h_2)U_-/2$
antidiagonal	$(-h_1+h_2)U_+/2$	$(h_1-h_2)U_-/2$
difference	$-h_1U_+$	$-h_2U_-$

These restrictions determine the class on W. For the proper map $\nu : P_+ \sqcup P_- \to W$ the difference of restrictions to B gives the exact sequence $0 \to \mathbb{Q}_W \to \nu_*\mathbb{Q}_{P_+\sqcup P_-} \to \mathbb{Q}_B \to 0$. There are no higher direct images since the fibers are finite. Now $H^3(B) = 0$, so restriction of $H_c^4(W)$ to the two branches is injective.

For the positive class $(h_1U_+, 0)$, fix one point on the first base factor, trivialize its hyperplane line there, and fix a nonzero amplitude a_*. Over the remaining base $\mathbb{P}^1$, take the complex point class of a_* in the amplitude line and extend by zero from a disk avoiding zero. Pushing forward from the base slice gives precisely $(h_1U_+, 0)$ by the Thom isomorphism. Its support misses the other branch. The exchanged construction represents $(0, h_2U_-)$. It remains to compute the universal image of the class on each base slice.

The class of a branch. We treat the first branch. The family over the remaining $B' = \mathbb{P}^1$ has first summand L and second summand $M \otimes \mathcal{O}_{B'}(1)$. Write $S = \mathrm{Tot}_C L$. Deform the threefolds as affine line bundles over S, retaining the projection to S. Choose independent sections σ_0, σ_1 of $\mathcal{O}_{B'}(1)$. Let z_0 be the simple zero of σ_0. Choose a nonzero class

$$c \in H^1(C, ML^{-1}) \simeq \mathbb{C}.$$

Cover C by a disk V avoiding the zeros of s_L, and the complement U of its center. These two Stein sets and their intersection compute this cohomology. Represent c on $U \cap V$. For $t \in [0,1]$, add the following translation to the usual transition of the second fiber coordinate, written intrinsically in the common line over the overlap. The added term is

$$t\, c\, u \left[\sigma_0 + (u^2/s_L - a_*)\sigma_1\right].$$

This is holomorphic on the whole overlap, including $u = 0$. The two-set gluing with inverse translation defines a holomorphic family, also for complex t. The relative tangent determinant in each threefold is the pullback of $TC \otimes L \otimes M$, up to a constant line on the fiber, so the cycles stay semi-Fano.

Fix a in a small disk about a_* avoiding zero. Any cycle of degree two with this amplitude projects onto the smooth spectral divisor $D_a \subset S$ with degree one. Indeed its projected cycle is determined by $u^2 = as_L$, and no component can lie in an affine line fiber. The finite degree-one map to the smooth divisor is an isomorphism on the normalization and the lift is a section of the affine line bundle over D_a. Conversely every such section gives the embedded cycle. Write $\pi : D_a \to C$. The identity $\pi_* \mathcal{O}_{D_a} = \mathcal{O}_C \oplus L^{-1}$ gives

$$H^0(D_a, \pi^* M) = 0, \qquad H^1(D_a, \pi^* M) = H^1(C, ML^{-1}).$$

The lifting obstruction is $t[\sigma_0(z) + (a - a_*)\sigma_1(z)][c\, u]$. Here multiplication by u identifies c with its nonzero class in the odd summand of the displayed direct image. Hence a lift exists precisely when this expression vanishes. When it does, the translations on the divisor are themselves zero, and the zero section in the two local charts gives the unique lift.

Pull back the Thom class of $a - a_*$ on this amplitude disk. It defines a supported class throughout the prism $B' \times [0,1]$. Its zero set consists of the section D_{a_*} over all of B' at $t = 0$, together with D_{a_*} over z_0 for every t. This set is compact. The amplitude is continuous by fiber evaluation in the projection to S. On the zero-obstruction locus the explicit local zero graphs vary continuously by their spectral equations. Thus the parametrization above is also a homeomorphism over the disk. Extension by zero and the oriented prism relation identify the endpoint insertions. At $t = 1$, the relative cycle space near the support is the graph

$$a = a_* - \sigma_0(z)/\sigma_1(z).$$

The function on the right has nonzero derivative at z_0. The insertion has local degree $+1$ on this graph. Projection of the graph to the base is a complex orientation preserving local isomorphism, so the Gysin image from the fiber over z_0 of the unit on the isolated curve is exactly this class.

For regularity in that fiber, the normal sequence of $D = D_{a_*}$ has subbundle $\pi^* M$ and quotient $\pi^* L^2$. The latter is the normal line to D in S, by its divisor equation. Its space of sections is generated by the amplitude variation $da = 1$, since $H^0(L^2) = \mathbb{C} s_L$ and $H^0(L) = 0$. The normal connecting map sends this variation to $\pm\sigma_1(z_0)[c\, u]$, by differentiating the transition on D_a at $a = a_*$. It is an isomorphism. Also $H^1(D, \pi^* L^2) = H^1(L^2) \oplus H^1(L) = 0$. It follows that $H^0(N_{D/Y_{z_0,1}}) = H^1(N_{D/Y_{z_0,1}}) = 0$.

Finally deform the ambient germ along this smooth curve to the total space of its normal bundle N. One model is the blow-up of $Y_{z_0,1} \times \mathbb{C}$ along $D \times 0$, with the strict transform of the central fiber removed. The copy of D persists and is isolated and regular in every fiber, including the normal bundle fiber. Near a smooth embedded curve of multiplicity one the relative cycles are embedded deformations, and vanishing of both normal cohomology groups makes the relative space locally the base. Thus the section over a closed parameter interval is compact and open in the relative cycle space there, and its unit continues to the zero section of N. Every degree-one compact cycle in N is a holomorphic section, so its degree-one cycle space is the single point $H^0(D, N) = 0$. In [Par25, Definition 4.7] one can take Borel stage zero and the empty tuple, since the whole degree-one locus is compact and its class is the unit. To identify the indices, adjunction gives $\deg N = 2g(D) - 2 = 6$, since the fiber threefold has trivial canonical bundle. Thus $c_1(T \operatorname{Tot} N)[D] = 0$. Pardon's scalar construction applies to any rank-two holomorphic bundle on a smooth proper curve of this genus and Chern number. This includes the possibly nonsplit N obtained from the normal sequence. With $m = 1$ the empty tuple at stage zero therefore gives $+x_{4,1}$, with the sign fixed by the local degree on the graph [Par25, Remark 4.1 and Definition 4.7].

Exchanging L and M preserves the assumptions, including nontriviality of the degree-zero ratio, so the second positive branch class has the same image. Substitution in the last row of the table gives $-2x_{4,1}$ for the difference. □

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