

# INITIAL CONDITIONS AND ASYMPTOTIC PHASE FOR HERMITIAN–YANG–MILLS FLOW

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ABSTRACT. We prove that the initial Hermitian metric selects a finite-dimensional trajectory that tracks Hermitian–Yang–Mills flow with exponentially decaying relative metric error, uniformly as the positive relaxation rates separate or tend to zero. The result holds for a nearby stable family of deformations of a specified sum of four line bundles on the product of three projective lines, along a fixed polarization path. We construct an exact invariant graph over the full integrable harmonic deformation space. Relative to this chosen graph, the asymptotic phase is continuously differentiable in the initial metric at fixed bundle and polarization. The local theorem permits bounded diagonal metric coordinates and sufficiently small transverse displacement. A prescribed smooth determinant-normalized metric enters this regime after a common restart time, chosen before the deformation, polarization and nearby-metric neighborhoods. The proof uses a fixed complementary spectral gap and metric order, without inverting a small relaxation rate. For a three-layer family with two independent positive slope parameters, we also prove all-future comparison with a small, time-independent relative error.

## 1. INTRODUCTION

A finite-dimensional approximation to a geometric flow must record more than its possible limiting directions. It must identify which finite trajectory belongs to a prescribed initial metric. This distinction becomes quantitative near a stability wall: several finite decay rates can tend to zero at unrelated speeds, while the directions complementary to the finite model retain a fixed gap. An initial displacement in those complementary directions can change the finite trajectory even after the displacement itself has decayed.

We prove an initial-data-sensitive finite reduction for Hermitian–Yang–Mills flow near a specified polystable bundle on  $(\mathbb{P}^1)^3$ . The finite coordinates are the complete local deformation coordinates, including composable extensions and their analytic obstructions. An exact invariant graph produces actual metric-flow solutions. A separate phase construction then selects the graph solution that tracks each nearby actual flow with exponentially decreasing relative error. The constants do not involve the inverse of any small finite eigenvalue.

**1.1. Attraction does not determine the finite initial value.** The distinction already occurs in the system

$$(1) \quad \dot{y} = -\varepsilon y + av, \quad \dot{v} = -sv, \quad 0 < \varepsilon < s.$$

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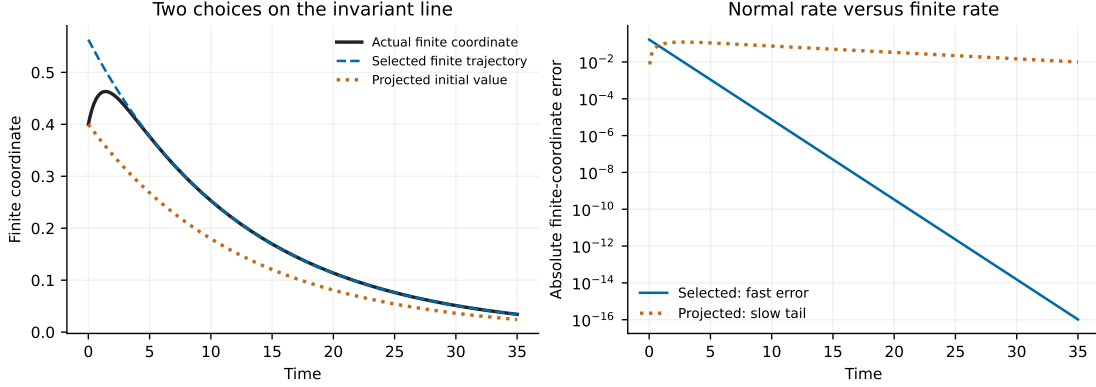


FIGURE 1. Phase selection in (1), with  $\varepsilon = 0.08$ ,  $s = 1$ ,  $a = 0.3$ ,  $y_0 = 0.4$ ,  $v_0 = 0.5$ . The selected finite solution differs from the actual finite coordinate by a fast-decaying term. Starting instead at the projected value leaves a slow tail. This exactly solved linear system illustrates the distinction; it is not a simulation of the geometric flow.

The invariant line is  $v = 0$ . From  $(y_0, v_0)$  its solution is

$$v(t) = v_0 e^{-st}, \quad y(t) = e^{-\varepsilon t} \left( y_0 + \frac{av_0}{s - \varepsilon} \right) - \frac{av_0}{s - \varepsilon} e^{-st}.$$

The finite solution started at the projected initial value  $y_0$  misses a term that decays at the slow rate  $\varepsilon$ . The selected initial value is instead  $y_{\text{ph}} = y_0 + av_0/(s - \varepsilon)$ . Its exact finite trajectory has error  $-av_0 e^{-st}/(s - \varepsilon)$ . The denominator is controlled by separation from the normal rate, not by  $\varepsilon^{-1}$ . Figure 1 illustrates these two choices.

For a nonlinear finite field with flow  $\varphi_t$ , the analogous phase is the limit of  $\varphi_{-t}(Y(t))$ . Exponential normal decay makes that limit converge when the finite derivative is small relative to the fixed normal gap. To prove differentiability of the phase one must also control the variations of this infinite-time limit. Finite-time smooth dependence alone does not provide that conclusion.

**1.2. The geometric result.** The reference bundle is

$$E_0 = \mathcal{O}(-2, -2, 4) \oplus \mathcal{O}(1, -2, 1) \oplus \mathcal{O}(-2, 1, 1) \oplus \mathcal{O}$$

on  $X = (\mathbb{P}^1)^3$ . Its factors have equal slope for the product polarization. We vary the first two product coefficients along a signed parameter  $l$ , with centered slope differences  $l(1, -2/5, 0, -3/5)$ . The stable-family assertions concern  $l > 0$ . The determinant metric is fixed. Time is normalized by

$$H^{-1} \dot{H} = -2(i\Lambda_l F_H - \kappa_E(l)I).$$

Spectra in this paper are those of the complex trace-free endomorphism Dolbeault Laplacian, with no extra factor two.

The complete harmonic deformation space  $V$  has complex dimension 83 and the obstruction space has dimension five. A fixed Hodge contraction reconstructs  $d + a(x)$ , with integrability exactly the convergent equation  $\kappa(x) = 0$ . We retain all higher terms. The nearby stable HE classes are represented by the corrected balanced locus  $\mathcal{B}_l^{\text{st}}$  of Section 2. The complete local quotient and its coverage are imported, with their hypotheses, from [Wue26c]; they specify the actual geometric family.

Theorem 3.2 constructs a chosen finitely differentiable graph  $u = \Psi(l, x)$  in complementary logarithmic metric coordinates. Its finite equation is exact. On each fixed stable holomorphic orbit, write  $x = e^Y \bar{x}$ , where  $\bar{x} \in \mathcal{B}_l^{\text{st}}$  and  $Y$  is a real trace-free diagonal. The resulting equation  $\dot{Y} = f_{l, \bar{x}}(Y)$  has three real variables. Its reconstruction  $e^{Y*}(h_0 e^{\Psi(l, e^Y \bar{x})})$  solves the metric PDE exactly, rather than to a finite asymptotic order.

For bounded  $Y$  and small complementary displacement, Theorem 5.1 selects a unique trajectory in a specified fast-tracking class. If  $D$  is the initial normal distance in  $h^{17/8}$  and  $r$  bounds  $|\bar{x}|$ , then

$$(2) \quad \begin{aligned} e^{-\eta(t)} H_*(t) &\leq H(t) \leq e^{\eta(t)} H_*(t), & \eta(t) &= CD e^{-ct}, \\ \left| \log \frac{\lambda_j(H(t))}{\nu_j(l, x_*(t))} \right| &\leq Cr^2 + CD e^{-ct}, & 1 \leq j \leq 3, & \quad \lambda_4(H(t)) \geq \gamma > 0. \end{aligned}$$

Here  $\nu_j$  are the positive eigenvalues of the finite commutator form on parallel trace-free diagonals. The constants are independent of their positive ratios. The relative metric logarithm has the same decaying estimate in  $h^{17/8}$ , and the comparison between  $H$  and  $H_*$  transfers to every positive Dolbeault eigenvalue. The  $Cr^2$  term is a separate static comparison with the harmonic finite operator.

The phase is  $C^1$  in the initial metric at fixed polarization and balanced base, with uniform derivative bounds in the declared domains. It is relative to the chosen invariant graph; neither a canonical graph nor a phase derivative in polarization is asserted. At equilibrium Proposition 7.2 identifies the finite linearized rates exactly as  $2\lambda_j(H_{\text{HE}})$ . Thus fast attraction to the finite trajectory is compatible with arbitrarily slow motion along it.

**1.3. Prescribed metrics and independent parameter ranges.** The local initial class can be reached from any prescribed smooth metric on the fixed underlying bundle. Theorem 6.3 has the following order of choices. Fix that metric, a graph and a requested entry accuracy. First choose a finite reference restart. Then choose sufficiently small polarization and deformation neighborhoods and a strong neighborhood of the prescribed metric. Every stable balanced point and every initial metric in those neighborhoods enter the phase theorem at that same restart. The phase remains  $C^1$  in the initial metric. Only the finite ODE is continued backward if its selected initial value is to be expressed at the original time zero.

The reference entry proof needs both metric convergence in a fixed presentation and strong finite-time dependence. Connection convergence modulo gauge does not directly imply the former. For the split reference bundle, metric order bounds the convergence gauges; elliptic compactness then identifies a limiting positive diagonal metric. The unsliced logarithmic equation supplies finite-time dependence in the full  $h^{19/8}$  domain. These arguments are given in Section 6.

A second application retains two independent positive slope parameters  $k, l$  on a three-layer diamond. Its four adjacent arrows compose nontrivially, and a quadratic central Hodge primitive is necessary for integrability. The total arrow energy confines its finite dynamics by a height identity. Theorem 8.2 gives all-future relative metric and three-mode comparison from an open initial class, without a lower bound on  $k/l$ . Its explicit example has one eigenvalue asymptotic to  $k$  and two of order  $l$  as  $k/l \rightarrow 0$ .

For this fixed diamond bundle, Theorem 9.2 allows arbitrary smooth prescribed initial metrics, after a common restart on a full positive parameter triangle. Its error is a controllably small constant. The proof recovers the three filtration layers from normalized holomorphic maps, removes exact adjacent and central entries, and proves exact diagonal-orbit matching. It does not produce the differentiable phase of the complete-family theorem. Section 10

identifies the overlap of the two descriptions along  $k = 3l/5$ , including their actual metric and finite spectral comparisons.

**1.4. Relation to earlier work and proof structure.** Finite-dimensional reductions and initial-data-dependent asymptotic trajectories have an established role in geometric flow. Haiden, Katzarkov, Kontsevich and Pandit study finite gradient flows and iterated logarithmic asymptotics and prove the metric-order principle used here [HKKP18]. Lam gives a finite reduction and bounded-distance comparison for HYM flow of semistable bundles with locally free graded object, together with an equivariant Kuranishi description [Lam26, Proposition 8.6 and Theorem 8.7]. These results are direct predecessors. Our assertion is a decaying near-one relative metric comparison, with differentiable phase and uniformity through the stated collapsing finite scales.

The invariant-graph input is Simonett’s theorem for parabolic equations [Sim94, Theorem 2.1]. Its attraction estimate concerns distance to the graph over the current finite coordinate; we separately select and differentiate the tracking trajectory. The proof is an application of established invariant-manifold and asymptotic-phase ideas, with the quantitative estimates needed for the present geometric degeneration printed in full. Shao–Simonett supplies the normally elliptic realization and continuous maximal regularity [SS14], while Simonett’s local semiflow theorem includes the strong initial topology needed for finite-time entry [Sim95].

For two-step extensions, [Wue26b, Sections 6 and 8] already constructs exact invariant-graph flows and proves prescribed-metric matching with all-future comparison after restart. The additions here are the complete composable deformation family, the phase with decaying relative error and its differentiability, and the three-layer geometric matching needed for the independent two-parameter application. Stationary metric and spectral estimates are imported only in their stated scope from [Wue26a, Wue26c]. They do not supply parabolic invariance or phase selection.

Sections 2–3 establish the geometric interface and exact finite graph. Section 4 proves the reusable phase estimate, and Sections 5–6 give its geometric realization and prescribed entry. Section 7 separates nonstationary spectral comparison from equilibrium rates. Sections 8–10 retain the two-parameter application and compare the overlapping descriptions.

## 2. THE COMPLETE DEFORMATION FAMILY

The finite coordinates used below describe holomorphic bundles, whereas the finite evolution stays on one holomorphic orbit. We first specify that distinction and the geometric neighborhood in which the estimates will be uniform.

**2.1. The split bundle and the polarization.** Let  $X = (\mathbb{P}^1)^3$ , and let  $\omega_i$  be the product Fubini–Study forms, normalized by  $\int_{\mathbb{P}^1} \omega_i = 1$ . Set

$$(3) \quad F_1 = \mathcal{O}(-2, -2, 4), \quad F_2 = \mathcal{O}(1, -2, 1), \quad F_3 = \mathcal{O}(-2, 1, 1), \quad F_4 = \mathcal{O}, \quad E_0 = \bigoplus_{i=1}^4 F_i.$$

Write  $h_0$  for the orthogonal product metric,  $A_0$  for its Chern connection, and  $d$  for the split Dolbeault operator and its induced operators on endomorphism-valued forms. A block  $ij$  maps  $F_j$  to  $F_i$ . The determinant line is  $\mathcal{O}(-3, -3, 6)$ . Throughout the paper its metric is fixed to be  $\det h_0$ .

For the complete family, the polarization follows the path

$$(4) \quad \omega_l = h_1(l)\omega_1 + h_2(l)\omega_2 + \omega_3, \quad h_1(l) = \left(1 + \frac{7l}{30\pi}\right)^{-1}, \quad h_2(l) = \left(1 + \frac{l}{6\pi}\right)^{-1}.$$

It is Kähler on an open interval about  $l = 0$ . The parabolic construction uses this signed interval; the stable-family conclusions use  $l > 0$ . All squared  $L^2$  norms are divided by  $\text{Vol}_l = h_1 h_2$ . The resulting spatial measure is independent of  $l$ . If  $\kappa_i(l) = i\Lambda_l F_{F_i, h_i}$  and  $\kappa_E = \frac{1}{4} \sum_i \kappa_i$ , then

$$(5) \quad b(l) = (\kappa_E - \kappa_i)_{i=1}^4 = l(1, -2/5, 0, -3/5).$$

The product determinant metric has constant contracted curvature  $4\kappa_E(l)$  even though its line bundle is nontrivial.

Let

$$\mathfrak{t} = \{\xi \in \mathbb{C}^4 : \sum_i \xi_i = 0\}, \quad \mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \cap \mathbb{R}^4, \quad |\xi|^2 = \sum_i |\xi_i|^2.$$

These are the parallel trace-free diagonal endomorphisms and their Hermitian part. There are no other split holomorphic endomorphisms: every nonzero difference of the degree triples in (3) has sum zero and a negative entry, so its product line bundle has no section. Let  $\mathcal{V}$  be the real bundle of  $h_0$ -Hermitian trace-free endomorphisms. On its sections let  $P_0$  be normalized  $L^2$  projection onto  $\mathfrak{t}_{\mathbb{R}}$ , and set  $Q_0 = 1 - P_0$ . The projections are fixed as  $l$  varies.

**2.2. Reconstruction and the actual HE neighborhood.** Use the reference geometry at  $l = 0$  to define harmonic projections  $P_j$ , harmonic inclusion  $\iota$ , and the Hodge homotopy  $K = d^*G$ . Put  $V = H^1(X, \text{End}_0 E_0)$ . Its nine nonzero blocks are

$$(6) \quad \begin{array}{c|cccc|cccc|ccc} ij & 12 & 32 & 34 & 13 & 23 & 24 & 21 & 31 & 41 \\ \hline \dim_{\mathbb{C}} V_{ij} & 8 & 8 & 4 & 8 & 8 & 4 & 8 & 8 & 27 \\ \text{differential factor} & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{array}$$

Thus  $\dim_{\mathbb{C}} V = 83$ , while  $H^2(X, \text{End}_0 E_0)$  has dimension five and is supported in block 14. These dimensions follow from Künneth and line-bundle cohomology, as computed in [Wue26c, Sections 3 and 5]. The representatives in (6) remain harmonic for every  $\omega_l$ . Their normalized squared norm is

$$(7) \quad \begin{aligned} q_l(x) = \|\iota x\|_l^2 &= h_1^{-1}(\|x_{12}\|_0^2 + \|x_{32}\|_0^2 + \|x_{34}\|_0^2) \\ &\quad + h_2^{-1}(\|x_{13}\|_0^2 + \|x_{23}\|_0^2 + \|x_{24}\|_0^2) \\ &\quad + \|x_{21}\|_0^2 + \|x_{31}\|_0^2 + \|x_{41}\|_0^2. \end{aligned}$$

Write  $|x| = \|\iota x\|_0$ .

We recall precisely the stationary geometric input. A constant invertible diagonal  $g$  acts by  $(gx)_{ij} = g_i g_j^{-1} x_{ij}$ . Its infinitesimal action is

$$(8) \quad C_x \xi = ((\xi_i - \xi_j)x_{ij})_{ij}.$$

**Proposition 2.1** (Complete local model, [Wue26c]). *On a fixed coefficient ball there is a convergent holomorphic reconstruction*

$$(9) \quad a(x) = \iota x - K(a(x)^2), \quad \kappa(x) = P_2(a(x)^2), \quad d_x = d + a(x).$$

*The operator  $d_x$  is integrable exactly when  $\kappa(x) = 0$ . The five-component equation retains all higher terms. The map  $a$  is equivariant under constant diagonals wherever both arguments lie in its domain, and every holomorphic map between sufficiently small integrable reconstructed objects is such a constant intertwiner.*

*For the centered curvature*

$$(10) \quad \mathcal{M}(l, x, u) = e^{u/2} i\Lambda_l F_{d_x, h_0 e^u} e^{-u/2} - \kappa_E(l)I$$

there is a unique small complementary solution  $w(l, x)$  of  $Q_0\mathcal{M}(l, x, w) = 0$ ,  $P_0w = 0$ . It is smooth and  $O(|x|^2)$  in every fixed finite spatial norm. Define

$$(11) \quad m_l(x) = P_0\mathcal{M}(l, x, w(l, x)), \quad \mathcal{B}_l(r) = \{x : |x| < r, \kappa(x) = 0, m_l(x) = 0\}.$$

For  $0 \leq l \leq l_0$ , its compact diagonal quotient identifies with an open neighborhood of actual fixed-determinant HE classes. For fixed  $s \geq 6$  there are uniform  $c_0, C_0 > 0$  such that this image contains every such class having a unitary representative  $\|A - A_0\|_{H^s} < c_0r$ , and every point of the image has a representative with  $\|A - A_0\|_{H^s} < C_0r$ .

The proposition is [Wue26c, Sections 6 and 9], including its geometric coverage and compact-orbit uniqueness. The reconstruction uses a fixed reference contraction and is independent of  $l$ . The complementary equation extends to signed  $l$  by the same parameter implicit-function theorem. We do not infer a parabolic invariance assertion from that elliptic equation.

Here are two estimates from the reconstruction that will be used explicitly. With  $J_x = D_x a$ ,

$$(12) \quad J_x v = \iota v + B_x v, \quad \langle \iota v, B_x v' \rangle_l = 0, \quad \|B_x v\|_{H^s} \leq C_s |x| |v|.$$

Indeed differentiating (9) puts the correction in the reference coexact space. Pairing with any harmonic block for  $\omega_l$  multiplies its reference pairing by the corresponding scalar in (7); it remains zero. Integration along a radial segment gives the same orthogonality for  $a(x) - \iota x$ . Equivariance also gives, exactly,

$$(13) \quad d_x^{\text{End}} \xi = -J_x C_x \xi.$$

The support graph has vertices  $1, 2, 3, 4$  and an undirected edge between  $i, j$  whenever  $x_{ij}$  or  $x_{ji}$  is nonzero. Let  $\mathcal{B}_l^{\text{st}}(r)$  be the connected-support part of (11). Its bundles are stable: they are HE, and the constant-intertwiner assertion makes their trace-free holomorphic kernel zero. Conversely this is the stable part of the declared local HE quotient. No smoothness or rank condition on  $\kappa^{-1}(0)$  is imposed.

**2.3. Metric and spectral conventions.** Each flow fixes  $l$ , its holomorphic bundle, and the determinant. Its time normalization is

$$(14) \quad H^{-1} \partial_t H = -2(i\Lambda_l F_H - \kappa_E(l)I).$$

All spectral statements concern  $\Delta_{\bar{\partial}, H} = \bar{\partial}_{\text{End}_0}^* \bar{\partial}_{\text{End}_0}$  on complex trace-free endomorphism zero-forms, without a factor two, with complex multiplicities and zeros counted. The finite Hermitian operator on  $\mathfrak{t}$  is

$$(15) \quad \langle \xi, L_{\text{fin}, l}(x) \xi \rangle = \|C_x \xi\|_l^2 = \sum_{ij} \|x_{ij}\|_l^2 |\xi_i - \xi_j|^2.$$

Its three ordered eigenvalues are  $\nu_1(l, x), \nu_2(l, x), \nu_3(l, x)$ . They are positive precisely when the support is connected.

For positive metrics on the same bundle define

$$(16) \quad d_\infty(H, K) = \sup_{p \in X} \max_j |\log \lambda_j(H_p^{-1} K_p)|.$$

Equivalently  $d_\infty(H, K) \leq a$  means  $e^{-a}K \leq H \leq e^a K$ . This distance is unchanged by simultaneous bundle pullback and satisfies the triangle inequality. The metric-order principle [HKKP18, Proposition 4.3] preserves such a sandwich between two exact solutions of (14). Scalar constant multiples of a stationary HE metric are stationary as well. We use this comparison in arbitrary complex dimension, rather than any curve-specific asymptotic conclusion.

Metrics across the family are compared in the displayed smooth presentations  $d_x$  on  $E_0$ . A prescribed metric means the same positive metric matrix in those presentations. Changing presentation requires pulling the metric back too. The uniform statements below do not concern arbitrary unbounded changes of presentation.

### 3. EXACT EVOLUTION ON A FINITE INVARIANT GRAPH

The finite equation is obtained from the metric heat equation before any expansion is made. Its higher terms matter because its reconstruction must be an exact solution over arbitrarily long times.

**3.1. The sliced metric equation.** For  $u \in Q_0\Gamma(\mathcal{V})$ , the unitary connection compatible with  $h_0e^u$  and the prescribed  $(0, 1)$  operator  $d + a(x)$  has  $(1, 0)$  part

$$(17) \quad \nabla_0^{1,0} + e^{-u}\nabla_0^{1,0}(e^u) - e^{-u}a(x)^{*h_0}e^u.$$

The middle term is the connection derivative of the endomorphism  $e^u$ . This defines (10) on the ambient coefficient ball, including nonintegrable  $x$ . Its pointwise trace is zero: the determinant connection remains the fixed product connection.

Define the pointwise commutator functions

$$(18) \quad C_u = \cosh(\text{ad}_u/2), \quad S_u = \frac{\sinh(\text{ad}_u/2)}{\text{ad}_u/2}, \quad B_u = S_u^{-1}C_u, \quad R(u) = P_0B_u|_{t_{\mathbb{R}}}.$$

The quotient is one at zero. In a unitary eigenbasis of  $u$ , the multipliers of  $S_u$  and  $B_u$  are  $\sinh a/a$  and  $a \coth a$ , where  $a$  is half an eigenvalue difference. They are positive;  $a \coth a \geq 1$ . Consequently  $R(u)$  is positive and invertible. These operators preserve Hermitian fields and trace, and  $S_u - I$ ,  $B_u - I$  and  $R(u) - I$  are  $O(\|u\|^2)$  near zero.

Set

$$(19) \quad \begin{aligned} \beta(l, x, u) &= -R(u)^{-1}P_0S_u^{-1}\mathcal{M}(l, x, u), \\ \dot{x} &= C_x\beta(l, x, u), \\ \dot{u} &= -2S_u^{-1}\mathcal{M}(l, x, u) - 2B_u\beta(l, x, u). \end{aligned}$$

**Proposition 3.1.** *System (19) preserves  $P_0u = 0$ , pointwise trace zero, the complete integrability locus  $\kappa = 0$ , and the positive diagonal orbit of  $x$ . If  $x(t) = g(t)x^*$ , with  $\dot{g}g^{-1} = \beta$  and  $\det g = 1$ , then*

$$(20) \quad H(t) = g(t)^*(h_0e^{u(t)})$$

*is an exact solution of (14) on the fixed bundle  $d+a(x^*)$  for as long as the coordinates remain in their domain.*

*Proof.* Applying  $P_0$  to the last line of (19) gives zero by the definition of  $R$  and  $\beta$ . Trace is preserved by the commutator functions. Each block coefficient is multiplied by  $\exp \int (\beta_i - \beta_j)$ , so diagonal equivariance of (9) transforms the entire  $a(x)$  and preserves the full equation  $\kappa = 0$ .

In the moving diagonal frame, the centered metric velocity of (20) is  $2C_u\beta + S_u\dot{u}$ . The two gauge derivatives are  $e^{-u/2}\beta e^{u/2}$  and  $e^{u/2}\beta e^{-u/2}$ , while

$$e^{-u/2}(\partial_t e^u)e^{-u/2} = \int_{-1/2}^{1/2} e^{s\text{ad}_u}\dot{u} ds = S_u\dot{u}.$$

Equation (19) makes their sum  $-2\mathcal{M}$ , which is precisely (14). The trace conditions preserve the determinant.  $\square$

**3.2. A common parabolic domain.** On Hermitian trace-free zero-forms the split real metric linearization is

$$(21) \quad L_l = \frac{1}{2}(\Delta_{\partial, l} + \Delta_{\bar{\partial}, l}).$$

It is self-adjoint and nonnegative; its kernel is  $\mathfrak{t}_{\mathbb{R}}$ . Indeed its quadratic form is half the sum of the two derivative energies, and a zero is a parallel Hermitian endomorphism. The cross-Hom vanishing in Section 2 identifies that kernel. Ellipticity and compactness of a small signed  $l$  interval give a fixed positive gap on  $Q_0$ . The curvature commutator in the Kähler identities is included in (21).

The exact curvature expression has the decomposition

$$(22) \quad S_u^{-1}\mathcal{M}(l, x, u) = L_l u - b(l) + G_l(x, u, \nabla_0 u), \quad G_l(0) = 0, \quad DG_l(0) = 0.$$

To verify the order assertion, take a contracted second jet  $W$  of  $u$  in (17). Its centered coefficient is

$$e^{-u/2}(d \exp)_u(W)e^{-u/2} = S_u W.$$

Multiplication by  $S_u^{-1}$  removes this varying principal coefficient. All other terms use at most first spatial derivatives of  $u$ . The linear coefficient term vanishes because  $\iota x$  is harmonic for  $\omega_l$ ;  $a(x) - \iota x$  is quadratic. Subtracting the complete linearization gives (22) on the ambient ball, without using integrability.

Since  $P_0 L_l u = 0$ , the sliced equations become

$$(23) \quad \begin{aligned} \beta &= R(u)^{-1}(b - P_0 G_l), \\ \dot{u} &= -2L_l u + \mathcal{N}_l(x, u), \\ \mathcal{N}_l &= -2G_l + 2(I - B_u R(u)^{-1})b + 2B_u R(u)^{-1}P_0 G_l. \end{aligned}$$

For every nearby  $l$ ,  $D\mathcal{N}_l(0, 0) = 0$  and  $D_u \beta(l, 0, 0) = 0$ . In particular no linear  $bu$  term remains. The finite linearization is  $(\Lambda_l x)_{ij} = (b_i - b_j)x_{ij}$ .

For nonintegral  $s$ , let  $h^s$  denote the little-Hölder space, the closure of smooth sections in the  $C^s$  norm. Use

$$(24) \quad \begin{aligned} \mathcal{X}_0 &= V_{\mathbb{R}} \oplus Q_0 h^{3/8}(\mathcal{V}), & \mathcal{X}_1 &= V_{\mathbb{R}} \oplus Q_0 h^{19/8}(\mathcal{V}), \\ \mathcal{X}_{\beta} &= (\mathcal{X}_0, \mathcal{X}_1)_{5/8, \infty}^0 = V_{\mathbb{R}} \oplus Q_0 h^{13/8}(\mathcal{V}), \\ \mathcal{X}_{\alpha} &= (\mathcal{X}_0, \mathcal{X}_1)_{7/8, \infty}^0 = V_{\mathbb{R}} \oplus Q_0 h^{17/8}(\mathcal{V}). \end{aligned}$$

Here  $\dim_{\mathbb{R}} V_{\mathbb{R}} = 166$  and the superscript zero denotes continuous real interpolation. The finite-rank smooth projection commutes with this interpolation. One derivative of  $h^{13/8}$  belongs to the multiplication algebra  $h^{5/8} \subset h^{3/8}$ . Thus the first-jet nonlinearities and the finite component in (23) are smooth  $\mathcal{X}_{\beta} \rightarrow \mathcal{X}_0$  maps, jointly in  $l$ .

The common-domain operator  $\text{diag}(0, 2L_l|_{Q_0}) : \mathcal{X}_1 \rightarrow \mathcal{X}_0$  has continuous maximal regularity. We use [SS14, Theorem 3.6 and Proposition 3.7]. For completeness, the bundle realization follows by complexifying, adding a smooth complementary bundle, and extending by a connection Laplacian with the same scalar principal symbol on that complement. The resulting system can be trivialized; the localization argument and Remark 3.3 of that source apply. The original bundle is reducing. Since  $P_0$  commutes with  $L_l$ , uniqueness restricts the linear problem to  $Q_0$  with its full differential domain. Adding the finite-dimensional zero block preserves maximal regularity. Smooth coefficients and uniform ellipticity on a compact parameter interval give uniform bounds. The second-order variation with  $l$  is retained in  $L_l$ , rather than treated as a first-jet perturbation.

### 3.3. The graph and its exact finite field.

**Theorem 3.2.** *For every prescribed finite integer  $m \geq 6$ , after shrinking the signed parameter interval and coefficient ball, there is a jointly  $C^m$  map*

$$(25) \quad \begin{aligned} \Psi(l, x) &\in Q_0 h^{19/8}(\mathcal{V}), \quad \Psi(l, 0) = 0, \quad D_x \Psi(l, 0) = 0, \\ \|\Psi(l, x)\|_{h^{19/8}} &\leq C|x|^2, \quad \|D_x \Psi(l, x)\| \leq C|x|. \end{aligned}$$

*Its graph is locally invariant for (19) and contains every equilibrium sufficiently small in the full domain. It may be chosen equivariant under constant diagonal unitary gauges. In particular  $\Psi(l, \bar{x}) = w(l, \bar{x})$  for every sufficiently small  $\bar{x} \in \mathcal{B}_l(r)$ .*

*The exact finite equation is*

$$(26) \quad \dot{x} = C_x \beta^\Psi(l, x), \quad \beta^\Psi(l, x) = -R(\Psi)^{-1} P_0 S_\Psi^{-1} \mathcal{M}(l, x, \Psi(l, x)).$$

*Its integrable trajectories reconstruct (14). On each positive diagonal orbit this is an equation in at most three real variables, and in exactly three when the support is connected.*

*Proof.* At  $l = 0$ , the center space is  $V_{\mathbb{R}}$  and the normal linearization is  $-2L_0|_{Q_0}$  with a fixed negative gap. Equations (22)–(24) verify the smoothness, common-domain and maximal-regularity hypotheses of [Sim94, Theorem 2.1]. Its graph has values in the full stable domain  $\mathcal{X}_1^s$ , has any prescribed finite regularity after shrinking, and contains small entire trajectories.

The parameterwise tangency needs a further check. Put  $T_l = D_x \Psi(l, 0)$ . Differentiating invariance gives

$$T_l \Lambda_l = -2L_l T_l.$$

Every eigenvalue of  $\Lambda_l$  tends to zero, while the normal spectrum stays bounded away from zero. Applying this identity to each finite eigenvector gives  $T_l = 0$ . The zero solution and all small equilibria are small entire trajectories. At a balanced point, the stationary complementary solution gives  $\mathcal{M} = 0$ , so it is an equilibrium of the exact slice. Its  $O(|x|^2)$  full-domain bound ensures the required smallness. Taylor's theorem gives (25).

Choose invariant norms and a diagonal-unitary-invariant finite cutoff in the fixed-point construction of [Sim94, Section 2]. The exact system, its projections and its linear operator commute with this compact action. Transforming the fixed-point graph gives another fixed point in the same uniqueness class, hence the same graph. This proves equivariance without averaging graphs. Local invariance and Proposition 3.1 give the exact reconstruction. The graph's  $h^{19/8}$  values have continuous second spatial derivatives, sufficient for the classical metric equation.  $\square$

We fix one such graph from now on. It is not asserted canonical, or  $C^\infty$  on one unchanged neighborhood. Away from equilibrium it need not equal  $w$ : its invariance equation contains  $D_x \Psi \dot{x}$ , which is absent from the stationary equation.

**Lemma 3.3.** *Let  $\mu_l(x) = \sum_{ij} \|x_{ij}\|_l^2 (e_i - e_j)$ , with all nine blocks included. On the ambient coefficient ball,*

$$(27) \quad \beta^\Psi(l, x) = b(l) - \mu_l(x) + E_l(x), \quad |D_x^j E_l(x)| \leq C|x|^{4-j} \quad (0 \leq j \leq 2).$$

*Proof.* Transport the compatible connection isometrically to  $h_0$  by  $e^{\Psi/2}$ . Its  $(0, 1)$  perturbation has expansion

$$\alpha = \iota x + B + O(|x|^3), \quad B = a(x) - \iota x - \frac{1}{2} d\Psi(l, x) = O(|x|^2).$$

The weighted harmonic orthogonality in (12), and the orthogonality of  $d\Psi$  to harmonic forms, give  $\langle \nu v, B \rangle_l = 0$  in each block and after every constant diagonal character weighting. Pairing the curvature of  $A_0 + \alpha - \alpha^*$  with a parallel real diagonal  $\xi$  gives the exact identity

$$(28) \quad \langle P_0 \mathcal{M}(l, x, \Psi), \xi \rangle = -\langle b(l), \xi \rangle + \sum_{ij} (\xi_i - \xi_j) \|\alpha_{ij}\|_l^2.$$

The linear curvature term integrates to zero. Orthogonality annihilates the cubic cross terms in the squared norms, while blocks with no linear harmonic part contribute fourth order. Thus  $P_0 \mathcal{M} = -b + \mu_l + O(|x|^4)$ . Since  $\Psi = O(|x|^2)$ , the factors  $S_\Psi^{-1} - I$  and  $R(\Psi)^{-1} - I$  are fourth order. The curvature is bounded by  $C(|l| + |x|^2)$ , so these factors preserve the stated remainder. The vanished jets through degree three and the  $C^m$  regularity give the differentiated estimates by Taylor's theorem.  $\square$

The cancellation here uses harmonic orthogonality. There is no diagonal sign acting negatively on all nine blocks: the characters of 12, 23, 31 multiply to one. Likewise equal reverse-arrow energies can give  $\mu_l(x) = 0$  while  $q_l(x) > 0$ . Confinement on the complete family will therefore use an equilibrium metric, not an ambient lower bound for  $|\mu_l|/q_l$ .

#### 4. SELECTION OF A FINITE TRAJECTORY

Attraction to an invariant graph controls a transverse error. It does not specify which trajectory on the graph has the correct initial condition. The next argument selects that trajectory by pulling the perturbed finite coordinate backward under the unperturbed finite flow. The construction uses an upper bound on the finite derivative, so its constants persist when finite decay rates vanish.

**4.1. The normal semigroup.** Write

$$(29) \quad \mathcal{E}_0 = Q_0 h^{3/8}(\mathcal{V}), \quad \mathcal{E}_1 = Q_0 h^{19/8}(\mathcal{V}), \quad \mathcal{E}_\alpha = Q_0 h^{17/8}(\mathcal{V}), \quad \alpha = 7/8, \quad A_l = 2L_l|_{Q_0}.$$

Scale equivalent norms so that  $\mathcal{E}_1 \rightarrow \mathcal{E}_\alpha \rightarrow C^0$  have norm at most one. There are uniform  $M \geq 1$  and  $\sigma_0 > 0$  such that  $S_l(t) = e^{-tA_l}$  satisfies

$$(30) \quad \|S_l(t)\|_{\mathcal{E}_\alpha \rightarrow \mathcal{E}_\alpha} \leq M e^{-\sigma_0 t}, \quad \|S_l(t)\|_{\mathcal{E}_0 \rightarrow \mathcal{E}_\alpha} \leq M(1 + t^{-\alpha}) e^{-\sigma_0 t} \quad (t > 0).$$

Here the constant in a Hölder norm need not be one. Uniform ellipticity gives uniform large-parameter sectorial estimates. Elliptic regularity identifies the spectrum in  $\mathcal{E}_0$  with the smooth, hence  $L^2$ , spectrum. Removing  $P_0$  leaves the fixed positive gap. On a bounded part of a common resolvent contour, continuity of  $A_l : \mathcal{E}_1 \rightarrow \mathcal{E}_0$ , compactness of the parameter interval, and the resolvent identity give uniform bounds. The contour formula and continuous interpolation then prove (30); the sectorial and interpolation inputs are [SS14, Proposition 2.2(c), Theorems 3.2 and 3.4–3.6].

**4.2. A finite equation coupled to a decaying normal component.** Consider, for fixed  $l$ ,

$$(31) \quad \dot{Y} = f(Y) + h(Y, v), \quad \dot{v} = -A_l v + F(Y, v), \quad Y \in \mathbb{R}^d, \quad v \in \mathcal{E}_\alpha.$$

Suppose  $f$  is bounded and  $C^2$ , with bounded derivatives,  $\|Df\|_\infty \leq L$ , and  $\|D^2 f\|_\infty \leq K_f$ . Let  $h : \mathbb{R}^d \times B_{\mathcal{E}_\alpha}(\rho) \rightarrow \mathbb{R}^d$  and  $F : \mathbb{R}^d \times B_{\mathcal{E}_\alpha}(\rho) \rightarrow \mathcal{E}_0$  be  $C^2$  with uniformly bounded derivatives through order two. Assume

$$(32) \quad \begin{aligned} h(Y, 0) = F(Y, 0) = 0, \quad \|D_v h\| + \|D_v F\| &\leq \epsilon, \\ \|D_Y h(Y, v)\| + \|D_Y F(Y, v)\| &\leq K \|v\|_{\mathcal{E}_\alpha}. \end{aligned}$$

For the refined estimate suppose additionally

$$(33) \quad \|D_v h(Y, v)\| \leq K(a + \|v\|_{\mathcal{E}_\alpha}), \quad a \geq 0.$$

The numbers  $K$  and the  $C^2$  bounds are fixed. The parameters  $\epsilon$  and  $L$  will be small.

Set

$$c = \sigma_0/8, \quad \varkappa = c/2, \quad J = \int_0^\infty (1 + s^{-\alpha}) e^{-(\sigma_0 - 4c)s} ds < \infty.$$

Choose  $\epsilon$  so that  $M\epsilon J \leq 1/2$ , decreasing it further for the variation estimates below, and choose  $L < \varkappa/4$ . Finally choose a small  $\delta_0$  depending only on these fixed bounds, with  $2M\delta_0 < \rho/2$ .

**Proposition 4.1.** *For every  $(Y_0, v_0)$  with  $D = \|v_0\|_{\mathcal{E}_\alpha} < \delta_0$ , system (31) has a global mild solution and*

$$(34) \quad \|v(t)\|_{\mathcal{E}_\alpha} \leq 2MD e^{-4ct}.$$

Let  $\phi_t$  be the complete flow of  $f$ . The limit

$$(35) \quad \Pi(Y_0, v_0) = \lim_{T \rightarrow \infty} \phi_{-T}(Y(T)), \quad Y_*(t) = \phi_t(\Pi(Y_0, v_0))$$

exists and defines a  $C^1$  map with uniform derivative bounds. It satisfies

$$(36) \quad |Y_*(t) - Y(t)| \leq C\epsilon D e^{-4ct}.$$

Under (33), the stronger bounds are

$$(37) \quad |Y_*(t) - Y(t)| \leq C(a + D) D e^{-4ct}, \quad \|D_{v_0} \Pi(Y_0, v_0)\| \leq C(a + \delta) \quad (\|v_0\| < \delta < \delta_0).$$

One has  $\Pi(Y_0, 0) = Y_0$ . For every  $q$  with  $L < q \leq 4c$ ,  $Y_*$  is the unique forward-complete finite trajectory whose distance from  $Y(t)$  is  $O(e^{-qt})$ .

*Proof.* Local existence and  $C^1$  dependence follow by contraction on continuous paths, using the integrable kernel in (30). The mild normal equation is

$$v(t) = S_t(t)v_0 + \int_0^t S_t(t-s)F(Y(s), v(s)) ds.$$

As long as  $\|v\| < \rho$ , multiplication by  $e^{4ct}$  gives

$$\sup_{t \leq T} e^{4ct} \|v(t)\| \leq MD + M\epsilon J \sup_{t \leq T} e^{4ct} \|v(t)\|.$$

Absorption proves (34) with a strict margin. The finite velocity is bounded. The same local convolution argument continues the solution across every finite time: split the Duhamel integral at a short terminal interval, whose norm is bounded by the integral of  $1 + s^{-\alpha}$ , and use strong continuity on the remaining interval. This proves the Cauchy property at a finite endpoint without compactness of an  $\mathcal{E}_\alpha$  ball. For  $\mathcal{E}_1$  initial data the usual strong regularity holds; general  $\mathcal{E}_\alpha$  data have positive-time regularity. The bound (34) itself includes time zero.

Put  $H(t) = h(Y(t), v(t))$ . The hypotheses and normal decay give  $|H(t)| \leq 2M\epsilon D e^{-4ct}$ , or, under (33),  $|H(t)| \leq C(a + D) D e^{-4ct}$ . Since  $\|D\phi_t\| \leq e^{L|t|}$ ,

$$\frac{d}{dt} \{\phi_{-t}(Y(t))\} = D\phi_{-t}(Y(t))H(t)$$

is integrable. Consequently

$$(38) \quad \Pi(Y_0, v_0) = Y_0 + \int_0^\infty D\phi_{-s}(Y(s))H(s) ds.$$

For  $T \geq t$ , applying the same identity to  $\phi_{t-s}(Y(s))$  gives

$$|\phi_{t-T}(Y(T)) - Y(t)| \leq \int_t^T e^{L(s-t)} |H(s)| ds.$$

Letting  $T \rightarrow \infty$  proves the trajectory estimates, with denominator  $4c - L$  bounded below by a fixed positive number. If two finite trajectories track  $Y$  at rate  $q > L$ , their initial separation is at most  $e^{Lt}$  times their separation at time  $t$ . Letting  $t \rightarrow \infty$  proves uniqueness. No inverse of a finite decay rate occurs.

It remains to justify differentiating the infinite-time formula. For a variation of the initial data write  $W = \delta Y$ ,  $V = \delta v$ . They satisfy

$$(39) \quad \begin{aligned} \dot{W} &= (Df + D_Y h)W + D_v h V, \\ V(t) &= S_t(t)V_0 + \int_0^t S_t(t-s)(D_Y F W + D_v F V)(s) ds. \end{aligned}$$

On  $[0, T]$  put  $U_T = \sup e^{-\varkappa t} |W(t)|$  and  $V_T = \sup e^{(4c-\varkappa)t} \|V(t)\|$ . The normal equation, (34), and the semigroup bound yield, after absorbing the  $\epsilon$  term,

$$V_T \leq C\|V_0\| + CKDU_T.$$

Integrating the finite equation yields

$$U_T \leq |W_0| + (L/\varkappa + CKD)U_T + C\epsilon V_T.$$

For example,  $D_Y h W$  is bounded by  $CKDe^{-(4c-\varkappa)t}U_T$ , an integrable forcing. Decrease  $\delta_0$  and  $\epsilon$  so both inequalities absorb. Uniformly in  $T$ ,

$$(40) \quad \begin{aligned} |W(t)| &\leq Ce^{\varkappa t}(|W_0| + \|V_0\|), \\ \|V(t)\| &\leq Ce^{-(4c-\varkappa)t}(\|V_0\| + D|W_0|). \end{aligned}$$

In particular  $|\delta H(t)| \leq Ce^{-(4c-\varkappa)t}(|W_0| + \|V_0\|)$ . Twice differentiating the finite ODE gives  $\|D^2\phi_{-s}\| \leq K_f se^{2Ls}$ . The derivative of the integrand in (38) is

$$D^2\phi_{-s}(Y(s))[W(s), H(s)] + D\phi_{-s}(Y(s))\delta H(s).$$

The two terms are bounded by constants times  $se^{-(4c-2L-\varkappa)s}$  and  $e^{-(4c-L-\varkappa)s}$ , respectively, times the initial variation norm. Both are uniformly integrable. Finite-time derivatives are continuous, so dominated convergence proves that  $\Pi$  is  $C^1$  with the stated bounds.

For the refined normal derivative set  $W_0 = 0$ . Along a trajectory with  $\|v_0\| < \delta$ , (33) gives  $\|D_v h\| \leq C(a + \delta)$ . The same two variation inequalities give  $U_T \leq C(a + \delta)\|V_0\|$  and  $V_T \leq C\|V_0\|$ . Both differentiated integrand terms retain this small factor. Integration gives the second estimate in (37). If  $v_0 = 0$ , then  $v = H = 0$  and (38) gives  $\Pi(Y_0, 0) = Y_0$ .  $\square$

**4.3. The response to normal initial data.** The phase retains the integrated forcing of the normal initial component. At an on-graph point let  $y(t) = \phi_t(Y_0)$ , and let  $U(t, 0)$  solve

$$\dot{V} = -A_l V + D_v F(y(t), 0)V, \quad V(0) = V_0.$$

It has the same uniform fast decay. Since  $D_Y F(Y, 0) = D_Y h(Y, 0) = 0$ , differentiating (38) gives

$$(41) \quad D_{v_0}\Pi(Y_0, 0)V_0 = \int_0^\infty D\phi_{-s}(y(s))D_v h(y(s), 0)U(s, 0)V_0 ds.$$

Under (33) its norm is at most  $Ca$ . It need not vanish. This explains why replacing the initial metric by its finite projection can choose a different long-time trajectory.

Invariant-manifold attraction and local semiflow regularity are classical [Sim94, Theorem 2.3], [Sim95, Theorem 3.1]. The proof above records the trajectory selection and infinite-time derivative bounds in the uniform form required here. The geometric application must still verify its hypotheses and remove the finite-coordinate cutoff.

## 5. ASYMPTOTIC PHASE FOR THE GEOMETRIC FLOW

We apply the preceding selection argument in diagonal metric coordinates. These coordinates retain a uniform meaning even when a positive orbit singular value is very small. Equilibrium metric order will keep them in the region where the exact equations apply.

**5.1. The initial class and the statement.** Fix  $A > 0$ . For  $\bar{x} \in \mathcal{B}_l^{\text{st}}(r)$ , write

$$(42) \quad \begin{aligned} |Y_0|_{\max} &< A, & x_0 &= e^{Y_0} \bar{x}, & u_0 &\in \mathcal{E}_1, & \|u_0 - \Psi(l, x_0)\|_{\mathcal{E}_1} &< \delta_A, \\ H^{\text{in}} &= e^{Y_0 *}(h_0 e^{u_0}), & v_0 &= u_0 - \Psi(l, x_0), & D &= \|v_0\|_{\mathcal{E}_\alpha}. \end{aligned}$$

Pullback by  $e^{Y_0}$  has matrix  $e^{Y_0} e^{u_0} e^{Y_0}$  in  $h_0$ -unitary frames. The determinant is fixed. Smooth initial metrics may be used; the displayed strong initial class suffices. For this balanced base put

$$(43) \quad f_{l, \bar{x}}(Y) = \beta^\Psi(l, e^Y \bar{x}), \quad \dot{Y} = f_{l, \bar{x}}(Y).$$

The graph is the fixed graph of Theorem 3.2, chosen before the initial metric.

**Theorem 5.1.** *There are  $l_0, r_A, \delta_A, C_A, \gamma_A, c > 0$  such that, for every  $0 < l \leq l_0$ ,  $0 < r \leq r_A$ ,  $\bar{x} \in \mathcal{B}_l^{\text{st}}(r)$  and initial metric (42), the following hold for all  $t \geq 0$ .*

*The actual solution of (14) has exact coordinates  $H(t) = e^{Y(t)*}(h_0 e^{u(t)})$ , with  $|Y(t)|_{\max} < A + 1/4$ . There is a selected solution  $Y_*(t)$  of (43), with  $|Y_*(t)|_{\max} < A + 1/2$ , whose reconstruction*

$$(44) \quad x_*(t) = e^{Y_*(t)} \bar{x}, \quad H_*(t) = e^{Y_*(t)*}(h_0 e^{\Psi(l, x_*(t))})$$

*is an exact flow on the same holomorphic bundle. Moreover,*

$$(45) \quad \begin{aligned} \|u(t) - \Psi(l, e^{Y(t)} \bar{x})\|_{\mathcal{E}_\alpha} &\leq C_A D e^{-4ct}, \\ |Y_*(t) - Y(t)| &\leq C_A (r + D) D e^{-4ct}, \\ e^{-\eta(t)} H_*(t) &\leq H(t) \leq e^{\eta(t)} H_*(t), \quad \eta(t) = C_A D e^{-4ct}. \end{aligned}$$

*The logarithm of the relative metric in the specified frame*

$$(46) \quad \mathcal{E}(t) = \log \left( e^{-\Psi_*/2} e^{-Y_*} (h_0^{-1} H(t)) e^{-Y_*} e^{-\Psi_*/2} \right), \quad \Psi_* = \Psi(l, x_*(t)),$$

*has  $h^{17/8}$  norm at most  $\eta(t)$ , including  $t = 0$ .*

*The phase  $Y_{\text{ph}} = Y_*(0)$  is  $C^1$  in the initial metric at fixed  $(l, \bar{x})$ , with uniform derivative bounds. It equals  $Y_0$  for on-graph initial data. For the same  $Y_0$  and two normal data with  $\mathcal{E}_\alpha$  norms below  $\delta < \delta_A$ ,*

$$(47) \quad |Y_{\text{ph}}^{(1)} - Y_{\text{ph}}^{(2)}| \leq C_A (r + \delta) \|u_0^{(1)} - u_0^{(2)}\|_{\mathcal{E}_\alpha}.$$

*The selected trajectory is unique among solutions defined for all  $t \geq 0$ , with  $|Y(t)|_{\max} < A + 1$ , whose reconstructed metrics have  $d_\infty$  distance  $O(e^{-ct})$  from the actual flow.*

Let  $\lambda_j(t)$  and  $\lambda_j^*(t)$  be the trace-free Dolbeault eigenvalues of  $H(t)$  and  $H_*(t)$ . Both kernels vanish, and

$$(48) \quad \begin{aligned} \left| \log \frac{\lambda_j(t)}{\lambda_j^*(t)} \right| &\leq 4C_A D e^{-4ct} \quad (j \geq 1), \\ \left| \log \frac{\lambda_j(t)}{\nu_j(l, x_*(t))} \right| &\leq C_A r^2 + C_A D e^{-4ct} \quad (1 \leq j \leq 3), \\ \lambda_4(t) &\geq \gamma_A. \end{aligned}$$

The constants are uniform in time and in the balanced point. No positive ratio of the three finite eigenvalues is bounded below.

The phase is relative to the chosen graph. The exponentially decaying comparison with its exact reconstruction is distinct from the static  $O(r^2)$  approximation by the harmonic finite operator. The theorem asserts neither a polarization derivative of the phase nor a uniform fast relaxation to equilibrium.

**5.2. Normal coordinates and removal of the cutoff.** Set  $x = e^Y \bar{x}$  and  $v = u - \Psi(l, x)$ . In the notation of (23), define

$$(49) \quad \begin{aligned} h(Y, v) &= \beta(l, x, \Psi + v) - \beta^\Psi(l, x), \\ F(Y, v) &= \mathcal{N}_l(x, \Psi + v) - \mathcal{N}_l(x, \Psi) - D_x \Psi(l, x) C_x h(Y, v). \end{aligned}$$

Subtracting graph invariance gives exactly

$$\dot{Y} = f_{l, \bar{x}}(Y) + h(Y, v), \quad \dot{v} = -A_l v + F(Y, v).$$

Both forcing terms vanish at  $v = 0$ . Since the balanced point is an equilibrium, (27) gives  $l \leq C|\bar{x}|^2$ . Therefore, on  $|Y|_{\max} \leq A + 3$ ,

$$(50) \quad \|f_{l, \bar{x}}\|_{C^2} \leq C_A r^2.$$

For  $\|v\|_{\mathcal{E}_\alpha} < \rho$ , the smooth first-jet maps and their vanished derivatives at  $(x, u) = (0, 0)$  give

$$(51) \quad \begin{aligned} \|D_v h\| + \|D_v F\| &\leq C_A(r + \rho), \\ \|D_v h(Y, v)\| &\leq C_A(r + \|v\|_{\mathcal{E}_\alpha}), \\ \|D_Y h(Y, v)\| + \|D_Y F(Y, v)\| &\leq C_A \|v\|_{\mathcal{E}_\alpha}. \end{aligned}$$

All second derivatives are bounded. For the first two lines, use  $|x| \leq C_A r$ ,  $\|\Psi\| \leq C_A r^2$  and  $\|D_x \Psi C_x\| \leq C_A r^2$  in (49). For the third, use vanishing at  $v = 0$  and bounded mixed derivatives. Differentiating  $x = e^Y \bar{x}$  introduces  $C_x$ , never its inverse.

Choose a smooth cutoff  $\chi(Y)$  equal to one on  $|Y|_{\max} \leq A + 1$  and supported in  $|Y|_{\max} < A + 2$ . Replace  $f, h, F$  by  $\chi f, \chi h, \chi F$ , extended by zero. First choose  $\rho$  small, then  $r_A$  and  $\delta_A$  small enough for Proposition 4.1, with finite Lipschitz bound  $L < c/8$ . The proposition gives a global cutoff solution and  $\|v(t)\|_{\mathcal{E}_\alpha} \leq 2MD e^{-4ct}$ .

We now prove that the cutoff is inactive. Before a possible first exit from  $|Y|_{\max} < A + 1$ , the reconstructed metric is the exact geometric flow. Its balanced equilibrium is  $\bar{H} = h_0 e^{\Psi(l, \bar{x})}$ . Comparing the separate exponentials with the identity gives initially

$$e^{-a} \bar{H} \leq H^{\text{in}} \leq e^a \bar{H}, \quad a = 2A + C_A r^2 + D.$$

Metric order preserves these inequalities while the reconstruction is exact. On the same interval,  $\|u(t)\|_\infty \leq C_A r^2 + 2MD$ , so

$$e^{-\|u(t)\|_\infty} e^{2Y(t)} \leq h_0^{-1} H(t) \leq e^{\|u(t)\|_\infty} e^{2Y(t)}.$$

Together with  $e^{-Cr^2}h_0 \leq \bar{H} \leq e^{Cr^2}h_0$ , this gives

$$(52) \quad |Y(t)|_{\max} \leq A + C_A(r^2 + D) < A + 1/4$$

after fixed shrinkages. This excludes the first exit. Positive-time strong regularity follows from the first-jet parabolic realization; continuity at zero in  $\mathcal{E}_\alpha$  passes metric order to the initial data. Hence this is the actual flow for every future time, and its coefficients stay in the common graph domain.

Apply the refined phase estimate with  $a = C_A r$ . The selected trajectory differs from  $Y(t)$  by  $C_A(r + D)De^{-4ct}$ , and thus stays in  $|Y_*|_{\max} < A + 1/2$ . Its cutoff is inactive too. It solves the original finite equation, preserves every integrability relation, and reconstructs the exact flow (44). Altering the auxiliary cutoff does not alter this phase: both candidate trajectories stay where the original field acts and track at a rate faster than its common Lipschitz bound.

**5.3. Initial metric coordinates and differentiability.** The strict conditions (42) define an open class of determinant-normalized metrics. To see this, recover  $Y$  from a metric matrix  $Q = h_0^{-1}H$  by

$$(53) \quad P_0 \log(e^{-Y} Q e^{-Y}) = 0.$$

If  $u = \log(e^{-Y} Q e^{-Y})$ , differentiating the centered exponential gives, for  $\xi \in \mathfrak{t}_\mathbb{R}$ ,

$$D_Y \{P_0 \log(e^{-Y} Q e^{-Y})\} \xi = -2R(u)\xi.$$

The lower bound  $R(u) \geq I$  proves local invertibility. Integrating this derivative on a diagonal line segment and pairing with its direction proves uniqueness of a root. The inverse coordinate map is smooth with uniform bounds on bounded diagonal sets and small complementary fields. It uses no orbit-energy bound.

Proposition 4.1 now makes the phase a  $C^1$  function of the initial metric, rather than merely of unspecified coordinates. It extends to a neighborhood in the  $\mathcal{E}_\alpha$  topology; (42) is its strong-topology restriction. The refined normal derivative proves (47) by integration along the normal segment. Formula (41), with  $D_v h = D_u \beta$  along the graph, gives its response to complementary initial data.

**5.4. Relative metrics, spectra, and uniqueness.** Put  $z = Y_* - Y$ . The positive matrix inside (46) is

$$e^{-\Psi_*/2} e^{-z} e^{\Psi(l, e^Y \bar{x}) + v} e^{-z} e^{-\Psi_*/2}.$$

Bounded diagonal coordinates and (25) give

$$|e^Y \bar{x} - e^{Y_*} \bar{x}| \leq C_A r |z|, \quad \|\Psi(l, e^Y \bar{x}) - \Psi_*\|_{\mathcal{E}_\alpha} \leq C_A r^2 |z|.$$

Exponential, multiplication and logarithm are smooth on the relevant positive subsets of the  $h^{17/8}$  algebra. The relative logarithm vanishes at  $z = v = 0$ . Normal decay and the phase estimate therefore give  $\|\mathcal{E}(t)\|_{h^{17/8}} \leq C_A D e^{-4ct}$ . This proves the strong and order estimates in (45). It does not use order monotonicity of exponentiation on noncommuting logarithms.

The induced endomorphism squared norms in degrees zero and one are within  $e^{\pm 2\eta(t)}$ . Their differential is unchanged, so min-max compares every positive eigenvalue within  $e^{\pm 4\eta(t)}$ . Lemma 7.1 below applies to the graph metric at  $x_*(t)$ , with logarithmic error  $C_A r^2$  and a fixed fourth gap. Pullback by  $e^{Y_*}$  is an exact isometry, introducing no condition-number constant. Connected support persists at finite times, so the trace-free kernel is zero. This proves (48).

Finally, on  $|Y|_{\max} < A + 1$  the map

$$Y \longmapsto P_0 \log \left( e^Y e^{\Psi(l, e^Y \bar{x})} e^Y \right)$$

has derivative  $2I + O_A(r^2)$ . Integrating along a line segment gives a lower Lipschitz bound independent of weak orbit energies. Two graph reconstructions within  $O(e^{-ct})$  in  $d_\infty$  have diagonal coordinates within  $O(e^{-ct})$ : their metrics lie in a fixed bounded positive range, where the logarithm is Lipschitz. The finite field has Lipschitz bound  $L < c$ , so the backward distance inequality in Proposition 4.1 forces the two phases to agree. This completes the proof of Theorem 5.1.

## 6. ENTRY FROM A PRESCRIBED INITIAL METRIC

The local theorem requires a small complementary metric component. For any one prescribed smooth metric we now verify that condition at a common finite time for all sufficiently small deformations and polarizations. The time is chosen first. The neighborhood of parameters and initial metrics is chosen afterwards.

**6.1. A metric limit for the split reference flow.** Fix a smooth positive metric  $H^0$  on the smooth  $E_0$ , with  $\det H^0 = \det h_0$ . At  $(l, x) = (0, 0)$ , the metric  $h_0$  is stationary. Choose  $M$  so that  $e^{-M}h_0 \leq H^0 \leq e^M h_0$ . Global metric-flow existence [Kob87, Theorem 6.8.25] and metric order give

$$(54) \quad e^{-M}h_0 \leq H_0(t) \leq e^M h_0 \quad (t \geq 0).$$

The cited book uses velocity  $-(i\Lambda F - \kappa_E I)$ ; its time is  $2t$  in (14).

**Lemma 6.1.** *There are  $T_n \rightarrow \infty$  and  $Y_\infty \in \mathfrak{t}_\mathbb{R}$ ,  $|Y_\infty|_{\max} \leq M/2$ , such that*

$$H_0(T_n) \longrightarrow e^{Y_\infty} * h_0 \quad \text{in } C^\infty.$$

*Proof.* The split bundle is polystable at  $\omega_0$ , and its graded sheaf is itself locally free. The corrected [SW15, Theorem 1.1(2)] gives smooth global connection convergence modulo unitary gauge, since its algebraic singular set is empty. Only a subsequence is needed. The metric-to-connection normalization can be checked as follows. A moving isometry  $B : (E_0, H_0(t)) \rightarrow (E_0, h_0)$  with  $B^{-1}\dot{B} = -i\Lambda F_{H_0}$  transports the Chern connection to a connection satisfying

$$\dot{A}^{0,1} = \bar{\partial}_A(i\Lambda F_A), \quad \dot{A}^{1,0} = -\partial_A(i\Lambda F_A), \quad \dot{A} = -d_A^* F_A.$$

The last identity is Kähler. Thus the time convention agrees with the connection convergence theorem. The distinct line factors and uniqueness of their HE connections identify the limit with  $A_0$ . Composing the isometries with the convergence gauges gives smooth complex maps  $B_n$  such that

$$(55) \quad B_n^* h_0 = H_0(T_n), \quad B_n d B_n^{-1} = d + b_n, \quad b_n \longrightarrow 0 \quad \text{smoothly.}$$

This connection assertion alone does not imply metric convergence.

By (54), the singular values of  $B_n$  belong to  $[e^{-M/2}, e^{M/2}]$ . The maps and their inverses are uniformly bounded in  $C^0$ . Holomorphicity gives  $d_{\text{End}} B_n = -b_n B_n$ . For every integer  $j \geq 0$ , the fixed overdetermined elliptic zero-form estimate gives

$$\|B_n\|_{H^{j+1}} \leq C_j(\|b_n B_n\|_{H^j} + \|B_n\|_{L^2}) \leq C'_j(\|B_n\|_{H^j} + 1).$$

Induction and Sobolev compactness give a smoothly convergent subsequence  $B_n \rightarrow B_\infty$ . The lower singular-value bound makes  $B_\infty$  invertible, and its limit equation makes it a holomorphic endomorphism of the split bundle. It is therefore constant diagonal. Taking positive

diagonal moduli defines  $e^{Y_\infty}$ ; phases disappear from  $B_\infty^* h_0$ . The determinant condition gives  $\sum_i Y_{\infty,i} = 0$ , and the singular-value bound gives  $|Y_\infty|_{\max} \leq M/2$ .  $\square$

No rate, full-time metric convergence, or differentiable dependence of  $Y_\infty$  on  $H^0$  is required. Initial-data differentiability will instead be obtained at a fixed finite time.

**6.2. Strong finite-time dependence before entry.** Write  $H = h_0 e^W$ , without requiring  $W$  to be small. The centered metric equation is  $\dot{W} = -2S_W^{-1}\mathcal{M}(l, x, W)$ . For Hermitian  $W$ ,  $S_W$  is invertible: its multipliers  $\sinh(a/2)/(a/2)$  are at least one. On bounded ranges of  $W$ , its inverse and all fixed finite derivatives are bounded.

The equation has a fixed scalar elliptic principal symbol for each  $l$ , also when  $W$  is large. For a contracted second jet  $Z$ , the principal term in  $\bar{\partial}(H^{-1}\partial H)$  is  $e^{-W}(d\exp)_W(Z)$ . Centering gives  $e^{-W/2}(d\exp)_W(Z)e^{-W/2} = S_W Z$ . Multiplication by  $S_W^{-1}$  removes this coefficient. All remaining terms use at most first derivatives of  $W$ . Thus, in a neighborhood of any fixed finite reference trajectory,

$$(56) \quad \dot{W} + \mathcal{A}_l W = \mathcal{F}(l, x, W, \nabla_0 W),$$

where  $\mathcal{A}_l$  can be taken to be  $2L_l$ , with lower-order terms absorbed in  $\mathcal{F}$ . This asserts equality of principal symbols and smooth first-jet regularity, not smallness of the nonlinearity along that trajectory. Trace of the metric equation is the scalar heat equation for  $\log(\det H / \det h_0)$ , so the trace-free space is preserved.

For nonintegrable  $x$ , use the compatible unitary connection defined by (17). Its contracted  $(1,1)$  curvature gives the same smooth ambient equation. The symbol calculation is local and requires no holomorphic frame for this extension. Parameters can therefore vary in an open ball before restriction to  $\kappa(x) = 0$ .

Use the unprojected spaces

$$Z_0 = h^{3/8}(\mathcal{V}), \quad Z_1 = h^{19/8}(\mathcal{V}), \quad Z_\beta = h^{13/8}(\mathcal{V}), \quad \beta = 5/8.$$

The first-jet map is smooth  $Z_\beta \rightarrow Z_0$ . The normally elliptic bundle realization has continuous maximal regularity including weight one [SS14, Theorem 3.6 and Proposition 3.7]. No normal projection or spectral gap is needed here; a harmless scalar shift of  $\mathcal{A}_l$  can be moved into  $\mathcal{F}$ . Append the finite parameters  $(l, x)$  as variables with zero derivative. Their operator dependence is smooth in  $\mathcal{L}(Z_1, Z_0)$ . The local semiflow theorem [Sim95, Theorem 3.1], with its allowed endpoint  $\alpha = 1$  and  $\beta = 5/8$ , applies.

For a fixed finite  $T$ , the reference logarithmic trajectory has compact image in  $Z_1$ . Cover it by finitely many local semiflow neighborhoods and compose their time maps. This gives an open neighborhood of  $(0, 0, W^0)$ , where  $W^0 = \log(h_0^{-1}H^0)$ , on which solutions exist through  $T$  and

$$(57) \quad (l, x, W^{\text{in}}) \mapsto W_{l,x,W^{\text{in}}}(T) \quad \text{is } C^1 \text{ into } Z_1.$$

Its derivative is uniformly bounded on a smaller neighborhood, by continuity at that base point. The bound may depend on  $T$  and  $H^0$ . No uniform infinite-time parameter estimate has been used.

**Proposition 6.2.** *There is  $A > 0$ , depending on  $H^0$  but not on the accuracy, such that for every  $\varepsilon > 0$  there are a finite  $T$ , positive  $l_\varepsilon, \rho_\varepsilon$ , and an open neighborhood  $\mathcal{U}_\varepsilon$  of  $W^0$  in  $Z_1$  with the following property. For*

$$|l| < l_\varepsilon, \quad |x| < \rho_\varepsilon, \quad \kappa(x) = 0, \quad W^{\text{in}} \in \mathcal{U}_\varepsilon,$$

the flow at time  $T$  has a normalization

$$(58) \quad H(T) = e^{Y_T *}(h_0 e^{u_T}), \quad |Y_T|_{\max} < A, \quad P_0 u_T = 0, \quad \|u_T\|_{h^{19/8}} < \varepsilon.$$

The time is selected before the parameter and initial neighborhoods. The normalization and time map are  $C^1$  in the initial metric with uniform derivative bounds on those neighborhoods. No stability hypothesis is needed for this finite-time statement.

*Proof.* Near  $H_\infty = e^{Y_\infty *} h_0$  from Lemma 6.1, solve  $P_0 \log(e^{-Y}(h_0^{-1} H)e^{-Y}) = 0$ . Its  $Y$  derivative at  $(Y_\infty, H_\infty)$  is  $-2I$ . The implicit-function theorem gives  $C^1$  maps  $Y(H)$  and  $u(H)$  in  $h^{19/8}$ ; trace zero follows from the determinant and  $\sum Y_i = 0$ . Choose  $A = M/2 + 2$  and fix a normalization neighborhood in which  $|Y(H)|_{\max} < A$ .

First select  $T = T_n$  so large that  $H_0(T)$  belongs to this neighborhood and

$$\|u(H_0(T))\|_{h^{19/8}} < \varepsilon/2.$$

Next apply (57) and continuity of the normalization to shrink the parameter and initial neighborhoods so that every  $H(T)$  lies there and has normal norm below  $\varepsilon$ . The derivative assertion follows by composition of the fixed-time maps.  $\square$

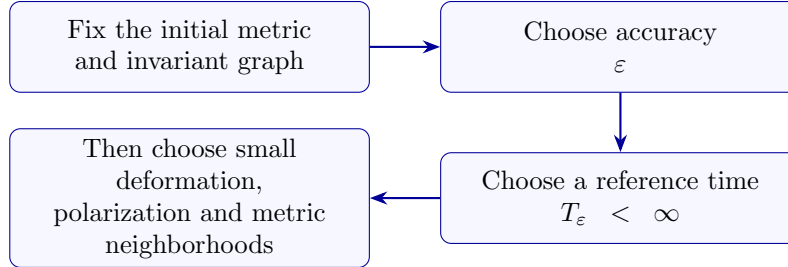


FIGURE 2. The order of choices in Theorem 6.3. The restart may depend on the prescribed metric and accuracy. It is chosen before the small parameters, not uniformly over all initial metrics.

### 6.3. Common restart and phase for the full stable family.

**Theorem 6.3.** Fix  $H^0$  as above and the graph of Theorem 3.2. There are  $A, C, c, \gamma, \varepsilon_0 > 0$  such that for each  $0 < \varepsilon < \varepsilon_0$  one can choose  $T_\varepsilon, l_\varepsilon, \rho_\varepsilon > 0$  and an open neighborhood  $\mathcal{U}_\varepsilon$  of  $W^0$  in  $h^{19/8}$  with the following conclusions. For every

$$(59) \quad 0 < l < l_\varepsilon, \quad \bar{x} \in \mathcal{B}_l^{\text{st}}(\rho_\varepsilon), \quad W^{\text{in}} \in \mathcal{U}_\varepsilon,$$

the flow from  $h_0 e^{W^{\text{in}}}$  has a common restart  $T = T_\varepsilon$  and a selected graph trajectory  $H_*(s)$ ,  $s \geq 0$ , on the same bundle, such that

$$(60) \quad \begin{aligned} H(T) &= e^{Y_T *}(h_0 e^{u_T}), \quad |Y_T|_{\max} < A, \\ v_T &= u_T - \Psi(l, e^{Y_T} \bar{x}), \quad \|v_T\|_{\mathcal{E}_1} < \varepsilon, \quad D_T = \|v_T\|_{\mathcal{E}_\alpha} \leq \varepsilon, \\ e^{-\eta(s)} H_*(s) &\leq H(T+s) \leq e^{\eta(s)} H_*(s), \quad \eta(s) = C D_T e^{-4cs}, \\ |Y_{\text{ph}} - Y_T| &\leq C(|\bar{x}| + D_T) D_T. \end{aligned}$$

The relative logarithm in the explicit frame (46), with actual time  $T + s$ , has  $h^{17/8}$  norm at most  $\eta(s)$ . The uniqueness class is that of Theorem 5.1.

For the actual and selected trace-free Dolbeault eigenvalues,

$$(61) \quad \begin{aligned} \left| \log \frac{\lambda_j(T+s)}{\lambda_j^*(s)} \right| &\leq 4CD_T e^{-4cs} \quad (j \geq 1), \\ \left| \log \frac{\lambda_j(T+s)}{\nu_j(l, e^{Y_*(s)} \bar{x})} \right| &\leq C|\bar{x}|^2 + CD_T e^{-4cs} \quad (1 \leq j \leq 3), \\ \lambda_4(T+s) &\geq \gamma. \end{aligned}$$

At fixed  $(l, \bar{x})$ , the restart phase is  $C^1$  in  $W^{\text{in}}$  with derivative bounded uniformly over (59). After a further shrinkage of  $\rho_\varepsilon$ , the finite trajectory extends back to original time zero within the reconstruction domain, and its value there has the same  $C^1$  initial-data dependence.

The fixed constants may depend on  $H^0$  and the chosen graph. The restart, neighborhoods and derivative bound may also depend on  $\varepsilon$ . None requires a positive lower bound on  $\nu_1(l, \bar{x})/\nu_3(l, \bar{x})$ . The comparison is asserted from  $T_\varepsilon$  onward, including when the finite trajectory has been continued to an earlier time.

*Proof.* Choose the bound  $A$  from Proposition 6.2, then fix the local phase constants for this  $A$ . Take  $\varepsilon_0 < \delta_A/2$ . For a given  $\varepsilon$ , apply the entry proposition with tolerance  $\varepsilon/2$ . It selects  $T_\varepsilon$  before the neighborhoods. Retaining that time, shrink the coefficient neighborhood so

$$\|\Psi(l, e^{Y_T} \bar{x})\|_{\mathcal{E}_1} \leq C e^{4A} |\bar{x}|^2 < \varepsilon/2,$$

and so that  $2\rho_\varepsilon$  lies within the local phase radius and the larger bounded diagonal translates lie within the graph domain. This proves the first two lines of (60). The stable balanced domain is nonempty arbitrarily near zero: the stationary three-step family of [Wue26a] provides connected balanced representatives of size  $O(\sqrt{l})$ . The stationary equation itself forces  $l \leq C|\bar{x}|^2$ , without constraining individual positive spectral ratios.

Apply Theorem 5.1 with radius  $2|\bar{x}|$ . Its exact reconstruction and uniqueness identify the actual flow after restart, giving all the metric and spectral assertions. The strong relative matrix is exactly (46) with this shifted time origin. The fixed-time restart map is  $C^1$  by (57) and normalization; its composition with the local phase map gives the stated dependence. No limiting reference diagonal has been differentiated in the initial metric.

On  $|Y|_{\max} \leq A+2$  the exact finite field obeys  $\|f_{l, \bar{x}}\|_{C^1} \leq C_{A+2} |\bar{x}|^2$ . The selected restart phase lies in  $|Y_{\text{ph}}|_{\max} < A+1/2$ . Shrink  $\rho_\varepsilon$  so that  $C_{A+2} \rho_\varepsilon^2 T_\varepsilon < 1/4$ . A first-exit argument then continues the finite ODE backward for time  $T_\varepsilon$ , within  $|Y|_{\max} < A+1$ . This is backward continuation of a finite ODE, not a backward parabolic initial-value problem. Its derivative is bounded by  $\exp(C_{A+2} \rho_\varepsilon^2 T_\varepsilon)$ . Composing with the restart phase proves the original-time assertion, without extending the small-error comparison before entry.  $\square$

## 7. NONSTATIONARY SPECTRA AND EQUILIBRIUM RATES

The spectral comparison used in the phase theorem does not require the moving metric to be HE. The identification of finite decay rates, by contrast, is an equilibrium assertion. We prove the two statements separately.

**7.1. A relative estimate along the whole trajectory.** Let  $D_x = d + [a(x), \cdot]$  on complex  $\text{End}_0 E_0$  zero-forms. Extend  $P_0, Q_0$  complex linearly. The norms in this subsection are the normalized spatial norms for  $\omega_l$  and  $h_0$ .

**Lemma 7.1.** *There are uniform  $C, \gamma, r_0 > 0$  such that, whenever  $|l| < l_0$ ,  $|x| < r_0$  and  $\kappa(x) = 0$ ,*

$$(62) \quad (1 - C|x|^2)\nu_j(l, x) \leq \lambda_j(D_x^* D_x) \leq (1 + C|x|^2)\nu_j(l, x) \quad (1 \leq j \leq 3), \quad \lambda_4 \geq \gamma.$$

*Zeros are included, and the kernel is exactly  $\ker C_x$ , represented by constant diagonals. The same assertions hold with relative error  $O(|x|^2)$  for the metric  $h_0 e^{\Psi(l, x)}$ . More generally a change of metric with logarithmic order distance at most  $\epsilon$  changes each positive eigenvalue by a factor between  $e^{-4\epsilon}$  and  $e^{4\epsilon}$ .*

*Proof.* The split Poincaré inequality on  $Q_0$  and the smallness of  $a$  give, by absorption,

$$\|\eta\|_{W^{1,2}} \leq C\|D_x \eta\|, \quad \eta \in Q_0 W^{1,2}.$$

For  $\xi \in \mathfrak{t}$ , equivariance gives  $D_x \xi = -J_x C_x \xi$ . The harmonic orthogonality in (12) implies

$$\|J_x v\|^2 = (1 + O(|x|^2))\|lv\|^2, \quad |\langle J_x v, D_x \eta \rangle| \leq C|x|\|v\| \|D_x \eta\|.$$

For the second estimate, the harmonic term pairs trivially with  $d\eta$ . Its pairing with  $[a, \eta]$  is bounded by  $C|x|\|v\|\|\eta\|_{L^2}$ . The correction  $B_x v$  is  $O(|x|)\|v\|$  in a fixed smooth norm and pairs with  $d\eta$  by the same bound. All constants are uniform in  $l$ , since the weighted harmonic orthogonality is exact.

Minimize  $\|D_x(\xi + \eta)\|^2$  over complementary  $\eta$ . Complementary coercivity gives a unique linear minimizer  $\eta_x \xi$ , and the variational equation and preceding pairing show

$$\|\eta_x \xi\|_{W^{1,2}} \leq C|x|\|C_x \xi\| \leq C|x|^2|\xi|.$$

The minimized form  $s_x$  consequently satisfies

$$(1 - C|x|^2)\|C_x \xi\|^2 \leq s_x(\xi) \leq (1 + C|x|^2)\|C_x \xi\|^2.$$

If  $C_x \xi = 0$ , the minimizer is zero. Conversely a zero of the minimized form has  $C_x \xi = 0$ , and complementary coercivity then forces  $\eta = 0$ . Thus the kernel assertion is exact.

On the form domain use  $T_x(\xi + \eta) = \xi + \eta_x \xi + \eta$ . It preserves that domain, differs from the identity by  $O(|x|^2)$  in  $L^2$ , and makes the derivative-energy form the orthogonal sum of  $s_x$  and  $\|D_x \eta\|^2$ . The mass form changes by  $1 + O(|x|^2)$ . Min-max, first on this three-dimensional finite space and then on its uniformly coercive complement, proves (62). This argument never divides by a positive eigenvalue of  $C_x^* C_x$ .

If  $e^{-\epsilon} H_1 \leq H_2 \leq e^\epsilon H_1$ , the induced endomorphism norms, on both zero-forms and one-forms, compare by  $e^{\pm 2\epsilon}$ . The holomorphic differential is unchanged. Its Rayleigh quotients and then its ordered positive eigenvalues compare by  $e^{\pm 4\epsilon}$ . The holomorphic kernel is unchanged as well. Apply this with  $\|\Psi(l, x)\|_{C^0} \leq C|x|^2$  for the graph metric.  $\square$

This proof also explains why an additive estimate for the three eigenvalues would be insufficient: its error could exceed the smallest mode when the three scales separate.

**7.2. Exact rates at a balanced point.** Let  $\bar{x} \in \mathcal{B}_l^{\text{st}}(r)$  and  $\bar{u} = \Psi(l, \bar{x}) = w(l, \bar{x})$ . The metric  $\bar{H} = h_0 e^{\bar{u}}$  is HE and  $f_{l, \bar{x}}(0) = 0$ . Write

$$L_{\text{dec}} = -D_Y f_{l, \bar{x}}(0).$$

Its eigenvalues describe linear decay of the exact finite equation.

**Proposition 7.2.** *The operator  $L_{\text{dec}}$  is self-adjoint for a positive inner product on  $\mathfrak{t}_{\mathbb{R}}$  and has three positive eigenvalues  $d_1 \leq d_2 \leq d_3$  satisfying*

$$(63) \quad d_j = 2\lambda_j(\bar{H}) = 2\nu_j(l, \bar{x})(1 + O(|\bar{x}|^2)), \quad 1 \leq j \leq 3.$$

The first equality is exact. The constants in the second are uniform without a lower bound on any ratio of the  $\nu_j$ .

*Proof.* Differentiate the graph metric  $e^Y * (h_0 e^{\Psi(l, e^Y \bar{x})})$  at  $Y = 0$ . In the  $\bar{H}$ -isometric frame its tangent map is

$$(64) \quad W\xi = 2C_{\bar{u}}\xi + S_{\bar{u}}D_x\Psi(l, \bar{x})C_{\bar{x}}\xi = 2\xi + O(|\bar{x}|^2)|\xi|.$$

The error estimate holds in  $h^{19/8}$ , so  $W$  is injective. Differentiating exact reconstruction at this stationary solution gives

$$WD_Y f_{l, \bar{x}}(0) = -2\tilde{\Delta}_{\bar{\delta}, \bar{H}}W.$$

Here the tilde denotes the isometric expression in the fixed Hermitian bundle. At an HE metric the curvature commutator vanishes; the real metric linearization is exactly  $-2\Delta_{\bar{\delta}}$ , which preserves Hermitian endomorphisms.

By (27), the finite derivative is  $O(|\bar{x}|^2)$ : balance gives  $|b(l)| \leq C|\bar{x}|^2$ . The image of  $W$  is therefore a three-dimensional invariant subspace with eigenvalues of this size. Lemma 7.1 gives exactly three small complex eigenvalues and a fixed fourth gap. Adjoint conjugation commutes with this HE Laplacian, so its small complex spectral subspace is the complexification of its three-dimensional Hermitian part. The image of  $W$  is that part. Pull back its  $L^2$  inner product by  $W$ . The displayed intertwining identity proves self-adjointness and the exact eigenvalue assertion. The relative comparison follows from Lemma 7.1.  $\square$

Thus  $d_1^{-1}$  is the weakest linearized relaxation time. Uniform exponential attraction to the finite graph in Theorem 5.1 coexists with an arbitrarily large  $d_1^{-1}$ . A linearized equilibrium rate alone does not specify the time at which a nonlinear trajectory reaches a prescribed tolerance.

**7.3. A larger initial class with constant error.** For comparison, one can dispense with proximity to the graph if a small constant error is sufficient. Suppose  $|Y_0|_{\max} < A$ ,  $\|u_0\|_{\mathcal{E}_1} < \delta$  and  $\bar{x} \in \mathcal{B}_l^{\text{st}}(r)$ . Start the finite graph flow at exactly  $Y_0$ , without selecting its phase. At time zero its metric and the prescribed metric have order distance at most  $\eta = C(\delta + e^{4A}r^2)$ . Comparison with the equilibrium metric confines this graph flow just as in the proof of Theorem 5.1; hence it exists for all future time. Order preservation between the two exact flows gives

$$e^{-\eta}H_{\text{red}}(t) \leq H(t) \leq e^{\eta}H_{\text{red}}(t), \quad \left| \log \frac{\lambda_j(H(t))}{\nu_j(l, e^{Y_{\text{red}}(t)}\bar{x})} \right| \leq 4\eta + C_A r^2, \quad 1 \leq j \leq 3.$$

The fourth eigenvalue stays above a fixed positive constant. This comparison does not identify a phase: a fixed nonzero error can hide the slow tail produced by an incorrect finite initial value. The selection argument of Section 4 is what replaces this constant by an exponentially decreasing discrepancy between exact reconstructed metrics.

## 8. TWO INDEPENDENT SLOPE PARAMETERS

The complete-family phase theorem follows the path  $k = 3l/5$ . There is also a three-layer family on which both slope parameters vary independently. Its total arrow energy supplies a confinement argument that is unavailable on the complete 83-dimensional space. We retain this application separately.

**8.1. The exact three-layer presentation.** Use the same four line bundles and put  $p = (k, l)$ ,  $\sigma = k + l$ ,

$$(65) \quad h_1 = \left(1 + \frac{k + 5l}{24\pi}\right)^{-1}, \quad h_2 = \left(1 + \frac{5k + l}{24\pi}\right)^{-1}, \quad h_3 = 1.$$

The local construction again uses an open signed neighborhood of zero. Its stable application has  $k, l > 0$ . Let  $V_\diamond = V_{12} \oplus V_{13} \oplus V_{24} \oplus V_{34}$ , of complex dimensions 8, 8, 4, 4, and write its entries as  $z = (A, C, B, D)$ . In harmonic polynomial bases write  $A = (a_0, a_1)$  and  $C = (c_0, c_1)$  as pairs of cubic polynomials, and  $B = (b_0, b_1)$  and  $D = (d_0, d_1)$  as pairs of linear polynomials in the coordinates  $[s : t]$  of the third  $\mathbb{P}^1$ . The obstruction is

$$(66) \quad \Omega(z) = a_0 b_0 + a_1 b_1 - c_0 d_0 - c_1 d_1 \in H^0(\mathbb{P}^1, \mathcal{O}(4)).$$

For the current product metric define

$$\Gamma_p(z) = -d_{14,p}^* G_{2,p}(A \wedge B + C \wedge D).$$

Let  $N_p(z)$  have these four adjacent entries and central entry  $\Gamma_p(z)$  in block 14. The Hodge decomposition gives  $d\Gamma_p = -(A \wedge B + C \wedge D)$  exactly when  $\Omega(z) = 0$ . There are no longer composable paths, so  $d + N_p(z)$  is then integrable with no further correction.

The harmonic representatives and their cohomological projections are fixed as  $p$  varies. The difference  $\Gamma_p - \Gamma_0$  is closed, and  $H^1(F_1 F_4^{-1}) = 0$  makes it exact. The unique coexact zero-form primitive  $f_p$  gives the determinant-one map

$$(67) \quad df_p(z) = \Gamma_p(z) - \Gamma_0(z), \quad \Phi_p(z) = I - f_p(z)E_{14}, \quad (d + N_p(z))\Phi_p(z) = \Phi_p(z)(d + N_0(z)).$$

Here  $f_p$  denotes a section of the indicated line bundle;  $E_{14}$  records its block. It is quadratic in  $z$ , smooth in  $p$ , and equivariant under constant diagonal gauges. This is an exact map, since its square and all adjacent products with it vanish. Write  $E_{z^*}$  for the fixed reference bundle  $d + N_0(z^*)$ .

The central primitive is essential. For example choose equal normalized harmonic bases on the first two factors and

$$(68) \quad A = C = (s^3, 0), \quad B = D = (s, 0).$$

Both path cohomology products equal the nonzero quartic  $s^4$  and cancel with the sign in (66). Their differential-form sum need not vanish. In the usual representatives, if  $\theta_i$  is the harmonic representative of  $\mathcal{O}(-2)$ ,  $\eta_i$  one of  $\mathcal{O}(-3)$ , and  $\tau_i$  the dual  $\mathcal{O}(1)$  section, that sum has the factor  $(\tau_1 \eta_1 \wedge \theta_2 - \theta_1 \wedge \tau_2 \eta_2)$ . The form  $\tau_i \eta_i$  has harmonic projection  $\theta_i$  but is not equal to it. Evaluating at a zero of  $\tau_1$  and a point where the second term is nonzero proves nonvanishing. Thus  $\Gamma_p$  is nonzero, as in [Wue26a, Section 6.4].

**8.2. Graph construction and confinement.** Apply the sliced construction of Section 3 with  $a(x)$  replaced by  $N_p(z)$  and both parameters adjoined with zero evolution. Its hypotheses remain valid on the ambient 24-dimensional coefficient ball: the adjacent entries are current-harmonic, the central entry is quadratic and co-closed, the principal second jet is still removed by  $S_u^{-1}$ , and the fixed differential domain and split normal gap are unchanged. The same parameterwise tangency equation gives a chosen  $C^m$  graph  $\Psi_\diamond(p, z) \in Q_0 h^{19/8}$  with  $\Psi_\diamond = O(q)$ , where

$$w = (\|A\|_p^2, \|C\|_p^2, \|B\|_p^2, \|D\|_p^2), \quad q = \sum_e w_e.$$

It is chosen before any initial metric. Its exact field  $\beta_\diamond$  is the expression in (26) with this curvature and graph. In particular

$$(69) \quad \dot{z}_{ij} = (\beta_i - \beta_j)z_{ij}, \quad \beta_\diamond = b - Dw + E_p(z), \quad |E_p(z)| \leq Kq^2,$$

where  $D$  has columns  $e_1 - e_2, e_1 - e_3, e_2 - e_4, e_3 - e_4$  and

$$(70) \quad b = \frac{1}{8}(5\sigma, 3k - 5l, -5k + 3l, -3\sigma).$$

The quartic estimate follows either from the orthogonality proof of Lemma 3.3, now using the current homotopy, or from the diagonal unitary sign  $\text{diag}(-1, 1, 1, -1)$ : it negates every adjacent entry, preserves the central quadratic entry, and makes the projected field even. The exact field, rather than its displayed truncation, is used throughout.

Let  $a = (1, 0, 0, -1)$ . Each incidence column has pairing one with  $a$ , so

$$(71) \quad a \cdot Dw = q, \quad \frac{1}{2}q^2 \leq |Dw|^2 \leq 2q^2.$$

The lower bound is Cauchy–Schwarz; the upper bound follows by the triangle inequality and  $|D_e| = \sqrt{2}$ .

**Lemma 8.1.** *There is a fixed  $q_* > 0$  such that every full-support integrable trajectory of (69) with initial energy  $q_0$  and*

$$(72) \quad 0 < q_0 < q_*, \quad |b| \leq q_0/(8\sqrt{2})$$

*exists in the graph domain for all  $t \geq 0$ , and  $0 < q(t) \leq q_0$ . Every initially nonzero block remains nonzero at finite time.*

*Proof.* The exact multiplicative equation gives

$$\dot{q} = 2(Dw) \cdot \beta_\diamond \leq -q^2 + 2\sqrt{2}|b|q + 2\sqrt{2}Kq^3.$$

Choose  $q_*$  with  $2\sqrt{2}Kq_* \leq 1/2$ , and small enough that  $q \leq q_*$  and  $\Psi_\diamond = O(q)$  lie strictly inside the graph and cutoff domains. At  $q = q_0$  the right side is at most  $-q_0^2/4$ , which excludes a first upward crossing. The finite smooth field on the resulting closed coefficient ball has no finite continuation obstruction. Each block is multiplied by the exponential of a finite real integral and cannot vanish.  $\square$

Only the upper barrier for the total energy is used. Individual weights may increase. No positive lower bound on a cut energy is part of the argument.

**8.3. An open initial class and all-future comparison.** For any full-support  $z^*$  with  $\Omega(z^*) = 0$ , take a positive determinant-one diagonal  $g_{\text{in}}$  such that  $z_0 = g_{\text{in}}z^*$  satisfies the strict energy conditions above. Allow smooth metrics

$$(73) \quad H^{\text{in}} = (g_{\text{in}}\Phi_p(z^*))^*(h_0e^{u_{\text{in}}}), \quad P_0u_{\text{in}} = 0, \quad \text{tr } u_{\text{in}} = 0, \quad \|u_{\text{in}}\|_{h^{19/8}} < \delta.$$

As the admissible diagonal varies, this is an open class of determinant-normalized metrics in the strong topology. Indeed the map  $(Y, u) \mapsto \log(e^Y e^u e^Y)$  has derivative  $2Y + u$  at zero, whose inverse is the  $P_0, Q_0$  splitting. The inverse-function theorem and translation by the fixed bundle maps give the claim; the strict energy inequalities persist. The class contains infinitely many normal directions off the graph.

Integrate  $\dot{g}g^{-1} = \beta_\diamond(p, gz^*)$  from  $g_{\text{in}}$  and reconstruct

$$(74) \quad H_{\text{red}}(t) = (g(t)\Phi_p(z^*))^*(h_0e^{\Psi_\diamond(p, g(t)z^*)}).$$

Let  $\nu_j(w)$  be the three positive eigenvalues of the four-vertex Laplacian with energy  $\sum_e w_e |\xi_i - \xi_j|^2$  and mass  $\sum_i |\xi_i|^2$  on  $\sum_i \xi_i = 0$ .

For  $k, l > 0$ , the *quadratic balance* is the positive diagonal orbit representative for which  $Dw = b$ . It exists and is unique on each full-support orbit, by [Wue26a, Section 7.1]. We use the subscript eq for its finite weights and eigenvalues.

**Theorem 8.2.** *For sufficiently small fixed  $q_*, \delta_* > 0$ , every initial metric (73) with  $\delta < \delta_*$  has, for all  $t \geq 0$ ,*

$$(75) \quad \begin{aligned} e^{-\eta} H_{\text{red}}(t) &\leq H(t) \leq e^{\eta} H_{\text{red}}(t), \quad \eta = \delta + C_{\Psi} q_0, \\ \left| \log \frac{\lambda_j(H(t))}{\nu_j(w(t))} \right| &\leq 4\eta + C_s q_0 \quad (1 \leq j \leq 3), \\ \lambda_4(H(t)) &\geq \gamma. \end{aligned}$$

*All constants are independent of the full-support orbit, positive  $k, l$ , time and positive eigenvalue ratios. The interval  $0 \leq t \leq 2/\nu_{1,\text{eq}}$  is therefore covered with these same constants, where the subscript denotes the quadratic balance on the same orbit.*

*Proof.* Lemma 8.1 and exact reconstruction give an all-future graph metric. Comparing  $e^{u_{\text{in}}}$  and  $e^{\Psi_{\diamond}(p, z_0)}$  separately with the identity gives the initial sandwich with  $\eta$ ; no operator monotonicity of the matrix exponential is invoked. Global metric-flow existence and order preservation propagate it.

For the current co-closed reconstruction  $N_p$ , the proof of Lemma 7.1 applies verbatim to its derivative: its harmonic inclusion and coexact correction are orthogonal, and  $\|DN_p - \iota\| \leq C|z|$ . It gives relative error  $O(q)$  and a fixed complementary gap for  $h_0$ . Changing to the graph metric adds  $O(q)$ , and its bundle pullback is an isometry. The metric sandwich contributes  $4\eta$ . Since  $q(t) \leq q_0$ , these are (75). Connected support and the exact kernel assertion make all three finite modes positive. Shrinking  $q_*, \delta_*$  makes the transferred fourth gap uniform.

The stationary theorem [Wue26a, Corollary 7.1] gives  $\lambda_{1,\text{eq}}/\nu_{1,\text{eq}} = 1 + O(\sigma)$ . For small  $\sigma$  this ratio is at least  $1/2$ , so the stated interval contains  $[0, 1/\lambda_{1,\text{eq}}]$ . The linearized metric rate is  $2\lambda_{1,\text{eq}}$  by our time normalization. This is a covered time range, not a nonlinear equilibration-time assertion.  $\square$

To see that the initial class has no implicit ratio restriction, let  $z_{\text{eq}} = g_{\text{eq}} z^*$  be its positive quadratic balance. The stationary diamond result gives  $q_{\text{eq}} = \sigma$ , and (71) gives  $|b| \leq \sqrt{2}\sigma$ . The height gauge

$$g_{\text{ht}} = \text{diag}(\sqrt{32}, 1, 1, 1/\sqrt{32})$$

multiplies all adjacent coefficients by  $\sqrt{32}$  and the central primitive by 32. Thus  $g_{\text{in}} = g_{\text{ht}} g_{\text{eq}}$  has  $q_0 = 32\sigma$  and satisfies the energy condition strictly. It has an open admissible neighborhood. Taking  $\delta = Bq_0$  makes every error in (75)  $O(\sigma)$ . The condition number of  $g_{\text{eq}}$  may diverge; all bundle pullbacks are exact isometries in the estimates.

**8.4. An explicit separation of time scales.** For (68), identical normalized bases give the cycle invariant  $\rho = w_{12}w_{24}/(w_{13}w_{34}) = 1$ . The balancing weights and nonzero characteristic polynomial are

$$(76) \quad \begin{aligned} w_{\text{eq}} &= \frac{1}{8}(5l, 5k, 3k, 3l), \\ s^3 - 2(k+l)s^2 + \left(\frac{15}{16}(k^2 + l^2) + 3kl\right)s - \frac{15}{16}kl(k+l). \end{aligned}$$

The weights satisfy  $Dw = b$  and the cycle identity. Expanding the weighted Laplacian determinant gives the polynomial. At  $k = 0$  the graph has two components, with positive

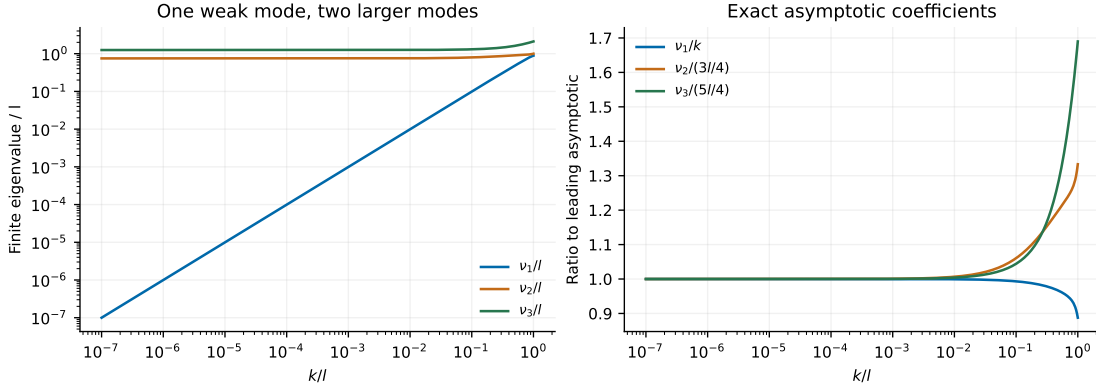


FIGURE 3. The three positive finite eigenvalues for  $l = 1$  as  $k/l$  decreases, and their normalized asymptotic ratios. Homogeneity gives every positive overall scale. The curves are eigenvalues of the explicit four-vertex matrix, not a numerical integration of HYM flow. Equation (76) and Theorem 8.2 supply their analytic meaning.

eigenvalues  $3l/4$  and  $5l/4$ . Continuity and the product of the three roots therefore give, as  $k/l \rightarrow 0$ ,

$$\nu_1/k \rightarrow 1, \quad \nu_2/l \rightarrow 3/4, \quad \nu_3/l \rightarrow 5/4.$$

The all-future comparison covers order  $1/k$  even when the other two inverse scales have order  $1/l$ . Figure 3 plots this exact finite calculation.

For a general full-support tuple, the projective direction in each block and the positive cycle invariant  $\rho$  are retained by the exact multiplicative field. The preceding confinement and metric comparison apply to all such tuples; the explicit polynomial is specific to (68). In particular, slopes alone do not determine a trajectory or its finite initial value.

## 9. PRESCRIBED METRICS ON A FIXED DIAMOND BUNDLE

We now hold the holomorphic bundle (68) fixed and remove the small-normal assumption on its initial metric. The reference polarization is the wall  $p = 0$ , at which this bundle is semistable and its graded object is the locally free sum  $E_0$ . The conclusion is a small constant all-future comparison after a common restart on a full positive  $(k, l)$  triangle. It is separate from the differentiable, exponentially accurate phase theorem.

**9.1. Exact diagonal matching at the reference polarization.** Write  $E = E_{z^*}$  for (68). Let  $H_0(t)$  be its flow at  $p = 0$  from any smooth metric with determinant  $\det h_0$ .

**Lemma 9.1.** *There are  $T_n \rightarrow \infty$ , positive constant diagonal matrices  $g_n$  with determinant one, and smooth Hermitian trace-free fields  $v_n$  such that*

$$(77) \quad H_0(T_n) = g_n^*(h_0 e^{v_n}), \quad v_n \rightarrow 0 \text{ in } C^\infty, \quad z_n = g_n z^* \rightarrow 0.$$

*Each  $z_n$  has full support and its central entry is exactly  $\Gamma_0(z_n)$ . No bound on  $g_n$ , convergence rate, or differentiable selection of this sequence is asserted.*

*Proof.* The holomorphic flag with successive factors  $F_1, F_2, F_3, F_4$  is locally free. All four slopes are zero at  $p = 0$ , so the bundle is semistable and this is a Jordan–Hölder filtration. Indeed degree inequalities for a subsheaf follow by intersecting it with the flag and adding the inequalities for its images in the line quotients. The graded sheaf itself is  $E_0$ . The corrected

convergence theorem [SW15, Theorem 1.1(2) and Section 2.2] therefore has empty algebraic singular set.

The metric-to-connection isometry used in Lemma 6.1, followed by its unitary convergence gauges, gives complex gauges  $B_n$  with

$$(78) \quad B_n^* h_0 = H_0(T_n), \quad \mathcal{D}_n = B_n(d + N_0(z^*))B_n^{-1} \longrightarrow d \quad \text{in } C^\infty.$$

Here the  $B_n$  need not be bounded. To obtain the much stronger diagonal assertion, we recover the filtration and normalize its extension representatives.

Let  $i : F_1 \rightarrow E$  and  $p_4 : E \rightarrow F_4$  be the canonical maps, and  $S_2 = \ker p_4$ . Cross-Hom vanishing gives

$$\text{Hom}(F_1, E) = \mathbb{C}i, \quad \text{Hom}(E, F_4) = \mathbb{C}p_4, \quad S_2/F_1 \simeq F_2 \oplus F_3.$$

For example  $E/F_1$  has a filtration by the other three factors, so it has no map from  $F_1$ . Similarly no map from  $S_2$  to  $F_4$  exists. Thus the first line and last kernel are intrinsic.

Transport  $i$  and  $p_4$  by  $B_n$  and normalize their  $L^2$  norms. After phases and a subsequence the resulting maps  $i_n, p_n$  converge smoothly to the product inclusion  $\iota_1$  and projection  $\pi_4$ . To justify this, their zero-form Dolbeault Laplacians have smoothly convergent coefficients by (78). Elliptic estimates bound each normalized kernel vector in every spatial norm. Each limit has norm one in the one-dimensional split kernel, identified by cross-Hom vanishing. This argument bounds the normalized maps, not the gauges  $B_n$ .

For large  $n$  the maps have constant rank. Put

$$j_{4,n} = p_n^*(p_n p_n^*)^{-1}, \quad K_n = \ker p_n \cap (\text{im } i_n)^\perp.$$

These converge to the product fourth inclusion and middle subbundle. Projection identifies  $K_n$  smoothly with  $F_2 \oplus F_3$ , and its quotient Dolbeault structure tends to the split one. The quotient is holomorphically isomorphic to  $S_2/F_1$ . Normalize the generators of  $\text{Hom}(F_i, K_n)$  as quotient maps, for  $i = 2, 3$ . The same elliptic argument makes their lifts  $j_{2,n}, j_{3,n}$  converge to the middle inclusions. They are holomorphic modulo  $\text{im } i_n$ , which is sufficient.

The map  $W_n = (i_n, j_{2,n}, j_{3,n}, j_{4,n})$  tends to the identity. Its transported structure has split diagonal and only entries 12, 13, 24, 34, 14, all tending to zero. The four adjacent entries  $\alpha_{ij}$  are closed. Write  $\alpha_{ij} = \zeta_{ij} + df_{ij}$  by the reference Hodge decomposition, with  $\zeta_{ij}$  harmonic and all  $f_{ij} \rightarrow 0$  smoothly. Put  $f = \sum f_{ij} E_{ij}$  and  $U_n = I - f$ . Its inverse is exactly  $I + f + f^2$  because  $f^3 = 0$ . The transformed adjacent entries are  $\zeta_{ij}$ , with no new block 23; all new products are central. If  $\chi$  denotes the central entry after this transformation, integrability gives

$$d\chi = -(\zeta_{12} \wedge \zeta_{24} + \zeta_{13} \wedge \zeta_{34}).$$

Hence  $\Omega(\zeta) = 0$  and  $\chi - \Gamma_0(\zeta)$  is closed. Since  $H^1(F_1 F_4^{-1}) = 0$ , it equals  $df_{14}$  with  $f_{14} \rightarrow 0$  smoothly. The further change  $V_n = I - f_{14} E_{14}$  removes it. Thus  $S_n = W_n U_n V_n \rightarrow I$  and

$$(79) \quad S_n^{-1} \mathcal{D}_n S_n = d + N_0(\zeta_n), \quad \zeta_n \rightarrow 0.$$

It remains to identify the resulting isomorphism  $C_n = S_n^{-1} B_n$  from the original fixed bundle. Any such isomorphism preserves the intrinsic first line, last kernel and the two distinct middle quotient factors. Its matrix therefore has diagonal constants  $c_i \neq 0$ , and only the four adjacent and central off-diagonal entries. The adjacent holomorphic-map equations have harmonic part  $\zeta_{ij} = (c_i/c_j) z_{ij}^*$ . Their remaining part says that the off-diagonal  $C_{ij}$  are holomorphic sections, so they vanish by cross-Hom vanishing. Diagonal covariance of  $\Gamma_0$  then makes the central map equation  $dC_{14} = 0$ , which vanishes as well. Thus  $C_n = c_n$  is exactly constant diagonal and  $\zeta_n = c_n z^*$ .

Write  $c_n = J_n d_n$  with  $J_n$  unitary diagonal and  $d_n$  positive, put  $a_n = (\det d_n)^{1/4}$  and  $g_n = a_n^{-1} d_n$ . Then

$$H_0(T_n) = g_n^* \tilde{h}_n, \quad \tilde{h}_n = a_n^2 J_n^* S_n^* h_0.$$

The determinant identity in (78) has  $|\det B_n| = 1$  pointwise. Since  $S_n \rightarrow I$ , it implies  $a_n \rightarrow 1$ . Compactness of the unitary diagonals gives  $\tilde{h}_n \rightarrow h_0$  smoothly without choosing a limit of  $J_n$ . Its determinant is exactly  $\det h_0$ . Take  $v_n = \log(h_0^{-1} \tilde{h}_n)$ , which is trace-free. Finally  $z_n = g_n z^* = J_n^{-1} \zeta_n \rightarrow 0$ ; its central entry is exactly the quadratic primitive by covariance.  $\square$

The three-layer recovery and the removal of the central exact entry are essential additions to the two-step matching argument of [Wue26b, Lemma 8.1]. Smooth connection convergence alone would not identify these metric coordinates.

**9.2. Changing polarization at a fixed finite time.** Let  $H_p^{\text{in}}$  be any smooth family of determinant-normalized metrics on this fixed bundle, extending to  $p = 0$ , and let  $H_p(t)$  be its flow. Fix a finite  $T$  first. Relative to the  $p$ -flow, the reference solution has metric residual

$$(80) \quad \begin{aligned} \mathcal{R}_p(t) &= H_0^{-1} \dot{H}_0 + 2(i\Lambda_p F_{H_0} - \kappa_E(p)I) \\ &= 2\{i(\Lambda_p - \Lambda_0)F_{H_0} - (\kappa_E(p) - \kappa_E(0))I\}. \end{aligned}$$

Its operator norm relative to  $H_0(t)$  on  $[0, T]$  is at most  $r_T(p) \leq C_T |p|$ . Put  $\alpha(p) = d_\infty(H_p^{\text{in}}, H_0^{\text{in}}) \rightarrow 0$ . Multiplication of a metric by a spatially constant scalar leaves its curvature unchanged, so  $e^{\pm(\alpha(p) + tr_T(p))} H_0(t)$  are respectively upper and lower barriers for the  $p$ -equation. The tensor maximum principle in [HKKP18, Proposition 4.3 and equation (4.9)] therefore gives

$$(81) \quad d_\infty(H_p(t), H_0(t)) \leq \alpha(p) + tr_T(p), \quad 0 \leq t \leq T.$$

This uses the metric residual directly with its factor two, not a conversion from a gauge-velocity error. The constant may depend on the already chosen  $T$ ; no infinite-time parameter continuity is claimed.

### 9.3. Restart first, then the full positive triangle.

**Theorem 9.2.** *Fix the graph  $\Psi_\diamond$  and any smooth initial-metric family  $H_p^{\text{in}}$  just described. For every  $\epsilon > 0$  there are a finite time  $T_\epsilon$ , a positive determinant-one diagonal  $g_\epsilon$ , and  $\sigma_\epsilon > 0$  such that, for every  $k, l > 0$  with  $k + l < \sigma_\epsilon$ , the exact finite trajectory started at  $g_\epsilon z^*$  at time  $T_\epsilon$  has reconstruction  $H_p^\epsilon(t)$  satisfying, for all  $t \geq T_\epsilon$ ,*

$$(82) \quad \begin{aligned} e^{-\epsilon} H_p^\epsilon(t) &\leq H_p(t) \leq e^\epsilon H_p^\epsilon(t), \\ \left| \log \frac{\lambda_j(H_p(t))}{\nu_j(w_p^\epsilon(t))} \right| &\leq \epsilon, \quad 1 \leq j \leq 3. \end{aligned}$$

*There are also fixed  $T_*, \sigma_*, \gamma > 0$  for this family such that  $\lambda_4(H_p(t)) \geq \gamma$  for  $t \geq T_*$  throughout  $k, l > 0$ ,  $k + l < \sigma_*$ . Neither conclusion imposes a restriction on  $k/l$ .*

*Proof.* Choose the sequence in Lemma 9.1 for  $H_0^{\text{in}}$  and put  $q_n = \sum_e \|(g_n z^*)_e\|_0^2 > 0$ ,  $\eta_n = \|v_n\|_{\text{op}, \infty}$ . Both tend to zero. Choose any  $\delta_n \downarrow 0$ . For fixed  $n$  the Hodge conversion has error

$$c_n(p) = d_\infty((g_n \Phi_p(z^*))^* h_0, g_n^* h_0) \rightarrow 0.$$

This is a fixed- $n$  continuity statement and needs no uniform bound on  $g_n$ . By (81) at  $T_n$  and the matching lemma, the order distance from  $H_p(T_n)$  to  $(g_n\Phi_p(z^*))^*h_0$  is at most  $\eta_n + \theta_n(p)$ , where

$$\theta_n(p) = \alpha(p) + T_n r_{T_n}(p) + c_n(p) \longrightarrow 0.$$

For all sufficiently large  $n$ ,  $q_n < q_*/4$ . Shrink a full positive triangle  $k + l < \sigma_n$  until

$$\frac{1}{2}q_n \leq q_{n,p} \leq 2q_n, \quad |b(p)| \leq q_n/(32\sqrt{2}), \quad \theta_n(p) \leq \delta_n, \quad q_{n,p} = q_p(g_n z^*).$$

Each condition follows from continuity at zero and  $b(0) = 0$ . None singles out a direction of approach. These conditions give strictly the energy hypotheses of Lemma 8.1.

Start the graph trajectory with  $g_n$  at  $T_n$ . Its energy is at most  $2q_n$  for all future time. Comparing its initial graph metric with  $h_0$ , then using the preceding order bound, gives the all-future sandwich with exponent

$$B_n = \eta_n + \delta_n + 2C_\Psi q_n \longrightarrow 0.$$

The proof of Theorem 8.2 transfers the spectrum with error  $E_n = 4B_n + 2C_s q_n \rightarrow 0$  and fourth gap  $e^{-4B_n}\gamma_1$ . Select one index  $n_*$  with  $B_{n_*} < 1$ . It supplies  $T_* = T_{n_*}$ ,  $\sigma_* = \sigma_{n_*}$  and  $\gamma = e^{-4}\gamma_1$ . For the requested  $\epsilon$ , choose  $n \geq n_*$  with  $E_n \leq \min(\epsilon, 1)$  and  $B_n \leq \epsilon$ . Set  $T_\epsilon = T_n$ ,  $g_\epsilon = g_n$  and  $\sigma_\epsilon = \min(\sigma_n, \sigma_*)$ . All comparisons then hold on the claimed common triangle.  $\square$

The theorem allows metrics outside the initial class of Section 8. To see this, take a smooth real  $f$  whose range contains  $-1$  and  $1$ , and put

$$H^{\text{in}} = h_0 \exp(Mf \operatorname{diag}(1, -1, 0, 0)),$$

independently of  $p$ . The first column of  $\Phi_p$  is the identity column. For any metric (73) with  $\|u_{\text{in}}\|_{\text{op},\infty} < \delta_*$ , the oscillation of the logarithm of its first diagonal entry, after subtracting the fixed reference entry, is at most  $2\delta_*$ . In the example it is  $2M$ . Taking  $M > \delta_*$  excludes that entire initial class while preserving the theorem's hypotheses.

The restart depends on the prescribed metric and requested accuracy. It is selected before the parameter neighborhood. The theorem supplies no effective value of  $T_\epsilon$ , no  $O(\sigma)$  error at one fixed restart, and no differentiable phase or decaying discrepancy for this two-parameter matching sequence. Those stronger conclusions belong to Theorem 6.3 on its complete-family domain.

## 10. COMPATIBILITY OF THE TWO FINITE DESCRIPTIONS

On  $k = 3l/5$ , the diamond lies in the complete deformation family. Its current-Hodge reconstruction and the fixed-reference reconstruction are related by an exact bundle map, but two chosen parabolic graphs need not coincide. We identify a nonempty common domain and compare their finite trajectories there.

**10.1. Bundle maps and initial-metric normalization.** On  $V_\diamond$ , the full reconstruction terminates at its quadratic central term, so it is  $N_0(z)$ . Along the stated path, (67) gives

$$(83) \quad \|\Phi_l(z) - I\|_{h^{19/8}} \leq Cl|z|^2, \quad \|D_z \Phi_l(z)\| \leq Cl|z|, \quad \Phi_l(gz) = g\Phi_l(z)g^{-1}.$$

These estimates follow by differentiating the smoothly parameter-dependent Hodge primitive at  $l = 0$ ; it is quadratic in  $z$ . The polynomial formula defines the map at a fixed original  $z^*$  even when that coordinate is not small.

Let  $z_{\text{eq}} = g_{\text{eq}} z^*$  be a full-support quadratic balance. Here  $\sigma = 8l/5$  and  $q_l(z_{\text{eq}}) = \sigma$ . The stationary theorem supplies an actual HE metric  $\widehat{H}_{\text{eq}}$  on  $d + N_l(z_{\text{eq}})$  whose logarithm relative to  $h_0$  is  $O(\sigma)$  in  $h^{19/8}$ . Pull it back by  $\Phi_l(z_{\text{eq}})$  and normalize by a positive diagonal  $S = e^{Y_{\text{eq}}}$ :

$$(84) \quad P_0 \log \left( S^{-1} (h_0^{-1} \Phi_l(z_{\text{eq}})^* \widehat{H}_{\text{eq}}) S^{-1} \right) = 0, \quad |Y_{\text{eq}}| \leq C\sigma.$$

The derivative in  $Y_{\text{eq}}$  at the identity is  $-2I$ ; the small metric logarithm and the implicit-function theorem give the uniform bound. Thus  $\bar{x} = Sz_{\text{eq}} \in \mathcal{B}_l^{\text{st}}$  and  $|\bar{x}| \leq C_0\sqrt{\sigma}$ . The fixed holomorphic map  $F = Sg_{\text{eq}}$  identifies  $E_{z^*}$  with  $E_{\bar{x}}$ . This is a bundle isomorphism, with the corresponding metric pullback; no estimate for the condition number of  $g_{\text{eq}}$  is used.

Take a diamond initial metric  $(g_{\text{old}} \Phi_l(z^*))^* (h_0 e^v)$  with  $P_0 v = 0$ . Put  $z_{\text{old}} = g_{\text{old}} z^*$  and  $d_{\text{in}} = g_{\text{old}} g_{\text{eq}}^{-1} S^{-1}$ . On  $E_{\bar{x}}$  that same metric is exactly

$$(85) \quad d_{\text{in}}^* (\Phi_l(z_{\text{old}})^* (h_0 e^v)).$$

Let  $e^w = \Phi_l(z_{\text{old}})^* e^v \Phi_l(z_{\text{old}})$  in  $h_0$  frames, and choose a real constant diagonal  $Z$  with

$$u = \log(e^{-Z} e^w e^{-Z}), \quad P_0 u = 0.$$

Because  $P_0 v = 0$ , (83) and the same normalization derivative give

$$(86) \quad |Z| \leq Clq_0, \quad \|u\|_{h^{19/8}} \leq C(\|v\|_{h^{19/8}} + lq_0), \quad g_{\text{in}} = e^Z d_{\text{in}}, \quad q_0 = q_l(z_{\text{old}}).$$

Equations (85)–(86) place exactly the same initial metric in the full-family coordinates of (42).

**10.2. A common domain without a spectral-ratio bound.** Fix  $A_0 > \log \sqrt{32}$  and  $B > 0$ . Impose

$$(87) \quad \begin{aligned} |\log(g_{\text{old}} g_{\text{eq}}^{-1})|_{\max} &< A_0, & q_0 &< q_*, & |b(l)| &< q_0/(8\sqrt{2}), \\ \|v\|_{h^{19/8}} &< B\sigma. \end{aligned}$$

Then  $q_0 \leq e^{4A_0}\sigma$ . Choose  $A = A_0 + 1$  and  $r = 2C_0\sqrt{\sigma}$ . The graph and (86) give  $\|u - \Psi(l, g_{\text{in}} \bar{x})\|_{\mathcal{E}_1} \leq C_{A_0, B}\sigma$ . For sufficiently small  $\sigma$ , these are the hypotheses of Theorem 5.1. Every bound is independent of the cycle invariant  $\rho$  and the positive finite spectral ratios. The height choice  $g_{\text{old}} = g_{\text{ht}} g_{\text{eq}}$  has  $q_0 = 32\sigma$  and lies strictly inside (87), so this is a nonempty open common domain.

Both time conventions are (14); the normalized harmonic norms agree on the selected subspace. The full-family phase theorem therefore gives the same  $O(\sigma)$  spectral and metric comparison on these actual bundles, with its additional decaying comparison to the selected exact trajectory. At equilibrium (84) changes the finite form only by relative  $1 + O(\sigma)$ . Proposition 7.2 then gives

$$d_j = 2\nu_j(z_{\text{eq}})(1 + O(\sigma)), \quad 1 \leq j \leq 3,$$

and the same estimates cover  $0 \leq t \leq 2/\nu_1(z_{\text{eq}})$ .

**10.3. Comparing the chosen graph trajectories.** Convert the diamond graph flow to  $E_{\bar{x}}$  by  $F$ . At each time normalize it as in (86), now with  $v = \Psi_{\diamond}(l, z_{\text{old}}(t))$ . Its exact metric becomes

$$\tilde{g}(t)^* (h_0 e^{\tilde{u}(t)}), \quad \tilde{x}(t) = e^{Z(t)} z_{\text{old}}(t) = \tilde{g}(t) \bar{x}.$$

Diamond confinement gives  $q_l(z_{\text{old}}(t)) \leq q_0$ , hence  $\|\tilde{u}(t)\|_{C^0} \leq Cq_0$  and  $|Z(t)| \leq Clq_0$ . The selected full-family graph metric and this converted graph metric are both exact flows on  $E_{\bar{x}}$ . Their initial order distance is  $O_{A_0, B}(\sigma)$ , by comparison with (85); order preservation propagates that bound for all future time.

Both complementary metric logarithms are  $O(\sigma)$ . Comparing their positive diagonal squares with  $h_0$  therefore gives

$$(88) \quad |\log g_*(t) - \log \tilde{g}(t)|_{\max} \leq C_{A_0, B} \sigma, \quad x_*(t) = e^{\Theta(t)} z_{\text{old}}(t), \quad |\Theta(t)|_{\max} \leq C_{A_0, B} \sigma.$$

Indeed insert the separate estimates  $e^{-C\sigma} h_0 \leq h_0 e^\Psi$ ,  $h_0 e^{\tilde{u}} \leq e^{C\sigma} h_0$  into the metric sandwich. The diagonal squares commute, and their scalar logarithms differ by  $O(\sigma)$ . No derivative of  $Z$  divided by a weak mode, and no inverse orbit form, occurs. Every edge energy and the entire finite form consequently compare by  $e^{\pm C\sigma}$ . Min–max gives that comparison for every positive finite eigenvalue.

The result is compatibility of exact bundle presentations, time normalizations, actual flows and finite spectra on their common domain. It does not identify the two independently chosen invariant graphs. The prescribed-metric theorem of Section 9 retains its larger independent two-parameter domain; it is not a consequence of restriction to the single path used by the complete-family phase theorem.

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