

HERMITIAN–YANG–MILLS FLOW AT AND BEYOND A STABILITY WALL

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ABSTRACT. We prove all-time confinement and two-sided decay estimates for Hermitian–Yang–Mills flow whose initial metric is a sufficiently small transverse perturbation of a metric on a fixed finite-dimensional invariant graph. The bundles are small integrable deformations of a specified sum of four line bundles on the product of three projective lines. At the stability wall the directed extension graph is acyclic and connected; beyond it we treat a connected triangular class with locally free Harder–Narasimhan line quotients. In a moving diagonal presentation, the harmonic coefficients remain small although the diagonal metric factors become unbounded. Their energy decays reciprocally at the wall and crosses to exponential decay on the unstable side, uniformly in the ratio of wall distance to initial energy. This confinement yields an asymptotic phase relative to a chosen invariant graph: each such initial metric selects a unique reduced trajectory with exponentially decaying relative metric error. The phase is continuously differentiable in the initial metric at fixed bundle and polarization. At each fixed nearby unstable polarization, subtracting the linear diagonal drift leaves a limiting residual that is locally invertible in the three real diagonal initial coordinates.

1. INTRODUCTION

Near a stable bundle, an equilibrium metric can confine a Hermitian–Yang–Mills trajectory by metric order. At a stability wall the corresponding extension may be strictly semistable, and on the other side it may be unstable. The metric can then leave every bounded set in its fixed holomorphic presentation. Nevertheless, its extension coefficients in a moving diagonal presentation can converge to zero. The distinction between these two behaviors makes it possible to retain a quantitative finite-dimensional description beyond the stable chamber.

This paper proves such a description for two classes of bundles on $(\mathbb{P}^1)^3$. The main step is an all-time energy estimate that controls the coefficients without bounding the diagonal metric factors. It yields a differentiable asymptotic phase, relative spectral estimates and a common entry result for prescribed initial metrics. On the strictly unstable side, the reduced motion has a residual after subtraction of its linear slope drift. That residual retains the three diagonal initial conditions on a chosen invariant graph.

1.1. The geometric classes. The reference bundle is the sum of the four product line bundles

$$F_1 = \mathcal{O}(-2, -2, 4), \quad F_2 = \mathcal{O}(1, -2, 1), \quad F_3 = \mathcal{O}(-2, 1, 1), \quad F_4 = \mathcal{O}.$$

The product polarization at the wall gives them equal slope. Their full trace-free first deformation space has dimension 83, and their second deformation space has dimension five. We

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use the complete analytic integrability equations on this harmonic parameter space. Write x_{ij} for a harmonic coefficient in $\text{Hom}(F_j, F_i)$; its directed support has the arrow $j \rightarrow i$ when $x_{ij} \neq 0$. The determinant-one diagonal torus acts by $x_{ij} \mapsto g_i g_j^{-1} x_{ij}$.

At the wall, our domain consists of the nonzero integrable tuples with connected acyclic support. Equivalently, these are the connected integrable points in the torus nullcone: zero belongs to the closure of their diagonal orbit. Each resulting bundle is strictly semistable and has the locally free Jordan–Hölder graded bundle $E_0 = \bigoplus_i F_i$. The domain includes all acyclic support patterns in the full harmonic space. Acyclicity alone does not impose the integrability equations.

For the unstable side, use the signed polarization path in (8) and write $l = -\tau$, with $\tau > 0$. We restrict to the five harmonic blocks

$$12, \quad 13, \quad 32, \quad 24, \quad 34.$$

This is a 32-dimensional harmonic subspace, subject to the full integrability equations. Its order 1, 3, 2, 4 is triangular. The associated subbundle flag has line quotients whose centered mean curvatures, in that order, are

$$\tau, \quad 0, \quad -2\tau/5, \quad -3\tau/5.$$

Thus it is the actual Harder–Narasimhan filtration for every $\tau > 0$. We require connected support, while allowing its nonzero weights to be arbitrarily unequal. Composable arrows are present: an explicit integrable example has a nonzero quadratic correction in the endpoint block. The class is therefore more than a sum of independent two-factor extensions.

The wall domain is larger than the triangular unstable domain. Neither is asserted to exhaust all nearby semistable or unstable bundles.

1.2. Energy and the transition across the wall. Fix one local parabolic invariant graph $\Psi(l, x)$ for the metric flow. Its precise analytic properties are recalled in Section 2. On it the exact finite equation is

$$\dot{x}_{ij} = (\beta_i^\Psi - \beta_j^\Psi) x_{ij}, \quad \beta^\Psi(l, x) = b(l) - \mu_l(x) + O(|x|^4).$$

Here $b(l)$ is minus the centered curvature of the split reference, and $\mu_l(x)$ is the diagonal moment, the sum of squared arrow norms times their incidence vectors. This is the expansion of the exact field, not the definition of a truncated evolution. Its trajectories reconstruct exact Hermitian–Yang–Mills metric flows.

Let $q_l(x) = \sum_{ij} \|x_{ij}\|_l^2$ be the total harmonic energy, with the spatial L^2 normalization specified below. Along a trajectory put $q(t) = q_l(x(t))$ and $q_0 = q(0)$. For a graph trajectory at the wall, Theorem 3.3 gives

$$(1) \quad \frac{q_0}{1 + 5q_0 t} \leq q(t) \leq \frac{q_0}{1 + q_0 t/5}, \quad t \geq 0.$$

The diagonal coordinate Y , defined by $x = e^Y x_b$ for a fixed base tuple x_b , grows logarithmically in norm. The decay of the coefficients thus coexists with divergence of the metric.

On the unstable side, define

$$\mathcal{L}_{a,d}(q, t; \tau) = \frac{q e^{-a\tau t}}{1 + \frac{dq}{a\tau}(1 - e^{-a\tau t})} \quad (\tau > 0), \quad \mathcal{L}_{a,d}(q, t; 0) = \frac{q}{1 + dqt}.$$

Theorem 4.2 proves the two-sided bounds

$$(2) \quad \mathcal{L}_{14/5,5}(q_0, t; \tau) \leq q_{-\tau}(x(t)) \leq \mathcal{L}_{2/5,1/5}(q_0, t; \tau).$$

They hold for every future time, uniformly in the independent scales τ and q_0 . When $\tau \ll q_0$, they display a reciprocal interval before exponential decay becomes decisive; when τ is larger, the linear drift acts earlier. No assumption orders these scales. The integral estimate

$$(3) \quad \int_0^\infty q_{-\tau}(x(t)) dt \leq 5 \log \left(1 + \frac{q_0}{2\tau} \right) \quad (\tau > 0)$$

will determine the residual after subtracting that drift.

1.3. The actual metric and its initial condition. The energy estimates extend to metrics off the invariant graph. Write a normalized initial metric in the form $e^{Y_0*}(h_0 e^{u_0})$ on the fixed bundle d_{x_b} . The coefficient $x_0 = e^{Y_0} x_b$ must be small, and the complementary displacement $v_0 = u_0 - \Psi(l, x_0)$ must be small in $h^{19/8}$, where h^s denotes the little Hölder space specified in Section 2. There is no bound on Y_0 , and no comparison between the size of v_0 and the size of x_0 .

Theorem 5.3 selects a unique graph trajectory $x_*(t) = e^{Y_*(t)} x_b$ with exponentially fast relative metric tracking. If $D = \|v_0\|_{h^{17/8}}$, its reconstructed metric $H_*(t)$ and the actual flow satisfy

$$(4) \quad e^{-CD e^{-4ct}} H_*(t) \leq H(t) \leq e^{CD e^{-4ct}} H_*(t).$$

The constants use the fixed complementary parabolic gap. They are uniform in the absolute diagonal coordinates and in all positive ratios of the three small finite eigenvalues. The phase $Y_*(0)$ depends C^1 on the initial metric in the stated relative coordinates, at each fixed bundle and polarization. It is relative to the chosen graph, which need not be canonical.

Let $\lambda_j(H(t))$ be the eigenvalues of the complex trace-free Dolbeault Laplacian. Let $\nu_j(l, x)$ be the three eigenvalues of the finite quadratic form

$$\xi \mapsto \sum_{ij} \|x_{ij}\|_l^2 |\xi_i - \xi_j|^2, \quad \sum_i \xi_i = 0, \quad \|\xi\|^2 = \sum_i |\xi_i|^2.$$

Connected support makes all three positive. Along the selected trajectory,

$$(5) \quad \left| \log \frac{\lambda_j(H(t))}{\nu_j(l, x_*(t))} \right| \leq C q_l(x_*(t)) + CD e^{-4ct} \quad (1 \leq j \leq 3), \quad \lambda_4(H(t)) \geq \gamma > 0.$$

The coefficient term also tends to zero here. The largest small eigenvalue is comparable to total energy; the two smaller ones remain represented by the full finite form. The total-energy bounds do not prescribe individual asymptotic exponents for those modes.

Theorem 6.1 reaches this description from every prescribed smooth determinant-normalized initial metric. For a requested small normal tolerance it chooses one finite restart, then coefficient, polarization and nearby-metric neighborhoods. All their allowed flows satisfy the estimates from that time onward. The initial-metric phase map remains C^1 . The restart is common on the chosen neighborhoods; it may depend on the prescribed metric and requested tolerance.

1.4. Residual drift and the graded limit. For every fixed $\tau > 0$, Theorem 7.1 proves that the graph coordinate has a limit after its linear term is removed:

$$Z_\infty = \lim_{t \rightarrow \infty} (Y(t) - b(-\tau)t), \quad |Z_\infty - Y(0)| \leq C \log \left(1 + \frac{q_0}{2\tau} \right).$$

For a fixed base tuple and graph, $Y(0) \mapsto Z_\infty$ is a C^1 local diffeomorphism on the three real diagonal coordinates. The selected phase and the actual metric have the same residual. Its

initial-metric dependence is obtained by composition with the phase map. These statements fix $\tau > 0$; neither the residual bound nor its derivative bound is uniform at the wall.

There is also a direct quantitative connection estimate. In the moving isometry $U(t) = e^{u(t)/2}e^{Y(t)}$, Proposition 7.2 shows convergence to the split product connection $A_{0,l}$, with

$$\|A(t) - A_{0,l}\|_{h^{9/8}} + \|F_{A(t)} - F_{A_{0,l}}\|_{h^{1/8}} \leq C(q_l(x(t))^{1/2} + De^{-4ct}).$$

The limit is the Hermitian–Einstein connection on the Jordan–Hölder graded bundle at the wall and the Yang–Mills connection on the Harder–Narasimhan graded bundle for $\tau > 0$. The estimate specifies the isometries and the dependence on the evolving finite energy.

1.5. Prior work and proof strategy. Haiden, Katzarkov, Kontsevich and Pandit prove canonical filtered asymptotics for metric flows on bundles over curves, including linear slope drift and iterated logarithms with bounded remainder [HKKP18, Theorem 1.1 and Section 4.3]. Their discussion of polystable Harder–Narasimhan quotients already exhibits exponentially decaying triangular coefficients. Their metric-order comparison, unlike that curve asymptotic theorem, applies in every complex dimension. We use this comparison in the present threefold setting.

For semistable bundles with locally free Jordan–Hölder graded object, Lam proves higher-dimensional finite-reduction and asymptotic results. In the setting of his Section 8, Proposition 8.6 gives bounded distance from the reduced flow, and Theorem 8.7 supplies the equivariant Kuranishi description [Lam26]. Thus finite reduction itself, and the importance of initial conditions in asymptotic descriptions, precede the present results.

Smooth connection convergence modulo gauge to the Harder–Narasimhan–Seshadri graded connection is also known for each fixed polarization in our classes. Their graded bundles are locally free, so the algebraic singular set in the corrected [SW15, Theorem 1.1(2)] is empty. Our final estimate gives a particular moving isometry and a quantitative bound in the two specified Hölder norms. It does not improve the regularity of that classical qualitative convergence theorem.

The two-factor analysis in [Wue26c, Theorem 5.4] already has a uniform profile through stable, wall and unstable regimes. Its Sections 6 and 8 construct exact graph flows and comparison after a prescribed-metric restart. The common local reduction, the abstract differentiable phase theorem and the signed nonstationary spectral estimate needed here are established in [Wue26b]. Section 2 states their hypotheses and conclusions. The full geometric family and its exact all-morphism statement come from [Wue26d, Sections 6 and 9].

The new global argument begins with height coercivity on acyclic support. It bounds the moment from below by total energy uniformly in the support weights. On the triangular unstable class, each allowed arrow also has a strictly negative slope character. Together with the quartic remainder of the exact field, these facts prove the two-sided differential inequalities behind (1) and (2).

To use the local phase theorem with unbounded Y , we cut off by coefficient energy, rather than by the size of Y . Derivatives of the energy cutoff remain bounded because every diagonal derivative of an arrow is bounded by its own norm. Its finite Lipschitz constant is controlled by $\tau + r^2$, with no condition on τ/r^2 . The perturbed energy inequalities then remove the cutoff for both the actual and selected trajectories. Cancelling their common large diagonal factor gives the relative metric estimate. An inverse coordinate estimate on convex energy sublevel sets proves uniqueness without inverting a weak orbit quadratic form.

Finally, for fixed $\tau > 0$, the integral (3) makes the nonlinear correction to linear drift integrable. The derivative of the finite field along the trajectory is integrable as well. Its variational matrix and inverse therefore converge, which proves local invertibility of the residual map. These arguments retain the initial condition while permitting all finite spectral ratios to degenerate.

Sections 3 and 4 establish the two geometric domains and their energy estimates. Section 5 proves phase tracking and relative spectra; Section 6 proves common entry from prescribed metrics. Section 7 treats the residual and the graded connection.

2. GEOMETRY AND THE COMMON LOCAL REDUCTION

We state the finite reduction and analytic phase criterion with the hypotheses needed below. The new global arguments will verify those hypotheses without a bounded diagonal metric or an HE equilibrium on the evolving bundle.

2.1. The complete local deformation model. On $X = (\mathbb{P}^1)^3$, normalize the product Fubini–Study forms by $\int_{\mathbb{P}^1} \omega_i = 1$, and set

$$(6) \quad \begin{aligned} F_1 &= \mathcal{O}(-2, -2, 4), & F_2 &= \mathcal{O}(1, -2, 1), & F_3 &= \mathcal{O}(-2, 1, 1), & F_4 &= \mathcal{O}, \\ E_0 &= F_1 \oplus F_2 \oplus F_3 \oplus F_4. \end{aligned}$$

Let h_0, A_0, d be the product metric, Chern connection and Dolbeault differential. Block ij maps F_j to F_i . The determinant line is $\mathcal{O}(-3, -3, 6)$; its product metric will remain fixed.

The reference harmonic space $V = H^1(X, \text{End}_0 E_0)$ has nine nonzero blocks:

ij	12	32	34	13	23	24	21	31	41
$\dim_{\mathbb{C}} V_{ij}$	8	8	4	8	8	4	8	8	27.

Thus $\dim_{\mathbb{C}} V = 83$. The second cohomology has dimension five, in block 14. With the reference harmonic inclusion ι , projection P_2 and homotopy $K = d^*G$, the complete reconstruction is

$$(7) \quad a(x) = \iota x - K(a(x)^2), \quad \kappa(x) = P_2(a(x)^2), \quad d_x = d + a(x).$$

It is analytic on a fixed coefficient ball. The operator d_x is integrable exactly when all five components of κ vanish, with all higher terms included. These facts and the co-closed local model are proved in [Wue26d, Sections 6 and 9].

Constant determinant-one complex diagonals act by $(gx)_{ij} = g_i g_j^{-1} x_{ij}$. The reconstruction obeys $a(gx) = ga(x)g^{-1}$ whenever both coefficient points lie in its domain. Write

$$\mathfrak{t} = \{\xi \in \mathbb{C}^4 : \sum_i \xi_i = 0\}, \quad \mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \cap \mathbb{R}^4, \quad C_x \xi = ((\xi_i - \xi_j)x_{ij})_{ij}.$$

All nonconstant cross-Hom sections vanish: each nonzero difference of the four degree triples has sum zero and a negative component. The same co-closedness and complementary coercivity used in [Wue26d] show that holomorphic endomorphisms of a small integrable d_x are exactly its constant diagonal intertwiners.

2.2. Polarizations, metrics and spectra. Use the signed family

$$(8) \quad \omega_l = h_1(l)\omega_1 + h_2(l)\omega_2 + \omega_3, \quad h_1(l) = \left(1 + \frac{7l}{30\pi}\right)^{-1}, \quad h_2(l) = \left(1 + \frac{l}{6\pi}\right)^{-1}.$$

It is positive on an interval about zero. In the unstable regime put $l = -\tau$, $\tau \geq 0$. The centered slope demands are

$$(9) \quad b(l) = l(1, -2/5, 0, -3/5).$$

Thus the mean curvature of F_i , minus the average mean curvature $\kappa_E(l)$, is $-b_i(l)$. The determinant metric has constant mean curvature $4\kappa_E(l)$ for every l .

Divide squared spatial L^2 norms by $h_1 h_2$. The resulting measure and the diagonal zero-form mass $|\xi|^2 = \sum_i |\xi_i|^2$ are fixed. Write

$$(10) \quad q_l(x) = \sum_{ij} \|x_{ij}\|_l^2, \quad \mu_l(x) = \sum_{ij} \|x_{ij}\|_l^2 (e_i - e_j).$$

Relative to the reference norm $|x|$, the block weights are h_1^{-1} on 12, 32, 34, h_2^{-1} on 13, 23, 24 and one on 21, 31, 41. They are uniformly positive. Put $q = q_0$ at the wall.

Every flow below fixes l and d_{x_b} and solves

$$(11) \quad H^{-1} \dot{H} = -2(i\Lambda_l F_{d_{x_b}, H} - \kappa_E(l)I), \quad \det H = \det h_0.$$

For positive metrics on the same bundle, $d_\infty(H_1, H_2)$ is the least η such that $e^{-\eta}H_2 \leq H_1 \leq e^\eta H_2$ pointwise. Metric order is preserved by two exact solutions [HKKP18, Proposition 4.3]. Global smooth existence follows from [Kob87, Theorem 6.8.25], with its time equal to twice our time.

The finite Hermitian form on $\mathfrak{t}$ is

$$(12) \quad \langle \xi, L_{\text{fin}, l}(x)\xi \rangle = \|C_x \xi\|_l^2 = \sum_{ij} \|x_{ij}\|_l^2 |\xi_i - \xi_j|^2.$$

Its ordered eigenvalues are ν_1, ν_2, ν_3 , with zeros included. The geometric eigenvalues λ_j refer throughout to $\bar{\partial}_{\text{End}_0}^* \bar{\partial}_{\text{End}_0}$ on complex zero-forms, with complex multiplicity and no factor two.

2.3. Exact sliced evolution and the graph. Let $\mathcal{V}$ be the real bundle of h_0 -Hermitian trace-free endomorphisms, let P_0 be the fixed normalized projection onto $\mathfrak{t}_{\mathbb{R}}$, and put $Q_0 = 1 - P_0$. For nonintegral s , h^s denotes the closure of smooth sections in C^s . Define

$$\mathcal{E}_0 = Q_0 h^{3/8}(\mathcal{V}), \quad \mathcal{E}_1 = Q_0 h^{19/8}(\mathcal{V}), \quad \mathcal{E}_\alpha = Q_0 h^{17/8}(\mathcal{V}), \quad \alpha = 7/8.$$

All these norms use the fixed background. Scale equivalent norms so $\mathcal{E}_1 \rightarrow \mathcal{E}_\alpha \rightarrow C^0$ have norm at most one. For a complementary logarithmic metric u , set

$$(13) \quad \begin{aligned} \mathcal{M}(l, x, u) &= e^{u/2} i\Lambda_l F_{d_x, h_0 e^u} e^{-u/2} - \kappa_E(l)I, \\ S_u &= \frac{\sinh(\text{ad}_u/2)}{\text{ad}_u/2}, \quad C_u = \cosh(\text{ad}_u/2), \quad R(u) = P_0 S_u^{-1} C_u|_{\mathfrak{t}_{\mathbb{R}}}, \\ \beta(l, x, u) &= -R(u)^{-1} P_0 S_u^{-1} \mathcal{M}(l, x, u). \end{aligned}$$

The quotient has value one at zero, and $R(u)$ is positive invertible. The compatible unitary connection defines this curvature also on ambient nonintegrable coefficients.

We use the following precise consequences of [Wue26b, Sections 3, 4 and 7]. They hold on a common signed parameter interval and coefficient neighborhood. For any fixed finite $m \geq 6$ one can choose a C^m invariant graph $\Psi(l, x) \in \mathcal{E}_1$ with

$$(14) \quad \Psi(l, 0) = D_x \Psi(l, 0) = 0, \quad \|\Psi(l, x)\|_{\mathcal{E}_1} \leq C|x|^2, \quad \|D_x \Psi(l, x)\| \leq C|x|.$$

Its exact reduced vector field is $\beta^\Psi(l, x) = \beta(l, x, \Psi(l, x))$, with expansion

$$(15) \quad \beta^\Psi(l, x) = b(l) - \mu_l(x) + \mathcal{R}_l(x), \quad |D_x^j \mathcal{R}_l(x)| \leq C|x|^{4-j} \quad (0 \leq j \leq 2).$$

Fix a uniform constant C_E such that $|\mathcal{R}_l(x)| \leq C_E q_l(x)^2$ on this signed neighborhood; the equivalence of $q_l(x)$ and $|x|^2$ gives such a constant. The equation and reconstruction are

$$(16) \quad \dot{Y} = \beta^\Psi(l, e^Y x_b), \quad x = e^Y x_b, \quad H_* = e^{Y*}(h_0 e^{\Psi(l, x)}).$$

Every integrable trajectory staying in the graph domain gives an exact solution of (11) on d_{x_b} . The graph is fixed from now on; it is not required to be positive-diagonal equivariant. The equivariance of a is what identifies the holomorphic bundle in this reconstruction.

Off the graph the exact slice has the form

$$(17) \quad \dot{x} = C_x \beta(l, x, u), \quad \dot{u} = -A_l u + \mathcal{N}_l(x, u), \quad A_l = 2L_l|_{Q_0}, \quad L_l = \frac{1}{2}(\Delta_{\partial, l} + \Delta_{\bar{\partial}, l}).$$

The first-jet maps extend smoothly from $V \times Q_0 h^{13/8}$ to the finite space and $\mathcal{E}_0$. For every signed l , $D_u \beta(l, 0, 0) = 0$ and $D_{(x, u)} \mathcal{N}_l(0, 0) = 0$. The common-domain semigroup satisfies

$$(18) \quad \|e^{-tA_l}\|_{\mathcal{E}_\alpha \rightarrow \mathcal{E}_\alpha} \leq M e^{-\sigma_0 t}, \quad \|e^{-tA_l}\|_{\mathcal{E}_0 \rightarrow \mathcal{E}_\alpha} \leq M(1 + t^{-7/8})e^{-\sigma_0 t}.$$

The constants are uniform. The principal operator includes the signed curvature commutator; no extra linear bu term is to be added to (17).

The same reference harmonic orthogonality and complementary coercivity give, for every small integrable x and signed l ,

$$(19) \quad (1 - C|x|^2)\nu_j(l, x) \leq \lambda_j(h_0, x; l) \leq (1 + C|x|^2)\nu_j(l, x), \quad 1 \leq j \leq 3, \quad \lambda_4 \geq \gamma_0.$$

The kernel is exactly the constant $\ker C_x$. Changing to $h_0 e^{\Psi(l, x)}$ preserves the relative order $O(q_l(x))$. The proof in [Wue26b, Lemma 7.1] eliminates only the uniformly coercive complement, before min-max on the three finite directions. It uses positivity of the product coefficients and bounds by $C|l|$, so its signed scope does not follow merely by extrapolating an estimate stated on the stable side.

2.4. The analytic phase criterion. For clarity we also state the analytic input [Wue26b, Proposition 4.1]. Suppose

$$\dot{Y} = f(Y) + h(Y, v), \quad \dot{v} = -A_l v + F(Y, v)$$

has a bounded C^2 reduced vector field with bounded second derivative and sufficiently small $\|Df\|_\infty$. On a fixed open normal ball in $\mathcal{E}_\alpha$, suppose h, F are C^2 with uniformly bounded derivatives, vanish at $v = 0$, and satisfy

$$\|D_v h\| + \|D_v F\| \leq \epsilon, \quad \|D_Y h\| + \|D_Y F\| \leq K\|v\|_{\mathcal{E}_\alpha}.$$

Here ϵ is sufficiently small relative to (18). If also $\|D_v h\| \leq K(a + \|v\|_{\mathcal{E}_\alpha})$, then for small initial normal size D there is a selected solution of the reduced equation with

$$(20) \quad \|v(t)\|_{\mathcal{E}_\alpha} \leq C D e^{-4ct}, \quad |Y_*(t) - Y(t)| \leq C(a + D) D e^{-4ct}, \quad c = \sigma_0/8.$$

Its phase $\Pi(Y_0, v_0)$ is C^1 with uniform derivative bounds, equals Y_0 on graph data, and obeys $\|D_{v_0} \Pi\| \leq C(a + \delta)$ on $\|v_0\| < \delta$. It is obtained as $\lim_{t \rightarrow \infty} \phi_{-t}(Y(t))$ for the complete reduced flow ϕ . The proof differentiates this improper limit using integrable variation estimates. It neither assumes an equilibrium nor inverts a finite rate. The new work below is to produce a global reduced vector field to which this criterion applies, then remove its cutoff on both trajectories.

3. THE INTEGRABLE NULLCONE AT THE WALL

At $l = 0$ the reduced vector field has no constant drift. Its total coefficient energy is coercive on acyclic supports, even though the corresponding nonsplit bundles have no HE metric.

3.1. Support and actual semistability. Let $T_{\mathbb{C}}$ be the determinant-one complex diagonal torus and define its linear nullcone

$$\mathcal{N} = \{x \in V : 0 \in \overline{T_{\mathbb{C}}x}\}.$$

The directed support has an arrow $j \rightarrow i$ when $x_{ij} \neq 0$; the undirected support forgets orientation.

Lemma 3.1. *A tuple belongs to $\mathcal{N}$ exactly when its directed support has no cycle. For every such support there is a real trace-free diagonal h , $|h|^2 = 5$, with $h_i - h_j \geq 1$ on every active arrow. Consequently*

$$(21) \quad q(x)/\sqrt{5} \leq |\mu_0(x)| \leq \sqrt{2}q(x).$$

The constants are independent of the support and all positive block-norm ratios.

Proof. On a directed cycle choose one nonzero linear coordinate from each vector block. Their product is a torus-invariant monomial: its characters telescope. It is nonzero at x and zero at the origin, so the orbit closure cannot contain the origin. Conversely assign heights $0, 1, 2, 3$ in a topological ordering of an acyclic support and subtract their mean. The resulting h is a permutation of $(-3/2, -1/2, 1/2, 3/2)$. Every active block of $e^{-th}x$ decays at least as e^{-t} . This also proves equivalence of the usual and affine algebraic closures in the nullcone definition. Finally $\langle h, \mu_0 \rangle \geq q$ gives the lower estimate, and the triangle inequality with $|e_i - e_j| = \sqrt{2}$ gives the upper. $\square$

Proposition 3.2. *Every sufficiently small $x \in \mathcal{N}$ with $\kappa(x) = 0$ defines a semistable bundle with locally free graded object E_0 . On an acyclic support its reconstruction is exactly*

$$(22) \quad a = a_1 + a_2 + a_3, \quad a_1 = \iota x, \quad a_2 = -K(a_1^2), \quad a_3 = -K(a_1 a_2 + a_2 a_1).$$

If $x \neq 0$ it is not polystable. If the undirected support has b connected components, its trace-free holomorphic endomorphism space has dimension $b - 1$.

Proof. Order the four summands by decreasing height. Every active block is strictly upper triangular. Both K and harmonic inclusion preserve matrix blocks. A product contributing to a_j has upper-triangular displacement at least j ; four such displacements vanish. Substitution of (22) into (7) therefore verifies the fixed-point equation exactly. Uniqueness of the small solution identifies it with the complete reconstruction, also on ambient nonintegrable tuples.

On $\kappa = 0$, the initial partial sums in this ordering are holomorphic subbundles with line quotients among the F_i . All slopes are zero at the wall. Intersecting a subsheaf with this filtration and adding the degree inequalities for its images in the line quotients proves semistability. The filtration is Jordan–Hölder and its graded sheaf itself is E_0 . The exact kernel assertion in (19) makes every holomorphic endomorphism constant on each undirected component, giving the stated dimension. For $x \neq 0$ there is at least one edge, so the full endomorphism dimension is less than four. A polystable bundle with this graded object would be isomorphic to E_0 , whose dimension is four. This proves nonpolystability. $\square$

Integrability is still the full equation $\kappa = 0$; acyclicity alone does not imply it. The wall domain for the connected theorem is

$$(23) \quad \mathcal{N}^{\text{con}}(\rho) = \{x \in \mathcal{N} : 0 < |x| < \rho, \kappa(x) = 0, \text{ supp}(x) \text{ connected}\}.$$

This is a specified family of strictly semistable bundles, without a smoothness assumption on its integrable locus. It is not identified with every nearby semistable deformation.

3.2. Exact graph flow and reciprocal decay.

Theorem 3.3. *There are uniform $q_*, C_g > 0$ such that every integrable $x_0 \in \mathcal{N}$ with $0 < q_0 = q(x_0) < q_*$ has an exact graph flow for all $t \geq 0$. Its support and full integrability are preserved, and*

$$(24) \quad \frac{q_0}{1 + 5q_0 t} \leq q(x(t)) \leq \frac{q_0}{1 + q_0 t/5}.$$

Writing $x(t) = e^{Y(t)} x_0$, $Y(0) = 0$, one has

$$(25) \quad \frac{1}{4} \log(1 + q_0 t/5) \leq |Y(t)|_{\max}, \quad |Y(t)| \leq C_g \log(1 + q_0 t/5).$$

The reconstruction in (16) is an exact metric flow on d_{x_0} throughout this interval.

Proof. The exact multiplicative field preserves every nonzero block on finite time intervals. Hence the support remains acyclic and diagonal equivariance preserves $\kappa = 0$. At the wall (15) gives

$$\dot{q} = 2\langle \mu_0, \beta^\Psi \rangle = -2|\mu_0|^2 + 2\langle \mu_0, \mathcal{R}_0(x) \rangle.$$

Using (21), choose q_* with $2\sqrt{2} \max(C_E, 1)q_* \leq 1/5$. On that region,

$$-5q^2 \leq \dot{q} \leq -q^2/5.$$

Choose it also strictly inside the common graph and reconstruction domain. The energy decreases and cannot reach its boundary. The finite smooth field continues on the smaller closed ball for every finite time. Integrating the inequalities for $1/q$ proves (24) and positivity.

The reduced velocity satisfies $|\beta^\Psi| \leq Cq$, while $\int_0^t q(s) ds \leq 5 \log(1 + q_0 t/5)$. This proves the upper diagonal bound. Conversely each squared block norm is at least $e^{-4|Y|_{\max}}$ times its initial norm, so $q(t) \geq e^{-4|Y(t)|_{\max}} q_0$. The upper energy bound gives the lower diagonal estimate. All diagonal matrices are finite on every finite interval, allowing exact reconstruction throughout. The proof uses no stationary metric on this bundle. $\square$

On connected support, (19) and the exact bundle isometry give

$$\left| \log \frac{\lambda_j(H_*(t))}{\nu_j(x(t))} \right| \leq Cq(x(t)) \quad (1 \leq j \leq 3), \quad \lambda_4 \geq \gamma.$$

Since $\text{tr } L_{\text{fin}} = 2q$, its largest eigenvalue satisfies $2q/3 \leq \nu_3 \leq 2q$. Thus λ_3 is comparable to $q_0/(1 + q_0 t)$. This total-energy argument does not determine individual decay powers for λ_1 or λ_2 . Their smaller scales remain in the exact finite evolution.

4. A TRIANGULAR CLASS ON THE UNSTABLE SIDE

The entire wall nullcone need not have decreasing slope along every arrow after the polarization changes. We therefore specify the unstable class before estimating its flow.

4.1. The filtration and a nonlinear example. Let

$$W = V_{12} \oplus V_{13} \oplus V_{32} \oplus V_{24} \oplus V_{34}, \quad \dim_{\mathbb{C}} W = 32,$$

and define

$$(26) \quad \mathcal{Z}_{\Delta}^{\text{con}}(\rho) = \{x \in W : |x| < \rho, \kappa(x) = 0, \text{supp}(x) \text{ connected}\}.$$

The number 32 is the dimension of the ambient harmonic subspace, not a dimension assertion for its integrable locus. All five obstruction components remain imposed.

Proposition 4.1. *For $\tau > 0$, every sufficiently small integrable $x \in W$ has Harder–Narasimhan line quotients F_1, F_3, F_2, F_4 in that order and is unstable for $\omega_{-\tau}$. At $\tau = 0$ it is semistable with locally free graded object E_0 ; nonzero points are not polystable. The connected locus contains arbitrarily small points with nonzero quadratic reconstruction term.*

Proof. All five allowed blocks are strictly upper triangular in the order 1, 3, 2, 4. Formula (22) therefore applies. On $\kappa = 0$ the initial partial sums are holomorphic subbundles with those line quotients. By (9), their centered mean curvatures are

$$(27) \quad \tau, \quad 0, \quad -2\tau/5, \quad -3\tau/5.$$

Slope is a positive common multiple of mean curvature, so these are strictly decreasing for $\tau > 0$. A filtration with stable quotients and decreasing slopes is Harder–Narasimhan: intersect a subsheaf with the filtration and add degree inequalities for its images in the line quotients. Its slope is at most the first quotient slope, and equality can occur only in that first piece. Repeating in the quotient identifies the successive maximal destabilizing subobjects. In particular F_1 destabilizes the bundle. At zero the same reasoning gives Jordan–Hölder; Proposition 3.2 gives the nonpolystability assertion.

For a nonlinear example set $x_{32} = 0$ and use identical Serre–dual bases on the first two projective factors. On either factor, let $e_0 \in H^0(\mathcal{O}(1))$ be the first chosen basis section, $\eta^0 \in H^1(\mathcal{O}(-3))$ its dual harmonic form, and θ the harmonic generator of $H^1(\mathcal{O}(-2))$, normalized so that the harmonic projection of $e_0\eta^0$ is θ . Writing $[s : t]$ on the third factor, take the four coefficient pairs

$$(28) \quad (x_{12}, x_{13}, x_{24}, x_{34}) = ((s^3, 0), (s^3, 0), (s, 0), (s, 0)).$$

The first two pairs represent $H^1(\mathcal{O}(-3)) \otimes H^0(\mathcal{O}(3))$, and the last two $H^0(\mathcal{O}(1)) \otimes H^1(\mathcal{O}(-2)) \otimes H^0(\mathcal{O}(1))$, on their indicated product factors. Their path cohomology products are the same nonzero s^4 , with opposite differential signs, so the obstruction vanishes. Their differential-form sum is nonzero: it is the antisymmetrized tensor comparing $e_0\eta^0$ with the harmonic generator θ of $H^1(\mathcal{O}(-2))$. The former has a zero where e_0 vanishes and nonzero harmonic projection θ ; θ has no zero. They are not proportional. The coexact block-14 primitive of the nonzero exact sum is therefore nonzero. No longer paths occur when $x_{32} = 0$. Scaling the four harmonic blocks by ϵ scales this primitive by ϵ^2 and gives arbitrarily small connected integrable examples. This is the canonical diamond construction of [Wue26a, Section 6.4]. $\square$

4.2. The two-parameter scalar comparison. For $a, d, q > 0$ define

$$(29) \quad \mathcal{L}_{a,d}(q, t; \tau) = \begin{cases} \frac{qe^{-a\tau t}}{1 + \frac{dq}{a\tau}(1 - e^{-a\tau t})}, & \tau > 0, \\ \frac{q}{1 + dqt}, & \tau = 0. \end{cases}$$

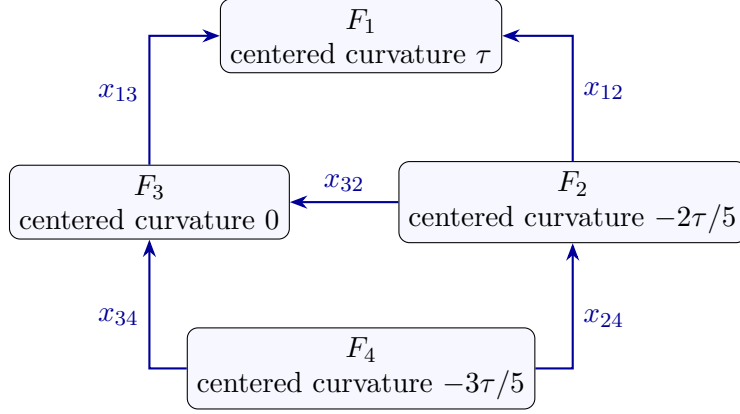


FIGURE 1. The five allowed harmonic blocks on the unstable side. Arrows run from source to target. For $\tau > 0$ their line quotients have the decreasing slope order 1, 3, 2, 4. The full reconstruction also contains induced nonlinear entries; the diagram does not replace the integrability equation $\kappa = 0$.

It solves $\dot{z} = -a\tau z - dz^2$, $z(0) = q$. The expression is continuous at $\tau = 0$ for each finite time. Its denominator retains the independent initial energy and distance from the wall.

Theorem 4.2. *There are fixed $\tau_0, q_* > 0$ such that, for $0 \leq \tau \leq \tau_0$ and integrable $x_0 \in W$ with $0 < q_0 = q_{-\tau}(x_0) < q_*$, the exact graph flow exists for all $t \geq 0$ and satisfies*

$$(30) \quad \mathcal{L}_{14/5,5}(q_0, t; \tau) \leq q_{-\tau}(x(t)) \leq \mathcal{L}_{2/5,1/5}(q_0, t; \tau).$$

It preserves support and full integrability and reconstructs an exact solution of (11). The same assertion holds on d_{x_b} from any $x_0 = e^{Y_0}x_b$ when both coefficient presentations lie in the reconstruction domain. No bound on Y_0 or ratio between τ and q_0 is required.

Proof. For $h = (3/2, -1/2, 1/2, -3/2)$ one has $|h|^2 = 5$ and $h_i - h_j \geq 1$ on the five allowed blocks. Thus $q_l/\sqrt{5} \leq |\mu_l| \leq \sqrt{2}q_l$. The five characters of $b(-\tau)$, in order 12, 13, 32, 24, 34, are

$$-\tau(7/5, 1, 2/5, 1/5, 3/5).$$

Consequently $-7\tau q_l/5 \leq \langle \mu_l, b \rangle \leq -\tau q_l/5$. The exact energy derivative and the quartic remainder give

$$-\frac{14}{5}\tau q - 4q^2 - 2\sqrt{2}C_E q^3 \leq \dot{q} \leq -\frac{2}{5}\tau q - \frac{2}{5}q^2 + 2\sqrt{2}C_E q^3.$$

Choose q_* uniformly inside the signed graph domain and with $2\sqrt{2}\max(C_E, 1)q_* \leq 1/5$. Then

$$(31) \quad -\frac{14}{5}\tau q - 5q^2 \leq \dot{q} \leq -\frac{2}{5}\tau q - \frac{1}{5}q^2.$$

The energy decreases, excluding exit from the smaller closed coefficient ball. Smooth finite-dimensional continuation gives every finite time; scalar exponential multipliers preserve support and equivariance preserves $\kappa = 0$. Comparison with the two exact scalar solutions gives (30). The reduced velocity is bounded on this region, so Y and e^Y exist on every finite interval, allowing exact reconstruction. $\square$

At $\tau = 0$ this is the reciprocal estimate of Theorem 3.3; at fixed positive τ it gives exponential decay. The transition is uniform even when τ/q_0 tends to zero or infinity. Integrating

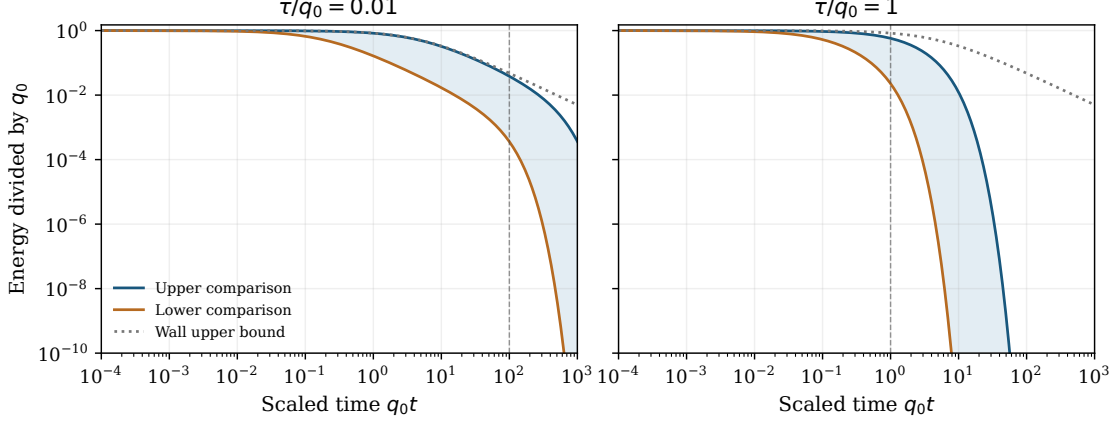


FIGURE 2. Exact scalar comparison profiles in (30). The horizontal variable is $q_0 t$; the two panels use different values of the independent ratio τ/q_0 . Brown and blue curves are the lower and upper bounds; the gray dotted curve is the wall upper profile. The vertical dashed line marks $\tau t = 1$. These are solutions of scalar comparison equations, not computed HYM trajectories.

the upper profile also gives

$$(32) \quad \int_0^T q_{-\tau}(x(t)) dt \leq \begin{cases} 5 \log \left(1 + \frac{q_0}{2\tau} (1 - e^{-2\tau T/5}) \right), & \tau > 0, \\ 5 \log(1 + q_0 T/5), & \tau = 0. \end{cases}$$

The logarithmic derivative of the first denominator is one fifth of the upper profile. Its infinite-time value is finite for $\tau > 0$, but can diverge as $\tau \rightarrow 0$. This distinction will determine the residual diagonal drift.

5. PHASE WITH UNBOUNDED DIAGONAL COORDINATES

We now allow complementary initial metric components. There are two geometric domains: the full connected integrable nullcone at $l = 0$, and the connected triangular locus at $l = -\tau$, $0 \leq \tau \leq \tau_0$. The argument is shared, but the domain at the wall is larger.

5.1. Initial metrics and the energy cutoff. Fix a small base tuple x_b in one of these domains. For $0 < r \leq r_0$ take

$$(33) \quad \begin{aligned} Y_0 &\in \mathfrak{t}_{\mathbb{R}}, \quad x_0 = e^{Y_0} x_b, \quad 0 < q_0 = q_l(x_0) < r^2/16, \\ u_0 &\in \mathcal{E}_1, \quad v_0 = u_0 - \Psi(l, x_0), \quad \|v_0\|_{\mathcal{E}_1} < \delta, \quad D = \|v_0\|_{\mathcal{E}_\alpha}, \quad H^{\text{in}} = e^{Y_0 *}(h_0 e^{u_0}). \end{aligned}$$

Both x_b and x_0 lie in the reconstruction domain. There is no bound on $|Y_0|$, and no condition comparing D with r . Uniform initial-data derivatives below concern (Y_0, v_0) with fixed l, x_b , not an unweighted matrix norm on the original metric at an unbounded Y_0 .

Put $x = e^Y x_b$ and $v = u - \Psi(l, x)$. Subtracting graph invariance from (17) gives exactly

$$\begin{aligned}
 (34) \quad & f(Y) = \beta^\Psi(l, x), \\
 & h(Y, v) = \beta(l, x, \Psi(l, x) + v) - \beta^\Psi(l, x), \\
 & F(Y, v) = \mathcal{N}_l(x, \Psi + v) - \mathcal{N}_l(x, \Psi) - D_x \Psi(l, x) C_x h(Y, v), \\
 & \dot{Y} = f(Y) + h(Y, v), \quad \dot{v} = -A_l v + F(Y, v).
 \end{aligned}$$

Choose a smooth $\chi : [0, \infty) \rightarrow [0, 1]$, equal to one on $[0, 1]$ and zero on $[2, \infty)$. Multiply f, h, F by $\chi(q_l(e^Y x_b)/r^2)$ and extend them by zero. Take $\sqrt{2}r$ within the common coefficient domain, using uniform equivalence of q_l and $|x|^2$, so extension occurs before any reconstruction boundary. No Banach-norm cutoff in v is needed.

The diagonal characters give, for $j \leq 3$, $|D_Y^j x| \leq C_j |x|$ and $|D_Y^j q_l| \leq C_j q_l$. On the transition region $r^2 \leq q_l \leq 2r^2$, the first two derivatives of the cutoff are consequently bounded independently of r, x_b, Y . The exact first jets in Section 2 give, on $\|v\|_{\mathcal{E}_\alpha} < \rho$,

$$\begin{aligned}
 (35) \quad & \|f\|_{C^2} \leq C(\tau + r^2), \\
 & \|D_v h\| + \|D_v F\| \leq C(r + \rho), \quad \|D_v h(Y, v)\| \leq C(r + \|v\|_{\mathcal{E}_\alpha}), \\
 & \|D_Y h\| + \|D_Y F\| \leq C\|v\|_{\mathcal{E}_\alpha}, \quad h(Y, 0) = F(Y, 0) = 0.
 \end{aligned}$$

Here $\tau = 0$ for the full wall domain. All second derivatives are bounded. For the normal bounds use the vanishing first jets and $D_x \Psi C_x = O(r^2)$. Mixed derivatives and vanishing at $v = 0$ give the last line. Differentiating the cutoff adds bounded factors multiplying maps already vanishing at $v = 0$. The first line must include τ : cutoff derivatives multiply the constant drift $b(-\tau)$.

Choose ρ, r_0, τ_0 small enough for the criterion in Section 2, with $\|Df\|_\infty < c/8$, and then choose δ uniformly for its initial and variation bounds. For every smaller r the same choices work. The cutoff system has a global solution and a C^1 phase satisfying

$$\begin{aligned}
 (36) \quad & \|v(t)\|_{\mathcal{E}_\alpha} \leq CDe^{-4ct}, \quad |h(Y(t), v(t))| \leq C(r + D)De^{-4ct}, \\
 & Y_{\text{ph}} = \lim_{T \rightarrow \infty} \phi_{-T}(Y(T)), \quad Y_*(t) = \phi_t(Y_{\text{ph}}), \quad |Y_*(t) - Y(t)| \leq C(r + D)De^{-4ct}.
 \end{aligned}$$

The construction has used only bounded coefficients, with no bound on the absolute diagonal coordinates. We next show that both trajectories avoid the cutoff region.

5.2. Perturbed energy confinement.

Lemma 5.1. *After uniform reductions of r_0, δ , both coefficient trajectories stay in $q_l < r^2$ for all $t \geq 0$ and retain their initial support and integrability. For $B \leq C(r + D)D \leq \log 2$, the actual coefficient energy satisfies*

$$(37) \quad e^{-B} \mathcal{L}_{14/5, 5}(q_0, t; \tau) \leq q_l(x(t)) \leq e^B \mathcal{L}_{2/5, 1/5}(q_0, t; \tau).$$

For $q_*^0 = q_l(x_*(0))$ the selected energy has the unforced bounds (30), and $|\log(q_*^0/q_0)| \leq C(r + D)D$.

Proof. Before a first exit from $q_l < r^2$, the cutoff is one. Diagonal equivariance preserves support and the full equation $\kappa = 0$. Theorems 3.3 and 4.2, with $r^2 < q_*$, give the corresponding unforced energy inequality. Since $|\mu_l| \leq \sqrt{2}q_l$, the perturbation adds at most $\zeta(t)q_l$ in absolute value, where $\zeta(t) = C(r + D)De^{-4ct}$. Thus

$$-\frac{14}{5}\tau q - 5q^2 - \zeta q \leq \dot{q} \leq -\frac{2}{5}\tau q - \frac{1}{5}q^2 + \zeta q.$$

Let $B(t) = \int_0^t \zeta(s) ds \leq B$. For $z = e^{-B(t)}q$ and $w = e^{B(t)}q$,

$$\dot{z} \leq -\frac{2}{5}\tau z - \frac{1}{5}e^{B(t)}z^2 \leq -\frac{2}{5}\tau z - \frac{1}{5}z^2, \quad \dot{w} \geq -\frac{14}{5}\tau w - 5e^{-B(t)}w^2 \geq -\frac{14}{5}\tau w - 5w^2.$$

Scalar comparison proves (37). In particular $q_l(x(t)) < 2q_0 < r^2/8$, excluding first exit. Reduce r_0, δ so $|Y_* - Y|_{\max} < 1/4$. The diagonal character identity gives $q_l(x_*) \leq e^{4|Y_* - Y|_{\max}} q_l(x) < er^2/8 < r^2$. The selected trajectory therefore also sees no cutoff. The exact reduced velocity remains bounded on each finite time interval; no block disappears at finite time. Applying the unforced theorem to that trajectory proves its energy bounds. The same character estimate at time zero gives the initial energy-ratio bound. $\square$

Exact reconstruction now identifies both solutions with actual metric flows on the original d_{x_b} :

$$(38) \quad H(t) = e^{Y(t)*}(h_0 e^{\Psi(l, x(t)) + v(t)}), \quad H_*(t) = e^{Y_*(t)*}(h_0 e^{\Psi(l, x_*(t))}).$$

The $\mathcal{E}_1$ initial class gives strong solutions, and the first-jet parabolic realization gives positive-time regularity. The nonlinear domain needs only the controlled $\mathcal{E}_\alpha$ norm.

At the wall, integration of $|\dot{Y}| \leq Cq_l + |h|$ and (37) gives

$$(39) \quad |Y(t) - Y_0| + |Y_*(t) - Y_{ph}| \leq C \log(1 + q_0 t/5) + C(r + D)D, \\ |Y(t) - Y_0|_{\max} \geq \frac{1}{4} \{\log(1 + q_0 t/5) - B\}.$$

For the selected trajectory the lower bound has $B = 0$ and q_0 replaced by q_*^0 . Thus both diagonal increments diverge, even though their difference tends to zero exponentially.

5.3. Relative metrics and uniqueness. Set $\Psi_* = \Psi(l, x_*(t))$ and define the relative positive matrix in local h_0 -unitary frames by

$$(40) \quad J_*(t) = e^{-\Psi_*/2} e^{-Y_*} (h_0^{-1} H(t)) e^{-Y_*} e^{-\Psi_*/2}.$$

This congruence sends $h_0^{-1} H_*$ to the identity. Writing $z = Y_* - Y$, the absolute diagonal cancels exactly:

$$J_* = e^{-\Psi_*/2} e^{-z} e^{\Psi(l, x) + v} e^{-z} e^{-\Psi_*/2}.$$

The function $Y \mapsto q_l(e^Y x_b)$ is a sum of positive exponentials and is convex. The segment between the two diagonal coordinates therefore remains in $q_l < r^2$. On it $\|D_x \Psi C_x\|_{\mathcal{E}_\alpha} \leq Cr^2$, so $\|\Psi(l, x) - \Psi_*\|_{\mathcal{E}_\alpha} \leq Cr^2|z|$. Smooth matrix functional calculus in the $h^{17/8}$ multiplication algebra now gives

$$(41) \quad \|\log J_*(t)\|_{h^{17/8}} \leq \eta(t), \quad e^{-\eta(t)} H_*(t) \leq H(t) \leq e^{\eta(t)} H_*(t), \quad \eta(t) = CD e^{-4ct}.$$

The logarithm is zero when $v = z = 0$, and its derivatives are bounded on the remaining small positive matrix neighborhood. The estimate contains no condition number of e^Y .

Lemma 5.2. *For graph metrics $H_i = e^{Y_i*}(h_0 e^{\Psi(l, e^{Y_i} x_b)})$ with both coefficient energies below r^2 ,*

$$(42) \quad |Y_1 - Y_2| \leq Cd_\infty(H_1, H_2).$$

The constant is independent of the absolute Y_i .

Proof. Put $\Delta = Y_1 - Y_2$ and $\Psi_i = \Psi(l, e^{Y_i} x_b)$. Pull the order sandwich back by e^{-Y_2} and take each diagonal entry. If the order distance is at most η , this gives

$$|2\Delta_i + \log(e^{\Psi_1})_{ii} - \log(e^{\Psi_2})_{ii}| \leq \eta.$$

The coefficient segment stays in $q_l < r^2$ by convexity. Differentiating the scalar logarithm on that segment bounds its difference by $Cr^2|\Delta|$. Taking the maximum and using $|\Delta| \leq 2|\Delta|_{\max}$ absorbs the error after shrinking r_0 . These scalar diagonal entries are independent of the choices of unitary line frames. $\square$

The exact uncut field has derivative bounded by Cr^2 on this sublevel set: $b(l)$ is constant and the remaining derivative is $O(q_l)$. Convexity gives the same Lipschitz bound between any two endpoints there. If two graph flows in this domain have order distance $O(e^{-ct})$ from H , Lemma 5.2 bounds their coordinate separation by the same exponential. Backward Gronwall on each finite interval, with $Cr^2 < c$, forces their initial separation to be zero. This proves uniqueness in the stated class and independence from the auxiliary cutoff, while retaining dependence on the chosen graph.

5.4. Initial-data regularity and spectra. The strict inequalities in (33) define an open metric class. Its normalization solves $P_0 \log(e^{-Y}(h_0^{-1}H)e^{-Y}) = 0$. At a normalized u the finite derivative is $-2R(u)$. The pointwise multiplier of $S_u^{-1}C_u$ is $a \coth a \geq 1$, so $R(u) \geq I$. The implicit-function theorem gives a smooth local inverse. At an arbitrary Y_0 , first pull back by e^{-Y_0} ; the same calculation has uniform bounds in that relative metric chart. Integrating the positive derivative along a diagonal segment proves uniqueness of the normalization.

The analytic phase criterion and (35) show that Y_{ph} is C^1 in (Y_0, v_0) at fixed l, x_b , with uniform bounds and

$$|Y_{\text{ph}} - Y_0| \leq C(r + D)D, \quad \|D_{v_0}\Pi\| \leq C(r + \delta), \quad \Pi(Y_0, 0) = Y_0.$$

These become initial-metric derivatives in the relative chart. They do not assert uniformity for absolute matrix variations at an unbounded diagonal.

Theorem 5.3. *There are fixed $r_0, \tau_0, \delta, c, C, \gamma > 0$ such that, on either of the two domains stated at the start of this section, every initial metric (33) selects a reduced trajectory $x_*(t) = e^{Y_*(t)}x_b$ on the fixed graph Ψ . Its initial value $Y_*(0)$ is C^1 in the relative initial-metric coordinates at fixed l, x_b , and for all $t \geq 0$*

$$|Y_*(t) - Y(t)| \leq C(r + D)De^{-4ct}, \quad e^{-CDe^{-4ct}}H_*(t) \leq H(t) \leq e^{CDe^{-4ct}}H_*(t).$$

The normal, energy and strong relative-logarithm estimates are (36)–(41). The selected trajectory is unique among same-orbit graph trajectories in $q_l < r^2$ with d_∞ error $O(e^{-ct})$.

For the geometric spectra of H, H_* and the finite spectrum at $x_*(t)$, the kernels vanish and

$$(43) \quad \begin{aligned} \left| \log \frac{\lambda_j(H(t))}{\lambda_j(H_*(t))} \right| &\leq 4CDe^{-4ct} & (j \geq 1), \\ \left| \log \frac{\lambda_j(H(t))}{\nu_j(l, x_*(t))} \right| &\leq Cq_l(x_*(t)) + CDe^{-4ct} & (1 \leq j \leq 3), \\ \lambda_4(H(t)) &\geq \gamma. \end{aligned}$$

All constants are uniform in time, absolute diagonal coordinates, support weights and positive spectral ratios.

Proof. Only the spectral assertions remain. Connected support and the exact kernel in (19) give zero trace-free holomorphic kernel. The sandwich (41) compares squared endomorphism norms in degrees zero and one by $e^{\pm 2\eta}$; the holomorphic differential is fixed. Min-max gives the first line with 4η . Pulling the graph metric to its small-coefficient presentation is an exact isometry, even when its diagonal is unbounded. Equation (19) and the graph bound then give the second line and fourth gap. Shrinking r_0, δ makes that gap uniform. $\square$

At the wall λ_3 is comparable to $q_*^0/(1+q_*^0t)$; on the unstable side it has the corresponding two-sided scalar bounds, up to fixed factors. No individual power or common exponential rate is inferred for the smaller modes. For a nonzero disconnected support the same proof applies with its exact $b-1$ dimensional kernel: count the zeros, compare only positive indices, and retain the fourth gap.

6. ENTRY FROM PRESCRIBED METRICS

The local theorem imposes smallness on a complementary metric component, but not on the diagonal factors. To reach that class from a prescribed metric, we use a finite-time entry result at the split reference object. The time is fixed before the parameter and deformation neighborhoods are chosen.

6.1. The finite-time input and its scope. We use [Wue26b, Proposition 6.2] in the following form. Let H^0 be any smooth positive metric on E_0 with $\det H^0 = \det h_0$, and put $W^0 = \log(h_0^{-1}H^0)$. There is $A > 0$, depending on H^0 but not on the requested tolerance, such that for every $\epsilon > 0$ one can first choose a finite time T , then positive signed-parameter and coefficient radii and an open neighborhood $\mathcal{U}$ of W^0 in $h^{19/8}(\mathcal{V})$, for which

$$(44) \quad H(T) = e^{Y_T *}(h_0 e^{u_T}), \quad |Y_T|_{\max} < A, \quad P_0 u_T = 0, \quad \|u_T\|_{h^{19/8}} < \epsilon.$$

This holds for every small signed l , every complete integrable x_b in the resulting coefficient ball, and every $W^{\text{in}} \in \mathcal{U}$, with initial metric $h_0 e^{W^{\text{in}}}$ on d_{x_b} . The coordinates are normalized near a chosen split-reference metric limit. They depend C^1 on the initial metric, with uniform derivative bounds on these finite-time neighborhoods. No stability hypothesis occurs.

The two ingredients matter here. On the split reference bundle, order against h_0 bounds both the evolving metric and its inverse. Smooth connection convergence modulo gauge, as in the corrected [SW15, Theorem 1.1(2)], then combines with elliptic estimates for the bounded holomorphic convergence gauges to produce a metric subsequence converging to a positive diagonal constant metric. Second, the unsliced logarithmic metric equation has a common normally elliptic principal part and smooth first-jet remainder. Its strong $h^{19/8}$ finite-time map continues along the fixed reference trajectory to the selected T . The proof in [Wue26b, Section 6] verifies the full-domain semiflow endpoint and the normalization derivative $-2R(u)$. It neither differentiates an infinite-time limiting metric nor requires a small positive finite spectral gap.

6.2. The common-restart theorem.

Theorem 6.1. *Fix H^0 and the graph Ψ . There are $A, C, c, \gamma, \epsilon_0 > 0$ such that, for every $0 < \epsilon < \epsilon_0$, one can choose a common finite restart $T = T_\epsilon$, radii $\rho_\epsilon, r_\epsilon > 0$, a parameter bound $\tau_\epsilon > 0$, and an open $h^{19/8}$ neighborhood $\mathcal{U}_\epsilon$ of W^0 with the following properties.*

The allowed data are either

$$(45) \quad \begin{aligned} & l = 0, \quad x_b \in \mathcal{N}^{\text{con}}(\rho_\epsilon), \\ & \text{or } l = -\tau, \quad 0 \leq \tau \leq \tau_\epsilon, \quad x_b \in \mathcal{Z}_\Delta^{\text{con}}(\rho_\epsilon), \end{aligned}$$

and $W^{\text{in}} \in \mathcal{U}_\epsilon$ in either case. The flow from $H(0) = h_0 e^{W^{\text{in}}}$ has at T the normalization (44) and, with $x_T = e^{Y_T} x_b$,

$$(46) \quad \begin{aligned} & 0 < q_T = q_l(x_T) < r_\epsilon^2/16, \quad v_T = u_T - \Psi(l, x_T), \\ & \|v_T\|_{h^{19/8}} < \epsilon, \quad D_T = \|v_T\|_{h^{17/8}} \leq \epsilon. \end{aligned}$$

The time is chosen before the parameter, coefficient and initial-metric neighborhoods.

For $s = t - T \geq 0$, there are actual sliced coordinates and a selected graph trajectory, both in $q_l < r_\epsilon^2$, such that every conclusion of Theorem 5.3 holds with (Y_0, u_0, D, q_0, t) replaced by (Y_T, u_T, D_T, q_T, s) . In particular

$$(47) \quad \begin{aligned} |Y_{\text{ph}} - Y_T| &\leq C(\rho_\epsilon + D_T)D_T, & \|\log J_*(s)\|_{h^{17/8}} &\leq CD_T e^{-4cs}, \\ e^{-CD_T e^{-4cs}} H_*(s) &\leq H(T+s) \leq e^{CD_T e^{-4cs}} H_*(s), \\ \left| \log \frac{\lambda_j(H(T+s))}{\nu_j(l, x_*(s))} \right| &\leq Cq_l(x_*(s)) + CD_T e^{-4cs} \quad (1 \leq j \leq 3), & \lambda_4(H(T+s)) &\geq \gamma. \end{aligned}$$

The phase is unique in the same-orbit fast relative-metric tracking class of that theorem. At each fixed (l, x_b) it is C^1 in the initial metric on $\mathcal{U}_\epsilon$. These derivative bounds may depend on H^0, ϵ but are uniform over the chosen coefficient neighborhood.

All positive spectral ratios are unrestricted. In the unstable domain no ratio between τ and coefficient energy is restricted. No uniform restart over all prescribed metrics is asserted.

Proof. Fix the local constants in Theorem 5.3 and choose $\epsilon_0 < \delta/2$, with its other strict normal margins. Apply the finite-time input to H^0 with tolerance $\epsilon/2$. It first gives T and $|Y_T|_{\max} < A$, then entry neighborhoods with $\|u_T\|_{h^{19/8}} < \epsilon/2$.

Let C_n bound $q_l(x) \leq C_n|x|^2$ uniformly on the signed entry interval. Shrink the parameter interval to include the closed $[-\tau_\epsilon, 0]$ within that interval and within the local phase interval. Next choose $\rho = \rho_\epsilon$ so small that, with

$$(48) \quad r = 8\sqrt{C_n}e^{2A}\rho, \quad r \leq r_0, \quad C_\Psi C_n e^{4A}\rho^2 < \epsilon/2,$$

the coefficient ball also lies inside the entry neighborhood. The diagonal action gives $q_l(e^{Y_T}x_b) \leq C_n e^{4A}|x_b|^2 < r^2/64$. The graph norm is below $\epsilon/2$, proving (46). Keep or shrink the metric neighborhood provided by the entry theorem.

Diagonal action preserves both geometric domains, their support and the complete obstruction equation. The local tracking theorem therefore applies at T . Its exact reconstruction agrees with the prescribed metric there; uniqueness of the parabolic flow identifies the all-future solution. Its phase and coordinate bounds $C(r + D_T)D_T$ become $C_{H^0}(\rho + D_T)D_T$ by (48). The energy estimates are precisely (37) and the unforced bounds at the selected initial energy. This proves all assertions in (47), including the strong norm at $s = 0$. The all-positive spectral comparison of the local theorem also applies.

At the fixed finite T , the restart map is C^1 uniformly on the chosen ambient neighborhoods. Subtracting $\Psi(l, e^{Y_T}x_b)$ adds the derivative $D_x \Psi C_x = O(q_T)$. Compose with the C^1 phase map already proved on the coefficient sublevel set. This gives the initial-metric derivative assertion without differentiating the split-reference limit or an inverse weak eigenvalue. $\square$

The theorem compares actual flows from T onward. The finite entry result controls the preceding interval; a backward parabolic problem is never solved. For smooth initial metrics the solution is smooth. The stated strong neighborhood also permits $h^{19/8}$ initial data with the full-domain local regularity and positive-time regularity used in the proof. At $\tau = 0$, both descriptions agree on their common triangular subspace by uniqueness of the selected phase for the fixed graph.

7. RESIDUAL DRIFT AND THE GRADED CONNECTION

On the unstable side, integrability of the coefficient energy allows subtraction of the linear diagonal drift. The remaining constant retains finite initial data. At the wall the same energy integral diverges, and this residual need not exist.

7.1. The residual for an exact graph trajectory.

Theorem 7.1. *Fix $0 < \tau \leq \tau_0$ and a connected integrable triangular base x_b in the reconstruction domain. For an exact graph trajectory from Y_0 with $q_0 = q_{-\tau}(e^{Y_0}x_b) < q_*$, the limit*

$$(49) \quad Z_\infty = \lim_{t \rightarrow \infty} (Y(t) - b(-\tau)t) = Y_0 + \int_0^\infty \{\beta^\Psi(-\tau, x(t)) - b(-\tau)\} dt$$

exists. With uniform C_β ,

$$(50) \quad |Z_\infty - Y_0| \leq 5C_\beta \log(1 + q_0/(2\tau)),$$

$$|Z_\infty - (Y(t) - b(-\tau)t)| \leq 5C_\beta \log(1 + q_{-\tau}(x(t))/(2\tau)) \leq \frac{5C_\beta}{2\tau} q_{-\tau}(x(t)).$$

On the open initial set $q_{-\tau}(e^{Y_0}x_b) < q_$, the map $Y_0 \mapsto Z_\infty$ is C^1 and a local diffeomorphism, with*

$$(51) \quad \|D_{Y_0}Z_\infty\| + \|(D_{Y_0}Z_\infty)^{-1}\| \leq 2(1 + q_0/(2\tau))^C.$$

Every active harmonic block has the nonzero limit

$$(52) \quad e^{-(b_i - b_j)t} x_{ij}(t) \longrightarrow e^{(Z_\infty)_i - (Z_\infty)_j} (x_b)_{ij}.$$

Proof. The exact field satisfies $|\beta^\Psi - b| \leq C_\beta q_l$. Equation (32) gives absolute convergence of (49) and its first bound. Apply that same estimate to the autonomous trajectory restarted at time t to bound the tail; $\log(1 + s) \leq s$ gives the last inequality. The block limit is the exact character identity $x_{ij} = e^{Y_i - Y_j}(x_b)_{ij}$.

Write $f(Y) = \beta^\Psi(-\tau, e^Y x_b)$. The moment derivative is $O(q_l)$, and the differentiated quartic remainder, composed with $D_Y x = C_x = O(|x|)$, is $O(q_l^2)$. Therefore $\|Df(Y)\| \leq C q_l(e^Y x_b)$ along the invariant region, independently of $|Y|$. For the finite variational matrix $J(t) = D_{Y_0}Y(t)$,

$$\dot{J} = Df(Y(t))J, \quad J(0) = I, \quad \|J(t)\|, \|J(t)^{-1}\| \leq \exp\left(C \int_0^t q_l(x(s)) ds\right).$$

At fixed $\tau > 0$ this integral is finite. Both $\dot{J}$ and the derivative of J^{-1} are integrable, so the matrices have limits whose product is the identity. The integrated energy bound yields (51).

To identify the limit as a derivative, take a small compact neighborhood of Y_0 within the open energy region. There its initial energy is at most some $\bar{q} < q_*$, and $q_l(x(t)) \leq \bar{q}e^{-2\tau t/5}$ uniformly. The residual integrals and their first variations thus converge locally uniformly. Their limits are C^1 and have the invertible derivative just obtained. The inverse-function theorem proves local invertibility. Connected support removes the trace-free diagonal stabilizer, so these are three independent initial diagonal coordinates. $\square$

For an actual solution of Theorem 5.3, apply this theorem to its selected trajectory. Since $Y_* - Y_{\text{act}} \rightarrow 0$ exponentially, both residuals have the same limit. In the prescribed-entry notation,

$$|Z_\infty - Y_T| \leq C(\rho_\epsilon + D_T)D_T + 5C_\beta \log(1 + q_*^0/(2\tau)).$$

The actual tail has, in addition to the selected tail in (50), the term $C(\rho_\epsilon + D_T)D_T e^{-4cs}$. Both rescaled actual and selected blocks have the limits (52), with time measured from restart. Composition with the phase and restart maps proves C^1 dependence on the prescribed initial metric at fixed $\tau > 0, x_b$. The local invertibility assertion concerns the three real diagonal initial coordinates of the reduced flow.

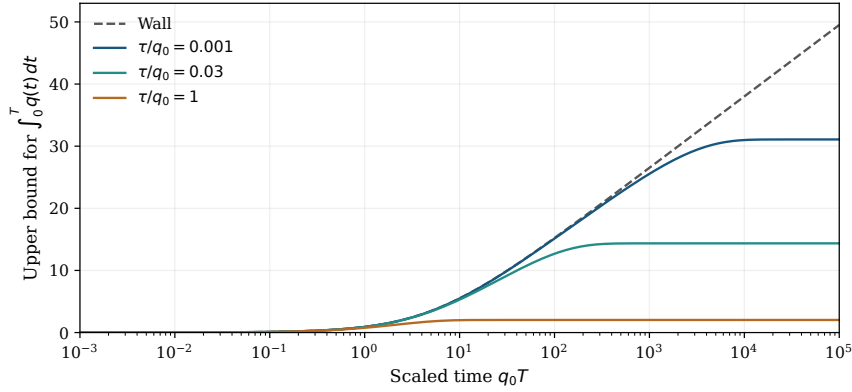


FIGURE 3. The integral of the upper scalar energy profile in (32), as a function of $q_0 T$. Positive values of τ/q_0 give finite limits; the wall curve grows logarithmically. Multiplication by C_β bounds displacement after subtracting linear drift. The curves are exact comparison integrals, not actual residual trajectories.

Neither the residual nor its derivative is bounded uniformly as $\tau \rightarrow 0$ by these estimates. At the wall, (39) forces unbounded diagonal displacement for nonzero data. Figure 3 shows the integral bound that distinguishes the two regimes.

7.2. An explicit topology for convergence of connections. Let $A_H(t)$ be the Chern connection on $(d_{x_b}, H(t))$, and write its actual coordinates as $u(t) = \Psi(l, x_{\text{act}}(t)) + v(t)$. The isometry from this evolving metric to h_0 is

$$(53) \quad U(t) = e^{u(t)/2} e^{Y_{\text{act}}(t)}, \quad U(t)^* h_0 U(t) = H(t).$$

Here the first factor is the positive isometry in the small-coefficient presentation; the order of factors matters.

Proposition 7.2. *For every local trajectory above, or from the common restart in Theorem 6.1, the transported connection $\tilde{A}(t)$ satisfies*

$$(54) \quad \|\tilde{A}(t) - A_0\|_{h^{9/8}} + \|F_{\tilde{A}(t)} - F_{A_0}\|_{h^{1/8}} \leq C(q_l(x_{\text{act}}(t))^{1/2} + D e^{-4ct}) \rightarrow 0.$$

For $\tau > 0$, A_0 is the split HN graded connection; at the wall it is the equal-slope HE graded connection. There is no curvature concentration in these flows.

Proof. Transport first by $e^{Y_{\text{act}}}$. Exact reconstruction equivariance gives the Chern connection of $(d_x, h_0 e^u)$; there is no norm or spatial derivative of the large diagonal left to estimate. Relative to A_0 , its $(1, 0)$ part is

$$e^{-u} \nabla_0^{1,0} e^u - e^{-u} a(x)^{*h_0} e^u.$$

The additional isometry $e^{u/2}$ involves a smooth matrix function and one spatial derivative of u . Hence its connection difference B has $\|B\|_{h^{9/8}} \leq C(|x| + \|u\|_{h^{17/8}})$. The reconstruction has bounded higher spatial norms on its coefficient ball, and $\|u\|_{h^{17/8}} \leq C q_l(x) + C D e^{-4ct}$. Uniform norm equivalence and $q_l(x) < 1$ give the first bound in (54).

For the curvature use $F_{A_0+B} - F_{A_0} = d_{A_0} B + B \wedge B$. A derivative sends $h^{9/8}$ to $h^{1/8}$ and multiplication is continuous in these spaces. The same bound follows for the curvature. The

energy estimates tend to zero in both regimes. Proposition 4.1 and Proposition 3.2 identify the limiting graded bundle. Each factor's product metric is HE for ω_l , though the four mean curvatures are unequal when $\tau > 0$. Uniform curvature convergence in these isometries excludes concentration. $\square$

After the usual positive identification of $H(t)$ with h_0 , the difference from (53) is unitary. Thus this also gives convergence modulo unitary gauge for that fixed-metric connection family. The assertion is in the displayed Hölder topologies and uses a time-dependent unitary gauge.

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