INTEGRAL GENERATORS FOR CURVE COUNTING ON THREEFOLDS

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Abstract. Pardon's integral curve-counting algebra has geometric generators whose local models may depend on the covering multiplicity. In first Chern number and virtual dimension zero we give a single model in each genus for all multiplicities. On a genus-g curve take two line bundles of degree $g-1$ with canonical determinant and act on them with opposite weights. The equivariant classes of the multiple zero sections freely generate the integral algebra. The comparison with an isolated holomorphic multiple is triangular in multiplicity, with leading coefficient one given by the complex orientation of the embedded curve. The resulting factors give a unique product with integral exponents for effective cycles of Chern number zero on every compact Kähler threefold whose first Chern class is nonnegative on curves.

1. The integral local factors

The universal count of effective curves on a compact threefold records the entire space of cycles in each integral homology class. Even when these spaces carry integral classes, the factors used to express their generating series can have fractional exponents. On a smooth projective Calabi–Yau threefold, for example, a genus-ten factor occurs with exponent in $\mathbb{Z}+2/3$ in three times a primitive class [Wue26a, Theorem 1.1], [Wue26h, Theorem 1.2]. This factor comes from scaling the two normal directions to a curve by the same weight. Which local factors recover the integral structure?

Pardon constructed an algebra of effective one-cycles in families of complex threefolds. In its semi-Fano part every irreducible component has nonnegative first Chern number. Write $H_{\mathbb{Z}}$ for the summand in first Chern number and virtual dimension zero. Multiplication is external product on disjoint targets. Pardon proved that integral geometric elements obtained by isolating multiples of embedded curves, one for each genus and multiplicity, freely generate this algebra. The normal neighborhood used to isolate a multiple may depend on its multiplicity. A uniform construction by scalar dilation gives free generators over $\mathbb{Q}$ [Par25, Theorem 1.2]. We identify a uniform equivariant construction which gives free generators over $\mathbb{Z}$.

Let C be a smooth compact curve of genus g. Choose a line bundle L of degree $g-1$ and give the two summands of $L\oplus(K_C\otimes L^{-1})$ weights $+1$ and -1. We call this the balanced antidiagonal model. Write $a_{g,m}$ for the normalized equivariant class supported on the m-fold zero section and $A_g(q)=1+\sum_{m\geq 1}a_{g,m}q^m$. This class is independent of the choices of C and L. The normalization divides a supported Thom class by a product of weights, so its construction gives an element of $H=H_{\mathbb{Z}}\otimes\mathbb{Q}$. For equal weights write $x_{g,m}$ for the scalar class. The two classes agree in multiplicity one.

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Theorem 1.1. *The map to the rationalization embeds $H_{\mathbb{Z}}$ as the polynomial ring $\mathbb{Z}[a_{g,m} : g \geq 0,\ m \geq 1]$. For each (g,m) one can choose an integral geometric local curve element $y_{g,m}$ of this type so that*

$$y_{g,m} - a_{g,m} \ \in \ \mathbb{Z}[a_{h,j} : h \geq 0,\ 1 \leq j < m].$$

There is a geometric obstruction to comparing the two sorts of generators directly in the normal bundle. For $g \geq 2$ choose L outside the theta divisor. Both normal summands are acyclic, so the embedded zero section is regular and isolated. Yet $h^0(C, L^2) \geq g-1$. A nonzero section s cuts out the compact double cycles $w^2 = ts$ in the first summand. They converge to the doubled zero section as t tends to zero. Isolation of the embedded curve says little about isolation of its doubles.

For a prescribed multiplicity m we start in a holomorphic neighborhood in which all multiples through m are isolated. We place a tube of this neighborhood in a closed symplectic six-torus. Bounded-area compactness keeps the cycles in the tube away from its boundary during a small deformation. Excision therefore lets us apply the compact cluster comparison to its insertion, giving a product of contributions of elementary curves, the regular local models of Ionel and Parker [IP18]. A genus-h elementary curve contributes the balanced series A_h by the geometric comparison of [Wue26b, Proposition 3.1]. In degree one, testing by genus separates the multiplicity-one classes: their fixed-genus series have distinct leading powers. The embedded curve has its complex orientation, so its genus occurs with coefficient $+1$ and every other genus with coefficient zero. The remaining factors have core class a proper multiple of the original class. In the coefficient of q^m these corrections therefore involve only generators of multiplicity less than m. This gives the integral triangular comparison in Theorem 1.1.

For a smooth compact Kähler threefold X, write $\Gamma_0(X)$ for zero and the effective integral classes of first Chern number zero. Let $U_{X,\beta}$ be the characteristic function of the entire space of effective cycles of class β, and take $U_{X,0} = 1$.

Theorem 1.2. *If $c_1(TX)$ is nonnegative on every compact irreducible curve, then*

$$\sum_{\beta \in \Gamma_0(X)} U_{X,\beta} Q^\beta = \prod_{\substack{\beta \in \Gamma_0(X) \\ \beta \neq 0}} \prod_{h \geq 0} A_h(Q^\beta)^{n_{\beta,h}} \quad \text{in } H_{\mathbb{Z}}[[\Gamma_0(X)]]. \tag{1}$$

The exponents are unique integers and only finitely many occur for each β. The completion is by Kähler area, and the integers are independent of the Kähler form. When X is smooth projective with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, the exponent $n_{\beta,h}$ is the signed multiplicity of elementary clusters of class β and genus h in the bounded-energy cluster decomposition.

Two identities for cycles explain why the basis theorem yields a product. Addition records every decomposition of a cycle, and division by an integer records every integral root of its class. After the lower-area factors have been removed, the first remaining coefficient is primitive for addition and vanishes under every proper division. It is therefore an integral combination of multiplicity-one classes. Their coefficients give the exponents at that homology label. The product retains the full integral homology label, including torsion. Its local factors also have a numerical interpretation. Bryan and Pandharipande computed the local Gromov–Witten residues [BP08], and Ionel and Parker identified the antidiagonal residue of an elementary curve [IP18, Proposition 3.3]. Doan, Ionel and Walpuski proved the requisite finiteness for the symplectic cluster expansion [DIW26]. The balanced universal classes and a conditional numerical factorization were constructed in [Wue26c]. The geometric comparison of

[Wue26b] established the projective cluster formula. Here its use on a tube gives the integral presentation and the compact Kähler factorization. In the compact example above, the primitive genus-three count is -14. Changing back to scalar factors subtracts $(4720/3)(-14)$ from the genus-ten exponent, giving the stated fractional part [Wue26a, Wue26h].

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2. Comparing local curve classes

2.1. **Two normal weights.** A degree-m cycle in the total space of $L_+ \oplus L_-$, with $L_+ = L$ and $L_- = K_C L^{-1}$, meets a fiber of the projection to C in a multiset of length m. No fiber contains a compact curve. The elementary symmetric functions of either coordinate in this multiset are sections of $L_\pm^j$, $1 \le j \le m$. For instance, on the normalization of a component the characteristic polynomial of multiplication by the fiber coordinate on its finite flat direct image extends the symmetric functions across branch values. Multiply the polynomials with their cycle multiplicities. Evaluation at more than $\max(0, m(g-1))$ distinct points therefore gives a tuple with common zero precisely $m[C]$. Its weights are the integers $\pm j$.

For clarity, a tuple of k nonzero weights $r_1, \dots, r_k$ on the cycle space gives, over the finite Borel base $\mathbb{P}^k$, a section of the pullback of $\bigoplus_i \mathcal{O}(r_i)$. Pull back its relative Thom class and divide by $\prod_i r_i$. The support on the relative cycle space is compact. Adding an equivariant test after passage to the next projective space gives the hyperplane pushforward of this class, since the normalized Euler class of the added line is the hyperplane class. Concatenating two tuples proves independence of the tests. Varying the curve and line of degree $g-1$ gives the same universal class $a_{g,m}$ [Wue26c, §§2–3]. The division in this prescription explains why integrality of its answer requires proof.

The first two weights can already distinguish the prescriptions. Over a curve of genus two the scalar and balanced doubles satisfy $a_{2,2} - x_{2,2} = -2x_{4,1}$. In the holomorphic double-cover calculation two smooth branches of genus four move away from the zero section. Their contribution fixes the sign in the comparison [Wue26b, §8].

There are two operations on the resulting coefficients. Pullback by addition of effective cycles is the coproduct Δ. Pullback by multiplication of a cycle by l is denoted ρ_l. With $u_0 = 1$, the scalar and the antidiagonal classes of a fixed genus both satisfy

$$\Delta u_d = \sum_{i+j=d} u_i \otimes u_j, \qquad \rho_l u_d = \begin{cases} u_{d/l} & l \mid d, \\ 0 & l \nmid d. \end{cases} \tag{2}$$

These identities include the diagonal of Borel bases in the coproduct. In each shifted term the two relative cohomological degrees are d and $-d$, with $d \neq 0$. In Chern number zero the factor of positive relative cohomological degree has negative virtual dimension and vanishes [Par25, Lemma 4.13 and Theorem 6.13], [Wue26c, Proposition 3.1]. The augmentation is restriction to the empty cycle.

The laws (2) alone leave a choice of integral coordinates. For instance one may replace $A_g(q)$ by $A_g(q)A_h(q^2)$ for $h \neq g$ and invert the change integrally. The unit of a positive elementary curve specifies the particular coordinates used in Theorem 1.2.

2.2. **Elementary curves and multiplicity one.** We use the comparison $\Phi : H \to S$ of [Wue26b, §2], where S is the rational cycle algebra for smooth families of almost-complex six-manifolds. Over a point the characteristic function of a compact open set of cycles gives an insertion. Open extension in the target, external product for disjoint targets, and the two

endpoint restrictions of a compactly supported class over an interval are the operations we will need. The geometric topology records the support and the integration current of the cycle. In integrable families it gives the topology of compact analytic cycles.

We recall three results from [Wue26b] and their roles in the comparison. First, on the elementary genus-h curve of Ionel and Parker the unit at its j-fold cycle is $\Phi(a_{h,j})$ [Wue26b, Proposition 3.1]. On the sum of two copies of a spin line, real rotations commute with the Lee deformation to a super-rigid normal operator; in complex eigenlines those rotations have weights $+1, -1$. The normalized Thom class extends on the finite Borel family, restricting at the elementary end to the unit of the isolated cycle. Second, a bounded-energy partition into clusters gives a *product* of these units and their inverses [Wue26b, §4, Proposition 4.1]. It is essential here to partition disconnected cycles as well as connected maps. We explain its local use in §3.

Third, the classes $\Phi(x_{h,1})$ are linearly independent over $\mathbb{Q}$ [Wue26b, Theorem 5.1]. The argument tests a proposed finite relation at its least genus h. The image-cycle map on connected stable maps of that genus is proper over compact sets of cycles. A class supported on an embedded curve of larger genus has no preimage in multiplicity one. At genus h the remaining regular embedded curve has a single preimage with trivial automorphism group and complex local degree one. Stability excludes trees of contracted rational components there. The compact-support argument on a smooth parameter family representing a putative relation promotes this local orientation calculation to independence [Wue26b, Lemmas 5.2–5.3 and Proposition 6.1]. Thus we need injectivity of Φ only on $V = \bigoplus_h \mathbb{Q} x_{h,1}$. The argument uses a fixed genus for each test of a relation.

Lemma 2.1. *In H, the primitive elements annihilated by every ρ_l, $l > 1$, are precisely V. Two series with constant coefficient one satisfying (2) through q^m agree to that order if their images under Φ do.*

Proof. For the first assertion we recall the algebraic argument of [Wue26b, Lemma 7.1]. The scalar classes generate H rationally by [Par25, Theorem 6.13]. In the polynomial algebra on formal scalar generators change variables to the coefficients $v_{h,j}$ of the logarithm of $1 + \sum_{j \geq 1} x_{h,j} q^j$. These are primitive, and $\rho_l v_{h,j}$ is $v_{h,j/l}$ if l divides j, and zero otherwise. These formulas already hold formally. A polynomial relation of smallest total degree in the new variables has zero constant term. Its reduced coproduct lies in the tensor square of the space of polynomials of strictly smaller degree. Evaluation in H is injective on that space, hence also on its tensor square over $\mathbb{Q}$. This makes the relation a primitive polynomial, which is linear in characteristic zero by the identity $f(v+w) = f(v) + f(w)$. Applying division by its largest label j would give a nonzero relation among the $x_{h,1}$ (use the identity if $j = 1$). Their smooth images are independent. This proves polynomial independence of the scalar generators. The same linearity and largest-label argument in H now identifies its joint primitive kernel with V. For the second assertion, the first coefficient of a difference is primitive by addition and killed by every proper division. Its smooth image vanishes, and Φ is injective on V. □

3. An isolated multiple and the integral basis

Fix a genus g and a multiplicity m. We will compare two representatives associated with these integers. On the equivariant side the curve and its balanced bundle are fixed throughout the construction. On the geometric side the entire holomorphic neighborhood may depend on m.

Lemma 3.1. *There is a fixed complex threefold germ about a genus-g curve C with acyclic normal bundle in which every $j[C]$, $1 \le j \le m$, is isolated. Its unit J_m is an integral geometric local curve element of type $(g, m, 0)$, and $J_1 = x_{g,1}$.*

Proof. For $g \ge 2$ take the divisorial deformation of a bundle neighborhood constructed in the proof of [Par25, Lemma 6.7]. Its embedded curve is unobstructed and has normal bundle of index zero, so it has no infinitesimal deformations in the fixed threefold. The m-multiple lies in the interior semi-regular locus where the semi-chart of multiples of deformations of C is open [Par25, Lemma 6.3 and Definition 6.6]. At Chern number zero its fixed-target dimension is zero. After restriction to a tube retracting onto C, every component has Chern number zero. The point $m[C]$ is therefore open in the entire cycle space of the germ. The inverse image of that point under addition of $(m-j)[C]$ is $\{j[C]\}$, proving the lower-degree assertions. The complex point orientation gives the geometric generator. Deformation to the normal bundle preserves the regular embedded curve in degree one. Its local chart is a section over the deformation parameter by the implicit function theorem. On the normal bundle the unit in degree one is the class with empty evaluation tuple, namely $x_{g,1}$.

In genus zero take $\mathcal{O}(-1)^{\oplus 2}$. In genus one take $L \oplus L^{-1}$ with L of infinite order in $\mathrm{Pic}^0(C)$. A nonzero fiber coordinate on a finite cover is impossible by negative degree in genus zero. In genus one it would trivialize a pulled-back line, whose norm would make L torsion. In both cases every compact curve is supported on the zero section, and $J_j = a_{g,j} = x_{g,j}$ by the empty-tuple construction. □

Proposition 3.2. *For a germ in Lemma 3.1 of genus $g \ge 2$, there exist integers $n_{e,h}$ of finite support, $1 \le e \le m$, for which*

$$1 + \sum_{j=1}^{m} J_j q^j = \prod_{e,h} A_h (q^e)^{n_{e,h}} \pmod{q^{m+1}}, \qquad n_{1,h} = \delta_{gh}. \tag{3}$$

Proof. We place a small tube of the germ in a closed symplectic six-torus. In a product of two symplectic elliptic curves take one horizontal torus and $g-1$ distinct vertical translates. Smoothing the $g-1$ transverse positive intersections gives a symplectic genus-g surface. Its class pairs once with a vertical torus, hence stays primitive in the product with a third elliptic curve. Denote the resulting class by A and take an integral symplectic form. An orientation-preserving identification of the surfaces may be used to identify their complex tangent structures along the core by choosing an appropriate compatible complex structure on its symplectic tangent plane. Their complex normal bundles are isomorphic, since both have rank two and degree $2g-2$. Choose a tubular identification with this normal derivative, taking the complex normal complement to the symplectic orthogonal. The transported integrable structure is compatible along the core and tame on a small tube. Extending a compatible structure along the surface and then using a cutoff in a chart of tame structures realizes the given germ inside the closed target.

Set $b = \omega(A) > 0$ and bound energy by mb. There is a tube V with compact closure in this realization such that the only bounded nonempty cycles in $\overline{V}$ are $j[C]$ for $j \le m$. Otherwise a sequence of tubes shrinking to C contains counterexamples. Retraction to C and positivity of area give class jA with $1 \le j \le m$. Take a subsequence of fixed class. Bounded-area cycle compactness [DIW26, Theorem 2.11] forces a limit $j[C]$. Convergence of currents and supports in the integrable tube contradicts isolation in the holomorphic cycle space. Inside such an isolating tube take V with compact closure and with C in its interior. Every bounded cycle contained in $\overline{V}$ lies in the larger isolating tube, so none can meet ∂V. This property

persists under small perturbations by the same compactness argument. Containment in V therefore defines a compact open and closed locus over a short closed path to a generic tame structure. Its unit transports $\Phi(J_j)$ in each class jA.

At the generic endpoint, the bounded connected supports are disjoint embedded curves. Partition the supports in V into finitely many clusters, including the exterior as a separate part [DIW26, Propositions 3.18 and 3.21]. In each part the union of the underlying curves is compact by closed incidence. Distinct parts have disjoint unions, and hence disjoint open target neighborhoods. A disconnected cycle then decomposes uniquely into cycles in the respective neighborhoods, including empty cycles. The unit for the union is the product of the corresponding series, with the convolution performed in integral homology. This is the excision argument of [Wue26b, §4]. When a path is considered in the ambient threefold, we keep only the finitely many labels occurring below the energy cutoff. Indeed the bounded cycle space over the closed path is compact and the integral homology label is locally constant. Products and inverses of these series still have finite support at the cutoff, since their positive degrees have a positive minimum.

For one of these clusters, of class eA, apply the cluster isotopy theorem [DIW26, Theorem 3.25] in the geometric form of [Wue26b, §4, proof of Proposition 4.1]. On a regular arc a subdivision changes the chosen factor only by factors in proper multiples. At a fold, compact continuation of the outer insertion gives the product of the two inner factors equal to one modulo such corrections. Consequently a crossing inverts the distinguished factor, modulo a finite product of corrections. At the elementary endpoint its factor is $\Phi(A_h(q^e))$. Iterate on the correction clusters; their core classes are keA, $k > 1$, and $\lfloor m/(ke) \rfloor < \lfloor m/e \rfloor$. This gives a finite signed product in S at the required cutoff. The use of disjoint target neighborhoods in excision is what allows these geometric identities to include the cycles with several components.

Addition and division for the isolated units satisfy (2) up to m: the decompositions of $j[C]$ are precisely $i[C] + (j-i)[C]$. The product on the right of (3) obeys the same laws because substitution, multiplication and inversion preserve them. Lemma 2.1 lifts the identity from S to H. In degree one it gives $x_{g,1} = \sum_h n_{1,h} x_{h,1}$, so independence makes the asserted leading exponent positive and equal to one. □

Proof of Theorem 1.1. Take the coefficient of q^m in (3). The unique unshifted factor is $A_g(q)$. Any other factor starts in degree $e \geq 2$ and only uses coefficients of multiplicities at most $\lfloor m/e \rfloor < m$. The binomial coefficients in all integer powers, including negative powers, are integers. Hence $J_m = a_{g,m} + P_{g,m}$ with $P_{g,m}$ an integral polynomial in lower multiplicities. Choose one germ for each pair (g, m) as in Lemma 3.1. Pardon's integral generation theorem permits *any one such choice for each type* [Par25, Theorem 6.13]. Nonnegative Chern numbers exclude other types from this summand. Induction in m, simultaneously for all g, identifies the subalgebra of H generated by these integral images with $\mathbb{Z}[a_{g,m}]$.

To see that it is a polynomial ring, apply the least-degree relation argument of Lemma 2.1 to formal balanced generators. Their logarithmic coordinates are primitive and have the same division law, and their degree-one images are again the independent $x_{g,1}$. There is no rational polynomial relation. The triangular equations with leading coefficient one imply that the chosen J_m for all types also have algebraically independent images over $\mathbb{Q}$. Since those classes generate the original integral summand, its rationalization map has zero kernel. This proves the stated integral identification. □

4. Integral products on compact threefolds

Let Γ be an additive commutative monoid with an additive degree $E : \Gamma \to \mathbb{R}_{\geq 0}$, positive away from zero, with finite sublevel sets. Multiplication in $H[[\Gamma]]$ is the convolution for addition in Γ. An element has only finitely many decompositions into nonzero summands: at its degree there are finitely many possible sublabels and these have a positive minimum degree. For a series $P = \sum P_\beta Q^\beta$ with $P_0 = 1$, compatibility with the two geometric operations means

$$\Delta P_\beta = \sum_{\gamma+\delta=\beta} P_\gamma \otimes P_\delta, \qquad \rho_l P_\beta = \sum_{l\gamma=\beta} P_\gamma \quad (l > 1). \tag{4}$$

The second sum ranges over every root in Γ. Distinct integral classes may have the same multiple.

Proposition 4.1. *A compatible series has all its coefficients in $H_\mathbb{Z}$ if and only if it has a product expansion in the factors $A_h(Q^\beta)$ with integral exponents, finitely supported in h at each $\beta \neq 0$. In that case the exponents are unique.*

Proof. A single factor satisfies (4) by (2). Equivalently the second identity says that applying ρ_l coefficientwise gives the substitution $Q^\gamma \mapsto Q^{l\gamma}$; this identity and the coproduct identity persist under products and inversion. All factors and their inverses have integral coefficients by Theorem 1.1.

Conversely divide out the factors already found at smaller degree. At a first remaining degree the coefficient R_β of the residual integral series is primitive. Indeed in the first sum of (4) every proper summand has smaller degree. For the same reason every term on the right of the second identity vanishes. Lemma 2.1 puts R_β in V. By the integral polynomial basis and $a_{h,1} = x_{h,1}$ we have $V \cap H_\mathbb{Z} = \bigoplus_h \mathbb{Z} x_{h,1}$. The coefficients in that direct sum are precisely the required exponents at β. Dividing out the corresponding factors changes no other label of this degree. Finite sublevel sets give the expansion and the same recursion proves uniqueness. □

Proof of Theorem 1.2. On a compact Kähler threefold the mass of an effective cycle for the Kähler metric is its area. Bounded-area cycles form a compact space and their integral homology label is locally constant in the geometric topology [DIW26, Theorem 2.11 and Remark 2.13]. The image of a compact space in this discrete group is finite, even when the group has torsion. The cycles of fixed class give a compact open and closed locus. A nonzero effective class has positive area. Nonnegativity of $c_1(TX)$ implies that all components of a cycle in $\Gamma_0(X)$ have zero first Chern number. In particular the unit of the whole locus is an integral semi-Fano insertion.

Addition of cycles pulls this unit back to the sum of the exterior products of the units on the class loci for all splittings of β. Multiplication by l pulls it back to the units on all class loci with $l\gamma = \beta$. These are precisely (4), so Proposition 4.1 proves (1). On a compact manifold the areas of complex tangent lines under two Kähler metrics are uniformly comparable. This identifies the two completions, and uniqueness gives the same exponents. Under the projective hypotheses, the geometric product of [Wue26b, Theorem 1.1] has exactly these local factors and signed cluster multiplicities. Its rationalization agrees with (1); uniqueness identifies the integers. □

5. Rational curves on a contracted face

For a genus-zero curve the scalar and balanced factors coincide. Write $A_0(q) = \sum_{m\geq 0} x_{0,m} q^m$. A family of resolved nodes shows how a negative power of this factor can arise

even though every cycle is effective. We give the family first because it also computes the contribution of a moving exceptional ruling.

Let E be the nonsplit extension of $\mathcal{O}$ by $\mathcal{O}(-2)$ over $\mathbb{P}^1$. Then $E \simeq \mathcal{O}(-1)^{\oplus 2}$, since the boundary map from $H^0(\mathcal{O})$ to $H^1(\mathcal{O}(-2))$ is an isomorphism. Let $u : \operatorname{Tot} E \to \mathbb{C}$ be the quotient coordinate. For a smooth complex function f on a closed disc, nonzero on the boundary, form the fiber product $u = f(t)$ over its interior. A complex-horizontal right inverse h to du gives this threefold an almost-complex structure: in the splitting of a tangent vector as $(a, k + h(df(a)))$ use multiplication by i on a and the vertical complex structure on k. The projection to the disc is holomorphic and each surface fiber retains its original complex structure. For holomorphic f this is the complex fiber product.

Lemma 5.1. *The series of compact cycles in this family, graded by their degree over $\mathbb{P}^1$, is $\Phi(A_0(q))^{\deg(f,0)}$ in the smooth algebra. For holomorphic f it is $A_0(q)^{\deg(f,0)}$ in H.*

Proof. A compact irreducible curve projects constantly to the disc. The only compact irreducible curve in $\operatorname{Tot} E$ is its zero section, by negativity after pullback to the normalization. Thus over a zero of f there is one rational curve, and there are no others. The full degree-m cycle locus is $\operatorname{Sym}^m(f^{-1}(0))$: the parametrization by sums is continuous in support and current and is a continuous bijection from a compact space to a Hausdorff space. The same construction over a compact path without boundary zeros transports its unit. For separated zero sets one can excise into disjoint targets, so their series multiply.

A positive simple zero can be deformed to the identity map, and there the family is an open neighborhood of the zero section of E. Its factor is $\Phi(A_0)$. At a negative regular zero the germ deforms to complex conjugation by linearizing at the zero and moving within the orientation-reversing linear isomorphisms. The local homotopies avoid boundary zeros on a small disc. In the family $f_s(x + iy) = x^2 - s + iy$ there are no zeros for $s < 0$ and two regular zeros of opposite signs for $s > 0$. The two factors therefore multiply to one. Translation of any f by a sufficiently small regular value counts its simple zeros with Brouwer signs, proving the assertion. For holomorphic f the small translations are a holomorphic family and the simple zeros are positive; the same computation is made directly on complex cycles. □

For $f(t) = t^k - s$, continuation around $s = 0$ permutes the k supports cyclically. The limiting single support has factor A_0^k. In bundle coordinates its transition at $s = 0$ is $v' = z^2 v + zt^k$, $t' = t$. Thus for distinct $k, l > r$ the rth infinitesimal neighborhoods of the zero curves agree while their series differ already in degree one. This is a dependence on the nonlinear neighborhood that persists however large the prescribed jet order is.

Proposition 5.2. *Let $p : X \to W$ be a birational contraction from a smooth projective threefold with $K_X \simeq \mathcal{O}_X$ to a projective variety. Suppose p is an isomorphism off a smooth divisor $D = \mathbb{P}_B(V)$, and contracts exactly its ruling fibers to the smooth connected curve $B \simeq p(D)$. If $b = g(B)$ and ℓ is a fiber class, the complete universal series on the contracted face is $A_0(Q^\ell)^{2b-2}$.*

Proof. The pullback of an ample divisor on W exposes the contracted face. Every irreducible component there is a ruling fiber, and in class $m\ell$ the cycle locus is $\operatorname{Sym}^m B$. All fibers have the same integral class because B is connected. An ample degree on X distinguishes its multiples, even when ℓ is divisible in $H_2(X; \mathbb{Z})$. Deform D to its normal bundle in X by blowing up $D \times 0$ in $X \times \mathbb{C}$ and deleting the strict transform of the central fiber. The normal bundle is K_D. A compact curve on the special fiber with contracted pushforward lies over one point of B. On that ruling the normal bundle is $\mathcal{O}(-2)$, so the curve is the zero section.

The relative fixed-label locus over a closed parameter interval remains $\mathrm{Sym}^m B$ times that interval, by the same compact continuous parametrization as in Lemma 5.1.

Use the convention of lines in V. The relative Euler sequence gives

$$0 \longrightarrow K_D \longrightarrow \mathcal{O}_D(-1) \otimes \pi^*(V^\vee \otimes K_B) \xrightarrow{u} \pi^* K_B \longrightarrow 0, \qquad \pi : D \to B.$$

Replace the zero section on the right by a transverse smooth section of K_B. Locally this is the family in Lemma 5.1, since on each ruling the Euler extension is nonsplit. A complex-linear horizontal splitting gives the same fiberwise structures as in that lemma and can be exchanged with a local product splitting by a homotopy. Globally the projection to B remains pseudoholomorphic. Each curve of zero base degree therefore stays in one fiber; along the homotopy of sections the relative symmetric powers of the compact zero loci give compact support in every degree. The signed number of zeros of the section is $2b-2$. Multiplying the local factors proves the assertion after applying Φ. Both complex series satisfy addition and division in the integer label m, since every effective summand or root of a contracted cycle is contracted. Lemma 2.1 lifts the equality. For $b \geq 1$ one can equally use a nonzero holomorphic differential and count its zeros holomorphically. The cotangent obstruction and its Euler number in the numerical type III theory are due to Wilson [Wil99, §4]. □

6. Specialization across a conifold transition

Suppose a flat projective family over a smooth curve has smooth algebraic total space of dimension four and central fiber with only ordinary double points. Let X_s be a smooth nearby fiber and let $Y \to X_0$ be a projective small resolution. Assume the two smooth threefolds have trivial canonical bundles and torsion-free second integral homology. Fix the common-complement marking and a relatively ample line bundle. Write $\pi : H_2(Y;\mathbb{Z}) \to H_2(X_s;\mathbb{Z})$ for the resulting quotient by the exceptional curve classes $[C_i]$. Writing U_X for the series in (1), the universal specialization identity of [Wue26i, Theorem 3.2] reads

$$U_{X_s} = \pi_* \left(U_Y \prod_i A_0(Q^{[C_i]})^{-1} \right). \tag{5}$$

Here π_* adds coefficients of classes with the same image. Finiteness requires the exceptional normalization inside the parentheses. In the blowup $\phi : \widetilde{Y} \to Y$ along the C_i, put $j = \mathrm{PD}_{\widetilde{Y}}^{-1} \phi^* \mathrm{PD}_Y$. After normalization all nonzero labels belong to $F = \{\gamma \neq 0 : j\gamma \text{ is effective on } \widetilde{Y}\}$. This set is closed under addition, avoids $\ker \pi$, and has finite sublevel sets for the projected ample degree [Wue26i, Proposition 3.1]. Indeed the ample degree of $j\gamma$ for a suitable polarization on the blowup is a fixed positive multiple of the projected degree, since $j\gamma$ pairs to zero with every exceptional divisor. The support assertion in (5) in that generality uses the numerical reconstruction of [Wue26i, §2]. It assumes that the indicated connected unmarked Gromov–Witten evaluations descend to the relevant relations in the rational cycle algebra, including those arising from families of complex targets. The analytic-family input to that descent is discussed in [Par25, Remark 1.8]. There is a direct argument using only fixed-target counts when both endpoints also have $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$.

The normalized series has integral balanced exponents $\widetilde{n}_{\gamma,h}$, obtained by subtracting one at each exceptional factor with its actual class. On the latter projective domain let $G_h(q,u)$ denote the connected numerical series of the elementary genus-h curve. The signed-cluster comparison of [Wue26b, §7.4] identifies the connected invariants of each fixed endpoint with the sum of $n_{\gamma,h} G_h(Q^\gamma, u)$. Write $w = (2\sin(u/2))^2$; the coefficient of q in G_h is w^{h-1}. The contracted face consists exactly of sums of the disjoint resolved nodes, so subtracting their G_0

removes all connected contributions on that face. For $\pi\gamma \neq 0$ the ordinary degeneration of the resolution gives its connected invariant as the empty-contact relative invariant of $j\gamma$ on the blowup. The nonempty contact strata have negative virtual dimension on the blowup side. In particular every connected coefficient at $\gamma \notin F$ vanishes after normalization [LLW18, §2.1], [Wue26i, §3].

Induct on an ample degree on the resolution. If $\gamma \notin F$, every proper root of γ is outside F, since F is closed under multiplication by positive integers. After the lower-degree terms have vanished, the connected coefficient at γ is $\sum_h \widetilde{n}_{\gamma,h} w^{h-1} = 0$. This is a finite Laurent polynomial and its least power has a distinct leading order in u. All its coefficients therefore vanish. Thus every remaining exponent is supported in F, and its product may be pushed forward in finite sums. The connected smoothing formula in the total-space class adds the fixed-target resolution counts with that image, also for classes ineffective on the smoothing [Wue26i, §3, proof of Theorem 3.2], [LW25, Theorem 2.16 and §3]. Comparing with the pushed-forward product, the same induction now in the smoothing degree identifies its exponents. This proves (5) on the stated projective domain without a theory of evaluations on general families. With the numerical reconstruction assumed in the broader case, the support theorem in [Wue26i, §3] and successive division by factors in F give the same exponent formula. In either case, for every nonzero effective β on the smoothing,

$$n_{\beta,h}(X_s) = \sum_{\substack{\gamma \in F \\ \pi\gamma = \beta}} \widetilde{n}_{\gamma,h}.$$

The proof applies in the monoid generated by the effective smoothing classes and $\pi(F)$, extending the series by zero. This also describes the cancellation when an image class is ineffective. Two small resolutions have the same blowup and their normalized series agree under its common zero-contact identification [Wue26i, §3]. The same summation for scalar product exponents preserves their integrality whenever the source exponents are integral [Wue26i, Corollary 3.3]. The scalar exponent in §1 explains the need for this hypothesis.

One can bound the sum explicitly on the classical transition of a quintic containing a plane [Ros06]. Let f, g be general quartics on $\mathbb{P}^4$. The central quintic $x_3 f + x_4 g = 0$ has sixteen nodes on the plane $x_3 = x_4 = 0$. Indeed the two restricted quartics meet transversely, and at each intersection x_3, x_4, f, g are local coordinates. Away from the plane smoothness follows from Bertini. In $\mathbb{P}^1_{[s:t]} \times \mathbb{P}^4$ the equations $sx_3 + tx_4 = 0$ and $-sg + tf = 0$ give its projective small resolution Y. The projection selects the kernel line of the two-by-two coefficient matrix, with an entire ruling over each node. Adjunction gives $K_Y = \mathcal{O}_Y$. Adding a general multiple of a quintic nonvanishing at all sixteen nodes gives a smoothing with smooth total space near the central fiber. Write B, H for the hyperplane classes from the two factors. Integral Lefschetz for the two ample divisors gives coordinates $(a, d) = (B \cdot \gamma, H \cdot \gamma)$ on $H_2(Y; \mathbb{Z})$, and the smoothing map is $(a, d) \mapsto d$. Each exceptional curve has class $(1, 0)$.

In this example F satisfies $0 \leq a \leq 4d$, $d \geq 1$. To see the upper bound, write $u = f/s = g/t$ and $v = x_4/s = -x_3/t$. These sections have classes $4H - B$ and $H - B$. The sections u, vh, with h ranging over cubics, generate the ideal of the exceptional curves after twisting by $4H - B$. Indeed their common ideal is locally (u, v), the two normal coordinates in the small resolution of the node. Thus on the blowup $\phi^*(4H - B) - E$ is generated by global sections, where E is the sum of the exceptional divisors. Pairing this nef divisor and $\phi^* B$ with $j\gamma$ proves both inequalities. A line in the plane missing the nodes lifts to the graph of the degree-four map $[f : g]$ on the line. Its proper transform has class $j(4, 1)$, so the upper bound is attained in F. After division by $A_0(q)^{16}$ the coefficient in smoothing degree $d > 0$ is

therefore a polynomial in the exceptional variable q of degree at most $4d$. Membership in this support bound alone does not establish that a coefficient is nonzero.

In the four-node toric transition let $\beta = (1, 1, 0)$ be the primitive line class of the smoothing plane. In the resolution basis of [Wue26i, §4] the marking takes $(n_1, \dots, n_6)$ to (n_2, n_3, n_4). The following individual source and smoothing coefficients use the stable-pair evaluation of geometric local classes assumed in [Wue26i, §§5–6]; see also [Par25, Remark 1.8]. The conifold specialization proved above uses the fixed-target Gromov–Witten comparison. Of the 37 candidate lifts of β, eight contribute. Their complete coefficients are given in [Wue26i, (21)–(24)]. The reconstruction also uses a bound on the Euler characteristic of a pure curve. The quadratic terms cancel upon specialization, so the eight polynomials sum to $3x_{0,1}$. On the smoothing the minimal pair spaces for lines and conics are the dual plane and $\mathbb{P}^5$. Their signed Euler characteristics 3 and -6 give

$$U_{X_s,\beta} = 3x_{0,1}, \qquad U_{X_s,2\beta} = 3x_{0,2} + 3x_{0,1}^2 - 6x_{0,1} \qquad \text{[Wue26i, Proposition 6.1]}.$$

The line spans an extremal ray and the only effective splitting of its double is $\beta + \beta$. Thus the first factor on the ray is $A_0(Q^\beta)^3$. At 2β its contribution is $3x_{0,2} + 3x_{0,1}^2$. The remaining $-6x_{0,1}$ is the leading coefficient of $A_0(Q^{2\beta})^{-6}$.

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