EXTENDING THE KEMPF–NESS FUNCTION TO COMPLEXES OF QUIVER REPRESENTATIONS

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Abstract. For every bounded complex of representations of a finite quiver with vertex charges in the upper half-plane, the superconnection endomorphism of Haiden, Katzarkov, Kontsevich and Pandit defines a closed one-form on the space of Hermitian metrics. We construct a primitive by transgression along the rescaling of the differentials. For nonzero total charge Z, rotating the primitive by $\overline{Z}/|Z|$ and taking its imaginary part gives a real potential. On representations this potential is a positive multiple of the Kempf–Ness function and is nonincreasing along the flow defined by the argument of the unitary polar factor of the endomorphism. For a two-term complex in degrees zero and one whose differential surjects onto a term supported at a loopless vertex i, the potential is nonincreasing if the vertex charge z_i satisfies $\arg z_i \geq \arg Z$ and either $\arg Z \geq \pi/3$ or the differential has rank one. On a three-term Kronecker quiver complex the potential can increase even when the induced metric on its stable cohomology is harmonic. Quillen's superconnection gives the analogous potential for two-term complexes of holomorphic bundles on a compact Riemann surface. On a single bundle it is a positive multiple of Donaldson's functional.

1. Introduction

Haiden, Katzarkov, Kontsevich and Pandit [HKKP26] propose a notion of Kähler geometry on a triangulated category with a stability condition in the sense of Bridgeland [Bri07]. In their framework an object carries a space of metrics, a complex-valued function $\mathcal{S}_{\mathbb{C}}$ on it whose differential is given by an endomorphism Ω, a phase $\arg\Omega$ defined by a positivity condition, a mass $\operatorname{tr}|\Omega|$, and a flow $h^{-1}\dot{h} = -\arg\Omega$ of the metrics, called the minimizing flow in [HKKP26, §4.1]; a metric is *harmonic* if $\arg\Omega$ is a scalar. For a representation of a quiver with central charges z_i and Hermitian metrics, Ω is z_i shifted by the moment map at each vertex, and the function $\mathcal{S}_{\mathbb{C}}$ is explicit [HKKP26, §5.1]: after rotation by the argument of the central charge of the representation, its imaginary part is a positive multiple of the Kempf–Ness function. For a complex of representations an endomorphism Ω is defined in [HKKP26, §5.4] through a superconnection in a graded algebra attached to the quiver; its trace is the central charge of the complex. No function $\mathcal{S}_{\mathbb{C}}$ is given there for complexes.

For a representation the flow is not the gradient flow of the Kempf–Ness function, but the Kempf–Ness function does not increase along it (Proposition 4.1); that is, it is a Lyapunov function for the flow. The question of this paper is whether the Kempf–Ness function of a representation extends to complexes. Two properties of the Kempf–Ness function [KN79, Kin94] are meant: its differential is the moment map shifted by the charges, and it does not increase along the flow. As a function with differential Ω it extends to every bounded complex of representations of every finite quiver (Theorem 1.1). As a function that does not

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increase along the flow it extends to two classes of complexes (Theorem 1.2) and not to all (Theorem 1.3). Even on a complex with polystable cohomology, the phase potential can be unbounded below and strictly concave along a geodesic in the space of metrics (Corollary 4.4).

In [HKKP26, §5.4] the phase of Ω is written with the functional calculus, which presupposes that Ω is normal. For complexes this can fail (Example 2.3). In the general framework of [HKKP26, §4.2] the argument is a self-adjoint element $\arg\Omega$ such that $\Omega(e^{-\sqrt{-1}\arg\Omega}x^*x)\geq 0$ for all elements x of the algebra attached to the metrized object, Ω being a functional there; for the functional $x\mapsto\operatorname{tr}(\Omega x)$ on endomorphisms this says that $\Omega e^{-\mathrm{i}\arg\Omega}$ is positive semidefinite. We therefore define $\arg\Omega$ as the Hermitian endomorphism Φ, with spectrum in the interval $(-\pi n,\pi(1-n))$ in degree n, the branch of [HKKP26, (5.15)], for which $\Omega e^{-\mathrm{i}\Phi}$ is positive definite; $e^{\mathrm{i}\Phi}$ is the unitary factor of the polar decomposition of Ω. For normal Ω this is the argument of the functional calculus. All statements about the flow of a complex refer to this definition. We call the resulting flow the *phase flow*; we do not use the name minimizing flow because on complexes the mass can increase along it [Wue26a, Theorem 1.1]. It is the flow of [Wue26b, Wue26c], whose titles keep the name of [HKKP26].

The phase flow was studied in [Wue26a, Wue26b] for two-term complexes of representations of a generalized Kronecker quiver that are quasi-isomorphic to a representation: the acyclic part decays, and for suitable charges the flow converges on the cohomology when the cohomology is polystable [Wue26b, Theorems 1.1 and 1.2]; these two notions are defined in Section 2. For these complexes a function with differential Ω was found by direct computation [Wue26a, Proposition 7.1], and for a range of charges its rotated imaginary part decreases along the flow [Wue26b, Theorem 7.6].

1.1. **Results.** Let Q be a finite quiver with central charges z_i in the upper half-plane, and E a bounded complex of representations of Q with spaces E_i^n at the vertex i in degree n, differentials d_i, arrows $\rho(a)$ and a Hermitian metric h_i^n on each E_i^n (Section 2). The endomorphism Ω preserves each E_i^n; we write Ω_i^n for its restriction to E_i^n, and $\|\cdot\|_{\mathrm{HS}}$ for the Hilbert–Schmidt norm of a linear map for the metrics. An integral formula for Ω is given in Proposition 2.1. Its trace is the central charge $Z=\sum_{i,n}(-1)^n z_i\dim E_i^n$ of the complex.

Theorem 1.1 (Primitive)**.** *For every bounded complex of representations of Q with a metric:*

(i) *The one-form $\xi\mapsto\operatorname{tr}(\Omega\,\xi)$ on the space of metrics, $\xi=h^{-1}\delta h$, is closed.*

(ii) *For $x\in[0,1]$ let $\Omega^{(x)}$ be the endomorphism of the complex with differentials $\sqrt{x}\,d_i$ and the same arrows and metrics. Then*

$$\mathcal{S}_{\mathbb{C}}(h)=\sum_n(-1)^n\Big[\sum_i z_i\log\det h_i^n-\sum_a\|\rho^n(a)\|_{\mathrm{HS}}^2\Big]+\int_0^1\sum_{i,n}n\,\operatorname{tr}\big((\Omega^{(x)})_i^n-(\Omega^{(0)})_i^n\big)\,\frac{dx}{x}$$

satisfies $d\mathcal{S}_{\mathbb{C}}(\xi)=\operatorname{tr}(\Omega\,\xi)$. Here $\rho^n(a)$ is the restriction of $\rho(a)$ to degree n, and the determinants are Gram determinants in fixed bases.

(iii) *If Z is not zero, then for any real φ with $Z=|Z|e^{\mathrm{i}\varphi}$, along the phase flow*

$$\frac{d}{dt}\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}\big)=-\sum_{i,n}\operatorname{tr}\Big(|\Omega_i^n|\,(\Phi_i^n-\varphi)\sin(\Phi_i^n-\varphi)\Big),$$

where $\Omega_i^n=|\Omega_i^n|e^{\mathrm{i}\Phi_i^n}$ is the polar decomposition with the spectrum of Φ_i^n in the interval $(-\pi n,\pi(1-n))$.

We call $\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})$ the *phase potential*. For a representation in degree zero, with $\varphi\in(0,\pi)$, it is $\sin\varphi$ times the Kempf–Ness function $\sum_a\|\rho(a)\|_{\mathrm{HS}}^2+\sum_i\theta_i\log\det h_i$ with $\theta_i=$

$\operatorname{Im}(e^{-\mathrm{i}\varphi} z_i)/\sin\varphi$, and each trace in the sum in (iii) is nonnegative (Proposition 4.1). For complexes they can have either sign. The last term of $\mathcal{S}_{\mathbb{C}}$ is a transgression by the number operator along the rescaling of the differentials, of the kind used for Bott–Chern forms of Hermitian complexes [BGS88], here taken on the space of metrics and in the algebra attached to the quiver.

Let K_m be the generalized Kronecker quiver, with vertices $0, 1$ and m arrows from 0 to 1. We write a complex of representations of K_m as a row of its terms, each term being a representation written as (space at the vertex 0 $\to$ space at the vertex 1); the maps between consecutive terms are the differentials.

Theorem 1.2 (Monotonicity in two classes)**.** *In each of the following cases the phase potential is nonincreasing along the phase flow.*

(i) *i_0 is a vertex of Q without loops; the complex is a vector space E_i^0 in degree zero at every vertex $i \neq i_0$ and $V \to U$ in degrees $0, 1$ at i_0, with surjective differential and nonzero cohomology; and, for $\varphi = \arg Z \in (0, \pi)$, one has $\arg z_{i_0} \geq \varphi$ and either $\varphi \geq \frac{\pi}{3}$ or $\dim U = 1$.*

(ii) *$Q = K_m$ and $\arg z_0 \geq \arg z_1$; the complex is $(0 \to U_1) \to (V \to W) \to (U_0 \to 0)$ in degrees $-1, 0, 1$, that is, with U_1 at the vertex 1 in degree -1 and U_0 at the vertex 0 in degree 1 (the subscript is the vertex), with injective differential $U_1 \to W$, surjective differential $V \to U_0$ and nonzero cohomology; and $\dim U_0 \leq 1$, $\dim U_1 \leq 1$.*

Theorem 1.3 (Failure of monotonicity)**.** *Let $Q = K_m$, $z_0 = -1 + \mathrm{i}$ and $z_1 = 1 + \mathrm{i}$.*

(i) *Let E be the orthogonal direct sum of a representation with spaces of equal dimension $k \geq 1$ at the two vertices, with a harmonic metric, and of the complex $\mathbb{C} \to \mathbb{C}$ at the vertex 1 in degrees $0, 1$ with a nonzero differential f. Then the phase potential increases along the whole trajectory, at the rate $\pi e^{-|f|^2}$, where $|f|$ is the norm of f for the metrics at the time considered, and it is not bounded below on the metrics of E.*

(ii) *There is a three-term complex as in Theorem 1.2(ii) with $\dim U_0 = 2$, $\dim U_1 = 1$ and stable cohomology, with a metric at which the phase potential increases along the flow.*

Theorem 1.2 is proved as Theorems 6.1 and 6.3, and Theorem 1.3 as Corollary 4.4 and Theorem 6.4.

Sharpness. For $Q = K_m$, $i_0 = 0$ and cohomology at both vertices, φ lies between $\arg z_1$ and $\arg z_0$, so the condition $\arg z_0 \geq \varphi$ is $\arg z_0 \geq \arg z_1$. For $\arg z_0 > \arg z_1$, strict decrease when $\varphi \geq 2\pi/5$ or $\dim U = 1$ is [Wue26b, Theorem 7.6]. The stronger bound $\pi/3$ follows from the elementary trace estimate in Section 2. Theorem 1.2(i) adds arbitrary quivers, the case of equality, and cohomology at a single vertex. In Theorem 1.3(i) the differential is at one vertex and $\arg z_1 < \varphi$; so the condition $\arg z_{i_0} \geq \varphi$ of Theorem 1.2(i) cannot be dropped. For $\dim U \geq 2$ a lower bound on φ cannot be dispensed with either: for K_2, the charges $z_0 = 10^{-6}(-1 + \mathrm{i})$, $z_1 = 1 + \frac{\mathrm{i}}{500}$ and cohomology of dimension vector $(1, 1)$, where $\varphi = 0.0020\ldots < \frac{\pi}{3}$, there is a complex with $\dim U = 2$ and a metric at which the potential increases [Wue26b, Proposition 10.2]. The optimal lower bound on φ for nonnegative total dissipation at every metric remains open. Theorem 1.3(ii) shows that the condition on the dimensions in Theorem 1.2(ii) cannot be dropped; for $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$ and cohomology with equal dimensions at the two vertices, the pairs $(\dim U_0, \dim U_1)$ for which the phase potential is nonincreasing at all metrics are determined in Corollary 6.5.

Method. The closedness in Theorem 1.1 needs only two odd derivations of a graded algebra with the relations of Lemma 3.2, a functional that vanishes on supercommutators and on the images of the derivations, and a formula for the variation of the curvature element under a change of metric; the same lemma serves the algebra of the quiver and, for bundles, the algebra of endomorphism-valued forms, where it is the classical argument by which Bott–Chern forms do not depend, modulo ∂- and $\bar\partial$-exact forms, on the path of metrics. Theorem 1.2(i) uses the trace estimate proved after Theorem 2.6, which strengthens [Wue26b, Theorem 3.1]. The entries of the Schur multiplier are the second divided differences of e^{-x} at $\ell_i, 0, \ell_k$, where the ℓ_i are the eigenvalues of the differential times its adjoint. In the three-term complexes the point 0 is replaced by the eigenvalues of the other differential times its adjoint, and no such inequality holds uniformly in these eigenvalues (Remark 6.6); this is the source of the example in Theorem 1.3(ii). The numerical values in that example are enclosures obtained with interval arithmetic [Joh17].

Theorem 1.4 (Duality)**.** *Let $\mathsf{E}^\vee$ be the dual complex over the opposite quiver, with $(E^\vee)_i^n = (E_i^{-n})^*$, the transposed maps, the dual metrics and the charges $-\overline{z_i}$. Then $\Omega^\vee = -(\Omega^\dagger)^{\mathsf{T}}$, the phases satisfy $\Phi^\vee = \pi - \Phi^{\mathsf{T}}$, the masses agree, and $h \mapsto e^{-\pi t}(h^{-1})^{\mathsf{T}}$ maps trajectories of the phase flow of E to trajectories of the phase flow of $\mathsf{E}^\vee$. If $Z(\mathsf{E}) \neq 0$ and the argument of $Z(\mathsf{E}^\vee)$ is taken as $\pi - \varphi$, this map preserves the phase potential up to a constant.*

Duality is used as a tool: it halves the cases in Theorem 6.1, and it carries statements about complexes with a surjective differential in degrees $0, 1$ over to complexes with an injective differential in degrees $-1, 0$ (Remark 5.2).

1.2. **Bundles.** The last section gives the corresponding formulation for a two-term complex $E^0 \to E^1$ of holomorphic vector bundles with Hermitian metrics on a compact Riemann surface, with $-\frac{\mathrm{i}}{2\pi}\Lambda F_h$ in the place of the moment map, following the dictionary of [HKKP26, Remark 5.14] and using Quillen's superconnection [Qui85]. We record the formal properties, which for the closedness follow the theory of Bott–Chern forms [BC65, Don85, BGS88]: the one-form defined by Ω on the space of metrics is closed, with a primitive given by the same transgression; for a single bundle the phase potential is a positive multiple of Donaldson's functional [Don85]; the derivative formula of Theorem 1.1(iii) holds along smooth solutions; the linearization is parabolic in the sense of Petrovskii for the convention $\Delta_\omega = -d^*d$; and for a section s of a line bundle the sum $(\Omega^0 + \Omega^1)\,\omega$ of the two blocks of Ω, times the Kähler form, is a representative of the Euler class of the kind constructed by Mathai and Quillen [MQ86], and the integrand of the phase potential decreases at every point of the surface (Propositions 7.2–7.5).

1.3. **Open questions.** Convergence of the phase flow on the cohomology and decay of the acyclic part are not treated here, except for the contractible summands of Proposition 4.3. For two-term complexes over K_m with a surjective differential at the vertex 0, the decay is proved in [Wue26b] for all charges and the convergence for strictly ordered charges and polystable cohomology. For the corresponding complexes on acyclic quivers, [Wue26c] proves decay of the exact part and exponential convergence of the rescaled metrics on polystable Harder–Narasimhan quotients of the cohomology. Remark 5.2 carries these results to injective differentials. For the three-term complexes of Theorem 1.2(ii) both are open. Beyond Theorems 6.1 and 6.3 and Corollaries 4.4 and 6.5 we do not know when the phase potential is monotone. For bundles, existence of solutions of the flow is not proved, and neither is the convergence along the flow, for a section of a line bundle, of the form $(\Omega^0 + \Omega^1)\,\omega$ to the current of integration over the zeros of the section.

1.4. **Organization.** Section 2 recalls the definition of [HKKP26], gives the integral formula for Ω, fixes the terminology, and sharpens one of the two matrix inequalities recalled from [Wue26b]. Section 3 proves Theorem 1.1(i),(ii), and Section 4 proves (iii) and Theorem 1.3(i). Section 5 proves Theorem 1.4. Section 6 proves Theorems 1.2 and 1.3(ii). Section 7 is on bundles.

2. The endomorphism Ω of a complex of representations

2.1. **Complexes of representations and metrics.** Let Q be a finite quiver with set of vertices Q_0 and set of arrows Q_1; we write $a : i \to j$ for an arrow from i to j. Loops and oriented cycles are allowed. We write K_m for the generalized Kronecker quiver with vertices $0, 1$ and $m \geq 1$ arrows $a_1, \dots, a_m$ from 0 to 1. Fix *central charges* z_i, $i \in Q_0$, in the open upper half-plane, and put $\alpha_i = \arg z_i \in (0, \pi)$; they define the central charge of a stability condition on the bounded derived category of finite-dimensional representations of Q whose heart is the category of representations [Bri07, Propositions 2.4 and 5.3]. Phases in this paper are arguments, in radians; in the normalization of [Bri07] they are divided by π, and a shift by one degree changes them by π. A *complex of representations* E consists of finite-dimensional complex vector spaces E_i^n ($i \in Q_0$, $n \in \mathbb{Z}$, almost all zero), differentials $d_i : E_i^n \to E_i^{n+1}$ with $d_i^2 = 0$, and for each arrow $a : i \to j$ a chain map $\rho(a) : E_i^\bullet \to E_j^\bullet$ of degree zero. Its central charge is

$$Z(\mathsf{E}) = \sum_{i,n} (-1)^n z_i \dim E_i^n .$$

A *metric* h on E is a Hermitian metric h_i^n on each E_i^n; a complex with a metric is called metrized. The adjoint of a linear map for the metrics is written †, inner products are conjugate-linear in the first argument, and $\|T\|_{\mathrm{HS}} = (\operatorname{tr} T^\dagger T)^{1/2}$ is the Hilbert–Schmidt norm. For an endomorphism T of a Hermitian space, $\frac{1}{2}(T + T^\dagger)$ is its Hermitian part and $\operatorname{Im} T = \frac{1}{2\mathrm{i}}(T - T^\dagger)$ its imaginary part. We write $E_i = \bigoplus_n E_i^n$ and $E = \bigoplus_i E_i$. The *number operator* N is the endomorphism of E equal to n on E_i^n, and $\varepsilon = (-1)^{\mathsf{N}}$ is the grading. A tangent vector to the space of metrics at h is an endomorphism $\xi = h^{-1}\delta h$ of E that preserves each E_i^n and is Hermitian for h. In the algebra $\mathcal{A}(Q)$ below we write $\sqrt{-1}$ for the imaginary unit, as in [HKKP26]; elsewhere it is i, and the italic i is a vertex.

2.2. **The definition of Haiden, Katzarkov, Kontsevich and Pandit.** The graded $*$-algebra $\mathcal{A}(Q)$ of [HKKP26, Definition 5.13] has the basis e_i, f_i ($i \in Q_0$) in degree 0, $a \in Q_1$ in degree 1 and a^* in degree -1, and the nonzero products of basis elements are

$$e_i^2 = e_i, \qquad e_i f_i = f_i e_i = f_i, \qquad e_j a = a e_i = a, \qquad e_i a^* = a^* e_j = a^*,$$
$$a a^* = \sqrt{-1}\, f_j, \qquad a^* a = -\sqrt{-1}\, f_i$$

for $a : i \to j$. In the graded tensor product $\mathcal{A}(Q) \otimes \operatorname{End}(E)$ the product is $(x \otimes T)(y \otimes S) = (-1)^{|T||y|} xy \otimes TS$. The involution of $\mathcal{A}(Q)$ fixes the e_i and f_i and exchanges a and a^*; on $\operatorname{End}(E)$ it is $T^* = T^\dagger$ for even T and $T^* = \sqrt{-1}\, T^\dagger$ for odd T, the adjoints being those of the metric. Let $\mathcal{B} \subset \mathcal{A}(Q) \otimes \operatorname{End}(E)$ be the subalgebra

$$\mathcal{B} = \bigoplus_i \big(e_i \otimes \operatorname{End}(E_i) \oplus f_i \otimes \operatorname{End}(E_i)\big) \oplus \bigoplus_{a : i \to j} \big(a \otimes \operatorname{Hom}(E_i, E_j) \oplus a^* \otimes \operatorname{Hom}(E_j, E_i)\big),$$

with unit $\sum_i e_i \otimes 1_{E_i}$. Put

$$A = \sum_i e_i \otimes d_i + \sum_a a \otimes \rho(a), \qquad F = (A - A^*)^2 + \sqrt{-1} \sum_i f_i \otimes z_i 1_{E_i}.$$

Here $A^* = \sqrt{-1}\sum_i e_i \otimes d_i^\dagger + \sum_a a^* \otimes \rho(a)^\dagger$, and this is the only use of the involution. These are elements of $\mathcal{B}$. Following [HKKP26, §5.4] we put

$$\Omega := \varepsilon\left[\exp(-\sqrt{-1}\,F)\right]_f \in \mathrm{End}(E), \tag{1}$$

where the exponential is taken in $\mathcal{B}$ and, for $X = \sum_i (e_i \otimes X_{e_i} + f_i \otimes X_{f_i}) + \cdots \in \mathcal{B}$, the endomorphism $[X]_f = \bigoplus_i X_{f_i}$ is the component along the f_i. With

$$\Delta_i = d_i d_i^\dagger + d_i^\dagger d_i, \qquad M_a = d_j^\dagger \rho(a) - \rho(a) d_i^\dagger, \qquad \Gamma_i = z_i + \sum_{a: i\to\cdot} \rho(a)^\dagger \rho(a) - \sum_{a:\cdot\to i} \rho(a)\rho(a)^\dagger, \tag{2}$$

where $M_a : E_i^\bullet \to E_j^{\bullet-1}$ for $a : i \to j$, one has [HKKP26, display after Remark 5.15]

$$-\sqrt{-1}\,F = -\sum_i e_i \otimes \Delta_i + \sum_a a \otimes M_a + \sqrt{-1}\sum_a a^* \otimes M_a^\dagger + \sum_i f_i \otimes \Gamma_i. \tag{3}$$

For a representation, that is, a complex in degree zero, $\Omega = \Gamma$: at each vertex Ω is z_i shifted by the moment map of the action of the unitary group of E_i.

In [HKKP26, Remark 5.14] the algebra $\mathcal{A}(Q)$ is compared with the differential forms on a Riemann surface: the e_i correspond to functions, a and a^* to forms of type $(1,0)$ and $(0,1)$, and the f_i to two-forms. In this dictionary $A - A^*$ plays the role of a superconnection in the sense of Quillen [Qui85] on the graded space E, F that of its curvature shifted by the central term, $[\,\cdot\,]_f$ that of the component of top degree, and the functional τ of Section 3 that of the integral of the supertrace of this component. Section 7 carries the dictionary out for bundles.

2.3. **An integral formula.** For Hermitian $\Delta \geq 0$ and an operator G put $K_\Delta(G) = \int_0^1 e^{-(1-x)\Delta} G e^{-x\Delta} dx$, and for operators $X : E' \to E''$, $Y : E'' \to E'$ and Hermitian $\Delta', \Delta'' \geq 0$ on E', E'',

$$\mathcal{J}_{\Delta'',\Delta'}(X,Y) = \int_{\substack{x,y\geq 0\\ x+y\leq 1}} e^{-x\Delta''} X\, e^{-(1-x-y)\Delta'}\, Y\, e^{-y\Delta''}\, dx\, dy.$$

In eigenbases of Δ, Δ'' and Δ', with eigenvalues ℓ_i, ℓ_i'' and ℓ_μ', these are given by divided differences of the function $\mathsf{f}(x) = e^{-x}$, written $\mathsf{f}[\,\cdot\,]$, by the Hermite–Genocchi formula:

$$K_\Delta(G)_{ik} = -\mathsf{f}[\ell_i, \ell_k]\, G_{ik}, \qquad \mathcal{J}_{\Delta'',\Delta'}(X,Y)_{ik} = \sum_\mu \mathsf{f}[\ell_i'', \ell_\mu', \ell_k'']\, X_{i\mu} Y_{\mu k},$$

with $-\mathsf{f}[a,b] > 0$ and $\mathsf{f}[a,b,c] > 0$, by the mean value form of divided differences. The expansion $e^{X+Y} = e^X + \int_0^1 e^{(1-x)X} Y e^{xX} dx + \ldots$ is called the Duhamel expansion below.

Proposition 2.1. *The endomorphism Ω preserves each E_i^n, and for every vertex i*

$$\varepsilon\,\Omega\big|_{E_i} = K_{\Delta_i}(\Gamma_i) + \sum_{a:k\to i} \mathcal{J}_{\Delta_i,\Delta_k}(M_a, M_a^\dagger) - \sum_{a:i\to j} \mathcal{J}_{\Delta_i,\Delta_j}(M_a^\dagger, M_a).$$

Proof. Write the right side of (3) as $X_e + X_a + X_{a^*} + X_f$. In $\mathcal{A}(Q)$ every product of three elements of $\{a, a^*, f_i\}$ vanishes, and so do the products $f_i f_k$, $f_i a$, $a f_i$, $f_i a^*$, $a^* f_i$, aa', $a^* a'^*$, and aa'^*, $a^* a'$ for $a \neq a'$. So in the Duhamel expansion of $\exp(X_e + Y)$, $Y = X_a + X_{a^*} + X_f$, the component along the f_i receives only the terms with one factor X_f, or with one factor X_a and one factor X_{a^*} of the same arrow, in either order. The element X_e is even and the e_i act as units, so $e^{xX_e} = \sum_i e_i \otimes e^{-x\Delta_i}$ and no signs arise from it. By the sign rule, for $a : i \to j$,

$$(a \otimes M_a)(\sqrt{-1}\, a^* \otimes M_a^\dagger) = \sqrt{-1}\,(-1)^{|M_a||a^*|} aa^* \otimes M_a M_a^\dagger = f_j \otimes M_a M_a^\dagger,$$

$$(\sqrt{-1}\, a^* \otimes M_a^\dagger)(a \otimes M_a) = \sqrt{-1}\,(-1)^{|M_a^\dagger||a|} a^* a \otimes M_a^\dagger M_a = -f_i \otimes M_a^\dagger M_a.$$

The second-order term of the expansion is $\int e^{xX_e} Y e^{(1-x-y)X_e} Y e^{yX_e}\,dx\,dy$ over the simplex; inserting the heat factors gives the three terms. Each term preserves the degree, because Δ_i, Γ_i have degree 0, M_a has degree -1 and $M_a^\dagger$ degree $+1$. □

The formula is the expansion to second order of the display for $-\sqrt{-1}F$ in [HKKP26, §5.4]; no higher terms occur.

Proposition 2.2. *For every metrized complex:*

(a) *All terms of Proposition 2.1 other than the summand $K_{\Delta_i}(z_i) = z_i e^{-\Delta_i}$ of $K_{\Delta_i}(\Gamma_i)$ are Hermitian. Hence, on each E_i^n, $\operatorname{Im}\big((-1)^n\Omega\big) = (\operatorname{Im} z_i)\, e^{-\Delta_i} > 0$.*

(b) $\operatorname{tr}\Omega = Z(\mathsf{E})$ [HKKP26, Lemma 5.17].

(c) *On each E_i^n there is a unique Hermitian Φ_i^n with spectrum in $(-\pi n, \pi(1-n))$ such that $\Omega|_{E_i^n} e^{-\mathrm{i}\Phi_i^n}$ is positive definite.*

(d) *The flow $h^{-1}\dot h = -\Phi$, $\Phi = \bigoplus \Phi_i^n$, has a unique solution for all $t \geq 0$ from every initial metric.*

Proof. (a) $K_\Delta(G)^\dagger = K_\Delta(G^\dagger)$ by the substitution $x \mapsto 1-x$, and $\mathcal{J}(X, X^\dagger)^\dagger = \mathcal{J}(X, X^\dagger)$ by exchanging x and y. (b) This is [HKKP26, Lemma 5.17]; the argument is as follows. With $\tau(X) = \operatorname{tr}(\varepsilon\,[X]_f)$ one has $\operatorname{tr}\Omega = \tau(e^{-\sqrt{-1}F})$. Replace all structure maps by their multiples by $u \in [0,1]$; then F becomes $-u^2[A, A^*] + F_z$ with $F_z = \sqrt{-1}\sum_i f_i \otimes z_i$, the element A commutes with it, and the derivative of $\tau(e^{-\sqrt{-1}F})$ in u is $2\sqrt{-1}\,u\,\tau\big([A, A^* e^{-\sqrt{-1}F}]\big) = 0$, because τ vanishes on supercommutators of elements of $\mathcal{B}$ (Lemma 3.1 below). At $u = 0$, $e^{-\sqrt{-1}F_z} = \sum_i (e_i + f_i \otimes z_i)$ and $\tau = \sum_{i,n} (-1)^n z_i \dim E_i^n$. (c) Let T be a matrix with $\operatorname{Im} T$ positive definite. Then T is invertible, since $Tv = 0$ gives $\langle v, (\operatorname{Im} T)v\rangle = \operatorname{Im}\langle v, Tv\rangle = 0$. Let $T = PU$ be its polar decomposition, with $P = (TT^\dagger)^{1/2} > 0$ and U unitary. If $Uv = \lambda v$, then $\langle v, (\operatorname{Im} T)v\rangle = (\operatorname{Im}\lambda)\,\langle v, Pv\rangle$, so $\operatorname{Im}\lambda > 0$. Hence $U = e^{\mathrm{i}\Psi}$ for a unique Hermitian Ψ with spectrum in $(0, \pi)$, and $Te^{-\mathrm{i}\Psi} = P > 0$; conversely such a Ψ gives a polar decomposition, which is unique. Apply this to $(-1)^n\Omega|_{E_i^n}$ and put $\Phi_i^n = \Psi - \pi n$. (d) The vector field is real-analytic on the space of metrics, and $\left\|h^{-1}\dot h\right\| \leq \pi(1 + \max|n|)$, so solutions cannot leave compact sets in finite time. □

The computation in (a) is that of [HKKP26, proof of Lemma 5.16]. That lemma also states that Ω is normal. By (a), a block $\Omega|_{E_i^n}$ is normal if and only if its Hermitian part commutes with $e^{-\Delta_i}$, and this does not hold for all complexes:

Example 2.3 (Ω need not be normal)**.** Let $Q = K_1$, with charges $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$, and consider the complex with the spaces $V = \mathbb{C}^2$ in degree 0 and $U_0 = \mathbb{C}^2$ in degree 1 at the vertex 0, the spaces $U_1 = \mathbb{C}$ in degree -1 and $W = \mathbb{C}$ in degree 0 at the vertex 1, the standard metrics, the differentials $d_0 = \operatorname{diag}(1, 4) : V \to U_0$ and $d_1 = 3 : U_1 \to W$, and the arrow $\rho(a_1) = (\rho_1, \rho_2) = (4, 100) : V \to W$. On V one has $d_0^\dagger d_0 = \operatorname{diag}(\ell_1, \ell_2)$ with $(\ell_1, \ell_2) = (1, 16)$ and $\Gamma_0 = z_0 + \rho(a_1)^\dagger\rho(a_1)$; the operator M_{a_1} of (2) restricts to $3\rho(a_1) : V \to U_1$, and $\Delta_1 = 9$ on U_1. By Proposition 2.1 (see Proposition 6.2 for the general formula), the block of Ω on V is

$$(\Omega_V)_{ik} = z_0 e^{-\ell_i}\delta_{ik} + \rho_i\rho_k\big(-\mathsf{f}[\ell_i, \ell_k] - 9\,\mathsf{f}[\ell_i, 9, \ell_k]\big),$$

where the factor 9 is $|d_1|^2$ and the point 9 is the eigenvalue of $d_1^\dagger d_1$. Its imaginary part $\operatorname{diag}(e^{-1}, e^{-16})$ has distinct eigenvalues, and its Hermitian part has the off-diagonal entry $400\,(-\mathsf{f}[1, 16] - 9\,\mathsf{f}[1, 9, 16]) = -1.2183\ldots \neq 0$; so the two do not commute. This complex is used again in Theorem 6.4. For two-term complexes over K_m see [Wue26a, Example 2.3].

2.4. **Terminology.**

Definition 2.4. Let E be a metrized complex.

(i) The *blocks* of Ω are the endomorphisms $\Omega_i^n = \Omega|_{E_i^n}$. The *phase* is $\Phi = \bigoplus \Phi_i^n$ with Φ_i^n as in Proposition 2.2(c), and $|\Omega_i^n| := \Omega_i^n e^{-\mathrm{i}\Phi_i^n} = (\Omega_i^n \Omega_i^{n\dagger})^{1/2}$. The word phase is also used, as usual, for the argument of a central charge.

(ii) The *phase flow* is the flow $h^{-1}\dot{h} = -\Phi$ of Proposition 2.2(d). The *mass* is $\mathfrak{m}(h) = \sum_{i,n} \operatorname{tr} |\Omega_i^n|$, and a metric is *harmonic* if Φ is a scalar.

(iii) Let i be a vertex. The complex of vector spaces $E_i^\bullet$ splits orthogonally as $\ker \Delta_i \oplus (\ker \Delta_i)^\perp$; the differential vanishes on the first summand, which maps isomorphically onto the cohomology, and the second summand is an acyclic subcomplex. We call $(\ker \Delta_i)^\perp$ the *acyclic part of* E *at* i and the rank of d_i its *rank*. The metrics *induced on the cohomology* are the restrictions of the metrics to the spaces $\ker \Delta_i$.

(iv) A *contractible summand* is a direct summand of E, as a complex of representations, that is contractible. Those used below are complexes $C^n \to C^{n+1}$ at one vertex, with invertible differential, on which no arrow acts.

(v) Suppose that the cohomology of E is a representation in degree zero, with $Z(\mathsf{E}) = |Z| e^{\mathrm{i}\varphi}$, $\varphi \in (0, \pi)$. Along a trajectory of the phase flow, the acyclic part *decays* if the least nonzero eigenvalue of each Δ_i tends to infinity and Ω minus the endomorphism of the cohomology representation with its induced metrics, extended by zero to the acyclic parts, tends to zero in the trace norm $\operatorname{tr}(T^\dagger T)^{1/2}$ for the metrics $h(t)$ (this is called cancellation of the acyclic part in [Wue26b] and decay of the exact part in [Wue26c]); and the flow *converges on the cohomology* if the metrics induced on the cohomology, multiplied by $e^{\varphi t}$, converge to a harmonic metric of the cohomology representation.

Remark 2.5. (1) The phase of Definition 2.4 satisfies the positivity condition of [HKKP26, §4.2] recalled in the Introduction, with the branches of [HKKP26, (5.15)], and the phase flow is the minimizing flow of [HKKP26, §4.1] for this phase; for normal Ω the phase is the argument in the sense of the functional calculus used in [HKKP26, §5.4].

(2) Since the intervals $(-\pi n, \pi(1-n))$ are pairwise disjoint, a harmonic metric can exist only on a complex concentrated in one degree n, that is, on a shifted representation; then Φ is the scalar $\arg Z(\mathsf{E})$ taken in $(-\pi n, \pi(1-n))$, the metric is a zero of the moment map shifted by the stability parameter (Proposition 4.1), and one exists if and only if the representation is polystable [HKKP26, Theorem 5.1], [Kin94, Proposition 6.5]. For a complex with terms in several degrees, 'harmonic' is used below only for the metrics induced on the cohomology.

(3) The acyclic part at a vertex is in general not preserved by the arrows, and an acyclic direct summand need not be contractible (Theorem 6.4).

Changing the metric at fixed structure maps is the same as moving the structure maps in their orbit under $\prod \mathrm{GL}(E_i^n)$ at a fixed metric. Precisely, if $h = \overline{g}^{\mathsf{T}} g$ in fixed bases, the matrix of a structure map $T \in \{d_i, \rho(a)\}$ in the orthonormal frames given by g is $g_{\text{target}} T g_{\text{source}}^{-1}$; along the phase flow one can choose g with $\dot{g} g^{-1} = -\frac{1}{2}\Phi$, where Φ now denotes the Hermitian matrix of the phase in these frames, and then

$$\dot{T} = \tfrac{1}{2}\big(T\,\Phi_{\text{source}} - \Phi_{\text{target}}\,T\big) \tag{4}$$

for the matrices in orthonormal frames, where Φ_{source}, Φ_{target} are the phases on the source and the target of T. The endomorphism Ω, the phases and the mass depend only on the matrices of the structure maps in orthonormal frames, up to unitary changes of the frames.

For $Q = K_m$ the charges are called *ordered* if $\alpha_0 \geq \alpha_1$ and *strictly ordered* if $\alpha_0 > \alpha_1$; that is, the simple representation at the source vertex has phase at least, respectively larger than, that of the simple representation at the target vertex. We write a complex of representations of K_m as a row of its terms, each term being the representation $(E_0^n \to E_1^n)$ with the space at the vertex 0 on the left; the maps between consecutive terms are the differentials (d_0, d_1).

2.5. **Two matrix inequalities.** We recall two matrix inequalities of [Wue26b] and strengthen the first. For Hermitian matrices $L > 0$ and M put

$$J_L(M) = \int_{\substack{x,y\geq 0\\ x+y\leq 1}} e^{-xL} M e^{-yL}\, dx\, dy,$$

so that $J_L(XY) = \mathcal{J}_{L,0}(X,Y)$; in an eigenbasis of L with eigenvalues ℓ_i, $J_L(M)_{ik} = \mathsf{f}[\ell_i, 0, \ell_k]\, M_{ik}$. We write $M_\pm \geq 0$ for the positive and negative parts of a Hermitian matrix $M = M_+ - M_-$.

Theorem 2.6 ([Wue26b, Theorems 3.1 and 4.1])**.** *The following hold.*

(a) *For Hermitian matrices $L > 0$ and $M \geq 0$,*

$$\operatorname{tr} J_L(M)_- \leq \tfrac{2}{5} \operatorname{tr} J_L(M). \tag{5}$$

(b) *Let $T(s)$ be a smooth family of matrices with constant positive definite imaginary part, and $\Psi(s)$ the Hermitian matrices with spectrum in $(0,\pi)$ and $Te^{-\mathrm{i}\Psi} > 0$. Then $\operatorname{tr}(\dot T\, \dot\Psi) < 0$ wherever $\dot T \neq 0$.*

For 1×1 matrices, (a) is empty because $J_L(M) \geq 0$, and (b) says that $\arg(x+\mathrm{i}y)$ decreases in x for fixed $y > 0$. In [Wue26b], (a) is reduced to an inequality in one real variable, proved with the help of interval arithmetic, and (b) is proved by expressing $\operatorname{tr}(\dot T\, \dot\Psi)$ through a Sylvester equation as a quadratic form whose kernels are positive semidefinite by an integral representation [Wue26b, Section 4].

We shall use the stronger estimate $\operatorname{tr} J_L(M)_- \leq \frac{1}{3} \operatorname{tr} J_L(M)$, which admits a direct proof. In an eigenbasis of L, write $J_L(M)_{ik} = M_{ik} j(\ell_i, \ell_k)$, where $j(a,b)$ is the integral of e^{-ax-by} over $x, y \geq 0$, $x + y \leq 1$. For $s \in I = [0, 1/2]$ put $F_a(s) = \int_0^s e^{-ax}\, dx$, $A_a = F_a(1/2)$ and $V_a(s) = e^{-a(1-s)}$. Splitting the triangle at $x = 1/2$ and $y = 1/2$ gives $j(a,b) = A_a A_b + \int_I (F_a V_b + V_a F_b)\, ds$. Thus both Q and $j + Q$ are positive semidefinite kernels, where

$$Q(a,b) = \tfrac{2}{3} \int_I (F_a - \tfrac{3}{4} V_a)(F_b - \tfrac{3}{4} V_b)\, ds.$$

Indeed, $j + Q$ is $A_a A_b$ plus the same integral with both minus signs replaced by plus signs. On the diagonal suppress the subscript a. Since $sV \leq F \leq A$ and V^2 is increasing, $\int_I FV \geq \int_I sV^2 \geq \frac{1}{4} \int_I V^2$. Expanding the square gives

$$Q(a,a) \leq \tfrac{1}{3} A^2 + \tfrac{1}{2} \int_I FV\, ds \leq \tfrac{1}{3} j(a,a).$$

By the Schur product theorem, the matrices with entries $M_{ik}(j+Q)(\ell_i,\ell_k)$ and $M_{ik} Q(\ell_i,\ell_k)$ are positive semidefinite. Taking the trace on the negative spectral subspace of their difference bounds $\operatorname{tr} J_L(M)_-$ by $\sum_i M_{ii} Q(\ell_i,\ell_i)$, which is at most $\frac{1}{3} \operatorname{tr} J_L(M)$. This estimate enters Theorem 6.1(b),(c) in rank at least two and hence the one-sided cases of Corollary 6.5. Proposition 7.4 uses Theorem 2.6(b).

3. Closedness and a primitive

Consider the complex-valued one-form

$$\vartheta_h(\xi) := \operatorname{tr}(\Omega\,\xi)$$

on the space of metrics; here $\xi = h^{-1}\delta h$ is a tangent vector at h. For representations $\Omega = \Gamma$, and ϑ is the differential of $\sum_i z_i \log\det h_i - \sum_a \|\rho(a)\|^2_{\mathrm{HS}}$ [HKKP26, §5.1]. For two-term complexes over K_m with a surjective differential at the vertex 0, a function with differential ϑ was found by direct computation in [Wue26a, Proposition 7.1]. Here closedness is proved for all complexes by a computation in the algebra $\mathcal{B}$ of Section 2.

The algebra $\mathcal{B}$ contains A, A^*, F and the elements $\sum_i e_i \otimes \xi_i$, with which we identify tangent vectors ξ. For $X \in \mathcal{B}$ let $\tau(X) = \operatorname{tr}\big(\varepsilon\,[X]_f\big)$, so that $\vartheta_h(\xi) = \tau(e^{-\sqrt{-1}F}\xi)$. All brackets below are supercommutators, $[X,Y] = XY - (-1)^{|X||Y|}YX$.

Lemma 3.1. *τ vanishes on supercommutators of elements of $\mathcal{B}$.*

Proof. The products of basis elements with a component along some f_i are those of the types (e_i, f_i), (f_i, e_i), (a, a^*) and (a^*, a). For the first two the claim is that the supertrace on E_i vanishes on supercommutators. For $a : i \to j$, $T \in \operatorname{Hom}(E_i, E_j)$ and $S \in \operatorname{Hom}(E_j, E_i)$ homogeneous with $|T| + |S| = 0$,

$$\tau\big([a \otimes T, a^* \otimes S]\big) = \sqrt{-1}\big((-1)^{|T|}\operatorname{str}_{E_j}(TS) - \operatorname{str}_{E_i}(ST)\big) = 0,$$

because $\operatorname{str}_{E_j}(TS) = (-1)^{|T|}\operatorname{str}_{E_i}(ST)$; here $\operatorname{str} = \operatorname{tr}(\varepsilon\,\cdot\,)$. □

The restriction to $\mathcal{B}$ matters. On the whole algebra $\mathcal{A}(Q) \otimes \operatorname{End}(E)$, with the projection onto $\bigoplus_i f_i \otimes \operatorname{End}(E_i)$ in the place of $[\,\cdot\,]_f$, the functional does not vanish on all supercommutators, for instance on $[e_i \otimes T, f_i \otimes S]$ with $T : E_j \to E_i$, $S : E_i \to E_j$, $i \neq j$.

The closedness of ϑ uses only the structure described in the next lemma. For the endomorphism-valued forms of Section 7 the lemma is the classical computation by which the one-form $\xi \mapsto \operatorname{str}(\xi\, e^{-\mathfrak{F}})$ on the space of metrics is closed modulo ∂- and $\bar\partial$-exact forms, and by which Bott–Chern forms do not depend, modulo such forms, on the path of metrics [BC65, Don85, BGS88]. We isolate the identities used in that argument because they also hold in the algebra $\mathcal{B}$.

Lemma 3.2. *Let $\mathcal{C}$ be a $\mathbb{Z}/2$-graded associative algebra over $\mathbb{C}$ with a linear functional τ that vanishes on supercommutators. For each point h of a manifold $\mathcal{M}$ let $\mathfrak{d}$, $\mathfrak{d}^\star$ be odd derivations of $\mathcal{C}$ and $F \in \mathcal{C}$ an even element, depending smoothly on h, such that*

$$\tau\circ\mathfrak{d} = 0 = \tau\circ\mathfrak{d}^\star, \qquad \mathfrak{d}F = 0 = \mathfrak{d}^\star F, \qquad \mathfrak{d}\mathfrak{d}^\star + \mathfrak{d}^\star\mathfrak{d} = -[F,\,\cdot\,].$$

Suppose that to each tangent vector of $\mathcal{M}$ at h an even element $\xi \in \mathcal{C}$ is assigned, linearly, such that the derivative of F in this direction is $\delta_\xi F = -\mathfrak{d}\mathfrak{d}^\star\xi$, and that for a two-parameter family $h(u,v)$, with the elements ξ, η assigned to $\partial_u h$, $\partial_v h$, one has $\partial_v\xi - \partial_u\eta = [\xi,\eta]$. Suppose finally that $\mathsf{e} := e^{-\sqrt{-1}F}$ and the elements

$$\Xi_Y := \int_0^1 e^{-\sqrt{-1}(1-x)F}\,Y\,e^{-\sqrt{-1}xF}\,dx \qquad (Y \in \mathcal{C})$$

are defined, with $\mathfrak{d}$ and $\mathfrak{d}^\star$ vanishing on the exponentials and with the derivative of e given by the Duhamel formula $\delta_\xi\mathsf{e} = -\sqrt{-1}\,\Xi_{\delta_\xi F}$. Then the one-form $\vartheta(\xi) = \tau(\mathsf{e}\,\xi)$ on $\mathcal{M}$ is closed.

Proof. We use three identities, for $\mathfrak{b} \in \{\mathfrak{d}, \mathfrak{d}^\star\}$ and homogeneous $X, Y \in \mathcal{C}$:

(I1) $\tau\big((\mathfrak{b}X)\,Y\big) = -(-1)^{|X|}\tau\big(X\,\mathfrak{b}Y\big)$, because $\mathfrak{b}(XY) = (\mathfrak{b}X)Y + (-1)^{|X|}X\,\mathfrak{b}Y$ and $\tau\circ\mathfrak{b} = 0$;
(I2) $\mathfrak{b}\,\Xi_Y = \Xi_{\mathfrak{b}Y}$, because $\mathfrak{b}$ vanishes on the exponentials of F;
(I3) $\tau(\Xi_Y Y') = \tau(Y\,\Xi_{Y'})$ for even Y, Y', by the cyclicity of τ on even elements and the substitution $x \mapsto 1-x$.

Moreover $\frac{d}{dx}\big(e^{-\sqrt{-1}(1-x)F}Ye^{-\sqrt{-1}xF}\big) = \sqrt{-1}\,[F, e^{-\sqrt{-1}(1-x)F}Ye^{-\sqrt{-1}xF}]$ gives $[F,\Xi_Y] = \sqrt{-1}\,[\mathsf{e}, Y]$. From $\delta_\xi F = -\mathfrak{d}\mathfrak{d}^\star\xi = [F,\xi] + \mathfrak{d}^\star\mathfrak{d}\xi$ and (I2),

$$\delta_\xi \mathsf{e} = -\sqrt{-1}\,[F,\Xi_\xi] - \sqrt{-1}\,\mathfrak{d}^\star\mathfrak{d}\,\Xi_\xi = [\mathsf{e},\xi] - \sqrt{-1}\,\mathfrak{d}^\star\mathfrak{d}\,\Xi_\xi.$$

For a two-parameter family,

$$\partial_v\vartheta(\xi) - \partial_u\vartheta(\eta) = \tau(\delta_\eta\mathsf{e}\,\xi) - \tau(\delta_\xi\mathsf{e}\,\eta) + \tau(\mathsf{e}\,[\xi,\eta]).$$

The terms with the first summand of $\delta\mathsf{e}$, together with the last term, give

$$\tau([\mathsf{e},\eta]\xi) - \tau([\mathsf{e},\xi]\eta) + \tau(\mathsf{e}[\xi,\eta]) = -\tau(\mathsf{e}[\xi,\eta]),$$

by cyclicity. For the remaining terms put $\beta(\eta,\xi) = \tau(\Xi_\eta\,\mathfrak{d}\mathfrak{d}^\star\xi)$ and $\beta'(\eta,\xi) = \tau(\Xi_\eta\,\mathfrak{d}^\star\mathfrak{d}\xi)$. By (I1), applied twice,

$$\tau\big((\mathfrak{d}^\star\mathfrak{d}\,\Xi_\eta)\,\xi\big) = \tau\big((\mathfrak{d}\,\Xi_\eta)(\mathfrak{d}^\star\xi)\big) = -\tau\big(\Xi_\eta\,\mathfrak{d}\mathfrak{d}^\star\xi\big) = -\beta(\eta,\xi),$$

and by (I2), (I3) and cyclicity,

$$\tau\big((\mathfrak{d}^\star\mathfrak{d}\,\Xi_\eta)\,\xi\big) = \tau\big(\Xi_{\mathfrak{d}^\star\mathfrak{d}\eta}\,\xi\big) = \tau\big((\mathfrak{d}^\star\mathfrak{d}\eta)\,\Xi_\xi\big) = \beta'(\xi,\eta).$$

So $\beta(\eta,\xi) = -\beta'(\xi,\eta)$. Since $\mathfrak{d}\mathfrak{d}^\star\xi + \mathfrak{d}^\star\mathfrak{d}\xi = -[F,\xi]$,

$$\beta(\eta,\xi) + \beta'(\eta,\xi) = -\tau(\Xi_\eta[F,\xi]) = \tau([F,\Xi_\eta]\,\xi) = \sqrt{-1}\,\tau([\mathsf{e},\eta]\xi) = -\sqrt{-1}\,\tau(\mathsf{e}[\xi,\eta]).$$

Therefore

$$\begin{aligned}-\sqrt{-1}\big[\tau\big((\mathfrak{d}^\star\mathfrak{d}\,\Xi_\eta)\xi\big) - \tau\big((\mathfrak{d}^\star\mathfrak{d}\,\Xi_\xi)\eta\big)\big] &= \sqrt{-1}\big[\beta(\eta,\xi) - \beta(\xi,\eta)\big]\\ &= \sqrt{-1}\big[\beta(\eta,\xi) + \beta'(\eta,\xi)\big] = \tau(\mathsf{e}[\xi,\eta]),\end{aligned}$$

and the two contributions cancel. □

Theorem 3.3 (Closedness)**.** *The one-form ϑ is closed. Hence there is a function $\mathcal{S}_{\mathbb{C}}$ on the space of metrics, unique up to an additive constant, with $d\mathcal{S}_{\mathbb{C}}(\xi) = \operatorname{tr}(\Omega\,\xi)$.*

Proof. We apply Lemma 3.2 to $\mathcal{C} = \mathcal{B}$, the functional τ of Lemma 3.1, the space of metrics, and $\mathfrak{d} = [A,\,\cdot\,]$, $\mathfrak{d}^\star = [A^*,\,\cdot\,]$. The functional vanishes on the images of these inner derivations by Lemma 3.1. Work in fixed bases. Adjoints depend on h by $T^\dagger = h^{-1}\overline{T}^{\mathsf{T}}h$, with $\overline{T}^{\mathsf{T}}$ the conjugate transpose of the matrix, so for a variation ξ one has $\delta_\xi T^\dagger = [T^\dagger,\xi]$, hence $\delta_\xi A^* = [A^*,\xi]$ and $\delta_\xi A = 0$. The elements A, A^* are odd, ξ and F are even. From $A^2 = 0$, and hence $(A^*)^2 = 0$ for every metric, we get $F_0 := (A - A^*)^2 = -[A, A^*]$ and, by the graded Jacobi identity, $[A, F_0] = 0 = [A^*, F_0]$ and $\mathfrak{d}\mathfrak{d}^\star + \mathfrak{d}^\star\mathfrak{d} = [[A, A^*],\,\cdot\,] = -[F_0,\,\cdot\,]$. The element $F_z = \sqrt{-1}\sum_i f_i \otimes z_i 1_{E_i}$ is central in $\mathcal{B}$ and does not depend on the metric. So $F = F_0 + F_z$ satisfies $\mathfrak{d}F = \mathfrak{d}^\star F = 0$, $\mathfrak{d}\mathfrak{d}^\star + \mathfrak{d}^\star\mathfrak{d} = -[F,\,\cdot\,]$ and $\delta_\xi F = -[A,[A^*,\xi]] = -\mathfrak{d}\mathfrak{d}^\star\xi$. For a two-parameter family of metrics with $\xi = h^{-1}\partial_u h$ and $\eta = h^{-1}\partial_v h$ one has $\partial_v\xi - \partial_u\eta = [\xi,\eta]$. The algebra is finite-dimensional, so the exponentials exist and the Duhamel formula holds. Hence ϑ is closed, and the space of metrics is contractible. □

Theorem 3.4 (Primitive)**.** *For $x \in [0,1]$ let $\Omega^{(x)}$ be the endomorphism (1) of the complex with differentials $\sqrt{x}\, d_i$ and the same arrows and metrics. Then*

$$\mathcal{S}_{\mathbb{C}}(h) = \sum_n (-1)^n \Big[\sum_i z_i \log\det h_i^n - \sum_a \|\rho^n(a)\|_{\mathrm{HS}}^2 \Big] + \int_0^1 \sum_{i,n} n\, \mathrm{tr}\big((\Omega^{(x)})_i^n - (\Omega^{(0)})_i^n \big) \frac{dx}{x}$$

satisfies $d\mathcal{S}_{\mathbb{C}}(\xi) = \mathrm{tr}(\Omega\, \xi)$. Here $\rho^n(a) = \rho(a)|_{E_i^n}$, $(\Omega^{(x)})_i^n$ is the block of $\Omega^{(x)}$ on E_i^n, and the determinants are Gram determinants in fixed bases.

Proof. Rescaling the differentials by $\sqrt{x}$ at a fixed metric is the same, up to unitary equivalence of the data, as the change of metric $h_i^n \mapsto x^n h_i^n$ at fixed differentials: it multiplies each $d_i^\dagger$ by x and leaves $\rho(a)^\dagger$ unchanged. Along this path of metrics the velocity is $\frac{1}{x}\mathsf{N}$, and N commutes with every tangent vector ξ. By Theorem 3.3, applied to the family obtained from $he^{u\xi}$ by this rescaling, $\delta_\xi \big[\frac{1}{x} \mathrm{tr}(\Omega^{(x)}\mathsf{N}) \big] = \partial_x \mathrm{tr}(\Omega^{(x)}\xi)$. Since $\Omega^{(x)}$ is analytic in x, the integrand $\frac{1}{x}\mathrm{tr}\big((\Omega^{(x)} - \Omega^{(0)})\mathsf{N}\big)$ is bounded and δ_ξ of the integral is $\mathrm{tr}(\Omega^{(1)}\xi) - \mathrm{tr}(\Omega^{(0)}\xi)$; here $\mathrm{tr}(\Omega^{(0)}\mathsf{N})$ does not depend on the metric, because for zero differentials $\Omega^{(0)} = \varepsilon\, \Gamma$ and the arrow terms of $\sum_i \mathrm{tr}\, \Gamma_i$ cancel in each degree. For zero differentials Ω is that of a representation in each degree, up to the sign ε, with the primitive given by the first sum. □

For two-term complexes over K_m with a surjective differential at the vertex 0 the formula of Theorem 3.4 reduces to the primitive of [Wue26a, Proposition 7.1]. The integral in Theorem 3.4 is $\int_0^1 \mathrm{tr}\big(\mathsf{N}\,(\Omega^{(x)} - \Omega^{(0)})\big)\frac{dx}{x}$: a transgression by the number operator along the rescaling of the differential, as in the construction of Bott–Chern forms of Hermitian complexes by Bismut, Gillet and Soulé [BGS88], where the regularized integral runs over $(0,\infty)$. Here the rescaling stops at $x = 1$, and the identity is on the space of metrics, in the algebra $\mathcal{B}$.

4. The derivative of the phase potential along the flow

Assume $Z(\mathsf{E}) \neq 0$ and fix $\varphi \in \mathbb{R}$ with $Z(\mathsf{E}) = |Z(\mathsf{E})|e^{\mathrm{i}\varphi}$. The *phase potential* of the metrized complex is $\mathrm{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})$, with $\mathcal{S}_{\mathbb{C}}$ as in Theorem 3.4. Put

$$\mathcal{D} := \sum_{i,n} \mathcal{D}_i^n, \qquad \mathcal{D}_i^n := \mathrm{tr}\Big(|\Omega_i^n|\, (\Phi_i^n - \varphi) \sin(\Phi_i^n - \varphi) \Big).$$

We call $\mathcal{D}_i^n$ the contribution of the block Ω_i^n. By Proposition 4.2 below, the derivative of the phase potential along the flow is $-\mathcal{D}$.

Proposition 4.1 (Representations)**.** *Let E be a representation in degree zero, with $\varphi \in (0,\pi)$, and put $\theta_i = \mathrm{Im}(e^{-\mathrm{i}\varphi} z_i)/\sin\varphi$ and $\mu_i = \sum_{a: i\to\cdot} \rho(a)^\dagger\rho(a) - \sum_{a:\cdot\to i} \rho(a)\rho(a)^\dagger$, the moment map. Then $\sum_i \theta_i \dim E_i = 0$ and*

$$\frac{\mathrm{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})}{\sin\varphi} = \sum_a \|\rho(a)\|_{\mathrm{HS}}^2 + \sum_i \theta_i \log\det h_i =: \mathcal{K}_\theta(h), \qquad d\mathcal{K}_\theta(\xi) = -\sum_i \mathrm{tr}\big(\xi_i(\mu_i - \theta_i)\big).$$

This is the Kempf–Ness function of the representation for the character $-\theta$ in the convention of King [Kin94], for which a representation is semistable when $\sum_i \theta_i \dim E_i' \leq 0$ for all subrepresentations E', that is, when no subrepresentation has a central charge of argument larger than φ. The metric is harmonic if and only if $\mu_i = \theta_i$ for all i. All contributions $\mathcal{D}_i^0$ are nonnegative, and $\mathcal{D} = 0$ exactly at the harmonic metrics.

Proof. Here $\mathcal{S}_{\mathbb{C}} = \sum_i z_i \log\det h_i - \sum_a \|\rho(a)\|^2_{\mathrm{HS}}$, $\operatorname{Im}(e^{-\mathrm{i}\varphi} z_i) = \theta_i \sin\varphi$ and $\operatorname{Im}(e^{-\mathrm{i}\varphi}) = -\sin\varphi$; and $\sum_i \theta_i \dim E_i = \operatorname{Im}(e^{-\mathrm{i}\varphi} Z(\mathsf{E}))/\sin\varphi = 0$. The block at i is $z_i + \mu_i$, so $\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\operatorname{tr}(\Omega\xi)\big) = -\sin\varphi \sum_i \operatorname{tr}\big((\mu_i - \theta_i)\xi_i\big)$. For a real number x, $\arg(z_i + x) = \varphi$ exactly for $x = \theta_i$. The phases lie in $(0,\pi)$, so $\Phi_i - \varphi$ has its spectrum in $(-\pi,\pi)$, where $x\sin x \geq 0$ with equality only at $x = 0$. □

Proposition 4.2. *Along the phase flow, $\frac{d}{dt}\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}\big) = -\mathcal{D}$. The sum $\mathcal{D}$ does not depend on the choice of φ modulo 2π.*

Proof. By Theorem 3.3 and the flow equation, $\frac{d}{dt}\mathcal{S}_{\mathbb{C}} = -\operatorname{tr}(\Omega\Phi)$, and $e^{-\mathrm{i}\varphi}\operatorname{tr}\Omega = |Z(\mathsf{E})|$ is real (Proposition 2.2(b)). Hence $\frac{d}{dt}\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}) = -\operatorname{Im}\operatorname{tr}\big(e^{-\mathrm{i}\varphi}\Omega(\Phi - \varphi)\big)$, and on each block $e^{-\mathrm{i}\varphi}\Omega^n_i(\Phi^n_i - \varphi) = |\Omega^n_i|\,(\Phi^n_i - \varphi)e^{\mathrm{i}(\Phi^n_i - \varphi)}$; the trace of a product of two Hermitian matrices is real. Replacing φ by $\varphi + 2\pi$ changes $\mathcal{D}$ by $-2\pi\sum\operatorname{tr}(|\Omega^n_i|\sin(\Phi^n_i - \varphi)) = -2\pi\operatorname{Im}(e^{-\mathrm{i}\varphi}Z(\mathsf{E})) = 0$. □

For a fixed φ the summand $\mathcal{D}^n_i$ is nonnegative when the spectrum of $\Phi^n_i - \varphi$ lies in $\bigcup_{l\geq 0} \pm[2l\pi, (2l+1)\pi]$. Since the phase on E^n_i lies in $(-\pi n, \pi(1-n))$, a block can have either sign as soon as the complex has terms in more than one degree.

4.1. **A contractible summand that increases the potential.** The simplest instance of a negative contribution is a contractible summand at one vertex.

Proposition 4.3. *Let $\mathsf{E} = \mathsf{E}' \oplus \mathsf{C}$ be an orthogonal direct sum of metrized complexes, where $\mathsf{C} = (C^n \xrightarrow{f} C^{n+1})$ is concentrated at one vertex i in degrees $n, n+1$, with f invertible, and no arrow acts on C. The phase flow preserves the orthogonal splitting. The blocks of Ω on C are $(-1)^n z_i e^{-f^\dagger f}$ and $(-1)^{n+1} z_i e^{-ff^\dagger}$, with constant phases $\alpha_i - \pi n$ and $\alpha_i - \pi(n+1)$, $\alpha_i = \arg z_i$; in orthonormal frames $f(t) = e^{\pi t/2} f(0)$; and the contribution of C to $\mathcal{D}$ is*

$$\pi\,|z_i|\,\operatorname{tr} e^{-ff^\dagger}\,\sin(\alpha_i - \pi n - \varphi).$$

Proof. For an orthogonal direct sum of metrized complexes Ω is the direct sum of the endomorphisms of the summands, by Proposition 2.1. On C, $\Delta = f^\dagger f \oplus ff^\dagger$, $\Gamma = z_i$ and all M_a vanish. By (4), $\dot f = \frac{1}{2}(f\Phi^n - \Phi^{n+1}f) = \frac{\pi}{2}f$. With $x = \alpha_i - \pi n - \varphi$ the two blocks contribute $|z_i|\operatorname{tr} e^{-f^\dagger f}\, x\sin x + |z_i|\operatorname{tr} e^{-ff^\dagger}(x-\pi)\sin(x-\pi) = \pi|z_i|\operatorname{tr} e^{-ff^\dagger}\sin x$. □

So the blocks of Ω on a contractible summand of this kind always tend to zero, at the rate $e^{-\lambda e^{\pi t}}$ with λ the least eigenvalue of $f^\dagger f$ at time 0; but the summand makes the potential decrease only if $\sin(\alpha_i - \pi n - \varphi) > 0$.

Corollary 4.4. *Let $Q = K_m$ with the charges $z_0 = -1+\mathrm{i}$, $z_1 = 1+\mathrm{i}$. Let R be a representation $(R_0 \xrightarrow{\beta_j} R_1)$ with $\dim R_0 = \dim R_1 \geq 1$, with a metric such that $\sum_j \beta_j^\dagger\beta_j = 1$ and $\sum_j \beta_j\beta_j^\dagger = 1$, placed in degree 0, and let $\mathsf{E} = \mathsf{R} \oplus \mathsf{C}$ with $\mathsf{C} = (\mathbb{C} \xrightarrow{f} \mathbb{C})$ at the vertex 1 in degrees $0, 1$, $f \neq 0$. Then $\varphi = \frac{\pi}{2}$, the metric of R is harmonic and does not move modulo rescaling, and along the whole trajectory*

$$\frac{d}{dt}\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}\big) = +\pi e^{-|f|^2} > 0.$$

The phase potential is unbounded below on the metrics of E and is strictly concave along a geodesic in this space.

Proof. The blocks of R are z_0+1 and z_1-1, both equal to $\mathrm{i}\,1$, of phase $\frac{\pi}{2}=\varphi$. Proposition 4.3 with $i=1$, $n=0$, $\alpha_1=\frac{\pi}{4}$, $|z_1|=\sqrt{2}$ gives the rate $-\pi\sqrt{2}\,e^{-|f|^2}\sin(\frac{\pi}{4}-\frac{\pi}{2})$. By Theorem 3.4, for C with metrics h^0,h^1 and $b=|f|^2$ in orthonormal frames, $\mathcal{S}_{\mathbb{C}}=z_1\big(\log h^0-\log h^1+\mathrm{Ein}(b)\big)$ with $\mathrm{Ein}(b)=\int_0^1(1-e^{-xb})\,dx/x=\gamma+\log b+\mathrm{E}_1(b)$, where γ is Euler's constant and $\mathrm{E}_1(b)=\int_1^\infty e^{-xb}\,dx/x$ is the exponential integral. Since b is a constant multiple of h^1/h^0, this is $z_1\mathrm{E}_1(b)$ up to a constant, and its contribution to $\mathrm{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})$ is $-\mathrm{E}_1(b)$ up to a constant, which tends to $-\infty$ as $b\to 0$. Vary only the metric of the target line of C by $h^1(s)=e^s h^1(0)$. This is a geodesic for the usual symmetric space metric and $b(s)=e^s b(0)$. Since $\mathrm{E}_1'(b)=-e^{-b}/b$, the first and second derivatives of the phase potential along this geodesic are e^{-b} and $-be^{-b}$, respectively. □

For two-term complexes over K_m with the acyclic part at the vertex 0 in degrees $0,1$ and cohomology that is nonzero at the vertex 1, as in [Wue26a, Wue26b], one has $\sin(\alpha_0-\varphi)>0$ for strictly ordered charges. In general, for cohomology $\mathsf{R}[-n_*]$ with R a representation of phase $\varphi_{\mathsf{R}}\in(0,\pi)$, so that $\varphi=\varphi_{\mathsf{R}}-\pi n_*$, a contractible summand at the vertex i in degrees $n,n+1$ decreases the potential if and only if $(-1)^{n-n_*}\sin(\alpha_i-\varphi_{\mathsf{R}})>0$.

5. Duality

Let E be a metrized complex of representations of Q with charges z_i. The *dual complex* $\mathsf{E}^\vee$ is the complex of representations of the opposite quiver Q^{op} with

$$(E^\vee)_i^n=(E_i^{-n})^*,\qquad d_i^\vee=d_i^{\mathsf{T}},\qquad \rho^\vee(a^{\mathrm{op}})=\rho(a)^{\mathsf{T}},$$

the dual metrics, and the charges $z_i^\vee=-\overline{z_i}$, which again lie in the upper half-plane. Here X^{T} is the transposed map between the dual spaces; in orthonormal frames and their dual frames it is the transposed matrix, and $\overline{X}$ denotes the entrywise complex conjugate.

Theorem 5.1 (Duality)**.** *For every metrized complex* E*:*

(a) *As endomorphisms of* E^*, $\Omega^\vee=-(\Omega^\dagger)^{\mathsf{T}}$, $|\Omega^\vee|=|\Omega|^{\mathsf{T}}$ *and* $\Phi^\vee=\pi-\Phi^{\mathsf{T}}$*; in dual orthonormal frames,* $\Omega^\vee=-\overline{\Omega}$ *and* $\Phi^\vee=\pi-\overline{\Phi}$*. Hence* $\mathfrak{m}(\mathsf{E}^\vee)=\mathfrak{m}(\mathsf{E})$ *and* $Z(\mathsf{E}^\vee)=-\overline{Z(\mathsf{E})}$*.*

(b) *If* $h(t)$ *is a trajectory of the phase flow of* E*, then* $e^{-\pi t}(h(t)^{-1})^{\mathsf{T}}$ *is a trajectory of the phase flow of* $\mathsf{E}^\vee$*. In orthonormal frames, transposition of all structure maps intertwines the equations* (4) *of the two complexes.*

(c) *If* $Z(\mathsf{E})\neq 0$ *and* $\varphi^\vee=\pi-\varphi$*, the phase potentials* $\mathrm{Im}(e^{-\mathrm{i}\varphi^\vee}\mathcal{S}_{\mathbb{C}}^\vee)$ *and* $\mathrm{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})$ *agree, up to an additive constant, on corresponding metrics* h *and* $c\,(h^{-1})^{\mathsf{T}}$*,* $c>0$*, and the contributions* $\mathcal{D}_i^n$ *of corresponding blocks agree.*

Proof. (a) Use Proposition 2.1 for $\mathsf{E}^\vee$ in dual orthonormal frames, where $(d^\vee)^\dagger=\overline{d}$ and $(\rho^\vee)^\dagger=\overline{\rho}$. Then $\Delta_i^\vee=d_i^{\mathsf{T}}\overline{d_i}+\overline{d_i}d_i^{\mathsf{T}}=\Delta_i^{\mathsf{T}}$. The arrows of Q^{op} out of i are the arrows of Q into i, so

$$\Gamma_i^\vee=-\overline{z_i}+\Big(\sum_{a:k\to i}\rho(a)\rho(a)^\dagger-\sum_{a:i\to j}\rho(a)^\dagger\rho(a)\Big)^{\mathsf{T}}=-\overline{\Gamma_i},$$

the bracket being Hermitian. For $a:i\to j$ the arrow a^{op} goes from j to i, and $M^\vee_{a^{\mathrm{op}}}=\overline{d_i}\,\rho(a)^{\mathsf{T}}-\rho(a)^{\mathsf{T}}\overline{d_j}=-M_a^{\mathsf{T}}$, $(M^\vee_{a^{\mathrm{op}}})^\dagger=-\overline{M_a}$. In Proposition 2.1 for Q^{op} the term of a^{op} at the vertex i is of incoming type,

$$+\mathcal{J}_{\Delta_i^{\mathsf{T}},\Delta_j^{\mathsf{T}}}\big(M_a^{\mathsf{T}},\overline{M_a}\big)=\big(\mathcal{J}_{\Delta_i,\Delta_j}(M_a^\dagger,M_a)\big)^{\mathsf{T}}=\overline{\mathcal{J}_{\Delta_i,\Delta_j}(M_a^\dagger,M_a)},$$

which enters E at i with the sign $-$; likewise the term at j is $-\overline{\mathcal{J}_{\Delta_j,\Delta_i}(M_a, M_a^\dagger)}$, which enters E with $+$. Finally $K_{\Delta^\mathsf{T}}(-\overline{\Gamma}) = -\big(K_\Delta(\Gamma^\dagger)\big)^\mathsf{T} = -\overline{K_\Delta(\Gamma)}$. Altogether $\varepsilon\,\Omega^\vee = -\overline{\varepsilon\,\Omega}$, and the degrees n and $-n$ have the same parity. If $\Omega = |\Omega| e^{\mathrm{i}\Phi}$ on a block of degree n, then $-\overline{\Omega} = \overline{|\Omega|}\, e^{\mathrm{i}(\pi-\overline{\Phi})}$ with $\overline{|\Omega|} > 0$, and the spectrum of $\pi - \overline{\Phi}$ lies in $(\pi n, \pi(1+n))$, the branch of degree $-n$.

(b) Transposing (4) and using $\Phi^\mathsf{T} = \overline{\Phi}$ gives $\frac{d}{dt}T^\mathsf{T} = \frac{1}{2}(\overline{\Phi_{\text{source}}}\,T^\mathsf{T} - T^\mathsf{T}\,\overline{\Phi_{\text{target}}})$. In $\mathsf{E}^\vee$ the map T^T goes from the dual of the target to the dual of the source, with phases $\pi - \overline{\Phi_{\text{target}}}$ and $\pi - \overline{\Phi_{\text{source}}}$, and its equation (4) is the same expression. For metrics: if $g = c\,(h^{-1})^\mathsf{T}$, then $g^{-1}\dot{g} = \dot{c}/c + \Phi^\mathsf{T}$, and $-\Phi^\vee = \Phi^\mathsf{T} - \pi$ gives $c = e^{-\pi t}$ up to a constant.

(c) For corresponding variations $\xi^\vee = -\overline{\xi}$, $\operatorname{tr}(\Omega^\vee\xi^\vee) = \overline{\operatorname{tr}(\Omega\xi)}$, so $\mathcal{S}_\mathbb{C}^\vee = \overline{\mathcal{S}_\mathbb{C}}$ up to a constant on the metrics $h \leftrightarrow (h^{-1})^\mathsf{T}$, and $\operatorname{Im}(e^{-\mathrm{i}(\pi-\varphi)}\overline{\mathcal{S}_\mathbb{C}}) = \operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_\mathbb{C})$. For a common factor $c > 0$ the endomorphism $\Omega^\vee$ is unchanged and $\mathcal{S}_\mathbb{C}^\vee(cg) = \mathcal{S}_\mathbb{C}^\vee(g) + Z(\mathsf{E}^\vee)\log c$; since $e^{-\mathrm{i}\varphi^\vee}Z(\mathsf{E}^\vee)$ is real, the phase potential is invariant under common rescaling. □

Remark 5.2 (Injective differentials). Duality exchanges complexes that are $V \xrightarrow{D} U$ in degrees $0,1$ at a vertex i_0, with D surjective, and a vector space in degree zero at the other vertices, with complexes that are $U' \xrightarrow{D'} V'$ in degrees $-1,0$ at i_0, with D' injective. The cohomology representation of the dual is the dual of the cohomology representation, with the dual metrics (the quotient metric on $V'/D'U'$ is the metric of the orthogonal complement of $D'U'$); subrepresentations correspond to quotients and arguments ψ of central charges to $\pi - \psi$, so semistability, stability and polystability correspond. By Theorem 5.1, every statement about the flow of the first kind of complex therefore gives one for the second kind, with the charges $-\overline{z_i}$, the argument $\pi - \varphi$ of the central charge, and the metrics $e^{-\pi t}(h^{-1})^\mathsf{T}$; in particular the rescaled metrics $e^{\varphi t}h(t)$ of the complex correspond to the rescaled metrics $e^{(\pi-\varphi)t}h^\vee(t)$ of the dual, so that the flow converges on the cohomology for one exactly when it does for the other, and the acyclic part decays for one exactly when it does for the other. This is how Theorem 6.1(c) follows from Theorem 6.1(b), and how the decay and convergence theorems of [Wue26b, Wue26c] pass to complexes with an injective differential.

For $Q = K_m$ one has $Q^{\mathrm{op}} \cong Q$ by exchanging the two vertices, and the dual charges are $(-\overline{z_1}, -\overline{z_0})$. The pair $(-1+\mathrm{i}, 1+\mathrm{i})$ is self-dual, and strictly ordered charges are mapped to strictly ordered charges. For these self-dual charges the sufficient conditions of Theorem 6.1(b) and (c) cover different dimension vectors when the acyclic part has rank at least two: for cohomology of dimension vector (k,l), part (b) applies to a surjective differential at the vertex 0 when $\arg((l-k)+\mathrm{i}(k+l)) \geq \frac{\pi}{3}$, and part (c) to an injective differential at the vertex 1 when this argument is at most $\frac{2\pi}{3}$; for $k = l$ both apply.

6. Where the phase potential is monotone

By Corollary 4.4 the phase potential is not monotone on all complexes. This section treats two families: complexes with cohomology in degree zero whose acyclic part is at one vertex of an arbitrary quiver, where the monotonicity known for K_m [Wue26b, Theorem 7.6] persists, and complexes over K_m with acyclic parts at both vertices, where it persists for ranks at most one and fails in general for larger ranks (Theorem 6.4, Corollary 6.5). We use $\mathcal{D}$ and $\mathcal{D}_i^n$ from Section 4, K_Δ, $\mathcal{J}$ and the divided differences $\mathsf{f}[\,\cdot\,]$ from Section 2, $J_L(M)$, $M_\pm$ and the trace estimate proved after Theorem 2.6, and the terms of Definition 2.4.

6.1. **An acyclic part at one vertex.** Let Q be a finite quiver, i_0 a vertex without loops, and consider complexes E with $E_i = E_i^0$ for $i \neq i_0$ and $E_{i_0} = V \oplus U$ in degrees $0,1$, with

$d_{i_0} = D : V \to U$ surjective and $r = \dim U \geq 1$. The chain-map condition says that every arrow into i_0 maps into $\ker D$, and is empty for the arrows out of i_0. The cohomology is a representation $\mathcal{H}$ of Q in degree 0, with $\ker D$ at i_0; we assume $\mathcal{H} \neq 0$. Then $Z = Z(\mathsf{E}) = \sum_i z_i \dim \mathcal{H}_i = |Z| e^{\mathrm{i}\varphi}$ with $\varphi \in (0, \pi)$. Let $\sigma = \sin\varphi$, $\alpha_i = \arg z_i$, $c_0 = |z_{i_0}| \sin(\alpha_{i_0} - \varphi)$, $B = D^\dagger D$, $L = DD^\dagger$,

$$C_{\mathrm{out}} = \sum_{a: i_0 \to j} \rho(a)^\dagger \rho(a), \qquad G_{\mathrm{in}} = \sum_{a: k \to i_0} \rho(a) \rho(a)^\dagger, \qquad J = J_L(DC_{\mathrm{out}} D^\dagger),$$

and $R(b) = (1 - e^{-b})/b$ for $b > 0$, with $R(0) = 1$.

Theorem 6.1 (Acyclic part at one vertex)**.** *For the complexes just described:*

(a) *The blocks of Ω are*

$$\Omega_V = z_{i_0} e^{-B} + K_B(C_{\mathrm{out}}) - G_{\mathrm{in}}, \qquad \Omega_U = J - z_{i_0} e^{-L},$$

and for $j \neq i_0$

$$\Omega_j = z_j + \sum_{a: j \to \cdot} \rho(a)^\dagger \rho(a) - \sum_{\substack{a: k \to j \\ k \neq i_0}} \rho(a) \rho(a)^\dagger - \sum_{a: i_0 \to j} \rho(a) R(B) \rho(a)^\dagger.$$

(b) *If $\alpha_{i_0} \geq \varphi$, then for every metric*

$$\mathcal{D} \geq \varphi c_0 \operatorname{tr} e^{-L} + \sigma\big(\varphi \operatorname{tr} J - \pi \operatorname{tr} J_-\big) \geq \varphi c_0 \operatorname{tr} e^{-L} + \sigma\big(\varphi - \tfrac{\pi}{3}\big) \operatorname{tr} J.$$

Hence the phase potential is nonincreasing along the flow if $\alpha_{i_0} \geq \varphi$ and either $\varphi \geq \frac{\pi}{3}$ or $r = 1$, and strictly decreasing if moreover $\alpha_{i_0} > \varphi$.

(c) *Let instead $E_i = E_i^0$ for $i \neq i_0$ and $E_{i_0} = U \oplus V$ in degrees $-1, 0$, with $d_{i_0} = D : U \to V$ injective, $r = \dim U \geq 1$, and nonzero cohomology, so that $\varphi = \arg Z(\mathsf{E}) \in (0, \pi)$. If $\alpha_{i_0} \leq \varphi$ and either $\varphi \leq \frac{2\pi}{3}$ or $r = 1$, the phase potential is nonincreasing along the phase flow.*

Proof. (a) Proposition 2.1. For $a : i_0 \to j$, $M_a = -\rho(a) D^\dagger : U \to E_j$. For $a : k \to i_0$, $M_a = d_{i_0}^\dagger \rho(a) = 0$, because $\rho(a)$ takes values in V, on which $d_{i_0}^\dagger$ vanishes. For arrows not meeting i_0, $M_a = 0$. So at i_0, $\Delta = B \oplus L$ and $\Gamma = (z_{i_0} + C_{\mathrm{out}} - G_{\mathrm{in}}) \oplus z_{i_0}$; $K_B(G_{\mathrm{in}}) = G_{\mathrm{in}}$ because G_{in} is supported on $\ker B$; and $M_a^\dagger M_a = D\rho(a)^\dagger \rho(a) D^\dagger$ is supported on U, with $\Delta_j = 0$. At $j \neq i_0$, $\Delta_j = 0$ and the mixed term is $\rho(a) D^\dagger \big[\int_0^1 (1-x) e^{-xL} dx\big] D \rho(a)^\dagger = \rho(a)(1 - R(B))\rho(a)^\dagger$.

(b) All blocks other than U are in degree 0, with phases in $(0, \pi)$, and contribute nonnegative terms to $\mathcal{D}$. Let (u_k) be an orthonormal eigenbasis of Φ_U with eigenvalues $\psi_k \in (-\pi, 0)$, $q_k = \langle u_k, |\Omega_U| u_k \rangle$, $\iota_k = \langle u_k, J u_k \rangle$ and $e_k = \langle u_k, e^{-L} u_k \rangle > 0$. Since $\langle u_k, \Omega_U u_k \rangle = q_k e^{\mathrm{i}\psi_k}$, $q_k \sin(\psi_k - \varphi) = \operatorname{Im}(e^{-\mathrm{i}\varphi} \langle u_k, \Omega_U u_k \rangle) = -(\sigma \iota_k + c_0 e_k)$. If $\sigma\iota_k + c_0 e_k \geq 0$, then $\psi_k - \varphi \in [-\pi, -\varphi)$ and the summand $q_k(\psi_k - \varphi)\sin(\psi_k - \varphi)$ of the contribution of U is at least $\varphi(\sigma\iota_k + c_0 e_k)$; otherwise $\psi_k - \varphi \in (-\pi - \varphi, -\pi)$ and the summand is at least $(\pi + \varphi)(\sigma\iota_k + c_0 e_k)$. In both cases it is at least $\varphi(\sigma\iota_k + c_0 e_k) - \pi\sigma(\iota_k)_-$, because $c_0 e_k \geq 0$; here $(\iota_k)_- = \max(-\iota_k, 0)$. Summing, with $\sum_k (\iota_k)_- \leq \operatorname{tr} J_-$, gives the first inequality; the second follows from the 1/3 estimate after Theorem 2.6, since $DC_{\mathrm{out}} D^\dagger \geq 0$. For $r = 1$ we have $J \geq 0$. Thus if $r = 1$ or $\varphi \geq \pi/3$, the strict inequality $\alpha_{i_0} > \varphi$ gives $c_0 > 0$ and $\mathcal{D} > 0$. (c) follows from (b) by Theorem 5.1: the dual complex has an acyclic part at i_0 in degrees $0, 1$ with surjective differential, the charge $-\overline{z_{i_0}}$ of argument $\pi - \alpha_{i_0}$ at i_0, and central charge of argument $\pi - \varphi$ (Remark 5.2). □

For $Q = K_m$, $i_0 = 0$ and cohomology at both vertices, φ lies between α_1 and α_0, and [Wue26b, Proposition 7.4 and Theorem 7.6] gives strict decrease when $\alpha_0 > \alpha_1$ and either $\varphi \geq 2\pi/5$ or $r = 1$. Part (b) also allows equality and cohomology at one vertex. For $r \geq 2$ a lower bound on φ cannot be dispensed with: for K_2, the charges $z_0 = 10^{-6}(-1+\mathrm{i})$, $z_1 = 1 + \frac{\mathrm{i}}{500}$ and cohomology of dimension vector $(1,1)$, where $\varphi = 0.0020\ldots$, there is a complex with $r = 2$ and a metric with $\mathcal{D} < 0$ [Wue26b, Proposition 10.2]. The hypothesis on α_{i_0} cannot be dropped: by Proposition 4.3 a contractible summand at i_0 in degrees $0, 1$ contributes $\pi|z_{i_0}| \operatorname{tr} e^{-ff^\dagger} \sin(\alpha_{i_0} - \varphi)$ to $\mathcal{D}$, and Corollary 4.4 is an instance with a negative sign. Decay of the acyclic part and convergence on the cohomology (Definition 2.4(v)) for the complexes of Theorem 6.1 are the subject of [Wue26b], for K_m with a surjective differential at the vertex 0, and of [Wue26c], for quivers without oriented cycles.

6.2. **Acyclic parts at both vertices of a generalized Kronecker quiver.** Let $Q = K_m$. Consider the three-term complexes

$$(0 \to U_1) \xrightarrow{D_1} (V \xrightarrow{A_j} W) \xrightarrow{D_0} (U_0 \to 0) \qquad \text{in degrees } -1, 0, 1, \tag{6}$$

with $D_1 : U_1 \to W$ injective and $D_0 : V \to U_0$ surjective; the chain-map conditions are empty. The cohomology is the representation $\ker D_0 \to W/D_1U_1$ in degree 0. Put $r_0 = \dim U_0$, $r_1 = \dim U_1$, $n_0 = \dim \ker D_0$, $n_1 = \dim W - r_1$, assume $n_0 + n_1 \geq 1$, and let $Z = n_0 z_0 + n_1 z_1 = |Z| e^{\mathrm{i}\varphi}$ with $\varphi \in (0, \pi)$, $\sigma = \sin\varphi$, $c_0 = |z_0| \sin(\alpha_0 - \varphi)$, $c_1 = |z_1| \sin(\varphi - \alpha_1)$, and

$$B_0 = D_0^\dagger D_0, \quad L_0 = D_0 D_0^\dagger, \quad P_1 = D_1 D_1^\dagger, \quad L_1 = D_1^\dagger D_1, \quad C = \sum_j A_j^\dagger A_j, \quad G = \sum_j A_j A_j^\dagger.$$

Here D_0 is the differential at the vertex 0 and D_1 the differential at the vertex 1; the subscripts are vertices, not degrees. For $r_1 = 0$ these are complexes as in Theorem 6.1(b) with $i_0 = 0$, and for $r_0 = 0$ as in Theorem 6.1(c) with $i_0 = 1$.

Proposition 6.2. *The blocks of Ω for a complex (6) are*

$$\begin{aligned}
\Omega_V &= z_0 e^{-B_0} + K_{B_0}(C) - \sum_j \mathcal{J}_{B_0,L_1}(A_j^\dagger D_1, D_1^\dagger A_j),\\
\Omega_W &= z_1 e^{-P_1} - K_{P_1}(G) + \sum_j \mathcal{J}_{P_1,L_0}(A_j D_0^\dagger, D_0 A_j^\dagger),\\
\Omega_{U_0} &= \mathcal{J}_{U_0} - z_0 e^{-L_0}, \qquad \mathcal{J}_{U_0} := \sum_j \mathcal{J}_{L_0,P_1}(D_0 A_j^\dagger, A_j D_0^\dagger),\\
\Omega_{U_1} &= -z_1 e^{-L_1} - \mathcal{J}_{U_1}, \qquad \mathcal{J}_{U_1} := \sum_j \mathcal{J}_{L_1,B_0}(D_1^\dagger A_j, A_j^\dagger D_1),
\end{aligned}$$

with phases in $(0,\pi)$ on V, W, in $(-\pi, 0)$ on U_0 and in $(\pi, 2\pi)$ on U_1. If the charges are ordered, then for every metric

$$\mathcal{D} \geq \varphi c_0 \operatorname{tr} e^{-L_0} + \sigma\big(\varphi \operatorname{tr} \mathcal{J}_{U_0} - \pi \operatorname{tr}(\mathcal{J}_{U_0})_-\big) + (\pi - \varphi) c_1 \operatorname{tr} e^{-L_1} + \sigma\big((\pi - \varphi) \operatorname{tr} \mathcal{J}_{U_1} - \pi \operatorname{tr}(\mathcal{J}_{U_1})_-\big).$$

Proof. Proposition 2.1 with $E_0 = V \oplus U_0$, $d_0 = D_0$, $E_1 = U_1 \oplus W$, $d_1 = D_1$, $\Delta_0 = B_0 \oplus L_0$, $\Delta_1 = L_1 \oplus P_1$, and the operator $M_{a_j} = d_1^\dagger \rho(a_j) - \rho(a_j) d_0^\dagger$ of (2), which maps $V \to U_1$ by $D_1^\dagger A_j$ and $U_0 \to W$ by $-A_j D_0^\dagger$. The bound for the block U_0 is the argument of Theorem 6.1(b) with J replaced by the Hermitian matrix $\mathcal{J}_{U_0}$. For U_1, with (u_k) an eigenbasis of Φ_{U_1}, eigenvalues $\psi_k \in (\pi, 2\pi)$, $q_k = \langle u_k, |\Omega_{U_1}| u_k\rangle$, $\iota_k = \langle u_k, \mathcal{J}_{U_1} u_k\rangle$ and $e_k = \langle u_k, e^{-L_1} u_k\rangle$: $q_k \sin(\psi_k - \varphi) = \sigma\iota_k + c_1 e_k$. If this is nonnegative, $\psi_k - \varphi \in (\pi - \varphi, \pi]$ and the summand is at least

$(\pi-\varphi)(\sigma\iota_k + c_1 e_k)$; otherwise $\psi_k - \varphi \in (\pi, 2\pi-\varphi)$ and it is at least $(2\pi-\varphi)(\sigma\iota_k + c_1 e_k)$. In both cases it is at least $(\pi-\varphi)(\sigma\iota_k + c_1 e_k) - \pi\sigma(\iota_k)_-$. □

In eigenbases of L_0, with eigenvalues ℓ_i, and of P_1, with eigenvalues p_μ ($p_\mu = 0$ on the orthogonal complement of $D_1 U_1$),

$$(\mathcal{J}_{U_0})_{ik} = \sum_{j,\mu} (D_0 A_j^\dagger)_{i\mu} \overline{(D_0 A_j^\dagger)_{k\mu}}\, \mathsf{f}[\ell_i, p_\mu, \ell_k],$$

with the second divided differences of e^{-x}. If all p_μ vanish this is the Schur multiplier of (5); in general the interpolation point 0 is replaced by the eigenvalues p_μ.

Theorem 6.3 (Ranks at most one)**.** *Let $\alpha_0 \geq \alpha_1$ (ordered charges) and $r_0 \leq 1$, $r_1 \leq 1$. Then for every metric on a complex (6)*

$$\mathcal{D} \geq \varphi\big(c_0 \operatorname{tr} e^{-L_0} + \sigma \operatorname{tr} \mathcal{J}_{U_0}\big) + (\pi-\varphi)\big(c_1 \operatorname{tr} e^{-L_1} + \sigma \operatorname{tr} \mathcal{J}_{U_1}\big) \geq 0,$$

where a term with $r_i = 0$ is omitted. The phase potential is nonincreasing along the phase flow, and strictly decreasing if the charges are strictly ordered, $n_0, n_1 \geq 1$ and $r_0 + r_1 \geq 1$.

Proof. For $r_0 = 1$, $\mathcal{J}_{U_0}$ is the nonnegative number

$$\sum_j \int_{\substack{x,y\geq 0\\ x+y\leq 1}} e^{-(x+y)L_0}\, D_0 A_j^\dagger e^{-(1-x-y)P_1} A_j D_0^\dagger \, dx\, dy;$$

likewise $\mathcal{J}_{U_1} \geq 0$ for $r_1 = 1$. Apply Proposition 6.2. If $\alpha_0 > \alpha_1$ and $n_0, n_1 \geq 1$, then $c_0, c_1 > 0$. □

Theorem 6.4 (Ranks $(2,1)$)**.** *Let $z_0 = -1+\mathrm{i}$, $z_1 = 1+\mathrm{i}$, and let $V = \mathbb{C}^3$, $W = \mathbb{C}^2$, $U_0 = \mathbb{C}^2$, $U_1 = \mathbb{C}$ with the standard metrics,*

$$D_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 100 \end{pmatrix}, \qquad A_j = 0 \ (j \geq 2).$$

The cohomology is the representation $\mathbb{C} \xrightarrow{1} \mathbb{C}$, which is stable, and the induced metric on it is harmonic. For these metrics

$$\mathcal{D} = -1.7280352\ldots < 0, \qquad \textit{that is,} \qquad \frac{d}{dt} \operatorname{Im}\big(e^{-\mathrm{i}\varphi} \mathcal{S}_{\mathbb{C}}\big) = +1.728\ldots > 0.$$

Proof. The complex is the orthogonal direct sum of the representation $\mathbb{C} \xrightarrow{1} \mathbb{C}$, whose two blocks z_0+1 and z_1-1 equal i and have phase $\varphi = \frac{\pi}{2}$, and of the acyclic complex $(0 \to U_1) \to (\mathbb{C}^2 \xrightarrow{\rho} \mathbb{C}) \to (U_0 \to 0)$ with $\rho = (\rho_1, \rho_2) = (4, 100)$ of Example 2.3. With $(\ell_1, \ell_2) = (1, 16)$ the eigenvalues of B_0 on $\mathbb{C}^2$ and $p = 9$, Proposition 6.2 gives for the acyclic complex the blocks

$$\begin{aligned}
(\Omega_V)_{ik} &= z_0 e^{-\ell_i} \delta_{ik} + \rho_i \rho_k \big(-\mathsf{f}[\ell_i, \ell_k] - p\, \mathsf{f}[\ell_i, p, \ell_k]\big), \\
\Omega_W &= (z_1 - |\rho|^2) e^{-p} + \sum_i \rho_i^2 \ell_i\, \mathsf{f}[p, \ell_i, p], \\
(-\Omega_{U_0})_{ik} &= z_0 e^{-\ell_i} \delta_{ik} - \sqrt{\ell_i \ell_k}\, \rho_i \rho_k\, \mathsf{f}[\ell_i, p, \ell_k], \\
-\Omega_{U_1} &= z_1 e^{-p} + p \sum_i \rho_i^2\, \mathsf{f}[p, \ell_i, p],
\end{aligned}$$

with the divided differences of e^{-x}. The four block contributions, on V, W, U_0 and U_1, are $3.4635\ldots$, $2.0009\ldots$, $-10.6257\ldots$ and $3.4332\ldots$; they and their sum were enclosed with

interval arithmetic [Joh17],[1] with radius below 10^{-18}. The 2×2 polar decompositions are explicit, and the phases on U_0 are $-3.1005\ldots$ and $-0.0347\ldots$ □

Corollary 6.5. *For the charges* $(-1+\mathrm{i}, 1+\mathrm{i})$ *and* $n_0 = n_1 \geq 1$*, the phase potential is nonincreasing along the phase flow for all metrics on all complexes* (6) *with ranks* (r_0, r_1) *if and only if* $\min(r_0, r_1) = 0$ *or* $r_0 = r_1 = 1$*.*

Proof. Here $\varphi = \frac{\pi}{2}$ and $\alpha_1 < \varphi < \alpha_0$. If $r_0 = r_1 = 0$ the complex is a representation and $\mathcal{D} \geq 0$ by Proposition 4.1. If exactly one of r_0, r_1 vanishes, this is Theorem 6.1(b) or (c) for $Q = K_m$; if $r_0 = r_1 = 1$ it is Theorem 6.3. Conversely, by Theorem 5.1, which exchanges r_0 and r_1 and fixes these charges, we may assume $r_0 \geq 2$ and $r_1 \geq 1$. Take the orthogonal direct sum of n_0 copies of $\mathbb{C} \xrightarrow{1} \mathbb{C}$, of the acyclic complex of Theorem 6.4, and of contractible summands at the vertex 0 in degrees $0, 1$ and at the vertex 1 in degrees $-1, 0$, of ranks $r_0 - 2$ and $r_1 - 1$, with differentials $x\,1$. By Proposition 4.3 the latter contribute $\pi(r_0 + r_1 - 3)e^{-x^2}$ to $\mathcal{D}$, which tends to 0 as $x \to \infty$. □

Remark 6.6. The example of Theorem 6.4 comes from the following computation. Let $r_0 = 2$, let P_1 have one nonzero eigenvalue p, and let there be one nonzero arrow, which maps into the corresponding line. Then $(\mathcal{J}_{U_0})_{ik} = v_i\overline{v_k}\,\mathsf{f}[\ell_i, p, \ell_k]$ for a vector $v \in \mathbb{C}^2$, and for a 2×2 matrix of this form the supremum over v of $\operatorname{tr}(\mathcal{J}_{U_0})_- / \operatorname{tr}\mathcal{J}_{U_0}$ is the positive part of $\frac{1}{2}\big(\mathsf{f}[\ell_1, p, \ell_2]\,(\mathsf{f}[\ell_1, p, \ell_1]\mathsf{f}[\ell_2, p, \ell_2])^{-1/2} - 1\big)$, attained for $|v_1|^2\mathsf{f}[\ell_1, p, \ell_1] = |v_2|^2\mathsf{f}[\ell_2, p, \ell_2]$. As $\ell_1 \to 0$ and $\ell_2 \to \infty$ this tends to

$$\frac{1}{2}\Big(\frac{R(p)\,e^{p/2}}{\sqrt{(e^{-p} - 1 + p)/p^2}} - 1\Big),$$

which is $(\sqrt{2} - 1)/2$ at $p = 0$ and tends to infinity with p. So (5) has no analogue that is uniform in p.

Whether the acyclic parts decay along the phase flow of the complexes (6), in the sense of Definition 2.4(v), is open.

7. Complexes of bundles on a curve

The dictionary of [HKKP26, Remark 5.14] recalled in Section 2 suggests the following formulation for two-term complexes of holomorphic vector bundles on a curve, through the superconnection of Quillen [Qui85]; the remark states the analogy between the algebra of the quiver and the forms on a Riemann surface, and the definition for bundles below is ours. A two-term complex $E^0 \to E^1$ is a holomorphic triple in the sense of Bradlow and García-Prada [BG96], and for $E^0 = \mathcal{O}$ a holomorphic pair [Bra91]; the equations studied there are the vortex equations. No equation for the pair of metrics is proposed here: the phases Φ^0 and Φ^1 defined below take values in the disjoint intervals $(0, \pi)$ and $(-\pi, 0)$, so that, as for complexes of representations (Remark 2.5), a metric of constant phase exists only on a single bundle, where it is a Hermitian–Einstein metric. The object of this section is the flow (8). We give the endomorphism Ω, a primitive and the derivative of the phase potential along the flow, show that the linearization of the flow is parabolic, and work out the case of a section of a line bundle. In this section † denotes the adjoint for the metrics, also for endomorphism-valued forms; K_Δ and the divided differences $\mathsf{f}[\,\cdot\,]$ of e^{-x} (not the bundle map f) are as in

[1] The program `theorem_6_4_dissipation.py`, which uses the library Arb, and its output accompany this paper as supplementary files. The program also checks that $\operatorname{tr}\Omega = 0$ for the acyclic complex and compares the blocks, in floating point, with an evaluation of the definition (1), with the exponential taken in $\mathcal{A}(K_1)\otimes\operatorname{End}(E)$.

Section 2, and E_1 and Ein as in Section 4. The results of the earlier sections are cited where they are used.

7.1. **The endomorphism Ω.** Let X be a compact Riemann surface with a Kähler form ω, $\int_X \omega = 1$, and z a complex number with $\operatorname{Im} z > 0$. For a holomorphic vector bundle E put $Z(E) = z \operatorname{rk} E - \deg E$; for $z = \mathrm{i}$ this is the usual central charge $-\deg E + \mathrm{i} \operatorname{rk} E$. A *metrized two-term complex* is $E^0 \xrightarrow{f} E^1$ with f a holomorphic bundle map and Hermitian metrics h^0, h^1. Let ∇ be the Chern connections, F_{h^n} their curvatures,

$$\mathfrak{k}^n = \tfrac{\mathrm{i}}{2\pi} \Lambda_\omega F_{h^n} \in \operatorname{End}(E^n), \qquad B = f^\dagger f, \quad L = f f^\dagger,$$

and, in a local holomorphic coordinate w with $\omega = \frac{\mathrm{i}}{2} g \, dw \wedge d\bar{w}$ and $f_w = \nabla_{\partial_w} f$,

$$\mathfrak{q}^0(T) = \frac{f_w^\dagger \, T \, f_w}{\pi g}, \qquad \mathfrak{q}^1(T) = \frac{f_w \, T \, f_w^\dagger}{\pi g},$$

which do not depend on the coordinate. The endomorphisms $\mathfrak{k}^n$ are Hermitian, and $\int_X \operatorname{tr} \mathfrak{k}^n \, \omega = \deg E^n$. Let ε be the grading of $E^0 \oplus E^1$, let $\mathbb{A} = \nabla + \varkappa(f + f^\dagger)$ be the superconnection on $E^0 \oplus E^1$ with $\varkappa^2 = -2\pi\mathrm{i}$, and

$$\mathfrak{F}_h = \tfrac{\mathrm{i}}{2\pi} \mathbb{A}^2 - z\,\omega = (B \oplus L) + \tfrac{\mathrm{i}\varkappa}{2\pi} \big(\nabla^{1,0} f + \nabla^{0,1} f^\dagger \big) + \tfrac{\mathrm{i}}{2\pi} F_h - z\,\omega.$$

We define $\Omega = \Omega^0 \oplus \Omega^1 \in \operatorname{End}(E^0) \oplus \operatorname{End}(E^1)$ by

$$\Omega \, \omega = \varepsilon \, \big[\exp(-\mathfrak{F}_h) \big]^{(2)}, \tag{7}$$

the two-form part of the exponential in the algebra of $\operatorname{End}(E^0 \oplus E^1)$-valued forms with the sign rule of graded tensor products. Up to the normalization $\varkappa$, $\mathbb{A}$ is Quillen's superconnection of the Hermitian complex, and the supertrace of $\exp(-\mathfrak{F}_h)$ is $e^{z\omega}$ times its Chern character form in the normalization $\operatorname{str} \exp(-\frac{\mathrm{i}}{2\pi}\mathbb{A}^2)$, whose part of degree two represents $-c_1(E^0) + c_1(E^1)$. In (7) the operator $\bar\partial + \varkappa f$, which does not depend on the metric, takes the place of the element A of Section 2; so the arrows correspond to forms of type $(0,1)$, whereas [HKKP26, Remark 5.14] compares them with forms of type $(1,0)$.

Proposition 7.1. *The blocks of Ω are*

$$\Omega^0 = K_B(z - \mathfrak{k}^0) + \mathcal{J}^0, \qquad \Omega^1 = \mathcal{J}^1 - K_L(z - \mathfrak{k}^1),$$

with $\mathcal{J}^0 = \int e^{-xB} \, \mathfrak{q}^0\big(e^{-(1-x-y)L}\big) \, e^{-yB} \, dx \, dy$ and $\mathcal{J}^1 = \int e^{-xL} \, \mathfrak{q}^1\big(e^{-(1-x-y)B}\big) \, e^{-yL} \, dx \, dy$ over $\{x, y \geq 0,\ x+y \leq 1\}$. The endomorphisms $\mathcal{J}^n$ are Hermitian, so $\operatorname{Im} \Omega^0 = (\operatorname{Im} z) e^{-B} > 0$ and $\operatorname{Im}(-\Omega^1) = (\operatorname{Im} z) e^{-L} > 0$ at every point of X. Moreover

$$\int_X \operatorname{tr}(\Omega) \, \omega = Z(E^0) - Z(E^1).$$

Proof. The Duhamel expansion of $\exp(-(B \oplus L) - Y)$ with $Y = \beta + \frac{\mathrm{i}}{2\pi} F_h - z\omega$, $\beta = \frac{\mathrm{i}\varkappa}{2\pi}(\nabla^{1,0} f + \nabla^{0,1} f^\dagger)$, has the two-form part $K_{B\oplus L}(z\omega - \frac{\mathrm{i}}{2\pi} F_h) + \int e^{-x(B\oplus L)} \beta e^{-(1-x-y)(B\oplus L)} \beta e^{-y(B\oplus L)}$, and $\beta \, T \, \beta = \frac{\mathrm{i}}{2\pi}(dw \otimes f_w + d\bar{w} \otimes f_w^\dagger) \, T \, (dw \otimes f_w + d\bar{w} \otimes f_w^\dagger)$. By the sign rule, on E^0: $(d\bar{w} \otimes f_w^\dagger) \, T \, (dw \otimes f_w) = dw \wedge d\bar{w} \otimes f_w^\dagger T f_w$, and on E^1: $(dw \otimes f_w) \, T \, (d\bar{w} \otimes f_w^\dagger) = -dw \wedge d\bar{w} \otimes f_w T f_w^\dagger$. Since $\frac{\mathrm{i}}{2\pi} dw \wedge d\bar{w} = \frac{1}{\pi g} \omega$, the two-form parts are $[K_B(z - \mathfrak{k}^0) + \mathcal{J}^0]\omega$ on E^0 and $[K_L(z - \mathfrak{k}^1) - \mathcal{J}^1]\omega$ on E^1; the latter is multiplied by -1. Hermiticity is as in Proposition 2.2(a). For the integral, replace f by uf, $u \in [0,1]$: the derivative in u of the supertrace of $[\exp(-\mathfrak{F})]^{(2)}$ is exact, by the transgression formula for superconnections [Qui85], so the integral equals its value at $u = 0$, which is $z(\operatorname{rk} E^0 - \operatorname{rk} E^1) - (\deg E^0 - \deg E^1)$. □

The formula has the shape of Ω for a two-term complex of representations, with $-\mathfrak{k}^0$ in the place of the arrow term. The phases Φ^0, with spectrum in $(0,\pi)$, and Φ^1, with spectrum in $(-\pi,0)$, are defined pointwise on X as in Proposition 2.2(c), and the *phase flow* is

$$(h^n)^{-1}\partial_t h^n = -\Phi^n(h), \qquad n = 0, 1. \tag{8}$$

For a single bundle, $E^1 = 0$, it is $h^{-1}\partial_t h = -\arg(z - \mathfrak{k}_h)$, with $z - \mathfrak{k}_h$ a normal endomorphism; its fixed points modulo rescaling are the metrics with $\mathfrak{k}_h$ a scalar, the Hermitian–Einstein metrics. Donaldson's heat flow [Don85] has velocity proportional to $\mu - \mathfrak{k}_h$, with μ the slope; the present flow has velocity $-\arg(z - \mathfrak{k}_h)$, and the two have the same stationary points modulo rescaling. For a line bundle this is a flow of the metric by an angle, of the same kind as the flow of Jacob and Yau for the deformed Hermitian–Yang–Mills equation [JY17, (1.2)], and the condition that the phase be constant is of the type of the equation for Z-critical connections [DMS24, Definition 2.22].

7.2. **The primitive and the derivative along the flow.** For a bundle E with a reference metric let M be the function on the space of metrics with $dM(\xi) = \int_X \operatorname{tr}(\mathfrak{k}_h \xi)\,\omega$ and $M(h^{\mathrm{ref}}) = 0$; this one-form is closed [Don85], which is also the case $f = 0$, $E^1 = 0$ of the proposition below.

Proposition 7.2. *On the space of smooth metrics the one-form $\xi \mapsto \int_X \operatorname{tr}(\Omega\,\xi)\,\omega$, $\xi = h^{-1}\delta h$, is closed. A primitive is*

$$\mathcal{S}_{\mathbb{C}}(h) = z\int_X \Big(\log\det\big((h^{0,\mathrm{ref}})^{-1}h^0\big) - \log\det\big((h^{1,\mathrm{ref}})^{-1}h^1\big)\Big)\omega - M(h^0) + M(h^1) \\ + \int_0^1\!\!\int_X \operatorname{tr}\big(\Omega^1_{[x]} + z - \mathfrak{k}^1\big)\,\omega\,\frac{dx}{x},$$

where $\Omega^1_{[x]}$ is the block on E^1 of the endomorphism (7) for $\sqrt{x}\,f$, the analogue of the $\Omega^{(x)}$ of Theorem 3.4; for $x = 0$ this block is $-(z - \mathfrak{k}^1)$.

Proof. We apply Lemma 3.2. Let $\mathcal{C}$ be the algebra of smooth $\operatorname{End}(E^0 \oplus E^1)$-valued forms on X, graded by the form degree plus the degree of the endomorphism, with the sign rule of graded tensor products, and let $\tau(\alpha) = \int_X \operatorname{str}\alpha^{(2)}$, the integral of the supertrace of the two-form part; it vanishes on supercommutators. Let $\mathbb{A}'' = \bar\partial + \varkappa f$ and $\mathbb{A}' = \nabla^{1,0} + \varkappa f^\dagger$ be the two parts of the superconnection, and put

$$\mathfrak{d} = (2\pi)^{-1/2}[\mathbb{A}'', \cdot], \qquad \mathfrak{d}^\star = -(2\pi)^{-1/2}[\mathbb{A}', \cdot], \qquad F = \tfrac{1}{2\pi}\mathbb{A}^2 + \sqrt{-1}\,z\,\omega = -\sqrt{-1}\,\mathfrak{F}_h.$$

These are odd derivations of $\mathcal{C}$: first-order operators acting on endomorphism-valued forms by supercommutators. One has $(\mathbb{A}'')^2 = 0$, because $\bar\partial^2 = 0$, $\bar\partial f = 0$ and $f^2 = 0$, and $(\mathbb{A}')^2 = 0$ likewise; so $\mathbb{A}^2 = [\mathbb{A}', \mathbb{A}'']$, and by the graded Jacobi identity $\mathfrak{d}$ and $\mathfrak{d}^\star$ annihilate $\mathbb{A}^2$ and satisfy $\mathfrak{d}\mathfrak{d}^\star + \mathfrak{d}^\star\mathfrak{d} = -\frac{1}{2\pi}[\mathbb{A}^2, \cdot] = -[F, \cdot]$; the form ω is closed and central. By Stokes' theorem and the vanishing of the supertrace on supercommutators, $\tau\circ\mathfrak{d} = 0 = \tau\circ\mathfrak{d}^\star$. For a variation $\xi = h^{-1}\delta h$ of the metrics, $\delta_\xi\mathbb{A}'' = 0$ and $\delta_\xi\mathbb{A}' = [\mathbb{A}', \xi]$ for the Chern connection and the adjoint $f^\dagger$, hence $\delta_\xi F = \frac{1}{2\pi}[\mathbb{A}'', [\mathbb{A}', \xi]] = -\mathfrak{d}\mathfrak{d}^\star\xi$; and $\partial_v\xi - \partial_u\eta = [\xi, \eta]$ as before. The exponential of $-\sqrt{-1}F = -\mathfrak{F}_h$ is the finite Duhamel expansion around the form-degree zero part $-(B \oplus L)$, so the remaining hypotheses hold, and $\tau(e^{-\sqrt{-1}F}\xi) = \int_X \operatorname{tr}(\Omega\,\xi)\,\omega$ by (7). All terms in the proof of the lemma are integrals over X of polynomials in ξ, η and their covariant derivatives up to order two, with smooth coefficients. The primitive is obtained as in Theorem 3.4 from the path of metrics (h^0, xh^1), $x \in (0, 1]$, along which the velocity is $\frac{1}{x}$ times the number operator and which corresponds to the rescaling $\sqrt{x}\,f$; the integrand

of the x-integral is bounded near $x = 0$ because $\Omega_{[x]}$ is analytic in x. For $f = 0$ one has $\Omega = (z - \mathfrak{k}_{h^0}) \oplus -(z - \mathfrak{k}_{h^1})$, with the primitive given by the first three terms. □

In the terms of the theory of Bott–Chern forms: the integral of the form-valued one-form $\xi \mapsto \operatorname{str}\big(\xi\, e^{-\mathfrak{F}_h}\big)$ along a path of metrics from h^{ref} to h is the secondary form of the pair (h^{ref}, h) for the character form $\operatorname{str}\exp(-\mathfrak{F}_h)$, and $\mathcal{S}_{\mathbb{C}}(h) - \mathcal{S}_{\mathbb{C}}(h^{\mathrm{ref}})$ is the integral over X of its part of degree two. As in Section 4, for $Z = Z(E^0) - Z(E^1) \neq 0$ and $Z = |Z|e^{\mathrm{i}\varphi}$ the function $\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})$ is called the *phase potential*.

For a single bundle of slope $\mu = \deg E/\operatorname{rk} E$, with $Z(E) = |Z|e^{\mathrm{i}\varphi}$, one has $\operatorname{Im}(e^{-\mathrm{i}\varphi}z) = -\mu\sin\varphi$, and

$$\frac{\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})}{\sin\varphi} = M(h) - \mu\int_X \log\det\big((h^{\mathrm{ref}})^{-1}h\big)\,\omega.$$

This is Donaldson's functional [Don85], in the normalization given by $\mathfrak{k}_h$; its critical points are the Hermitian–Einstein metrics $\mathfrak{k}_h = \mu$.

Proposition 7.3. *Let $Z = Z(E^0) - Z(E^1)$, let φ be a real number with $\operatorname{Im}(e^{-\mathrm{i}\varphi}Z) = 0$, and let $h(t)$ be a smooth solution of (8) on an interval. Then*

$$\frac{d}{dt}\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}(h(t))\big) = -\int_X \sum_{n=0,1} \operatorname{tr}\Big(|\Omega^n|(\Phi^n - \varphi)\sin(\Phi^n - \varphi)\Big)\omega.$$

(a) *For a single bundle, with $\varphi = \arg Z(E)$, the right side is nonpositive and vanishes only at Hermitian–Einstein metrics.*
(b) *If $\operatorname{rk} E^0 = \operatorname{rk} E^1$ and $\deg E^1 > \deg E^0$, then $Z > 0$. With $\varphi = 0$ the right side is negative, so $\operatorname{Im}\mathcal{S}_{\mathbb{C}}$ is strictly decreasing along the flow.*

Proof. Apply Proposition 7.2 to the flow equation and use the polar decomposition in each fibre as in Proposition 4.2. Under the integral one may replace Φ^n by $\Phi^n - \varphi$ in the factor multiplying the sine, since the integral of $\operatorname{Im}(e^{-\mathrm{i}\varphi}\operatorname{tr}\Omega)$ is $\operatorname{Im}(e^{-\mathrm{i}\varphi}Z) = 0$ by Proposition 7.1. (a) Φ and φ lie in $(0,\pi)$, and $\Phi = \varphi$ exactly when $\mathfrak{k}_h = \mu$. (b) $x\sin x > 0$ on $(0,\pi)$ and on $(-\pi, 0)$. □

For $\operatorname{rk} E^0 > \operatorname{rk} E^1$ one has $\varphi \in (0,\pi)$, the term of degree zero is nonnegative and the sign of the term of degree one is not controlled in general, as for complexes of representations; for $\operatorname{rk} E^0 < \operatorname{rk} E^1$ the roles are exchanged.

7.3. **Parabolicity.** A system $\partial_t\xi = \mathcal{P}\,\Delta_\omega\xi +$ terms of lower order, with $\mathcal{P}$ a smooth field of endomorphisms of the fibre and $\Delta_\omega = -d^*d \leq 0$ the Laplace–Beltrami operator of ω, is parabolic in the sense of Petrovskii if all eigenvalues of $\mathcal{P}$ have positive real part at every point.

Proposition 7.4. *The linearization of (8) at a smooth metric is $\partial_t\xi^n = \frac{1}{4\pi}\mathcal{P}^n\,\Delta_\omega\xi^n +$ terms of lower order, where $\mathcal{P}^n = -\mathcal{V}^n \circ K^n$, $K^0 = K_B$, $K^1 = K_L$, and $\mathcal{V}^n$ is the derivative of the phase Φ^n with respect to the Hermitian part of $(-1)^n\Omega^n$ at fixed imaginary part. All eigenvalues of $\mathcal{P}^n$ have positive real part: the system is parabolic in the sense of Petrovskii. Moreover $\mathcal{P}^n$ has a positive definite symmetric part for the fibre metric $\operatorname{tr}(K^n(\xi)\eta)$ on Hermitian endomorphisms.*

Proof. Ω depends on second derivatives of h only through $\mathfrak{k}$, and for a variation ξ of the metric the variation of $\mathfrak{k}^n$ is $-\frac{1}{4\pi}\Delta_\omega\xi^n$ up to terms of lower order. So the second-order part of the variation of $(-1)^n\Omega^n$ is the Hermitian endomorphism $\frac{1}{4\pi}K^n(\Delta_\omega\xi^n)$, with the imaginary part unchanged, and the variation of the phase is $\mathcal{V}^n$ applied to it. The operators K_B and K_L are positive definite and self-adjoint on Hermitian matrices: in an eigenbasis of B, or of L, with eigenvalues ℓ_i, they multiply the entries by the positive numbers $-\mathsf{f}[\ell_i,\ell_k]$. By Theorem 2.6(b), $\operatorname{tr}(\eta\,\mathcal{V}^n\eta) < 0$ for Hermitian $\eta \neq 0$. Hence $\operatorname{tr}\big(K(\xi)\,\mathcal{P}\xi\big) = -\operatorname{tr}\big(K(\xi)\,\mathcal{V}(K(\xi))\big) > 0$, which is the statement on the symmetric part for the inner product $\operatorname{tr}(K(\xi)\eta)$; an operator with positive definite symmetric part for some inner product has its eigenvalues in the open right half-plane. □

We do not prove short-time existence of smooth solutions of (8). For a line bundle the flow is the scalar equation $\partial_t u = -\arg\big(z - \mathfrak{k}_{h^{\mathrm{ref}}} + \frac{1}{4\pi}\Delta_\omega u\big)$ for $h = e^u h^{\mathrm{ref}}$, whose right side is increasing in $\Delta_\omega u$, and $\mathcal{P} = \operatorname{Im} z/|z-\mathfrak{k}_h|^2$. This tends to zero where the phase approaches 0 or π, so the parabolicity is not uniform.

7.4. **A section of a line bundle.** Let the complex be $\mathcal{O} \xrightarrow{s} \mathcal{L}$, with $\mathcal{L}$ a line bundle of degree $d \geq 1$ and s a nonzero holomorphic section with divisor Σ. It is quasi-isomorphic to $\mathcal{O}_\Sigma[-1]$, and $Z = Z(\mathcal{O}) - Z(\mathcal{L}) = d > 0$, so $\varphi = 0$. Let $\mathsf{h} = h^1/h^0$, a metric on $\mathcal{L}$, $b = |s|^2_{\mathsf{h}}$, $\mathfrak{k}^n = \frac{\mathrm{i}}{2\pi}\Lambda_\omega F_{h^n}$ as before and $\nu = |\nabla_w s|^2_{\mathsf{h}}/(\pi g) \geq 0$. All blocks are scalar, and Proposition 7.1 gives

$$\Omega^0 = e^{-b}\big(z - \mathfrak{k}^0 + \tfrac{\nu}{2}\big), \qquad \Omega^1 = -e^{-b}\big(z - \mathfrak{k}^1 - \tfrac{\nu}{2}\big),$$

$$\Phi^0 = \arg\big(z - \mathfrak{k}^0 + \tfrac{\nu}{2}\big) \in (0,\pi), \qquad \Phi^1 = \arg\big(z - \mathfrak{k}^1 - \tfrac{\nu}{2}\big) - \pi \in (-\pi, 0).$$

The sum is

$$(\Omega^0 + \Omega^1)\,\omega = e^{-|s|^2_{\mathsf{h}}}\big[\tfrac{\mathrm{i}}{2\pi}F_{\mathsf{h}} + \tfrac{\mathrm{i}}{2\pi}\nabla s \wedge \overline{\nabla s}\big] = \tfrac{\mathrm{i}}{2\pi}F_{\mathsf{h}} + \tfrac{\mathrm{i}}{2\pi}\partial\bar\partial\,\mathrm{Ein}\big(|s|^2_{\mathsf{h}}\big),$$

a representative of the Euler class of $\mathcal{L}$ of the kind constructed by Mathai and Quillen [MQ86], pulled back by s; its integral is d, and when s is replaced by cs with $c \to \infty$, it converges to the current of integration over Σ, because $\mathrm{Ein}(c^2 b) = \gamma + \log c^2 + \log b + \mathrm{E}_1(c^2 b)$, with γ Euler's constant, and $\frac{\mathrm{i}}{2\pi}\partial\bar\partial \log b = \delta_\Sigma - \frac{\mathrm{i}}{2\pi}F_{\mathsf{h}}$ by the Poincaré–Lelong formula.

With $u^n = \log(h^n/h^{n,\mathrm{ref}})$ the flow is $\partial_t u^n = -\Phi^n$, so $\partial_t \log b = \Phi^0 - \Phi^1 \in (0, 2\pi)$: b increases at every point outside Σ. By Proposition 7.2 the phase potential is

$$\operatorname{Im}\mathcal{S}_{\mathbb{C}} = (\operatorname{Im} z)\,\mathcal{W}(h), \qquad \mathcal{W}(h) = \int_X \big(u^0 - u^1 + \mathrm{Ein}(b)\big)\omega = \int_X \mathrm{E}_1(b)\,\omega + \mathrm{const.}$$

It involves no derivatives of the metrics, and it is bounded below, with a bound that is not attained.

Proposition 7.5. *Along every solution of the flow, at every point of X,*

$$\partial_t\big(u^0 - u^1 + \mathrm{Ein}(b)\big) = -e^{-b}(\Phi^0 - \Phi^1) < 0, \qquad \frac{d}{dt}\mathcal{W} = -\int_X e^{-b}\big(\Phi^0 + |\Phi^1|\big)\,\omega.$$

Consequently $\int_0^T\!\int_X e^{-b}(\Phi^0 + |\Phi^1|)\,\omega\,dt \leq \int_X \mathrm{E}_1(b|_{t=0})\,\omega < \infty$ on the interval of existence.

Proof. $\partial_t(u^0 - u^1) = -(\Phi^0 - \Phi^1)$ and $\partial_t\,\mathrm{Ein}(b) = \frac{1-e^{-b}}{b}\,\partial_t b = (1-e^{-b})(\Phi^0 - \Phi^1)$. The function $\mathrm{E}_1(b)$ has a logarithmic singularity along Σ and is integrable; $\mathcal{W} - \int_X \mathrm{E}_1(b)\,\omega$ is constant along the flow and $\mathrm{E}_1 \geq 0$. □

So in this case the potential is monotone for every z in the upper half-plane, in accordance with Proposition 7.3(b). If instead $d = 0$ and s has no zeros, the complex is acyclic and $Z = 0$; for constant metrics on a trivialization $\Omega^0 = ze^{-b} = -\Omega^1$ and $b(t) = b(0)e^{\pi t}$, as for the contractible summand of Proposition 4.3.

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