MASS INCREASE AND ACYCLIC CANCELLATION FOR PHASE FLOWS ON QUIVER COMPLEXES

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Abstract. Trace-norm mass can increase along the Haiden–Katzarkov–Kontsevich–Pandit phase flow defined by positivity, giving a negative answer to the proposed mass-monotonicity question with the prescribed degree branches. For two-term complexes over a two-vertex parallel-arrow quiver, with degree-one term supported at the source and surjective source differential, we nevertheless prove quantitative acyclic cancellation. For fixed upper-half-plane charges, arbitrary surviving tuples and bounded initial arrow norms, a sufficiently large least acyclic singular value gives exponential comparison with instantaneous cohomology. The estimates allow unequal surviving dimensions and are independent of arrow count, stabilizers and acyclic condition numbers. The proof controls the polar phase using only the nonsingular cohomology block. For equal surviving dimensions and charges $-1+\mathrm{i}$, $1+\mathrm{i}$, a finite infimum of the cohomology Kempf–Ness function forces its moment-map defects to tend to zero, without initial balance or attainment of the infimum. When the source differential is invertible, cancellation holds from every positive metric without a large-scale assumption, and the normalized cohomology metric converges. At fixed external differentials, static mass converges in C^1 in bounded arrow coordinates. An explicit family converges in C^2 while every positive Hölder seminorm of its Hessian diverges. An exact metric primitive also identifies a positive cohomology potential and a negative ambient Levi direction.

1. Introduction

Haiden, Katzarkov, Kontsevich and Pandit associate a mass and a metric phase flow to a complex of quiver representations [HKKP26, §§4.2 and 5.4]. Their construction uses a coefficient of a superconnection exponential. Acyclic summands should disappear when the differential becomes large in the evolving metric, leaving the cohomology representation. Question 5.18 of [HKKP26] asks whether the mass is nonincreasing along the flow. We give a negative answer for the phase determined by their positivity condition and degree branches. We then prove that quantitative acyclic cancellation can hold despite this failure.

The distinction between dissipation and cancellation is essential. The first concerns the sign of a derivative at a finite metric. The second requires the differential to escape to infinity under the actual equation, together with estimates on the coupled arrows. Prescribing a large differential and computing a static limit does not establish that a trajectory enters this regime.

1.1. **The complexes and their phase flow.** Fix a quiver with two vertices and k arrows from the first vertex to the second. We consider the two-term complex

$$(V \xrightarrow{(A_1,\dots,A_k)} W) \xrightarrow{(D,0)} (U \longrightarrow 0), \tag{1}$$

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in degrees zero and one. The chain condition is automatic. All vector spaces are finite dimensional over $\mathbb{C}$; the Hermitian metrics vary, whereas the complex maps are fixed. The central charges z_0, z_1 have strictly positive imaginary parts.

Section 2 derives the coefficient $\Omega = \Omega_V \oplus \Omega_U \oplus \Omega_W$ explicitly. On a block of degree q, the matrix $T = (-1)^q\Omega$ has strictly positive imaginary part. If $T = U_T|T|$ is its polar decomposition, set

$$\Phi = \operatorname{Arg} U_T - \pi q I, \qquad m(h) = \|\Omega_V\|_1 + \|\Omega_U\|_1 + \|\Omega_W\|_1, \qquad h^{-1}\dot{h} = -\Phi. \tag{2}$$

Here $\operatorname{Arg} U_T$ has spectrum in $(0, \pi)$ and $\|M\|_1 = \operatorname{tr}(M^\dagger M)^{1/2}$ is the trace norm. For a single matrix, an unadorned norm denotes the operator norm. Squared lengths of arrow tuples are sums of squared Hilbert–Schmidt norms, denoted by the subscript HS on each arrow. This is the unique phase in the specified branch for which $\Omega e^{-\mathrm{i}\Phi}$ is positive definite. In particular, it realizes the positivity definition of [HKKP26, §4.2].

The coefficient need not be normal, contrary to the normality assertion in [HKKP26, Lemma 5.16]. We exhibit a small example in Section 2. This issue is separate from the mass counterexample: positivity still defines a smooth Hermitian phase, and the metric equation exists for every finite positive time. Throughout the paper, the phase means (2), not an analytic logarithm of a nonnormal coefficient.

1.2. **Mass increase and cancellation.** The following two statements describe different properties of the same equation.

Theorem 1.1. *Let $V = U = \mathbb{C}^2$, $W = \mathbb{C}$, $k = 1$, and take standard initial metrics with*

$$D = \begin{pmatrix} 1/5 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \end{pmatrix}, \qquad z_0 = 1 + \mathrm{i}\epsilon, \quad z_1 = 4 + \mathrm{i}\epsilon.$$

For every sufficiently small $\epsilon > 0$, the initial derivative of m along (2) is strictly positive.

The proof in Section 5 reduces the sign to a finite real calculation with a positive interval bound. The endpoint $\epsilon = 0$ is used only as a one-sided limit; it is not an admissible central charge. No particular positive upper endpoint for ϵ is needed for the theorem.

Theorem 1.2. *Suppose that $D : V \to U$ is invertible, $\dim V = \dim U = n \geq 1$ and $\dim W = m_0 \geq 1$. Allow arbitrary k arrows, arbitrary positive initial metrics, and arbitrary z_0, z_1 in the strict upper half-plane. Put $a = \arg z_1$. The phase flow exists for all $t \geq 0$. In moving orthonormal frames, with $X_j = A_j D^{-1}$, there are constants $C, C' > 0$ depending on the initial data and charges such that*

$$\sum_j \|X_j(t)\|_{\mathrm{HS}}^2 \leq e^{-at} \sum_j \|X_j(0)\|_{\mathrm{HS}}^2, \tag{3}$$

$$\lambda_{\min}(D^\dagger D)(t) \geq at - C, \tag{4}$$

$$0 \leq m(h(t)) - m_0|z_1| \leq C' e^{-at}. \tag{5}$$

The acyclic coefficient blocks tend to zero and $\Omega_W \to z_1 I_W$ at the same exponential rate in the evolving trace norms. In fixed coordinates on W, the metric $e^{at} h_W(t)$ converges exponentially to a positive definite limit.

The proof uses the exact contraction equation for $A_j D^{-1}$. A weighted polar estimate then forces the least singular value of D to grow. Neither step assumes that the mass decreases. No simultaneous diagonalization of $D^\dagger D$ and $\sum_j A_j^\dagger A_j$ is imposed.

1.3. **Cancellation with arbitrary surviving cohomology.** When $\ker D$ is nonzero, the cohomology has both a source and a target. The ratio equation acquires an additional term, which can still be controlled without a stability or balance assumption on the surviving representation. Let $H = \mathbb{C}^{n_s}$, $W = \mathbb{C}^{n_w}$, $U = \mathbb{C}^r$ and $V = H \oplus \mathbb{C}^r$, with $n_s, n_w, r \geq 1$. In initial adapted orthonormal frames take arbitrary arrows $B_j : H \to W$ and set

$$D(0) = [0, F_0], \qquad A_j(0) = [B_j, T_j], \qquad F_0 \in \mathrm{GL}(r, \mathbb{C}), \quad \sum_j \|T_j\|_{\mathrm{HS}}^2 \leq M^2. \tag{6}$$

Fix any charges z_0, z_1 in the strict upper half-plane and put $a = \arg z_1$. Only the least singular value of F_0 will be constrained. Its other singular values may be arbitrarily large.

Along the flow use the orthogonal splitting $V = \ker D \oplus (\ker D)^\perp$ and write $D = [0, F]$, $A_j = [H_j, T_j]$. Let Ω_H be the ordinary degree-zero coefficient of the instantaneous surviving arrows H_j, and let m_H be its trace-norm mass. Its inclusion into the full graded space places zero in the acyclic blocks. Put

$$q = \sum_j \|H_j\|_{\mathrm{HS}}^2, \qquad s = \sum_j \left\|T_j F^{-1}\right\|_{\mathrm{HS}}^2, \qquad \lambda = \sigma_{\min}(F)^2, \quad \lambda_0 = \lambda(0),$$

and set

$$Q = \tfrac{1}{2}\big(m_H(0) - n_s \operatorname{Re} z_0 + n_w \operatorname{Re} z_1\big) + 1.$$

The elementary mass bound in Section 6 gives $Q \geq q(0) + 1$.

Theorem 1.3. *There are $\Lambda_*(Q, M, z_0, z_1)$ and $C(n_s, n_w, r, Q, M, z_0, z_1)$ such that every initial complex (6) with $\sigma_{\min}(F_0) \geq \Lambda \geq \Lambda_*$ has a global phase flow satisfying $q(t) \leq Q - 3/4$ and*

$$s(t) \leq M^2 \Lambda^{-2} e^{-at/2}, \qquad \lambda(t) \geq \max\{\lambda_0, at/4 - C\}. \tag{7}$$

Moreover,

$$\|\Omega(t) - (\Omega_H(t) \oplus 0_{\mathrm{ac}})\|_1 + |m(h(t)) - m_H(t)| \leq C\Lambda^{-2} e^{-at/8}. \tag{8}$$

The constants are independent of arrow count, the surviving stabilizer and the condition number of F_0. They can be chosen uniformly when the surviving dimensions, acyclic rank, charges and total initial arrow-norm bounds are fixed. No balance or stability hypothesis is required.

Section 6 gives explicit constants and a threshold independent of acyclic rank. The charges are fixed in these uniformity statements; no uniformity as their imaginary parts vanish is asserted. For example, acyclic singular values Λ and $R\Lambda$ are permitted with the same constants for every $R \geq 1$.

The theorem compares with evolving cohomology. It does not assert that cohomology itself approaches balance. The proof uses the ordinary cohomology mass to bound q, and a polar estimate using only the nonsingular cohomology block to control the additional coupling. A separate weighted phase estimate forces growth of the acyclic differential. These arguments make no use of full-complex mass dissipation or of a positive orbit Hessian gap.

1.4. **Relaxation from finite cohomology energy.** Now suppose $n_s = n_w = n$ and $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$. Define the moment-map defects

$$\mu_s = \sum_j H_j^\dagger H_j - I, \qquad \mu_w = \sum_j H_j H_j^\dagger - I, \qquad \|\mu\|^2 = \|\mu_s\|_{\mathrm{HS}}^2 + \|\mu_w\|_{\mathrm{HS}}^2.$$

In fixed holomorphic coordinates let h_s, h_w be the induced metrics on $\ker D, W$, both multiplied by $e^{\pi t/2}$. Using the initial orthonormal determinant volume forms, define the cohomology Kempf–Ness function on all positive metric pairs by

$$\mathcal{K} = q + \log\det h_s - \log\det h_w.$$

Theorem 1.4. *Under the hypotheses of Theorem 1.3, with these equal dimensions and charges, suppose $\mathcal{K}_* := \inf \mathcal{K} > -\infty$ and put $E_0 = \mathcal{K}(0) - \mathcal{K}_*$. Set $J_Q = 1 + (Q-1)^2$. Then*

$$0 \le \mathcal{K}(t) - \mathcal{K}_*,$$

$$\mathcal{K}(t) - \mathcal{K}_* + \frac{1}{2J_Q}\int_0^t \|\mu(u)\|^2\, du \le E_0 + C_{\mathrm{en}}(Q,M)\Lambda^{-4}. \tag{9}$$

Both moment-map defects tend to zero and $m(h(t)) \to 2n$. The infimum need not be attained, and initial balance is not required.

A balanced metric pair attains a finite minimum by geodesic convexity, but its value depends on the chosen determinant normalization. Section 6 also gives a finite, unattained example. A uniform energy bound across initial metrics retains a bound on E_0. Neither this theorem nor the cancellation estimate asserts a uniform final relaxation rate or convergence of normalized surviving metrics.

The initially balanced case has a particularly simple normalization. Suppose

$$\sum_j B_j^\dagger B_j = \sum_j B_j B_j^\dagger = I_n. \tag{10}$$

Then $q(0) = n$, $m_H(0) = 2n$, $Q = 2n+1$ and $\mathcal{K}_* = \mathcal{K}(0) = n$.

Corollary 1.5. *For (10), the preceding conclusions hold with threshold $\Lambda_*(n,M)$ and constants $C(n,r,M)$, uniformly over all balanced tuples. In particular $q(t) \le 2n + 1/4$, the cancellation rate in (8) is $\pi/32$, and (9) holds with $E_0 = 0$ and $J_Q = 1 + 4n^2$.*

No individual arrow is required to be invertible, and the surviving representation need not be irreducible.

The following family makes the stabilizer uniformity explicit. Let $n = r = 2$, $k = 3$, and put

$$B_1 = cI, \qquad B_2 = c\sigma_z, \qquad B_3 = c\epsilon\sigma_x, \qquad c = (2+\epsilon^2)^{-1/2}, \quad 0 \le \epsilon \le 1, \tag{11}$$

where $\sigma_z = \mathrm{diag}(1,-1)$ and $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Corollary 1.6. *For (11), $F_0 = \Lambda I_2$ and every mixing tuple with $\sum_j \|T_j\|_{\mathrm{HS}}^2 \le M^2$, the conclusions of Corollary 1.5 hold for $\Lambda \ge \Lambda_*(2,M)$, uniformly in $\epsilon \in [0,1]$. The cohomology automorphism algebra is scalar for $\epsilon > 0$ and diagonal for $\epsilon = 0$.*

Proof. Both Gram sums in (10) are I_2. Since $B_1 = cI$ is invertible, an automorphism is a common source and target matrix commuting with σ_z and $\epsilon\sigma_x$. The first condition makes it diagonal, and the second makes it scalar when $\epsilon > 0$. The estimates follow from Corollary 1.5. □

The parameter ϵ is fixed along each trajectory, but may approach zero independently of Λ across the family.

1.5. **Static regularity and potentials.** The coefficient formulas yield an exact complex primitive in metric directions. For an arbitrary fixed differential with least acyclic singular value Λ, the relative primitive and its bounded-arrow-coordinate derivatives have error $O(\Lambda^{-2})$. The mass has a C^1 limit with first-derivative error $O(\Lambda^{-1})$. These bounds do not give arbitrary derivative estimates for the polar positive factor or its phase.

On the balanced family (11), with fixed nonzero mixing, the mass converges in C^2 but its Hessian has no uniform positive Hölder bound. A shrinking-scale profile also implies divergence of third derivatives at points approaching $\epsilon = 0$, and of the fourth derivative at $\epsilon = 0$. Thus the static cancellation limit does not have unrestricted smooth parameter dependence. The real phase combination of the primitive also has a negative ambient Levi direction. Section 8 therefore separates the positive limiting cohomology potential from a more restricted finite-scale positive construction. In that construction the quotient comparison is an exact scalar isometry. It is not a metric on the full space of arbitrary mixed complexes.

1.6. **Relation to earlier work.** In the zero-arrow case, the differential rescaling and cancellation mechanism already appear in [HKKP26, Example 4.1]. The motivating large-differential discussion in [HKKP26, §5.4] is a static comparison. Our results concern the actual coupled phase equation and prove that its solutions enter the required regime. The ordinary quiver metric flow studied in [HKKP23, §5] has a different vector field; its asymptotic results do not by themselves apply to (2).

Superconnection localization to cohomology and heat-transgression primitives have classical precedents. Bismut–Lott [BL95, Theorems 2.13 and 2.16] treat cohomology localization and transgression under a prescribed graded metric rescaling. Paradan–Vergne [PV09, §3.2, Proposition 4 in the preprint] give heat-decay estimates away from the noninvertibility locus for a prescribed superconnection rescaling. We use elementary finite-dimensional instances of these analytic ideas. The additional estimates here control the noncommuting polar vector field, including coupling to a surviving representation without a positive orbit gap. The quotient constructions in the last section use classical quiver and Kähler reduction [King94, Sj95, SL91]; their role is to identify precisely the geometry supplied by the explicit primitive.

Sections 2–3 establish the coefficient and polar estimates. Section 4 proves invertible-source cancellation, Section 5 proves mass increase, and Section 6 treats surviving source cohomology. Sections 7–8 concern static regularity and potentials.

2. The coefficient and the positive phase

We first compute the coefficient for (1), allowing an arbitrary rectangular differential D. Adjoint symbols always refer to the current Hermitian metrics. Put

$$B = D^\dagger D, \quad L = DD^\dagger, \quad C = \sum_j A_j^\dagger A_j, \quad E = DCD^\dagger, \quad P = e^{-B}, \quad Q = e^{-L}. \tag{12}$$

For a nonnegative Hermitian matrix B, define

$$R(B) = \int_0^1 e^{-sB}\, ds, \tag{13}$$

$$K_B(C) = \int_0^1 e^{-(1-s)B} C e^{-sB}\, ds, \tag{14}$$

$$J_L(E) = \int_{\substack{x,z\geq 0\\ x+z\leq 1}} e^{-xL} E e^{-zL}\, dx\, dz. \tag{15}$$

In particular $R(0) = 1$; no inverse of B is implicit in this notation.

Proposition 2.1. *The superconnection coefficient of* [HKKP26, §5.4] *on* (1)*, including its degree sign, is*

$$\begin{aligned} \Omega_V &= z_0 P + K_B(C), \\ \Omega_U &= J_L(E) - z_0 Q, \\ \Omega_W &= z_1 I_W - \sum_j A_j R(B) A_j^\dagger. \end{aligned} \tag{16}$$

If $n = \dim V$*,* $r = \dim U$ *and* $m_0 = \dim W$*, then*

$$\operatorname{tr}\Omega = (n-r)z_0 + m_0 z_1. \tag{17}$$

Moreover,

$$\operatorname{Im}\Omega_V = (\operatorname{Im} z_0)P, \qquad \operatorname{Im}(-\Omega_U) = (\operatorname{Im} z_0)Q, \qquad \operatorname{Im}\Omega_W = (\operatorname{Im} z_1) I_W. \tag{18}$$

Proof. We specify the graded conventions to fix the signs. In the quiver algebra of [HKKP26, Definition 5.13], an arrow $\alpha : s \to t$ and its adjoint have degrees 1 and -1, while the vertex elements e_i, f_i have degree zero. The nontrivial two-arrow products are $\alpha\alpha^* = \mathrm{i} f_t$ and $\alpha^*\alpha = -\mathrm{i} f_s$ for a matching arrow; cross-arrow products vanish. Products involving an f_i and an arrow, and products of two f_i's, vanish. Tensor multiplication has the Koszul sign

$$(x \otimes T)(y \otimes S) = (-1)^{|T||y|} xy \otimes TS.$$

The involution on an even endomorphism is the ordinary adjoint, and on an odd endomorphism it is i times that adjoint. The source superconnection and curvature are

$$\mathsf{A} = \sum_i e_i \otimes d_i + \sum_j \alpha_j \otimes A_j, \qquad F = (\mathsf{A} - \mathsf{A}^*)^2 + \mathrm{i}\sum_i f_i \otimes z_i.$$

The coefficient is $\Omega = (-1)^{\deg} \exp(-\mathrm{i}F)_f$, where the subscript means coefficient extraction after exponentiation.

Write $X = -\mathrm{i}F$. Its vertex part is $-B$ on V, $-L$ on U and zero on W. Its f coefficients before exponentiation are $C + z_0 I$ on V, $z_0 I$ on U, and $z_1 I - \sum_j A_j A_j^\dagger$ on W. Its mixed arrow terms are $-A_j D^\dagger : U \to W$ and $-\mathrm{i} D A_j^\dagger : W \to U$, with their respective algebra factors. Their paired f contribution is $-DCD^\dagger$ on U and $\sum_j A_j D^\dagger D A_j^\dagger$ on W.

The Duhamel expansion contributing to an f coefficient terminates at one f insertion or two arrow insertions. There are no mixed paths on V, giving $K_B(C) + z_0 P$. On U, the two-insertion term is $-J_L(E)$ before multiplication by the degree-one sign. On W the two-insertion contribution is

$$\sum_j \int_0^1 (1-s) A_j D^\dagger e^{-sL} D A_j^\dagger\, ds.$$

Using $D^\dagger e^{-sL} D = B e^{-sB}$ and

$$-I + \int_0^1 (1-s) B e^{-sB}\, ds = -R(B)$$

proves (16).

The substitutions $s \mapsto 1-s$ and $(x,z) \mapsto (z,x)$ show that $K_B(C)$ and $J_L(E)$ are Hermitian. This proves (18); neither integral operator is asserted to preserve positivity. Cyclicity of trace gives

$$\operatorname{tr} K_B(C) = \operatorname{tr}(CP), \qquad \operatorname{tr} J_L(E) = \operatorname{tr}\left(C \int_0^1 sBe^{-sB}\, ds\right) = \operatorname{tr}(C(R(B) - P)).$$

The arrow traces cancel. The nonzero spectra of B and L agree, so $\operatorname{tr} P - \operatorname{tr} Q = n - r$, proving (17). □

Lemma 2.2. *If $\operatorname{Im} T > 0$ and $T = U_T|T|$ is its polar decomposition, then every eigenvalue of U_T lies in the open upper semicircle. On a block of degree q, the matrix $\Phi = \operatorname{Arg} U_T - \pi q I$ is the unique Hermitian matrix with spectrum in $(-\pi q, \pi(1-q))$ such that $(-1)^q T e^{-\mathrm{i}\Phi}$ is positive definite.*

Proof. For a unit eigenvector v of U_T with eigenvalue $e^{\mathrm{i}\theta}$,

$$0 < \langle v, \operatorname{Im}(T) v\rangle = \sin\theta\ \langle v, |T| v\rangle\,.$$

Thus $0 < \theta < \pi$. With the stated branch, $(-1)^q T e^{-\mathrm{i}\Phi} = TU_T^\dagger = (TT^\dagger)^{1/2} > 0$. Conversely positivity identifies $e^{\mathrm{i}\Phi}$ as the polar unitary of $(-1)^q T$. Uniqueness of polar decomposition and of the branch give uniqueness of Φ. □

For the trace functional $\mathcal{L}(a) = \operatorname{tr}(\Omega a)$, the positivity requirement $\mathcal{L}(e^{-\mathrm{i}\Phi} a^\dagger a) \geq 0$ for all a is exactly the positivity of $\Omega e^{-\mathrm{i}\Phi}$. Lemma 2.2 therefore identifies the phase required by [HKKP26, §4.2] without a normality hypothesis. The degree convention also respects shifts: the shift [1] moves degree q to $q-1$ and increases the phase by π. Orthogonal direct sums give direct sums of coefficients and phases, and additive mass.

Example 2.3. Take $V = U = W = \mathbb{C}^2$, $z_0 = z_1 = \mathrm{i}$, $D = \operatorname{diag}(1, \sqrt{2})$ and $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Writing $p = e^{-1}$ and $q = e^{-2}$, formula (16) gives

$$\Omega_V = \begin{pmatrix} (1+\mathrm{i})p & p-q \\ p-q & (2+\mathrm{i})q \end{pmatrix}, \qquad [\Omega_V, \Omega_V^\dagger] = 2\mathrm{i}(p-q)^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0.$$

Thus coefficient extraction does not preserve the normality of an element in the graded algebra. This example contradicts the normality clause of [HKKP26, Lemma 5.16], while retaining the strict imaginary-part property and the central-charge trace.

2.1. **Variation and global existence.** At an instant choose orthonormal frames. Along the metric phase flow, the fixed maps have moving-frame derivatives

$$\dot{D} = \tfrac{1}{2}(D\Phi_V - \Phi_U D), \qquad \dot{A}_j = \tfrac{1}{2}(A_j \Phi_V - \Phi_W A_j). \tag{19}$$

For example, a variation $\xi_q = h_q^{-1}\delta h_q$ gives $\delta D = (\xi_U D - D\xi_V)/2$ and $\delta A_j = (\xi_W A_j - A_j \xi_V)/2$ in such frames. Substituting $\xi = -\Phi$ gives (19).

All coefficient variations can be computed without denominators involving eigenvalue differences. Define block matrices

$$N_V = \begin{pmatrix} -B & C \\ 0 & -B \end{pmatrix}, \quad N_U = \begin{pmatrix} -L & E & 0 \\ 0 & 0 & I \\ 0 & 0 & -L \end{pmatrix}, \quad N_R = \begin{pmatrix} -B & I \\ 0 & 0 \end{pmatrix}.$$

Then $(e^{N_V})_{11} = P$, $(e^{N_V})_{12} = K_B(C)$, $(e^{N_U})_{11} = Q$, $(e^{N_U})_{13} = J_L(E)$ and $(e^{N_R})_{12} = R(B)$. The Fréchet derivative

$$\mathrm{D}\exp_N(\dot N) = \int_0^1 e^{(1-s)N}\dot N e^{sN}\,ds \tag{20}$$

and the product rule therefore determine every $\dot\Omega_q$. In particular,

$$\dot\Omega_W = -\sum_j (\dot A_j R A_j^\dagger + A_j \dot R A_j^\dagger + A_j R \dot A_j^\dagger).$$

Lemma 2.4. *Let $\Omega_q = U_q|\Omega_q|$ on each homogeneous block. Then*

$$\dot m = \sum_q \mathrm{Re}\,\mathrm{tr}(U_q^\dagger \dot\Omega_q). \tag{21}$$

The coefficients and phase define a smooth metric vector field, whose solutions exist for all $t \geq 0$.

Proof. For an invertible matrix $\Omega = UH$, differentiation of $H^2 = \Omega^\dagger\Omega$ and taking trace after multiplication by H^{-1} gives $\mathrm{tr}\,\dot H = \mathrm{Re}\,\mathrm{tr}(U^\dagger\dot\Omega)$. This proves (21). The polar decomposition and its upper-semicircle logarithm are smooth on the domain in Lemma 2.2. If a phase derivative is needed, put $S = U^\dagger\dot U$ and $Z = U^\dagger\dot\Omega$; it is determined by

$$HS + SH = Z - Z^\dagger, \qquad S^\dagger = -S,$$

followed by differentiation of the chosen unitary logarithm. The Sylvester operator is invertible because $H > 0$.

The branches give $0 < \Phi_V, \Phi_W < \pi I$ and $-\pi I < \Phi_U < 0$. In fixed coordinates the metric forms satisfy

$$e^{-\pi t}h_V(0) \leq h_V(t) \leq h_V(0), \qquad h_U(0) \leq h_U(t) \leq e^{\pi t}h_U(0),$$

and the same degree-zero bounds hold on W. On every finite time interval these inequalities confine the metrics to a compact subset of the positive cone. The smooth ordinary differential equation therefore extends past every finite time. □

3. Polar estimates uniform under degeneration of a complementary block

A bound for the full polar factor based on the least singular values, such as [Li95], can deteriorate when a singular value approaches zero. Polar approximation and trace-norm variational principles are classical [High86, Lew95]. The acyclic coefficient has exponentially small singular values, so such a bound would be unsuitable here. The following estimates use only the surviving block or one specified reference direction. All Hilbert–Schmidt norms in this section are for fixed orthonormal frames.

Lemma 3.1. *Let*

$$T = \begin{pmatrix} A & E \\ E^\dagger & C \end{pmatrix}, \qquad \mathrm{Im}\,T > 0,$$

and suppose that the least singular value of A is at least $b > 0$. Let $U = \mathrm{polar}(T)$, $U_A = \mathrm{polar}(A)$ and $e = \|E\|_{\mathrm{HS}}$. Then

$$\|U_{12}\|_{\mathrm{HS}} = \|U_{21}\|_{\mathrm{HS}} \leq 4e/b, \qquad \|U_{11} - U_A\|_{\mathrm{HS}} \leq 2e/b. \tag{22}$$

Writing $\Psi = \mathrm{Arg}\,U$ and $\psi_A = \mathrm{Arg}\,U_A$, one has

$$\|\Psi_{12}\|_{\mathrm{HS}} \leq 2\pi e/b. \tag{23}$$

If $e/b \le 1/8$, then also

$$\|\Psi_{11} - \psi_A\|_{\mathrm{HS}} \le 4\pi e/b. \tag{24}$$

There is no lower bound on the singular values of C or T in these estimates.

Proof. Put $v = \|U_{12}\|_{\mathrm{HS}} = \|U_{21}\|_{\mathrm{HS}}$, where equality follows by taking traces in the unitary row and column identities. Pinching is contractive for the trace norm, so $\|T\|_1 \ge \|A\|_1 + \|C\|_1$. In the maximizing pairing $\|T\|_1 = \operatorname{Re}\operatorname{tr}(U^\dagger T)$, the C contribution is at most $\|C\|_1$ because U_{22} is a contraction, and the two off-diagonal contributions have sum at most $2ve$. If $A = U_A|A|$ and $d = \|U_{11} - U_A\|_{\mathrm{HS}}$, the deficit in the A contribution is

$$\operatorname{tr}\big(|A|[I - \operatorname{Re}(U_{11}^\dagger U_A)]\big) \ge \frac{b}{2}(d^2 + v^2).$$

The bracket is positive semidefinite, and its trace is $(d^2 + v^2)/2$ by the first-column unitary identity. It follows that $d^2 + (v - 2e/b)^2 \le (2e/b)^2$, proving (22).

For $\alpha, \beta \in [0, \pi]$,

$$|\alpha - \beta| \le \frac{\pi}{2}|e^{\mathrm{i}\alpha} - e^{\mathrm{i}\beta}|. \tag{25}$$

Expanding the commutators of U and $\operatorname{Arg} U$ with the first-block orthogonal projection in an eigenbasis of U proves $\|\Psi_{12}\|_{\mathrm{HS}} \le (\pi/2)v$, hence (23). The same scalar inequality, expanded in eigenbases of two normal matrices, gives the Hilbert–Schmidt estimate

$$\|\operatorname{Arg} U - \operatorname{Arg} U_0\|_{\mathrm{HS}} \le \frac{\pi}{2}\,\|U - U_0\|_{\mathrm{HS}} \tag{26}$$

whenever both unitary spectra lie in the closed upper semicircle.

By Lemma 2.2, $\operatorname{Im} U > 0$; its compression U_{22} is invertible and its polar unitary V also has upper-semicircle spectrum. Since $U_{22}^\dagger U_{22} = I - U_{12}^\dagger U_{12}$,

$$\|U_{22} - V\|_{\mathrm{HS}} = \Big\|(I - U_{12}^\dagger U_{12})^{1/2} - I\Big\|_{\mathrm{HS}} \le \Big\|U_{12}^\dagger U_{12}\Big\|_{\mathrm{HS}} \le v^2.$$

Take $U_0 = \operatorname{diag}(U_A, V)$. If $e/b \le 1/8$, then $v \le 1/2$ and

$$\|U - U_0\|_{\mathrm{HS}}^2 = d^2 + 2v^2 + \|U_{22} - V\|_{\mathrm{HS}}^2 \le 40(e/b)^2 < 64(e/b)^2.$$

Equation (26) proves (24). □

Lemma 3.2. *Let $T_0 = U_0 H_0$ be an invertible reference matrix in polar form, and suppose that $T = T_0 + E$ has $\operatorname{Im} T > 0$ with polar unitary U. If v is a unit vector satisfying $H_0 v = pv$ and $U_0 v = e^{\mathrm{i}\theta} v$, where $p > 0$ and $0 < \theta < \pi$, then*

$$|\langle v, (\operatorname{Arg} U - \theta I)v\rangle| \le \pi\sqrt{\frac{\|E\|_1}{p}}. \tag{27}$$

Proof. The trace-norm variational principle and triangle inequality imply

$$\operatorname{tr}\big(H_0[I - \operatorname{Re}(U^\dagger U_0)]\big) \le 2\,\|E\|_1\,.$$

The bracket is positive semidefinite, and $H_0 - pvv^\dagger \ge 0$. Its expectation on v is $1 - \operatorname{Re}(e^{-\mathrm{i}\theta}\langle v, Uv\rangle)$. Writing $\Psi = \operatorname{Arg} U$ and using $1 - \cos x \ge 2x^2/\pi^2$ for $|x| \le \pi$ therefore gives

$$\frac{2p}{\pi^2}\,\big\langle v, (\Psi - \theta I)^2 v\big\rangle \le 2\,\|E\|_1\,.$$

Cauchy–Schwarz proves the assertion. □

The frequently used special case is $T = zP + E$, $P > 0$, $\operatorname{Im} z > 0$. For an eigenvector $Pv = p_0 v$, (27) becomes

$$|\langle v, (\operatorname{Arg}\operatorname{polar}(T) - \arg z)v\rangle| \le \pi\sqrt{\frac{\|E\|_1}{|z|p_0}}. \tag{28}$$

The estimate is useful even when no uniform lower bound for all eigenvalues of P is available. Later it is applied to a direction realizing the least singular value of the differential, for which p_0 is known exactly.

4. Cancellation for an invertible differential

We prove Theorem 1.2. The surviving cohomology is W in degree zero. Put $a = \arg z_1 \in (0, \pi)$ and, in moving orthonormal frames, define

$$X_j = A_j D^{-1}, \qquad H_X = \sum_j X_j^\dagger X_j, \qquad s = \operatorname{tr} H_X, \qquad \lambda = \lambda_{\min}(D^\dagger D). \tag{29}$$

Lemma 4.1. *Along the phase flow,*

$$\dot{X}_j = \tfrac{1}{2}(X_j\Phi_U - \Phi_W X_j), \qquad s(t) \le s(0)e^{-at}. \tag{30}$$

Proof. Differentiate $A_j D^{-1}$ using (19); the source phase cancels. Since $A_j = X_j D$,

$$\Omega_W = z_1 I - \sum_j X_j(I - e^{-L})X_j^\dagger. \tag{31}$$

The subtracted matrix is positive semidefinite. This target block is normal, with scalar strictly positive imaginary part, and $\arg(z_1 - r) \ge \arg z_1$ for $r \ge 0$. Thus $\Phi_W \ge aI$, whereas $\Phi_U < 0$ on the degree-one branch. Consequently

$$\dot{s} = \sum_j \operatorname{tr}(X_j^\dagger X_j \Phi_U) - \operatorname{tr}\left(\sum_j X_j X_j^\dagger \Phi_W\right) \le -as. \qquad \square$$

Lemma 4.2. *Let $\kappa = \pi/(2e)$. For an invertible differential of rank n, the coefficient terms in (16) satisfy*

$$\|K_B(C)\|_1 \le n\kappa s, \qquad \|J_L(E)\|_1 \le ns, \qquad \|\Omega_W - z_1 I\|_1 \le s. \tag{32}$$

The constants do not depend on any singular-value ratio of D.

Proof. Write $D = U_D B^{1/2}$ and $S = U_D^\dagger H_X U_D$. Then $C = B^{1/2} S B^{1/2}$, $S \ge 0$ and $\operatorname{tr} S = s$. For $u > 0$, $\left\|B^{1/2}e^{-uB}\right\| \le (2eu)^{-1/2}$. Integration in (14), using $\int_0^1 [s(1-s)]^{-1/2} ds = \pi$, gives $\|K_B(C)\| \le \kappa s$, hence the first trace-norm bound.

On U, $E = LH_X L$. In an eigenbasis of L with eigenvalues l_i,

$$(J_L(E))_{ij} = k_{ij}(H_X)_{ij}, \qquad k_{ij} = l_i l_j \int_{\substack{x,z\ge 0\\ x+z\le 1}} e^{-xl_i - zl_j}\, dx\, dz.$$

Enlarging the integration region to the nonnegative quadrant gives $0 \le k_{ij} \le 1$. Positivity of H_X implies $|(H_X)_{ij}| \le \sqrt{(H_X)_{ii}(H_X)_{jj}}$. Therefore

$$|\,\langle v, J_L(E)v\rangle\,| \le \left(\sum_i |v_i|\sqrt{(H_X)_{ii}}\right)^2 \le s\,\|v\|^2\,.$$

Since $J_L(E)$ is Hermitian, this proves the second estimate. Finally the positive semidefinite matrix subtracted in (31) has trace at most s. □

Lemma 4.3. *There is a constant C depending on the initial data and charges such that $\lambda(t) \geq at - C$ for every $t \geq 0$.*

Proof. The least eigenvalue λ is locally Lipschitz. At almost every time, choose a unit minimum eigenvector v realizing its derivative, and set $u = Dv/\sqrt{\lambda}$. Then $Lu = \lambda u$ and

$$\dot{\lambda} = \lambda\big(\langle v, \Phi_V v\rangle - \langle u, \Phi_U u\rangle\big) > 0. \tag{33}$$

The same lower estimates hold on every vector in a repeated minimum eigenspace, and hence for the corresponding one-sided eigenvalue derivatives. In particular $\lambda \geq \lambda_0 := \lambda(0) > 0$.

Apply (28) to $\Omega_V = z_0 e^{-B} + K_B(C)$ and $-\Omega_U = z_0 e^{-L} - J_L(E)$. On v and u the reference heat eigenvalue is $e^{-\lambda}$. The reference phases differ by π after the degree-one branch shift. Lemmas 4.1 and 4.2 yield

$$\dot{\lambda} \geq \lambda\left[\pi - C_0 e^{(\lambda - at)/2}\right], \qquad C_0 = \pi\sqrt{\frac{ns(0)}{|z_0|}}(\sqrt{\kappa} + 1). \tag{34}$$

Let $L_0 = \max\{0, 2\log(2C_0/\pi)\}$, taking $L_0 = 0$ when $C_0 = 0$. Whenever $\lambda \leq at - L_0$, the bracket in (34) is at least $\pi/2$.

For completeness, choose

$$M_0 = \max\{\lambda_0, 4a/\pi, 1\}, \quad T_0 = (M_0 + L_0)/a, \quad T_1 = T_0 + \frac{2}{\pi}\log(M_0/\lambda_0).$$

For $t \geq T_0$ while $\lambda \leq M_0$, $\dot{\lambda} \geq \pi\lambda/2$, so λ reaches M_0 by T_1. The line $b(t) = M_0 + a(t - T_1)$ satisfies $b(t) - at = M_0 - aT_1 \leq -L_0$. At a contact point $\lambda = b$ for $t \geq T_1$, $\dot{\lambda} \geq \pi b/2 \geq 2a > \dot{b}$. Scalar comparison gives $\lambda \geq b$ thereafter. Before T_1 we have $\lambda > 0$, so $\lambda(t) \geq at - aT_1$ for all $t \geq 0$. □

Proof of Theorem 1.2. Global existence is Lemma 2.4. The contraction and gap statements follow from Lemmas 4.1 and 4.3. The latter also gives $\|e^{-B}\| = \|e^{-L}\| \leq e^{aT_1}e^{-at}$. Thus

$$\begin{aligned} \|\Omega_V\|_1 + \|\Omega_U\|_1 &\leq \big(2n|z_0|e^{aT_1} + n(\kappa+1)s(0)\big)e^{-at}, \\ \|\Omega_W - z_1 I\|_1 &\leq s(0)e^{-at}. \end{aligned}$$

The central-charge identity is $\operatorname{tr}\Omega = m_0 z_1$. The triangle inequality gives the lower mass bound $m(h(t)) \geq m_0|z_1|$ and, with the last two estimates, the upper bound (5).

It remains to identify convergence of a metric in fixed coordinates. For $r \geq 0$,

$$0 \leq \arg(z_1 - r) - a = \int_0^r \frac{\operatorname{Im} z_1}{|z_1 - u|^2}\, du \leq \frac{r}{\operatorname{Im} z_1}.$$

Spectral calculus in (31) yields $0 \leq \Phi_W - aI \leq s(0)e^{-at}(\operatorname{Im} z_1)^{-1}I$. For $\widehat{h}_W = e^{at}h_W$, comparison of Hermitian forms gives

$$e^{-s(0)/(a\operatorname{Im} z_1)}h_W(0) \leq \widehat{h}_W(t) \leq h_W(0).$$

The normalized metric is decreasing, and hence has a positive definite limit. Its derivative in any fixed coordinate norm is $O(e^{-at})$, by the same comparison and the phase bound. Integration proves exponential convergence. □

Remark 4.4. Zero-dimensional cases have direct descriptions. If $V = U = 0$, the only coefficient is $z_1 I_W$ and $h_W(t) = e^{-at} h_W(0)$. If $W = 0$, all arrows vanish; writing $\theta_0 = \arg z_0$ gives $\Phi_V = \theta_0 I$ and $\Phi_U = (\theta_0 - \pi)I$. Consequently $D(t) = e^{\pi t/2} D(0)$ in moving frames and $m(h(t)) = 2|z_0| \operatorname{tr} e^{-e^{\pi t} D(0)^\dagger D(0)}$. Thus the acyclic mass decays faster than every fixed exponential.

5. A mass-increasing trajectory

We prove Theorem 1.1 by computing a positive one-sided limit of the exact mass derivative. All metric and phase conventions remain those of Section 2. For the initial matrices in that theorem, $D^\dagger D = \operatorname{diag}(1/25, 4)$ and $A^\dagger A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right)$ do not commute.

5.1. **The real limiting coefficient.** Put

$$b_1 = 1/25, \quad b_2 = 4, \quad p = e^{-b_1}, \quad q = e^{-b_2}, \quad R_i = \frac{1 - e^{-b_i}}{b_i}, \quad f_i = R_i - e^{-b_i},$$

and define

$$\ell = \frac{p - q}{b_2 - b_1}, \qquad s_* = \frac{\sqrt{b_1 b_2}(R_1 - R_2)}{b_2 - b_1}.$$

At $\epsilon = 0$ the coefficient blocks have the real symmetric limits

$$H = \begin{pmatrix} 2p & 2\ell \\ 2\ell & 5q \end{pmatrix}, \quad G = \begin{pmatrix} f_1 - p & 2s_* \\ 2s_* & 4f_2 - q \end{pmatrix}, \quad w = 4 - R_1 - 4R_2 = 25p + q - 22. \tag{35}$$

The elementary bounds below give

$$\begin{gathered} -0.05060 < \det H < -0.05059, \qquad -0.85991 < \det G < -0.85990, \\ 2.03805 < w < 2.03806. \end{gathered} \tag{36}$$

Thus H and G each have one negative and one positive eigenvalue. For a real symmetric 2 by 2 matrix M with this property,

$$P_-(M) = \frac{1}{2}\left(I - \frac{2M - (\operatorname{tr} M) I}{\sqrt{(\operatorname{tr} M)^2 - 4 \det M}} \right). \tag{37}$$

Write $P_V = P_-(H)$ and $P_U = P_-(G)$.

All limiting blocks are invertible. The polar unitary is continuous near them. On the degree-zero branch its eigenvalues approach 1 and -1 through the upper semicircle; the corresponding phases approach 0 and π. On the degree-one block apply the same observation to $-\Omega_U$ and subtract π. It follows that

$$\Phi_V \longrightarrow \pi P_V, \qquad \Phi_U \longrightarrow -\pi P_U, \qquad \Phi_W \longrightarrow 0 \quad (\epsilon \downarrow 0). \tag{38}$$

The argument is continuous on the closed upper semicircle with these endpoint values. This assertion does not use continuity across a branch cut from the lower semicircle.

5.2. **A finite real formula for the derivative.** Divide the limiting map velocities by π and denote this directional derivative by a prime. Equation (19) becomes

$$D' = \tfrac{1}{2}(D P_V + P_U D), \qquad A' = \tfrac{1}{2} A P_V. \tag{39}$$

Since the central-charge trace is constant, the trace-norm derivative formula gives

$$\lim_{\epsilon \downarrow 0} \frac{\dot{m}_\epsilon(0)}{\pi} = -2 \operatorname{tr}(P_V H' + P_U G'). \tag{40}$$

Indeed, at the real invertible blocks the polar unitaries are their sign matrices $I-2P_V$, $I-2P_U$, and 1. The identity contributions sum to the derivative of (17), which vanishes.

Here is a scalar-arithmetic evaluation of the right side of (40). It avoids differentiating eigenvectors of the coefficient. Let $B=\operatorname{diag}(b_1,b_2)$ and $C=\left(\begin{smallmatrix}1&2\\2&4\end{smallmatrix}\right)$. Set

$$B'=\tfrac12(P_VB+BP_V)+DP_UD,\quad \gamma=\frac{B'_{12}}{b_2-b_1},\quad K=\begin{pmatrix}0&\gamma\\-\gamma&0\end{pmatrix}. \tag{41}$$

Use instantaneous singular-vector frames for D. In the source frame, $B'+[B,K]$ is diagonal and $b'_i=B'_{ii}$. The arrow Gram derivative is

$$\widetilde{C}'=\tfrac12(P_VC+CP_V)+[C,K]. \tag{42}$$

The corresponding target frame makes D diagonal to first order. Frame rotations do not change the traces $\operatorname{tr}(P_VH')$ and $\operatorname{tr}(P_UG')$, since the projections commute with the respective limiting blocks.

The remaining derivatives are

$$R'_i=-\frac{f_i}{b_i}b'_i,\qquad f'_i=\left(-\frac{f_i}{b_i}+e^{-b_i}\right)b'_i, \tag{43}$$

$$\ell'=\frac{(-pb'_1+qb'_2)(b_2-b_1)-(p-q)(b'_2-b'_1)}{(b_2-b_1)^2}, \tag{44}$$

$$\begin{aligned}s'_*&=\frac{s_*}{2}\left(\frac{b'_1}{b_1}+\frac{b'_2}{b_2}\right)\\&\quad+\sqrt{b_1b_2}\,\frac{(R'_1-R'_2)(b_2-b_1)-(R_1-R_2)(b'_2-b'_1)}{(b_2-b_1)^2}.\end{aligned} \tag{45}$$

In these frames,

$$H'_{ii}=e^{-b_i}\big(\widetilde{C}'_{ii}-(1+C_{ii})b'_i\big),\qquad H'_{12}=\ell\widetilde{C}'_{12}+C_{12}\ell', \tag{46}$$

$$G'_{ii}=f_i\widetilde{C}'_{ii}+\left[C_{ii}\left(-f_i/b_i+e^{-b_i}\right)+e^{-b_i}\right]b'_i, \tag{47}$$

$$G'_{12}=s_*\widetilde{C}'_{12}+C_{12}s'_*.$$

Equations (35)–(47) are a finite expression using rational operations, two exponentials and positive square roots.

5.3. **Strict positivity.** The two exponentials lie in the rational intervals

$$\begin{aligned}&0.96078943915232320<p<0.96078943915232322,\\&0.01831563888873417<q<0.01831563888873419.\end{aligned} \tag{48}$$

To verify them, use the Taylor sums of degrees 31 and 30 for $e^{-1/25}$, and of degrees 81 and 80 for e^{-4}. The odd and even sums bound the exponential from below and above, respectively, by the sign of the Taylor remainder. Each comparison with the displayed decimal endpoint is a rational comparison.

Substitution in the finite formulas above, with interval operations, gives the enclosures in Table 1. Each printed matrix entry differs from its exact value by less than 10^{-8}. These deliberately coarse enclosures suffice to give

$$0.2395<-2\operatorname{tr}(P_VH'+P_UG')<0.2397. \tag{49}$$

One may verify the table using only rational endpoints: every positive square root can be enclosed by rational lower and upper bounds whose squares enclose its argument. Thus

the sign calculation does not depend on a floating-point eigensolver or a finite-difference derivative.[1]

TABLE 1. Enclosures for the sign calculation. For each symmetric matrix, the entries are listed in the order $(11, 12, 22)$; each entry has error less than 10^{-8}.

Matrix	11	12	22
H	1.92157888	0.47599687	0.09157819
G	−0.94131486	0.14845312	0.89010617
P_V	0.05643063	−0.23075142	0.94356937
P_U	0.99355619	−0.08001432	0.00644381
H'	0.01353176	−0.24643619	−0.29654199
G'	0.06216737	0.10644365	0.12306918

Proof of Theorem 1.1. The coefficients, their directional derivatives, and the polar unitaries extend continuously to the invertible real limiting blocks. Equation (38) gives convergence of the actual flow directions from $\epsilon > 0$. The trace-norm derivative is consequently continuous in this one-sided limit. Equations (40) and (49) show that the limit is positive. It follows that $\dot{m}_\epsilon(0) > 0$ for all sufficiently small $\epsilon > 0$, when both central charges are admissible. □

The example disproves universal mass dissipation for the specified positive phase, rather than merely exposing a failure of normality. It does not preclude eventual cancellation, which Theorem 1.2 proves for these same complexes. No change of vector field is used in that theorem.

6. Uniform cancellation with a surviving source

We prove Theorems 1.3 and 1.4. The extra source cohomology prevents direct use of (30). The block estimate of Section 3 controls the resulting term without an inverse of the surviving orbit operator.

6.1. **Equations in an adapted orthonormal splitting.** For any surjective D, choose orthonormal frames adapted to $V = \ker D \oplus (\ker D)^\perp$, and write

$$D = [0, F], \qquad A_j = [H_j, T_j], \qquad \Phi_V = \begin{pmatrix} \Phi_{HH} & \Phi_{HA} \\ \Phi_{AH} & \Phi_{AA} \end{pmatrix}.$$

Add to the source frame velocity the skew-Hermitian generator

$$K_V = \frac{1}{2} \begin{pmatrix} 0 & \Phi_{HA} \\ -\Phi_{AH} & 0 \end{pmatrix}.$$

The off-diagonal blocks have sizes n_s by r and r by n_s; no equality of their sizes is needed. The kernel component of $\dot{D}$ remains zero: its unrotated term $F\Phi_{AH}/2$ is canceled by $F(K_V)_{AH}$.

[1]The source distribution includes two independent interval computations: `figures/code/compute05_interval.py` and `figures/code/compute05_independent.py`. The latter uses only integer interval arithmetic from the Python standard library. They enclose the one-sided limiting derivative, not a specified positive value of ϵ.

Equation (19) then becomes

$$\dot F = \tfrac12(F\Phi_{AA} - \Phi_U F), \qquad \dot H_j = \tfrac12(H_j\Phi_{HH} - \Phi_W H_j), \tag{50}$$

$$\dot T_j = H_j\Phi_{HA} + \tfrac12(T_j\Phi_{AA} - \Phi_W T_j). \tag{51}$$

In particular, with $X_j = T_jF^{-1}$,

$$\dot X_j = \tfrac12(X_j\Phi_U - \Phi_W X_j) + H_j\Phi_{HA}F^{-1}. \tag{52}$$

The second term is the coupling absent from the invertible-source argument.

For this subsection put $B = F^\dagger F$, $L = FF^\dagger$, $C_T = \sum_j T_j^\dagger T_j$, $C_s = \sum_j H_j^\dagger H_j$ and $C_w = \sum_j H_jH_j^\dagger$. The exact coefficients are

$$\begin{aligned}
\Omega_V &= \begin{pmatrix} z_0 I + C_s & E_R \\ E_R^\dagger & z_0 e^{-B} + K_B(C_T) \end{pmatrix}, \qquad E_R = \sum_j H_j^\dagger T_j R(B), \\
\Omega_U &= J_L(FC_TF^\dagger) - z_0 e^{-L}, \\
\Omega_W &= z_1 I - C_w - Y, \qquad Y = \sum_j X_j(I - e^{-L})X_j^\dagger \ge 0.
\end{aligned} \tag{53}$$

Define

$$q = \sum_j \|H_j\|_{\mathrm{HS}}^2, \qquad s = \sum_j \|X_j\|_{\mathrm{HS}}^2, \qquad \lambda = \lambda_{\min}(F^\dagger F). \tag{54}$$

Since $\|FR(B)\| \le \lambda^{-1/2}$, Cauchy–Schwarz gives

$$\|E_R\|_{\mathrm{HS}} \le \sqrt{qs/\lambda}. \tag{55}$$

Also $\|Y\|_{\mathrm{HS}} \le \operatorname{tr} Y \le s$. The source compression Φ_{AA} is positive and Φ_U is negative, so the least-singular-value calculation (33), now applied to F, shows that λ is nondecreasing.

6.2. **Uniform bounds for the surviving arrows.** Allow arbitrary surviving tuples and charges in the strict upper half-plane. Put $b_i = \operatorname{Im} z_i > 0$, $a = \arg z_1 \in (0,\pi)$ and define the ordinary cohomology phases

$$\phi_s = \arg(z_0 I + C_s), \qquad \phi_w = \arg(z_1 I - C_w), \qquad \delta_s = \Phi_{HH} - \phi_s, \quad \delta_w = \Phi_W - \phi_w.$$

These ordinary coefficients are normal. Their source singular values are at least b_0. Lemma 3.1 gives

$$\|\delta_s\|_{\mathrm{HS}} \le \frac{4\pi}{b_0}\sqrt{qs/\lambda}, \qquad \|\delta_w\|_{\mathrm{HS}} \le s/b_1 \tag{56}$$

when $\sqrt{qs/\lambda}/b_0 \le 1/8$. For the target inequality, the scalar function $g(x) = \arg(z_1 - x)$ has derivative at most $1/b_1$. Expanding two Hermitian matrices in their separate eigenbases shows $\|g(C_w + Y) - g(C_w)\|_{\mathrm{HS}} \le \|Y\|_{\mathrm{HS}}/b_1$. No operator monotonicity assertion is needed.

The unconditional off-diagonal estimate $\|\Phi_{HA}\|_{\mathrm{HS}} \le (2\pi/b_0)\sqrt{qs/\lambda}$, together with $\Phi_W \ge aI$, gives from (52)

$$s' \le \left(-a + \frac{4\pi q}{b_0\lambda}\right) s. \tag{57}$$

To bound q, use the ordinary cohomology mass

$$\mathcal{M}_H = \|z_0 I + C_s\|_1 + \|z_1 I - C_w\|_1. \tag{58}$$

The scalar inequalities $|z_0 + x| \geq x + \operatorname{Re} z_0$ and $|z_1 - x| \geq x - \operatorname{Re} z_1$, together with $\operatorname{tr} C_s = \operatorname{tr} C_w = q$, imply

$$q \leq \tfrac{1}{2}(\mathcal{M}_H - n_s \operatorname{Re} z_0 + n_w \operatorname{Re} z_1). \tag{59}$$

For the ordinary degree-zero phase equation, its derivative in source and target phase eigenbases is

$$\sum_{j,i,l} |(H_j)_{li}|^2 (\cos\alpha_i - \cos\beta_l)(\alpha_i - \beta_l) \leq 0,$$

because cosine decreases on $(0, \pi)$. Equivalently, its arrow gradient is $2(H_j \cos\phi_s - \cos\phi_w H_j)$, with total Hilbert–Schmidt norm at most $4\sqrt{q}$. The additional arrow velocity in (50) is $(H_j\delta_s - \delta_w H_j)/2$, so

$$\mathcal{M}'_H \leq 2q(\|\delta_s\|_{\mathrm{HS}} + \|\delta_w\|_{\mathrm{HS}}). \tag{60}$$

Only this auxiliary ordinary mass, not the full complex mass, has been used in a dissipation inequality.

Let $M_0 = \mathcal{M}_H(0)$ and put

$$Q = \tfrac{1}{2}(M_0 - n_s \operatorname{Re} z_0 + n_w \operatorname{Re} z_1) + 1. \tag{61}$$

Then $Q \geq q(0) + 1 \geq 1$.

Proposition 6.1. *Suppose $\lambda_0 = \sigma_{\min}(F_0)^2$ satisfies*

$$\lambda_0 \geq L_* := \max\left\{1, \frac{8\pi Q}{a b_0}, \frac{8\sqrt{Q}M}{b_0}, \frac{64\pi Q^{3/2} M/b_0 + 8QM^2/b_1}{a}\right\}. \tag{62}$$

For all $t \geq 0$, one has $q(t) \leq Q - 3/4$ and

$$s(t) \leq \frac{M^2}{\lambda_0} e^{-at/2}, \tag{63}$$

$$\|\delta_s\|_{\mathrm{HS}} \leq \frac{4\pi\sqrt{Q}M}{b_0\lambda_0} e^{-at/4}, \qquad \|\delta_w\|_{\mathrm{HS}} \leq \frac{M^2}{b_1\lambda_0} e^{-at/2}. \tag{64}$$

Proof. Initially $q \leq Q - 1$ and $s \leq M^2/\lambda_0$, since $\left\|F_0^{-1}\right\| = \lambda_0^{-1/2}$. While $q \leq Q$, monotonicity of λ and (62) give $s' \leq -as/2$. This proves (63) on that interval and ensures $\sqrt{qs/\lambda}/b_0 \leq 1/8$. Equations (56) give (64). Integrating (60) yields

$$\mathcal{M}_H(t) \leq M_0 + \frac{32\pi Q^{3/2} M/b_0 + 4QM^2/b_1}{a\lambda_0} \leq M_0 + \tfrac{1}{2}.$$

Equation (59) gives $q \leq Q - 3/4 < Q$, closing the bootstrap by continuity. Global existence extends the bounds to all $t \geq 0$. □

Take $\Lambda_* = \sqrt{L_*}$. If $q(0) \leq R$, the triangle inequality gives $M_0 \leq 2R + n_s|z_0| + n_w|z_1|$. The same proof therefore works with Q replaced by the uniform upper bound

$$Q_R = R + \tfrac{1}{2}\left[n_s(|z_0| - \operatorname{Re} z_0) + n_w(|z_1| + \operatorname{Re} z_1)\right] + 1. \tag{65}$$

This gives a threshold uniform on initial surviving-arrow norm balls, independent of acyclic rank, arrow count and the largest singular value of F_0. Charge dependence is explicit in (62); no uniformity near the boundary of the upper half-plane is asserted.

6.3. **Growth and cancellation of the acyclic sector.** Put $\sigma = a/4$ and $\kappa = \pi/(2e)$. For the trace-norm constants put

$$d = \min(n_s, r), \qquad A_0 = 2\sqrt{dQ}M + r\kappa M^2. \tag{66}$$

Compare the source coefficient in (53) with

$$\Omega_{V,0} = \operatorname{diag}(z_0 I + C_s, z_0 e^{-B}).$$

Its error has trace norm at most

$$\|\Omega_V - \Omega_{V,0}\|_1 \le \frac{A_0}{\lambda_0} e^{-\sigma t}. \tag{67}$$

Indeed the two off-diagonal blocks contribute at most $2\sqrt{d}\,\|E_R\|_{\mathrm{HS}}$, since each has rank at most d. Lemma 4.2, with rank r, applies to $T_j = X_j F$ on the acyclic block and gives $\|K_B(C_T)\|_1 \le r\kappa s$. Similarly the degree-one error relative to $-z_0 e^{-L}$ is at most $rM^2\lambda_0^{-1}e^{-at/2}$.

Apply Lemma 3.2 to a vector in the acyclic source block realizing λ, and to its image under $F/\sqrt{\lambda}$. The reference singular value is $|z_0|e^{-\lambda}$ in both cases. The reference phase on that source direction is $\arg z_0$, regardless of the phase on the surviving source block. Consequently

$$\lambda' \ge \lambda\left[\pi - C_1 e^{(\lambda-\sigma t)/2}\right], \tag{68}$$

where the following constant has no condition-number dependence:

$$C_1 = \frac{\pi}{\sqrt{|z_0|}}\left(\sqrt{A_0} + \sqrt{r}M\right).$$

We used $\lambda_0 \ge 1$ to discard a favorable factor. The same estimate holds for every vector in a repeated least eigenspace, so it also controls the one-sided derivatives of λ. Set $L_0 = \max\{0, 2\log(2C_1/\pi)\}$, with $L_0 = 0$ if $C_1 = 0$. Because $\lambda_0 \ge 1$, the line $\sigma t - L_0$ lies below λ up to time $(1 + L_0)/\sigma$. At a later contact point, (68) gives $\lambda' \ge \pi\lambda/2 \ge \pi/2 > 2\sigma = a/2$. The same barrier comparison as before gives

$$\lambda(t) \ge \max\{\lambda_0, \sigma t - L_0\}. \tag{69}$$

In particular,

$$\left\|e^{-B}\right\| = \left\|e^{-L}\right\| \le \min(e^{-\lambda_0}, e^{L_0}e^{-\sigma t}) \le \frac{2e^{L_0/2}}{e\lambda_0} e^{-\sigma t/2}. \tag{70}$$

For the last step, take the geometric mean and use $\lambda_0 e^{-\lambda_0/2} \le 2/e$.

Define the instantaneous cohomology coefficient to have blocks $z_0 I + C_s$ and $z_1 I - C_w$. Equations (53), (63), (67) and (70) show that

$$C_{\mathrm{can}} = 2\left[A_0 + (r+1)M^2 + \frac{4r|z_0|}{e} e^{L_0/2}\right] \tag{71}$$

satisfies

$$\|\Omega - (\Omega_H \oplus 0_{\mathrm{ac}})\|_1 \le \tfrac{1}{2} C_{\mathrm{can}} \lambda_0^{-1} e^{-at/8}.$$

Here the two acyclic heat blocks each have rank r, and the target error has trace norm at most s. These account for every term in (71). The trace norm is 1-Lipschitz with respect to itself, so the same bound holds for $|m - \mathcal{M}_H|$. Adding the bounds and using $\lambda_0 \ge \Lambda^2$ proves the quantitative cancellation assertion in Theorem 1.3. Taking $C = \max\{C_{\mathrm{can}}, L_0\}$ supplies one constant for its growth and cancellation conclusions. All estimates use tuple norms and only the least acyclic singular value; no stability hypothesis has entered.

6.4. **Relaxation from finite initial energy.** For the rest of this section assume $n_s = n_w = n$ and $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$, so $a = \pi/4$ and $b_0 = b_1 = 1$. The surviving tuple need not be initially balanced. In fixed holomorphic coordinates, $\ker D$ is fixed and $A_j|_{\ker D} = B_j$. Its induced metrics are the restriction of h_V and the metric h_W. The restriction's metric velocity is $-\Phi_{HH}$, as represented by (50).

Multiply both induced metrics by $e^{\pi t/2}$ and call them h_s, h_w. This leaves the arrow norms unchanged. The common scalar contributions to the logarithmic determinants cancel because the surviving dimensions agree. Using the initial orthonormal determinant volume forms, define on all positive metric pairs

$$\mathcal{K} = q + \log\det h_s - \log\det h_w, \qquad \mu_s = C_s - I, \quad \mu_w = C_w - I. \tag{72}$$

Assume $\mathcal{K}_* := \inf \mathcal{K} > -\infty$ and put $E_0 = \mathcal{K}(0) - \mathcal{K}_*$. The infimum need not be attained.

The normalized metric velocities are

$$\xi_s = \arctan \mu_s - \delta_s, \qquad \xi_w = -\arctan \mu_w - \delta_w.$$

Variation of (72) gives

$$\mathcal{K}' = -\operatorname{tr}(\mu_s \arctan \mu_s + \mu_w \arctan \mu_w) + \operatorname{tr}(\mu_s \delta_s - \mu_w \delta_w).$$

Here $M_0 \geq 2n$, so $Q \geq 2n + 1 \geq 3$. Since $q \leq Q$, the moment spectra lie in $[-1, Q-1]$. Set $J_Q = 1 + (Q-1)^2$. Integrating $(\arctan x)' = (1+x^2)^{-1}$ from zero gives $x \arctan x \geq x^2/J_Q$ on this interval. With squared norms summing source and target Hilbert–Schmidt squares, Young's inequality yields

$$\mathcal{K}' \leq -\frac{1}{2J_Q} \|\mu\|^2 + \frac{J_Q}{2} \|\delta\|^2 . \tag{73}$$

By (64),

$$\int_0^\infty \|\delta(t)\|^2 \, dt \leq \frac{32\pi^2 Q M^2 + M^4}{a\lambda_0^2}.$$

Consequently, with

$$C_{\mathrm{en}}(Q, M) = \frac{J_Q}{2a}(32\pi^2 Q M^2 + M^4), \tag{74}$$

one has

$$0 \leq \mathcal{K}(t) - \mathcal{K}_*,$$

$$\mathcal{K}(t) - \mathcal{K}_* + \frac{1}{2J_Q} \int_0^t \|\mu(u)\|^2 \, du \leq E_0 + C_{\mathrm{en}}(Q, M)\lambda_0^{-2} \leq E_0 + C_{\mathrm{en}}(Q, M)\Lambda^{-4}. \tag{75}$$

Only the lower bound for $\mathcal{K}$ is used. A bound on E_0 is needed for a uniform numerical energy bound across initial metrics; its finiteness suffices for the following qualitative conclusion.

The adapted equations and phase bounds give

$$\sum_j \left\| \dot{H}_j \right\|_{\mathrm{HS}}^2 \leq \pi^2 q, \qquad \|\dot{\mu}_s\|_{\mathrm{HS}}, \ \|\dot{\mu}_w\|_{\mathrm{HS}} \leq 2\pi q.$$

Thus μ is uniformly continuous, independently of k. An integrable nonnegative squared norm with this property tends to zero: otherwise disjoint intervals of a fixed positive length would each contribute a fixed positive amount to its integral. It follows that $\mu_s, \mu_w \to 0$, hence $\mathcal{M}_H \to 2n$. The cancellation estimate then gives $m(h(t)) \to 2n$. This proves Theorem 1.4. When $M = 0$, the forcing δ vanishes, but μ need not vanish at finite times for nonbalanced initial metrics.

6.5. **Balanced minima and an unattained infimum.** Suppose (h_s^*, h_w^*) is a balanced metric pair. In orthonormal frames for that pair, a geodesic of positive metrics has the form $(e^{\tau S_s}, e^{\tau S_w})$. After diagonalizing S_s, S_w, the squared arrow norm is a sum of nonnegative multiples of $e^{\tau(\lambda_w-\lambda_s)}$; the logarithmic terms are affine. Thus $\mathcal{K}$ is convex along the geodesic, and balance makes its derivative zero at $\tau = 0$. Every positive pair is the other endpoint of such a geodesic, so this balanced pair attains the global minimum

$$\mathcal{K}_* = n + \log\det h_s^* - \log\det h_w^*.$$

In particular, initially balanced orthonormal metrics give $M_0 = 2n$, $Q = 2n+1$, $J_Q = 1+4n^2$ and $E_0 = 0$. Equations (62) and (75) recover Corollary 1.5, with threshold depending only on n, M. No strict convexity or positive stabilizer gap is required.

The finite-infimum hypothesis also allows nonattainment. Take $B_1 = I_2$ and $B_2 = e_{12}$, the matrix with a single nonzero entry 1 in position $(1,2)$. For arbitrary positive metrics put $X = h_s^{-1/2} h_w h_s^{-1/2}$. Then

$$\mathcal{K} = \operatorname{tr} X - \log\det X + \|B_2\|^2_{\mathrm{HS},h} > 2,$$

since $x - \log x \geq 1$ and $B_2 \neq 0$. Along $h_s = h_w = \operatorname{diag}(e^{-t}, e^t)$, however, $\mathcal{K} = 2 + e^{-2t}$. Its infimum is 2 and is not attained. Theorem 1.4 applies to this surviving representation when the initial acyclic scale is sufficiently large.

Cancellation alone cannot imply relaxation to balance: for zero surviving arrows with these charges, $\mathcal{M}_H = 2\sqrt{2}\,n$ and $\mu_s = \mu_w = -I$, while $\mathcal{K}$ has no finite lower bound. The exponential comparison with instantaneous cohomology remains valid. Even under the finite-infimum hypothesis, (75) supplies neither a uniform final relaxation rate nor convergence of normalized surviving metrics. It is not an inverse estimate for the Hessian of $\mathcal{K}$ at a changing stabilizer.

7. Static cancellation, regularity and a metric primitive

The estimates in this section concern fixed external differential scales and varying arrow coordinates. They do not differentiate solutions of the phase equation with respect to their initial data.

7.1. **An exact primitive in metric directions.** Fix reference inner products and determinant volume forms on V, U, W, and represent positive metrics by h_V, h_U, h_W. The matrices in (12) are computed in orthonormal frames for those metrics. Define the entire function

$$\operatorname{Ein}(b) = \int_0^1 \frac{1 - e^{-sb}}{s}\, ds, \qquad b\operatorname{Ein}'(b) = 1 - e^{-b}. \tag{76}$$

Proposition 7.1. *For arbitrary rectangular D, the function*

$$\begin{aligned}\mathcal{S}_{\mathbb{C}}(h) = {} & z_0(\log\det h_V - \log\det h_U) + z_1 \log\det h_W \\ & + z_0 \operatorname{tr}\operatorname{Ein}(B) - \operatorname{tr}(CR(B))\end{aligned} \tag{77}$$

satisfies, for $\xi_q = h_q^{-1}\delta h_q$,

$$d\mathcal{S}_{\mathbb{C}}(\xi) = \operatorname{tr}(\Omega_V \xi_V) + \operatorname{tr}(\Omega_U \xi_U) + \operatorname{tr}(\Omega_W \xi_W). \tag{78}$$

Proof. Compute at orthonormal frames, where ξ_q is Hermitian and the map variations are those preceding (19). The Ein term has coefficients $-z_0(I-P)$ on V and $z_0(I-Q)$ on U. Adding the logarithmic terms gives $z_0 P$ and $-z_0 Q$.

The Fréchet derivative $\mathrm{D}R_B$ is self-adjoint in the trace pairing, since R is real analytic on Hermitian matrices. Variation of $-\operatorname{tr}(CR(B))$ on V gives

$$\tfrac{1}{2}\{C, R(B)\} + \tfrac{1}{2}\{B, \mathrm{D}R_B(C)\} = K_B(C).$$

To verify the identity, diagonalize B. Twice the multiplier on the left is

$$R(b_i) + R(b_j) + (b_i + b_j)\frac{R(b_i) - R(b_j)}{b_i - b_j} = 2\frac{e^{-b_j} - e^{-b_i}}{b_i - b_j},$$

with continuous values at repeated eigenvalues. On U the coefficient is $-D\,\mathrm{D}R_B(C)D^\dagger = J_L(DCD^\dagger)$, as follows by differentiating the integral defining R. On W it is $-\sum_j A_j R(B) A_j^\dagger$. These are exactly (16). □

Equation (78) specifies a primitive on the space of metrics of each fixed object. By itself it allows an additive constant depending on that object. Such dependence can change a transverse moduli-space Levi form. Whenever potentials on an arrow space are used below, their normalization is the explicit formula (77) at the declared reference metrics, with only the stated acyclic subtraction. Arbitrary object-dependent functions are not part of that normalization. Locally pluriharmonic additions do not change the compared forms.

7.2. **Nonscalar static cancellation and first derivatives of mass.** Fix reference metrics, an orthogonal decomposition $V = H \oplus C$, and an invertible map $F : C \to U$. Set $D = [0, F]$ and $A_j = [B_j, T_j]$. The differential F is fixed when the arrows vary. Write

$$P_F = F^\dagger F, \qquad L_F = FF^\dagger, \qquad \lambda = \sigma_{\min}(F)^2 \geq \Lambda^2 \geq 1.$$

The limiting cohomology coefficient Ω_H has blocks $z_0 I + \sum_j B_j^\dagger B_j$ and $z_1 I - \sum_j B_j B_j^\dagger$.

Use the Euclidean norm on the real and imaginary arrow entries, whose square is the sum of squared Hilbert–Schmidt norms. The C^ℓ norm uses the corresponding operator norms of real multilinear derivatives. No derivative in F or the reference metrics is included.

Proposition 7.2. *On every bounded arrow-coordinate ball, for fixed charges in the strict upper half-plane and every fixed integer $\ell \geq 0$,*

$$\|\Omega_F - (\Omega_H \oplus 0_{\mathrm{ac}})\|_{C^\ell} \leq C_\ell \Lambda^{-2}, \qquad \|m_F - m_H\|_{C^0} \leq C_0 \Lambda^{-2}. \tag{79}$$

Moreover,

$$\|dm_F - dm_H\|_{C^0} \leq C\Lambda^{-1}. \tag{80}$$

Constants may depend on the fixed dimensions, charges, ball radius and ℓ, but not on arrow count, the surviving stabilizer or the condition number of F.

Proof. The nonscalar blocks are given by (53), with $H_j = B_j$. Spectral calculus gives

$$\|R(P_F)\| \leq \lambda^{-1}, \qquad \left\|e^{-P_F}\right\| = \left\|e^{-L_F}\right\| \leq e^{-\lambda}, \qquad \|K_{P_F}(C)\|_{\mathrm{HS}} \leq e^{-\lambda}\,\|C\|_{\mathrm{HS}}\,.$$

For the degree-one term take a singular-value decomposition of F, with squared singular values $b_i \geq \lambda$. The multiplier on C_{ij} in $J_{L_F}(FCF^\dagger)$ is

$$\sqrt{b_i b_j}\int_{\substack{s,t\geq 0\\ s+t\leq 1}} e^{-sb_i - tb_j}\,ds\,dt \leq \frac{1}{\sqrt{b_i b_j}} \leq \lambda^{-1}.$$

This bounds its Hilbert–Schmidt operator norm independently of the largest singular value. All arrow expressions are quadratic. Tuple Cauchy–Schwarz bounds their first two derivatives independently of the number of arrows, and higher derivatives vanish. Norm conversion in

the fixed output dimensions proves the coefficient bound in (79), first with λ^{-1} in place of Λ^{-2}. The trace-norm triangle inequality gives the mass bound.

Every block is invertible at finite F: its signed imaginary part is a positive scalar times I, e^{-P_F} or e^{-L_F}. The trace norm $N(Z) = \|Z\|_1$ is smooth on invertible matrices and convex [Lew95]. Its first derivative satisfies $|DN_Z(H)| \le \|H\|_1$, while $D^2N_Z(H,H) \ge 0$. Along any real unit arrow direction,

$$\frac{d^2}{dt^2} N(\Omega(y+tv)) \ge -\left\|\Omega''(y+tv)\right\|_1 .$$

The second derivatives of the coefficients are uniformly bounded on a ball of radius $R+1$. The cohomology mass has a bounded Hessian there, because its blocks have uniformly positive imaginary parts. Thus $f_F = m_F - m_H$ satisfies $D^2 f_F \ge -L_R I$, uniformly in F. For $|y| \le R$, $|v| = 1$ and $0 < h \le 1$, integration gives

$$df_F(y)v \le \frac{f_F(y+hv) - f_F(y)}{h} + \frac{L_R h}{2} \le \frac{2C_0}{h\lambda} + \frac{L_R h}{2}.$$

Take $h = \lambda^{-1/2}$ and replace v by $-v$ for the other sign. This proves (80). No uniform upper Hessian bound near a singular acyclic coefficient has been used. □

The map-dependent part of the primitive is exactly

$$z_0 \operatorname{tr} \mathrm{Ein}(P_F) - \sum_j \|B_j\|_{\mathrm{HS}}^2 - \sum_j \operatorname{tr}\big(T_j R(P_F) T_j^\dagger\big). \tag{81}$$

The first term is independent of the arrows at fixed F. After its declared subtraction, the primitive has the same C^ℓ error $O_\ell(\Lambda^{-2})$ for every fixed ℓ. Section 8 states the relative normalization including the metric determinant terms. This subtraction is not constant along a full metric orbit where the norm of D changes. At phase $\pi/2$ and the charges $-1+\mathrm{i}$, $1+\mathrm{i}$, the map-dependent real function at fixed F is $\sum_j \|B_j\|_{\mathrm{HS}}^2 + \sum_j \operatorname{tr}(T_j R(P_F) T_j^\dagger)$, a positive Hermitian quadratic form in the arrows. Positivity when the differential varies is a separate question, addressed below.

7.3. **A shrinking-scale profile and failure of higher derivatives.** For the rest of this subsection let $H = W = C = U = \mathbb{C}^2$, $D = [0, \Lambda I]$, $z_0 = -1+\mathrm{i}$ and $z_1 = 1+\mathrm{i}$. Put

$$p = e^{-\Lambda^2}, \qquad \rho_\Lambda = R(\Lambda^2) = \frac{1-p}{\Lambda^2}.$$

Return to the balanced family (11) and choose the constant nonzero mixing

$$T_1 = I, \qquad T_2 = -\sigma_z, \qquad T_3 = I. \tag{82}$$

All three are invertible and $\sum_j \|T_j\|_{\mathrm{HS}}^2 = 6$. Here $\sum_j T_j^\dagger T_j = \sum_j T_j T_j^\dagger = 3I$, $\sum_j B_j^\dagger T_j = c\epsilon\sigma_x$ and $m_H = 4$. The source coefficient is a direct sum, after a constant unitary change and permutation, of two unitarily conjugate matrices

$$Z_\pm = \begin{pmatrix} \mathrm{i} & \pm u \\ \pm u & p(2+\mathrm{i}) \end{pmatrix}, \qquad u = \rho_\Lambda c\epsilon.$$

Since $\det Z_\pm = -p - u^2 + 2\mathrm{i}p$ and $\operatorname{tr}(Z_\pm^\dagger Z_\pm) = 1 + 5p^2 + 2u^2$,

$$\begin{aligned} g_p(y) &= \sqrt{1 + 5p^2 + 2y + 2\sqrt{5p^2 + 2py + y^2}}, \\ m_\Lambda(\epsilon) &= 2g_p\left(\frac{\rho_\Lambda^2 \epsilon^2}{2+\epsilon^2}\right) + 2\sqrt{(3\rho_\Lambda - 2p)^2 + p^2} + 2\sqrt{1 + 9\rho_\Lambda^2}. \end{aligned} \tag{83}$$

All ϵ derivatives occur in the first summand.

Proposition 7.3. *Let $s_\Lambda = \sqrt{2p}/\rho_\Lambda$. For each compact real interval I and each fixed non-negative integer k,*

$$\frac{m_\Lambda(s_\Lambda\xi) - m_\Lambda(0)}{2p} \longrightarrow P(\xi) := \xi^2 + \sqrt{(1+\xi^2)^2+4} - \sqrt{5} \quad \text{in } C^k(I). \tag{84}$$

In particular,

$$\partial_\epsilon^3(m_\Lambda - m_H)(s_\Lambda) \sim \frac{3}{4}\frac{e^{\Lambda^2/2}}{\Lambda^6}, \tag{85}$$

$$\partial_\epsilon^4(m_\Lambda - m_H)(0) \sim \frac{24}{5\sqrt{5}}\frac{e^{\Lambda^2}}{\Lambda^8}. \tag{86}$$

Thus uniform C^3 convergence fails on every fixed interval $[0,\epsilon_0]$ with $\epsilon_0 > 0$.

Proof. Put $\eta_\Lambda = p/\rho_\Lambda^2$, $w_\Lambda(\xi) = \xi^2/(1+\eta_\Lambda\xi^2)$ and $H(w) = w + \sqrt{5+2w+w^2}$. At $\epsilon = s_\Lambda\xi$ the argument of g_p is pw_Λ. Rationalizing the outer square root in (83) gives the exact expression

$$P_\Lambda(\xi) := \frac{m_\Lambda(s_\Lambda\xi) - m_\Lambda(0)}{2p} = \frac{2H(w_\Lambda(\xi)) + 5p}{\sqrt{1 + 2pH(w_\Lambda(\xi)) + 5p^2} + 1} - \sqrt{5}. \tag{87}$$

Since $p \to 0$ and $\eta_\Lambda \to 0$, the right side is a smooth function of (p, η_Λ, ξ) near $(0,0,I)$. Its square-root arguments stay bounded away from zero. This proves (84); in fact $\|P_\Lambda - P\|_{C^k(I)} \le C_{I,k}(p+\eta_\Lambda) = O_{I,k}(\Lambda^4 e^{-\Lambda^2})$.

The exact derivative relation is

$$\partial_\epsilon^k m_\Lambda(s_\Lambda\xi) = 2p\, s_\Lambda^{-k} P_\Lambda^{(k)}(\xi), \qquad k \ge 1.$$

Direct differentiation gives $P'''(1) = 3\sqrt{2}/4$. For example, for $Q(\xi) = \xi^4 + 2\xi^2 + 5$, its value and first three derivatives at 1 are $8, 8, 16, 24$; substituting them in the third derivative of $\sqrt{Q}$ gives the stated value. Therefore the third derivative is asymptotic to $3\rho_\Lambda^3/(4\sqrt{p})$ at $\epsilon = s_\Lambda$, which proves (85). This point tends to zero through positive values.

For the derivative at zero set $d = 1 + \sqrt{5}p$ and $A_0 = 1 + 1/\sqrt{5}$. At fixed $p > 0$, expansion of the exact formula gives

$$g_p(y) = d + \frac{A_0}{d}y + \left(\frac{2}{5\sqrt{5}pd} - \frac{A_0^2}{2d^3}\right)y^2 + O_p(y^3).$$

Since $y = \rho_\Lambda^2\epsilon^2/(2+\epsilon^2)$, it follows exactly that

$$\partial_\epsilon^4 m_\Lambda(0) = 12\left[-\frac{A_0}{d}\rho_\Lambda^2 + \left(\frac{2}{5\sqrt{5}pd} - \frac{A_0^2}{2d^3}\right)\rho_\Lambda^4\right].$$

Using $\rho_\Lambda \sim \Lambda^{-2}$ and $p = e^{-\Lambda^2}$ proves (86). □

Corollary 7.4. *For the family (82), $m_\Lambda \to 4$ in $C^2(I)$ on every compact real interval I. More precisely,*

$$\|m_\Lambda - 4\|_{C^2(I)} = O_I(\Lambda^{-2}), \qquad \|m_\Lambda'\|_{C^0(I)} + \|m_\Lambda''\|_{C^0(I)} = O_I(\Lambda^{-4}).$$

For every $\epsilon_0 > 0$ and $0 < \alpha \le 1$, however, the Hölder seminorm $[m_\Lambda'']_{C^{0,\alpha}([0,\epsilon_0])}$ tends to infinity.

Proof. Put $f(\epsilon) = \epsilon^2/(2+\epsilon^2)$, $y = \rho_\Lambda^2 f(\epsilon)$ and $h(y) = \sqrt{(y+p)^2+4p^2}$. For $y \geq 0$ set $A(y) = 1 + (y+p)/h(y)$. Then $g_p \geq 1$, $1 \leq A \leq 2$ and

$$g_p' = A/g_p, \qquad g_p'' = \frac{4p^2}{g_p h^3} - \frac{A^2}{g_p^3}.$$

Moreover,

$$(f')^2 = \frac{16f}{(2+\epsilon^2)^3} \leq 2f, \qquad \frac{p^2 y}{h^3} \leq \frac{p^2 y}{(p+y)^3} \leq 1.$$

Only $2g_p(y)$ varies with ϵ in (83). Therefore

$$|m_\Lambda'| \leq 4\rho_\Lambda^2 |f'|, \qquad |m_\Lambda''| \leq 2\rho_\Lambda^4 (f')^2 \left(\frac{4p^2}{h^3} + 4\right) + 4\rho_\Lambda^2 |f''| \leq C_I \rho_\Lambda^2.$$

These are direct estimates on the fixed interval I. Together with (79), they prove the C^2 assertions.

For the Hölder claim, the exact rescaling gives $m_\Lambda''(s_\Lambda \xi) = \rho_\Lambda^2 P_\Lambda''(\xi)$. Direct differentiation of P gives

$$P''(0) = 2 + \frac{2}{\sqrt{5}}, \qquad P''(1) = 2 + \frac{3}{\sqrt{2}}, \qquad \Delta := \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{5}} > 0.$$

Since $s_\Lambda \to 0$, both 0 and s_Λ lie in $[0, \epsilon_0]$ for all large Λ. Hence

$$[m_\Lambda'']_{C^{0,\alpha}([0,\epsilon_0])} \geq \frac{|m_\Lambda''(s_\Lambda) - m_\Lambda''(0)|}{s_\Lambda^\alpha} \sim 2^{-\alpha/2} \Delta \, \frac{e^{\alpha\Lambda^2/2}}{\Lambda^{4+2\alpha}} \longrightarrow \infty. \tag{88}$$

The asymptotic equivalence concerns the displayed two-point quotient. □

This is a derivative obstruction in the actual parameter of (11), with fixed reference metrics and fixed mixing. The family extends smoothly to negative ϵ, so derivatives at zero are unambiguous. It is not asserted that ϵ is a smooth coordinate on a coarse moduli quotient. The analytic singular scale comes from the acyclic coefficient approaching zero; a changing stabilizer alone is not an explanation of trace-norm singularities. The fixed-interval C^2 bound is specific to this family. It does not settle a general uniform C^2 estimate on the full arrow-coordinate ball.

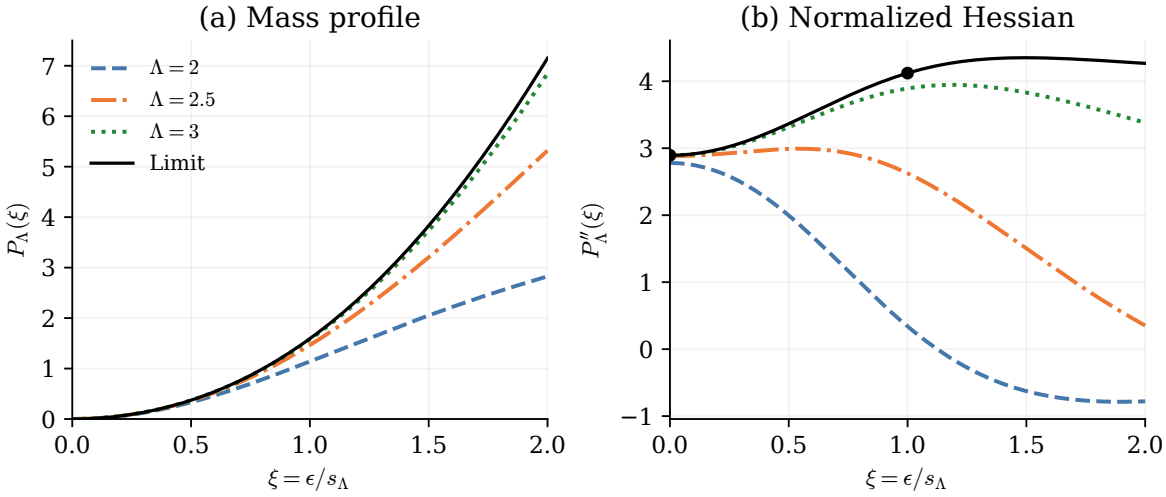


FIGURE 1. The exact normalized mass difference P_Λ and normalized Hessian P''_Λ, with their limits P and P''. The marked limiting Hessian values at $\xi = 0, 1$ differ by Δ; their original parameter separation tends to zero. The coordinate scale is $s_\Lambda = \sqrt{2p}/\rho_\Lambda \sim \sqrt{2}\Lambda^2 e^{-\Lambda^2/2}$. The curves illustrate a proved static limit; they are not numerical phase-flow trajectories.

7.4. **A negative ambient Levi direction.** For a phase angle $\theta \in (0, \pi)$, the real phase combination in [HKKP26, §5.1] is $-\operatorname{Re}\mathcal{S}_{\mathbb{C}} + \cot\theta \operatorname{Im}\mathcal{S}_{\mathbb{C}}$. On a complex gauge orbit the logarithmic determinant terms are locally pluriharmonic. Its map-dependent candidate potential is therefore

$$\mathcal{K}_\theta(A, D) = \operatorname{tr}(CR(B)) + c_\theta \operatorname{tr}\operatorname{Ein}(B), \qquad c_\theta = -\operatorname{Re} z_0 + \cot\theta \operatorname{Im} z_0. \tag{89}$$

This formula is a construction from the explicit primitive; it is not an assertion that [HKKP26, §5.4] proves its ambient positivity.

Proposition 7.5. *For $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$ and $\theta = \pi/2$, the potential (89) has a negative Levi direction on a complex gauge orbit of the mixed family, for every $\Lambda > 0$ and every $\epsilon \in [0, 1]$.*

Proof. On one scalar acyclic summand put $a = |A|^2$, $b = |D|^2$. The restriction of the potential is $aR(b) + c\operatorname{Ein}(b)$. Its $A\overline{A}$ Levi entry is $R(b) > 0$. The Schur complement of that entry is

$$e^{-b}[c - a\rho(b)], \qquad \rho(b) = \frac{b - 1 + e^{-b}}{b(1 - e^{-b})}. \tag{90}$$

Indeed the $D\overline{D}$ entry is $a(R' + bR'') + ce^{-b}$, and the squared mixed entry divided by R is $ab(R')^2/R$; using $bR' = e^{-b} - R$ gives (90). For $b > 0$, $\rho(b) > 1/2$: the sign of $2\rho - 1$ is the sign of $b - 2 + (b + 2)e^{-b}$, whose derivative is $1 - (b + 1)e^{-b} > 0$ and whose value at zero is zero. Thus $c = 1$, $a = 4$ gives a negative direction for every $b > 0$.

Realize this scalar restriction by taking $T_1 = 2I$ and $T_2 = T_3 = 0$ while keeping the balanced B_j fixed. Vary one acyclic diagonal entry of D and the corresponding entry of T_1. Independent nonzero complex rescalings of that acyclic source line and its degree-one target line realize these two local variables within one complex gauge orbit. Their potential is the scalar expression above plus a constant, proving the assertion. □

An object-dependent additive constant cannot repair this negative direction along the same metric orbit. The proposition obstructs a generally plurisubharmonic ambient potential with the prescribed phase primitive. It does not exclude a positive limiting cohomology metric or every possible construction made directly on a quotient.

7.5. **Quantities and derivative domains.** For clarity, the static estimates concern the following distinct objects, at fixed external $D = [0, F]$, $\sigma_{\min}(F) \geq \Lambda$, and fixed reference metrics. The profile rows further specialize to $F = \Lambda I$ and the indicated family:

Quantity	Coordinate derivatives	Conclusion
Coefficient Ω_F	bounded real B, T coordinates	C^k error $O_k(\Lambda^{-2})$
Relative primitive	the same coordinates	C^k error $O_k(\Lambda^{-2})$
Trace-norm mass	the same coordinates	C^0 error $O(\Lambda^{-2})$, C^1 error $O(\Lambda^{-1})$
Mass in (82)	surviving parameter ϵ	C^2 convergence; $C^{2,\alpha}$ seminorms diverge
Normalized profile	$\xi = \epsilon/s_\Lambda$ on compact intervals	C^k convergence for every fixed k

No all-orders derivative conclusion for $(\Omega_\Lambda \Omega_\Lambda^\dagger)^{1/2}$, the polar phase, or an actual flow trajectory is contained in this table.

8. The limiting quotient and a restricted positive construction

We first identify the limiting primitive under specified changes of presentation. We then give a finite-scale positive construction with additional hypotheses. Proposition 7.5 rules out extending its ambient positivity to all mixed presentations by the same primitive.

8.1. **Relative primitives and elementary cancellations.** Fix Hermitian reference data on a surviving source H and target W. Let C be an acyclic source sector and $D = [0, F] : H \oplus C \to U$, where $F : C \to U$ is invertible. Evaluate the primitive on metrics $h_V = h_H \oplus h_C$ and write $A_j = [B_j, T_j]$. The metrics on C, U and the map F are fixed throughout the static arrow-coordinate comparison. Put $P_F = F^\dagger F$ and $\lambda = \lambda_{\min}(P_F) > 0$. The zero-arrow acyclic primitive is

$$\mathcal{S}_{\mathrm{ac}} = z_0(\log\det h_C - \log\det h_U) + z_0 \operatorname{tr} \mathrm{Ein}(P_F). \tag{91}$$

With fixed determinant volume forms, subtracting this declared expression from (77) gives

$$\begin{aligned} \mathcal{S}_{\mathrm{rel}} &= z_0 \log\det h_H + z_1 \log\det h_W - q_h(B) \\ &\quad - \sum_j \operatorname{tr}\big(T_j R(P_F) T_j^\dagger\big), \qquad q_h(B) = \sum_j \|B_j\|^2_{\mathrm{HS},h}\,. \end{aligned} \tag{92}$$

For fixed F and acyclic metric data, $0 \leq R(P_F) \leq \lambda^{-1} I$. Thus the last term has absolute value at most $\lambda^{-1} \sum_j \|T_j\|^2_{\mathrm{HS}}$ and has $O(\lambda^{-1})$ bounds for every fixed derivative in bounded real B, T coordinates. This is a relative static comparison, not a constant subtraction on the full metric orbit.

Several elementary compatibilities follow from this exact formula. For mutually orthogonal acyclic sectors with $F = \mathrm{diag}_a(\Lambda_a I)$, the error is

$$-\sum_a R(\Lambda_a^2) \sum_j \Big\| T_j^{(a)} \Big\|^2_{\mathrm{HS}}\,. \tag{93}$$

The subtractions and removals of these sectors commute. Their bound depends on $\min_a \Lambda_a$, not on the ratios of the scales. This is an assertion for the displayed block decomposition, not for interacting acyclic differentials without such a decomposition.

An elementary source shear $g_V = \left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right)$, where $S : C \to H$, leaves D unchanged and replaces T_j by $T_j - B_j S$. The cohomology maps remain B_j. For bounded data the two error terms in (92) both tend to zero, so their limits agree. The same holds for every fixed derivative after composition with a holomorphic parameter chart whose shear has bounded derivatives. The two reference presentations must retain the same cohomology Hermitian data, or transport them by a specified holomorphic isometry. This does not assert equality at finite scale under a nonunitary shear with unchanged ambient metrics.

Constant unitary frame changes preserve the comparison when they transport F and T_j together. With unequal scales, a rotation of acyclic frames generally changes the displayed matrix of F. Rotating only the arrows while holding that matrix fixed is not the claimed invariance.

Now specialize to $\dim H = \dim W = n > 0$, $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$. The total cohomology charge is $2n\mathrm{i}$, of phase $\pi/2$. At fixed reference metrics, the limiting real arrow potential is

$$K_H(B) = q(B) = \sum_j \|B_j\|_{\mathrm{HS}}^2 . \tag{94}$$

Its Levi form is the usual positive flat form. The limiting metric-direction function is $q_h(B) + \log\det h_H - \log\det h_W$. Its critical equations are

$$\sum_j B_j^\dagger B_j = I_H, \qquad \sum_j B_j B_j^\dagger = I_W. \tag{95}$$

The equality of dimensions is necessary by taking traces. The resulting orbit-type Kähler quotients are the ordinary quiver quotients. Their existence is classical; the calculation above identifies the limiting potential and its normalization for these presentations.

8.2. **An explicit finite-scale restriction.** Let $X = \mathrm{Hom}(H, W)^k$, $G = \mathrm{GL}(H) \times \mathrm{GL}(W)$ and $K = \mathrm{U}(H) \times \mathrm{U}(W)$, acting by $B_j \mapsto g_W B_j g_H^{-1}$. Take finitely many acyclic copies $C_a = U_a = H$. Fix positive external scales Λ_a and constant matrices $L^{(a)}$ on the arrow-index space $\mathbb{C}^k$ such that

$$(L^{(a)})^\dagger L^{(a)} = \kappa_a^2 I_k, \qquad \kappa_a \geq 0. \tag{96}$$

At least one κ_a may be positive, so mixing need not vanish. Define the holomorphic family of presentations

$$T_j^{(a)} = \sum_l L_{jl}^{(a)} B_l, \qquad D = [0, \mathrm{diag}_a(\Lambda_a I_H)]. \tag{97}$$

Its equivariance is for the embedding

$$g_V = \mathrm{diag}(g_H, g_H, \ldots, g_H), \qquad g_U = \mathrm{diag}(g_H, \ldots, g_H), \qquad g_W = g_W.$$

It is not equivariant as a family with fixed graph data under the entire presentation gauge group.

Transport h_H to every C_a and U_a, and retain h_W on W. All scales, arrow-index matrices and these fixed copy identifications are held constant in the following tangent and metric comparisons. Then $D^\dagger D$ on the ath copy is $\Lambda_a^2 I$ and

$$\sum_j \left\| T_j^{(a)} \right\|_{\mathrm{HS},h}^2 = \kappa_a^2 q_h(B), \qquad \log\det h_V - \log\det h_U = \log\det h_H.$$

Consequently the pullback of (77), after subtraction of the constant $z_0 n \sum_a \mathrm{Ein}(\Lambda_a^2)$, is exactly

$$\mathcal{S}_{\mathbb{C},\Lambda} = z_0 \log\det h_H + z_1 \log\det h_W - \alpha_\Lambda q_h(B), \qquad \alpha_\Lambda = 1 + \sum_a \kappa_a^2 R(\Lambda_a^2) > 0. \tag{98}$$

This subtraction is constant on the specified tied-metric family, as well as in B at fixed scales. In contrast, it is not constant on the full metric space of the complex. The real arrow potential at phase $\pi/2$ is $K_\Lambda = \alpha_\Lambda q$. Its metric and form are $\alpha_\Lambda g_{\mathrm{flat}}$ and $\alpha_\Lambda \omega_{\mathrm{flat}}$. Their positivity follows directly from this formula.

Proposition 8.1. *For each fixed collection of data in (96)–(97), the map*

$$\sigma_\Lambda : B \longmapsto \alpha_\Lambda^{-1/2} B \tag{99}$$

induces an isometry on each connected smooth orbit-type stratum from the ordinary balanced Kähler quotient to the quotient defined by K_Λ at the same central moment level. It is the identity on the corresponding polystable cohomology isomorphism classes. The isometry preserves lengths, intrinsic distances and metric completions on identically specified path domains.

Proof. Identify the Lie algebra dual with pairs of Hermitian matrices using the fixed moment-map normalization

$$\mu_\alpha(B) = \left(-\alpha \sum_j B_j^\dagger B_j,\ \alpha \sum_j B_j B_j^\dagger \right), \qquad \eta = (-I_H, I_W).$$

Let $Z = \mu_1^{-1}(\eta)$ and $Z_\Lambda = \mu_{\alpha_\Lambda}^{-1}(\eta)$. The diagonal scalar circle acts trivially; it may be retained as an ineffective subgroup, and η annihilates its Lie algebra since the dimensions are equal. We do not assume freeness of the action. These finite-dimensional holomorphic Hamiltonian reductions have the usual smooth Kähler structures on connected orbit-type strata [SL91, Sj95]. The quiver Kempf–Ness correspondence identifies their points with polystable representations [King94, §6]. The hypotheses apply here: the ambient space is flat Kähler, K is compact, the level is central, and the source Gram component controls $q(B)$, making the moment map proper.

The map σ_Λ is a K-equivariant bijection from Z to Z_Λ, preserving every stabilizer subgroup. It satisfies

$$\sigma_\Lambda^*(\alpha_\Lambda g_{\mathrm{flat}}) = g_{\mathrm{flat}}, \qquad \sigma_\Lambda^*(\alpha_\Lambda \omega_{\mathrm{flat}}) = \omega_{\mathrm{flat}}.$$

For clarity, if v is a tangent lift and ξ_X an infinitesimal K-orbit vector, then

$$d\sigma_\Lambda(v) = \alpha_\Lambda^{-1/2} v, \qquad \xi_X(\sigma_\Lambda B) = \alpha_\Lambda^{-1/2} \xi_X(B).$$

It follows directly that

$$\inf_{\xi \in \mathfrak{k}} \alpha_\Lambda \left\| d\sigma_\Lambda(v) + \xi_X(\sigma_\Lambda B) \right\|^2 = \inf_{\xi \in \mathfrak{k}} \left\| v + \xi_X(B) \right\|^2 .$$

These are the squared reduced tangent norms. Stabilizer directions contribute zero orbit vectors and cause no inverse estimate. Furthermore σ_Λ is the action of $(I_H, \alpha_\Lambda^{-1/2} I_W) \in G$, so it leaves the representation isomorphism class unchanged.

For the path statement, fix a K-invariant subset $\mathcal{P} \subset Z$ and use $\sigma_\Lambda(\mathcal{P})$ on the other side. Choose piecewise smooth paths within connected orbit-type strata, with any allowed concatenations specified identically via σ_Λ. The equality of tangent norms preserves every admissible path length and hence their infimum on each finite-distance component. If this convention first produces a pseudometric, divide by zero distance on both sides. The resulting

metric spaces, and therefore their metric completions, are isometric. No identification with a separate algebraic compactification is used. □

The finite-scale identity is elementary after (98); it is not a general relative estimate for arbitrary mixing. For the rank-two family (11), scalar rescaling preserves its stabilizer jump exactly, so this restricted quotient identity remains valid without an orbit gap.

As a concrete example, take one copy and $L = \kappa I_k$, $\kappa > 0$. Then $T_j = \kappa B_j$, and the off-diagonal block of the source arrow Gram matrix is $E_H = \kappa C_H$. It equals κI on Z and $(\kappa/\alpha_\Lambda)I$ on Z_Λ. In either case it is nonzero, so the full source arrow Gram matrix does not commute with $D^\dagger D = \operatorname{diag}(0, \Lambda^2 I)$. This is mixing in the specified orthogonal Hermitian presentation. The nonunitary shear $S = \kappa I$ removes it algebraically, so the example is not asserted to be an intrinsically nonsplit complex.

For a set I of retained acyclic copies, put $\alpha_I = 1 + \sum_{a\in I} \kappa_a^2 R(\Lambda_a^2)$. The calibrated map between the I and J levels is multiplication by $\sqrt{\alpha_I/\alpha_J}$. These factors telescope under successive removals, proving order independence for this construction. Constant unitary changes of frames transport the differential and arrows together as above.

8.3. **Limits of the comparison.** The comparison fixes each Λ_a and each $L^{(a)}$. Varying Λ_a in a tangent direction differentiates $\alpha_\Lambda^{-1/2}$ and introduces an additional term; Proposition 8.1 makes no metric assertion in such total-parameter directions. It also does not claim that the full phase flow preserves the tied-metric family. Indeed that family keeps the norm of D fixed, whereas the cancellation theorems prove its growth.

Corollary 1.5 applies to ordinarily balanced initial tuples $B \in Z$ with any finite positive number ℓ of these acyclic copies. Indeed the combined acyclic rank is $r = \ell n$, and

$$\sum_{a,j} \left\| T_j^{(a)} \right\|_{\mathrm{HS}}^2 = n \sum_a \kappa_a^2 =: M^2, \qquad \sigma_{\min}(F_0) = \min_a \Lambda_a.$$

If $\min_a \Lambda_a \geq \Lambda_*(n, M)$, the actual phase flow has the theorem's cancellation and relaxation properties, independently of the ratios of the initial scales. This applies for arbitrary k and every stabilizer of the ordinarily balanced tuple. The evolving metrics are those of the full phase flow, not metrics constrained to remain in the family used for the quotient isometry. The rescaled representative $\alpha_\Lambda^{-1/2} B$ is generally not ordinarily balanced at the reference metrics. It does admit balanced cohomology metrics: multiply the target metric by α_Λ and leave the source metric fixed. Theorem 1.4 therefore also applies to this representative under its corresponding large-scale bound, with the initial energy excess retained.

Thus the general result of this section is the relative static primitive limit under the declared elementary changes. The finite positive metric is a restricted construction. Neither result asserts a general metric on the moduli of all complexes or an isometry under arbitrary derived equivalences.

9. Further questions

The mass counterexample and the cancellation theorems concern the same positive phase equation. They show that acyclic cancellation need not come from trace-norm mass dissipation. For an invertible differential, the contracting quotient $A_j D^{-1}$ supplies the missing control from every initial metric. With a surviving source, estimates using only the nonsingular cohomology block close the argument at large initial acyclic scale. They allow arbitrary surviving tuples, unequal dimensions and fixed upper-half-plane charges, uniformly in arrow count, stabilizers and acyclic condition numbers.

Cancellation and relaxation have different hypotheses. For equal surviving dimensions and charges $-1+\mathrm{i}$, $1+\mathrm{i}$, a finite infimum of the cohomology Kempf–Ness function gives relaxation from nonbalanced initial metrics. Its quantitative energy bound retains the initial excess $\mathcal{K}(0)-\inf\mathcal{K}$. An arrow-norm bound alone need not bound this excess: for a scalar one-arrow representation $B=b\neq 0$ at standard metrics, it is $|b|^2-1-\log|b|^2$, which diverges as $b\to 0$. Lower boundedness, and not attainment, is the analytic hypothesis used in the proof. For arbitrary surviving representations that hypothesis can fail even though large-scale cancellation still holds.

Removing the large-initial-scale assumption remains open for the mixed theorem. The inequality $s'\leq(-a+4\pi q/(b_0\lambda))s$ requires control of q/λ before it becomes contractive; monotonicity of λ alone supplies no such bound. It is also natural to ask whether the mechanism extends to longer complexes while retaining independence from positive surviving orbit eigenvalues.

The integrated moment-map estimate gives no uniform final relaxation rate, normalized surviving-metric limit, or parameter derivatives of actual trajectories. These questions require additional control of the surviving dynamics. The static results allow arbitrary fixed invertible acyclic differentials but do not differentiate that differential. For the explicit scalar-scale family, mass converges in C^2 and every positive Hölder seminorm of its Hessian diverges. Optimal general second-derivative estimates on the full arrow-coordinate space remain open.

The explicit primitive separates metric-direction exactness from transverse positivity. Its negative ambient Levi direction excludes one direct positive-potential construction. The limiting cohomology metric and restricted finite-scale example leave open a general quotient-only construction with its own normalization and compatibility laws. Extensions to bundle complexes and singular geometric limits require additional elliptic analysis; none follows from these finite-dimensional results.

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