

METRIC RESPONSE AND HERMITIAN YANG–MILLS FLOW FOR TWO-STEP EXTENSIONS

BERND JOHANNES WUEBBEN

ABSTRACT. For two-step extensions of pairwise nonisomorphic stable bundles on a compact Kähler manifold, we quantify the degeneration of Hermitian–Einstein metrics and Hermitian Yang–Mills flow near intersecting slope-stability walls. We assume equal slopes at the reference form and connected support for the extension class. When the centered slopes provide full local parameters, every singular value of the trace-free part of $H^{-1}dH$, for the Hermitian–Einstein metric H , is relatively asymptotic to the reciprocal of a nonzero eigenvalue of a weighted extension-graph Laplacian. The comparison uses rank-weighted slope and normalized L^2 norms and remains uniform for arbitrarily separated positive stability margins. We also construct a finite-dimensional invariant manifold of the flow and compare its solutions with flows from any smooth family of initial metrics. For each tolerance, one late time and then one parameter neighborhood give multiplicative metric comparison, up to a constant scalar, and relative accuracy for every small endomorphism Dolbeault eigenvalue for all subsequent times. The comparison applies in the stable, semistable and unstable regimes. In the two-factor case, we determine a uniform asymptotic profile for the smallest eigenvalue across the wall, including exponential decay on the unstable side. An application to explicit weak Fano toric tangent bundles connects nonsimplicial stability chambers, metric response and flow on the same bundles.

1. INTRODUCTION

The Kobayashi–Hitchin correspondence decides the existence of a Hermitian–Einstein metric from slope stability. Near a stability wall the limiting graded bundle has additional automorphisms, and the metric equation loses invertibility in the corresponding directions. We ask for a quantitative description that remains valid when several wall distances vanish at unrelated rates. The relevant finite operator is a weighted extension-graph Laplacian. This paper determines the singular values of the intrinsic metric derivative, separates the singular part of the linear inverse, and proves actual-flow comparisons with a finite evolution. A final theorem connects these statements on one family of genuine toric tangent bundles.

Consider a fixed holomorphic two-step extension

$$0 \longrightarrow \bigoplus_{i \in I} F_i \longrightarrow E \longrightarrow \bigoplus_{j \in J} F_j \longrightarrow 0$$

on a compact connected Kähler manifold X of complex dimension n . The m factors are locally free, pairwise nonisomorphic, and stable with equal slope at the reference form ω_0 . There is an edge $j \rightarrow i$ for every nonzero component of its fixed extension class; assume the underlying bipartite graph is connected. Let ω_s be a smooth finite-dimensional Kähler family, $r_v = \text{rk } F_v$, $M = \text{diag}(r_v)$, and $D_e = e_i - e_j$. With smoothly continued factor HE

Date: October 27, 2026.

2020 Mathematics Subject Classification. 53C07 (primary); 53C55, 58J35, 58J50 (secondary).

Key words and phrases. Hermitian–Einstein metric, Hermitian Yang–Mills flow, slope stability, vector bundle extension, graph Laplacian, metric variation.

metrics and harmonic representatives $\beta_{e,s}$, put

$$(1) \quad \begin{aligned} V_s &= \int_X \omega_s^n / n!, & \kappa_v(s) &= \frac{2\pi c_1(F_v)[\omega_s]^{n-1}}{(n-1)!V_s r_v}, & \kappa_E &= \frac{\sum_v r_v \kappa_v}{\sum_v r_v}, \\ b_v &= -r_v(\kappa_v - \kappa_E), & c_e(s) &= \|\beta_{e,s}\|_{L^2}^2 / V_s > 0. \end{aligned}$$

For $b \in D\mathbb{R}_{>0}^{E(\Gamma)}$, exponential balancing means

$$(2) \quad k_e = c_e(s)e^{(D^T x^0)_e}, \quad Dk = b, \quad \sum_v r_v x_v^0 = 0, \quad \sigma = \sum_e k_e.$$

It has a unique solution. Denote the positive eigenvalues of $M^{-1/2}D \operatorname{diag}(k)D^T M^{-1/2}$, in increasing order, by $\nu_1(k), \dots, \nu_{m-1}(k)$. For a nonempty proper vertex subset C , let ∂C be the edges joining C to its complement, and define its cut capacity and normalized capacity by

$$\operatorname{cap}_k(C) = \sum_{e \in \partial C} k_e, \quad \frac{\operatorname{cap}_k(C)}{\sum_e k_e}.$$

These definitions apply to any positive conductance array used below. The independent stationary theorem in [Wue26c] constructs actual HE metrics and gives every complex small endomorphism eigenvalue relatively by this operator. We restate its hypotheses and normalizations in Section 2.

Theorem 1.1 (Intrinsic metric response). *Suppose the centered slope map $s \mapsto b(s)$ has full rank $m-1$ at the wall. Use a smooth chart $s = s(b, \zeta)$, with bounded first derivatives near the wall, and hold ζ fixed. For the actual HE metric H_s in the local stable chamber, consider*

$$\dot{b} \mapsto \mathcal{D}_s \dot{b} = (H_s^{-1} \partial_{\dot{b}} H_s)_0.$$

The input norm is $\|\dot{b}\|_{M^{-1}}^2 = \sum_v \dot{b}_v^2 / r_v$. The output is a real H_s -Hermitian pointwise trace-free endomorphism, with norm squared $V_s^{-1} \int_X \operatorname{tr}(A^2) dV_s$. Its decreasing singular values obey

$$(3) \quad s_j(\mathcal{D}_s) = \frac{1 + O(\sigma)}{\nu_j(k(s, b))}, \quad 1 \leq j < m.$$

The coefficient is one. The error constant remains uniform as positive normalized cut capacities or ratios of positive graph eigenvalues tend to zero, including repeated eigenvalues and approaches to intersecting chamber faces. The metric tangent is intrinsic; it is not the derivative of a fixed-frame matrix logarithm.

Theorem 3.3 gives the accompanying exact inverse and positive-resolvent decomposition. Its finite Schur form agrees with the graph form up to a relative $O(\sigma)$ error, and its complementary inverse is uniformly regular in fixed Sobolev norms in the scaled harmonic presentation. Theorem 4.3 proves (3) from an exact differentiation identity, rather than differentiating a C^0 approximation. Neither conclusion is a nonlinear choice of gauge on a quotient moduli space. For the general two-step extension, Theorem 8.2 also gives metric and relative spectral comparison for prescribed HYM flows at stable, wall and unstable nearby parameters. Its parameter neighborhood is chosen after the comparison time.

One geometric family and the synthesis theorem. For $k \geq 1$, put $d = 2k$, $n = d + 2$, and, with quotient projectivization, set

$$B = (\mathbb{P}^1)^d, \quad L = \mathcal{O}_B(1^k, -1^k), \quad X_k = \mathbb{P}_B(\mathcal{O} \oplus L) \times_B \mathbb{P}_B(\mathcal{O} \oplus L).$$

Write $\pi : X_k \rightarrow B$. Let h_j be the pulled-back hyperplane class of the j -th base factor, and let ξ_i be the first Chern class of the tautological quotient line bundle from the i -th projective-bundle factor. Put $\ell = c_1(L)$, $\eta_i = \xi_i - \ell/2$, and define the line bundles

$$R_i = \mathcal{O}_{X_k}(2\xi_i) \otimes \pi^* L^{-1}, \quad Q_j = \pi^* \mathcal{O}_B(2h_j).$$

Their Chern classes, in vertex order $(R_1, R_2, Q_1, \dots, Q_d)$, are $C = (2\eta_1, 2\eta_2, 2h_1, \dots, 2h_d)$. The equality poset $K_{2,d}$ has two minimal elements, d maximal elements, and every minimal element below every maximal element. Its order ideals index the invariant saturated tangent subsheaves having the same slope as the tangent bundle at the wall. For a Hermitian metric h , let F_h denote its Chern curvature and $K_{h,\omega} = i\Lambda_\omega F_h$ its mean curvature. Use the smoothly varying line-factor metrics $h_{v,s}$, and put $h_{0,s} = \bigoplus_v h_{v,s}$. For the flow statement, write $u = \Psi(s, t)$ for the parabolic invariant graph constructed in Section 6, where $t = (t_{ij})$ is the vector of extension amplitudes. Its diagonal gauge velocity is denoted $A_s(t)$: the corresponding positive diagonal gauge solves $\dot{g}_s g_s^{-1} = A_s(t)$.

Theorem 1.2 (Geometry, spectrum, metric response, and flow). *For each $k \geq 1$, there is a unique $r_k > 1$ for which all the line factors have equal slope at the ample real class $H_0 = \eta_1 + \eta_2 + r_k \sum_j h_j$. On the single fixed bundle $E = T_{X_k}$, the following six statements hold.*

- (1) Geometry and filtration. *The variety is smooth projective weak Fano, is not Fano, and has $\dim X_k = \rho(X_k) = n$. Its actual tangent sequence is*

$$(4) \quad 0 \longrightarrow R_1 \oplus R_2 \longrightarrow T_{X_k} \longrightarrow \bigoplus_{j=1}^d Q_j \longrightarrow 0.$$

Its n line factors are distinct and stable at H_0 , all $2d$ component extension classes are nonzero, and all amplitude assignments remain integrable. The torus-invariant equality subbundles are the relative tangents of the smooth projections specified in Section 9.

- (2) Realized chamber and parameters. *The inclusion lattice of the torus-invariant equal-slope subbundles is the ideal lattice of $K_{2,d}$; its quotient at each covering relation has rank one. If $D_X = \sum_v C_v$ and $z_v = (C_v - D_X/n)H^{n-1}$, its analytic parameters are*

$$(5) \quad b_v = -\frac{2\pi z_v}{(n-1)!V(H)}, \quad x_i = b_{R_i} > 0, \quad y_j = -b_{Q_j} > 0, \quad \sigma = x_1 + x_2 = \sum_j y_j.$$

These are exactly the local stable inequalities; non-strict signs give semistability, and, locally, outside that cone the tangent bundle is unstable. Its pointed cone has dimension $n-1$, $4k$ rays, n facets and section $\Delta^1 \times \Delta^{d-1}$, with one free polarization coordinate. There is a full fixed-volume chart $H(s)$ with $b(H(s)) = s \in Z = \{\sum_v s_v = 0\}$ and controlled smooth representatives ω_s ; on the stable part $\|s\|_1 = 2\sigma$.

- (3) Finite operator and harmonic coefficients. *Choose only the reference form and factor metrics compatibly with exchange of the two projective factors. At the reference parameter, the harmonic coefficients satisfy $c_{ij}(0) = a_i q_j$, $a_i, q_j > 0$. Write χ_v for the*

mean-zero balancing potentials, denoted x_v^0 in (2). For the actual current balancing, $k_{ij} = c_{ij}(s)e^{\chi_{R_i} - \chi_{Q_j}}$, $Dk = b$, while

$$(6) \quad k_{ij}^0 = x_i y_j / \sigma, \quad k_{ij} / k_{ij}^0 = 1 + O(\sigma), \quad L^0 = D \operatorname{diag}(k^0) D^T.$$

The current array need not have rank one; the fixed tangent extension and its harmonic data are part of the statement.

- (4) Every actual small eigenvalue. In the strict chamber, the complex End_0 zero-form Dolbeault spectrum of the HE metric has

$$(7) \quad \lambda_j = \nu_j(L^0)(1 + O(\sigma)) \quad (1 \leq j < n), \quad \lambda_n \geq \gamma > 0.$$

The estimate is relative for every ordered eigenvalue, with complex multiplicities and arbitrary positive ratios among the coordinates x_i, y_j and among the positive graph eigenvalues. For $\delta = \min(x_1, x_2, y_1, \dots, y_d)$,

$$\delta \leq \nu_1(L^0) \leq \frac{n}{n-1} \delta, \quad \prod_{j=1}^{n-1} \lambda_j = \frac{nx_1 x_2 \prod_j y_j}{\sigma} (1 + O(\sigma)).$$

- (5) Actual metric, inverse, and response. Put

$$(8) \quad w_{R_i} = x_i / a_i, \quad w_{Q_j} = \sigma q_j / y_j, \quad W_v = w_v / \left(\prod_u w_u \right)^{1/n}, \quad h_{\operatorname{mod}, s} = \Phi_s^* \left(\bigoplus_v W_v h_{v, s} \right),$$

where Φ_s is the harmonic-representative identification. With $\det H_s = h_{\det, s} = \prod_v h_{v, s}$, $e^{-C\sigma} h_{\operatorname{mod}, s} \leq H_s \leq e^{C\sigma} h_{\operatorname{mod}, s}$, even when weight ratios diverge. The exact linear inverse and its positive-real resolvents have the finite Schur decomposition and uniform complementary $H^a \rightarrow H^{a+2}$ estimates of Theorem 3.3, in its specified identification. In particular its L^2 norm is $\nu_1(L^0)^{-1}(1 + O(\sigma))$. For Euclidean inputs $\dot{s} \in Z$ and normalized real Hermitian L^2 outputs, all $n-1$ decreasing singular values of $(H_s^{-1} dH_s)_0$ are $\nu_j(L^0)^{-1}(1 + O(\sigma))$.

- (6) Actual flow on the same bundle. Fix one finite-regularity invariant graph and its exact corrected field constructed in Section 6. For any smooth positive initial-metric family on E , consider $h_s^{-1} \partial_\tau h_s = -2(K_{h_s, \omega_s} - \kappa_E(s)I)$, with each ω_s fixed in time. For every $\varepsilon > 0$, there exist $T < \infty$, a neighborhood $U_\varepsilon \ni 0$, a scalar $a_T > 0$, and trajectories of the parameter-dependent field of that same graph, with common restart data $g_s(T) = g_T$, $\dot{g}_s g_s^{-1} = A_s(t)$, $\det g_s = 1$, $t_{ij} = g_{R_i} / g_{Q_j} > 0$, such that for every $s \in U_\varepsilon$ and $\tau \geq T$,

$$(9) \quad \begin{aligned} e^{-\varepsilon} a_T \Phi_s^* g_s^* h_{0, s} &\leq h_s(\tau) \leq e^{\varepsilon} a_T \Phi_s^* g_s^* h_{0, s}, \\ \left| \log \frac{\lambda_j(h_s(\tau), \omega_s)}{\nu_j(D \operatorname{diag}(\widehat{k}(\tau)) D^T)} \right| &\leq \varepsilon \quad (j < n), \\ \widehat{k}_{ij} = c_{ij}(0) t_{ij}^2 &= \frac{X_i Y_j}{\widehat{\sigma}}, \quad X_i = \sum_j \widehat{k}_{ij}, \quad Y_j = \sum_i \widehat{k}_{ij}, \quad \widehat{\sigma} = \sum_i X_i = \sum_j Y_j. \end{aligned}$$

The scalar is independent of s, τ . One fixed late time and neighborhood give a uniform positive λ_n gap for all future times. Stable, wall and unstable parameters are included. If initial determinants are $h_{\det, s}$, the scalar can be absorbed into the arbitrary-tolerance error. The dynamic margins X, Y are instantaneous conductance sums, not the fixed demands x, y ; their evolution retains the full corrected field and the initial-metric information.

The stationary parts concern only the strict chamber. They assert no HE metric on a nonsplit semistable boundary bundle. The flow conclusions have their own proof and quantifiers. None of these Dolbeault spectra is identified with the full Yang–Mills Hessian.

For the smallest example, $k = 1$, $r_1 = (3 + \sqrt{21})/6$. Thus the geometry includes an irrational real ample reference class and a square-cone chamber on a genuine fourfold tangent bundle. The geometric theorem is proved independently in [Wue26d]. Its spectral mechanism is the independent theorem of [Wue26c]. The inverse, response and actual-flow arguments in the present paper are the additional analytic content of their synthesis.

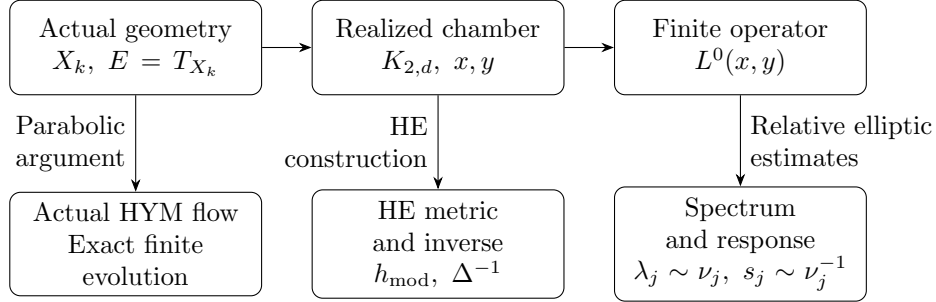


FIGURE 1. The common family in Theorem 1.2. All statements concern the same tangent extension. The finite operator includes harmonic extension data; the flow requires its separate parabolic argument. The lower assertions do not follow from the chamber alone.

Proof strategy and relation to prior work. The response proof begins with compact-unitary equivariance of the elliptic complementary solution. Differentiating it controls commutators relative to the energy of each weak graph direction. At actual HE, adjunction preserves the real operator; its Schur form is consequently a relative perturbation of the graph form. Denote this Schur operator by S . Let T be the block-scalar projection of the intrinsic metric velocity generated by a variation of the diagonal balancing potential. Exact differentiation factors the reduced Jacobian as MST . The metric velocity contains the same T , which cancels before the singular inverse is applied. The remaining background derivative is bounded, whereas every inverse graph singular value diverges; this proves all response singular values with coefficient one.

For several-factor flows the noncommuting exponential velocity is retained in the slice equation. Its exact principal-part cancellation gives a common-domain parabolic system. A finite-regularity invariant manifold, with its full correction, reconstructs exact flows. Bipartite mass coercivity confines their total conductance for all future time without requiring each edge to decrease. Qualitative reference matching and fixed-time parameter comparison supply a restart; order preservation then has zero future residual. This explains the quantifier order ε , then T , then U_ε . The signed two-factor theorem in Section 5 instead uses a radial integrable-error family and a calibrated scalar clock. Its unstable tail requires separation of variables rather than a fixed speed perturbation.

Local HYM wall-crossing and quantitative continuity are established prior work, notably Clarke–Tipler [CT24] and Deloche [Del25]; the latter already gives a quantitative connection estimate and treats nonabelian deformation theory. Sektnan–Tipler [ST24] provides a related adiabatic analytic framework. Earlier two-factor endomorphism spectra and flow matching are developed in [Wue26a, Wue26b]. Our claims concern the specified intrinsic derivative

and inverse spectrum at all positive ratios, the signed joint profile, and exact corrected all-future comparison in the stated two-step class, rather than the first quantitative continuity statement.

Finite reductions and iterated-logarithmic metric growth also have substantial predecessors. Haiden–Katzarkov–Kontsevich–Pandit develop the finite theory in [HKKP23], and the metric order and approximate-flow comparisons in [HKKP18, §4.2]; their geometric metric-growth theorem there is for curves. Lam [Lam26] treats semistable bundles with locally free graded object and bounded iterated-logarithmic remainder. Those bounded remainders are not asserted to converge. Our nearby-parameter relative small-spectrum theorem retains an exact corrected finite field instead of inferring near-one comparison from a bounded metric sandwich. The domain-valued parabolic graph uses Simonett’s parameter theorem [Sim94] with the explicit hypotheses checked in Section 6.

2. THE COMMON STATIONARY MODEL

Let X be a compact connected Kähler manifold of complex dimension n , and let ω_s be a smooth finite-dimensional family of Kähler forms near $s = 0$. Fix the holomorphic extension

$$(10) \quad 0 \longrightarrow S := \bigoplus_{i \in I} F_i \longrightarrow E \longrightarrow U := \bigoplus_{j \in J} F_j \longrightarrow 0.$$

The m factors are pairwise nonisomorphic, locally free and stable with equal slope at ω_0 . There is one edge $e : j \rightarrow i$ for each nonzero component of the fixed extension class. Assume this bipartite support is connected. Multiple basis vectors in one Ext space are part of that single component, not additional edges.

Write $r_v = \text{rk } F_v$, $M = \text{diag}(r_v)$, $R = \sum_v r_v$, and let the incidence column be $D_e = e_i - e_j$. Continue normalized factor HE metrics $h_{v,s}$ smoothly and represent the fixed classes by harmonic $\beta_{e,s}$. These are the usual local factor continuation and harmonic projection constructions; for the line factors in our geometric application they also follow directly from the scalar Poisson equation. Set

$$(11) \quad \begin{aligned} V_s &= \int_X \omega_s^n / n!, & \kappa_v &= \frac{2\pi}{(n-1)! V_s r_v} c_1(F_v) [\omega_s]^{n-1}, \\ \kappa_E &= R^{-1} \sum_v r_v \kappa_v, & b_v &= -r_v (\kappa_v - \kappa_E), & c_e(s) &= V_s^{-1} \|\beta_{e,s}\|_{L^2}^2 > 0. \end{aligned}$$

The demand vector b has ordinary sum zero. This name refers only to the displayed centered slope vector. Potentials, in contrast, will be rank-centered: $\sum_v r_v x_v = 0$.

On the fixed smooth direct sum put

$$(12) \quad \bar{\partial}_{s,t} = \bar{\partial}_0 + N_{s,t}, \quad N_{s,t} = \sum_e t_e \beta_{e,s}, \quad h_{0,s} = \bigoplus_v h_{v,s}.$$

Every amplitude assignment is integrable, including complex or zero amplitudes, because N maps U to S and vanishes on S . A positive diagonal gauge $g_x = \text{diag}(e^{x_v/2})$ sends the amplitude-one bundle to the amplitudes $t_e = e^{(D^T x)_e/2}$. With cycles, arbitrary independent amplitudes need not belong to this fixed orbit.

The amplitude-one presentation is identified with the original E by a smooth time-independent triangular map Φ_s . Indeed $\beta_s - \beta_{\text{ref}} = \bar{\partial} a_s$, with $a_s : U \rightarrow S$ chosen smoothly by the split Green operator. The relevant degree-zero cross-Hom kernels vanish at the wall and nearby. Since products of two such blocks vanish, $\Phi_s = I \pm a_s$ is an exact isomorphism with determinant one. We keep this identification in every metric formula on E .

We normalize the reference splitting so that β_{ref} has zero block on every pair (i, j) whose extension class vanishes. Such a block is $\bar{\partial}$ -exact and is removed by a triangular change of splitting. Blockwise harmonic projection and the split Green operator then keep these blocks zero for every s . Thus a_s , and hence $\Phi_s - I$, are supported only on edges of the fixed graph. This convention is used in Lemma 4.2.

Let

$$(13) \quad \mathcal{H} = \{X = \text{diag}(X_v I_{F_v}) : X_v \in \mathbb{R}, \sum_v r_v X_v = 0\}, \quad \langle X, Y \rangle_M = \sum_v r_v X_v Y_v.$$

This is the real parallel trace-free scalar space. On real Hermitian pointwise trace-free sections use the normalized inner product $V_s^{-1} \int_X \text{tr}(AB) dV_s$, which induces exactly this mass inner product on $\mathcal{H}$. Write P_s for its orthogonal projection and $Q_s = 1 - P_s$. Smooth block-diagonal isometries fix all scalar sections and identify these spaces uniformly as s varies. All spatial estimates below use these identifications.

For a metric h , write $K_h = i\Lambda_{\omega_s} F_h$. The centered curvature is

$$\mathcal{K}(s, t, u) = e^{u/2} K_{\bar{\partial}_{s,t}, h_{0,s} e^u} e^{-u/2} - \kappa_E I, \quad P_s u = 0,$$

where u is Hermitian and pointwise trace free. The determinant metric is unchanged and has constant curvature trace, so $\text{tr } \mathcal{K} = 0$ pointwise. At the split metric the real linearization is

$$(14) \quad L_s = \frac{1}{2}(\Delta_{\bar{\partial},s} + \Delta_{\partial,s}).$$

The Chern metric-variation identities are those of [Kob87, §4.2, equations (4.2.12)–(4.2.16)]. Its wall kernel is exactly $\mathcal{H}$; the stable distinct factors exclude cross-Hom sections. Its normal restriction has a uniform elliptic gap. The implicit-function theorem therefore solves $Q_s \mathcal{K} = 0$ uniquely for a small smooth $u = u(s, t)$, with

$$(15) \quad \|u(s, t)\|_{C^a} \leq C_a |t|^2, \quad \|D_t u(s, t)\|_{C^a} \leq C_a |t|$$

for each fixed spatial order. Harmonicity removes the first amplitude derivative for every nearby s . This construction, including complex amplitudes as real parameters, is the elliptic complementary construction used below. It differs from the parabolic invariant manifold of Section 6.

We state precisely the independent stationary input from [Wue26c, Theorem 1.1].

Theorem 2.1 (Stationary two-step comparison). *Locally E is stable exactly when $b \in D\mathbb{R}_{>0}^{E(\Gamma)}$. For such b the unique rank-centered balancing potential x^0 satisfies*

$$(16) \quad k_e = c_e(s) e^{(D^T x^0)_e}, \quad Dk = b, \quad \sigma = \sum_e k_e = \sum_{i \in I} b_i.$$

An actual HE metric is obtained with $x = x^0 + y$, $\|y\|_\infty \leq C\sigma$, and the complementary solution $u(s, t)$, where $t_e = e^{(D^T x)_e/2}$. Its determinant-normalized fixed-bundle metric satisfies

$$(17) \quad e^{-C\sigma} \Phi_s^* \left(\bigoplus_v e^{x_v^0} h_{v,s} \right) \leq H_s \leq e^{C\sigma} \Phi_s^* \left(\bigoplus_v e^{x_v^0} h_{v,s} \right).$$

If $\nu_1(k) \leq \dots \leq \nu_{m-1}(k)$ are the positive eigenvalues of $M^{-1/2} D \text{diag}(k) D^T M^{-1/2}$, the complex trace-free endomorphism Dolbeault spectrum has

$$(18) \quad \lambda_j = \nu_j(k)(1 + O(\sigma)) \quad (j < m), \quad \lambda_m \geq \gamma > 0.$$

All errors remain uniform as any positive normalized cut capacity or ratio of positive graph eigenvalues tends to zero. No fixed positive lower bound for these quantities is required.

The mechanism of the input is relevant to the new estimates. Exponential balancing minimizes the strictly convex coercive function $\sum_e c_e e^{(D^T x)_e - b \cdot x}$ on rank-centered potentials. The reduced curvature equation is $-b + D(c_e t_e^2) + \mathcal{R} = 0$, with amplitude jets through order three vanishing in $\mathcal{R}$, parameterwise. For every cut C , the exact splitting trace identity gives

$$|\mathcal{R}_C| \leq C|t|^2 \sum_{e \in \partial C} t_e^2.$$

The weighted graph inverse controls oscillation by $(m-1) \max_C |q_C| / \sum_{e \in \partial C} k_e$. It yields $y = O(\sigma)$ without a bound on the ordinary graph inverse. For the spectrum, harmonicity gives $\bar{\partial}_0^*[N_{s,t}, X] = 0$ on block scalars; the resulting complementary coupling is small relative to their graph energy. Schur elimination and min-max then compare every ordered mode, including repetitions. These are the cut and spectral arguments of [Wue26c, §§3–5], not a parameter-derivative conclusion from (17).

Constants throughout this paper concern fixed factors, support, nonzero extension classes and a controlled smooth background family. They are not uniform if classes vanish or the number of factors grows.

Lemma 2.2. *Let all amplitudes be positive and small, without requiring balance. For any metric $h_0 e^v$ with $\|v\|_{C^0} \leq C|t|^2$, the complex End_0 Dolbeault eigenvalues satisfy*

$$\lambda_j = \nu_j(c_e t_e^2)(1 + O(|t|^2)) \quad (j < m), \quad \lambda_m \geq \gamma > 0.$$

This is a complex spectral statement and requires no HE equation.

Proof. First use the split metric and decompose into the complex block-scalar kernel and its orthogonal complement. The scalar energy is exactly $q_t(X) = \sum_e c_e t_e^2 |(D^T X)_e|^2$. Harmonic cancellation gives the off-diagonal block bound $\|B_t X\|^2 \leq C|t|^2 q_t(X)$, and the complementary form has a fixed gap. At a spectral parameter $0 \leq \mu \leq C|t|^2$, Schur elimination subtracts $B_t^*(C_t - \mu)^{-1} B_t$, a nonnegative form at most $C|t|^2 q_t$. Inertia, or equivalently the min-max principle for this Schur family, places each ordered small eigenvalue between $(1 - C|t|^2)\nu_j$ and ν_j . The rank of the scalar space is $m-1$, while the complement keeps its gap. Finally e^v changes degree-zero and degree-one endomorphism norms multiplicatively by $e^{O(|t|^2)}$. Rayleigh quotients transfer the same relative comparison to the asserted metric. $\square$

3. THE REAL HE OPERATOR AND ITS INVERSE

Fix an actual HE zero from Theorem 2.1, at its corrected amplitudes t . Suppress s , and put

$$(19) \quad \tilde{k}_e = c_e(s)t_e^2, \quad q(X) = \sum_e \tilde{k}_e |(D^T X)_e|^2, \quad \mathcal{L} = M^{-1} D \text{diag}(\tilde{k}) D^T|_{\mathcal{H}}.$$

Thus $q = \langle X, \mathcal{L} X \rangle_M$, $|t|^2 \asymp \sigma$, $q(X) \leq C\sigma \|X\|_M^2$, and $q = (1 + O(\sigma))q_k$ as quadratic forms. No lower bound for this energy on a unit vector is used.

Use the unitary identification $\mathcal{U}(Y) = e^{u/2} Y e^{-u/2}$ from the metric $h_0 e^u$ to h_0 . The HE Dolbeault Laplacian becomes a self-adjoint operator Δ in this fixed realization. It preserves the real Hermitian subspace: adjunction exchanges $\Delta_{\bar{\partial}}$ and Δ_{∂} , while their difference on endomorphism zero-forms is the curvature commutator, which vanishes at HE. This use of the HE equation is essential for the real block argument.

Lemma 3.1. *For every fixed spatial order a ,*

$$(20) \quad \| [u, X] \|_{C^a} + \| u_x [X] \|_{C^a} \leq C_a \sqrt{\sigma} \sqrt{q(X)}, \quad X \in \mathcal{H},$$

where $u_x[X] = D_t u[(t_e(D^T X)_e/2)_e]$.

Proof. The complementary solution exists smoothly for complex amplitudes z and obeys $\|D_z u\|_{C^a} \leq C_a |z|$. For $U_\theta = e^{i\theta X}$, uniqueness gives

$$u(s, (e^{i\theta(D^T X)_e} z_e)_e) = U_\theta u(s, z) U_\theta^{-1}.$$

Differentiating at $\theta = 0$ gives

$$i[X, u] = D_z u[(i(D^T X)_e z_e)_e].$$

At the real amplitudes $z = t$,

$$\|D_z u\|_{\mathbb{R}^{2|E(\Gamma)|} \rightarrow C^a} \leq C_a |t| \leq C_a \sqrt{\sigma}, \quad \sum_e t_e^2 |(D^T X)_e|^2 \leq c_{\min}^{-1} q(X).$$

These bounds prove the commutator estimate. The ordinary chain rule with amplitude velocity $t_e(D^T X)_e/2$ gives the same bound for $u_x[X]$. $\square$

For later use define the analytic even operators

$$(21) \quad C_u = \cosh(\text{ad}_u/2), \quad S_u = \frac{\sinh(\text{ad}_u/2)}{\text{ad}_u/2}, \quad Z_u = \sinh(\text{ad}_u/2),$$

with their continuous values at zero. Two commutators and $u = O(\sigma)$ imply

$$(22) \quad \|Z_u X\|_{C^a} \leq C_a \sqrt{\sigma} \sqrt{q(X)}, \quad \|(C_u - I)X\|_{C^a} \leq C_a \sigma^{3/2} \sqrt{q(X)}.$$

Proposition 3.2. *In the orthogonal decomposition $\mathcal{H} \oplus Q$, write*

$$(23) \quad \Delta = \begin{pmatrix} G & B^* \\ B & C \end{pmatrix}.$$

The complement has $C \geq \gamma I$, and

$$(24) \quad \begin{aligned} &|\langle X, GX \rangle_M - q(X)| \leq C_0 \sigma q(X), \\ &\|BX\|_{L^2/V}^2 \leq C_0 \sigma q(X), \\ &S := G - B^* C^{-1} B = (1 + O(\sigma))\mathcal{L} \quad \text{as quadratic forms on } \mathcal{H}. \end{aligned}$$

Proof. For a physical block scalar X , harmonicity gives $\bar{\partial}_0^*[N, X] = 0$. Hence, in the split metric,

$$\|\Delta_{h_0, t} X\|_{L^2/V} = \|[N, \cdot]^*[N, X]\|_{L^2/V} \leq C \sqrt{\sigma} \sqrt{q(X)}.$$

Changing the inner products in degrees zero and one by e^u changes the first-order formal adjoint by coefficients $O(\sigma)$ in each required spatial norm. This follows by conjugating the split adjoint by the positive Gram matrices. Applied to $\bar{\partial}_t X$, it adds at most $C\sigma\sqrt{q(X)}$. In the centered realization the vector X corresponds to $e^{-u/2} X e^{u/2}$. Lemma 3.1 bounds the difference in C^2 by $C\sqrt{\sigma}\sqrt{q(X)}$; the operator has uniformly bounded second-order coefficients there. Projection proves the bound on B .

The physical scalar energy is $q(X)(1 + O(\sigma))$, by multiplicative metric comparison. Its centered image is $C_u X + Z_u X$, with Hermitian and skew-Hermitian parts. Since Δ preserves adjunction at HE, their real energies add orthogonally. Equation (22) gives energy at most $C\sigma q(X)$ for $Z_u X$ and at most $C\sigma^3 q(X)$ for $(C_u - I)X$. The energy triangle inequality and then Cauchy–Schwarz show that the energy of the centered X is $q(X)(1 + O(\sigma))$. This is the asserted bound on G .

The split complement has a uniform gap. Smooth coefficients, elliptic estimates and compactness preserve it on Q for the nearby centered operator. The bounded inverse C^{-1} and

the bound on B give $0 \leq B^*C^{-1}B \leq (C_0/\gamma)\sigma\mathcal{L}$. Subtracting proves the relative Schur-form estimate. $\square$

The complement is defined by its closed form on QH^1 , with operator domain QH^2 . Here and below the projections act on the real Hermitian pointwise trace-free section bundle. They are uniformly smooth finite-rank modifications of the identity, so the usual elliptic domain and regularity statements persist.

Theorem 3.3. *Let $J : \mathcal{H} \rightarrow \mathcal{H} \oplus Q$ be $JX = X - C^{-1}BX$. Then*

$$(25) \quad Q\Delta J = 0, \quad J^*\Delta J = S, \quad \Delta^{-1} = JS^{-1}J^* + QC^{-1}Q.$$

Moreover

$$(26) \quad \|JX - X\|_{L^2/V} \leq C\sqrt{\sigma}\sqrt{q(X)}, \quad J^*J = I + O(\sigma^2),$$

and for each fixed nonnegative integer a ,

$$(27) \quad \|C^{-1}v\|_{H^{a+2}} \leq C_a\|v\|_{H^a}, \quad \|(J - I)X\|_{H^{a+2}} \leq C_a\sqrt{\sigma}\sqrt{q(X)}.$$

The inverse norm is $\nu_1(k)^{-1}(1 + O(\sigma))$. For every real $z \geq 0$, set

$$C_z = C + zI, \quad J_zX = X - C_z^{-1}BX, \quad S_z = G + zI - B^*C_z^{-1}B.$$

Then the exact positive-resolvent formula and form estimates are

$$(28) \quad \begin{aligned} (\Delta + zI)^{-1} &= J_zS_z^{-1}J_z^* + QC_z^{-1}Q, \\ (1 - C_1\sigma)q(X) + z\|X\|_M^2 &\leq \langle X, S_zX \rangle_M \leq (1 + C_1\sigma)q(X) + z\|X\|_M^2, \\ \|J_zX - X\|_{L^2/V} &\leq \frac{\sqrt{C_0}\sqrt{\sigma}}{\gamma + z}\sqrt{q(X)}. \end{aligned}$$

Here $C_1 = C_0(1 + \gamma^{-1})$.

Proof. Solving the complementary block gives $Qv = C^{-1}(Qf - BPv)$. Substitution into the scalar block gives $SPv = J^*f$. Proposition 3.2 proves $S > 0$, so this elimination is legitimate and proves (25). The estimates on B give (26); in particular $J^*J - I = B^*C^{-2}B = O(\sigma^2)$ because $q(X) \leq C\sigma\|X\|_M^2$. Uniform elliptic estimates with the gap controlling the L^2 term give (27); use the same smooth harmonic cancellation at the required spatial order for BX . The complex spectrum in (18) agrees with the real Hermitian spectrum at HE, giving the inverse norm.

The identical elimination with $C + zI$ proves the resolvent identity. Its correction is nonnegative and bounded by $C_0\sigma q/(\gamma + z)$, uniformly for $z \geq 0$, proving its form estimates. Finally $\|C_z^{-1}\| \leq (\gamma + z)^{-1}$ gives the last bound. $\square$

These identities give a linear HE metric-equation inverse with its possibly large finite response separated from a uniformly regular complement. They are not a nonlinear gauge choice on a quotient moduli space. The spatial norms are in the scaled harmonic presentation and the unitary identification above; they are not uniform bounds in an arbitrary unbounded fixed-bundle frame. Only real nonnegative spectral shifts are asserted.

4. EVERY INTRINSIC METRIC-RESPONSE SINGULAR VALUE

Assume the slope map $s \mapsto b(s)$ has rank $m - 1$ at the reference point. Work in a smooth parameter chart $s = s(b, \zeta)$ with bounded derivatives near the wall, holding ζ fixed. On inputs use the dual mass norm

$$(29) \quad \|\dot{b}\|_{M^{-1}}^2 = \sum_v \dot{b}_v^2 / r_v, \quad f = M^{-1}\dot{b} \in \mathcal{H}.$$

On outputs use the real Hermitian normalized L^2 norm at the actual metric H_s . Define the intrinsic logarithmic tangent

$$(30) \quad \mathcal{D}_s \dot{b} = (H_s^{-1} \partial_b H_s)_0,$$

where the subscript is pointwise trace-free projection. It is H_s -Hermitian. This is the intrinsic tangent to the space of positive Hermitian forms, not $d(\log H_s)$ in a fixed matrix frame. Smooth scalar normalizations of H_s do not affect it.

Let $F(s, x)$ be the mass-weighted reduced curvature vector: at a complementary solution its component is $r_v(P_s \mathcal{K})_v$. At an HE zero use the operators of Section 3 and put

$$(31) \quad V_X = C_u X + S_u u_x[X], \quad TX = PV_X.$$

Lemma 4.1. *At every constructed HE zero,*

$$(32) \quad Q\Delta V_X = 0, \quad V_X = JTX, \quad T = I + O(\sigma^2), \quad F_x = MST.$$

Proof. Differentiate the metric $g_x^*(h_0 e^{u(s, t(x))})$ in the direction X , then transport to the centered unitary realization. Since $g_x = e^{x/2}$, its two diagonal-gauge terms sum to $C_u X$, and the exponential derivative gives $S_u u_x[X]$. Thus its centered velocity is exactly (31), with coefficient one. For a varying conjugation a , its derivative contributes $[\dot{a}a^{-1}, aKa^{-1}]$. Applied successively to g_x and $e^{u/2}$, both terms vanish at the scalar HE curvature. Here the second logarithmic derivative is $(\partial_X e^{u/2})e^{-u/2}$; no commutation of u and $u_x[X]$ is required. Also $\Delta_\partial = \Delta_{\bar{\partial}}$ at HE. Differentiating $Q\mathcal{K} = 0$ therefore gives $Q\Delta V_X = 0$. Complementary elimination gives $V_X = JPV_X$. Since $Pu_x = 0$, $C_u - I, S_u - I = O(u^2)$, and $u_x[X] = O(\sigma)\|X\|_M$, projection gives $T = I + O(\sigma^2)$. Finally differentiating $F = MP\mathcal{K}$ at the zero gives $F_x = MP\Delta V = MST$. $\square$

This also proves smooth dependence inside the stable chamber. Both S and T are invertible, so the ordinary finite implicit-function theorem gives a smooth root near each HE zero. The determinant-normalized HE metric on the simple stable E is unique. Hence these local metric branches agree. Their coordinates agree as well. Fix an interior parameter s_0 . For s sufficiently close to s_0 , continuity of $x^0(s)$ on the open chamber places the branches, relative to $g_{x^0(s_0)}$, in one small slice chart $(y, u) \mapsto e^{y/2} e^u e^{y/2}$, whose derivative at zero is an isomorphism from $\mathcal{H} \oplus Q$. This neighborhood may depend on s_0 . The branches patch without differentiating a choice in the stationary Brouwer construction. No smooth extension of the diverging x to the boundary is asserted.

Lemma 4.2. *At fixed x , the reduced curvature derivative and full metric derivative satisfy*

$$(33) \quad F_b \dot{b} = -\dot{b} + E_1 \dot{b}, \quad \|M^{-1} E_1 \dot{b}\|_M \leq C\sigma \|\dot{b}\|_{M^{-1}}, \quad \|H_b \dot{b}\|_{L^2/V} \leq C\|\dot{b}\|_{M^{-1}}.$$

Here H_b denotes the intrinsic trace-free fixed- x metric derivative, in the centered realization.

Proof. The reduced equation is $-b + D(c_e t_e^2) + \mathcal{R}$. Smooth harmonic coefficients with bounded derivatives give $D(c_e, b t_e^2) = O(\sigma)\|\dot{b}\|$. The amplitude jets of $\mathcal{R}$ through order three vanish for every background parameter. Taylor's formula therefore bounds its first background

derivative by $C|t|^4\|\dot{b}\|$. The derivative of $-b$ is exactly $-\dot{b}$, proving the first assertion. It is an ordinary forcing bound, not a cut-relative estimate; the exact inverse will be retained.

For the metric derivative one must account for the possibly unbounded g_x . In the triangular harmonic identification, $\Phi_b\Phi_s^{-1} = \pm a_b$ is still a sum of $U \rightarrow S$ blocks. Products of two such blocks vanish. Therefore

$$(34) \quad (g_x\Phi_b\Phi_s^{-1}g_x^{-1})_e = \pm t_e a_{e,b}, \quad \|g_x\Phi_b\Phi_s^{-1}g_x^{-1}\|_{C^a} \leq C_a\sqrt{\sigma}\|\dot{b}\|.$$

By the reference-splitting normalization in Section 2, a_b has no block on a pair without an edge, so no unscaled block appears. The background metric derivative is block diagonal and commutes with g_x . The derivative u_b at fixed t is $O(\sigma)$. Smooth centered exponential identities, these bounds, and the bounded parameter chart prove the assertion about H_b . Varying unitary background frames adds bounded terms; curvature conjugation derivatives vanish at HE. $\square$

Theorem 4.3. *Let $s_1(\mathcal{D}_s) \geq \dots \geq s_{m-1}(\mathcal{D}_s)$ be the singular values from (29) to the normalized real Hermitian L^2 output norm. For every sufficiently small stable parameter,*

$$(35) \quad s_j(\mathcal{D}_s) = \frac{1 + O(\sigma)}{\nu_j(k(s, b))}, \quad 1 \leq j < m.$$

The error remains uniform as positive normalized cut capacities or ratios of positive graph eigenvalues tend to zero, also when eigenvalues repeat. It uses the current harmonic coefficients.

Proof. Differentiate $F(s(b), x(b)) = 0$. The two lemmas give $ST\dot{x} = (I + E)f$, with $\|E\|_{M \rightarrow M} \leq C\sigma$. The full centered intrinsic velocity is $JT\dot{x} + H_b\dot{b}$, so the exact cancellation is

$$(36) \quad \mathcal{D}_s\dot{b} = JS^{-1}(I + E)f + H_b\dot{b}.$$

In particular, no ordinary error in T is multiplied through an ill-conditioned inverse.

Relative Schur-form comparison and min-max bracket each eigenvalue of S by $(1 + O(\sigma))\nu_j(k)$. Thus each decreasing singular value of S^{-1} is $(1 + O(\sigma))/\nu_j(k)$. Right multiplication by $I + E$ changes every singular value by $1 + O(\sigma)$, and left multiplication by J changes it by $1 + O(\sigma^2)$, since $J^*J = I + O(\sigma^2)$. These are singular-value min-max inequalities and require no commutation. Finally the trace identity gives

$$\sum_{j=1}^{m-1} \nu_j(k) = \sum_{e:j \rightarrow i} k_e(r_i^{-1} + r_j^{-1}) \leq 2\sigma,$$

so $\nu_{m-1} \leq 2\sigma$. Hence even the smallest singular value of this principal response is at least c/σ . The additive perturbation bound $|s_j(A + H) - s_j(A)| \leq \|H\|$, with the bounded H_b , changes every value only relatively by $O(\sigma)$. The input map $\dot{b} \mapsto M^{-1}\dot{b}$ is an isometry in the stated norms, proving the theorem with coefficient one. $\square$

For two factors with conductance $\beta > 0$, the positive graph eigenvalue is $\beta(r_1 + r_2)/(r_1r_2)$. The rank-centered metric potential difference is $\log(\beta/c)$. Its output norm divided by the dual-mass input norm is $r_1r_2/((r_1 + r_2)\beta)$, checking the coefficient in (35).

The theorem is not a directional relative expansion by the graph inverse: noncommuting form perturbations can rotate widely separated eigenspaces. It asserts the exact decomposition (36), the relative Schur form, and all singular values. If the slope map realizes only a proper subspace of the centered slope directions, the response is restricted to that subspace;

its singular values need not recover the entire graph spectrum. Replacing current $c(s)$ by $c(0)$ gives error $O(\sigma + |s|)$ unless $|s| = O(\sigma)$ is separately proved.

5. THE SIGNED TWO-FACTOR TRANSITION

There is a closed joint profile for two factors. Its proof uses the elliptic complementary family and an integrable future error. The several-factor theorem will instead use the exact parabolic graph. The distinction matters on the unstable side, where a fixed error in an exponential rate would destroy all-future relative control.

We use the following order principle in both arguments. Its arbitrary-dimensional form is the metric comparison of [HKKP18, Propositions 4.3 and 4.5].

Lemma 5.1. *Let h be an exact normalized HYM flow, and let a positive metric family H on the same holomorphic bundle and base form have full logarithmic residual*

$$\mathcal{E}_H = H^{-1}\dot{H} + 2(K_H - \kappa_E I), \quad \|\mathcal{E}_H\|_{\infty, \text{op}, H} \leq f(\tau).$$

If $e^{-\eta}aH(T) \leq h(T) \leq e^{\eta}aH(T)$, then for every future time of existence

$$(37) \quad e^{-\eta - \int_T^\tau f} aH(\tau) \leq h(\tau) \leq e^{\eta + \int_T^\tau f} aH(\tau).$$

For exact H , the integral is zero. A sandwich $e^{-\eta}aH \leq h \leq e^{\eta}aH$ changes every End_0 Dolbeault eigenvalue by factors between $e^{-4\eta}$ and $e^{4\eta}$.

Proof. Multiplying the approximate family by $\exp(\pm \int_T^\tau f)$ adds respectively $\pm fI$ to its logarithmic velocity and changes no curvature, since the scalar is spatially constant. Its residual is therefore nonnegative or nonpositive as a Hermitian endomorphism. The tensor maximum principle for HYM metric order, applied at the first contact of the largest relative eigenvalue, gives the two barriers. This uses the displayed full residual, so no half-gauge-velocity normalization is suppressed. On endomorphisms a metric sandwich changes squared norms in degrees zero and one by $e^{\pm 2\eta}$. The scalar a cancels, and min–max gives the spectral factors. $\square$

Now let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be nonsplit, with distinct stable locally free factors of equal slope at $s = 0$. Put $a = \text{rk } F$, $b = \text{rk } G$, $R = a + b$, and

$$(38) \quad q = \text{diag}((b/R)I_F, -(a/R)I_G), \quad \delta_s = \kappa_G(s) - \kappa_F(s), \quad c_s = \frac{\|\beta_s\|^2}{V_s ab/R}.$$

The complementary curvature is exactly $\mathcal{K}(s, t, u(s, t)) = k_s(t)q$, where $k_s(t) = -\delta_s + c_s t^2 + O(t^4)$. Conjugation by $\text{diag}(-I_F, I_G)$ makes this scalar even in t , although the whole correction u need not be even. Thus

$$(39) \quad z_s(w) = k_s(\sqrt{w}) + \delta_s = c_s w + O(w^2), \quad z'_s(w) = c_s + O(w) > 0$$

is smooth up to $w = 0$, jointly in s . Only this scalar is asserted smooth in w .

Choose $w(T) = w_0$ above the positive equilibrium if $\delta_s > 0$, and solve

$$(40) \quad \dot{w} = -2w(z_s(w) - \delta_s), \quad t = \sqrt{w}, \quad H_s^{[T]} = \Phi_s^*(t^q)^*(h_{0,s}e^{u(s,t)}).$$

The solution is positive at finite times, monotone, and tends to $w_\dagger$, where $z_s(w_\dagger) = \delta_s$ for $\delta_s > 0$ and $w_\dagger = 0$ otherwise. Lemma 2.2 gives $\lambda_1(H_s^{[T]})/z_s(w) = 1 + O(w)$ and a fixed next gap.

Lemma 5.2. *The metric in (40) has full residual bounded by $Cw|z_s(w) - \delta_s|$, and*

$$(41) \quad \int_T^\infty \|\mathcal{E}_{H_s^{[T]}}\|_{\infty, \text{op}} d\tau \leq C(w_0 - w_*) \leq Cw_0.$$

Proof. Write $k = k_s(t)$. The diagonal gauge velocity is $-kq$. After transport by t^q , the residual is exactly

$$-k(e^{-u}qe^u + q - 2e^{-u/2}qe^{u/2}) + e^{-u}\partial_\tau e^u.$$

The parenthesis is $(\mathcal{A}_u - I)^2q$, where $\mathcal{A}_u X = e^{-u/2}Xe^{u/2}$, so it is $O(w^2)$. The exponential derivative formula and $\|u_t\| \leq C|t|$, $\dot{t} = -tk$, bound the second term by $Cw|k|$. Transported and intrinsic norms differ by a bounded factor $e^{O(w)}$, irrespective of the diagonal gauge size. Finally $w|z_s(w) - \delta_s| = |\dot{w}|/2$; monotonicity proves the integrable estimate. The residual vanishes at equilibrium. $\square$

Define the positive signed profile

$$(42) \quad P_\delta(\tau) = \begin{cases} \frac{\delta}{1 - e^{-2\delta\tau}}, & \delta \neq 0, \\ \frac{1}{2\tau}, & \delta = 0. \end{cases}$$

Lemma 5.3. *Suppose $w_0 \rightarrow 0$ and $2Tz_0(w_0) \rightarrow 1$ along a reference restart sequence $T \rightarrow \infty$. For every $\varepsilon > 0$, one sufficiently late T and one neighborhood U give*

$$\sup_{s \in U, \tau \geq T} \left| \frac{z_s(w_s^{[T]}(\tau))}{P_\delta(\tau)} - 1 \right| < \varepsilon.$$

Both signs of δ_s and all sizes of $\delta_s\tau$ are included.

Proof. Writing $z = z_s(w)$, its exact equation is $\dot{z} = -2\alpha_s(z)z(z - \delta_s)$, where $\alpha_s = wz'_s/z_s = 1 + O(w) = 1 + O(z)$, uniformly. For $\delta \geq 0$, compare with the constant-speed solutions

$$Y_r(\tau) = \frac{\delta}{1 - (1 - \delta/z_0)e^{-2r\delta(\tau-T)}} \quad (\delta > 0), \quad Y_r(\tau) = \frac{z_0}{1 + 2rz_0(\tau - T)} \quad (\delta = 0),$$

using $r = 1 \pm Cw_0$. Put $\eta = |2Tz_0 - 1| + |r - 1| + 2\delta T$. For small η , we have $\delta < z_0$, so the trajectory starts above equilibrium. The offset $\theta = -T - \log(1 - \delta/z_0)/(2r\delta)$, with its continuous zero-gap value, has $|\theta|/T \leq C\eta$. The ratio to P_δ is $(1 - e^{-x})/(1 - e^{-r(1+\theta/\tau)x})$, $x = 2\delta\tau$. Monotonicity and concavity of $1 - e^{-x}$ bound this ratio by $1 + O(\eta)$, uniformly for every $x \geq 0$.

For $\delta = -d < 0$, constant speed perturbations would change the exponential rate. Instead put $Q = \log(1 + d/z)$, $Q_0 = \log(1 + d/z_0)$, and $L = 2d(\tau - T)$. Then $Q = Q_0 + L + E$, with $E = 2d \int_T^\tau (\alpha_s(z) - 1) dv$. The pointwise estimate gives $|E| \leq Cz_0L$. Separation of variables, using $\alpha_s \geq 1/2$, also gives

$$|E| \leq Cd \log \frac{z_0 + d}{z(\tau) + d} \leq Cz_0.$$

Thus $|E| \leq Cz_0 \min(Q_0 + L, 1)$. For small z_0 this is at most $(Q_0 + L)/2$, so every intermediate argument in the following mean-value estimate is at least $(Q_0 + L)/2$. Since $(\log(e^x - 1))' \leq 1 + 1/x$, the solution differs from its unit-speed Y_1 by logarithmic relative error $O(z_0)$ for the entire future interval. For that unit-speed solution the offset is $\theta = -T + \log(1 + d/z_0)/(2d)$. If $\eta = |2Tz_0 - 1| + 2dT$ is small, then $|\theta|/T \leq C\eta$; applying the same logarithmic derivative bound to exponents $2d\tau$ and $2d(\tau + \theta)$ gives logarithmic relative error $O(\eta)$ to P_{-d} , even at exponentially late times.

Choose T first so that the reference clock and w_0 errors are small. For that fixed T , continuity of $z_s(w_0)$ and $\delta_s \rightarrow 0$ gives a neighborhood where $|2Tz_s(w_0) - 1|$, $|\delta_s|T$, and $z_s(w_0)$ are small. Positive-gap trajectories start above equilibrium. The two estimates prove the assertion, with no comparison of $|s|$ to the possibly much smaller $|\delta_s|$. $\square$

Theorem 5.4. *For any smooth positive initial-metric family on the fixed two-factor extension, let $h_s(\tau)$ solve normalized HYM flow. For every $\varepsilon > 0$ there are a finite T and a neighborhood U of zero such that for all $s \in U$ and $\tau \geq T$,*

$$(43) \quad \left| \frac{\lambda_1(h_s(\tau), \omega_s)}{P_{\delta_s}(\tau)} - 1 \right| < \varepsilon.$$

There are also one fixed late time, neighborhood and positive lower bound for λ_2 throughout their future interval.

Proof. The qualitative matching lemma proved in Section 8 uses only reference convergence and the two-step holomorphic filtration. It gives $T \rightarrow \infty$, $w_T \rightarrow 0$, scalars $a_T > 0$, and $\eta_T \rightarrow 0$ with $h_0(T) \in e^{\pm \eta_T} a_T H_0(w_T)$. We first calibrate the clock without assuming a joint theorem. At the wall the radial solution has $\dot{z} = -2(1 + o(1))z^2$, so $\tau z \rightarrow 1/2$ for any fixed restart. Lemmas 5.1–5.2 compare the exact reference trajectory with it at all future times with logarithmic error $\eta_T + Cw_T$. Letting these restart errors tend to zero proves $2\tau\lambda_1(h_0(\tau)) \rightarrow 1$. Evaluating at T , and using the matching and model spectral estimate, gives $2Tz_0(w_T) \rightarrow 1$.

For each fixed late T , finite-time parameter continuity gives $h_s(T) \in e^{\pm o_T(1)} h_0(T)$ and $H_s(w_T) \in e^{\pm o_T(1)} H_0(w_T)$. One direct proof of the first assertion regards h_0 on $[0, T]$ as an approximate ω_s -flow: its residual is $2i(\Lambda_{\omega_s} - \Lambda_{\omega_0})F_{h_0} - 2(\kappa_E(s) - \kappa_E(0))I = O_T(|s|)$, and the initial metrics are close. Apply Lemma 5.1. After choosing T , shrink the neighborhood so that these restart errors and $|\delta_s|T$ are small. The residual integral is $O(w_T)$ uniformly for all future times. The scalar lemma, model spectral estimate and metric Rayleigh comparison prove (43). One fixed sufficiently late restart supplies the complementary gap as well. $\square$

For $\delta > 0$ the profile tends to δ ; at the wall it is $1/(2\tau)$; for $\delta < 0$ it is asymptotic to $|\delta|e^{-2|\delta|\tau}$. The theorem includes all these regimes and their joint transitions. Its spectrum is that of the endomorphism Dolbeault Laplacian; it does not identify the full Yang–Mills Hessian or a nonlinear contraction rate.

6. AN EXACT FINITE EVOLUTION FOR THE FLOW

The elliptic complementary metrics do not in general form an invariant manifold for HYM flow. We construct a separate parabolic graph and retain its exact finite field. Each Kähler parameter s remains constant along a trajectory.

For a general small normal u , retain $\mathcal{K}(s, t, u)$ from Section 2 and C_u, S_u from (21). Set

$$(44) \quad B_u = S_u^{-1}C_u, \quad \mathcal{R}_s(u) = P_s B_u|_{\mathcal{H}}.$$

In an eigenbasis of u , S_u has eigenvalue $\sinh a/a > 0$, and B_u has eigenvalue $a \coth a > 0$, where $a = (u_i - u_j)/2$. They preserve real Hermitian sections and trace. The finite operator $\mathcal{R}_s(u)$ is positive and invertible, with $\mathcal{R}_s(u) = I + O(\|u\|_\infty^2)$. Define

$$(45) \quad \begin{aligned} A(s, t, u) &= -\mathcal{R}_s(u)^{-1} P_s S_u^{-1} \mathcal{K}(s, t, u), \\ \dot{t}_e &= (A_i - A_j)t_e, \quad (e : j \rightarrow i), \\ \dot{u} &= -2S_u^{-1} \mathcal{K}(s, t, u) - 2B_u A(s, t, u). \end{aligned}$$

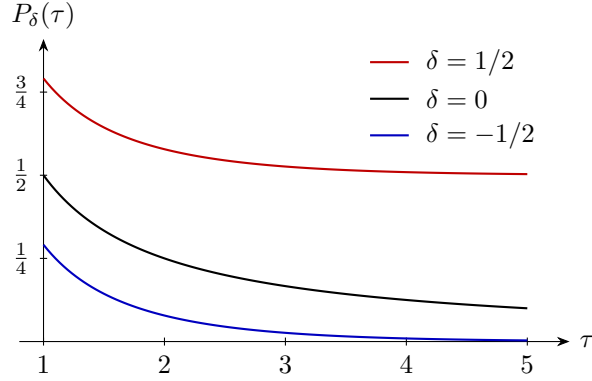


FIGURE 2. The proved two-factor profile on the stable side, at the wall, and on the unstable side. These are evaluations of formula (42), not measured HYM trajectories. Uniform relative comparison in Theorem 5.4 includes the entire future interval after its late restart.

Here $A \in \mathcal{H}$, so its components are rank-centered.

Proposition 6.1. *System (45) preserves $P_s u = 0$ and pointwise trace zero. Every classical solution reconstructs an exact normalized HYM flow on the bundle at its initial amplitudes. It preserves every zero edge, every nonzero amplitude sign, and its initial diagonal orbit.*

Proof. Integrate $\dot{g}g^{-1} = A$, starting from a positive diagonal gauge. Its determinant stays one. The amplitude equation gives $N_{t(\tau)} = g(\tau)N_{t(T)}g(\tau)^{-1}$, with the initial gauge included in this identity as appropriate. On that fixed bundle the reconstructed metric is $H = g^*(h_{0,s}e^u)$. Centering its velocity gives exactly

$$(46) \quad 2C_u A + S_u \dot{u} = -2\mathcal{K}.$$

Indeed the two gauge terms are $e^{-u/2}Ae^{u/2} + e^{u/2}Ae^{-u/2}$, and

$$e^{-u/2}(\partial_\tau e^u)e^{-u/2} = \int_{-1/2}^{1/2} e^{\theta \text{ad}_u} \dot{u} d\theta = S_u \dot{u}.$$

Solving (46) subject to $P_s \dot{u} = 0$ gives exactly (45). Its pointwise trace is zero because $\text{tr } \mathcal{K} = \text{tr } A = 0$ and the commutator functions preserve trace. The multiplicative amplitude equations preserve signs and zeroes. For $h \in \ker D$, they give $\partial_\tau \sum_e h_e \log |t_e| = h^T D^T A = 0$ on nonzero support. Thus the evolution stays on its initial diagonal orbit. $\square$

6.1. The normal equation and its domain. The highest-derivative coefficient in centered Chern curvature is the same S_u as in the velocity. At a contracted second jet W the metric-change formula gives

$$e^{-u/2}(d \exp)_u(W)e^{-u/2} = S_u W.$$

All derivatives of this coefficient and all extension terms use only first spatial jets. Consequently there is an exact identity

$$(47) \quad S_u^{-1} \mathcal{K} = L_s u + K_{\text{split},s} + \mathcal{G}_s(t, u, \nabla_s u), \quad K_{\text{split},s} = \text{diag}(-b_v/r_v),$$

where $\mathcal{G}_s$ depends only on the first jet and its value and first derivatives vanish at $(t, u) = (0, 0)$. The amplitude derivative vanishes by current harmonicity; the normal derivative is the

definition of the full real split linearization. Self-adjointness and $L_s \mathcal{H} = 0$ imply $P_s L_s u = 0$, so (45) takes the form

$$(48) \quad \begin{aligned} A &= -\mathcal{R}_s(u)^{-1}(K_{\text{split},s} + P_s \mathcal{G}_s), \\ \dot{u} &= -2L_s u + \mathcal{N}_s(t, u), \\ \mathcal{N}_s &= -2\mathcal{G}_s + 2(B_u \mathcal{R}_s(u)^{-1} - I)K_{\text{split},s} + 2B_u \mathcal{R}_s(u)^{-1} P_s \mathcal{G}_s. \end{aligned}$$

Both first derivatives of $\mathcal{N}_s$ vanish at the origin, for every nearby s . In particular no extra $b_s u$ normal linear term remains. The edge linearization there is

$$(49) \quad (\Lambda_s)_{ee} = b_i/r_i - b_j/r_j \longrightarrow 0.$$

We verify a concrete continuous-maximal-regularity realization of Simonett's parameter theorem [Sim94, Theorem 2.1]. Let h^q denote little-Hölder sections, the closure of smooth sections in the nonintegral Hölder norm, of the real Hermitian pointwise trace-free bundle $\mathcal{V}_s$. Smooth block isometries identify it with $\mathcal{V}_0$. The maps $T_s = Q_s|_{Q_0}$ and their inverses identify all normal spaces uniformly. Put $u = T_s v$ and take

$$(50) \quad \begin{aligned} X_0 &= \mathbb{R}^{E(\Gamma)} \oplus Q_0 h^{3/8}(\mathcal{V}_0), & X_1 &= \mathbb{R}^{E(\Gamma)} \oplus Q_0 h^{19/8}(\mathcal{V}_0), \\ X_\beta &= (X_0, X_1)_{5/8, \infty}^0 = \mathbb{R}^{E(\Gamma)} \oplus Q_0 h^{13/8}, & X_\alpha &= (X_0, X_1)_{7/8, \infty}^0 = \mathbb{R}^{E(\Gamma)} \oplus Q_0 h^{17/8}. \end{aligned}$$

The superscript zero denotes continuous real interpolation. All spatial exponents are non-integral. The bounded finite-rank projection preserves these interpolation identities. A first derivative of $u \in h^{13/8}$ lies in $h^{5/8} \subset h^{3/8}$. Hölder multiplication, smooth matrix functions near zero and the finite inverse in (48) therefore make every nonlinear term a smooth map from a neighborhood in X_β to X_0 , jointly with s .

The principal operator is the smooth common-domain family $\mathcal{A}(s) = \text{diag}(0, 2T_s^{-1}L_s T_s) : X_1 \rightarrow X_0$. It has continuous maximal regularity on this scale. One can apply the normally elliptic little-Hölder theorem [SS14, Remark 3.3 and Proposition 3.7] to the endomorphism bundle as follows. Embed the real bundle into a trivial bundle with an orthogonal complement, extend L_s by a connection Laplacian on that complement with the same scalar principal symbol, and apply the local matrix-valued elliptic resolvent proof on the trivial bundle. Restriction to the reducing original summand gives the bundle result. Since P_s commutes with L_s , uniqueness of the linear parabolic solution restricts maximal regularity to Q_s with its full domain. Conjugation by T_s and the finite-dimensional zero summand preserve it. The estimates are locally uniform in s .

At the wall the flow linearization is $\text{diag}(0, -2L_0|_{Q_0})$, with finite center and uniformly negative normal spectrum. Thus all hypotheses of the parameter-dependent invariant-manifold theorem hold. Keeping L_s in its common-domain operator family is necessary: treating $(L_s - L_0)u$ as a lower-order term on X_β would lose two derivatives.

Theorem 6.2. *Fix a finite integer $\ell \geq 4$. On a sufficiently small common parameter/amplitude neighborhood there is a jointly C^ℓ map $\Psi(s, t) \in Q_s h^{19/8}(\mathcal{V}_s)$ whose graph is locally invariant for (45), contains every equilibrium sufficiently small in the domain norm of X_1 and inside the construction's cutoff neighborhood, and satisfies*

$$(51) \quad \Psi(s, 0) = 0, \quad D_t \Psi(s, 0) = 0, \quad \|\Psi(s, t)\|_{C^2} \leq C|t|^2, \quad \|D_t \Psi(s, t)\|_{C^2} \leq C|t|.$$

It may be chosen equivariant under constant vertex signs. Substitution $u = \Psi(s, t)$ in (45), with its exact A , defines one finite corrected field. Its trajectories reconstruct exact classical HYM solutions while they remain in the domain.

Proof. The preceding verification and [Sim94, Theorem 2.1] give a graph valued in the full normal domain $h^{19/8}$, not merely in the interpolation space. Every $(s, 0, 0)$ is an equilibrium, so it lies on the graph. Differentiate the invariance identity at $t = 0$. If $H_s = D_t(T_s^{-1}\Psi)(s, 0)$, then $H_s\Lambda_s = B_sH_s$, where $B_s = -2T_s^{-1}L_sT_s$. Shrink so each eigenvalue of Λ_s has absolute value below the fixed normal gap. Each $B_s - (\Lambda_s)_{ee}$ is invertible, whence $H_s = 0$. Taylor's formula, jointly in s and t , gives the uniform quadratic bounds at every nearby split equilibrium.

For vertex signs ε_v , conjugation by $U_\varepsilon = \text{diag}(\varepsilon_v I)$ sends t_e to $\varepsilon_i \varepsilon_j t_e$ and u to $U_\varepsilon u U_\varepsilon^{-1}$. It preserves the exact equations and all projections. In the fixed-point construction of the invariant manifold choose invariant norms and a radial smooth cutoff in the finite amplitude space. The transformed graph solves the same integral fixed-point equation and belongs to the same uniqueness class. Hence that constructed graph is equivariant. This uses no averaging of graphs and no smooth Banach-space norm.

Local invariance and parabolic uniqueness identify a reduced trajectory with (45). Since the graph is C^1 into the full $h^{19/8}$ domain, its reconstructed metric is C^1 in time with continuous second spatial derivatives, and the parabolic equation holds pointwise. Proposition 6.1 then gives an exact classical HYM solution. $\square$

The graph is a chosen finite-regularity object and is not canonical. The construction can be made at any prescribed finite spatial order by shifting all exponents in (50). We neither identify graphs obtained on different scales nor assert a single infinitely differentiable graph. Equilibria on disconnected supports need not be HE metrics; the equilibrium-inclusion statement alone says no more.

7. CONFINEMENT WITHOUT A LOWER CUT BOUND

Fix the invariant graph of Theorem 6.2. On it write

$$(52) \quad A = M^{-1}(b - p) + E_s(t), \quad k_e = c_e(s)t_e^2, \quad p = Dk, \quad \sigma = \sum_e k_e.$$

The quadratic projected curvature term is $M^{-1}Dk$. The projected split linearization annihilates the normal variable exactly; substituting $\Psi = O(|t|^2)$ in the smooth equations therefore gives

$$(53) \quad \|E_s(t)\|_M \leq C|t|^3 \leq K\sigma^{3/2}.$$

In particular there is no suppressed $|b||t|^2$ term. The contributions of the split scalar curvature through $\mathcal{R}_s(u)^{-1} - I$ and $S_u^{-1} - I$ are $O(|b||t|^4)$, since both operators differ from the identity by $O(u^2)$. They are absorbed in $C|t|^3$ on the fixed bounded parameter neighborhood. The proof will only use this cubic bound, not a higher-order sign improvement.

Let $R_I = \sum_{i \in I} r_i$, $R_J = \sum_{j \in J} r_j$, and put

$$\rho_* = R_I^{-1} + R_J^{-1}, \quad C_*^2 = (\min_{i \in I} r_i)^{-1} + (\min_{j \in J} r_j)^{-1}.$$

Since all arrows go from J to I ,

$$(54) \quad \sum_{i \in I} p_i = \sigma, \quad \sum_{j \in J} p_j = -\sigma, \quad \rho_* \sigma^2 \leq \|p\|_{M^{-1}}^2 \leq C_*^2 \sigma^2.$$

The lower estimate is weighted Cauchy–Schwarz on each part. For the upper estimate, the components have one sign on each part, so the sum of their squares is at most the square of their sum. This estimate allows every nonnegative conductance ratio.

Proposition 7.1. *There is a fixed small threshold σ_* such that, if at a restart T the amplitudes are positive,*

$$(55) \quad 0 < \sigma(T) = \sigma_0 < \sigma_*, \quad \|b\|_{M^{-1}} \leq \frac{\rho_*}{4C_*} \sigma_0,$$

the exact finite trajectory exists for every $\tau \geq T$, stays in the graph domain, and satisfies $0 < \sigma(\tau) \leq \sigma_0$. At $b = 0$,

$$(56) \quad \sigma(\tau) \leq \frac{\sigma_0}{1 + \rho_* \sigma_0 (\tau - T)}.$$

Proof. Since s is fixed, the amplitude equations give exactly

$$\dot{\sigma} = 2p \cdot A = -2\|p\|_{M^{-1}}^2 + 2p^T M^{-1} b + 2p \cdot E_s(t).$$

Using (53)–(54) and shrinking the threshold to absorb $2C_* K \sigma^{5/2}$, we obtain

$$(57) \quad \dot{\sigma} \leq -\rho_* \sigma^2 + 2C_* \|b\|_{M^{-1}} \sigma.$$

At a first upward crossing of σ_0 this derivative would be at most $-\rho_* \sigma_0^2/2$, a contradiction. Choosing σ_* smaller if necessary keeps $|t|^2 \leq \sigma_0/c_{\min}$ inside a strictly smaller amplitude ball, and the quadratic graph bound keeps the normal variable inside its domain. Finite-dimensional continuation gives the all-future solution. Positivity persists at every finite time by the multiplicative equations. At $b = 0$, integrate $\dot{\sigma} \leq -\rho_* \sigma^2$ to obtain (56). $\square$

No individual edge must decrease. For unit masses, targets 1, 2, sources 3, 4, take leading conductances

$$(k_{13}, k_{14}, k_{23}, k_{24}) = (\epsilon, \epsilon^2, \epsilon^2, 1), \quad b = (2\epsilon, \epsilon, -2\epsilon, -\epsilon).$$

For the leading field the first edge grows because $A_1 - A_3 = 2\epsilon - 2\epsilon^2 > 0$, while $A_2 - A_4 = -2 + 2\epsilon - 2\epsilon^2 < 0$. Scaling both the conductances and b by a common small positive number places the example in an arbitrarily small amplitude neighborhood; for sufficiently small ϵ , (55) holds and the total decreases. This explains why the scalar total, rather than separate edge monotonicity, is the appropriate confinement quantity. The exact corrected field, rather than its leading term, is used in the proposition and all subsequent comparisons.

8. MATCHING PRESCRIBED ACTUAL FLOWS

Let h_s^{in} be any smooth family of positive Hermitian metrics on the fixed E in (10), on a common parameter neighborhood. Let

$$(58) \quad h_s^{-1} \partial_\tau h_s = -2(K_{h_s, \omega_s} - \kappa_E(s)I), \quad h_s(0) = h_s^{\text{in}}.$$

We use qualitative convergence only at $s = 0$, and finite-time continuity after fixing a restart. No nearby-parameter uniform infinite-time convergence theorem is assumed.

At the reference class the extension is semistable. Refining its two split layers gives a Jordan–Hölder filtration whose graded object is the locally free $\bigoplus_v F_v$. Thus the algebraic singular set in [SW15, §2.2] is empty. The corrected version of that theorem supplies smooth global reference connection convergence modulo unitary gauge [SW15, Theorem 1.1(2)]. Its erratum changes the algebraic-cycle multiplicity definition and leaves this convergence statement unchanged. We need only a late subsequence of this qualitative convergence.

The fixed E is simple. Indeed cross-Hom vanishing makes any endomorphism preserve S , and its induced actions on the distinct stable factors of S and U are scalar. The extension relation makes endpoint scalars agree on every nonzero edge. Connectedness makes them all equal, and $\text{Hom}(U, S) = 0$ removes the remaining off-diagonal map. Consequently the complex End_0 Laplacian has no kernel at any finite time, for any metric.

Lemma 8.1. *There are times $T \rightarrow \infty$, determinant-one positive diagonal gauges g_T , scalars $a_T > 0$, and errors $\eta_T \rightarrow 0$, such that*

$$(59) \quad h_0(T) \in e^{\pm\eta_T} a_T \Phi_0^* g_T^* h_{0,0}, \quad t_{T,e} = (g_T)_i / (g_T)_j > 0, \quad |t_T| \rightarrow 0.$$

Here $h \in e^{\pm\eta} H$ means $e^{-\eta} H \leq h \leq e^{\eta} H$ as Hermitian forms. The gauges lie on the fixed diagonal extension orbit, including when the support contains cycles.

Proof. We give the holomorphic matching step after qualitative convergence. For each $i \in I$, the space $\text{Hom}(F_i, E)$ is one-dimensional: the extension exact sequence and $\text{Hom}(F_i, U) = 0$ reduce it to its scalar inclusion into S . Similarly $\text{Hom}(E, F_j)$ is one-dimensional for each $j \in J$. Along the smooth connection-convergence sequence, normalize the generators of these holomorphic-map kernels. Elliptic estimates and convergence of the corresponding Dolbeault operators identify their kernels with the limiting one-dimensional kernels. After unitary phases their maps converge smoothly to the standard inclusions and projections of the distinct stable summands.

Write $i_T : S \rightarrow E$ for the combined normalized inclusions and $p_T : E \rightarrow U$ for the combined normalized projections in the connection-convergence realization. Then $p_T i_T = 0$, and p_T is surjective for late T , by convergence to the standard projection. Its orthogonal right inverse is

$$j_T = p_T^* (p_T p_T^*)^{-1}, \quad p_T j_T = I_U.$$

Thus $(i_T, j_T) : S \oplus U \rightarrow E$ is a smooth splitting converging to the standard orthogonal one. Its induced factor metrics are close to the fixed factor HE metrics after the positive scalar normalizations just used. In this splitting the extension entry $\beta_T : U \rightarrow S$ is $\bar{\partial}$ -closed and tends smoothly to zero. The two-step condition is what makes it a closed entry in every complex dimension; no vanishing of all $(0, 2)$ -forms is used. Apply the split Hodge decomposition entrywise. Its exact component is removed by a triangular change tending smoothly to the identity, because the cross-Hom Green operators have no degree-zero kernel.

Each harmonic component is a nonzero scalar multiple of its fixed class $\beta_{e,0}$. Indeed the convergence realization remains holomorphically isomorphic to the fixed E , and its inclusions $F_i \rightarrow E$ and projections $E \rightarrow F_j$ are unique up to scalars. Thus the extension class in the splitting (i_T, j_T) is the fixed class rescaled by these factor maps. These scalars are ratios of the vertex normalizations of the inclusions and projections: a change of those fixed holomorphic factor maps acts on an extension component by its target scalar divided by its source scalar. The multipliers therefore obey all the diagonal-orbit relations. Their phases can be absorbed by parallel unitary vertex gauges, leaving positive ratios $t_{T,e}$. Harmonic projection of $\beta_T \rightarrow 0$ shows that every $t_{T,e} \rightarrow 0$, since the original classes are nonzero. The metric in the resulting harmonic presentation tends smoothly to $h_{0,0}$.

The resulting holomorphic comparison with the original fixed extension differs by an automorphism of E ; simplicity leaves only an overall scalar. Its positive metric factor is a_T . Multiplying all diagonal vertex factors by one common positive number makes $\det g_T = 1$ without changing amplitudes, and absorbs the change into a_T . The metric convergence gives (59). No rate or smooth choice in T is required. $\square$

This lemma is independent of the radial clock calculation in Section 5. For two factors it supplies exactly the matching used there; for several factors it avoids assuming a universal reference spectral rate.

Choose once the graph of Theorem 6.2. At a matching time start its finite trajectory at the same t_T for every nearby s , with initial gauge g_T . Its reconstructed exact flow on the fixed bundle is

$$(60) \quad H_s^{[T]}(\tau) = \Phi_s^* g_s(\tau)^* (h_{0,s} e^{\Psi(s, t^{[T]}(s, \tau))}), \quad \dot{g}_s g_s^{-1} = A(s, t^{[T]}, \Psi), \quad g_s(T) = g_T.$$

At T , the extra quadratic graph correction changes the matching error by $O(|t_T|^2)$, which can be absorbed in η_T . Set $\Sigma_T = \sum_e c_e(0) t_{T,e}^2 \rightarrow 0$.

Theorem 8.2. *For every $\varepsilon > 0$ there are one finite matching time T , one neighborhood U_ε of zero, and trajectories of the parameter-dependent exact finite field of that graph, with the same restart data $g_s(T) = g_T$ for every s , such that for all $s \in U_\varepsilon$, $\tau \geq T$, and $j < m$,*

$$(61) \quad \begin{aligned} e^{-\varepsilon} a_T H_s^{[T]}(\tau) &\leq h_s(\tau) \leq e^{\varepsilon} a_T H_s^{[T]}(\tau), \\ \left| \log \frac{\lambda_j(h_s(\tau), \omega_s)}{\nu_j(M^{-1/2} D \operatorname{diag}(k^{[T]}(s, \tau)) D^T M^{-1/2})} \right| &\leq \varepsilon, \quad k_e^{[T]} = c_e(s) (t_e^{[T]})^2 > 0. \end{aligned}$$

The scalar $a_T > 0$ is independent of s and τ . There are also one fixed neighborhood U_* , time T_* , and $\gamma > 0$ such that $\lambda_m(h_s(\tau), \omega_s) \geq \gamma$ for $s \in U_*$ and $\tau \geq T_*$. Every tolerance neighborhood can be intersected with U_* . Stable, wall and unstable parameters are included. No fixed positive lower bound on normalized cut capacities or ratios of positive graph eigenvalues is required.

Proof. For a fixed T , finite-time parameter comparison gives $h_s(T) \in e^{\pm o_T(1)} h_0(T)$, by the residual/barrier argument in the proof of Theorem 5.4. The initial model (60) is continuous in s at this fixed positive amplitude vector. Choose any $\delta_T \downarrow 0$ and shrink a neighborhood U_T , after choosing T , so the two parameter errors total at most δ_T , and

$$(62) \quad \frac{1}{2} \Sigma_T \leq \sum_e c_e(s) t_{T,e}^2 \leq 2 \Sigma_T, \quad \|b(s)\|_{M^{-1}} \leq \frac{\rho_*}{8C_*} \Sigma_T.$$

Continuity at zero provides all these conditions. They imply (55), so confinement reconstructs an exact model for every future time with total conductance at most $2 \Sigma_T$.

At the restart, the actual metric is within $e^{\pm B_T}$ of a_T times the model, where $B_T = \eta_T + \delta_T \rightarrow 0$. Both are exact flows on the same bundle and form. The order lemma propagates precisely this error to every future time with zero future residual. Lemma 2.2 applies to the graph correction $\Psi = O(|t|^2)$ and gives model log spectral error at most $C \Sigma_T$, with a uniform next gap. The metric sandwich therefore gives

$$(63) \quad \left| \log \frac{\lambda_j(h_s(\tau), \omega_s)}{\nu_j(k^{[T]}(s, \tau))} \right| \leq 4B_T + C \Sigma_T.$$

Choose T late enough for the requested tolerance. One fixed late restart, together with the model complementary gap and its metric sandwich, gives U_*, T_*, γ . $\square$

If the initial determinants equal the factor determinant metric, the determinant heat equation preserves that identity. The model has this same determinant because $\det g_s = 1$ and $\operatorname{tr} \Psi = 0$. Taking determinants in the restart sandwich gives $|\log a_T| \leq B_T$, so absorbing this error removes a_T in the final arbitrary-tolerance assertion. For general initial determinants we retain a_T ; no time-dependent scalar renormalization of the prescribed flow is made.

The graph is noncanonical but is fixed before the tolerance and restart are chosen. The restart coordinates may depend on the initial family and tolerance. The theorem retains their exact corrected field and initial-orbit information. It gives no all-future replacement by the uncorrected field, no universal several-factor logistic profile, and no joint eigensection estimate.

9. THE COMMON TANGENT-BUNDLE DEGENERATION

We now prove the synthesis theorem stated in the introduction. The geometry is supplied independently by [Wue26d]; we specify all its objects and verify each analytic hypothesis on the same fixed tangent bundle.

For $k \geq 1$, set $d = 2k$, $n = d + 2$, and use quotient projectivization in

$$(64) \quad B = (\mathbb{P}^1)^d, \quad L = \mathcal{O}_B(1^k, -1^k), \quad X_k = \mathbb{P}_B(\mathcal{O} \oplus L) \times_B \mathbb{P}_B(\mathcal{O} \oplus L), \quad \pi : X_k \rightarrow B.$$

Let $\ell = c_1(L) = \sum_j \epsilon_j h_j$, with k signs of each kind, $\eta_i = \xi_i - \ell/2$, and $H_0 = \eta_1 + \eta_2 + r_k \sum_j h_j$. The number $r_k > 1$ is the unique root of

$$(65) \quad (kr + 1)F'_k(r) - k(2k + 2)F_k(r) = 0, \quad F_k(r) = \sum_{j=0}^k \frac{(-1)^j \binom{k}{j} r^{2k-2j}}{(2j+1)(j+1)}.$$

These are genuine ample classes. In a lattice basis $e_1, e_2, f_1, \dots, f_d$, the ray pairs are $\{\pm e_1\}, \{\pm e_2\}, \{f_j, -f_j + \epsilon_j(e_1 + e_2)\}$; maximal cones choose one from each pair. The full ample cone for $H = x\eta_1 + y\eta_2 + \sum_j u_j h_j$ is $x, y > 0, u_j > (x + y)/2$. Thus H_0 is interior, while $-K_{X_k} = 2(\eta_1 + \eta_2 + \sum_j h_j)$ is nef and big and not ample. The varieties are smooth projective weak Fano, not Fano.

The independent geometric theorem [Wue26d, Theorem 1.1 and Propositions 7.1–7.2] gives the actual sequence

$$(66) \quad 0 \longrightarrow R_1 \oplus R_2 \longrightarrow E = T_{X_k} \longrightarrow \bigoplus_{j=1}^d Q_j \longrightarrow 0, \quad R_i = \mathcal{O}(2\xi_i) \otimes \pi^* L^{-1}, \quad Q_j = \pi^* \mathcal{O}_B(2h_j).$$

The n Chern classes $C = (2\eta_1, 2\eta_2, 2h_1, \dots, 2h_d)$ form a basis of $N^1(X_k)_{\mathbb{R}}$, and their degrees at H_0 agree. Hence the factors are distinct stable lines of equal slope. All $2d$ extension components are nonzero. Explicitly, differentiating a projective transition gives $\pm d \log g_{ab} \otimes z_i \partial_{z_i}$. The Cartan field generates the $\mathcal{O}_B$ direct summand of $\pi_* R_i = L^{-1} \oplus \mathcal{O}_B \oplus L$, with no higher direct images. Its j -component is $\pm \epsilon_j$ times the nonzero generator of $H^1(B, \mathcal{O}_B(-2h_j))$, whose coefficient map remains injective. Thus this is an actual complete bipartite extension, not a poset used in place of a bundle. Its off-diagonal products vanish for every amplitude. Refinement gives a locally free graded bundle, meeting the reference flow-matching hypothesis.

The equality poset is $K_{2,d}$, with the R_i minimal and the Q_j maximal. To recall why it is complete, the ray set is the disjoint union of two bases. At H_0 every ray divisor has the same degree, so a rank- q subspace contains at most $2q$ rays. Equality forces a positive-basis coordinate subspace, and a selected base coordinate forces both fibre coordinates. These are exactly the ideals of $K_{2,d}$. Its specified equality subbundles are $R_i = T_{X_k/Y_{3-i}}$, where $Y_i = \mathbb{P}_B(\mathcal{O} \oplus L)$ denotes factor i . For $A \subseteq \{1, \dots, d\}$, put $B_{A^c} = \prod_{j \notin A} \mathbb{P}^1$, with the empty product understood as a point. The ideal containing both minima and the Q_j with $j \in A$ corresponds to $T_{X_k/B_{A^c}}$, where $X_k \rightarrow B_{A^c}$ is the natural projection. All these projections are

smooth and projective. This is the filtration geometry used here; no classification of arbitrary foliations is asserted.

Write $D_X = \sum_v C_v$ and

$$(67) \quad z_v(H) = (C_v - D_X/n)H^{n-1}, \quad b_v(H) = -\frac{2\pi z_v(H)}{(n-1)!V(H)}, \quad V(H) = H^n/n!.$$

The full local stable chamber is

$$(68) \quad x_i = b_{R_i} > 0, \quad y_j = -b_{Q_j} > 0, \quad \sigma = x_1 + x_2 = \sum_j y_j.$$

Non-strict signs give semistability and their complement gives instability, locally. The pointed cone has dimension $n-1$, $2d = 4k$ rays, n facets and section $\Delta^1 \times \Delta^{d-1}$, with one free polarization coordinate. It is nonsimplicial. These signs reverse the upward equality order, since extension incidence is $Q_j \rightarrow R_i$.

Lemma 9.1. *There is a full analytic chart $H = H(s)$ on the fixed-volume hypersurface at H_0 , with $V(H(s)) = V_0$, $b(H(s)) = s \in Z := \{\sum_v s_v = 0\}$. It admits smooth Kähler representatives with $\|\omega_s - \omega_0\|_{C^a} = O_a(|s|)$ and an exchange-invariant reference ω_0 . On the stable part, $\|s\|_1 = 2\sigma$, so all background derivatives are controlled and $|s| = O(\sigma)$.*

Proof. The centered C_v are independent modulo their single sum relation, and the real ample Hodge isomorphism makes the derivative of the power map invertible [GRT16, Theorem 6.4 and Corollary 6.5]. Thus dz_{H_0} has rank $n-1$, with kernel exactly the radial line $\mathbb{R}H_0$. Volume changes on that line: $dV_{H_0}[H_0] = nV_0 \neq 0$. The volume denominator contributes no derivative at $z = 0$, so db is invertible on the fixed-volume tangent hyperplane. The inverse-function theorem gives the full signed chart and bounded derivatives. Average a reference Kähler form over fibre exchange, then use bounded closed representatives of a divisor basis to realize the chart by forms. Positivity is open. The sign identity (68) gives $\|s\|_1 = 2\sigma$. This is also [Wue26d, Proposition 9.1]. $\square$

Choose factor HE metrics by their scalar elliptic equations, smoothly in this chart. Their product is the specified determinant metric $h_{\det,s}$; choose the fixed-bundle harmonic isomorphism Φ_s with determinant one as in Section 2. Only the reference form is required exchange-invariant. Invariance throughout would force equal target slopes and lose the full chart.

Lemma 9.2. *The reference factor metrics can be chosen compatibly with fibre exchange so that the positive harmonic coefficient array has $c_{ij}(0) = a_i q_j$, with $a_i, q_j > 0$. In particular, the coefficient ratio around every four-cycle is*

$$\frac{c_{1j}(0)c_{2\ell}(0)}{c_{1\ell}(0)c_{2j}(0)} = 1 \quad (j \neq \ell).$$

The current array $c_{ij}(s)$ need not have rank one, but $c_{ij}(s)/c_{ij}(0) = e^{O(|s|)}$.

Proof. The exchange involution identifies R_1 with R_2 , fixes each Q_j , and carries the actual Cartan extension component in one row to the corresponding component in the other. Choose the two relative factor metrics as pullbacks of one another and the quotient metrics invariantly, using line HE uniqueness after scalar normalization. At the invariant reference form the harmonic projection commutes with this isometry, so the two harmonic rows have equal norms. Their common positive entries give the q_j . Independent constant rescalings of the factor metrics only multiply rows and columns, giving the stated $a_i q_j$ form. Smoothness

and positivity give the logarithmic coefficient estimate. This argument makes no claim of exchange symmetry off the reference. $\square$

For stable s , define

$$(69) \quad k_{ij}^0 = \frac{x_i y_j}{\sigma}, \quad L^0 = \begin{pmatrix} \text{diag}(x_1, x_2) & -xy^T/\sigma \\ -yx^T/\sigma & \text{diag}(y_1, \dots, y_d) \end{pmatrix},$$

and set

$$(70) \quad w_{R_i} = x_i/a_i, \quad w_{Q_j} = \sigma q_j/y_j, \quad W_v = w_v/(\prod_u w_u)^{1/n}, \quad h_{\text{mod},s} = \Phi_s^* \left(\bigoplus_v W_v h_{v,s} \right).$$

Then $c_{ij}(0)W_{R_i}/W_{Q_j} = k_{ij}^0$, with $Dk^0 = b$ and $\prod_v W_v = 1$. The balancing problem with reference coefficients is thus explicit. The coefficient estimate from [Wue26c, Proposition 2.3], or its cut proof, gives uniformly at the same demand

$$(71) \quad k_{ij}(s)/k_{ij}^0 = e^{O(\sigma)}, \quad \|x^{0,\text{current}} - \log W\|_\infty \leq C\sigma.$$

For completeness, along a path of log coefficients at fixed b , differentiating balance gives a graph equation with cut forcing at most the log-coefficient derivative times that cut's capacity. The cut estimate for the graph inverse controls potential oscillation uniformly. Integrating, then centering, proves the potential estimate, and exponentiating proves the edge estimate. Lemma 9.1 changes its $O(|s|)$ bound into $O(\sigma)$.

Lemma 9.3. For $\delta = \min(x_1, x_2, y_1, \dots, y_d) > 0$,

$$(72) \quad \delta \leq \nu_1(L^0) \leq \frac{n}{n-1}\delta, \quad \prod_{j=1}^{n-1} \nu_j(L^0) = \frac{nx_1x_2 \prod_j y_j}{\sigma}.$$

Proof. Put $D_g = \text{diag}(x_1, x_2, y_1, \dots, y_d)$. The off-diagonal block of its normalized adjacency matrix is $\sqrt{x}\sqrt{y}^T/\sigma$, of rank one and singular value one. The normalized Laplacian therefore has eigenvalues 0, 1 (the latter $n-2$ times), and 2. For $\sum_v u_v = 0$, subtract the degree-weighted mean c . The normalized gap gives

$$u^* L^0 u \geq \sum_v (D_g)_{vv} |u_v - c|^2 \geq \delta \sum_v |u_v - c|^2 \geq \delta \sum_v |u_v|^2.$$

At a minimum-degree vertex the trial vector $e_v - \mathbf{1}/n$ has energy δ and norm squared $(n-1)/n$, proving the upper bound. Delete the second target row and column. Schur elimination of $\text{diag}(y)$ gives cofactor $x_1 x_2 \prod_j y_j / \sigma$. The null vector $\mathbf{1}$ has norm squared n , so its nonzero eigenvalue product is n times this cofactor, giving the second identity. $\square$

Proof of Theorem 1.2. The geometry and filtration in part (1) were specified in (64)–(66), with actual nonzero components and two-step integrability. The independent toric classification gives part (2), and Lemma 9.1 realizes its entire signed slope space by controlled ample forms. The factor metrics and harmonic representatives meet every hypothesis of the analytic setup. Lemma 9.2 and (69)–(71) prove part (3).

For part (4), apply Theorem 2.1 to the current balancing, then use the edgewise comparison (71) and min–max. This gives every ordered complex small eigenvalue relative to L^0 , including multiplicities and arbitrary intersecting-face approaches, with its fixed next gap. Lemma 9.3 transfers the minimum-margin bound and the cluster product. No full zeta determinant is involved.

For part (5), the current-coefficient metric sandwich (17) transfers by (71) to $h_{\text{mod},s}$, even if its weights diverge. All gauges, the complementary correction and the W have determinant one, so both metrics have the prescribed determinant after the permitted global scalar normalization of HE. The exact inverse, its complementary Sobolev estimates and positive resolvents are Theorem 3.3. Because the chart realizes every centered slope direction, we may apply Theorem 4.3 with all ranks one; the input norm is Euclidean on Z . Substituting (71) proves the stated real response singular values. Stationary assertions here are confined to the strict chamber.

For part (6), the locally free graded bundle, distinct factors and connected actual support meet the matching hypotheses of Theorem 8.2. For each tolerance it supplies one T , then U_T , and exact trajectories of the same chosen corrected field. Here $\rho_* = 1/2 + 1/d$ and $C_* = \sqrt{2}$. The confinement conditions give $\sigma_{\text{dyn}} \leq 2\Sigma_T$ for all future times and, since $b = s$, $|s| \leq C\Sigma_T$. The exact metric and relative spectral comparisons and fixed next gap follow with stable, wall and unstable parameters included.

To identify its explicit dynamic graph, freeze only the coefficients:

$$(73) \quad \hat{k}_{ij} = c_{ij}(0)t_{ij}^2, \quad X_i = \sum_j \hat{k}_{ij}, \quad Y_j = \sum_i \hat{k}_{ij}, \quad \hat{\sigma} = \sum_i X_i = \sum_j Y_j.$$

The fixed-orbit relation $t_{ij} = g_{R_i}/g_{Q_j}$ and $c_{ij}(0) = a_i q_j$ make this a positive rank-one array, so $\hat{k}_{ij} = X_i Y_j / \hat{\sigma}$ exactly. Its ratio to the current array is $e^{O(|s|)}$, uniformly for every future time and every positive ratio. Min–max replaces the current finite spectrum by this product graph at additional logarithmic error $C|s|$, absorbed by the late choice and shrink.

Finally $\Psi = O(\Sigma_T)$ allows the exact model metric to be replaced in the sandwich by $\Phi_s^* g_s^* h_{0,s}$. The diagonal weights are g_v^2 , of product one. Inserting $X, Y, \hat{\sigma}$ into (70) gives exactly these weights: each unnormalized expression has the same scalar multiplier $\sum_j q_j / g_{Q_j}^2$, which cancels under product normalization. The determinant assertion follows from Section 8. The margins X, Y are instantaneous conductance sums, not the fixed slope demands x, y . They need not be balanced at finite times; their time law is still the exact corrected field, including its initial-metric information. This completes all six parts on the same fixed $E = T_{X_k}$. $\square$

For $k = 1$, $r_1 = (3 + \sqrt{21})/6$, $H_0^4 = 16 + 4\sqrt{21}$, and every factor has raw degree $6 + 2\sqrt{21}$. This is the same fourfold in [Wue26c, §7]. On the symmetric ray $x_i = \sigma/2$, $y_j = \sigma/d$, the positive finite spectrum is

$$\sigma/2 \text{ (1 time)}, \quad \sigma/d \text{ (} d-1 \text{ times)}, \quad \sigma(1/2 + 1/d) \text{ (1 time)}.$$

For the fourfold it is $\sigma/2, \sigma/2, \sigma$. Away from that ray, the same theorem permits every positive coordinate ratio, metric-weight ratio and spectral ratio. The spectral paper's three-scale path is a particular instance; the flow statement supplies no universal time profile for those instantaneous several-factor margins.

10. SCOPE AND SUPPORTING CALCULATIONS

The geometric, combinatorial, spectral and metric descriptions in Theorem 1.2 concern one fixed family of actual tangent bundles. The chamber supplies which centered slope vectors occur; the tangent sequence supplies the actual nonzero extension classes; harmonic representatives and factor metrics supply the finite coefficients. The spectral and response theorems then quantify loss of rigidity. The exact finite flow retains its own correction, orbit

and initial data. This is a concrete two-step instance of the connection between stability geometry and analytic degeneration.

The necessary data do not reduce to an unweighted chamber in general. On cyclic extension supports harmonic coefficient ratios can change the leading spectrum at fixed ranks and slopes. The tangent family's reference symmetry is a proved additional simplification, and current coefficients are not silently made rank one. Likewise dynamic conductance margins cannot be replaced by fixed equilibrium demands. The several-factor theorem provides no universal closed time profile for them.

Supporting finite computations accompany the printed formulas. They check the exact block inverse and positive-resolvent identities, two-factor mass normalization, formal non-commuting exponential velocities, bipartite confinement inequalities, signed scalar profiles and restart comparisons, and the tangent product graph. Independent failure controls include an ordinary off-diagonal perturbation that overwhelms a weak graph direction and a fixed speed perturbation that changes an unstable exponential tail. The programs and figure generators are reproducible independently of source-project modules. These exact or high-precision finite checks support the algebra; the elliptic and parabolic assertions are proved in the text, not established by those computations.

The invariant graph has prescribed finite differentiability and depends on its construction. No canonical reduction, one fixed infinitely differentiable graph, or parameter derivative of a complex-parameter resolvent is claimed. All spatial inverse estimates use the named scaled harmonic and unitary identifications. The response is intrinsic $H^{-1}dH$; it is neither an ordinary directional relative graph-inverse expansion nor a moduli-metric comparison. The inverse is a linear HE metric-equation inverse, not a nonlinear global quotient gauge fixing.

Repeated stable factors require a nonabelian finite automorphism algebra. Composable arrows require separate integrability and actual HE calibration. Singular graded objects, full spectral determinants and quotient/moduli metric degeneration also remain outside the theorems here. The toric examples do not provide general Fano or arbitrary-poset realization. Extending the common mechanism to those settings would require further theorems, rather than a change of terminology in this two-step synthesis.

REFERENCES

- [CT24] A. Clarke and C. Tipler, *Semi-stability and local wall-crossing for Hermitian Yang–Mills connections*, J. Symplectic Geom. **22** (2024), no. 3, 673–694; arXiv:2304.05245v1.
- [Del25] R. Delloque, *Continuity of HYM connections with respect to metric variations*, J. Lond. Math. Soc. (2) **112** (2025), no. 1, e70219, doi:10.1112/jlms.70219; arXiv:2403.16814v4.
- [GRT16] D. Greb, J. Ross, and M. Toma, *Variation of Gieseker moduli spaces via quiver GIT*, Geom. Topol. **20** (2016), no. 3, 1539–1610; arXiv:1409.7564v2.
- [HKKP18] F. Haiden, L. Katzarkov, M. Kontsevich, and P. Pandit, *Iterated logarithms and gradient flows*, preprint, arXiv:1802.04123 [math.RT], 2018. <https://arxiv.org/abs/1802.04123>.
- [HKKP23] F. Haiden, L. Katzarkov, M. Kontsevich, and P. Pandit, *Semistability, modular lattices, and iterated logarithms*, J. Differential Geom. **123** (2023), no. 1, 21–66. <https://doi.org/10.4310/jdg/1679503804>.
- [Kob87] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Publications of the Mathematical Society of Japan, vol. 15, Kanô Memorial Lectures, vol. 5, Iwanami Shoten and Princeton University Press, 1987.
- [Lam26] S. T. Lam, *Semistability and asymptotics of geometric flows*, preprint, arXiv:2606.09235 [math.DG], 2026. <https://arxiv.org/abs/2606.09235>.
- [ST24] L. M. Sektnan and C. Tipler, *Hermitian Yang–Mills connections on pullback bundles*, Calc. Var. Partial Differential Equations **63** (2024), no. 1, Paper No. 13, 63 pp., doi:10.1007/s00526-023-02618-z; arXiv:2006.06453v2.

- [SS14] Y. Shao and G. Simonett, *Continuous maximal regularity on uniformly regular Riemannian manifolds*, J. Evol. Equ. **14** (2014), no. 1, 211–248. <https://doi.org/10.1007/s00028-014-0218-6>.
- [SW15] B. Sibley and R. A. Wentworth, *Analytic cycles, Bott–Chern forms, and singular sets for the Yang–Mills flow on Kähler manifolds*, Adv. Math. **279** (2015), 501–531. <https://doi.org/10.1016/j.aim.2015.04.009>. See also the erratum included in arXiv:1402.3808v3.
- [Sim94] G. Simonett, *Invariant manifolds and bifurcation for quasilinear reaction-diffusion systems*, Nonlinear Anal. **23** (1994), no. 4, 515–544. [https://doi.org/10.1016/0362-546X\(94\)90092-2](https://doi.org/10.1016/0362-546X(94)90092-2).
- [Wue26d] B. J. Wuebben, *Nonsimplicial local stability chambers of toric tangent bundles*, companion manuscript, 2026.
- [Wue26b] B. J. Wuebben, *Small Dolbeault eigenvalues along Hermitian Yang–Mills flow*, preprint, 2026; [Zenodo:23643223](https://zenodo.org/record/23643223).
- [Wue26a] B. J. Wuebben, *Small Dolbeault eigenvalues near stability walls*, preprint, 2026; [Zenodo:23682045](https://zenodo.org/record/23682045).
- [Wue26c] B. J. Wuebben, *Uniform Hermitian–Einstein spectra for bipartite two-step extensions*, companion manuscript, 2026.

BERND JOHANNES WUEBBEN, NEW YORK, NY, wuebben@gmail.com