

# NONSIMPLICIAL LOCAL STABILITY CHAMBERS OF TORIC TANGENT BUNDLES

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ABSTRACT. We construct smooth projective toric varieties with nonsimplicial local slope-stability chambers for their tangent bundles. In every even dimension  $2k + 2 \geq 4$ , a fibre product of two projective-line bundles over  $(\mathbb{P}^1)^{2k}$  admits an ample real class at which the tangent bundle is strictly semistable. Near this class, the semistable locus is analytically isomorphic to a pointed cone over  $\Delta^1 \times \Delta^{2k-1}$ , times a line, where  $\Delta^j$  denotes the  $j$ -simplex. The torus-invariant saturated subsheaves of equal slope form the ideal lattice of the complete bipartite poset with two minimal and  $2k$  maximal elements, and the stable line factors realize every centered slope direction. These varieties are weak Fano and not Fano. The fourfold has minimal dimension and Picard rank among toric varieties with such a chamber and nearby stable polarizations. We also prove a local normal form at any ample real class where the tangent bundle of a smooth projective toric variety is semistable. It retains precisely the slope directions realized by polarizations, requires no local-freeness assumption on equal-slope subsheaves, and localizes every nearby Harder–Narasimhan filtration to their equality lattice.

## 1. INTRODUCTION

Slope stability of a toric tangent bundle is determined by finitely many linear inequalities in the numerical curve class of the polarization. The inequalities are explicit, but their geometric occurrence is a separate question. A cone constructed from an abstract partial order need not occur as the local stability chamber of a tangent bundle. We construct a family for which the entire equality lattice, the accessible slope directions, and the tangent extension can all be determined. Its smallest member has a square-cone chamber.

Throughout, projectivization parametrizes one-dimensional quotients. For  $k \geq 1$ , put  $d = 2k$ ,  $n = d + 2$ , and set

$$(1) \quad B = (\mathbb{P}^1)^d, \quad L = \mathcal{O}_B(1^k, -1^k), \quad X_k = \mathbb{P}_B(\mathcal{O}_B \oplus L) \times_B \mathbb{P}_B(\mathcal{O}_B \oplus L).$$

Here  $1^k, -1^k$  means  $k$  entries equal to 1 followed by  $k$  entries equal to  $-1$ . Write  $h_j$  for the base hyperplane classes,  $\xi_i$  for the two tautological classes, and

$$\ell = c_1(L), \quad \eta_i = \xi_i - \ell/2, \quad A = \eta_1 + \eta_2, \quad B_0 = \sum_{j=1}^d h_j.$$

The poset  $K_{2,d}$  has two minimal elements,  $d$  maximal elements, and every minimal element is below every maximal element. An *order ideal* contains every predecessor of each of its elements. An invariant saturated tangent subsheaf is an *equality subsheaf* at a semistable polarization if its slope equals that of the tangent bundle. We include zero and the whole tangent bundle in their lattice.

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For this family put  $C_1 = 2\eta_1$ ,  $C_2 = 2\eta_2$ ,  $C_{j+2} = 2h_j$ , and  $D = \sum_p C_p = c_1(T_{X_k})$ . The centered line-factor degree vector is

$$z(H) = ((C_p - D/n)H^{n-1})_{p=1}^n, \quad \sum_p z_p(H) = 0.$$

Thus its coordinates are the line-factor slopes after subtracting their mean. Saying that every such direction is realized locally means that the differential of  $H \mapsto z(H)$  is onto the zero-sum hyperplane.

**Theorem 1.1.** *For every  $k \geq 1$ , there is a unique  $r_k > 1$  such that the line bundles with first Chern classes  $2\eta_1, 2\eta_2, 2h_1, \dots, 2h_d$  have equal slope at  $H_0 = A + r_k B_0$ . This class is ample, and  $T_{X_k}$  is strictly semistable there. The following assertions hold.*

- (1) *The equality subsheaves form the ideal lattice of  $K_{2,d}$ . Adjoining one poset element to an ideal increases the rank of its corresponding subsheaf by one. The differential of  $H \mapsto z(H)$  at  $H_0$  is surjective onto the zero-sum hyperplane.*
- (2) *In real analytic coordinates centered at  $H_0$ , the local semistable set is the germ of*

$$(2) \quad \left\{ z \in \mathbb{R}^{d+2} : \sum_p z_p = 0, \quad z_1, z_2 \leq 0, \quad z_3, \dots, z_{d+2} \geq 0 \right\} \times \mathbb{R}.$$

*The stable set is obtained by making all displayed sign inequalities strict. The pointed cone has dimension  $d+1$ ,  $2d$  rays,  $d+2$  facets, and a section  $\Delta^1 \times \Delta^{d-1}$ . It is not simplicial.*

- (3) *For  $\pi : X_k \rightarrow B$ , the actual tangent sequence is*

$$(3) \quad 0 \longrightarrow R_1 \oplus R_2 \longrightarrow T_{X_k} \longrightarrow \bigoplus_{j=1}^d Q_j \longrightarrow 0, \quad R_i = \mathcal{O}_{X_k}(2\xi_i) \otimes \pi^* L^{-1}, \quad Q_j = \pi^* \mathcal{O}_B(2h_j).$$

*These are pairwise nonisomorphic stable line factors at  $H_0$ . All  $2d$  components in  $\text{Ext}^1(Q_j, R_i)$  are nonzero. Every specified saturated equivariant equality subsheaf is a relative tangent bundle of a smooth projection.*

*The variety  $X_k$  is weak Fano and is not Fano. For  $k = 1$ ,  $r_1 = (3 + \sqrt{21})/6$ , and  $\dim X_1 = \rho(X_1) = 4$ . Both dimensions are minimal for a nonsimplicial pointed chamber at a strictly semistable class with a nearby stable locus.*

Sections 5–7 prove this theorem from the fan, intersections, and tangent transitions. No analytic existence or spectral comparison is used to establish the geometry.

We also prove a local normal form without assuming that equality subsheaves are subbundles. The equality lattice determines a poset  $P$ , positive integer block ranks  $r_p$ , and block divisor classes  $C_p$ . For  $D = c_1(T_X)$ , define the centered slope map

$$\mathcal{A}(\alpha)_p = (C_p - r_p D/n) \cdot \alpha, \quad L_{\text{acc}} = \text{im } \mathcal{A} \subset Z = \{z \in \mathbb{R}^P : \sum_p z_p = 0\}.$$

Theorem 3.2 identifies the local semistable germ with  $(\mathcal{C}(P) \cap L_{\text{acc}}) \times \mathbb{R}^{\rho - \dim L_{\text{acc}}}$ , where  $\mathcal{C}(P)$  is defined by nonpositive sums on order ideals. The accessible image is essential: different sections of the same poset cone can have different faces or no strict stable direction. Proposition 4.1 gives a character criterion for this failure for any fixed toric equality data.

Nonsimpliciality distinguishes differentiable chamber germs, rather than their ordinary topology. Under strict feasibility, the semistable and stable germs are ambiently bi-Lipschitz

equivalent to a closed and open halfspace pair (Proposition 4.2). The equality lattice also controls more than the stable set: after a single shrink, every nearby ordinary Harder–Narasimhan filtration uses only its original equality subsheaves. Theorem 8.1 identifies the region of each filtration flag and gives its product form. For  $X_k$  all flags occur; there are  $6F(2k)$ , where  $F$  is the ordered Bell number. For the fourfold there are eighteen flags, thirteen of full dimension in the three-dimensional centered slope space.

**Proof strategy.** The ray set of our family is the union of two lattice bases. At the equal-slope class every ray divisor has the same degree, so the two basis rank bounds prove semistability and classify all equality cases at once. To produce that class inside the ample cone, we compute the volume along a symmetric line. Its derivative gives a scalar equal-slope equation; a normalized ample affine line and the strict real Hodge inequality prove a unique simple root. Independent block divisor classes then give every centered slope direction.

For the general local model, positivity makes the tight ray sets a distributive lattice. The real numerical Hodge isomorphism turns ample powers into local coordinates, leaving exactly the ideal inequalities restricted to the actual slope image. Finally, we apply the finite ray-span argument simultaneously to every equality quotient. One common neighborhood excludes all initially strict competitors, so the HN problem becomes maximal-average selection on the original ideal lattice. Semistability of its successive intervals gives the stated product regions.

**Relation to earlier work.** Klyachko’s filtrations describe equivariant bundles and the distinction between invariant saturated subsheaves and genuine subbundles [Kly90]; see also [KD19, Proposition 4.1.1]. Hering–Nill–Süß give the finite tangent slope test and explicitly describe its curve-class inequalities [HNS22, Proposition 1.2, Remark 2.6, and §5]. Neumann’s and Pang’s work already supplies ordinary real-class Harder–Narasimhan theory, torus invariance, polyhedrality, and successive quotient computations [Neu09, §3.3][Pan15, §§1.3, 2.1–2.2]. Our local equality-interval description and its products refine these results near a fixed semistable class; a general HN algorithm or curve-class polyhedrality is not a new assertion here.

The equality-poset construction and a chamber normal form under a cone-wise equality-subbundle hypothesis appear in [Wue26]. We retain that construction, remove the hypothesis from the numerical normal form by keeping the actual slope image, and prove the realizability theorem above. The root-cone ray and simpliciality criteria are those of Boussicault–Féray–Lascoux–Reiner [BFLR12, §5], with the edge sign reversed. Our family realizes their nonsimplicial cone by actual toric tangent bundles.

This geometry is the independent algebraic input to a three-paper programme on stability and loss of analytic rigidity. The companion spectral work treats actual Hermitian–Einstein metrics on two-step extensions; the subsequent quantitative work adds inverse, response, and flow estimates and states the integrated theorem on the same  $T_{X_k}$ . Section 9 specifies the slope normalization and parameter chart supplied here. The harmonic metric coefficients required by that analysis are additional data.

## 2. REAL POLARIZATIONS AND THE EQUALITY LATTICE

Let  $X = X_\Sigma$  be a smooth projective toric variety of dimension  $n \geq 2$ , with lattice  $N$ , dual lattice  $M$ , primitive ray vectors  $v_\rho$ , and invariant prime divisors  $D_\rho$ . We use ordinary intersection degrees

$$\deg_H \mathcal{F} = c_1(\mathcal{F})H^{n-1}, \quad \mu_H(\mathcal{F}) = \deg_H \mathcal{F} / \operatorname{rk} \mathcal{F}$$

for a nonzero torsion-free sheaf and an ample real divisor class  $H \in N^1(X)_{\mathbb{R}}$ . Stability always means stability against all coherent subsheaves of smaller positive rank. It suffices to test saturated subsheaves, whose quotients are torsion free.

The torus trivialization of  $T_X$  identifies its equivariant generic fibre with  $N_{\mathbb{C}}$ . For a complex subspace  $V \subset N_{\mathbb{C}}$ , let  $\mathcal{F}_V$  be the unique saturated equivariant subsheaf with generic fibre  $V$ . The tangent filtrations give

$$(4) \quad \mathrm{rk} \mathcal{F}_V = \dim V, \quad c_1(\mathcal{F}_V) = \sum_{v_\rho \in V} D_\rho.$$

For example, at the divisor of a ray the tangent filtration has the single intermediate line  $\mathbb{C}v_\rho$ . Its induced filtration on  $V$  contributes one to the first Chern coefficient precisely when that line lies in  $V$ . This proves (4) from the codimension-one filtrations; see [HNS22, §2][Pan15, Theorem 2.1.8]. Neither saturation nor this calculation asserts local freeness.

**Lemma 2.1.** *For an ample real  $H$ , ordinary slope (semi)stability of  $T_X$  is equivalent to*

$$(5) \quad \frac{\sum_{v_\rho \in V} D_\rho H^{n-1}}{\dim V} (<) \leq \frac{\sum_\rho D_\rho H^{n-1}}{n} \quad (0 \neq V \subsetneq N_{\mathbb{C}}).$$

*Only spans of subsets of ray vectors need be tested.*

*Proof.* For an integral ample class this is the tangent criterion of [HNS22, Proposition 1.2 and Remark 2.6]. The reduction to equivariant subsheaves also works at a real class, including equality. Choose an auxiliary integral torus-linearized ample line bundle to form a projective Quot scheme. The closure of the torus orbit of a quotient  $T_X \rightarrow \mathcal{Q}$  contains a torus-fixed quotient: one can successively take limits along one-parameter subgroups of the torus in a projective embedding. In this connected flat family, the rank and numerical first Chern class of the quotient, and hence of its kernel, are constant. Indeed the determinant of the flat family defines a family of line bundles on  $X$ . Since  $\mathrm{Pic}(X)$  is discrete, its class is constant on the connected orbit closure. This fixes the full numerical first Chern class, not just its degree against the auxiliary polarization. Saturating the fixed kernel makes it equivariant and adds an effective divisorial class to its first Chern class. Such a class has nonnegative degree against  $H^{n-1}$ . Thus any ordinary subsheaf violating either stability test gives an equivariant one with at least its real slope. Equation (4) now gives (5).

Every  $D_\rho H^{n-1}$  is positive. If  $V$  contains a ray, its contained-ray span  $W$  has the same numerator as  $V$  and no larger dimension; its slope is at least that of  $V$ . If it contains no ray, its slope is zero, strictly smaller than  $\mu_H(T_X) > 0$ . This proves the finite reduction.  $\square$

Fix henceforth an ample real  $H_0$  at which  $T_X$  is semistable. Write

$$(6) \quad \alpha_0 = H_0^{n-1}, \quad D = \sum_\rho D_\rho, \quad w_\rho = D_\rho \alpha_0 > 0, \quad \tau = D \alpha_0 / n > 0.$$

A subspace  $V$  is *tight* if  $\sum_{v_\rho \in V} w_\rho = \tau \dim V$ ; zero and  $N_{\mathbb{C}}$  are tight. A *ray flat* is the full set of rays in a subspace spanned by rays. Tight subspaces are ray-spanned by the proof above, since reducing the dimension would otherwise violate semistability.

**Proposition 2.2.** *The ray sets of the tight subspaces form a finite distributive lattice under union and intersection. There is a finite poset  $P$  of ray blocks such that these sets are exactly its order ideals. There are positive integers  $r_p$  and divisor classes  $C_p = \sum_{\rho \text{ in block } p} D_\rho$  satisfying*

$$(7) \quad \sum_{p \in P} r_p = n, \quad C_p \alpha_0 = \tau r_p.$$

For an ideal  $I$ , its tight subspace  $V_I$  and equality subsheaf  $\mathcal{F}_I = \mathcal{F}_{V_I}$  have  $\dim V_I = r(I) := \sum_{p \in I} r_p$  and  $c_1(\mathcal{F}_I) = C_I := \sum_{p \in I} C_p$ .

*Proof.* Put  $w(V) = \sum_{v_\rho \in V} w_\rho$ . For tight  $V, W$ ,

$$w(V) + w(W) \leq w(V + W) + w(V \cap W) \leq \tau(\dim(V + W) + \dim(V \cap W)) = w(V) + w(W).$$

All inequalities are equalities. Positivity of the ray weights shows that  $V + W$  contains no ray outside the union of the two ray sets. The intersection ray set is already their set intersection. Both subspaces are tight and ray-spanned. Their ray sets therefore determine them, proving the distributive-lattice assertion.

Declare two rays equivalent when they belong to exactly the same tight sets, and call the equivalence classes blocks. For a block  $q$ , let  $J_q$  be the intersection of all tight sets containing it. Define  $p \leq q$  if block  $p$  lies in  $J_q$ . This is a partial order: antisymmetry is precisely the block equivalence, and transitivity follows because  $q \subset J_r$  implies  $J_q \subset J_r$ . By definition,  $J_q = \bigcup_{p \leq q} p$ . Each tight set is an ideal, while every ideal is the union of its  $J_q$ ; these unions are tight. Thus we have exactly the asserted ideal lattice.

Both  $J_p$  and  $J_p \setminus \{p\}$  are ideals. Their rank difference is the positive integer

$$r_p = \dim V_{J_p} - \dim V_{J_p \setminus \{p\}} = \frac{\sum_{\rho \text{ in block } p} w_\rho}{\tau}.$$

Telescoping along a linear extension of any ideal proves all the rank and Chern-class formulas, including (7).  $\square$

We call  $P$  the *equality poset*. This is the poset whose ideal lattice is the lattice of invariant saturated equality subsheaves; it does not specify a locally free associated graded bundle or its extension classes. The construction agrees with [Wue26, §3] and uses no additional cone-wise subbundle hypothesis.

### 3. THE ACCESSIBLE LOCAL NORMAL FORM

For a finite poset  $P$ , put

$$(8) \quad Z = \{z \in \mathbb{R}^P : \sum_p z_p = 0\}, \quad \mathcal{C}(P) = \{z \in Z : z(I) := \sum_{p \in I} z_p \leq 0 \text{ for every order ideal } I\}.$$

Write  $\mathcal{C}^{\text{str}}(P)$  for the locus where every proper nonempty ideal inequality is strict. For a one-element poset it is  $\{0\}$ . The cone is the upward poset root cone: if  $p \triangleleft q$  is a cover, let  $u_{pq} = e_q - e_p$ . Our sign is opposite to that of [BFLR12, §5]. Here  $e_p$  denotes the standard coordinate vector of  $\mathbb{R}^P$ , independently of the lattice basis used later to describe a toric fan.

**Lemma 3.1.** *We have  $\mathcal{C}(P) = \text{cone}\{u_{pq} : p \triangleleft q\}$ . Its span has dimension  $|P| - c(P)$ , where  $c(P)$  is the number of connected components of the undirected Hasse graph. Its extreme rays are exactly the cover vectors, and it is simplicial precisely when that graph is a forest. If  $P$  is connected,  $\mathcal{C}^{\text{str}}(P)$  is its interior in  $Z$ , and consists of positive combinations of all cover vectors. If  $P$  is disconnected with at least two elements, the strict locus is empty.*

*Proof.* An upward edge has ideal sum either zero or minus one, giving the first inclusion. A functional nonnegative on all edge vectors is an order-preserving potential, modulo a constant. Express it as a constant plus a nonnegative combination of upper-level-set indicators. Their complements are ideals, so it is nonnegative on  $\mathcal{C}(P)$ . Separation from the closed finitely generated edge cone gives the reverse inclusion. Incidence rank gives the dimension. The

extreme-ray and forest assertions are the root-cone statements of [BFLR12, Proposition 5.1(i) and Corollary 5.3].

For connected  $P$ , the sum of all cover vectors has a strictly negative sum on every proper nonempty ideal: such an ideal has an edge leaving it. Conversely, from any strict point one may subtract a sufficiently small positive multiple of this vector and remain in the cone. This gives a representation with every edge coefficient positive and proves the interior assertion. In the disconnected case a component and its complement are proper ideals; both sums cannot be negative. The relative interior still has positive edge representations in the component-zero-sum span, but it is not the strict stable locus.  $\square$

Return to the equality data of Section 2. The numerical curve space  $N_1(X)_{\mathbb{R}}$  is dual to  $N^1(X)_{\mathbb{R}}$ . Both have dimension  $\rho$ . Define

$$(9) \quad K_p = C_p - \frac{r_p}{n}D, \quad \mathcal{A} : N_1(X)_{\mathbb{R}} \longrightarrow Z, \quad \mathcal{A}(\alpha)_p = K_p\alpha, \quad L_{\text{acc}} = \text{im } \mathcal{A}, \quad a = \dim L_{\text{acc}}.$$

The name *accessible slope space* means precisely this linear image, not an independently prescribed subspace of  $Z$ . By (7),  $\mathcal{A}(\alpha_0) = 0$ .

**Theorem 3.2.** *Let  $X$  be smooth projective toric,  $n \geq 2$ , and let  $T_X$  be semistable at an interior ample real class  $H_0$ . In real analytic coordinates centered at  $H_0$ , the pair of local semistable and stable sets is the germ of*

$$(10) \quad ((\mathcal{C}(P) \cap L_{\text{acc}}) \times \mathbb{R}^{\rho-a}, \quad (\mathcal{C}^{\text{str}}(P) \cap L_{\text{acc}}) \times \mathbb{R}^{\rho-a}).$$

*In particular, stable ample classes approach  $H_0$  if and only if  $L_{\text{acc}}$  meets  $\mathcal{C}^{\text{str}}(P)$ . No local-freeness assumption on  $\mathcal{F}_I$  is needed.*

*Proof.* The Hodge–Riemann bilinear relations for the Kähler class  $H_0$ , restricted to  $N^1(X)_{\mathbb{R}}$ , give the real numerical Hodge index theorem. Thus  $E \mapsto H_0^{n-2}E$  is an isomorphism  $N^1(X)_{\mathbb{R}} \rightarrow N_1(X)_{\mathbb{R}}$  [GRT16, Theorem 6.4 and Corollary 6.5]. Consequently the derivative of the polynomial power map  $\Psi(H) = H^{n-1}$  is the isomorphism  $(n-1)H_0^{n-2}(-)$ . The analytic inverse-function theorem gives an open neighborhood of  $\alpha_0$  consisting entirely of powers of actual ample real classes.

There are finitely many ray-flat tests. Those strict at  $H_0$  remain strict on a common neighborhood. The remaining tests are exactly the tight ideals, and their centered degrees are

$$(C_I - r(I)D/n)\alpha = \sum_{p \in I} \mathcal{A}(\alpha)_p.$$

Thus the semistable and stable inequalities are exactly those of (8) and its strict locus there.

Choose a linear right inverse  $J : L_{\text{acc}} \rightarrow N_1(X)_{\mathbb{R}}$  of  $\mathcal{A}$ . The explicit coordinates are

$$(11) \quad z = \mathcal{A}(\alpha - \alpha_0), \quad t = \alpha - \alpha_0 - Jz \in \ker \mathcal{A}, \quad \alpha = \alpha_0 + Jz + t.$$

These prove (10). A strict  $z$  lifts along  $\alpha_0 + \varepsilon Jz$  for every sufficiently small  $\varepsilon > 0$ ; its inverse power is ample, preserves all the initially strict tests, and makes the remaining tests strict. The converse follows by applying  $\mathcal{A}$ .  $\square$

The cone-wise hypothesis called condition (15) in [Wue26] requires every tight subspace  $V$  to contain  $\dim V$  ray generators of each maximal cone. It is a genuine additional geometric hypothesis: the simultaneous bases then make the specified equality subsheaves subbundles. The preceding theorem removes it from the numerical local model by retaining  $L_{\text{acc}}$ ; it does

not remove it from a local-freeness claim. In particular, the classes  $C_p$  in general are divisor sums, not asserted Chern classes of stable vector-bundle factors.

**Example 3.3.** Sections can change the cone even for a tree poset. A four-edge star with upward coordinates  $a_1, \dots, a_4$  has cone  $\mathbb{R}_{\geq 0}^4$ . Its section  $a_1 + a_2 = a_3 + a_4$  is a three-dimensional cone with four rays generated by  $e_1 + e_3, e_1 + e_4, e_2 + e_3, e_2 + e_4$ , where these are the coordinate vectors of the nonnegative orthant. No toric realization of this section is asserted. Also, for the three-element star  $0 < 1, 0 < 2$ , the line  $\text{span}_{\mathbb{R}}(0, 1, -1) \subset Z$  misses the strict cone although the poset is connected. These examples explain why a Hasse-cycle criterion for the full cone is not a criterion for an arbitrary accessible section.

#### 4. FEASIBILITY, TOPOLOGY, AND DIMENSION BOUNDS

The image in (9) permits a precise alternative between an accessible stable direction and an order-preserving numerical relation. Write  $M_{\mathbb{R}} = M \otimes \mathbb{R}$ .

**Proposition 4.1.** *Suppose that  $P$  is connected and  $|P| \geq 2$ . Then  $L_{\text{acc}} \cap \mathcal{C}^{\text{str}}(P) = \emptyset$  if and only if there exist numbers  $a_p$ , not all zero, such that*

$$(12) \quad p \leq q \implies a_p \leq a_q, \quad \sum_p r_p a_p = 0, \quad \sum_p a_p C_p = 0 \quad \text{in } N^1(X)_{\mathbb{R}}.$$

*Equivalently there is a nonzero  $m \in M_{\mathbb{R}}$  whose values  $\langle m, v_{\rho} \rangle = a_p$  are constant on each ray block and order-preserving on  $P$ .*

*Proof.* The connected root cone is full-dimensional in  $Z$ . Separation of its interior from a subspace is equivalent to a nonzero dual class  $[a] \in \mathcal{C}(P)^* \cap L_{\text{acc}}^{\perp}$ , where potentials are considered modulo constants. Pairing with each upward cover vector says exactly that  $a$  is order-preserving. Orthogonality to  $\text{im } \mathcal{A}$  says

$$\sum_p a_p C_p - \frac{\sum_p r_p a_p}{n} D = 0.$$

Subtract the rank-weighted mean from every  $a_p$ . This gives (12) and preserves its nonzero class and order. Conversely those conditions give the separating dual class.

For a smooth complete toric variety the kernel of the real invariant divisor map to  $N^1(X)_{\mathbb{R}}$  consists of principal real divisors:  $m \mapsto \sum_{\rho} \langle m, v_{\rho} \rangle D_{\rho}$ . Here the divisor class group equals the Picard group, and numerical and real linear equivalence agree, so this is also the numerical kernel. Applying this sequence to  $\sum_p a_p C_p$  gives the character formulation. Pairing that relation with  $\alpha_0$  already forces the rank centering in (12). The rays span  $N_{\mathbb{R}}$ , so a nonzero coefficient vector gives a nonzero character and conversely.  $\square$

This decides feasibility for a specified pair of toric equality data. It neither asserts that such a nonzero obstruction occurs on a toric variety nor proves universal accessibility. Disconnected posets must be treated separately: their strict locus is empty even if the block classes are independent.

**Proposition 4.2.** *Assume that  $H_0$  is strictly semistable and that nearby stable classes exist. Put  $K = \mathcal{C}(P) \cap L_{\text{acc}}$  and  $f = \rho - \dim L_{\text{acc}}$ . Then  $K$  is a full-dimensional pointed cone in  $L_{\text{acc}}$ , and its interior is  $\mathcal{C}^{\text{str}}(P) \cap L_{\text{acc}}$ . The pair  $(K \times \mathbb{R}^f, \text{int } K \times \mathbb{R}^f)$  is ambiently and positively homogeneously bi-Lipschitz homeomorphic to a closed and open halfspace pair in  $\mathbb{R}^{\rho}$ . Its spherical semistable link is a closed  $(\rho - 1)$ -ball, with stable part its interior.*

*Proof.* Choose a strict  $v \in L_{\text{acc}}$ . For each proper nonempty ideal put  $\ell_I(z) = -z(I)$ , so  $\ell_I(v) > 0$ . These finitely many functionals cut out  $K$ . Strict inequalities are exactly its interior; adding  $\varepsilon v$  to any point proves that this interior has closure  $K$ . The full root cone is pointed: a strictly increasing potential has positive value on every nonzero positive combination of upward cover vectors. Its section is pointed as well. The poset is connected by Lemma 3.1.

Write the ambient vector as  $y + tv$ , where  $y$  ranges over a complement to  $\mathbb{R}v$  in  $L_{\text{acc}}$ , together with all free coordinates. The latter are ignored by every  $\ell_I$ . Then

$$y + tv \in K \times \mathbb{R}^f \iff t \geq \phi(y), \quad \phi(y) = \max_I \frac{-\ell_I(y)}{\ell_I(v)}.$$

The finite maximum  $\phi$  is Lipschitz and positively homogeneous. The ambient map  $(y, t) \mapsto (y, t - \phi(y))$  and its inverse are Lipschitz and carry strict epigraphs to open halfspaces. Radial normalization gives the link statement. Free directions lie on the boundary. Composing with the analytic coordinates of Theorem 3.2 gives the local toric assertion.  $\square$

A differentiable equivalence preserves more information. Its derivative at the vertex is an invertible linear map taking tangent cones to tangent cones. It therefore preserves lineality and the face lattice of the pointed quotient. A nonsimplicial pointed cone cannot become an orthant under such an equivalence, even though Proposition 4.2 removes its corners topologically. Here and below the lineality is the largest vector subspace contained in the cone.

**Proposition 4.3.** *Under the hypotheses of Proposition 4.2, a nonsimplicial pointed chamber requires  $n \geq 4$  and  $\rho \geq 4$ . If  $n = 4$ , then  $|P| = 4$ , every  $r_p = 1$ ,  $\dim L_{\text{acc}} = 3$ , and  $L_{\text{acc}} = Z$ . Its Hasse graph is a four-cycle, with the poset either a diamond or  $K_{2,2}$ . In either case the local semistable cone is linearly isomorphic to*

$$(13) \quad \text{cone}(\Delta^1 \times \Delta^1) \times \mathbb{R}^{\rho-3}.$$

*Proof.* Write  $s = |P|$ ,  $a = \dim L_{\text{acc}}$ . Positivity of the integer ranks gives  $s \leq n$ , and  $a \leq s - 1$ . Moreover  $0 \neq \alpha_0 \in \ker \mathcal{A}$ , so  $a \leq \rho - 1$ . A full-dimensional pointed cone of dimension at most two is simplicial. Thus nonsimpliciality forces  $a \geq 3$ , giving the two lower bounds. If  $n = 4$ , all the rank inequalities are equalities, so  $s = 4$ ,  $a = 3$ , all ranks are one, and  $L_{\text{acc}} = Z$ .

The full root cone is nonsimplicial only when its Hasse graph has an undirected cycle. A Hasse graph has no triangle. On four vertices the only connected cyclic possibility is a four-cycle, and no diagonal may be added. Among its acyclic orientations, the one with a directed path of length three would make the remaining edge transitive, so it is not a cover graph orientation. Up to relabeling, the remaining types are the diamond and the height-two crown. In both cases the four cover vectors have one relation with two unit terms on each side, and any three are independent. Their cone is therefore linearly the cone over a quadrilateral, indeed the square cone in (13).  $\square$

The fourfold constructed below attains both lower bounds with the crown poset. We do not realize the diamond here. In higher dimension a nonsimplicial section of a simplicial full root cone is also possible abstractly, as Example 3.3 shows.

## 5. A DOUBLE PROJECTIVE BUNDLE AND ITS AMPLE WALL

Return to (1). Put  $\epsilon_j = 1$  for  $j \leq k$  and  $\epsilon_j = -1$  otherwise. Thus  $\ell = \sum_j \epsilon_j h_j$ . In the lattice basis  $e_1, e_2, f_1, \dots, f_d$ , the primitive rays are

$$(14) \quad e_1, -e_1, \quad e_2, -e_2, \quad f_j, -f_j + \epsilon_j(e_1 + e_2) \quad (1 \leq j \leq d).$$

Every maximal cone chooses one ray from each of these  $n = d + 2$  pairs. These are the transition fan of the two split projective bundles. In particular the classes  $\xi_1, \xi_2, h_1, \dots, h_d$  form a basis of the real divisor space, so  $\rho(X_k) = n$ . We give direct fan and positivity checks to fix all conventions.

**Lemma 5.1.** *The fan (14) is smooth and projective. Its full ample cone, in coordinates  $H = x\eta_1 + y\eta_2 + \sum_j u_j h_j$ , is*

$$(15) \quad x > 0, \quad y > 0, \quad u_j > (x + y)/2 \quad (1 \leq j \leq d).$$

*The anticanonical class is  $-K_{X_k} = 2(A + B_0)$ ; it is nef and big and is not ample.*

*Proof.* For (15), consider the bounded polytope in the dual coordinates  $t_1, t_2, q_1, \dots, q_d$  given by

$$(16) \quad -x/2 \leq t_1 \leq x/2, \quad -y/2 \leq t_2 \leq y/2, \quad -u_j/2 \leq q_j \leq u_j/2 + \epsilon_j(t_1 + t_2).$$

Each base interval is strictly positive. At a vertex one endpoint of every  $q_j$ -interval is active, since otherwise  $q_j$  is free; both endpoints cannot be active. The remaining active inequalities fix the endpoints of  $t_1, t_2$ . Its  $2^n$  vertex normal cones are therefore exactly the chosen-pair cones above. Their ray matrices are triangular with diagonal entries  $\pm 1$ , so they are unimodular. This proves smoothness, completeness, projectivity, and sufficiency of (15).

For necessity, the invariant fibre curves have degrees  $x, y$ . The curves obtained by varying base coordinate  $j$  at one of the four fibre sections have degrees

$$u_j + \epsilon_j(\delta_1 x + \delta_2 y - (x + y)/2), \quad \delta_i \in \{0, 1\}.$$

Their minimum is  $u_j - (x + y)/2$ . Every invariant-curve type is on this list, because a codimension-one cone omits one ray pair. An ample class has positive intersection with these curves, proving necessity. Summing the invariant divisors gives  $-K = 2(A + B_0)$ , a limit of ample classes with a zero intersection on one of these curves. It is nef and not ample. Its positive top intersection, proved below, gives bigness.  $\square$

The projective-bundle relation and the base intersection ring give

$$(17) \quad h_j^2 = 0, \quad \xi_i^2 = \ell \xi_i, \quad \eta_i^2 = \ell^2/4, \quad \int_{X_k} \eta_1 \eta_2 h_1 \cdots h_d = 1.$$

Indeed the two tautological roots are  $0, \ell$ ; integration along a projective-line fibre takes the coefficient of  $\xi_i$  in the reduction modulo  $\xi_i(\xi_i - \ell)$ . Equivalently, in formal power series,

$$\pi_{i*} e^{\eta_i} = \frac{e^{\ell/2} - e^{-\ell/2}}{\ell} = \int_{-1/2}^{1/2} e^{t\ell} dt.$$

This identity is interpreted by its power series even when  $\ell$  is nilpotent; it involves no division in the Chow ring. Applying it twice, then using  $h_j^2 = 0$ , gives the normalized volume of  $H_r = A + rB_0$ :

$$(18) \quad \begin{aligned} \frac{H_r^n}{n!} &= \int_{[-1/2, 1/2]^2} \prod_{j=1}^d (r + \epsilon_j(t_1 + t_2)) dt_1 dt_2 \\ &= 2 \int_0^1 (1 - s)(r^2 - s^2)^k ds = F_k(r), \end{aligned}$$

$$(19) \quad F_k(r) = \sum_{j=0}^k \frac{(-1)^j \binom{k}{j}}{(2j+1)(j+1)} r^{2k-2j}.$$

The triangular density of  $t_1 + t_2$  proves the middle equality. In particular  $F_k(1) > 0$ , completing the weak-Fano assertion.

Let  $C_1 = 2\eta_1$ ,  $C_2 = 2\eta_2$ , and  $C_{j+2} = 2h_j$ . These are the sums of the prime divisors in the ray pairs. Fibre exchange makes the first two degrees equal. Permutations of the plus indices and of the minus indices make the base degrees equal within each group. Swapping the two groups replaces  $\ell$  by  $-\ell$ ; tensoring  $\mathcal{O} \oplus L^{-1}$  by  $L$  identifies its projectivization with that of  $\mathcal{O} \oplus L$  and preserves  $\eta_i$ . Thus all  $d$  base degrees agree. Homogeneity and differentiation of (18) now give

$$(20) \quad \begin{aligned} C_1 H_r^{n-1} &= C_2 H_r^{n-1} = (n-1)!(nF_k(r) - rF'_k(r)), \\ C_{j+2} H_r^{n-1} &= \frac{(n-1)!}{k} F'_k(r). \end{aligned}$$

For example  $\text{Vol}(xA + rB_0) = x^n F_k(r/x)$ ; its  $x$ -derivative is the sum of the two equal fibre derivatives. The  $r$ -derivative is the sum of the  $d$  equal base derivatives. This also checks the factorials in (20).

**Proposition 5.2.** *The polynomial*

$$(21) \quad G_k(r) = (kr + 1)F'_k(r) - k(2k + 2)F_k(r)$$

*has exactly one root  $r_k > 1$ , and that root is simple. All the classes  $C_p$  have the same positive degree at  $H_0 = A + r_k B_0$ .*

*Proof.* Write  $I_k = \int_0^1 (1 - s^2)^k ds$ . Direct integration gives

$$F_k(1) = 2I_k - \frac{1}{k+1}, \quad F'_k(1) = 4kI_{k-1} - 2, \quad I_k = \frac{2k}{2k+1} I_{k-1}.$$

It follows that

$$G_k(1) = 2(k+1) \left( I_k - \frac{1}{k+1} \right) > 0,$$

since  $(1 - s^2)^k > (1 - s)^k$  for  $0 < s < 1$ . The leading term of  $G_k$  is  $-2kr^{2k}$ , so there is a root beyond one.

For uniqueness set  $c = 1/k$  and normalize onto the ample affine line

$$\tilde{H}_r = \frac{A + rB_0}{r + c} = B_0 + t(A - cB_0), \quad t = (r + c)^{-1}.$$

Put  $E = A - cB_0$ . By the real Hodge index theorem, the second derivative of  $(\tilde{H}_r^n)^{1/n}$  as a function of  $t$  has the sign of

$$(E^2 \tilde{H}_r^{n-2})(\tilde{H}_r^n) - (E \tilde{H}_r^{n-1})^2 < 0.$$

The inequality is strict because the independent classes  $A, B_0$  make  $E$  never proportional to  $\tilde{H}_r$ . The equation  $G_k(r) = 0$  is precisely the critical-point equation for  $F_k(r)/(r + c)^n$ , hence for its positive volume root. Strict negative second derivative on the ample segment  $r > 1$  of this affine line proves at most one critical point and nondegeneracy there. Thus the root is unique and simple. Finally (20) makes the equality of all  $C_p$ -degrees equivalent to  $G_k = 0$ . Their common degree is positive because each  $C_p$  is an effective nonzero divisor and  $H_0$  is ample by  $r_k > 1$ .  $\square$

## 6. ALL EQUALITY SUBSPACES AND THE CHAMBER

Let  $\tau = C_p H_0^{n-1}$ , independent of  $p$  by Proposition 5.2. We first show that every one of the  $2n$  invariant prime divisors has degree  $\tau/2$ . The divisors of the two base rays in pair  $j$  are linearly equivalent to  $h_j$ , so this is already true for them. The principal relation from the character dual to  $e_i$  is

$$D_{e_i} - D_{-e_i} + \sum_j \epsilon_j D_{-f_j + \epsilon_j(e_1 + e_2)} = 0.$$

The last sum has degree zero because there are equally many plus and minus signs. Hence the two fibre divisors have equal degrees as well. In particular  $\mu_{H_0}(T_X) = \tau$ .

**Proposition 6.1.** *The tangent bundle is semistable at  $H_0$ . Its tight subspaces are exactly*

$$(22) \quad 0, \quad \mathbb{C}e_1, \quad \mathbb{C}e_2, \quad V_J = \text{span}_{\mathbb{C}}\{e_1, e_2, f_j : j \in J\} \quad (J \subset \{1, \dots, d\}).$$

*Their equality poset is  $K_{2,d}$ , with rank-one blocks the ray pairs in (14). Each specified saturated equivariant equality subsheaf is a vector subbundle.*

*Proof.* The ray set is the disjoint union of the two lattice bases

$$E^+ = \{e_1, e_2, f_1, \dots, f_d\}, \quad E^- = \{-e_1, -e_2, -f_j + \epsilon_j(e_1 + e_2) : 1 \leq j \leq d\}.$$

A complex subspace of dimension  $q$  contains at most  $q$  vectors from each basis. Its degree is therefore at most  $(2q)(\tau/2) = q\tau$ , which proves the full slope test.

Equality forces  $q$  vectors from each basis. The subspace must then be spanned by a subset of  $E^+$ . A negative base ray lies in that coordinate subspace precisely when its  $f_j$  and both  $e_1, e_2$  are selected. Thus a selected base coordinate forces both fibre coordinates if the negative basis is to contribute the same number of vectors. If no base coordinate is selected, any subset of the two fibre coordinates is allowed. This proves (22), including every equality case, not only the displayed natural candidates. These are exactly the ideals of  $K_{2,d}$ , and their rank increments are one.

On each maximal cone, the chosen rays in the selected pairs form a basis of the corresponding equality subspace: a selected base ray differs from  $\pm f_j$  by a combination of the already selected  $e_1, e_2$ . Thus the cone-wise tangent coordinate bases also make this subspace a coordinate subspace. The equivariant subbundle criterion [KD19, Proposition 4.1.1] applies. Alternatively, Proposition 7.2 identifies these subbundles directly as relative tangent bundles of smooth maps.  $\square$

The classes  $C_p$  form a basis of  $N^1(X_k)_{\mathbb{R}}$ . Their centered classes  $K_p = C_p - D/n$  therefore have exactly the one relation  $\sum_p K_p = 0$ . The slope map  $\mathcal{A}$  has rank  $n - 1$  and image  $Z$ . Theorem 3.2 consequently gives a full  $(n - 1)$ -dimensional cone and one free coordinate. The ideal inequalities reduce to

$$(23) \quad z_1, z_2 \leq 0, \quad z_{j+2} \geq 0 \quad (1 \leq j \leq d), \quad \sum_p z_p = 0.$$

Indeed the singleton minima and the ideals omitting one maximum give these inequalities. Every other proper ideal is either a subset of the two minima or the two minima together with some maxima; its inequality follows by summing the displayed ones. The strict versions are likewise equivalent.

**Proposition 6.2.** *The cone (23) has dimension  $d + 1$ ,  $2d$  extreme rays  $e_{j+2} - e_i$ , and  $d + 2$  facets. Its section  $\sum_{j=1}^d z_{j+2} = 1$  is  $\Delta^1 \times \Delta^{d-1}$ . Every nonzero face is indexed by nonempty subsets  $A' \subset \{1, 2\}$ ,  $B' \subset \{3, \dots, n\}$ , with dimension  $|A'| + |B'| - 1$ .*

*Proof.* Put  $x_i = -z_i$ ,  $y_j = z_{j+2}$ , with common total  $\sigma = \sum_i x_i = \sum_j y_j$ . For  $\sigma > 0$ , the nonnegative transport matrix  $x_i y_j / \sigma$  expresses  $z$  as a combination of the asserted edge generators. Each is extreme: its zero coordinates force every summand in a nonnegative decomposition to have the same pair of nonzero coordinates. At total one the coordinates  $x$  and  $y$  vary independently in the two simplices, proving the section and all its faces. Its facets are precisely the  $d + 2$  coordinate vanishings, since both simplex factors have positive dimension.  $\square$

Since  $2d > d + 1$  for  $d \geq 2$ , every member has a nonsimplicial chamber. The semistable germ cannot be carried to an orthant times free directions by a  $C^1$  or analytic diffeomorphism. In the fourfold there are five proper nonempty ideals, but only four facets: the ideal of both minima has equality only at the pointed apex. Its supporting hyperplane must not be counted as a fifth facet.

There are actual ample stable directions. By (20),

$$(24) \quad C_1 H_r^{n-1} - C_3 H_r^{n-1} = -\frac{(n-1)!}{k} G_k(r).$$

Immediately below  $r_k$ , still with  $r > 1$ , we have  $G_k(r) > 0$ . The two minimal block degrees are smaller than all maximal block degrees. Their centered slopes have exactly the strict signs in (23); all originally strict ray-flat tests remain strict by continuity. Thus  $H_r$  is stable there. Rational  $r$  can be chosen approaching  $r_k$ , and scaling gives integral ample stable polarizations. No claim about the entire interval  $1 < r < r_k$  is needed.

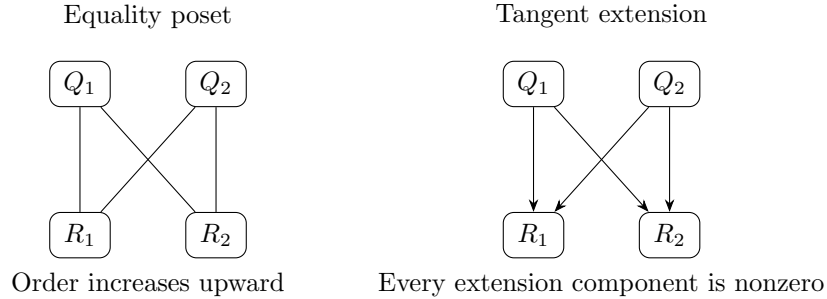


FIGURE 1. The crown fourfold: equality order and extension orientation. For general  $X_k$ , replace the two upper vertices by  $2k$  vertices; every upper vertex is joined to both lower vertices. The extension arrows point toward the subbundle factors.

## 7. THE ACTUAL TANGENT EXTENSION

The relative tangent of a quotient projective bundle  $\mathbb{P}_B(\mathcal{O} \oplus L)$  is  $\mathcal{O}(2\xi) \otimes L^{-1}$ , as follows from the rank-two relative Euler sequence. The relative tangent of a fibre product of smooth morphisms is the direct sum of the pulled-back relative tangents. The base tangent is  $\bigoplus_j \mathcal{O}_B(2h_j)$ . These observations prove the exact sequence (3). Its factors have first Chern classes  $C_p$ , hence are pairwise nonisomorphic. As line bundles they are stable, and their slopes at  $H_0$  are equal. Refining the subobject by its two summands and the quotient by its  $d$  summands gives a Jordan–Hölder filtration with locally free graded bundle.

**Proposition 7.1.** *Every component of the extension (3) in  $\text{Ext}^1(Q_j, R_i)$  is nonzero. It is the image of  $\pm\epsilon_j$  times a fixed generator of  $H^1(B, \mathcal{O}_B(-2h_j))$  under the coefficient map defined by the fibre vector field  $z_i\partial/\partial z_i$ .*

*Proof.* Trivialize  $L$  on base charts  $U_a$ , with transitions  $g_{ab}$ . On an affine projective fibre chart the coordinate transforms as  $z_{i,b} = g_{ab}^{\pm 1} z_{i,a}$ , with one consistent choice of sign. Differentiating gives the off-diagonal tangent transition

$$(25) \quad \pm d \log g_{ab} \otimes \left( z_i \frac{\partial}{\partial z_i} \right).$$

The vector field extends over the other projective chart and defines a global section of  $R_i$ . The  $i$ -th row is therefore the image of the Atiyah class of  $L$  in  $H^1(B, \Omega_B^1)$ . Since  $c_1(L) = \sum_j \epsilon_j h_j$ , its  $j$ -th component is the stated nonzero multiple before applying the coefficient map.

This map remains injective on that cohomology group. The three fibrewise vector fields give

$$(26) \quad \pi_* R_i = L^{-1} \oplus \mathcal{O}_B \oplus L, \quad R^q \pi_* R_i = 0 \quad (q > 0).$$

The middle summand is generated by  $z_i\partial/\partial z_i$ . On the active fibre the bundle is  $\mathcal{O}_{\mathbb{P}^1}(2)$ ; the other fibre contributes its structure sheaf. Both have no higher cohomology, proving the vanishing in (26). Leray and the projection formula identify the induced map on  $H^1$  with inclusion of the direct summand  $\mathcal{O}_B(-2h_j)$ . Its one-dimensional  $H^1$  has nonzero generator, and  $\epsilon_j = \pm 1$ . All  $2d$  components survive.  $\square$

Thus the actual extension support has arrows  $Q_j \rightarrow R_i$  for every pair  $(i, j)$ . Its underlying graph is the same complete bipartite graph as the equality poset, with the arrows reversed. The orientation distinction is recorded in Figure 1. After a smooth splitting, the off-diagonal Dolbeault entry  $\beta$  maps  $\bigoplus Q_j$  to  $R_1 \oplus R_2$  and vanishes on the latter. Hence  $\beta \wedge \beta = 0$ , as for every two-step extension of this form. Here integrability is already supplied by the actual tangent bundle; it is not inferred merely from a drawn graph. The graph also contains undirected cycles, so a tree-only spectral result would not cover this example.

We next identify the equality subbundles as foliations with explicit smooth leaf-space morphisms. Put  $Y_i = \mathbb{P}_B(\mathcal{O} \oplus L)$  for the two factors, and, for  $J \subset \{1, \dots, d\}$ , put  $B_{J^c} = \prod_{j \notin J} \mathbb{P}^1$ .

**Proposition 7.2.** *The equality line  $\mathcal{F}_{\mathcal{C}e_i} = R_i$  is  $T_{X_k/Y_{3-i}}$ . For the subspace  $V_J$  in (22),*

$$(27) \quad \mathcal{F}_{V_J} = T_{X_k/B_{J^c}}.$$

*All the morphisms here are smooth and projective, and these are equalities of saturated subbundles in  $T_{X_k}$ .*

*Proof.* Projection to the other projective-bundle factor is a smooth projective-line bundle, with relative tangent  $R_i$ . The map  $X_k \rightarrow B_{J^c}$  is the composition of the smooth  $(\mathbb{P}^1)^2$ -bundle  $X_k \rightarrow B$  with the smooth base projection. Its differential is surjective, so its kernel is a saturated subbundle. On the dense torus its generic tangent subspace is exactly  $V_J$ . A saturated subsheaf of a vector bundle is uniquely determined by its generic subspace: it is the intersection of that subspace with the bundle inside its rational generic fibre. This proves (27) and the line assertion.  $\square$

Together with Sections 5–6 and Proposition 4.3, these results complete the proof of Theorem 1.1. They concern the specified saturated equivariant equality objects. A modification supported in codimension two can have the same slope without being a subbundle; no assertion about all such modifications is intended.

## 8. LOCAL HARDER–NARASIMHAN FILTRATIONS

We again allow the general toric equality data of Section 2. For an ideal  $I \subset P$ , write  $r(I) = \sum_{p \in I} r_p$ ,  $z(I) = \sum_{p \in I} z_p$ , and  $\mathcal{F}_I$  for its saturated tangent subsheaf. No local-freeness assumption is made in this section.

Harder–Narasimhan filtrations exist for torsion-free sheaves at the real ample-power classes used here. More generally the real movable framework gives existence, saturation, uniqueness, and strictly decreasing semistable quotient slopes [Pan15, Theorem 1.3.20 and Remarks 1.3.22–1.3.26]. The unique inclusion-largest maximal destabilizer is preserved by a connected algebraic group acting trivially on numerical classes. Thus all HN terms here are equivariant. The finite curve-class decompositions and quotient-preimage procedure of Neumann and Pang [Neu09, Theorem 3.3.4 and Proposition 3.3.5] [Pan15, Proposition 2.1.19 and §2.2] are prior results. The following theorem identifies which finite objects suffice on one common neighborhood of a semistable class.

**Theorem 8.1.** *After shrinking once around  $H_0$ , every ordinary HN term of  $T_X$  is one of the original equality subsheaves  $\mathcal{F}_I$ . Given a current ideal  $I$ , the next term is the inclusion-largest ideal  $J \supsetneq I$  maximizing*

$$(28) \quad \frac{z(J) - z(I)}{r(J) - r(I)}.$$

*The common slope  $\mu_H(T_X)$  is subtracted in this formula. In particular the whole local HN flag decomposition is determined by  $(P, r, L_{\text{acc}})$  in the coordinates of Theorem 3.2.*

*Proof.* We prove the finite exclusion simultaneously in all equality intervals. Fix ideals  $I \subsetneq J$  and put  $\mathcal{Q} = \mathcal{F}_J/\mathcal{F}_I$ . It is torsion free and saturated in  $T_X/\mathcal{F}_I$ , since  $T_X/\mathcal{F}_J$  is torsion free. If  $\mathcal{G} \subset \mathcal{Q}$  is saturated, its preimage  $\tilde{\mathcal{G}} \subset T_X$  is saturated: the sequence

$$(29) \quad 0 \longrightarrow \mathcal{Q}/\mathcal{G} \longrightarrow T_X/\tilde{\mathcal{G}} \longrightarrow T_X/\mathcal{F}_J \longrightarrow 0$$

has torsion-free end terms. For an equivariant  $\mathcal{G}$ , the preimage is torus-invariant and saturated. Klyachko’s classification [Kly90] therefore identifies it with  $\mathcal{F}_W$  for a complex subspace  $V_I \subset W \subset V_J \subset N_{\mathbb{C}}$ . Its quotient slope is

$$(30) \quad \mu_H(\mathcal{F}_W/\mathcal{F}_I) = \frac{(C_W - C_I)H^{n-1}}{\dim W - r(I)}, \quad C_W = \sum_{v_\rho \in W} D_\rho.$$

At  $H_0$ , the whole interval  $\mathcal{Q}$  has slope  $\tau$  and is semistable: a subobject of greater slope would, by its preimage, contradict semistability of  $T_X$ .

Choose the equivariant inclusion-largest subobject of maximal slope in  $\mathcal{Q}$ , allowing  $\mathcal{Q}$  itself; its existence and equivariance follow from the real-class HN theorem above. Its slope is positive on a small neighborhood because it is at least  $\mu_H(\mathcal{Q})$ , which is close to  $\tau > 0$ . Replace its preimage subspace  $W$  by

$$W' = \text{span}_{\mathbb{C}}(V_I, \{v_\rho : v_\rho \in W\}).$$

The numerator in (30) is unchanged and nonnegative; positivity of the slope makes it positive. The denominator weakly decreases. A strict decrease would give a larger slope, contradicting maximality. Thus every relevant maximizer is in the finite list of spaces spanned by  $V_I$  and a subset of the ray vectors, inside  $V_J$ .

At  $H_0$ , such a quotient has slope  $\tau$  exactly when  $W$  is tight: add the degree  $\tau r(I)$  of  $\mathcal{F}_I$  in (30). Otherwise its slope is strictly smaller than that of the whole interval. There are finitely many equality intervals and finitely many candidates in each. Shrink once so all

whole-interval slopes stay positive and every initially smaller candidate stays smaller than its whole interval. Then every maximum comes from an original tight ideal between  $I$  and  $J$ . This is one neighborhood valid for every successive quotient; no iteration-dependent shrinking is needed.

Degrees and ranks are valuations on the ideal lattice. If two ideals containing  $I$  maximize the average, their union also maximizes it, by the valuation identities and the maximality bound on their intersection. The empty relative intersection contributes zero to both identities. Taking the union of all maximizers gives a unique inclusion-largest maximizer. The next maximal slope in the quotient is strictly smaller: an equal or larger one would contradict respectively inclusion maximality or slope maximality of the previous term. Each resulting quotient is semistable by the same finite test. The defining uniqueness characterization of the HN filtration proves the theorem. Finally  $C_J - C_I = (r(J) - r(I))D/n + \sum_{p \in J \setminus I} K_p$  gives exactly (28).  $\square$

The theorem localizes the ordinary filtration, including its quotients when they are merely torsion free. It is stronger than retaining only the tangent stability inequalities, but is a local refinement of the earlier quotient algorithm, rather than a replacement for it.

**Proposition 8.2.** *Fix an ideal flag  $\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m = P$ . Put  $S_j = I_j \setminus I_{j-1}$ , with its induced poset, and  $R_j = \sum_{p \in S_j} r_p$ . Decompose uniquely*

$$z_p = r_p \lambda_j + w_p \quad (p \in S_j), \quad \sum_{p \in S_j} w_p = 0.$$

*Its HN region in the full space  $Z$  is exactly*

$$(31) \quad \lambda_1 > \cdots > \lambda_m, \quad \sum_j R_j \lambda_j = 0, \quad w|_{S_j} \in \mathcal{C}(S_j) \quad (1 \leq j \leq m).$$

*It is linearly isomorphic to*

$$(32) \quad \mathbb{R}_{>0}^{m-1} \times \prod_{j=1}^m \mathcal{C}(S_j), \quad \dim = |P| - 1 - \sum_{j=1}^m (c(S_j) - 1).$$

*For actual polarizations intersect (31) with  $L_{\text{acc}}$  and add the free coordinates. If  $L_{\text{acc}} = Z$ , every flag occurs arbitrarily near  $H_0$ .*

*Proof.* The interval of ideals  $[I_{j-1}, I_j]$  is the ideal lattice of  $S_j$ . After subtracting the segment slope, its semistability inequalities are precisely the ideal sums of  $w|_{S_j}$ . Theorem 8.1 makes these tests complete in the common neighborhood. Strictly decreasing segment slopes are the remaining HN requirement, proving (31). Consecutive slope gaps give arbitrary positive coordinates; their weighted mean-zero condition fixes the common additive constant. Lemma 3.1 gives  $\dim \mathcal{C}(S_j) = |S_j| - c(S_j)$ , proving the dimension. For occurrence set  $w = 0$  and choose strictly decreasing  $\lambda_j$  with weighted mean zero. Scale toward the origin and use the inverse ample-power coordinates.  $\square$

Each region is convex and contractible. It includes any semistable boundary directions inside its individual HN quotients. In particular, it need not be relatively open, and its closure need not be a union of entire other HN regions. A finite central arrangement obtained by equating ideal-interval slopes refines the description, but the statement above is not a global HN fan theorem.

**Corollary 8.3.** *For the family  $X_k$ , every nearby HN term is one of the relative tangent subbundles in Proposition 7.2. Every ideal flag occurs. If  $F(m)$  denotes the ordered Bell number, the number of flags is  $6F(2k)$ . For  $k = 1$ , there are thirteen regions of dimension three, four of dimension two, and one of dimension one in  $Z$ , before the one free polarization coordinate.*

*Proof.* Full accessibility and the relative-tangent identification were proved in Sections 6–7. The count is a finite poset argument. In  $K_{a,b}$ , a flag either contains the full ideal of minima, or passes it in a unique mixed segment. In the first case it gives independent ordered partitions of the  $a$  minima and  $b$  maxima, counted by  $F(a)F(b)$ . In the second case splitting the mixed segment into its last minimal block followed by its first maximal block gives the same pair of ordered partitions. Conversely merging those two blocks recovers exactly one flag of the second type. Thus the total is  $2F(a)F(b)$ . Since  $F(2) = 3$ , our family has  $6F(2k)$ .

For the crown the nine flags that pass through the full ideal of minima have four, four, and one regions in dimensions three, two, and one. This follows by choosing either two ordered singleton blocks or one double block on each side: a double block is an antichain and lowers the dimension by one in (32). Each of the other nine flags has one mixed connected segment and any remaining pure segments are singletons. All nine are three-dimensional. Summing gives the stated counts.  $\square$

The quotients between these HN terms on  $X_k$  are locally free. Indeed the terms are nested relative tangent subbundles, and their generic inclusions are their inclusions as kernels of the indicated projections; on fibres these have constant ranks. This is a feature of this family, not a local-freeness consequence of the general numerical normal form.

## 9. THE FOURFOLD AND ITS ANALYTIC PARAMETERS

For  $k = 1$ , let  $B = \mathbb{P}^1 \times \mathbb{P}^1$  and  $L = \mathcal{O}_B(1, -1)$ . The complete ray matrix, in the order of (14), is

$$(33) \quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

One ray from each consecutive pair gives the sixteen maximal cones. The exact intersections simplify to

$$(34) \quad \begin{aligned} \eta_i^2 &= -\frac{1}{2}h_1h_2, & F_1(r) &= r^2 - \frac{1}{6}, & G_1(r) &= -2r^2 + 2r + \frac{2}{3}, \\ r_1 &= \frac{3 + \sqrt{21}}{6}, & H_0^4 &= 16 + 4\sqrt{21}, \\ C_p H_0^3 &= 6 + 2\sqrt{21}, & D_p H_0^3 &= 3 + \sqrt{21}. \end{aligned}$$

The irrationality is a property of this reference divisor ray; nearby stable integral polarizations still exist. At the rational ample test classes  $r = 5/4$  and  $r = 3/2$ , the block degrees  $(C_1 H_r^3, \dots, C_4 H_r^3)$  are respectively

$$(59/4, 59/4, 15, 15), \quad (23, 23, 18, 18).$$

An exact check of every ray flat gives strict stability at  $5/4$ , whereas the relative tangent line  $R_1$  destabilizes at  $3/2$ . At  $r = 1$  one of the base intervals in (16) collapses; the class is not ample. These checks distinguish the equal-slope construction from a merely formal solution in curve space. The all- $k$  statements rely on the preceding proofs.

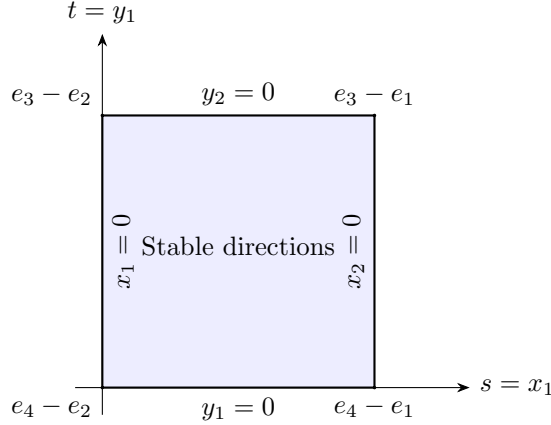


FIGURE 2. The square section of the fourfold chamber, in coordinates  $z = (-s, -(1-s), t, 1-t)$ . The four vertices represent extreme rays and the four edges represent facets. The proper ideal consisting of both minima has sum  $-1$  throughout this section; its equality occurs only at the apex of the full cone.

The fourfold has seven tight subspaces including zero and the whole space, with the five proper nonzero ones

$$\mathbb{C}e_1, \quad \mathbb{C}e_2, \quad \text{span}_{\mathbb{C}}(e_1, e_2), \quad \text{span}_{\mathbb{C}}(e_1, e_2, f_1), \quad \text{span}_{\mathbb{C}}(e_1, e_2, f_2).$$

The square in Figure 2 records the four rays and four facets of its centered chamber. The HN regions outside and on this chamber carry the additional flag information of Corollary 8.3; eighteen is a count of filtration flags, not a count of full-dimensional stable chambers.

**Parameters for the analytic application.** For every  $X_k$ , order the line factors as  $(R_1, R_2, Q_1, \dots, Q_d)$ , and define

$$(35) \quad \begin{aligned} V(H) &= H^n/n!, & z_v(H) &= (C_v - D/n)H^{n-1}, \\ \kappa_v(H) &= \frac{2\pi C_v H^{n-1}}{(n-1)!V(H)}, & b_v(H) &= -\left(\kappa_v(H) - \frac{1}{n} \sum_w \kappa_w(H)\right) = -\frac{2\pi z_v(H)}{(n-1)!V(H)}. \end{aligned}$$

The  $\kappa_v$  are the Hermitian–Einstein constants for the line factors when the Kähler form represents  $H$ . The minus sign reverses the upward poset orientation to the extension orientation  $Q_j \rightarrow R_i$ . For stable polarizations these centered constants satisfy

$$(36) \quad x_i = b_{R_i} > 0, \quad y_j = -b_{Q_j} > 0, \quad \sigma = x_1 + x_2 = \sum_j y_j.$$

The factor ranks are all one; no rank factors are suppressed in these equalities.

**Proposition 9.1.** *There is an analytic chart  $H = H(s)$  on a full neighborhood of  $H_0$  in its fixed-volume hypersurface, with*

$$(37) \quad H(0) = H_0, \quad V(H(s)) = V_0, \quad b(H(s)) = s \in \mathbb{Z}, \quad |H(s) - H_0| \leq C\|s\|.$$

*On its stable part  $\|s\|_1 = 2\sigma$ , hence  $|H(s) - H_0| = O(\sigma)$ . It admits smooth Kähler representatives  $\omega_s$  with  $\|\omega_s - \omega_0\|_{C^a} = O_a(\|s\|)$  for every fixed  $a$ . The reference representative  $\omega_0$  can be invariant under interchange of the two projective factors.*

*Proof.* Full accessibility gives  $\text{rk } dz_{H_0} = n - 1$ . Homogeneity and  $z(H_0) = 0$  put  $H_0$  in its kernel, which is therefore exactly  $\mathbb{R}H_0$ . Moreover  $dV_{H_0}[H_0] = nV_0 \neq 0$ . The restriction of  $dz$  to  $\ker dV$  is an isomorphism onto  $Z$ . The derivative of the volume denominator in (35) contributes zero at  $z = 0$ , so the same is true for  $db$ . The analytic inverse-function theorem on the fixed-volume hypersurface proves the chart and its derivative bound. Equation (36) gives the  $\ell^1$  identity.

The reference class is invariant under fibre exchange. Average any Kähler representative over this involution to choose  $\omega_0$ . Represent a basis of the real divisor space by smooth closed real  $(1, 1)$ -forms, and add the linear combination corresponding to  $H(s) - H_0$ . Positivity is open, so this supplies the claimed family and all derivative bounds after shrinking.  $\square$

The chart includes every sufficiently small centered constant vector, not only those on a symmetric path. Requiring every representative to be invariant under fibre exchange would impose equal fibre slopes and lose this property. Only the reference form is required to be invariant. The tangent extension, its actual nonzero components, and this full controlled parameter chart are the geometric inputs to the analytic companions. Neither the fan nor the chamber computes absolute harmonic extension norms. Such coefficients, and any symmetry used to simplify them at the reference form, belong to the metric analysis.

**Exact checks and further questions.** The printed fan, maximal-cone rule, and ring determine all the calculations above. Supporting exact computations use rational row reduction to enumerate ray flats, and compare Chow-ring intersections with (18) independently. For  $k = 1, 2, 3$  the counts are shown in Table 1; the finite calculation is not the proof for arbitrary  $k$ .

TABLE 1. Exact finite checks. Ray-flat and tight counts include zero and the full space. HN flags include regions of every dimension.

| $k$ | $n = \rho$ | Maximal cones | Ray flats | Tight flats | HN flags |
|-----|------------|---------------|-----------|-------------|----------|
| 1   | 4          | 16            | 34        | 7           | 18       |
| 2   | 6          | 64            | 274       | 19          | 450      |
| 3   | 8          | 256           | 2314      | 67          | 28098    |

The reproducible calculations also check deliberate failure cases. The ample test fails at  $r = 1$ . A positive class above the wall has an explicit destabilizing relative line. With all twists set to zero, the ray-flat computation gives all sixteen coordinate subspaces as tight in dimension four, an antichain rather than a crown. The HN enumeration compares the maximal-average rule with the independent flag inequalities (31) and records one direction for every crown flag. Computation sources accompany the manuscript, including the native figure descriptions.

The examples settle nonsimplicial tangent-chamber realization in the stated smooth-projective weak-Fano class. General Fano realizations, arbitrary equality posets, realizability of dependent accessible sections, and global chamber gluing remain open here. The low-dimensional restriction does not realize the diamond, and the relative-tangent result is not a classification of all foliations or Mori contractions. Singular graded sheaves and higher-step extension integrability require further analysis.

## REFERENCES

- [BFLR12] A. Boussicault, V. Féray, A. Lascoux, and V. Reiner, *Linear extension sums as valuations on cones*, J. Algebraic Combin. **35** (2012), no. 4, 573–610. [doi:10.1007/s10801-011-0316-2](https://doi.org/10.1007/s10801-011-0316-2); arXiv:1008.3278v2.
- [GRT16] D. Greb, J. Ross, and M. Toma, *Variation of Gieseker moduli spaces via quiver GIT*, Geom. Topol. **20** (2016), no. 3, 1539–1610. [doi:10.2140/gt.2016.20.1539](https://doi.org/10.2140/gt.2016.20.1539); arXiv:1409.7564v2.
- [HNS22] M. Hering, B. Nill, and H. Süß, *Stability of tangent bundles on smooth toric Picard-rank-2 varieties and surfaces*, in P. Aluffi, D. Anderson, M. Hering, M. Mustață, and S. Payne (eds.), *Facets of Algebraic Geometry*, Vol. II, London Math. Soc. Lecture Note Ser., vol. 473, Cambridge University Press, 2022, pp. 1–25. [doi:10.1017/9781108877855.001](https://doi.org/10.1017/9781108877855.001); arXiv:1910.08848v1.
- [KD19] B. Khan and J. Dasgupta, *Toric vector bundles on Bott tower*, Bull. Sci. Math. **155** (2019), 74–91. [doi:10.1016/j.bulsci.2019.04.003](https://doi.org/10.1016/j.bulsci.2019.04.003); arXiv:1802.08803v2.
- [Kly90] A. A. Klyachko, *Equivariant bundles on toral varieties*, Math. USSR-Izv. **35** (1990), no. 2, 337–375. [doi:10.1070/IM1990v035n02ABEH000707](https://doi.org/10.1070/IM1990v035n02ABEH000707).
- [Neu09] S. Neumann, *A decomposition of the Moving cone of a projective manifold according to the Harder–Narasimhan filtration of the tangent bundle*, Ph.D. thesis, Albert-Ludwigs-Universität Freiburg im Breisgau, November 2009. <https://freidok.uni-freiburg.de/data/7287>.
- [Pan15] T.-S. Pang, *The Harder–Narasimhan Filtrations and Rational Contractions*, Ph.D. thesis, Albert-Ludwigs-Universität Freiburg im Breisgau, 2015. [doi:10.6094/UNIFR/10375](https://doi.org/10.6094/UNIFR/10375).
- [Wue26] B. J. Wuebben, *Local stability chambers for toric tangent bundles*, preprint, 26 October 2026, revised 27 October 2026, version 2, 17 pp. [doi:10.5281/zenodo.23682140](https://doi.org/10.5281/zenodo.23682140).

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