CONVERGENCE OF THE MINIMIZING FLOW ON ACYCLIC QUIVERS

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Abstract. The minimizing flow of Haiden, Katzarkov, Kontsevich and Pandit recovers the Harder–Narasimhan filtration of every nonzero representation of a finite acyclic quiver from the exponential decay rates of its Hermitian metric $h(t)$. This holds for all vertex charges in the upper half-plane and every initial metric. If v lies in the kth filtration step outside the preceding one and Z_k is the central charge of the kth quotient, then $t^{-1}\log h(t)(v,v)$ tends to $-\arg Z_k$. The mass of the evolving metric tends to $\sum_k |Z_k|$. On each quotient the metric rescaled by $e^{t\arg Z_k}$ converges exponentially to a harmonic metric if the quotient is polystable. In general the rescaled metric stays at bounded distance from a Kempf–Ness gradient trajectory with vertex weights proportional to the imaginary parts of the charges. The iterated-logarithm asymptotics of Haiden, Katzarkov, Kontsevich and Pandit therefore apply up to bounded factors. For two-term complexes with nonzero cohomology concentrated in degree zero and differential supported at one vertex, the same results hold on cohomology when the phase is taken from the polar decomposition. For generalized Kronecker quivers the results extend to bounded complexes whose differentials vanish at one vertex, with the phases shifted by $-\pi n$ in degree n. Under Beilinson's equivalence this includes every object of $D^b(\mathbb{P}^1)$ in a minimal projective or injective Kronecker model.

1. Introduction

Haiden, Katzarkov, Kontsevich and Pandit [HKKP26] propose a notion of Kähler geometry on a triangulated category with a stability condition in the sense of Bridgeland [Bri07], in which an object carries a space of metrics, an endomorphism Ω depending on the metric whose trace is the central charge, a phase $\arg\Omega$ defined by a positivity condition, a mass $\operatorname{tr}|\Omega|$, and a flow $h^{-1}\dot{h} = -\arg\Omega$ of the metrics, called the minimizing flow in [HKKP26, §4.1]. A metric is *harmonic* if $\arg\Omega$ is a scalar, and harmonic metrics are expected to exist on polystable objects and to be the limits of the flow [HKKP26, §§1, 4.1], a categorical form of the theorem of Donaldson, Uhlenbeck and Yau [Don85, UY86]. This paper proves that on a quiver without oriented cycles the flow converges, up to a scalar function of the time, on every polystable representation, and that for complexes of representations with one differential the metrics induced on the cohomology converge in the same sense when the cohomology is polystable. When it is not polystable, the long-time behaviour is described through the Harder–Narasimhan filtration of the cohomology and, on its factors, through gradient flows of Kempf–Ness functions.

1.1. **Representations.** Let Q be a finite quiver without oriented cycles, with central charges z_i in the upper half-plane at its vertices, and let R be a representation of Q with Hermitian

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metrics h_i on its spaces. Write $\beta^\dagger$ for the adjoint of a linear map for the metrics. The *moment map* at the vertex i is the Hermitian endomorphism μ_i equal to the sum of $\beta^\dagger\beta$ over the arrows β out of i minus the sum of $\beta\beta^\dagger$ over the arrows into i. For a representation the endomorphism Ω of [HKKP26, §5.1] is $\bigoplus_i(z_i+\mu_i)$; it is normal, and the flow is

$$h_i^{-1}\dot{h}_i = -\arg(z_i+\mu_i),$$

with the argument taken by the functional calculus, in $(0,\pi)$. Let $Z(\mathsf{R}) = \sum_i z_i \dim \mathsf{R}_i = |Z|e^{\mathrm{i}\varphi}$ be the central charge of R, and

$$\theta_i = \frac{\operatorname{Im}(e^{-\mathrm{i}\varphi}z_i)}{\sin\varphi}.$$

A representation with central charge of argument φ is *semistable* if no subrepresentation has a central charge of argument larger than φ; this is King's stability [Kin94] for the real parameter $-\theta$, and a metric is harmonic exactly when $\mu_i = \theta_i$ for all i, that is, when it is a zero of the moment map shifted by the stability parameter (Section 5).

Theorem 1.1 (Representations)**.** *For all charges in the upper half-plane, every nonzero representation* R *of* Q *and every initial metric, along the flow:*

(i) *the Kempf–Ness function* $\sum_a \|\rho_a\|^2 + \sum_i \theta_i \log\det h_i$ *of* R *does not increase;*
(ii) *if* R *is semistable, then* $\mu_i(t) \to \theta_i$ *for all* i*, and the mass tends to* $|Z|$*;*
(iii) *if* R *is polystable, the metrics* $e^{\varphi t}h(t)$ *converge exponentially to a harmonic metric.*

Here ρ_a is the map of the arrow a and $\|\cdot\|$ the Hilbert–Schmidt norm for the metrics. The factor $e^{\varphi t}$ is needed because a harmonic metric is not stationary: it moves by the scalar $e^{-\varphi t}$. After multiplication of the metrics by $e^{\varphi t}$ the velocity of the flow at the vertex i is $\varphi - \arg(z_i+\mu_i)$, an increasing function of $\mu_i - \theta_i$ that vanishes at zero. The downward gradient flow of the Kempf–Ness function, which [HKKP26, (5.5)] describes as a simplification of this flow, has the velocity $\mu_i - \theta_i$ up to a positive factor; its exponential convergence on polystable representations is classical (see [HKKP23, §5.1], and [GRS21] for the general theory), and the gradient flow of the norm square of the moment map on the space of representations is studied in [HW11]. Theorem 1.1 concerns the flow $-\arg\Omega$ itself. The precise statement, with the rate, is Theorem 5.5.

1.2. **Complexes with one differential.** For complexes of representations Ω is defined through a superconnection [HKKP26, §5.4]; it is then not normal in general, and we take $\arg\Omega$ to be the argument of the unitary factor in the polar decomposition, which is our reading of the positivity condition of [HKKP26, §4.2]; it agrees with the functional calculus of [HKKP26, §5.4] when Ω is normal (Section 2). All statements below refer to this choice, and we call the flow the *phase flow*.

Fix a vertex i_0. A complex of representations E of Q has *a surjective differential at* i_0 if at each vertex $i \neq i_0$ it is a vector space in degree zero and at i_0 it is $V \xrightarrow{D} U$ in degrees $0, 1$ with D surjective (Definition 2.1). Its cohomology is a representation $\mathcal{H}$ of Q in degree zero, with $\ker D$ at i_0; we assume $\mathcal{H} \neq 0$, and put $Z = Z(\mathcal{H}) = |Z|e^{\mathrm{i}\varphi}$ and θ as above. For Hermitian metrics h on the spaces of E, the endomorphism Ω preserves each of these spaces. We write Ω_V, Ω_U and Ω_i $(i \neq i_0)$ for its blocks, $\mathfrak{m}(h) = \operatorname{tr}|\Omega_V| + \operatorname{tr}|\Omega_U| + \sum_{i\neq i_0}\operatorname{tr}|\Omega_i|$ for the mass, where $|T| = (TT^\dagger)^{1/2}$, and μ_i for the moment map of $\mathcal{H}$ with the induced metrics, so that the endomorphism Ω of the representation $\mathcal{H}$ is $\bigoplus_i(z_i+\mu_i)$. Let λ be the least eigenvalue of $DD^\dagger$ and $\|\cdot\|_1$ the trace norm. The quotient complex $\mathsf{E}/\mathcal{H}$ is supported at i_0 and is exact. We call its metric realization $(\ker D)^\perp \to U$ the *exact part*.

Theorem 1.2 (Convergence on the cohomology)**.** *Let* E *be a complex with a surjective differential at* i_0*. For all charges in the upper half-plane and every initial metric, along the phase flow:*

(i) *if* $\mathcal{H}$ *is semistable, then* $\mu_i(t) \to \theta_i$ *for all* i *and* $\mathfrak{m}(h(t)) \to |Z|$*;*
(ii) *if* $\mathcal{H}$ *is polystable, the metrics induced on the spaces of* $\mathcal{H}$*, multiplied by* $e^{\varphi t}$*, converge exponentially to a harmonic metric of* $\mathcal{H}$*.*

Theorem 1.3 (Decay of the exact part)**.** *Let* E *be a complex with a surjective differential at* i_0 *and* $U \neq 0$*. For all charges in the upper half-plane and every initial metric, along the phase flow,* $\lambda(t) \to \infty$ *and, for* $t \geq t_1$*,*

$$\|\Omega_V - (z_{i_0} + \mu_{i_0}) \oplus 0\|_1 + \|\Omega_U\|_1 + \sum_{i \neq i_0} \|\Omega_i - (z_i + \mu_i)\|_1 \leq C\, e^{-\varkappa(t-t_1)},$$

where $\oplus 0$ *refers to* $V = \ker D \oplus (\ker D)^\perp$*, the constants* $C, \varkappa > 0$ *depend only on the quiver, the vertex* i_0*, the charges and the dimensions, and* t_1 *depends in addition on the initial metric.*

Theorem 1.3 makes no assumption on $\mathcal{H}$. For $U \neq 0$ the complex itself carries no harmonic metric, since the phase has its spectrum in $(0, \pi)$ on V and in $(-\pi, 0)$ on U; what is proved is that the exact part decays and that the metrics induced on the cohomology converge. By duality [Wue26c, Theorem 5.1] the two theorems hold for complexes *with an injective differential at* i_0, which at i_0 are $U' \to V'$ in degrees $-1, 0$ with an injective differential (Proposition 5.8). The precise statements, with the constants, are Theorems 4.2 and 5.6, Proposition 4.3 and Remark 5.7.

1.3. **The method.** For a complex with $U \neq 0$ the Kempf–Ness function of the cohomology decreases only up to an error controlled by the exact part (Theorem 5.6(i)), so the proof cannot start from a Lyapunov function; it rests on an a priori bound for the arrows. Once the arrows are bounded, the exact part decays, the error becomes exponentially small, and the convexity of the Kempf–Ness function gives the convergence (Proposition 5.4). For an arrow a with map ρ_a let $e_a = \|\rho_a\|^2$ if a does not start at i_0 and $e_a = \operatorname{tr}(\rho_a R(D^\dagger D)\rho_a^\dagger)$, with $R(b) = (1 - e^{-b})/b$, if it does, and put $\mathcal{N} = \sum_a e_a$. For a representation $\mathcal{N}$ is the sum of the squared norms of the arrows.

Theorem 1.4 (A priori bound)**.** *Let* E *be a complex with a surjective differential at* i_0*. There are constants* $\gamma > 0$ *and* $c \geq 0$*, depending only on the quiver, the vertex* i_0*, the charges and the dimensions, such that for all metrics*

$$\frac{d\mathcal{N}}{dt} \leq -\gamma\, \mathcal{N} + c$$

along the phase flow. For a representation one can take $\gamma = \pi/\ell$*, with* ℓ *the number of arrows of a longest path in* Q*, taking* $\ell = 1$ *when* Q *has no arrows.*

The derivative $d\mathcal{N}/dt$ is a sum of three traces, one for V, one for U and one for the remaining vertices (Lemma 3.2). The trace for V is bounded by a trace inequality for the polar phase, and the trace for U by a bound for the negative part of a Hermitian matrix that occurs in Ω_U, both from [Wue26b] (Proposition 2.2(ii),(i)); the third by an elementary estimate (Lemma 3.3(c)), in which the blocks are normal and are indefinite perturbations of the charges. The estimates at the different vertices are combined with coefficients that increase along the arrows, and γ is the least increment; this is where the absence of oriented cycles is used, and for a quiver with a loop no such bound holds (Remark 3.5). See Theorem 3.4 and Proposition 3.6.

1.4. **Cohomology that is not polystable.** Every nonzero representation $\mathcal{H}$ has a unique filtration $0 = \mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n = \mathcal{H}$ by subrepresentations whose factors $\mathrm{gr}^k = \mathcal{H}^k/\mathcal{H}^{k-1}$ are semistable with $\arg Z(\mathrm{gr}^1) > \cdots > \arg Z(\mathrm{gr}^n)$, the Harder–Narasimhan filtration; we write $\phi_k = \arg Z(\mathrm{gr}^k)$. The sum $\sum_k |Z(\mathrm{gr}^k)|$ is the mass of $\mathcal{H}$ for the stability condition.

Theorem 1.5 (Harder–Narasimhan asymptotics)**.** *Let $\mathcal{H}$ be a nonzero representation of Q with the phase flow from an arbitrary initial metric, or the cohomology of a complex with a surjective or an injective differential at one vertex, with the metrics $h_i(t)$ induced by the phase flow of the complex. For all charges in the upper half-plane:*

(i) *for every nonzero vector v of $\mathcal{H}_i$, $\lim_{t\to\infty} \frac{1}{t} \log h_i(t)(v,v) = -\phi_k$, where k is the least index with $v \in \mathcal{H}_i^k$; so the Harder–Narasimhan filtration is the filtration of the spaces by the exponential rate of decay of the metric;*

(ii) *the eigenvalues of the argument of $z_i + \mu_i$, with μ the moment map of $\mathcal{H}$ for the metrics $h(t)$, converge to the ϕ_k with the multiplicities $\dim \mathrm{gr}_i^k$, and $\sum_i \operatorname{tr} |z_i + \mu_i|$ converges to $\sum_k |Z(\mathrm{gr}^k)|$, which is its infimum over all metrics on $\mathcal{H}$; along the phase flow of a representation this sum, the mass, is nonincreasing;*

(iii) *the second fundamental forms of the subrepresentations $\mathcal{H}^k$ tend to zero exponentially, and the metric induced on gr^k, multiplied by $e^{\phi_k t}$, has the properties stated in Theorem 1.2 and in Theorem 1.6 below for gr^k in the place of $\mathcal{H}$: its moment map tends to the stability parameter of gr^k, and it converges exponentially to a harmonic metric if gr^k is polystable.*

Theorem 1.6 (Semistable cohomology)**.** *Let $\mathcal{H}$ be as in Theorem 1.5 and semistable, with $Z(\mathcal{H}) = |Z| e^{\mathrm{i}\varphi}$. Then the metrics $e^{\varphi t} h(t)$ remain at bounded distance, in the space of metrics, from a trajectory of the gradient flow of the Kempf–Ness function of $\mathcal{H}$ for the Riemannian metric $\sum_i \mathsf{m}_i \operatorname{tr} \xi_i^2$ with the weights $\mathsf{m}_i = \operatorname{Im} z_i / \sin^2 \varphi$. Hence they are an asymptotic solution of this gradient flow in the sense of* [HKKP23]*, and by* [HKKP23, Corollary 5.10 and Theorem 5.11] *their growth and decay are given, up to bounded factors, by products of powers of iterated logarithms of t on the steps of the iterated weight filtration of $\mathcal{H}$, for the weight function $\sum_i \mathsf{m}_i \dim \mathsf{R}_i$, which equals $|Z(\mathsf{R})| / \sin\varphi$ on the subquotients R of $\mathcal{H}$ of phase φ.*

For a nonzero arrow $\mathbb{C} \to \mathbb{C}$ and two charges of equal argument, the metrics grow and decay like powers of t with exponents $|z_1|/|z_0 + z_1|$ and $-|z_0|/|z_0 + z_1|$. These exponents determine the ratio of the two weights (Proposition 6.6). The proof of Theorem 1.6 compares the velocity of the flow with the gradient of the Kempf–Ness function: the difference is quadratic in the shifted moment map, hence integrable in time, and gradient flows of convex functions on spaces of nonpositive curvature do not increase distances. The proof of Theorem 1.5 uses the mass as a Lyapunov function; its dissipation is bounded below by a multiple of the sum of the squared norms of the commutators of the arrows with the phases. See Theorems 6.3 and 6.11, Corollary 6.5 and Propositions 6.7 and 6.8.

1.5. **Kronecker quivers and the projective line.** Let K_m be the generalized Kronecker quiver, with two vertices and m arrows from the first to the second.

Theorem 1.7 (Differentials at one vertex)**.** *Let E be a bounded complex of representations of K_m whose differentials vanish at one of the two vertices, with any Hermitian metric.*

(i) *E is an orthogonal direct sum of subcomplexes E_n, one for each degree n, where E_n has cohomology $H^n(\mathsf{E})$ in degree n and, if $H^n(\mathsf{E}) \neq 0$, is a shift of a complex with a surjective or with an injective differential at one vertex. The phase flow preserves this decomposition and is the product of the flows of the E_n.*

(ii) *For all charges in the upper half-plane and every n, the blocks of the endomorphism Ω_{E_n} of E_n approach those of $(-1)^n\Omega_{H^n(\mathsf{E})}$, the endomorphism of the representation $H^n(\mathsf{E})$ with its induced metrics, extended by zero; the trace-norm distance decays exponentially from a time that depends on the initial metric. On nonzero $H^n(\mathsf{E})$ the induced metrics multiplied by $e^{-\pi n t}$ satisfy the conclusions of Theorem 1.5 and, in the semistable case, of Theorem 1.6. On a Harder–Narasimhan quotient of phase ϕ, the induced metric multiplied by $e^{(\phi-\pi n)t}$ converges exponentially to a harmonic metric if the quotient is polystable. The mass $\mathfrak{m}(h(t))$ converges to the sum over n of the masses of the representations $H^n(\mathsf{E})$ for the stability condition.*

(iii) *Minimal bounded complexes of projective representations of K_m, and minimal bounded complexes of injective representations, have differentials vanishing at one vertex. Hence (i) and (ii) hold for every object of the bounded derived category of K_2, that is, of the projective line, represented by such a complex with arbitrary metrics; the limit of the mass is the mass of the object for the stability condition with heart the category of representations.*

See Theorems 7.1, 7.2 and 7.4 and Proposition 7.3.

1.6. **Relation to earlier work and scope.** For $Q = K_m$ and a surjective differential at the first vertex, with $U \neq 0$ and cohomology that is nonzero at both vertices, Theorems 1.2, 1.3 and 1.4 are [Wue26b, Theorems 5.4, 5.7, 6.1 and 6.6], with the same proofs, up to two differences: here equal arguments of the two charges are included in Theorem 1.2, and the constant c of Theorem 1.4 is the one of [Wue26b] only after the refinement of Remark 3.5(2). Sections 4 and 5 follow [Wue26b]. What is added here is the a priori bound on a general quiver (Section 3), a reduction of the Kempf–Ness theorem for real stability parameters to the rational case of King (Lemma 5.3), the description of the flow for cohomology that is not polystable (Section 6), and Section 7. Theorem 1.1 is contained in [Wue26b] only for representations of K_m with both spaces nonzero and the charge of the first vertex of larger argument than that of the second, where it follows from the theorems on complexes by adding an exact summand.

Complexes with nonzero differentials at two or more vertices are open, in particular complexes of representations of K_m with differentials at both vertices. Quivers with oriented cycles are excluded. For a cohomology representation that is semistable and not polystable the induced metrics have no limit after any normalization by a scalar function of the time (Remark 7.5); Theorem 1.6 determines them up to a bounded factor, and whether this factor converges is open in general. It converges in the example of Proposition 6.6 and for a representation of K_2 for which the metrics are shown in [Wue26b, Proposition 6.9] to grow and decay like $t^{\pm 1/2}$. The time t_1 of Theorem 1.3 is explicit but very large (Remark 4.4).

1.7. **Organization.** Section 2 fixes the setting and states the results used from [Wue26a, Wue26b, Wue26c]. Section 3 proves Theorem 1.4, Section 4 Theorem 1.3, and Section 5 Theorems 1.1 and 1.2, including the Kempf–Ness theorem for real stability parameters in the form needed. Section 6 proves Theorems 1.5 and 1.6, and Section 7 proves Theorem 1.7.

2. Setting and results used

2.1. **Complexes with one differential.** Let Q be a finite quiver without oriented cycles, with set of vertices Q_0 and set of arrows Q_1; for an arrow a we write $\mathsf{s}(a)$ and $\mathsf{t}(a)$ for its source and target, and $a : i \to j$ if $\mathsf{s}(a) = i$, $\mathsf{t}(a) = j$. Fix *central charges* $z_i = x_i + \mathrm{i}y_i$, $i \in Q_0$, in the open upper half-plane, and put $\alpha_i = \arg z_i \in (0, \pi)$. Phases are arguments,

measured in radians; the phase of [Bri07] is the argument divided by π, and a shift by one degree changes arguments by π. A *complex of representations* of Q is a bounded complex in the abelian category of finite-dimensional representations: vector spaces E_i^n, differentials $d_i : E_i^n \to E_i^{n+1}$ and, for each arrow $a : i \to j$, a chain map $\rho_a : E_i^\bullet \to E_j^\bullet$. For a representation R with dimension vector (n_i) we write $Z(\mathsf{R}) = \sum_i n_i z_i$ for its central charge.

Definition 2.1. Let $i_0 \in Q_0$. A complex of representations E of Q is a *complex with a surjective differential at* i_0 if at every vertex $i \neq i_0$ it is a vector space W_i in degree zero, and at i_0 it is

$$V \xrightarrow{D} U \qquad \text{in degrees } 0, 1,$$

with D surjective. It is a *complex with an injective differential at* i_0 if at every vertex $i \neq i_0$ it is a vector space in degree zero, and at i_0 it is $U' \xrightarrow{D'} V'$ in degrees $-1, 0$ with D' injective. In both cases the cohomology is a representation in degree zero, and we assume that it is not zero.

So E has cohomology in degree zero and one differential, located at i_0, from degree 0 to degree 1 or from degree -1 to degree 0. For a complex with a surjective differential the arrows act by linear maps ρ_a between the spaces of degree zero, with $D\rho_a = 0$ for the arrows into i_0 and no condition on the arrows out of i_0. Its cohomology is the representation $\mathcal{H}$ with $\mathcal{H}_{i_0} = \ker D$, $\mathcal{H}_i = W_i$ for $i \neq i_0$ and the restricted arrows. Put $r = \dim U \geq 0$, $n_i = \dim \mathcal{H}_i$,

$$Z = Z(\mathcal{H}) = |Z| e^{\mathrm{i}\varphi}, \qquad \varphi \in (0, \pi), \qquad \sigma = \sin\varphi,$$

and write $z_* = x_* + \mathrm{i} y_*$ and α_* for the charge at i_0. We call φ the *total phase* of E; φ/π is the phase of $\mathcal{H}$ for the stability condition with central charge $Z(\,\cdot\,)$ and heart the category of representations [Bri07, Propositions 2.4 and 5.3]. For $r = 0$ the complex is a representation. For the generalized Kronecker quiver K_m, with vertices $0, 1$ and $m \geq 1$ arrows from 0 to 1, the complexes with a surjective differential at the vertex 0 with $r \geq 1$ and $n_0, n_1 \geq 1$ are those of [Wue26a, Wue26b]. Sections 3–5 treat complexes with a surjective differential; those with an injective differential are their duals (Proposition 5.8).

A *metric* on E consists of Hermitian metrics on V, U and the W_i; adjoints for the metrics are written †. The inclusion of $\mathcal{H}$ in degree zero gives an exact quotient complex $\mathsf{E}/\mathcal{H}$ supported at i_0. The metric identifies its degree-zero space $V/\ker D$ with $(\ker D)^\perp$. We call the resulting pair $(\ker D)^\perp \to U$ the *exact part* and r its *rank*. An outgoing arrow may be nonzero on $(\ker D)^\perp$. Put

$$B = D^\dagger D, \qquad L = DD^\dagger, \qquad \lambda = \lambda_{\min}(L), \qquad R(b) = \frac{1 - e^{-b}}{b},$$

$$C_{\mathrm{out}} = \sum_{a : i_0 \to \cdot} \rho_a^\dagger \rho_a, \qquad G_{\mathrm{in}} = \sum_{a : \cdot \to i_0} \rho_a \rho_a^\dagger,$$

and, for Hermitian $\Delta \geq 0$ and matrices G, M,

$$K_\Delta(G) = \int_0^1 e^{-(1-x)\Delta} G e^{-x\Delta} dx, \qquad J_L(M) = \int_{\substack{x,y \geq 0 \\ x+y \leq 1}} e^{-xL} M e^{-yL}\, dx\, dy,$$

and $J = J_L(DC_{\mathrm{out}}D^\dagger)$. We write $M_\pm \geq 0$ for the positive and negative parts of a Hermitian matrix M, $\|\cdot\|_1$ for the trace norm, $\|\cdot\|$ for the Hilbert–Schmidt norm and $\|\cdot\|_{\mathrm{op}}$ for the operator norm of matrices.

2.2. **The endomorphism Ω, the phase and the flow.** For a bounded complex of representations of Q with Hermitian metrics on all its spaces E_i^n, the endomorphism Ω of [HKKP26, §5.4] is defined through a superconnection in a graded algebra attached to Q: Ω is the grading times the component along the basis elements f_i of $\exp(-\sqrt{-1}F)$, where $F=(A-A^*)^2+\sqrt{-1}\sum_i f_i\otimes z_i$ and $A=\sum_i e_i\otimes d_i+\sum_a a\otimes\rho_a$ in the algebra of [HKKP26, Definition 5.13] tensored with the endomorphisms. Expanding the exponential in the terms that contain the arrows, which stops at second order, gives the following formula [Wue26c, Proposition 2.1], which may be taken as the definition of Ω here. Let ε be the grading, equal to $(-1)^n$ in degree n, and for $i\in Q_0$ and an arrow $a:i\to j$ put, on $E_i=\bigoplus_n E_i^n$,

$$\Delta_i=d_id_i^\dagger+d_i^\dagger d_i,\qquad \eta_a=d_j^\dagger\rho_a-\rho_a d_i^\dagger,\qquad \mu_i^{\mathsf{E}}=\sum_{a:i\to\cdot}\rho_a^\dagger\rho_a-\sum_{a:\cdot\to i}\rho_a\rho_a^\dagger.$$

Then Ω preserves each E_i^n, and on E_i

$$\begin{aligned}\varepsilon\,\Omega=K_{\Delta_i}\big(z_i+\mu_i^{\mathsf{E}}\big)&+\sum_{a:k\to i}\int e^{-x\Delta_i}\eta_a\,e^{-(1-x-y)\Delta_k}\eta_a^\dagger\,e^{-y\Delta_i}\\&-\sum_{a:i\to j}\int e^{-x\Delta_i}\eta_a^\dagger\,e^{-(1-x-y)\Delta_j}\eta_a\,e^{-y\Delta_i},\end{aligned}\tag{1}$$

the integrals over $\{x,y\ge 0,\ x+y\le 1\}$ with the measure $dx\,dy$. For a representation, that is, a complex in degree zero, this is $\Omega=\bigoplus_i(z_i+\mu_i^{\mathsf{E}})$: the charge at each vertex shifted by the moment map. All terms other than $z_ie^{-\Delta_i}$ are Hermitian, so $(-1)^n\Omega$ has the positive definite imaginary part $y_ie^{-\Delta_i}$ on E_i^n. A matrix T with positive definite imaginary part has a unique Hermitian matrix Ψ with spectrum in $(0,\pi)$ such that $Te^{-\mathrm{i}\Psi}$ is positive definite: in the polar decomposition $T=|T|\,\mathsf{W}$, with $|T|=(TT^\dagger)^{1/2}$ and W unitary, an eigenvalue $e^{\mathrm{i}\psi}$ of W with unit eigenvector v gives $\langle v,Tv\rangle=e^{\mathrm{i}\psi}\langle v,|T|v\rangle$, hence $\sin\psi>0$ (see [Wue26a, Lemma 2.2]). We call Ψ the *polar phase* of T and $|T|=Te^{-\mathrm{i}\Psi}$ its modulus. The *phase* Φ of the metrized complex is, on E_i^n, the polar phase of $(-1)^n\Omega$ minus πn, with spectrum in $(-\pi n,\pi-\pi n)$. This is how we read the positivity condition of [HKKP26, §4.2] on the branches of [HKKP26, (5.15)]; in [HKKP26, §5.4] the phase is written with the functional calculus, the two readings agree when Ω is normal, and for complexes Ω is not normal in general [Wue26a, Example 2.3]. Two properties are read off from (1):

- (O1) for an orthogonal direct sum of metrized complexes, Ω and Φ are the direct sums of those of the summands;
- (O2) if $\mathsf{F}[-n]$ denotes the complex F with all degrees raised by n, then $\Omega_{\mathsf{F}[-n]}=(-1)^n\Omega_{\mathsf{F}}$ and $\Phi_{\mathsf{F}[-n]}=\Phi_{\mathsf{F}}-\pi n$.

The *phase flow* is the flow

$$h^{-1}\dot h=-\Phi\tag{2}$$

of the metrics; it is the minimizing flow of [HKKP26, §4.1] for this phase, and we use a neutral name because the mass, defined below, can increase along it, already for two-term complexes over K_m [Wue26a, Theorem 1.1]. It exists for all $t\ge 0$ from every initial metric, because the vector field is real-analytic and the velocity $h^{-1}\dot h$ is bounded [Wue26c, Proposition 2.2]. The *mass* of a metric is $\mathfrak{m}(h)=\sum_{i,n}\operatorname{tr}|\Omega_i^n|$, where Ω_i^n is the restriction of Ω to E_i^n, and a metric is *harmonic* if Φ is a scalar. Let each vector $e_k(t)$ of a frame of E_i^n solve $\dot e_k=\frac12\Phi e_k$, starting from an $h(0)$-orthonormal frame; the frame is $h(t)$-orthonormal, because $h^{-1}\dot h=-\Phi$ and Φ

is self-adjoint for h. Differentiating the matrices in these frames shows that the matrix of each structure map $T \in \{d_i, \rho_a\}$ evolves by

$$\dot T = \tfrac12\big(T\,\Phi_{\text{source}} - \Phi_{\text{target}}\,T\big), \tag{3}$$

with the phases on the source and the target of T. By (O2) the trajectories of the flow of $\mathsf{F}[-n]$ are those of F multiplied by $e^{\pi n t}$, and (3) is the same for both.

For a complex with a surjective differential at i_0, (1) gives the blocks (4) below [Wue26c, Theorem 6.1(a)]. Indeed, at i_0 one has $\Delta_{i_0} = B \oplus L$ on $V \oplus U$, and $\Delta_j = 0$ for $j \neq i_0$; $\eta_a = -\rho_a D^\dagger$ on U for the arrows a out of i_0, and $\eta_a = 0$ for all other arrows; the matrix G_{in} has its range in $\ker D$, so that $K_B(G_{\text{in}}) = G_{\text{in}}$; and $\int e^{-(1-x-y)L}dx\,dy = (L - 1 + e^{-L})L^{-2}$ over the simplex, so that $D^\dagger(L-1+e^{-L})L^{-2}D = 1 - R(B)$. Hence

$$\begin{gathered}\Omega_V = z_* e^{-B} + M_V, \qquad M_V := K_B(C_{\text{out}}) - G_{\text{in}}, \qquad \Omega_U = J - z_* e^{-L},\\ \Omega_j = z_j + M_j, \qquad M_j := \sum_{a:j\to\cdot} \rho_a^\dagger\rho_a - \sum_{\substack{a:k\to j\\ k\neq i_0}} \rho_a\rho_a^\dagger - \sum_{a:i_0\to j} \rho_a R(B)\rho_a^\dagger \qquad (j\neq i_0).\end{gathered} \tag{4}$$

The matrices M_V, J and M_j are Hermitian and do not depend on the charges; the Ω_j are normal; and $\operatorname{tr}\Omega = Z$ [HKKP26, Lemma 5.17]. The phases are Φ_V and Φ_j, with spectrum in $(0,\pi)$, and Φ_U, with spectrum in $(-\pi, 0)$; the mass is $\mathfrak{m}(h) = \operatorname{tr}|\Omega_V| + \operatorname{tr}|\Omega_U| + \sum_{j\neq i_0}\operatorname{tr}|\Omega_j|$.

2.3. **The adapted splitting.** Let E have a surjective differential at i_0, and fix a metric. Split $V = \ker D \oplus (\ker D)^\perp$ orthogonally and write the blocks of Φ_V as Φ_{HH}, Φ_{HA}, Φ_{AH}, Φ_{AA}, the index H referring to $\ker D$ and A to its complement. In orthonormal frames adapted to the splitting,

$$D = (0, F), \qquad \rho_a = (H_a, T_a)\ \ (a: i_0\to j), \qquad \rho_a = \begin{pmatrix} R_a \\ 0\end{pmatrix}\ \ (a: k\to i_0),$$

with F invertible. Put $X_a = T_a F^{-1}$ for the arrows out of i_0. The splitting depends on the metric. As in [Wue26a, §6.1], the frame of V is kept adapted by the additional rotation with the anti-Hermitian generator $\mathsf{a} = \frac12\left(\begin{smallmatrix} 0 & \Phi_{HA}\\ -\Phi_{AH} & 0\end{smallmatrix}\right)$, and (3) becomes

$$\begin{aligned}\dot F &= \tfrac12(F\Phi_{AA} - \Phi_U F), \qquad \dot H_a = \tfrac12(H_a\Phi_{HH} - \Phi_j H_a),\\ \dot X_a &= \tfrac12(X_a\Phi_U - \Phi_j X_a) + H_a\Phi_{HA}F^{-1}, \qquad \dot R_a = \tfrac12(R_a\Phi_k - \Phi_{HH}R_a).\end{aligned} \tag{5}$$

Indeed, the rotation changes the right sides of (3) by $+D\mathsf{a}$, $+\rho_a\mathsf{a}$ for the maps out of V and by $-\mathsf{a}\rho_a$ for the maps into V; the first block of $\frac12(D\Phi_V - \Phi_U D) + D\mathsf{a}$ vanishes, the first two equations and $\dot T_a = \frac12(T_a\Phi_{AA} - \Phi_j T_a) + H_a\Phi_{HA}$ are its second block and the blocks of $\frac12(\rho_a\Phi_V - \Phi_j\rho_a) + \rho_a\mathsf{a}$, the third follows from $X_a = T_aF^{-1}$, and the second block of $\frac12(\rho_a\Phi_k - \Phi_V\rho_a) - \mathsf{a}\rho_a$ vanishes.

The arrows of the cohomology representation, for the induced metrics, have the matrices $\beta_a = H_a$, R_a or ρ_a according as a leaves i_0, enters i_0 or does not meet i_0. Its *moment map* is

$$\mu_i = \sum_{a:i\to\cdot}\beta_a^\dagger\beta_a - \sum_{a:\cdot\to i}\beta_a\beta_a^\dagger \qquad (i\in Q_0),$$

outgoing minus incoming arrows; its endomorphism Ω is $\bigoplus_i(z_i + \mu_i)$, which is normal, with phases $\arg(z_i + \mu_i)$. Put

$$q_{\text{out}} = \sum_{a:i_0\to\cdot}\|H_a\|^2, \qquad s = \sum_{a:i_0\to\cdot}\|X_a\|^2, \qquad Y_j = \sum_{a:i_0\to j} X_a(1-e^{-L})X_a^\dagger \geq 0.$$

Since $B = 0 \oplus F^\dagger F$ and $FR(F^\dagger F)F^\dagger = 1 - e^{-L}$, the blocks (4) are, in the adapted splitting,

$$\Omega_V = \begin{pmatrix} z_* + \mu_{i_0} & E_R \\ E_R^\dagger & z_* e^{-F^\dagger F} + K_{F^\dagger F}(C_T) \end{pmatrix}, \qquad \Omega_j = z_j + \mu_j - Y_j, \tag{6}$$

with $E_R = \sum_a H_a^\dagger T_a R(F^\dagger F)$ and $C_T = \sum_a T_a^\dagger T_a$, the sums over the arrows out of i_0; moreover $C_T = F^\dagger S F$ and $D C_{\mathrm{out}} D^\dagger = L\,S\,L$ with $S = \sum_a X_a^\dagger X_a$, $\operatorname{tr} S = s$. Let

$$\delta_{i_0} = \Phi_{HH} - \arg(z_* + \mu_{i_0}), \qquad \delta_j = \Phi_j - \arg(z_j + \mu_j) \quad (j \neq i_0)$$

be the differences between the phases of the complex on the cohomology and the phases of the cohomology representation. We say that the exact part *decays* along a trajectory if $\lambda \to \infty$ and

$$\|\Omega_V - (z_* + \mu_{i_0}) \oplus 0\|_1 + \|\Omega_U\|_1 + \sum_{j \neq i_0} \|\Omega_j - (z_j + \mu_j)\|_1 \to 0,$$

where $\oplus 0$ refers to $V = \ker D \oplus (\ker D)^\perp$: the blocks of Ω approach those of the cohomology representation with its induced metrics. If $s = 0$, then $E_R = 0$, $C_T = 0$, $J = 0$ and $Y_j = 0$: the complex is the orthogonal direct sum of its cohomology representation and of the complex $(\ker D)^\perp \to U$ at i_0, and all δ_i vanish.

2.4. **Results used.** We use the following results of [Wue26a, Wue26b]; they are statements about matrices.

Proposition 2.2. *The following inequalities hold.*

(i) [Wue26b, Theorem 3.1] *For Hermitian matrices $L > 0$ and $M \geq 0$, $\operatorname{tr} J_L(M)_- \leq \frac{2}{5} \operatorname{tr} J_L(M)$.*

(ii) [Wue26b, Corollary 4.2] *Let $P > 0$ and K be Hermitian matrices and z a complex number with $\operatorname{Im} z > 0$, and let Ψ be the polar phase of $zP + K$. Then $\operatorname{tr}(K\Psi) \leq (\arg z) \operatorname{tr} K$.*

(iii) [Wue26a, Lemma 3.1] *Let $T = \left(\begin{smallmatrix} A & E \\ E^\dagger & C \end{smallmatrix}\right)$ have positive definite imaginary part, let the least singular value of A be at least $b > 0$, and let Ψ and ψ_A be the polar phases of T and A. Then $\|\Psi_{12}\| \leq 2\pi \|E\| / b$, and if $\|E\| / b \leq \frac{1}{8}$, then $\|\Psi_{11} - \psi_A\| \leq 4\pi \|E\| / b$.*

(iv) [Wue26a, Lemma 3.2], *in the form of* [Wue26b, Proposition 2.3(v)]. *Let $T_\circ = |T_\circ|\, e^{\mathrm{i}\Psi_\circ}$ be an invertible matrix in polar form, with $\Psi_\circ$ Hermitian, and $T = T_\circ + M$ a matrix with $\operatorname{Im} T > 0$ and polar phase Ψ. If v is a unit vector with $|T_\circ| v = pv$ and $\Psi_\circ v = \psi v$, where $p > 0$ and $0 < \psi < \pi$, then $|\langle v, (\Psi - \psi) v \rangle| \leq \pi \sqrt{\|M\|_1 / p}$.*

(v) [Wue26a, Lemma 4.2 and §6.3], *in the form of* [Wue26b, Proposition 2.3(ii)]. *Let F be an invertible $r \times r$ matrix, $L = FF^\dagger$, and $S \geq 0$ a Hermitian $r \times r$ matrix. Then, with $\kappa_0 = \pi/(2e)$, $\left\|K_{F^\dagger F}(F^\dagger S F)\right\|_1 \leq r\kappa_0 \operatorname{tr} S$ and $\|J_L(LSL)\|_1 \leq r \operatorname{tr} S$.*

We also use two results of [Wue26c] on complexes.

Proposition 2.3 ([Wue26c])**.** (a) [Wue26c, Theorem 3.4] *Let E be a complex with a surjective differential at i_0, and for a metric h let*

$$\mathcal{S}_{\mathbb{C}}(h) = \sum_i z_i \log\det h_i - z_* \log\det h_U - \sum_a \|\rho_a\|^2 + \int_0^1 \left(\operatorname{tr} \Omega_U^{(x)} - \operatorname{tr} \Omega_U^{(0)}\right) \frac{dx}{x}, \tag{7}$$

where h_{i_0}, h_U and h_i are the metrics of V, U and W_i, the determinants are Gram determinants in fixed bases, and $\Omega^{(x)}$ denotes Ω for the complex with D replaced by $\sqrt{x}\, D$. Then $\mathcal{S}_{\mathbb{C}}$ is a complex-valued primitive of Ω: $d\mathcal{S}_{\mathbb{C}}(\xi) = \operatorname{tr}(\Omega_V \xi_V) + \operatorname{tr}(\Omega_U \xi_U) + \sum_j \operatorname{tr}(\Omega_j \xi_j)$ for $\xi = h^{-1}\delta h$.

(b) [Wue26c, Theorem 5.1] *Let $\mathsf{E}^\vee$ be the dual complex over the opposite quiver: the dual spaces in the opposite degrees, the transposed maps, the dual metrics, and the charges $-\overline{z_i}$. Then $h(t) \mapsto e^{-\pi t}(h(t)^{-1})^\mathsf{T}$ maps trajectories of the phase flow of E to trajectories of the phase flow of $\mathsf{E}^\vee$; in dual orthonormal frames Ω is replaced by $-\overline{\Omega}$ and Φ by $\pi - \overline{\Phi}$; and the mass is preserved.*

3. An a priori bound for the arrows

Definition 3.1. For an arrow a put

$$e_a = \|\rho_a\|^2 \quad \text{if } \mathsf{s}(a) \neq i_0, \qquad e_a = \operatorname{tr}\big(\rho_a R(B) \rho_a^\dagger\big) \quad \text{if } \mathsf{s}(a) = i_0.$$

The *arrow energy* of the metrized complex is $\mathcal{N} = \sum_{a \in Q_1} e_a$.

For an arrow a out of i_0 we split $e_a = e_a^K + e_a^J$ with

$$e_a^K = \operatorname{tr}\big(\rho_a^\dagger \rho_a \, e^{-B}\big) \geq 0, \qquad e_a^J = \operatorname{tr}\big(\rho_a^\dagger \rho_a \, (R(B) - e^{-B})\big) \geq 0.$$

Then

$$\operatorname{tr} K_B(C_{\mathrm{out}}) = \sum_{a : i_0 \to \cdot} e_a^K, \qquad \operatorname{tr} J = \sum_{a : i_0 \to \cdot} e_a^J, \qquad \mathcal{N} = \sum_a \|\beta_a\|^2 + \sum_{j \neq i_0} \operatorname{tr} Y_j. \tag{8}$$

The first identity holds because $\operatorname{tr} K_B(G) = \operatorname{tr}(G e^{-B})$. For the second, let $\omega(\ell) = \int_0^1 w e^{-w\ell} dw = (1 - e^{-\ell} - \ell e^{-\ell})/\ell^2$; then

$$\operatorname{tr} J = \operatorname{tr}\big(D C_{\mathrm{out}} D^\dagger \, \omega(L)\big), \qquad D^\dagger \omega(L) D = R(B) - e^{-B}.$$

The third follows from $\rho_a R(B) \rho_a^\dagger = H_a H_a^\dagger + X_a (1 - e^{-L}) X_a^\dagger$. In particular

$$\sum_a \|\beta_a\|^2 \leq \mathcal{N}, \qquad \operatorname{tr} J \leq \mathcal{N}, \qquad (1 - e^{-\lambda})\, s \leq \sum_j \operatorname{tr} Y_j \leq \mathcal{N}, \qquad \|\mu_i\|_1 \leq \mathcal{N}. \tag{9}$$

Lemma 3.2. *For all charges, along the phase flow,*

$$\frac{d\mathcal{N}}{dt} = \operatorname{tr}(M_V \Phi_V) + \operatorname{tr}(J \Phi_U) + \sum_{j \neq i_0} \operatorname{tr}(M_j \Phi_j).$$

Proof. In the function (7) of Proposition 2.3(a), by (4),

$$\operatorname{tr} \Omega_U^{(x)} = \operatorname{tr} J_{xL}(x D C_{\mathrm{out}} D^\dagger) - z_* \operatorname{tr} e^{-xL}, \qquad \operatorname{tr} J_{xL}(x D C_{\mathrm{out}} D^\dagger) = x \operatorname{tr}\big(D C_{\mathrm{out}} D^\dagger \omega(xL)\big).$$

Since $\int_0^1 \omega(x\ell)\, dx = (1 - R(\ell))/\ell$ and $D^\dagger \frac{1 - R(L)}{L} D = 1 - R(B)$, the part of $\mathcal{S}_\mathbb{C}$ that does not contain the charges is

$$-\sum_a \|\rho_a\|^2 + \operatorname{tr} C_{\mathrm{out}} - \operatorname{tr}\big(C_{\mathrm{out}} R(B)\big) = -\mathcal{N}.$$

Both sides of the identity for $d\mathcal{S}_\mathbb{C}$ are affine functions of the charges, so the terms without charges agree: $d\mathcal{N}(\xi) = -\operatorname{tr}(M_V \xi_V) - \operatorname{tr}(J \xi_U) - \sum_j \operatorname{tr}(M_j \xi_j)$. Put $\xi = -\Phi$. □

We use the following elementary inequality: for a complex number ζ in the upper half-plane and a real number $u \geq 0$,

$$u\big(\pi - \arg(\zeta - u)\big) \leq \max\{\pi \max(\operatorname{Re}\zeta, 0),\, 2 \operatorname{Im}\zeta\}. \tag{10}$$

It is trivial for $u = 0$; let $u > 0$. If $\operatorname{Re}\zeta \leq u/2$ then $\pi - \arg(\zeta - u) = \arctan \frac{\operatorname{Im}\zeta}{u - \operatorname{Re}\zeta} \leq \frac{2 \operatorname{Im}\zeta}{u}$, and otherwise $u < 2 \operatorname{Re}\zeta$ and $\pi - \arg(\zeta - u) \leq \frac{\pi}{2}$ for $u \geq \operatorname{Re}\zeta$, while for $u < \operatorname{Re}\zeta$ the left side is at most $\pi u < \pi \operatorname{Re}\zeta$.

Lemma 3.3. *For every metric and all charges:*

(a) $\operatorname{tr}(M_V\Phi_V) \le \alpha_* \operatorname{tr} M_V = \alpha_*\big(\sum_{a:i_0\to\cdot} e_a^K - \sum_{a:\cdot\to i_0} e_a\big)$;

(b) $\operatorname{tr}(J\Phi_U) \le \pi \operatorname{tr} J_- \le \frac{2\pi}{5}\operatorname{tr} J = \frac{2\pi}{5}\sum_{a:i_0\to\cdot} e_a^J$, *and* $\operatorname{tr}(J\Phi_U)\le 0$ *if* $r\le 1$;

(c) *for* $j\ne i_0$, *with the number* $b_j := \max\{\pi|x_j|,\, 2y_j\}$,

$$\operatorname{tr}(M_j\Phi_j) \le n_j b_j - \pi \operatorname{tr}(M_j)_-, \qquad \operatorname{tr}(M_j)_- \ge \Big(\sum_{a:\cdot\to j} e_a - \sum_{a:j\to\cdot} e_a\Big)_+.$$

Proof. (a) Proposition 2.2(ii) with $zP = z_* e^{-B}$ and $K = M_V$; and $\operatorname{tr} G_{\mathrm{in}} = \sum_{a:\cdot\to i_0} e_a$.

(b) Since $-\pi < \Phi_U < 0$, $\operatorname{tr}(J_+\Phi_U) \le 0$ and $-\operatorname{tr}(J_-\Phi_U) \le \pi \operatorname{tr} J_-$; then apply Proposition 2.2(i) to $M = DC_{\mathrm{out}}D^\dagger \ge 0$. If $r = 1$, J is a nonnegative number.

(c) The matrix $\Omega_j = z_j + M_j$ is normal and $\Phi_j = g_j(M_j)$ with $g_j(u) = \arg(z_j + u) \in (0,\pi)$ for real u. For $u \ge 0$, $\arg(z_j+u) = \pi - \arg(-\overline{z_j} - u)$, so (10) with $\zeta = -\overline{z_j}$ gives $u\, g_j(u) \le b_j$. For $u < 0$, $u\, g_j(u) = -\pi|u| + |u|\big(\pi - \arg(z_j - |u|)\big) \le -\pi|u| + b_j$ by (10) with $\zeta = z_j$. Summing over the eigenvalues of M_j gives the first inequality. Finally $\operatorname{tr}(M_j)_- \ge (-\operatorname{tr} M_j)_+$, and $-\operatorname{tr} M_j = \sum_{a:\cdot\to j} e_a - \sum_{a:j\to\cdot} e_a$ by (4). □

3.1. **Height functions.** The estimates of Lemma 3.3 at the different vertices are combined with coefficients that increase along the arrows. Let

$$\alpha_*^+ = \max\{\alpha_*, \tfrac{2\pi}{5}\} \ \text{ if } r \ge 2, \qquad \alpha_*^+ = \alpha_* \ \text{ if } r \le 1.$$

A *height function* is a function $w : Q_0 \setminus \{i_0\} \to [0,1]$ such that

$$w_{\mathsf{t}(a)} > w_{\mathsf{s}(a)} \ \text{ if } \mathsf{s}(a), \mathsf{t}(a) \ne i_0, \qquad \pi w_{\mathsf{t}(a)} > \alpha_*^+ \ \text{ if } \mathsf{s}(a) = i_0, \qquad \pi w_{\mathsf{s}(a)} < \alpha_* \ \text{ if } \mathsf{t}(a) = i_0.$$

Height functions exist. Let $\mathcal{A}$ be the set of vertices different from i_0 with a path to i_0, $\mathcal{D}$ the set of vertices different from i_0 reached by a path from i_0, and $\mathcal{R}$ the set of the remaining vertices different from i_0. Since Q has no oriented cycle, $\mathcal{A}\cap\mathcal{D} = \emptyset$, there is no arrow from $\mathcal{R}\cup\mathcal{D}$ to $\mathcal{A}$, and none from $\mathcal{D}$ to $\mathcal{R}$. So there is a linear order of $Q_0\setminus\{i_0\}$, compatible with the arrows, in which the vertices of $\mathcal{A}$ come first and those of $\mathcal{D}$ last; let w be increasing for this order, with values in $(0, \frac{\alpha_*}{2\pi})$ on $\mathcal{A}$, in $(\frac{\alpha_*}{2\pi}, \frac{\pi+\alpha_*^+}{2\pi})$ on $\mathcal{R}$ and in $(\frac{\pi+\alpha_*^+}{2\pi}, 1)$ on $\mathcal{D}$. For a height function w put

$$\gamma_w = \min\Big\{ \min_{\substack{a\in Q_1\\ \mathsf{s}(a),\mathsf{t}(a)\ne i_0}} \pi\big(w_{\mathsf{t}(a)} - w_{\mathsf{s}(a)}\big),\ \min_{a:i_0\to\cdot}\big(\pi w_{\mathsf{t}(a)} - \alpha_*^+\big),\ \min_{a:\cdot\to i_0}\big(\alpha_* - \pi w_{\mathsf{s}(a)}\big)\Big\} > 0, \tag{11}$$

$$c_Q = \sum_{j\ne i_0} n_j b_j,$$

where a minimum over an empty set is omitted; if Q has no arrows, put $\gamma_w = 1$.

Theorem 3.4 (A priori bound)**.** *Let* E *be a complex with a surjective differential at* i_0 *and* w *a height function. For all charges in the upper half-plane and every trajectory of the phase flow,*

$$\frac{d\mathcal{N}}{dt} \le -\gamma_w \mathcal{N} + c_Q, \qquad \mathcal{N}(t) \le \mathcal{N}(0)e^{-\gamma_w t} + \frac{c_Q}{\gamma_w}\big(1 - e^{-\gamma_w t}\big).$$

Hence $\mathcal{N}(t) \le \max\{\mathcal{N}(0), c_Q/\gamma_w\}$ *for all* $t \ge 0$, *and if* $c_Q > 0$,

$$\mathcal{N}(t) \le \bar{\mathcal{N}} := \frac{2c_Q}{\gamma_w} \qquad \text{for } t \ge t_0 := \frac{1}{\gamma_w}\log^+\frac{\gamma_w\mathcal{N}(0)}{c_Q},$$

where $\log^+ x = \max(\log x, 0)$ *for* $x > 0$ *and* $\log^+ 0 = 0$.

Proof. For $w_j \in [0,1]$ one has $\operatorname{tr}(M_j)_- \geq w_j\,(-\operatorname{tr} M_j)$, by Lemma 3.3(c). So Lemmas 3.2 and 3.3 give

$$\frac{d\mathcal{N}}{dt} \leq c_Q - \pi \sum_{j\neq i_0} w_j \Big(\sum_{a:\cdot\to j} e_a - \sum_{a:j\to\cdot} e_a \Big) + \alpha_* \sum_{a:i_0\to\cdot} e_a^K + \tfrac{2\pi}{5} \sum_{a:i_0\to\cdot} e_a^J - \alpha_* \sum_{a:\cdot\to i_0} e_a,$$

where for $r \leq 1$ the term with $\frac{2\pi}{5}$ is absent. Collect the terms arrow by arrow. An arrow a with $\mathsf{s}(a), \mathsf{t}(a) \neq i_0$ has the coefficient $-\pi(w_{\mathsf{t}(a)} - w_{\mathsf{s}(a)})$. An arrow a out of i_0 has $-\pi w_{\mathsf{t}(a)} e_a$ from the first sum and at most $\alpha_*^+(e_a^K + e_a^J) = \alpha_*^+ e_a$ from the next two. An arrow a into i_0 has $+\pi w_{\mathsf{s}(a)} e_a - \alpha_* e_a$. Hence $\frac{d\mathcal{N}}{dt} \leq c_Q - \gamma_w \sum_a e_a$. The second inequality follows by integration, and $\mathcal{N}(0) e^{-\gamma_w t} \leq c_Q/\gamma_w$ for $t \geq t_0$. □

Remark 3.5. (1) By (9) the theorem bounds the arrows of the cohomology representation, its moment map, $\operatorname{tr} J$, and $s \leq \mathcal{N}/(1-e^{-\lambda})$. The bound $\bar{\mathcal{N}}$ depends only on the charges, the dimensions n_j and the height function; the initial metric enters only through t_0. The height function constructed above is determined by the quiver, the vertex i_0 and the numbers α_*, α_*^+ once the linear order and the values are fixed by a rule, for instance values equally spaced in each of the three intervals; with such a rule the constants of Theorem 3.4 and of (13) depend only on the quiver, i_0, the charges and the dimensions, except κ_1 and the times t_0, t_1, t_2, which depend in addition on the initial metric.

(2) For $Q = K_m$ and $i_0 = 0$ the only condition on w is $\pi w_1 > \alpha_*^+$, and $w_1 = 1$ gives $\gamma_w = \pi - \alpha_*^+$, the constant of [Wue26b, Theorem 5.4]. For a vertex j that is a sink the number b_j may be replaced by $\max\{\pi \max(x_j, 0), 2y_j\}$, and for a source by $\max\{\pi \max(-x_j, 0), 2y_j\}$, since only one sign of the eigenvalues of M_j occurs.

(3) For a quiver with an oriented cycle there is no such bound in general: a loop acting by a multiple x of a unitary matrix has zero moment map for every x, the phases are then constant, and the arrow does not change along the flow.

For a representation no vertex is distinguished, and the bound takes the following form. It is used in Theorem 5.5.

Proposition 3.6 (Representations)**.** *Let* R *be a nonzero representation of a finite quiver* Q *without oriented cycles, with dimension vector* (n_i) *and any metric. Let* $\ell_Q \geq 1$ *be the number of arrows of a longest path in* Q*; if* Q *has no arrows, put* $\ell_Q = 1$*. For all charges in the upper half-plane, the sum* $\mathcal{N} = \sum_a \|\rho_a\|^2$ *of the squared norms of the arrows satisfies, along the phase flow,*

$$\frac{d\mathcal{N}}{dt} \leq -\frac{\pi}{\ell_Q}\mathcal{N} + c_{\mathsf{R}}, \qquad c_{\mathsf{R}} = \sum_i n_i \max\{\pi|x_i|, 2y_i\}.$$

Hence $\mathcal{N}(t) \leq \bar{\mathcal{N}}_{\mathsf{R}} := 2\ell_Q c_{\mathsf{R}}/\pi$ *for* $t \geq t_{\mathsf{R}} := \frac{\ell_Q}{\pi} \log^+ \frac{\pi \mathcal{N}(0)}{\ell_Q c_{\mathsf{R}}}$.

Proof. By (3), $\frac{d}{dt}\|\rho_a\|^2 = \operatorname{tr}(\rho_a^\dagger \rho_a \Phi_{\mathsf{s}(a)}) - \operatorname{tr}(\rho_a \rho_a^\dagger \Phi_{\mathsf{t}(a)})$, so $\frac{d\mathcal{N}}{dt} = \sum_i \operatorname{tr}(\mu_i \Phi_i)$ with the moment map $\mu_i = \sum_{a:i\to\cdot} \rho_a^\dagger \rho_a - \sum_{a:\cdot\to i} \rho_a \rho_a^\dagger$. The block $\Omega_i = z_i + \mu_i$ is normal and $\Phi_i = \arg(z_i + \mu_i)$, so the argument of Lemma 3.3(c) gives, for every vertex i and every $w_i \in [0,1]$,

$$\operatorname{tr}(\mu_i \Phi_i) \leq n_i \max\{\pi|x_i|, 2y_i\} - \pi \operatorname{tr}(\mu_i)_- \leq n_i \max\{\pi|x_i|, 2y_i\} - \pi w_i \Big(\sum_{a:\cdot\to i} \|\rho_a\|^2 - \sum_{a:i\to\cdot} \|\rho_a\|^2 \Big).$$

Let $\ell(i)$ be the number of arrows of a longest path ending at i, and $w_i = \ell(i)/\ell_Q$; then $w_{\mathsf{t}(a)} - w_{\mathsf{s}(a)} \geq 1/\ell_Q$ for every arrow a. Summing over i gives $\frac{d\mathcal{N}}{dt} \leq c_{\mathsf{R}} - \pi \sum_a (w_{\mathsf{t}(a)} - w_{\mathsf{s}(a)}) \|\rho_a\|^2 \leq c_{\mathsf{R}} - \frac{\pi}{\ell_Q}\mathcal{N}$, and the last statement follows by integration. □

4. Growth of the differential and decay of the exact part

In this section E is a complex with a surjective differential at i_0 and $r \geq 1$. The estimates of Lemma 4.1 and the proof of Theorem 4.2 follow those of [Wue26b] for K_m; what is new is that the block $z_* + \mu_{i_0}$ is no longer z_* plus a positive matrix, and that the phases at the targets of the arrows out of i_0 are bounded below only through the a priori bound. Recall $\lambda = \lambda_{\min}(L) = \lambda_{\min}(F^\dagger F)$, and let

$$\varpi_\star = \min\bigl\{\lambda_{\min}(\Phi_j) : \ j = \mathsf{t}(a) \text{ for an arrow } a \text{ out of } i_0\bigr\};$$

only targets with $W_j \neq 0$ enter this minimum. If there are none, then $s = 0$, Lemma 4.1(iii) reads $0 \leq 0$, and $\varpi_\star$, $\bar\varpi$ and the constants (13) are not used. Let $\mathrm{E}_1(b) = \int_1^\infty e^{-xb}\frac{dx}{x}$ be the exponential integral; for $b > 0$,

$$\frac{e^{-b}}{1+b} \leq \mathrm{E}_1(b) \leq e^{-b}\log\bigl(1+\tfrac{1}{b}\bigr), \qquad \mathrm{E}_1'(b) = -\frac{e^{-b}}{b}, \tag{12}$$

because the differences of the three functions tend to 0 at infinity and have derivatives of constant sign.

Lemma 4.1. *For all charges and every metric:*

(i) λ *is nondecreasing along the flow; its right derivative is the minimum of* $\lambda\bigl(\langle v, \Phi_{AA} v\rangle - \langle u, \Phi_U u\rangle\bigr)$ *over the unit vectors* v *with* $F^\dagger F v = \lambda v$, *where* $u = Fv/\sqrt{\lambda}$.

(ii) $\|E_R\| \leq \sqrt{q_{\mathrm{out}} s/\lambda}$ *and* $\sum_j \|Y_j\|_1 \leq s$; *hence, with* $\mathsf{d} = \min(n_{i_0}, r)$ *and* $\mathcal{E}_V := \Omega_V - \operatorname{diag}(z_* + \mu_{i_0},\ z_* e^{-F^\dagger F})$,

$$\|\mathcal{E}_V\|_1 \leq 2\sqrt{\mathsf{d}\, q_{\mathrm{out}} s/\lambda} + r\kappa_0 s.$$

(iii) $\dot s \leq \bigl(-\varpi_\star + \frac{4\pi q_{\mathrm{out}}}{y_*\lambda}\bigr)s$ *along the flow. Moreover* $\lambda_{\min}(\Phi_j) \geq \arg(z_j + \mathcal{N})$ *for each* $j \neq i_0$ *with* $W_j \neq 0$. *At every sink with* $W_j \neq 0$ *one has* $\lambda_{\min}(\Phi_j) \geq \alpha_j$.

(iv) *If* $\sqrt{q_{\mathrm{out}} s/\lambda} \leq y_*/8$, *then* $\|\delta_{i_0}\| \leq \frac{4\pi}{y_*}\sqrt{q_{\mathrm{out}} s/\lambda}$; *and* $\|\delta_j\| \leq \|Y_j\|/y_j \leq s/y_j$ *for* $j \neq i_0$.

(v) *Along the flow*

$$\frac{d}{dt}\operatorname{tr}\mathrm{E}_1(L) \leq -\operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr), \qquad \operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr) \geq \frac{y_*(\operatorname{tr} e^{-L})^2}{(y_* + |x_*|)\operatorname{tr} e^{-L} + \operatorname{tr} J_+}.$$

Proof. (i) By (5), $\frac{d}{dt}(F^\dagger F) = \frac{1}{2}(\Phi_{AA}F^\dagger F + F^\dagger F \Phi_{AA}) - F^\dagger \Phi_U F$. For a unit vector v with $F^\dagger F v = \lambda v$ this gives $\bigl\langle v, \frac{d}{dt}(F^\dagger F)v\bigr\rangle = \lambda\langle v, \Phi_{AA} v\rangle - \langle Fv, \Phi_U Fv\rangle$, which is positive because $\Phi_{AA} > 0 > \Phi_U$; and the right derivative of the least eigenvalue of a differentiable family of Hermitian matrices is the minimum of these numbers over the unit vectors of the eigenspace.

(ii) Since $E_R = \sum_a H_a^\dagger X_a F R(F^\dagger F)$ and $\bigl\|FR(F^\dagger F)\bigr\|_{\mathrm{op}} \leq \lambda^{-1/2}$, the Cauchy–Schwarz inequality gives the first bound. $\sum_j \operatorname{tr} Y_j = \operatorname{tr}\bigl((1 - e^{-L})S\bigr) \leq s$. The off-diagonal part of $\mathcal{E}_V$ has rank at most $2\mathsf{d}$ and trace norm $2\|E_R\|_1 \leq 2\sqrt{\mathsf{d}}\,\|E_R\|$, and its remaining block is $K_{F^\dagger F}(C_T) = K_{F^\dagger F}(F^\dagger S F)$; use Proposition 2.2(v).

(iii) By (5),

$$\dot s = \sum_a \operatorname{tr}(X_a^\dagger X_a \Phi_U) - \sum_a \operatorname{tr}(X_a^\dagger \Phi_{\mathsf{t}(a)} X_a) + 2\operatorname{Re}\sum_a \operatorname{tr}\bigl(X_a^\dagger H_a \Phi_{HA} F^{-1}\bigr).$$

The first sum is nonpositive, the second is at least $\varpi_\star s$, and the last term is at most $2\sqrt{s\, q_{\mathrm{out}}/\lambda}\,\|\Phi_{HA}\|$. In (6) the block $z_* + \mu_{i_0}$ is normal with imaginary part y_*, so its singular values are at least y_*, and Proposition 2.2(iii) gives $\|\Phi_{HA}\| \leq \frac{2\pi}{y_*}\|E_R\| \leq \frac{2\pi}{y_*}\sqrt{q_{\mathrm{out}} s/\lambda}$. For

the phases: $\Phi_j = g_j(M_j)$ with $g_j(u) = \arg(z_j + u)$ decreasing, and $M_j \le \sum_{a:j\to\cdot} \rho_a^\dagger \rho_a \le \mathcal{N}$ by (4); for a sink $M_j \le 0$.

(iv) The first statement is Proposition 2.2(iii) for (6). For $j \ne i_0$, $\Phi_j = g_j(\mu_j - Y_j)$ and $\arg(z_j + \mu_j) = g_j(\mu_j)$ by (6), $|g_j'| \le 1/y_j$, and a function with Lipschitz constant c satisfies $\|g(A) - g(A')\| \le c\,\|A - A'\|$ for Hermitian matrices A, A' in the Hilbert–Schmidt norm, as one sees by expanding in eigenbases of A and A'.

(v) By (5), $\dot L = F\Phi_{AA}F^\dagger - \frac12(\Phi_U L + L\Phi_U)$, so by (12)

$$\frac{d}{dt}\operatorname{tr}\mathrm{E}_1(L) = -\operatorname{tr}\bigl(e^{-F^\dagger F}\Phi_{AA}\bigr) + \operatorname{tr}\bigl(e^{-L}\Phi_U\bigr),$$

and both terms are nonpositive. Let (u_k) be an orthonormal eigenbasis of Φ_U with eigenvalues $\psi_k \in (-\pi, 0)$, and put $e_k = \langle u_k, e^{-L}u_k\rangle > 0$, $j_k = \langle u_k, Ju_k\rangle$. Since $-\Omega_U = z_* e^{-L} - J$ has the polar phase $\Phi_U + \pi$ and a positive definite modulus, $\langle u_k, (z_* e^{-L} - J)u_k\rangle$ is a positive multiple of $e^{\mathrm{i}(\psi_k+\pi)}$; hence $\cot(\psi_k + \pi) = (x_* e_k - j_k)/(y_* e_k)$ and

$$|\psi_k| = \operatorname{arccot}\frac{j_k - x_* e_k}{y_* e_k} \ge \operatorname{arccot}\frac{(j_k)_+ + |x_*|e_k}{y_* e_k} \ge \frac{y_* e_k}{(y_* + |x_*|)e_k + (j_k)_+},$$

because arccot is decreasing and $\operatorname{arccot} x \ge \frac{1}{1+x}$ for $x \ge 0$. Therefore

$$\operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr) = \sum_k |\psi_k| e_k \ge \sum_k \frac{y_* e_k^2}{(y_* + |x_*|)e_k + (j_k)_+} \ge \frac{y_*(\sum_k e_k)^2}{(y_* + |x_*|)\sum_k e_k + \sum_k (j_k)_+}$$

by the Cauchy–Schwarz inequality, and $\sum_k e_k = \operatorname{tr} e^{-L}$, $\sum_k (j_k)_+ \le \operatorname{tr} J_+$. □

4.1. **Constants.** Fix a height function w and let γ_w, c_Q, $\bar{\mathcal{N}}$, t_0 be as in Theorem 3.4. We assume $c_Q > 0$, that is, $W_j \ne 0$ for some $j \ne i_0$; otherwise all arrows vanish, Proposition 4.3 below applies, and we put $\bar{\mathcal{N}} = 0$ and $t_0 = 0$. Let $\bar\varpi$ be the least of the numbers

$$\arg(z_j + \bar{\mathcal{N}}) \ \ (j \text{ not a sink}), \qquad \alpha_j \ \ (j \text{ a sink}),$$

over the nonzero target spaces W_j of arrows out of i_0; by Lemma 4.1(iii) and Theorem 3.4, $\varpi_\star \ge \bar\varpi$ for $t \ge t_0$. Let $\lambda_0 = \lambda(0)$, $c_\lambda = 1/\log(1 + 1/\lambda_0)$, $\Xi_0 = \operatorname{tr}\mathrm{E}_1(L)|_{t=0}$, and

$$\begin{aligned}
\kappa_1 &= \frac{y_*\, c_\lambda^2}{\sqrt{2}\,|z_*|\,c_\lambda\,\Xi_0 + \frac75\bar{\mathcal{N}}}, \qquad \Lambda_1 = \max\Bigl\{1, \frac{8\pi\bar{\mathcal{N}}}{\bar\varpi\, y_*}\Bigr\},\\
t_1 &= t_0 + \frac{(1+\Lambda_1)e^{\Lambda_1}}{\kappa_1}, \qquad s_1 = \frac{\bar{\mathcal{N}}}{1 - e^{-1}}, \qquad \mathfrak{c}_1 = 2\sqrt{\mathsf{d}\,\bar{\mathcal{N}} s_1} + r\kappa_0 s_1,\\
\mathfrak{c}_2 &= \frac{\pi}{\sqrt{|z_*|}}\bigl(\sqrt{\mathfrak{c}_1} + \sqrt{r s_1}\bigr), \qquad \mathfrak{c}_3 = \max\Bigl\{0, 2\log\frac{2\mathfrak{c}_2}{\pi}\Bigr\},\\
C_1 &= \mathfrak{c}_1 + (r+1)s_1 + 2|z_*| r e^{\mathfrak{c}_3}, \qquad t_2 = t_1 + \frac{2}{\bar\varpi}\log^+\frac{8\bar\varpi\, s_1}{\pi y_*}.
\end{aligned} \tag{13}$$

In these constants the subscripts are ordinal labels and do not refer to vertices.

Theorem 4.2 (Decay of the exact part)**.** *Let E be a complex with a surjective differential at i_0 and $r \ge 1$, with at least one arrow from i_0 to a vertex j with $W_j \ne 0$; fix a height function w and let the constants be as in (13). For all charges in the upper half-plane and along every trajectory of the phase flow:*

(i) *$\lambda(t) \to \infty$; for $t \ge t_1$, $\lambda(t) \ge \max\{\Lambda_1,\ \frac{\bar\varpi}{4}(t - t_1) - \mathfrak{c}_3\}$ and $s(t) \le s_1 e^{-\frac{\bar\varpi}{2}(t-t_1)}$;*

(ii) *for* $t \ge t_1$,

$$\|\Omega_V - (z_* + \mu_{i_0}) \oplus 0\|_1 + \|\Omega_U\|_1 + \sum_{j \ne i_0} \|\Omega_j - (z_j + \mu_j)\|_1 \le C_1\, e^{-\frac{\bar\varpi}{4}(t-t_1)};$$

the difference between the mass and the mass $\sum_i \|z_i + \mu_i\|_1$ *of the cohomology representation with its induced metrics satisfies the same bound;*

(iii) *for* $t \ge t_2$, $\|\delta_{i_0}\|^2 \le \frac{2\pi\bar\varpi}{y_*}\, s$ *and* $\|\delta_j\| \le s/y_j$ *for* $j \ne i_0$.

In particular the exact part decays.

The theorem makes no assumption on the cohomology representation.

Proof. Growth of λ. Put $\Xi = \operatorname{tr} \mathrm{E}_1(L)$. By Lemma 4.1(i) all eigenvalues of L are at least λ_0 for $t \ge 0$, so (12) gives $\operatorname{tr} e^{-L} \ge c_\lambda \Xi$. For $t \ge t_0$, $\operatorname{tr} J_+ = \operatorname{tr} J + \operatorname{tr} J_- \le \frac{7}{5} \operatorname{tr} J \le \frac{7}{5}\bar{\mathcal{N}}$ by Proposition 2.2(i), (9) and Theorem 3.4, and $y_* + |x_*| \le \sqrt{2}|z_*|$. The function $x \mapsto x^2/(cx+d)$ is increasing for $x > 0$, and Ξ is nonincreasing; so Lemma 4.1(v) gives, for $t \ge t_0$,

$$\dot\Xi \le -\frac{y_* c_\lambda^2\, \Xi^2}{\sqrt{2}|z_*| c_\lambda\, \Xi + \frac{7}{5}\bar{\mathcal{N}}} \le -\kappa_1 \Xi^2,$$

hence $\mathrm{E}_1(\lambda(t)) \le \Xi(t) < 1/(\kappa_1(t-t_0))$ for $t > t_0$. For $t \ge t_1$ this is at most $e^{-\Lambda_1}/(1+\Lambda_1) \le \mathrm{E}_1(\Lambda_1)$ by (12), so $\lambda(t) \ge \Lambda_1$.

Decay of s. For $t \ge t_0$ we have $q_{\mathrm{out}} \le \bar{\mathcal{N}}$ and $\varpi_\star \ge \bar\varpi$, so for $t \ge t_1$, $\frac{4\pi q_{\mathrm{out}}}{y_* \lambda} \le \frac{\bar\varpi}{2}$ and $\dot s \le -\frac{\bar\varpi}{2} s$ by Lemma 4.1(iii); at t_1, $\lambda \ge 1$ and $s \le \bar{\mathcal{N}}/(1 - e^{-\lambda}) \le s_1$ by (9).

Linear growth of λ. By Lemma 4.1(ii) and Proposition 2.2(v), for $t \ge t_1$, $\|\mathcal{E}_V\|_1 \le \mathfrak{c}_1 e^{-\frac{\bar\varpi}{4}(t-t_1)}$ and $\|J\|_1 \le r s_1 e^{-\frac{\bar\varpi}{2}(t-t_1)}$. Let v be a unit vector of $(\ker D)^\perp$ with $F^\dagger F v = \lambda v$ and $u = Fv/\sqrt{\lambda}$. Apply Proposition 2.2(iv) to $T = \Omega_V$ with $T_\circ = \operatorname{diag}(z_* + \mu_{i_0},\ z_* e^{-F^\dagger F})$, whose first block is normal and invertible and for which $|T_\circ| v = |z_*| e^{-\lambda} v$ and $\Psi_\circ v = \alpha_* v$, and to $T = -\Omega_U = z_* e^{-L} - J$ with $T_\circ = z_* e^{-L}$ and the vector u:

$$|\langle v, (\Phi_V - \alpha_*) v\rangle| \le \pi \sqrt{\frac{\|\mathcal{E}_V\|_1\, e^{\lambda}}{|z_*|}}, \qquad |\langle u, (\Phi_U + \pi - \alpha_*) u\rangle| \le \pi \sqrt{\frac{\|J\|_1\, e^{\lambda}}{|z_*|}}.$$

By Lemma 4.1(i) the right derivative of λ satisfies $\dot\lambda \ge \lambda\big[\pi - \mathfrak{c}_2 e^{\lambda/2 - \frac{\bar\varpi}{8}(t-t_1)}\big]$ for $t \ge t_1$. Where λ is at most $\frac{\bar\varpi}{4}(t-t_1) - \mathfrak{c}_3$, the bracket is at least $\frac{\pi}{2}$, so $\dot\lambda \ge \frac{\pi}{2}\lambda \ge \frac{\pi}{2} > \frac{\bar\varpi}{4}$; since $\lambda(t_1) \ge 1 > -\mathfrak{c}_3$, the graph of λ cannot cross below this line. This proves (i).

(ii) By (6), $\|\Omega_V - (z_* + \mu_{i_0}) \oplus 0\|_1 \le \|\mathcal{E}_V\|_1 + |z_*| r e^{-\lambda}$, $\|\Omega_U\|_1 \le rs + |z_*| r e^{-\lambda}$ and $\sum_j \|\Omega_j - (z_j + \mu_j)\|_1 = \sum_j \operatorname{tr} Y_j \le s$; by (i), $e^{-\lambda} \le e^{\mathfrak{c}_3} e^{-\frac{\bar\varpi}{4}(t-t_1)}$. The statement on the masses follows from the triangle inequality for the trace norm.

(iii) For $t \ge t_2$, $s \le \frac{\pi y_*}{8\bar\varpi}$ by (i). Then $\frac{q_{\mathrm{out}} s}{\lambda y_*^2} \le \frac{\bar{\mathcal{N}}}{\Lambda_1} \frac{s}{y_*^2} \le \frac{\bar\varpi\, s}{8\pi y_*} \le \frac{1}{64}$, and Lemma 4.1(iv) gives $\|\delta_{i_0}\|^2 \le \frac{16\pi^2}{y_*^2} \frac{q_{\mathrm{out}} s}{\lambda} \le \frac{2\pi\bar\varpi}{y_*} s$. □

Proposition 4.3 (No arrow out of i_0)**.** *Let* E *have a surjective differential at* i_0 *and* $r \ge 1$, *and let* $\rho_a = 0$ *for every arrow* a *out of* i_0*; this holds if no arrow leaves* i_0, *and if* $W_j = 0$ *for all* $j \ne i_0$. *Then, for every metric, the complex is the orthogonal direct sum of its cohomology representation and of the complex* $(\ker D)^\perp \to U$ *at* i_0, *the flow preserves this decomposition,* $\delta_i = 0$ *for all* i, *the phases on* $(\ker D)^\perp$ *and* U *are* α_* *and* $\alpha_* - \pi$, *and*

$$F(t) = e^{\pi t/2} F(0), \qquad \Omega_V = (z_* + \mu_{i_0}) \oplus z_* e^{-F^\dagger F}, \qquad \Omega_U = -z_* e^{-L}$$

for all t. In particular $\|\Omega_V - (z_* + \mu_{i_0}) \oplus 0\|_1 + \|\Omega_U\|_1 \le 2|z_*| r \exp(-\lambda_0 e^{\pi t})$, and the exact part decays. This bound is at most $2|z_*| r\, e^{-(t-t_1)}$ for $t \ge t_1 := \frac{1}{\pi} \log^+ \frac{1}{\lambda_0}$.

Proof. Here $C_{\mathrm{out}} = 0$, so $J = 0$, $E_R = 0$, $C_T = 0$ and $Y_j = 0$ in (4) and (6). The blocks of Ω_V are $z_* + \mu_{i_0}$ and $z_* e^{-F^\dagger F}$, so $\Phi_{HA} = 0$, $\Phi_{AA} = \alpha_*$, $\Phi_U = \alpha_* - \pi$, and (5) gives $\dot F = \frac{\pi}{2} F$. □

Remark 4.4. (1) The time t_1 contains the factor e^{Λ_1}, and $\Lambda_1 \ge 8\bar{\mathcal{N}}/y_*$.

(2) When an arrow leaves i_0, the phases Φ_{AA} and Φ_U on the exact part need not tend to α_* and $\alpha_* - \pi$, in contrast with Proposition 4.3. In the proof of Theorem 4.2(i), the bound $\langle v, \Phi_V v\rangle - \langle u, \Phi_U u\rangle \ge \pi/2$ is used at a possible contact with the line $\lambda = \frac{\varpi}{4}(t - t_1) - \mathfrak{c}_3$. It gives the stated lower bound for $\lambda(t)$.

5. Stability, the Kempf–Ness function and convergence

5.1. **The stability parameter.** Let $\mathcal{H}$ be a nonzero representation of Q with dimension vector (n_i) and $Z(\mathcal{H}) = |Z| e^{\mathrm{i}\varphi}$, $\varphi \in (0, \pi)$, $\sigma = \sin\varphi$; in the applications it is the cohomology of a complex with a surjective differential, or the representation itself. For $i \in Q_0$ put

$$\theta_i = \frac{\mathrm{Im}(e^{-\mathrm{i}\varphi} z_i)}{\sigma} = \frac{|z_i| \sin(\alpha_i - \varphi)}{\sigma}.$$

Then $\sum_i n_i \theta_i = 0$, and for a real number x one has $\arg(z_i + x) = \varphi$ if and only if $x = \theta_i$; moreover $|z_i + \theta_i| = y_i/\sigma$. A representation R has $\arg Z(\mathsf{R}) \le \varphi$ if and only if $\sum_i \theta_i \dim \mathsf{R}_i \le 0$. A representation R with $\sum_i \theta_i \dim \mathsf{R}_i = 0$, such as $\mathcal{H}$, is *semistable* if $\sum_i \theta_i \dim \mathsf{R}'_i \le 0$ for each of its subrepresentations R', *stable* if the inequality is strict for the proper nonzero ones, and *polystable* if it is a direct sum of stable representations R'' with $\sum_i \theta_i \dim \mathsf{R}''_i = 0$. This is stability in the sense of King [Kin94, Definition 1.1] for the real parameter $-\theta$ (the sign comes from the convention, outgoing minus incoming, for the moment map), and it is the semistability of objects of phase φ/π for the stability condition of Section 2. Since $\arg(z_i + x) = \varphi$ only for $x = \theta_i$, a metric on $\mathcal{H}$ is harmonic exactly when $\mu_i = \theta_i$ for all i; we call such a metric a *zero of the shifted moment map* $\mu - \theta$, where $\mu = (\mu_i)$ is the moment map of Section 2.

5.2. **The Kempf–Ness function.** Let $\mathcal{X}$ be the space of metrics $\mathfrak{h} = (\mathfrak{h}_i)_{i \in Q_0}$ on the spaces $\mathcal{H}_i$. A tangent vector at $\mathfrak{h}$ is a tuple $\xi = (\xi_i)$ of $\mathfrak{h}$-Hermitian endomorphisms, $\delta\mathfrak{h} = \mathfrak{h}\xi$, and $\|\xi\|^2 = \sum_i \mathrm{tr}\, \xi_i^2$ defines an invariant Riemannian metric for which $\mathcal{X}$ is complete, simply connected and nonpositively curved, with geodesics $u \mapsto \mathfrak{h} e^{u\xi}$ and distance d. The *Kempf–Ness function* of $\mathcal{H}$ is

$$\mathcal{K}_{\mathcal{H}}(\mathfrak{h}) = \sum_a \mathrm{tr}\big(\mathfrak{h}_{\mathsf{s}(a)}^{-1} \beta_a^* \mathfrak{h}_{\mathsf{t}(a)} \beta_a\big) + \sum_i \theta_i \log\det \mathfrak{h}_i, \tag{14}$$

where β_a are the matrices of the arrows of $\mathcal{H}$ in fixed bases, β_a^* their conjugate transposes, and $\mathfrak{h}_i$ stands for the Gram matrix. The first sum is the sum of the squared norms of the arrows for the metric $\mathfrak{h}$. Since $\sum_i n_i \theta_i = 0$, $\mathcal{K}_{\mathcal{H}}$ is invariant under a common rescaling of the $\mathfrak{h}_i$. In terms of the function (7) for the representation $\mathcal{H}$, $\mathcal{K}_{\mathcal{H}} = \mathrm{Im}(e^{-\mathrm{i}\varphi} \mathcal{S}_{\mathbb{C}})/\sigma$.

Lemma 5.1. *The differential of the Kempf–Ness function is $d\mathcal{K}_{\mathcal{H}}(\xi) = -\sum_i \mathrm{tr}\big(\xi_i(\mu_i - \theta_i)\big)$, and along a geodesic*

$$\frac{d^2}{du^2} \mathcal{K}_{\mathcal{H}}(\mathfrak{h} e^{u\xi}) = \sum_a \big\| \xi_{\mathsf{t}(a)} \beta_a(u) - \beta_a(u) \xi_{\mathsf{s}(a)} \big\|^2 \ge 0, \qquad \beta_a(u) = e^{u\xi_{\mathsf{t}(a)}/2} \beta_a e^{-u\xi_{\mathsf{s}(a)}/2},$$

in an $\mathfrak{h}$-orthonormal frame. So $\mathcal{K}_{\mathcal{H}}$ is geodesically convex, its critical points are the zeros of the shifted moment map, and its gradient has the norm $\|\mu-\theta\| := (\sum_i \|\mu_i-\theta_i\|^2)^{1/2}$.

Proof. In an $\mathfrak{h}$-orthonormal frame the first sum of (14) at $\mathfrak{h}e^{u\xi}$ is $\sum_a \operatorname{tr}(e^{-u\xi_{\mathsf{s}(a)}}\beta_a^\dagger e^{u\xi_{\mathsf{t}(a)}}\beta_a)$, and $\log\det$ is affine along geodesics with derivative $\operatorname{tr}\xi_i$. Differentiate once and twice. □

For $\mathfrak{h}\in\mathcal{X}$ let $\mathfrak{a}(\mathfrak{h})$ be the space of tuples ξ of $\mathfrak{h}$-Hermitian endomorphisms with $\xi_{\mathsf{t}(a)}\beta_a = \beta_a\xi_{\mathsf{s}(a)}$ for all arrows a, the Hermitian infinitesimal automorphisms of $\mathcal{H}$, and let $\mathcal{Z}\subset\mathcal{X}$ be the set of zeros of the shifted moment map. The next lemma collects the convexity properties used; they are classical for Kempf–Ness functions (see [GRS21, Theorem 4.3] for actions on closed Kähler manifolds), and we include the short proofs for the present linear action.

Lemma 5.2. *Suppose $\mathcal{Z}\neq\emptyset$.*

(a) *For $p\in\mathcal{Z}$, $\mathcal{Z}=\{pe^\xi : \xi\in\mathfrak{a}(p)\}$; it is a closed totally geodesic submanifold.*

(b) *The number*

$$\nu_* = \min\Big\{\sum_a \big\|\xi_{\mathsf{t}(a)}\beta_a-\beta_a\xi_{\mathsf{s}(a)}\big\|^2 :\ \xi\perp\mathfrak{a}(p),\ \|\xi\|=1\Big\} > 0,$$

with the β_a in a p-orthonormal frame, does not depend on $p\in\mathcal{Z}$ (if $\mathfrak{a}(p)$ is the whole tangent space, which happens when all arrows vanish, put $\nu_*=\infty$, with the convention $\infty\cdot 0=0$ below; then $\mathcal{Z}=\mathcal{X}$ and $\mathcal{K}_{\mathcal{H}}$ is constant); and for every unit $\xi\perp\mathfrak{a}(p)$ and $0\le u\le\varrho_* := \log 2/\sqrt{2}$, $\frac{d^2}{du^2}\mathcal{K}_{\mathcal{H}}(pe^{u\xi})\ge\nu_*/2$.

(c) *For $\mathfrak{h}\in\mathcal{X}$ put $\mathfrak{d}=d(\mathfrak{h},\mathcal{Z})$ and $\mathcal{E}=\mathcal{K}_{\mathcal{H}}(\mathfrak{h})-\min\mathcal{K}_{\mathcal{H}}$. Then $\mathcal{E}\le\|\mu-\theta\|\,\mathfrak{d}$, $\mathcal{E}\ge\frac{\nu_*}{4}\min(\mathfrak{d}^2,\varrho_*\mathfrak{d})$, and if $\mathfrak{d}\le\varrho_*$, then $\|\mu-\theta\|\ge\frac{\nu_*}{2}\mathfrak{d}$ and $\mathcal{E}\le\frac{2}{\nu_*}\|\mu-\theta\|^2$.*

Proof. (a) If $\xi\in\mathfrak{a}(p)$, the second derivative of $\mathcal{K}_{\mathcal{H}}$ along $u\mapsto pe^{u\xi}$ vanishes identically by Lemma 5.1, and the first derivative vanishes at $u=0$; so $pe^\xi\in\mathcal{Z}$. Conversely, if $pe^\xi\in\mathcal{Z}$, the convex function $u\mapsto\mathcal{K}_{\mathcal{H}}(pe^{u\xi})$ takes its minimum at $u=0$ and $u=1$, hence is constant on $[0,1]$, and its second derivative at 0 gives $\xi\in\mathfrak{a}(p)$. So $\mathcal{Z}$ is the image of a linear subspace under the exponential map at each of its points.

(b) The quadratic form is positive on the orthogonal complement of its kernel $\mathfrak{a}(p)$. By Lemma 5.1, $\xi_{\mathsf{t}(a)}\beta_a(u)-\beta_a(u)\xi_{\mathsf{s}(a)} = e^{u\xi_{\mathsf{t}(a)}/2}(\xi_{\mathsf{t}(a)}\beta_a-\beta_a\xi_{\mathsf{s}(a)})e^{-u\xi_{\mathsf{s}(a)}/2}$ has norm at least $e^{-u(\|\xi_{\mathsf{t}(a)}\|_{\mathrm{op}}+\|\xi_{\mathsf{s}(a)}\|_{\mathrm{op}})/2}$ times that of $\xi_{\mathsf{t}(a)}\beta_a-\beta_a\xi_{\mathsf{s}(a)}$, and $\|\xi_{\mathsf{s}(a)}\|_{\mathrm{op}}+\|\xi_{\mathsf{t}(a)}\|_{\mathrm{op}}\le\sqrt{2}$ for a unit ξ, because $\mathsf{s}(a)\neq\mathsf{t}(a)$. So the second derivative is at least $e^{-\sqrt{2}u}\nu_*$. If $p'=pe^\zeta$ with $\zeta\in\mathfrak{a}(p)$, then $e^{-\zeta/2}$ is a tuple of automorphisms of $\mathcal{H}$ that maps p-orthonormal frames to p'-orthonormal frames and preserves the matrices β_a; conjugation by it carries $\mathfrak{a}(p)$ and its orthogonal complement isometrically to those of p'.

(c) Let $p\in\mathcal{Z}$ be a nearest point and $\mathfrak{h}=pe^{\mathfrak{d}\xi}$ with $\|\xi\|=1$; then $\xi\perp\mathfrak{a}(p)$. The function $f(u)=\mathcal{K}_{\mathcal{H}}(pe^{u\xi})-\min\mathcal{K}_{\mathcal{H}}$ is convex with $f(0)=f'(0)=0$. First, $0=f(0)\ge f(\mathfrak{d})-\mathfrak{d}f'(\mathfrak{d})$ and $f'(\mathfrak{d})\le\|\mu-\theta\|$. Secondly, $f''\ge\nu_*/2$ on $[0,\varrho_*]$ by (b), so $f(u)\ge\frac{\nu_*}{4}u^2$ there, and $f(u)\ge f(\varrho_*)u/\varrho_*$ for $u\ge\varrho_*$ by convexity. Thirdly, if $\mathfrak{d}\le\varrho_*$, then $\|\mu-\theta\|\ge f'(\mathfrak{d})=\int_0^{\mathfrak{d}}f''\ge\frac{\nu_*}{2}\mathfrak{d}$; combine with the first inequality. □

Lemma 5.3 (Kempf–Ness, King)**.** *Let $\mathcal{K}_*=\inf\mathcal{K}_{\mathcal{H}}\in[-\infty,\infty)$.*

(a) *$\mathcal{K}_*>-\infty$ if and only if $\mathcal{H}$ is semistable.*

(b) *A zero of the shifted moment map exists if and only if $\mathcal{H}$ is polystable.*

For a parameter that is a positive multiple of an integral vector these are the theorems of Kempf–Ness and King [KN79, Kin94]: (b) is [Kin94, Proposition 6.5], and (a) is stated in

[HKKP23, Theorem 5.1] and follows from [Kin94, Proposition 3.1, Lemmas 2.2 and 6.3]. The parameter θ is real, and for a quiver with more than two vertices it need not be a multiple of an integral vector; (b) is stated for real parameters in [HKKP26, Theorem 5.1], with reference to [Kin94], and the general theory of [GRS21] has no integrality hypothesis but is written for actions on closed Kähler manifolds. We give the reduction to integral parameters.

Proof. Let Θ be the real vector space of all $\theta' \in \mathbb{R}^{Q_0}$ with $\sum_i n_i\theta'_i = 0$, and write $\mathcal{K}^{\theta'}$ for the function (14) with θ' in the place of θ. The first sum of (14) does not depend on θ', so $\mathcal{K}^{\sum c_l\theta^{(l)}} = \sum c_l\mathcal{K}^{\theta^{(l)}}$ whenever $\sum c_l = 1$. For a positive number c, $\mathcal{K}^{c\theta'}$ at the arrows β is c times $\mathcal{K}^{\theta'}$ at the arrows $\beta/\sqrt{c}$, which have the same subrepresentations; so the cited theorems hold for all rational $\theta' \in \Theta$. Let $\mathfrak{S}$ be the finite set of dimension vectors of subrepresentations of $\mathcal{H}$.

(a) If a subrepresentation $\mathcal{H}'$ with dimension vector (n'_i) has $\sum_i n'_i\theta_i > 0$, let $\mathfrak{h}_i = 1 \oplus e^\tau$ on $\mathcal{H}'_i \oplus \mathcal{H}'^{\perp}_i$ for reference metrics. The first sum of (14) stays bounded as $\tau \to \infty$, because the arrows map $\mathcal{H}'$ to $\mathcal{H}'$, and $\sum_i \theta_i \log\det\mathfrak{h}_i = \tau\sum_i \theta_i(n_i - n'_i) = -\tau\sum_i n'_i\theta_i \to -\infty$. Conversely, the set of $\theta' \in \Theta$ for which $\mathcal{H}$ is semistable is the rational polyhedral cone $\{\theta' : \sum_i n'_i\theta'_i \le 0 \text{ for all } (n'_i) \in \mathfrak{S}\}$. A point of a rational polyhedral cone is a convex combination of rational points of the cone: the cone is generated by finitely many rational vectors v_j, and if $\theta = \sum_j \lambda_j v_j$ with $\lambda_j \ge 0$, choose rational numbers $q_j > 0$ with $\sum_j \lambda_j/q_j < 1$; then θ is a convex combination of the points $q_j v_j$ and 0. So $\theta = \sum_l c_l\theta^{(l)}$ with $c_l \ge 0$, $\sum c_l = 1$ and rational $\theta^{(l)}$ for which $\mathcal{H}$ is semistable, and $\mathcal{K}^\theta = \sum_l c_l\mathcal{K}^{\theta^{(l)}}$ is bounded below.

(b) Let $\mathfrak{h}$ be a zero of the shifted moment map, $\mathcal{H}'$ a subrepresentation and Π_i the $\mathfrak{h}$-orthogonal projection onto $\mathcal{H}'_i$. Since $\beta_a\Pi_{\mathsf{s}(a)} = \Pi_{\mathsf{t}(a)}\beta_a\Pi_{\mathsf{s}(a)}$,

$$\sum_i n'_i\theta_i = \sum_i \operatorname{tr}(\Pi_i\mu_i) = \sum_a \big(\|\beta_a\Pi_{\mathsf{s}(a)}\|^2 - \|\Pi_{\mathsf{t}(a)}\beta_a\|^2\big) = -\sum_a \|\Pi_{\mathsf{t}(a)}\beta_a(1-\Pi_{\mathsf{s}(a)})\|^2 \le 0.$$

So $\mathcal{H}$ is semistable, and if equality holds the orthogonal complement of $\mathcal{H}'$ is a subrepresentation, and $\mathfrak{h}$ restricts to a zero of the shifted moment map on both summands. By induction on the dimension, $\mathcal{H}$ is polystable. Conversely, an orthogonal direct sum of zeros of the shifted moment maps of the summands is a zero of the shifted moment map; so let $\mathcal{H}$ be stable. The set of $\theta' \in \Theta$ for which $\mathcal{H}$ is stable is open and defined by finitely many strict rational inequalities, so $\theta = \sum_l c_l\theta^{(l)}$ with $c_l > 0$, $\sum c_l = 1$ and rational $\theta^{(l)}$ for which $\mathcal{H}$ is stable. For each l the function $\mathcal{K}^{\theta^{(l)}}$ attains its minimum [Kin94, Proposition 6.5], and by Lemma 5.2(a), which holds for every parameter in Θ, its set of minimizers is $\{pe^\xi : \xi \in \mathfrak{a}(p)\}$; a stable representation has only scalar endomorphisms, so the minimizers form one orbit of the common rescalings. A geodesically convex function on the quotient of $\mathcal{X}$ by the common rescalings, which is again a complete simply connected manifold of nonpositive curvature, with a single minimizer is proper: it exceeds its minimum by a positive constant on the unit sphere around the minimizer, and then by that constant times d at distance $d \ge 1$. Hence $\mathcal{K}^\theta = \sum_l c_l\mathcal{K}^{\theta^{(l)}}$ is proper on this quotient and attains its minimum, at a critical point. □

5.3. **Convergence.** For $i \in Q_0$ let $G_i(u) = \varphi - \arg(z_i + u)$ for real u. These functions are increasing, $G_i(\theta_i) = 0$, and $0 < G'_i(u) = y_i/|z_i+u|^2 \le 1/y_i$. For a number $\bar{\mathcal{N}} \ge 0$ put

$$\bar{Q} = \max\big\{\bar{\mathcal{N}},\ \max_i |\theta_i|\big\}, \qquad \Gamma = \max_i \frac{(|z_i| + \bar{Q})^2}{y_i}. \tag{15}$$

The next proposition is the common part of the convergence theorems for representations and for complexes; for representations the perturbation δ in it is zero.

Proposition 5.4. *Let $\mathfrak{h}(t)$, $t \ge \tau_0$, be a differentiable curve in $\mathcal{X}$ whose velocity $\xi = \mathfrak{h}^{-1}\dot{\mathfrak{h}}$ has the form*

$$\xi_i = G_i(\mu_i) - \delta_i \qquad (i \in Q_0),$$

with $\mu = \mu(\mathfrak{h}(t))$ and $\mathfrak{h}$-Hermitian endomorphisms δ_i. Put $\|\delta\|^2 = \sum_i \|\delta_i\|^2$, and suppose that for $t \ge \tau_0$ the sum of the squared norms of the arrows for the metric $\mathfrak{h}(t)$ is at most $\bar{\mathcal{N}}$, and that $\|\delta(t)\| \le c_\delta\, e^{-\kappa(t-\tau_0)}$ with constants $c_\delta \ge 0$, $\kappa > 0$. Let Γ be as in (15). Then:

(i) $\frac{d}{dt}\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \le -\frac{1}{\Gamma}\|\mu-\theta\|^2 + \|\mu-\theta\|\,\|\delta\|$ *for $t \ge \tau_0$.*

(ii) *If $\mathcal{H}$ is semistable, then $\mu_i \to \theta_i$ for all i and $\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \to \mathcal{K}_*$ as $t \to \infty$.*

(iii) *If $\mathcal{H}$ is polystable, there is a zero $\mathfrak{h}_\infty$ of the shifted moment map with $d(\mathfrak{h}(t),\mathfrak{h}_\infty) \le \mathfrak{C}\, e^{-\varkappa t}$ for every $\varkappa < \min\{\nu_*/(8\Gamma), \kappa\}$, with a constant $\mathfrak{C}$ depending on $\varkappa$ and the curve; and $\mu - \theta$ tends to zero exponentially.*

Proof. (i) By Lemma 5.1,

$$\frac{d}{dt}\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) = -\sum_i \operatorname{tr}\big((\mu_i-\theta_i)G_i(\mu_i)\big) + \sum_i \operatorname{tr}\big((\mu_i-\theta_i)\delta_i\big).$$

The eigenvalues of μ_i lie in $[-\bar{\mathcal{N}},\bar{\mathcal{N}}]$; for real u with $|u| \le \bar{Q}$, $(u-\theta_i)G_i(u) = G_i'(\zeta)(u-\theta_i)^2$ with a point ζ between u and θ_i, and $G_i'(\zeta) \ge 1/\Gamma$.

(ii) Put $\mathcal{E}(t) = \mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) - \mathcal{K}_* \ge 0$ (Lemma 5.3(a)). By (i),

$$\frac{d}{dt}\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \le -\frac{1}{2\Gamma}\|\mu-\theta\|^2 + \frac{\Gamma}{2}\|\delta\|^2,$$

so $\|\mu-\theta\|^2$ is integrable on $[\tau_0,\infty)$. In $\mathfrak{h}$-orthonormal frames the arrows evolve by $\dot\beta_a = \frac12(\xi_{\mathsf{t}(a)}\beta_a - \beta_a\xi_{\mathsf{s}(a)})$, and β_a and ξ are bounded; so the derivatives of the μ_i are bounded, $\|\mu-\theta\|^2$ is uniformly continuous, and $\mu_i \to \theta_i$. The function $\mathcal{E}(t) + \frac{\Gamma}{2}\int_t^\infty \|\delta\|^2$ is nonincreasing, so $\mathcal{E}$ has a limit $\mathcal{E}_\infty \ge 0$. Suppose $\mathcal{E}_\infty > 0$ and choose $\mathfrak{h}'$ with $\mathcal{K}_{\mathcal{H}}(\mathfrak{h}') - \mathcal{K}_* < \mathcal{E}_\infty/2$. By convexity along the geodesic from $\mathfrak{h}(t)$ to $\mathfrak{h}'$, $\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) - \mathcal{K}_{\mathcal{H}}(\mathfrak{h}') \le \|\mu-\theta\|\, d(\mathfrak{h}(t),\mathfrak{h}')$, and the left side is at least $\mathcal{E}_\infty/4$ for large t. The speed of $\mathfrak{h}(t)$ is at most $\|\mu-\theta\| / \min_i y_i + \|\delta\|$, which is square integrable, so $d(\mathfrak{h}(t),\mathfrak{h}') \le c + c'\sqrt{t}$; then $\|\mu-\theta\|^2 \ge \mathcal{E}_\infty^2/(16(c+c'\sqrt{t})^2)$ for large t, which is not integrable. So $\mathcal{E} \to 0$.

(iii) By Lemma 5.3(b), $\mathcal{Z} \ne \emptyset$. By Lemma 5.2(c) and $\mathcal{E} \to 0$, $d(\mathfrak{h}(t),\mathcal{Z}) \le \varrho_*$ for large t, and then $\|\mu-\theta\|^2 \ge \frac{\nu_*}{2}\mathcal{E}$, so $\frac{d}{dt}\mathcal{E} \le -\frac{\nu_*}{4\Gamma}\mathcal{E} + \frac{\Gamma}{2}\|\delta\|^2$ and $\mathcal{E}(t) = O(e^{-2\varkappa t})$ for every $2\varkappa < \min\{\nu_*/(4\Gamma), 2\kappa\}$. Integrating the second inequality of the proof of (ii) over $[t,t+1]$ bounds $\int_t^{t+1}\|\mu-\theta\|^2$ by a multiple of $\mathcal{E}(t) + \int_t^{t+1}\|\delta\|^2$; by the bound on the speed and the Cauchy–Schwarz inequality the length of $\mathfrak{h}$ over $[t,t+1]$ is $O(e^{-\varkappa t})$, so $\mathfrak{h}(t)$ converges at this rate, and the limit is a zero of the shifted moment map by continuity. Finally μ is a locally Lipschitz function on $\mathcal{X}$ with $\mu(\mathfrak{h}_\infty) = \theta$. □

Theorem 5.5 (Representations)**.** *Let Q be a finite quiver without oriented cycles, with charges z_i in the upper half-plane, and let R be a nonzero representation of Q with $Z(\mathsf{R}) = |Z|e^{\mathrm{i}\varphi}$ and $\theta_i = \operatorname{Im}(e^{-\mathrm{i}\varphi}z_i)/\sin\varphi$. Let $h(t)$ be the solution of the phase flow $h_i^{-1}\dot h_i = -\arg(z_i+\mu_i)$ from an arbitrary initial metric, and $\mathfrak{h}(t) = e^{\varphi t}h(t)$. Then:*

(a) *the Kempf–Ness function $\mathcal{K}_{\mathsf{R}}(\mathfrak{h}(t)) = \mathcal{K}_{\mathsf{R}}(h(t))$ is nonincreasing;*

(b) *if R is semistable, then $\mu_i(t) \to \theta_i$ for all i, $\mathcal{K}_{\mathsf{R}}(h(t)) \to \inf \mathcal{K}_{\mathsf{R}}$, and the mass $\sum_i \operatorname{tr}|z_i + \mu_i(t)|$ tends to $|Z|$;*

(c) *if R is polystable, $\mathfrak{h}(t)$ converges to a harmonic metric $\mathfrak{h}_\infty$, that is, a metric with $\mu_i(\mathfrak{h}_\infty) = \theta_i$ for all i, and $d(\mathfrak{h}(t),\mathfrak{h}_\infty) \le \mathfrak{C}\, e^{-\varkappa t}$ for every $\varkappa < \nu_*/(8\Gamma)$, where ν_**

is the number of Lemma 5.2(b) and Γ is given by (15) with $\bar{\mathcal{N}} = \bar{\mathcal{N}}_{\mathsf{R}}$, the bound of Proposition 3.6; and $\mu - \theta$ and the mass minus $|Z|$ tend to zero exponentially.

Proof. The block of Ω at i is $z_i + \mu_i$, so the velocity of $\mathfrak{h}(t)$ is $\xi_i = \varphi - \arg(z_i + \mu_i) = G_i(\mu_i)$. By Lemma 5.1, $\frac{d}{dt}\mathcal{K}_{\mathsf{R}}(\mathfrak{h}(t)) = -\sum_i \operatorname{tr}\big((\mu_i - \theta_i)G_i(\mu_i)\big) \le 0$, because $(u - \theta_i)G_i(u) \ge 0$ for all real u; this is (a). By Proposition 3.6 the sum of the squared norms of the arrows is at most $\bar{\mathcal{N}}_{\mathsf{R}}$ for $t \ge t_{\mathsf{R}}$, and it does not change under a common rescaling of the metrics. So Proposition 5.4 applies with $\tau_0 = t_{\mathsf{R}}$, $\delta = 0$ and every $\kappa > 0$, and gives (b) and (c); for the mass, $\sum_i \operatorname{tr}|z_i + \mu_i| \to \sum_i n_i|z_i + \theta_i| = \sum_i n_i y_i/\sigma = |Z|$, exponentially in case (c) because $\operatorname{tr}|z_i + \mu_i|$ is a Lipschitz function of μ_i. □

Theorem 5.6 (Complexes with a surjective differential)**.** *Let E be a complex with a surjective differential at i_0 and $r \ge 1$, with at least one arrow from i_0 to a vertex j with $W_j \ne 0$; fix a height function and let the constants be as in (13), and Γ as in (15). Let*

$$\mathfrak{h}(t) = \big(e^{\varphi t} h_i(t)|_{\mathcal{H}_i}\big)_{i \in Q_0}$$

be the metrics induced on the cohomology representation $\mathcal{H}$, multiplied by $e^{\varphi t}$; at i_0 the induced metric is the restriction of the metric of V to $\ker D$. Put $c_\delta^2 = \frac{2\pi\bar{\varpi}\, s_1}{y_*} + s_1^2 \sum_{j \ne i_0} y_j^{-2}$. For all charges in the upper half-plane:

(i) *For $t \ge t_2$,*

$$\frac{d}{dt}\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \le -\frac{1}{\Gamma}\,\|\mu - \theta\|^2 + \|\mu - \theta\|\,\|\delta\|\,, \qquad \|\delta(t)\| \le c_\delta\, e^{-\frac{\bar{\varpi}}{4}(t - t_1)}.$$

(ii) *If $\mathcal{H}$ is semistable, then as $t \to \infty$: $\mu_i \to \theta_i$ for all i, $\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \to \mathcal{K}_*$ and $\mathfrak{m}(h(t)) \to |Z|$.*

(iii) *If $\mathcal{H}$ is polystable, $\mathfrak{h}(t)$ converges to a harmonic metric $\mathfrak{h}_\infty$ of $\mathcal{H}$, a zero of the shifted moment map, with $d(\mathfrak{h}(t), \mathfrak{h}_\infty) \le \mathfrak{C}\, e^{-\varkappa t}$ for every $\varkappa < \min\{\nu_*/(8\Gamma),\ \bar{\varpi}/4\}$; and $\mu - \theta$ and $\mathfrak{m} - |Z|$ tend to zero exponentially.*

Proof. By (2) the velocity of $\mathfrak{h}(t)$ at the vertex i is $\varphi - \Phi_i$, with Φ_{HH} in the place of Φ_{i_0}, that is, $\xi_i = G_i(\mu_i) - \delta_i$ with the δ_i of Section 2. For $t \ge t_0$ the sum of the squared norms of the arrows of $\mathcal{H}$ is at most $\mathcal{N} \le \bar{\mathcal{N}}$ by (9) and Theorem 3.4, and for $t \ge t_2$ Theorem 4.2(i),(iii) gives $\|\delta\|^2 \le \frac{2\pi\bar{\varpi}}{y_*} s + s^2 \sum_j y_j^{-2} \le c_\delta^2 e^{-\frac{\bar{\varpi}}{2}(t - t_1)}$. So Proposition 5.4 applies with $\tau_0 = t_2$ and $\kappa = \bar{\varpi}/4$. For the mass, $\sum_i \|z_i + \mu_i\|_1 \to \sum_i n_i|z_i + \theta_i| = |Z|$, exponentially in case (iii) because $\|z_i + \mu_i\|_1$ is a Lipschitz function of μ_i, and the difference to $\mathfrak{m}$ is bounded by Theorem 4.2(ii). □

Remark 5.7. Let E have a surjective differential at i_0 and $r \ge 1$, and let $\rho_a = 0$ for all arrows a out of i_0; this includes the case in which all W_j vanish. By Proposition 4.3 the complex is, for every metric, the orthogonal direct sum of its cohomology representation and an exact complex at i_0, and the flow preserves this decomposition. So the metrics induced on the cohomology follow the phase flow of the representation $\mathcal{H}$, Theorem 5.5 applies to them, and $\mathfrak{m}(h(t)) \to |Z|$ if $\mathcal{H}$ is semistable, exponentially if it is polystable.

For example, Theorems 4.2 and 5.6, Proposition 4.3 and Remark 5.7 apply to the quiver A_l with the linear orientation, with the differential at any vertex.

5.4. **Complexes with an injective differential.** Let E' be a complex with an injective differential at i_0 (Definition 2.1), with $U' \xrightarrow{D'} V'$ at i_0. The arrows out of i_0 vanish on $D'U'$, and the cohomology is a representation in degree zero with $V'/D'U'$ at i_0; the *exact part* is

the subcomplex $U' \to D'U'$, of rank $\dim U'$. The metric induced on $V'/D'U'$ is the quotient metric, that is, the metric of the orthogonal complement of $D'U'$ in V'.

Proposition 5.8 (Injective differential)**.** *Let E' be a complex with an injective differential at i_0 and $U' \neq 0$, with total phase φ, and let λ' be the least eigenvalue of $D'^{\dagger}D'$. Its dual $\mathsf{E}'^{\vee}$ over the opposite quiver, with the charges $-\overline{z_i}$, is a complex with a surjective differential at i_0 and total phase $\pi - \varphi$. Along the phase flow of E':*

(i) *$\lambda' \to \infty$, and the blocks of Ω approach those of the cohomology representation with its induced metrics, with the bound of Theorem 4.2(ii), or of Proposition 4.3, for $\mathsf{E}'^{\vee}$;*

(ii) *the statements of Theorem 5.6, or of Remark 5.7, hold for the metrics induced on the cohomology of E', multiplied by $e^{\varphi t}$, with the constants of $\mathsf{E}'^{\vee}$.*

Proof. The opposite quiver has no oriented cycles, and the transposed differential $D'^{\mathsf{T}} : V'^{*} \to U'^{*}$ is surjective in degrees $0, 1$. By Proposition 2.3(b), the trajectories correspond by $h \mapsto e^{-\pi t}(h^{-1})^{\mathsf{T}}$, the blocks of Ω correspond up to sign and complex conjugation in dual orthonormal frames, with the same trace norms, and the mass is preserved. The cohomology representation of $\mathsf{E}'^{\vee}$ is the dual of that of E', with the dual of the induced metrics; subrepresentations of one correspond to quotients of the other, and the argument ψ of a central charge corresponds to $\pi - \psi$, so that one is semistable, stable or polystable if and only if the other is. The normalized induced metrics $e^{(\pi-\varphi)t}e^{-\pi t}(h^{-1})^{\mathsf{T}}$ of $\mathsf{E}'^{\vee}$ converge exactly when $e^{\varphi t}h$ does. □

6. Semistable and unstable cohomology

This section describes the long-time behaviour when $\mathcal{H}$ is not polystable. For semistable $\mathcal{H}$ the curve of normalized metrics stays at bounded distance from a trajectory of a gradient flow of the Kempf–Ness function, whose asymptotics are known (Theorem 6.3). With no hypothesis on $\mathcal{H}$, the mass converges to the mass of the Harder–Narasimhan filtration, the filtration of the spaces by the exponential rate of decay of the metric is that filtration, and its factors, which are semistable, follow the semistable theory (Theorem 6.11).

Throughout the section Q is a finite quiver without oriented cycles, $\mathcal{H}$ is a nonzero representation of Q, with $Z(\mathcal{H}) = |Z|e^{\mathrm{i}\varphi}$, and $\mathfrak{h}(t)$, $t \geq \tau_0$, is a curve of metrics on $\mathcal{H}$ as in Proposition 5.4: it is continuously differentiable, its velocity is $\xi_i = G_i(\mu_i) - \delta_i$ with continuous $\mathfrak{h}$-Hermitian δ_i, $\|\delta(t)\| \leq c_\delta e^{-\kappa(t-\tau_0)}$ with $\kappa > 0$, and the sum of the squared norms of the arrows is at most $\bar{\mathcal{N}}$. We call this *hypothesis* (P). No stability of $\mathcal{H}$ is assumed in (P). It holds, with $\delta = 0$, for the normalized phase flow $\mathfrak{h} = e^{\varphi t}h$ on a representation of a quiver without oriented cycles (Proposition 3.6); and it holds for the normalized metrics induced on the cohomology of a complex with a surjective or an injective differential at one vertex, for every cohomology representation: the proof of Theorem 5.6(i) and Theorem 4.2 use no stability, and Remark 5.7 covers the complexes excluded there. For an injective differential one passes to the dual complex (Proposition 5.8): the normalized induced metrics of the two are dual to each other, so their velocities are ξ and $-\xi^{\mathsf{T}}$; the moment map changes sign under duality; and $\varphi - \arg(z_i + u) = -\big[(\pi - \varphi) - \arg(-\overline{z_i} - u)\big]$ for real u. So (P) for the dual gives (P).

We write $\Phi_i^{\mathcal{H}} = \arg(z_i + \mu_i) = \varphi - G_i(\mu_i)$ for the phase of the representation $\mathcal{H}$ with the metric $\mathfrak{h}$, a Hermitian matrix in an orthonormal frame; the function $u \mapsto \arg(z_i + u)$ is decreasing, with derivative in $[-1/y_i, 0)$. In orthonormal frames of $\mathfrak{h}(t)$ whose coordinate maps g_i, with $|g_i v|^2 = \mathfrak{h}_i(v, v)$, satisfy $\dot{g}_i g_i^{-1} = \frac{1}{2}\xi_i$, the arrows evolve by

$$\dot{\beta}_a = \tfrac{1}{2}\big(\beta_a(\Phi^{\mathcal{H}}_{\mathsf{s}(a)} + \delta_{\mathsf{s}(a)}) - (\Phi^{\mathcal{H}}_{\mathsf{t}(a)} + \delta_{\mathsf{t}(a)})\beta_a\big), \tag{16}$$

and the eigenvalues of all $\Phi_i^{\mathcal{H}}$ lie in a compact interval $[\psi_-, \psi_+] \subset (0, \pi)$ determined by the charges and $\bar{\mathcal{N}}$. Two facts on functions of Hermitian matrices are used. If g is a continuously differentiable function on $\mathbb{R}$ with $|g'| \le c$, then $\|g(A) - g(B)\| \le c\,\|A - B\|$ for Hermitian A, B and the Hilbert–Schmidt norm; and if g is nonincreasing, then $\operatorname{tr}\big((g(A) - g(B))(A - B)\big) \le 0$. Both follow from the identities $\langle u_j, (g(A) - g(B))v_l\rangle = (g(a_j) - g(b_l))\,\langle u_j, v_l\rangle$ and $\langle u_j, (A - B)v_l\rangle = (a_j - b_l)\,\langle u_j, v_l\rangle$ for orthonormal eigenvectors u_j of A and v_l of B with eigenvalues a_j, b_l.

6.1. **Semistable cohomology: comparison with the gradient flow.** Put

$$\mathsf{m}_i = \frac{1}{G_i'(\theta_i)} = \frac{y_i}{\sigma^2} \qquad (i \in Q_0),$$

and let d_{m} be the distance of the Riemannian metric $\|\xi\|_{\mathsf{m}}^2 = \sum_i \mathsf{m}_i \operatorname{tr} \xi_i^2$ on $\mathcal{X}$. With this metric $\mathcal{X}$ is the product of the symmetric spaces of positive definite Hermitian matrices, the i-th factor with its invariant metric multiplied by m_i; it is complete, simply connected and nonpositively curved, its geodesics are again $u \mapsto \mathfrak{h} e^{u\xi}$, and $d_{\mathsf{m}}(\mathfrak{h}, \mathfrak{h}')^2 = \sum_i \mathsf{m}_i \left\|\log(\mathfrak{h}_i^{-1/2}\mathfrak{h}_i'\mathfrak{h}_i^{-1/2})\right\|^2$. By Lemma 5.1 the gradient of $\mathcal{K}_{\mathcal{H}}$ for this metric is the tangent vector $(-(\mu_i - \theta_i)/\mathsf{m}_i)_i$. The *weighted gradient flow* is

$$\mathfrak{g}_i^{-1}\dot{\mathfrak{g}}_i = \frac{\mu_i(\mathfrak{g}) - \theta_i}{\mathsf{m}_i} \qquad (i \in Q_0). \tag{17}$$

Its solutions exist for all later times: the sum q of the squared norms of the arrows satisfies $\dot{q} = -\sum_i \operatorname{tr}(\mu_i \xi_i) \le \sum_i n_i \theta_i^2/(4\mathsf{m}_i)$ along (17), so the velocity is bounded on bounded intervals of time. The flow (17) is the gradient flow studied in [HKKP23]: it is [HKKP23, (5.11)], with the parameters m_i and θ_i there.

Lemma 6.1. *For all real u and all i, $\big|G_i(u) - (u - \theta_i)/\mathsf{m}_i\big| \le \frac{3\sqrt{3}}{16\,y_i^2}(u - \theta_i)^2$. Consequently $\|G_i(M) - (M - \theta_i)/\mathsf{m}_i\| \le \frac{3\sqrt{3}}{16\,y_i^2}\,\|M - \theta_i\|^2$ for every Hermitian matrix M.*

Proof. $G_i''(u) = -2y_i(x_i + u)/((x_i + u)^2 + y_i^2)^2$, and $2y|w|/(w^2 + y^2)^2$ has its maximum $3\sqrt{3}/(8y^2)$ at $w^2 = y^2/3$. Use Taylor's formula at θ_i, where $G_i = 0$ and $G_i' = 1/\mathsf{m}_i$. For a Hermitian matrix apply this to the eigenvalues u_k, and $(\sum_k (u_k - \theta_i)^4)^{1/2} \le \sum_k (u_k - \theta_i)^2$. □

Lemma 6.2. *Let $\mathcal{M}$ be a complete, simply connected Riemannian manifold of nonpositive sectional curvature, with distance d, and F a continuously differentiable function on $\mathcal{M}$ that is convex along geodesics. Let $y(t)$ be a solution of $\dot{y} = -\operatorname{grad} F(y)$ and $x(t)$ a continuously differentiable curve with $\dot{x} = -\operatorname{grad} F(x) + e(t)$ on an interval $[t_0, t_1]$. Then $d(x(t), y(t)) \le d(x(t_0), y(t_0)) + \int_{t_0}^t |e|$ for $t_0 \le t \le t_1$.*

Proof. On such a manifold any two points p, p' are joined by a unique geodesic $\gamma_{p,p'} : [0, 1] \to \mathcal{M}$, which depends smoothly on (p, p'), and $d(p, p')^2 = \int_0^1 |\gamma_{p,p'}'|^2$ is smooth. By the first variation formula, with $\gamma_t = \gamma_{x(t),y(t)}$ and $\rho(t) = d(x(t), y(t))$,

$$\frac{d}{dt}\frac{\rho^2}{2} = \big\langle \dot{y}, \gamma_t'(1)\big\rangle - \big\langle \dot{x}, \gamma_t'(0)\big\rangle = -\big\langle \operatorname{grad} F(y), \gamma_t'(1)\big\rangle + \big\langle \operatorname{grad} F(x), \gamma_t'(0)\big\rangle - \big\langle e, \gamma_t'(0)\big\rangle\,.$$

The function $u \mapsto F(\gamma_t(u))$ is convex and differentiable, so its derivative $\langle \operatorname{grad} F(\gamma_t(u)), \gamma_t'(u)\rangle$ is nondecreasing, and the right side is at most $|e|\,\rho$. For $\epsilon > 0$ this gives $\frac{d}{dt}\sqrt{\rho^2 + \epsilon} \le |e|$; integrate and let $\epsilon \to 0$. □

For $e = 0$ the lemma says that the gradient flow does not increase distances; Harada and Wilkin prove a related contraction estimate for the trace of the relative metric and its inverse [HW11, Theorem 3.18]; see also [HKKP23, §5.3].

Theorem 6.3 (Comparison with the gradient flow)**.** *Assume (P) and let $\mathcal{H}$ be semistable. Let $\mathfrak{g}(t)$, $t \geq \tau_0$, be the solution of the weighted gradient flow (17) with $\mathfrak{g}(\tau_0) = \mathfrak{h}(\tau_0)$. Then for all $t \geq \tau_0$*

$$d_{\mathsf{m}}\big(\mathfrak{h}(t), \mathfrak{g}(t)\big) \leq C_{\mathrm{e}} := \max_i \frac{3\sqrt{3}}{16\,\sigma\, y_i^{3/2}} \Big(2\Gamma\big(\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(\tau_0)) - \mathcal{K}_*\big) + \frac{\Gamma^2 c_\delta^2}{2\kappa}\Big) + \frac{c_\delta}{\kappa} \max_i \sqrt{\mathsf{m}_i},$$

with Γ as in (15); and, as Hermitian forms, $e^{-c}\mathfrak{g}_i(t) \leq \mathfrak{h}_i(t) \leq e^{c}\mathfrak{g}_i(t)$ with $c = C_{\mathrm{e}}/\min_i \sqrt{\mathsf{m}_i}$.

Proof. The velocity of $\mathfrak{h}$ is minus the gradient of $\mathcal{K}_{\mathcal{H}}$ for the metric $\|\cdot\|_{\mathsf{m}}$ plus the tangent vector $e_i = [G_i(\mu_i) - (\mu_i - \theta_i)/\mathsf{m}_i] - \delta_i$. By Lemma 6.1, $\|e\|_{\mathsf{m}} \leq \max_i \frac{3\sqrt{3}}{16\,\sigma y_i^{3/2}} \|\mu - \theta\|^2 + \max_i \sqrt{\mathsf{m}_i}\, \|\delta\|$. By Proposition 5.4(i) and Young's inequality, $\frac{d}{dt}\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(t)) \leq -\frac{1}{2\Gamma}\|\mu - \theta\|^2 + \frac{\Gamma}{2}\|\delta\|^2$, and $\mathcal{K}_{\mathcal{H}} \geq \mathcal{K}_* > -\infty$ because $\mathcal{H}$ is semistable (Lemma 5.3(a)); so $\int_{\tau_0}^{\infty} \|\mu - \theta\|^2 \leq 2\Gamma(\mathcal{K}_{\mathcal{H}}(\mathfrak{h}(\tau_0)) - \mathcal{K}_*) + \Gamma^2 c_\delta^2/(2\kappa)$. Lemma 6.2 on $\mathcal{X}$ with the metric $\|\cdot\|_{\mathsf{m}}$ and $F = \mathcal{K}_{\mathcal{H}}$, which is convex along the geodesics $u \mapsto \mathfrak{h} e^{u\xi}$ (Lemma 5.1), gives the first claim. The second follows from $\big\|\log(\mathfrak{g}_i^{-1/2}\mathfrak{h}_i\mathfrak{g}_i^{-1/2})\big\| \leq d_{\mathsf{m}}(\mathfrak{h}, \mathfrak{g})/\sqrt{\mathsf{m}_i}$. □

Corollary 6.4. *Two curves that satisfy (P) for the same semistable representation $\mathcal{H}$ and the same charges remain at bounded distance from each other. In particular this holds for the normalized metrics induced on the cohomology of two complexes with a surjective or injective differential at one vertex whose cohomology representations are identified by a fixed isomorphism, and for any two initial metrics.*

Proof. Each of the two curves is at bounded distance from a solution of (17) by Theorem 6.3, and two solutions of (17) do not move apart by Lemma 6.2 with $e = 0$. □

The asymptotics of the flow (17) on a semistable representation are the subject of [HKKP23]. In the notation of [HKKP23, §5.2] the trace is $\tau(b) = \sum_i \mathsf{m}_i \operatorname{tr} b_i$, the central element ρ has the components θ_i/m_i, the flow is [HKKP23, (5.28)], and the standing assumption [HKKP23, (5.59)] is the semistability of $\mathcal{H}$ in the sense of Section 5. A curve $\mathfrak{h}$ is called there an *asymptotic solution* if $C^{-1}\mathfrak{g} \leq \mathfrak{h} \leq C\mathfrak{g}$ for large times, for some, equivalently for every, solution $\mathfrak{g}$ [HKKP23, §5.3]. In [HKKP23, §5.4] the lattice of subrepresentations of $\mathcal{H}$ of the same phase carries the *iterated weight filtration*, a filtration labelled by finite sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ of real numbers and defined algebraically from that lattice and the function $\mathsf{R} \mapsto \sum_i \mathsf{m}_i \dim \mathsf{R}_i$ [HKKP23, §4 and (5.58)]. After the arrows have been conjugated by a suitable invertible tuple of matrices, with $1 = \sum_\lambda p_\lambda$ the orthogonal splitting given by the filtration, the function

$$\mathfrak{a}(t) = \sum_\lambda t^{\lambda_1} (\log t)^{\lambda_2} \cdots \big(\log^{(n-1)} t\big)^{\lambda_n} p_\lambda \tag{18}$$

is an asymptotic solution [HKKP23, Corollary 5.10], and every solution satisfies the bounds of [HKKP23, Theorem 5.11] on the steps of the filtration.

Corollary 6.5 (Iterated logarithms)**.** *Assume (P) and let $\mathcal{H}$ be semistable. Then $\mathfrak{h}$ is an asymptotic solution of the gradient flow [HKKP23, (5.28)] with the weights $\mathsf{m}_i = y_i/\sigma^2$. Hence there are an invertible tuple of matrices b and a constant C such that $C^{-1} b^* \mathfrak{a}(t) b \leq \mathfrak{h}(t) \leq$*

$Cb^*\mathfrak{a}(t)b$ for large t, where $\mathfrak{a}$ is the function (18) for the arrows of $\mathcal{H}$ conjugated by b; and the bounds of [HKKP23, Theorem 5.11] *hold for $\mathfrak{h}$. The weight function of the iterated weight filtration is*

$$\sum_i \mathsf{m}_i \dim \mathsf{R}_i = \frac{\operatorname{Im} Z(\mathsf{R})}{\sigma^2} = \frac{|Z(\mathsf{R})|}{\sigma}$$

on the subquotients R of $\mathcal{H}$ that are semistable of phase φ: a constant multiple of the mass.

Proof. The first statement is the last statement of Theorem 6.3; the others follow from the definition of an asymptotic solution and the results of [HKKP23] just quoted. For a projection p the inequalities $e^{-c}\mathfrak{g} \le \mathfrak{h} \le e^c\mathfrak{g}$ give $e^{-c}p\mathfrak{g}p \le p\mathfrak{h}p \le e^c p\mathfrak{g}p$. □

The first statement of the corollary can also be deduced from [HKKP23, Proposition 5.7], which says that a curve satisfying the equation of the gradient flow up to an absolutely integrable term is an asymptotic solution; the term is the vector e of the proof of Theorem 6.3. The bounded factor between $\mathfrak{h}$ and (18) is not shown to converge, also not in [HKKP23] for the gradient flow; it converges in Proposition 6.6 below and in the example of [Wue26b, Proposition 6.9]. In the following example the exponents determine the ratio of the vertex weights.

Proposition 6.6. *Let Q be the quiver $0 \to 1$, let z_0, z_1 have the same argument φ, and let $\mathcal{H} = (\mathbb{C} \xrightarrow{b} \mathbb{C})$ with $b \ne 0$; it is semistable and not polystable, a nonsplit extension of the simple representation at the vertex 0 by the simple representation at the vertex 1. For every initial metric the normalized phase flow $\mathfrak{h} = e^{\varphi t}h$ satisfies*

$$\log \mathfrak{h}_0(t) = \frac{|z_1|}{|z_0|+|z_1|}\log t + c_0 + o(1), \qquad \log \mathfrak{h}_1(t) = -\frac{|z_0|}{|z_0|+|z_1|}\log t + c_1 + o(1)$$

as $t \to \infty$, with constants c_0, c_1. The solutions of (17) with arbitrary positive weights $\mathsf{m}'_0, \mathsf{m}'_1$ have the exponents $\mathsf{m}'_1/(\mathsf{m}'_0+\mathsf{m}'_1)$ and $-\mathsf{m}'_0/(\mathsf{m}'_0+\mathsf{m}'_1)$.

Proof. Here $\theta = 0$ and $\mathsf{m}_i = |z_i|/\sigma$. The squared norm $q = |b|^2\mathfrak{h}_1/\mathfrak{h}_0$ of the arrow gives $\mu_0 = q$, $\mu_1 = -q$, so $(\log \mathfrak{h}_0)' = G_0(q)$, $(\log \mathfrak{h}_1)' = G_1(-q)$ and $(\log q)' = -(G_0(q) - G_1(-q)) < 0$. By Lemma 6.1, $G_0(q) - G_1(-q) = \varkappa q + O(q^2)$ with $\varkappa = 1/\mathsf{m}_0 + 1/\mathsf{m}_1$. So q decreases to zero, $\frac{d}{dt}(1/q) = \varkappa + O(q)$, hence $1/q = \varkappa t + O(\log t)$ and $q = 1/(\varkappa t) + O(t^{-2}\log t)$. Then $(\log \mathfrak{h}_0)' = q/\mathsf{m}_0 + O(q^2)$ integrates to $\frac{1}{\mathsf{m}_0\varkappa}\log t + c_0 + o(1)$, and $\frac{1}{\mathsf{m}_0\varkappa} = \frac{\mathsf{m}_1}{\mathsf{m}_0+\mathsf{m}_1}$; likewise for $\mathfrak{h}_1$. For (17) with weights m'_i the same computation applies with $(\log \mathfrak{g}_0)' = q/\mathsf{m}'_0$, $(\log \mathfrak{g}_1)' = -q/\mathsf{m}'_1$. □

So on this representation the metric grows like t to the power $|Z(\mathsf{S}_1)|/|Z(\mathcal{H})|$ on the quotient S_0 and decays like t to the power $-|Z(\mathsf{S}_0)|/|Z(\mathcal{H})|$ on the subrepresentation S_1, where S_i is the simple representation at the vertex i; and a gradient flow whose weights are not proportional to the masses $|z_i|$ is at logarithmically growing distance from the phase flow. For a generalized Kronecker quiver with $\alpha_0 \ne \alpha_1$ the weights do not matter, because the stable factors of a semistable representation then have proportional dimension vectors.

6.2. **The mass and the Harder–Narasimhan filtration.** A representation $\mathsf{R} \ne 0$ is semistable of phase $\phi = \arg Z(\mathsf{R})$, in the sense of Section 5 with the parameter θ formed with ϕ, if and only if $\arg Z(\mathsf{R}') \le \phi$ for all nonzero subrepresentations R'. Every nonzero representation $\mathcal{H}$ has a unique filtration

$$0 = \mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^{\mathsf{n}} = \mathcal{H}$$

by subrepresentations whose factors $\mathrm{gr}^k = \mathcal{H}^k/\mathcal{H}^{k-1}$ are semistable of phases $\phi_1 > \phi_2 > \cdots > \phi_{\mathsf{n}}$, the *Harder–Narasimhan filtration* for the stability condition of Section 2 [Bri07, Proposition 2.4]; a filtration by subrepresentations with semistable factors of strictly decreasing phases is this one, because there are no nonzero morphisms from a semistable representation to a semistable representation of smaller phase. We call $\mathfrak{m}_{\mathrm{HN}}(\mathcal{H}) = \sum_k |Z(\mathrm{gr}^k)|$ the *mass of the filtration*; it is the mass of the object $\mathcal{H}$ for the stability condition. For a tuple of metrics on $\mathcal{H}$ let $\mathfrak{m}(\mathfrak{h}) = \sum_i \operatorname{tr}|z_i + \mu_i|$ be the mass of the metrized representation, Π_k the orthogonal projections onto $\mathcal{H}^k$, and

$$\mathcal{Y}_k = \sum_a \left\| \Pi_{k,\mathsf{t}(a)} \beta_a (1 - \Pi_{k,\mathsf{s}(a)}) \right\|^2$$

the squared norm of the second fundamental form of $\mathcal{H}^k$.

Proposition 6.7 (Dissipation of the mass)**.** *Assume (P). Then for $t \geq \tau_0$*

$$\frac{d\,\mathfrak{m}(\mathfrak{h}(t))}{dt} \leq -\mathfrak{D} + 4\bar{\mathcal{N}} \left\|\delta\right\|, \qquad \mathfrak{D} \geq \min(\sin\psi_-, \sin\psi_+) \sum_a \left\| \beta_a \Phi^{\mathcal{H}}_{\mathsf{s}(a)} - \Phi^{\mathcal{H}}_{\mathsf{t}(a)} \beta_a \right\|^2,$$

where $\mathfrak{D} = \sum_a \sum_{j,l} |\langle v_l, \beta_a u_j\rangle|^2 (p_j - q_l)(\cos q_l - \cos p_j)$ for orthonormal eigenvectors u_j of $\Phi^{\mathcal{H}}_{\mathsf{s}(a)}$ and v_l of $\Phi^{\mathcal{H}}_{\mathsf{t}(a)}$ with eigenvalues p_j, q_l. In particular the mass is nonincreasing along the phase flow of a representation, for all charges.

Proof. The derivative of $u \mapsto |z_i + u|$ is $\cos\arg(z_i + u)$, so $\frac{d}{dt}\mathfrak{m} = \sum_i \operatorname{tr}(\cos \Phi^{\mathcal{H}}_i \dot{\mu}_i)$. Insert (16); the result is linear in $\Phi^{\mathcal{H}} + \delta$. The part with $\Phi^{\mathcal{H}}$ is, since $\Phi^{\mathcal{H}}_i$ commutes with its cosine,

$$\sum_a \sum_{j,l} |\langle v_l, \beta_a u_j\rangle|^2 \big(p_j \cos p_j - q_l \cos p_j - p_j \cos q_l + q_l \cos q_l\big) = -\mathfrak{D}.$$

The cosine decreases on $(0,\pi)$, and $(p-q)(\cos q - \cos p) \geq \min(\sin\psi_-, \sin\psi_+)(p-q)^2$ for $p, q \in [\psi_-, \psi_+]$. The part with δ consists of four traces for each arrow, each at most $\|\beta_a\|^2 \|\delta\|$. □

For Kronecker quivers the monotonicity of the mass on representations is [Wue26b, Proposition 9.1].

Proposition 6.8 (Mass of a metric and mass of the filtration)**.** *For every tuple of metrics on $\mathcal{H}$,*

$$\mathfrak{m}(\mathfrak{h}) \geq \mathfrak{m}_{\mathrm{HN}}(\mathcal{H}) + \sum_{k=1}^{\mathsf{n}-1} \big(\cos\phi_{k+1} - \cos\phi_k\big)\, \mathcal{Y}_k,$$

and the coefficients $\cos\phi_{k+1} - \cos\phi_k$ are positive.

Proof. Let $Q_k = \Pi_k - \Pi_{k-1}$. By the pinching inequality for the trace norm and $\operatorname{tr}|X| \geq |\operatorname{tr} X|$, $\mathfrak{m} \geq \sum_k \big|\sum_i \operatorname{tr}(Q_{k,i}(z_i + \mu_i))\big|$. For a subrepresentation with projections Π one has $\sum_i \operatorname{tr}(\Pi_i \mu_i) = \sum_a (\left\|\beta_a \Pi_{\mathsf{s}(a)}\right\|^2 - \left\|\Pi_{\mathsf{t}(a)} \beta_a\right\|^2) = -\sum_a \left\|\Pi_{\mathsf{t}(a)} \beta_a (1 - \Pi_{\mathsf{s}(a)})\right\|^2$, as in the proof of Lemma 5.3(b). So with $\mathcal{Y}_0 = \mathcal{Y}_{\mathsf{n}} = 0$, $\sum_i \operatorname{tr}(Q_{k,i}(z_i + \mu_i)) = Z(\mathrm{gr}^k) - (\mathcal{Y}_k - \mathcal{Y}_{k-1})$, and the modulus of this number is at least its component in the direction of $Z(\mathrm{gr}^k)$, which is $|Z(\mathrm{gr}^k)| - (\mathcal{Y}_k - \mathcal{Y}_{k-1})\cos\phi_k$. Sum over k and sum by parts. □

6.3. **Unstable cohomology.** Assume (P). The arrows $\beta(t) = (\beta_a(t))$ in the frames of (16) form a bounded curve. Let ω be the set of its limit points as $t \to \infty$; it is nonempty, compact and connected.

Lemma 6.9. *Assume* (P).

(a) *The mass* $\mathfrak{m}(\mathfrak{h}(t))$ *converges, and* $\beta_a \Phi^{\mathcal{H}}_{\mathsf{s}(a)} - \Phi^{\mathcal{H}}_{\mathsf{t}(a)} \beta_a$, $\dot\beta_a$ *and the derivatives of the* $\Phi^{\mathcal{H}}_i$ *tend to zero.*

(b) *There are numbers* $\pi > \phi'_1 > \cdots > \phi'_{\mathsf{n}'} > 0$ *and nonzero dimension vectors* $d_k = (n_{i,k})_i$ *with* $\sum_k d_k = (n_i)$ *and* $\arg Z(d_k) = \phi'_k$ *such that for every* $\beta^\infty \in \omega$ *the phase of* β^∞ *at the vertex* i *has the eigenvalues* ϕ'_k *with multiplicities* $n_{i,k}$*, and the representation with the arrows* β^∞ *is the orthogonal direct sum of representations* P_k *of dimension vectors* d_k*, supported on the eigenspaces, with the moment map of a harmonic metric of phase* ϕ'_k *on* P_k*. In particular* P_k *is polystable of phase* ϕ'_k*.*

(c) *The eigenvalues of* $\Phi^{\mathcal{H}}_i(t)$*, with multiplicity, converge to the* ϕ'_k *with multiplicities* $n_{i,k}$*, and* $\mathfrak{m}(\mathfrak{h}(t)) \to \sum_k |Z(d_k)|$.

Proof. (a) By Proposition 6.7, $\mathfrak{m} + 4\bar{\mathcal{N}} \int_t^\infty \|\delta\|$ is nonincreasing and nonnegative, so $\mathfrak{m}$ converges and $\mathfrak{D}$ is integrable. By (16) the $\dot\beta_a$ are bounded, hence so are the derivatives of the μ_i and, by the Lipschitz property, of the $\Phi^{\mathcal{H}}_i$. So $\sum_a \left\| \beta_a \Phi^{\mathcal{H}}_{\mathsf{s}(a)} - \Phi^{\mathcal{H}}_{\mathsf{t}(a)} \beta_a \right\|^2$ is Lipschitz and integrable and tends to zero; then $\dot\beta_a \to 0$ by (16), and the derivatives of μ_i and $\Phi^{\mathcal{H}}_i$ tend to zero.

(b) For $\beta^\infty \in \omega$, by (a) and continuity each arrow intertwines the phases at its source and target. So the eigenspaces of the phase for a fixed eigenvalue p form an orthogonal direct summand P_p, and on it $\arg(z_i + \mu_i) = p$ for all i: the metric is harmonic of phase p on P_p, which is therefore polystable (Lemma 5.3(b)), and $\arg Z(\mathsf{P}_p) = p$ because $\sum_i \operatorname{tr}(\Pi_i \mu_i) = 0$ for an orthogonal direct summand. Thus the eigenvalues of the phase of β^∞ lie in the finite set of arguments of the numbers $Z(d)$, $0 \neq d \leq (n_i)$. The ordered list of these eigenvalues at each vertex depends continuously on β, takes values in a finite set on ω, and ω is connected; so it is constant on ω.

(c) The distance from $\beta(t)$ to ω tends to zero, and on P_k all eigenvalues of $z_i + \mu_i$ lie on one ray. □

By (c) there is a time after which the spectrum of each $\Phi^{\mathcal{H}}_i(t)$ consists of separated clusters near the ϕ'_k. Let $P_k(t)$ be the spectral projection of the k-th cluster and $P_{\leq k} = P_1 + \cdots + P_k$. Since the derivatives of the $\Phi^{\mathcal{H}}_i$ tend to zero and the clusters stay separated,

$$\left\| \dot P_{\leq k}(t) \right\| \to 0, \qquad \|P_l \beta_a P_k\| \to 0 \quad (l \neq k), \tag{19}$$

the second because $X = P_l \beta_a P_k$ satisfies $X \Phi^{\mathcal{H}}_{\mathsf{s}(a)} - \Phi^{\mathcal{H}}_{\mathsf{t}(a)} X = P_l(\beta_a \Phi^{\mathcal{H}}_{\mathsf{s}(a)} - \Phi^{\mathcal{H}}_{\mathsf{t}(a)} \beta_a) P_k$ with separated spectra on the two sides.

Lemma 6.10. *Let* $\Psi(t)$*,* $t \geq T$*, be a continuous family of Hermitian* $n \times n$ *matrices and* $P(t)$ *a continuously differentiable family of orthogonal projections of constant rank. Suppose there are numbers* $a_+ > a_-$ *and* $\epsilon \geq 0$ *with* $12\epsilon \leq a_+ - a_-$ *such that for all* $t \geq T$ *and all vectors* w*, with* $w' = P(t)w$ *and* $w'' = w - w'$*,*

$$\operatorname{Re} \langle w', \Psi w \rangle \geq a_+ |w'|^2 - \epsilon |w'||w''|, \qquad \operatorname{Re} \langle w'', \Psi w \rangle \leq a_- |w''|^2 + \epsilon |w'||w''|, \qquad \left\| \dot P \right\| \leq \epsilon.$$

Then $\mathbb{C}^n = S^{\mathrm{f}} \oplus S^{\mathrm{s}}$ *with* $\dim S^{\mathrm{f}} = \operatorname{rank} P$*, such that for the solutions of* $\dot w = -\frac{1}{2}\Psi w$ *on* $[T, \infty)$*:*

(i) *if* $w(T) \in S^{\mathrm{f}}$, *then* $|w''(t)| \le \frac{6\epsilon}{a_+ - a_-}|w'(t)|$ *and* $|w(t)|^2 \le 2e^{-(a_+ - 3\epsilon)(t-T)}|w(T)|^2$ *for* $t \ge T$;
(ii) *if* $w(T) \in S^{\mathrm{s}}$, *then* $|w(t)|^2 \ge \frac{1}{2}e^{-(a_- + 3\epsilon)(t-T)}|w(T)|^2$ *for* $t \ge T$.

Proof. Since $P\dot{P}P = 0 = (1-P)\dot{P}(1-P)$, a solution satisfies $\frac{d}{dt}|w'|^2 \le -a_+|w'|^2 + 3\epsilon|w'||w''|$ and $\frac{d}{dt}|w''|^2 \ge -a_-|w''|^2 - 3\epsilon|w'||w''|$. Let $c_0 = 6\epsilon/(a_+ - a_-) \le \frac{1}{2}$. For $c \in [c_0, 1]$ with $c > 0$ and a nonzero solution, at a time with $|w''| = c|w'|$,

$$\frac{d}{dt}\big(c^2|w'|^2 - |w''|^2\big) \le c^2|w'|^2\big(-(a_+ - a_-) + 3\epsilon(c + c^{-1})\big) < 0,$$

because $3\epsilon c \le \frac{1}{4}(a_+ - a_-)$ and $3\epsilon c^{-1} \le \frac{1}{2}(a_+ - a_-)$. So a solution in the open cone $\{|w''| > c|w'|\}$ cannot reach its boundary: these cones are invariant forwards in time. Let $S^{\mathrm{s}} = \ker P(T)$. A solution starting there stays in the cone with $c = 1$, where $\frac{d}{dt}|w''|^2 \ge -(a_- + 3\epsilon)|w''|^2$; this gives (ii). For $\tau \ge T$ let S_τ be the space of the initial values $w(T)$ of the solutions with $w(\tau)$ in the range of $P(\tau)$. Such a solution satisfies $|w''| \le c_0|w'|$ on $[T, \tau]$, since otherwise it would lie, at the time τ, in one of the open cones with $c \in [c_0, 1]$, $c > 0$. Let S^{f} be a limit point of the spaces S_τ in the Grassmannian as $\tau \to \infty$. By continuity every solution starting in S^{f} satisfies $|w''| \le c_0|w'|$ for all $t \ge T$, and then $\frac{d}{dt}|w'|^2 \le -(a_+ - 3\epsilon)|w'|^2$ and $|w|^2 \le 2|w'|^2$; this is (i). The two growth estimates exclude a common nonzero vector, and the dimensions add up to n. □

Theorem 6.11 (Harder–Narasimhan asymptotics)**.** *Let* $\mathcal{H}$ *be a nonzero representation of* Q, *with Harder–Narasimhan filtration* $(\mathcal{H}^k)$, *factors* gr^k *and phases* $\phi_1 > \cdots > \phi_{\mathsf{n}}$, *and let* $\mathfrak{h}(t)$ *satisfy* (P). *Then:*

(i) *the eigenvalues of* $\Phi_i^{\mathcal{H}}(t)$, *with multiplicity, converge to the* ϕ_k *with multiplicities* $\dim \mathrm{gr}_i^k$, *and* $\mathfrak{m}(\mathfrak{h}(t))$ *converges to* $\mathfrak{m}_{\mathrm{HN}}(\mathcal{H})$, *the infimum of the mass over all metrics on* $\mathcal{H}$;
(ii) *for every nonzero* $v \in \mathcal{H}_i$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathfrak{h}_i(t)(v, v) = \varphi - \phi_k,$$

where k *is the least index with* $v \in \mathcal{H}_i^k$; *so the Harder–Narasimhan filtration is the filtration of the spaces by the exponential rate of decay of the metric;*
(iii) *for every* $\epsilon > 0$ *there is a constant* C *with* $\mathcal{Y}_k(t) \le Ce^{-2\varkappa_k t}$, $\varkappa_k = \min\{\frac{1}{2}(\phi_k - \phi_{k+1}) - \epsilon,\ \kappa\}$, *for* $k < \mathsf{n}$ *(with* $\kappa = \infty$ *if* $\delta = 0$*);*
(iv) *for each* k *the metric induced by* $\mathfrak{h}(t)$ *on* gr^k, *multiplied by* $e^{(\phi_k - \varphi)t}$, *satisfies* (P) *for the semistable representation* gr^k. *For one factor the original perturbation and its rate* κ *are retained. For two or more factors take* $0 < 2\epsilon < \min_{l<\mathsf{n}}(\phi_l - \phi_{l+1})$; *the perturbation then has rate* $\min_{l<\mathsf{n}} \varkappa_l$. *Hence Proposition 5.4, Theorem 6.3 and Corollary 6.5 apply to it: its moment map tends to the stability parameter of* gr^k, *it is an asymptotic solution of the weighted gradient flow of* gr^k, *and if* gr^k *is polystable it converges exponentially to a harmonic metric of* gr^k.

Proof. Step 1: the filtration by exponents. Fix $k < \mathsf{n}'$, a vertex i, and $\epsilon > 0$ with $60\epsilon \le \phi'_k - \phi'_{k+1}$. By Lemma 6.9 and (19) there is a time T after which the clusters are within ϵ of the ϕ'_l, $\|\delta\| \le \epsilon$ and $\left\|\dot{P}_{\le k}\right\| \le \epsilon$. For a vector $v \in \mathcal{H}_i$ the coordinates w of v in the frame satisfy $|w|^2 = \mathfrak{h}_i(v, v)$ and $\dot{w} = -\frac{1}{2}\Psi w$ with $\Psi = \Phi_i^{\mathcal{H}} - \varphi + \delta_i$. Since $P = P_{\le k}$ commutes with $\Phi_i^{\mathcal{H}}$, the hypotheses of Lemma 6.10 hold with $a_+ = \phi'_k - \varphi - 2\epsilon$ and $a_- = \phi'_{k+1} - \varphi + 2\epsilon$. In terms of vectors of $\mathcal{H}_i$ the lemma gives complementary subspaces, of which the first has the dimension

$\sum_{l\le k} n_{i,l}$, with $\mathfrak{h}_i(t)(v,v) \le C_v e^{-(\phi'_k-\varphi-5\epsilon)t}$ on the first and $\mathfrak{h}_i(t)(v,v) \ge c_v e^{-(\phi'_{k+1}-\varphi+5\epsilon)t}$, $c_v > 0$, for the nonzero vectors of the second. A vector with a nonzero component in the second subspace then satisfies $\liminf \frac{1}{t}\log \mathfrak{h}_i(t)(v,v) \ge -(\phi'_{k+1}-\varphi+5\epsilon)$. So the first subspace is the set of all v with $\limsup \frac{1}{t}\log \mathfrak{h}_i(t)(v,v) \le -(\phi'_k-\varphi-5\epsilon)$; these sets decrease with ϵ and have the same dimension, so they coincide with

$$F_{k,i} := \Big\{ v \in \mathcal{H}_i : \ \limsup_{t\to\infty} \frac{1}{t}\log \mathfrak{h}_i(t)(v,v) \le \varphi - \phi'_k \Big\}, \qquad \dim F_{k,i} = \sum_{l\le k} n_{i,l},$$

and for $v \in F_{k,i} \setminus F_{k-1,i}$ the limit of $\frac{1}{t}\log \mathfrak{h}_i(t)(v,v)$ exists and equals $\varphi - \phi'_k$ (with $F_0 = 0$ and $F_{\mathsf{n}'} = \mathcal{H}$). Here the lower bound for $k = 1$ and the upper bound for $k = \mathsf{n}'$, which do not come from a gap, follow from $\frac{d}{dt}|w|^2 = -\langle w, \Psi w\rangle$ and Lemma 6.9(c): the eigenvalues of Ψ lie eventually between $\phi'_{\mathsf{n}'} - \varphi - 2\epsilon$ and $\phi'_1 - \varphi + 2\epsilon$.

Step 2: the F_k are subrepresentations. For an arrow a of $\mathcal{H}$ and $v \in \mathcal{H}_{\mathsf{s}(a)}$, the norm of the image of v for $\mathfrak{h}_{\mathsf{t}(a)}$ is at most $\bar{\mathcal{N}}^{1/2}$ times the norm of v for $\mathfrak{h}_{\mathsf{s}(a)}$. So the arrows do not decrease the rate of decay, and they map F_k to F_k. The factor F_k/F_{k-1} has the dimension vector d_k, hence $\arg Z(F_k/F_{k-1}) = \phi'_k$.

Step 3. By Lemma 6.10(i), applied with ϵ arbitrarily small for large T, the coordinates of the vectors of F_k form the graph of a linear map from the range of $P_{\le k}(t)$ to its orthogonal complement whose norm tends to zero. So for the $\mathfrak{h}(t)$-orthogonal projections $\Pi'_k(t)$ onto F_k, in the frames, $\|\Pi'_k - P_{\le k}\| \to 0$; and with $Q_k = \Pi'_k - \Pi'_{k-1}$, by (19) and Lemma 6.9(c),

$$Q_l \beta_a Q_k \to 0 \ (l \ne k), \qquad \big\|Q_k(\Phi^{\mathcal{H}} - \phi'_k)Q_k\big\| \to 0, \qquad Q_l \Phi^{\mathcal{H}} Q_k \to 0 \ (l \ne k). \tag{20}$$

Step 4: the factors are semistable. In frames adapted to the decomposition $1 = \sum_k Q_k(t)$ the arrows are block upper triangular, and the k-th diagonal block is the tuple of arrows of the representation F_k/F_{k-1} in orthonormal frames of its induced metric, a point of the orbit of this representation under the changes of bases. Along a suitable sequence of times these frames and the arrows converge, and by (20) the k-th diagonal block converges to the arrows of the summand P_k of a point of ω. So the closure of the orbit of F_k/F_{k-1} contains a representation isomorphic to P_k, which is semistable of phase ϕ'_k. If F_k/F_{k-1} had a subrepresentation of a dimension vector d' with $\arg Z(d') > \phi'_k$, then so would every representation in the closure of its orbit, because the set of representations with a subrepresentation of dimension vector d' is the image of a closed set under a projection with compact fibres, a product of Grassmannians. This contradicts the semistability of P_k. So F_k/F_{k-1} is semistable of phase ϕ'_k.

Step 5. The filtration (F_k) has semistable factors of strictly decreasing phases. Hence it is the Harder–Narasimhan filtration: $\mathsf{n}' = \mathsf{n}$, $F_k = \mathcal{H}^k$, $\phi'_k = \phi_k$ and $d_k = \dim \mathrm{gr}^k$. With Lemma 6.9(c) and Step 1 this gives (i) and (ii); the mass of the filtration is the infimum of the mass by Proposition 6.8.

Step 6: the second fundamental forms. Fix $k < \mathsf{n}$ and write $\Pi = \Pi_k$. For an arrow a let $\upsilon_a = \Pi_{\mathsf{t}(a)}\beta_a(1 - \Pi_{\mathsf{s}(a)})$, and for a Hermitian matrix M at a vertex let $M^{\mathrm{off}} = \Pi M(1-\Pi) + (1-\Pi)M\Pi$. With $\tilde{\Phi} = \Phi^{\mathcal{H}} + \delta$ we claim that

$$\frac{d\mathcal{Y}_k}{dt} = \sum_a \Big[\mathrm{tr}\big(\upsilon_a^\dagger \upsilon_a \,\tilde{\Phi}_{\mathsf{s}(a)}\big) - \mathrm{tr}\big(\upsilon_a \upsilon_a^\dagger \,\tilde{\Phi}_{\mathsf{t}(a)}\big)\Big] + \sum_i \mathrm{tr}\big(\tilde{\Phi}_i^{\mathrm{off}} \mu_i^{\mathrm{off}}\big). \tag{21}$$

Indeed, the coordinate maps of the frames satisfy $\dot{g}g^{-1} = -\frac{1}{2}(\tilde{\Phi} - \varphi)$, a Hermitian matrix, and the range of Π is the image of a fixed subspace, so $\dot{\Pi} = -\frac{1}{2}\tilde{\Phi}^{\mathrm{off}}$. Differentiate υ_a with (16), use $(1 - \Pi_{\mathsf{t}(a)})\beta_a\Pi_{\mathsf{s}(a)} = 0$, and take the trace against $\upsilon_a^\dagger$: with $\beta'_a = \Pi_{\mathsf{t}(a)}\beta_a\Pi_{\mathsf{s}(a)}$ and

$\beta_a'' = (1-\Pi_{\mathsf{t}(a)})\beta_a(1-\Pi_{\mathsf{s}(a)})$,

$$\frac{d}{dt}\|v_a\|^2 = \operatorname{tr}\big(v_a^\dagger v_a \tilde\Phi_{\mathsf{s}(a)}\big) - \operatorname{tr}\big(v_a v_a^\dagger \tilde\Phi_{\mathsf{t}(a)}\big) - 2\operatorname{Re}\operatorname{tr}\big(\Pi\tilde\Phi_{\mathsf{t}(a)}(1-\Pi)\beta_a'' v_a^\dagger\big) + 2\operatorname{Re}\operatorname{tr}\big(\Pi\tilde\Phi_{\mathsf{s}(a)}(1-\Pi)v_a^\dagger\beta_a'\big),$$

and $(1-\Pi)\mu_i\Pi = \sum_{\mathsf{s}(a)=i} v_a^\dagger\beta_a' - \sum_{\mathsf{t}(a)=i}\beta_a'' v_a^\dagger$; summing over the arrows gives (21).

The term $\sum_i \operatorname{tr}((\Phi_i^{\mathcal{H}})^{\mathrm{off}}\mu_i^{\mathrm{off}})$ is nonpositive: the matrix $2\Pi-1$ is unitary and Hermitian, conjugation by it changes the sign of the off-diagonal parts, and $\Phi_i^{\mathcal{H}}$ is a decreasing function of μ_i, so four times the trace is $\operatorname{tr}((g(A)-g(B))(A-B)) \le 0$ with $A=\mu_i$, $B=(2\Pi-1)\mu_i(2\Pi-1)$ and $g=\arg(z_i+\cdot)$. By (20) and Step 5, for every $\epsilon>0$ and large t the eigenvalues of $\Pi\Phi^{\mathcal{H}}\Pi$ on the range of Π are at least $\phi_k-\epsilon$ and those of $(1-\Pi)\Phi^{\mathcal{H}}(1-\Pi)$ on the complement are at most $\phi_{k+1}+\epsilon$. The terms with δ are at most $2\|\delta\|\,\mathcal{Y}_k + 2\sqrt{2}\,\|\delta\|\,(\bar{\mathcal{N}}\mathcal{Y}_k)^{1/2}$. Hence

$$\frac{d\mathcal{Y}_k}{dt} \le -\big(\phi_k-\phi_{k+1}-2\epsilon-2\|\delta\|\big)\mathcal{Y}_k + 2\sqrt{2}\,\|\delta\|\,(\bar{\mathcal{N}}\mathcal{Y}_k)^{1/2}$$

for large t. With $\|\delta\| \le c_\delta e^{-\kappa(t-\tau_0)}$ this gives (iii); one applies the inequality with $\epsilon/2$ in the place of ϵ, so that a factor t arising when the two exponents coincide is absorbed.

Step 7: the factors. If $\mathsf{n}=1$, the factor is $\mathcal{H}$ and $\phi_1=\varphi$, so (iv) follows directly from (P). For $\mathsf{n}>1$, the restriction of a metric to a subspace and the quotient metric move with the compressed velocity. So in the adapted frames of Step 4 the metric induced on gr^k, multiplied by $e^{(\phi_k-\varphi)t}$, has the velocity $\phi_k - Q_k\Phi^{\mathcal{H}}Q_k - Q_k\delta Q_k$, and its arrows are the k-th diagonal block. Let $\mu^{(k)}$ be the moment map of this block, D the block-diagonal part of μ and $O=\mu-D$. Every block of O, and every block of $Q_kDQ_k-\mu^{(k)}$, is a sum of products of two blocks of the arrows of which at least one, respectively both, are off-diagonal; so with $Y^2 = \sum_a\sum_{l<k}\|Q_l\beta_aQ_k\|^2 \le \sum_k \mathcal{Y}_k$, $\|O\| \le 4\bar{\mathcal{N}}^{1/2}Y$ and $\big\|Q_kDQ_k-\mu^{(k)}\big\| \le Y^2$. By the Lipschitz property, and because a function of D is block diagonal,

$$\Big\|Q_k\Phi^{\mathcal{H}}Q_k - \arg(z+\mu^{(k)})\Big\| \le \frac{4\bar{\mathcal{N}}^{1/2}Y + Y^2}{\min_i y_i}.$$

So the velocity is $\phi_k - \arg(z+\mu^{(k)}) - \delta^{(k)}$ with $\delta^{(k)}$ exponentially small by (iii), and the squared norms of the arrows of the block sum to at most $\bar{\mathcal{N}}$: this is (P) for gr^k, which is semistable of phase ϕ_k. □

Remark 6.12. (1) For a subrepresentation of a representation, and $\delta=0$, the identity (21) holds at every time, with $\mathcal{Y}$ the squared norm of its second fundamental form. Its last term is nonpositive. So the second fundamental form decays as long as the phases on the subrepresentation exceed those on its orthogonal complement.

(2) The filtration by the rates of decay is a filtration by subrepresentations for the simple reason of Step 2; the theorem identifies the rates and the dimensions. For the gradient flow the limiting filtration is described in the same terms in [HKKP23, §1.1], and for unstable points of Kempf–Ness gradient flows the term linear in t is in [Lam26].

(3) By (i) and Proposition 6.7, along the phase flow of a representation of a quiver without oriented cycles the mass decreases to the mass of the object for the stability condition; the largest and the smallest eigenvalue of the phase tend to ϕ_1 and ϕ_{n}.

7. Kronecker quivers: complexes with differentials at one vertex

Let $Q=K_m$, with vertices $0,1$ and arrows $a_1,\dots,a_m$ from 0 to 1. A bounded complex of representations E has spaces V^n at the vertex 0 and W^n at the vertex 1, differentials

$d_0 : V^n \to V^{n+1}$ and $d_1 : W^n \to W^{n+1}$, and arrows $A_j^n : V^n \to W^n$ with $d_1 A_j^n = A_j^{n+1} d_0$. Its cohomology in degree n is a representation $H^n(\mathsf{E})$. The endomorphism Ω, the phase and the phase flow of E with a metric are those of Section 2, and we use the properties (O1) and (O2) stated there. In this section the charges z_0, z_1 are arbitrary in the upper half-plane unless stated otherwise; we call them *strictly ordered* if $\alpha_0 > \alpha_1$, that is, if the simple representation at the vertex 0 has the larger phase. For strictly ordered charges there are stable representations with both spaces nonzero; for $\alpha_0 \le \alpha_1$ the only stable representations are the two simple ones, because a representation $(\mathsf{R}_0 \to \mathsf{R}_1)$ with $\mathsf{R}_0, \mathsf{R}_1 \neq 0$ has the subrepresentation $(0 \to \mathsf{R}_1)$, whose phase α_1 is then not smaller than that of the representation.

7.1. **The decomposition.**

Theorem 7.1 (Decomposition)**.** *Let E be a bounded complex of representations of K_m with a metric.*

(a) *Suppose $d_1 = 0$. Put $U^n = d_0(V^{n-1}) \subset V^n$, and let $(U^n)^\perp$ be its orthogonal complement in V^n. For each n the spaces*

$$\big((U^n)^\perp \xrightarrow{A_j^n} W^n\big) \text{ in degree } n, \qquad (U^{n+1} \to 0) \text{ in degree } n+1,$$

with the differential $d_0 : (U^n)^\perp \to U^{n+1}$, which is surjective, form a subcomplex of representations E_n. The complex E is the orthogonal direct sum of the E_n. If $H^n(\mathsf{E}) \neq 0$, then $\mathsf{E}_n[n]$ is a complex with a surjective differential at the vertex 0 (Definition 2.1), with cohomology $H^n(\mathsf{E})$.

(b) *Suppose $d_0 = 0$. Let C^n be the orthogonal complement of $\ker d_1^n$ in W^n. For each n the spaces*

$$(0 \to C^{n-1}) \text{ in degree } n-1, \qquad \big(V^n \xrightarrow{A_j^n} \ker d_1^n\big) \text{ in degree } n,$$

with the differential $d_1 : C^{n-1} \to \ker d_1^n$, which is injective, form a subcomplex of representations E_n. The complex E is the orthogonal direct sum of the E_n. If $H^n(\mathsf{E}) \neq 0$, then $\mathsf{E}_n[n]$ is a complex with an injective differential at the vertex 1 (Definition 2.1), with cohomology $H^n(\mathsf{E})$.

(c) *In both cases, if $H^n(\mathsf{E}) = 0$, then E_n is a complex $C \to C'$ at one vertex with invertible differential, on which the arrows vanish.*

(d) *In both cases Ω and the phase of E are the direct sums of those of the E_n, and the phase flow preserves the decomposition: the subspaces remain orthogonal for the metric at every time, and the metric of each E_n evolves by the phase flow of E_n.*

The subspaces $(U^n)^\perp$ and C^n, and with them the subcomplexes E_n, depend on the metric.

Proof. (a) Since $A_j^{n+1} d_0 = d_1 A_j^n = 0$, the arrows vanish on U^{n+1}; so E_n is closed under the arrows, and it is closed under the differential because d_0 maps $(U^n)^\perp$ into U^{n+1} and vanishes on U^{n+1}. The map $d_0 : (U^n)^\perp \to U^{n+1} = d_0(V^n)$ is surjective because d_0 vanishes on U^n. The spaces of the E_n decompose those of E: W^n belongs to E_n, and $V^n = U^n \oplus (U^n)^\perp$ with U^n in E_{n-1} and $(U^n)^\perp$ in E_n, orthogonally. The kernel of d_0 on $(U^n)^\perp$ is isomorphic to $\ker d_0^n / d_0(V^{n-1})$, so the cohomology of E_n is $H^n(\mathsf{E})$ in degree n.

(b) Since $d_1 A_j^n = A_j^{n+1} d_0 = 0$, the arrows map V^n into $\ker d_1^n$. The differential maps C^{n-1} into $d_1(W^{n-1}) \subset \ker d_1^n$ and is injective on it. V^n belongs to E_n, and $W^n = \ker d_1^n \oplus C^n$ with $\ker d_1^n$ in E_n and C^n in E_{n+1}. The cohomology of E_n is V^n at the vertex 0 and $\ker d_1^n / d_1(W^{n-1})$ at the vertex 1.

(c) In (a), $H^n(\mathsf{E}) = 0$ means $W^n = 0$ and $\ker d_0 \cap (U^n)^\perp = 0$; in (b) it means $V^n = 0$ and $d_1(C^{n-1}) = \ker d_1^n$.

(d) The first statement is (O1). Hence at a metric for which the decomposition is orthogonal the vector field (2) is tangent to the set of such metrics, and the solution stays in this set; on it the flow is the product of the flows of the summands. □

The metric induced on $H^n(\mathsf{E})$ is the restriction of the metric to the orthogonal complement of the image of the differential in its kernel; it is the metric induced on the cohomology of $\mathsf{E}_n[n]$ in the sense of Section 5.

Theorem 7.2 (Long-time behaviour)**.** *Let E be a bounded complex of representations of K_m with $d_0 = 0$ or $d_1 = 0$, with any metric and arbitrary charges in the upper half-plane, and let E_n be as in Theorem 7.1. Then along the phase flow, for every n:*

(i) *if the differential f of E_n is not zero, the least nonzero eigenvalue of $ff^\dagger$ tends to infinity; and the trace-norm distance between the endomorphism Ω_{E_n} of E_n and $(-1)^n\Omega_{H^n(\mathsf{E})}$, the endomorphism of the representation $H^n(\mathsf{E})$ with its induced metrics, extended by zero, tends to zero exponentially for $t \geq t_1$; here t_1 is the time of Theorem 4.2 or of Proposition 4.3 for $\mathsf{E}_n[n]$ or its dual if $H^n(\mathsf{E}) \neq 0$, and $t_1 = \frac{1}{\pi}\log^+\frac{1}{\lambda_0}$, with λ_0 the least eigenvalue of $ff^\dagger$ at time 0, if $H^n(\mathsf{E}) = 0$;*

(ii) *if $H^n(\mathsf{E})$ is nonzero and semistable, with $\varphi_n = \arg Z(H^n(\mathsf{E})) \in (0,\pi)$, its moment map tends to its stability parameter θ (Section 5); if it is polystable, the metrics induced on $H^n(\mathsf{E})$, multiplied by $e^{(\varphi_n-\pi n)t}$, converge exponentially to a harmonic metric of $H^n(\mathsf{E})$;*

(iii) *if $H^n(\mathsf{E}) \neq 0$, with $\varphi_n = \arg Z(H^n(\mathsf{E}))$, the metrics induced on $H^n(\mathsf{E})$, multiplied by $e^{(\varphi_n-\pi n)t}$, satisfy the hypothesis (P) of Section 6 from some time on. So Theorem 6.11 applies to them with the Harder–Narasimhan filtration of the representation $H^n(\mathsf{E})$, and Theorem 6.3 and Corollary 6.5 apply to its factors.*

In all cases $\mathfrak{m}(h(t)) \to \sum_n \mathfrak{m}_{\mathrm{HN}}(H^n(\mathsf{E}))$, the sum of the masses of the Harder–Narasimhan filtrations of the cohomology representations; this is $\sum_n |Z(H^n(\mathsf{E}))|$ if they are semistable.

Proof. By Theorem 7.1 the flow is the product of the flows of the E_n, and by (O2) the flow of $\mathsf{E}_n[n]$ has the same trajectories up to the factor $e^{-\pi nt}$. If $H^n(\mathsf{E}) = 0$, then $\mathsf{E}_n[n]$ is a complex $C \xrightarrow{f} C'$ at one vertex i, in degrees $0,1$ or $-1,0$, on which the arrows vanish; by (1) its endomorphism is $\pm\big(z_i e^{-f^\dagger f} \oplus (-z_i e^{-ff^\dagger})\big)$, the phases on C and C' are scalars that differ by π, and by (3) $f(t) = e^{\pi t/2} f(0)$ in orthonormal frames. Let now $H^n(\mathsf{E}) \neq 0$ and $d_1 = 0$, so that $\mathsf{E}_n[n]$ has a surjective differential at the vertex 0. If this differential is zero, $\mathsf{E}_n[n]$ is the representation $H^n(\mathsf{E})$, and Theorem 5.5 applies. If it is not zero and $W^n \neq 0$, Theorems 4.2 and 5.6 apply, with $Q = K_m$ and $i_0 = 0$. If it is not zero and $W^n = 0$, all arrows vanish, and Proposition 4.3 and Remark 5.7 apply. For $d_0 = 0$ the complex $\mathsf{E}_n[n]$ has an injective differential at the vertex 1: if this differential is zero, $\mathsf{E}_n[n]$ is the representation $H^n(\mathsf{E})$ and Theorem 5.5 applies; otherwise Proposition 5.8 reduces it to the two remaining cases for its dual. In each of these cases the metrics induced on $H^n(\mathsf{E})$, normalized as in (iii), satisfy (P), as recalled at the beginning of Section 6; this is (iii). For the mass, the mass of E_n differs from the mass of the representation $H^n(\mathsf{E})$ with its induced metrics by a quantity that tends to zero, by (i) and because the trace norm of a difference bounds the difference of the trace norms, and the latter mass tends to $\mathfrak{m}_{\mathrm{HN}}(H^n(\mathsf{E}))$ by Theorem 6.11(i). □

7.2. **Complexes of projective and of injective representations.** The indecomposable projective representations of K_m are the simple representation $S_1 = (0 \to \mathbb{C})$ at the vertex 1 and $P_0 = (\mathbb{C} \xrightarrow{e_j} \mathbb{C}^m)$, in which the j-th arrow maps 1 to the j-th basis vector; the indecomposable injective representations are the simple representation $S_0 = (\mathbb{C} \to 0)$ and $I_1 = (\mathbb{C}^m \xrightarrow{e_j^*} \mathbb{C})$. One has $\mathrm{Hom}(P_0, S_1) = 0$ and $\mathrm{Hom}(S_0, I_1) = 0$, the endomorphisms of S_1, P_0, S_0, I_1 are the scalars, a homomorphism $S_1^a \to P_0^b$ is zero at the vertex 0, and a homomorphism $I_1^a \to S_0^b$ is zero at the vertex 1. A bounded complex of projective, or of injective, representations is *minimal* if no component of its differential between indecomposable summands is an isomorphism.

Proposition 7.3. *For bounded complexes of representations of K_m:*

(a) *A complex of projective representations has $d_0 = 0$ if and only if its differential has no component $P_0 \to P_0$ that is an isomorphism. In particular a minimal one has $d_0 = 0$, and its differential has only components $S_1 \to P_0$.*

(b) *A complex of injective representations has $d_1 = 0$ if and only if its differential has no component $I_1 \to I_1$ that is an isomorphism. In particular a minimal one has $d_1 = 0$, and its differential has only components $I_1 \to S_0$.*

(c) *Every bounded complex of representations is quasi-isomorphic to a minimal bounded complex of projective representations, and to a minimal bounded complex of injective ones.*

Proof. (a) Write the terms as $P_0^{b_n} \oplus S_1^{a_n}$. The components $P_0 \to S_1$ vanish, the components $P_0 \to P_0$ and $S_1 \to S_1$ are scalars, and the components $S_1 \to P_0$ are zero at the vertex 0. The space at the vertex 0 is $\mathbb{C}^{b_n}$, and d_0 is the matrix of the components $P_0 \to P_0$. (b) is dual. (c) The category of representations of K_m is hereditary with enough projectives and injectives, so a bounded complex has bounded resolutions by finite-dimensional projectives and by injectives; and a bounded complex of projectives, or of injectives, is the direct sum of a minimal one and of a contractible one, obtained by splitting off the isomorphisms between indecomposable summands. □

By Theorem 7.1, Theorem 7.2 therefore applies to every bounded complex of projective representations with $d_0 = 0$, and of injective representations with $d_1 = 0$, with any metric; in particular to the minimal ones. Thus a minimal complex of projective representations falls under Theorem 7.1(b), with injective differentials at the vertex 1, and a minimal complex of injective representations under Theorem 7.1(a), with surjective differentials at the vertex 0. An example, which is not minimal unless all β_j vanish, is the standard projective resolution of a representation $\mathsf{R} = (\mathsf{R}_0 \xrightarrow{\beta_j} \mathsf{R}_1)$ with $\mathsf{R}_0 \neq 0$,

$$\big(0 \to \mathsf{R}_0 \otimes \mathbb{C}^m\big) \longrightarrow \big(\mathsf{R}_0 \xrightarrow{v \mapsto (v \otimes e_j,\, 0)} (\mathsf{R}_0 \otimes \mathbb{C}^m) \oplus \mathsf{R}_1\big), \qquad v \otimes e_j \mapsto (v \otimes e_j,\, -\beta_j v),$$

in degrees $-1, 0$: it has $d_0 = 0$ and an injective differential at the vertex 1, with an exact part of rank $m \dim \mathsf{R}_0$.

7.3. **The derived category of the projective line.** Let $m = 2$. By Beilinson's theorem [Bei78] the bounded derived category of representations of K_2 is equivalent to the bounded derived category $D^b(\mathbb{P}^1)$ of coherent sheaves on the projective line, by the functor $R\mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1), \cdot\,)$, which sends a sheaf $\mathcal{F}$ with $\mathrm{Ext}^1(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{F}) = 0$ to the representation $\mathrm{Hom}(\mathcal{O}(1), \mathcal{F}) \rightrightarrows \mathrm{Hom}(\mathcal{O}, \mathcal{F})$, the arrows being the compositions with the two coordinate sections of $\mathcal{O}(1)$. Under this equivalence the simple representations S_1 and S_0 correspond to $\mathcal{O}$

and $\mathcal{O}(-1)[1]$ [Oka06, §2.4]. More generally, for $l \geq 0$ let $\mathsf{L}_l = (\mathbb{C}[x]_{<l} \rightrightarrows \mathbb{C}[x]_{\leq l})$ be the representation on the polynomials of degree less than l and at most l, with the two arrows given by the inclusion and by multiplication by x, and $\mathsf{L}_l^\vee$ its dual, of dimension vector $(l+1, l)$, regarded as a representation of K_2 through the isomorphism of K_2 with its opposite quiver that exchanges the vertices; so $\mathsf{L}_0 = S_1$ and $\mathsf{L}_0^\vee = S_0$. Computing the functor on sheaves, with Serre duality for the second, gives

$$\mathsf{L}_l \leftrightarrow \mathcal{O}(l), \qquad \mathsf{L}_l^\vee \leftrightarrow \mathcal{O}(-l-1)[1] \qquad (l \geq 0),$$

and the representations of dimension vector $(1,1)$ with a nonzero arrow correspond to the skyscraper sheaves of the points of $\mathbb{P}^1$. A pair of charges z_0, z_1 in the upper half-plane is the central charge of a stability condition on this category with heart the category of representations (by [Bri07, Propositions 2.4 and 5.3], since every object of the heart has finite length). An object X is isomorphic to the direct sum of the $H^n(X)[-n]$, with $H^n(X)$ its cohomology representations, because the category of representations is hereditary. Let gr_n^k be the factors of the Harder–Narasimhan filtration of the representation $H^n(X)$, semistable of phases $\phi_{n,1} > \phi_{n,2} > \dots$ in $(0, \pi)$. The objects $\operatorname{gr}_n^k[-n]$ are semistable of the phases $\phi_{n,k} - \pi n$, which decrease strictly in the lexicographic order of (n,k); so they are the Harder–Narasimhan factors of X, and the mass of X for the stability condition is $\sum_n \mathfrak{m}_{\mathrm{HN}}(H^n(X)) = \sum_{n,k} |Z(\operatorname{gr}_n^k)|$. If every nonzero $H^n(X)$ is semistable this is $\sum_n |Z(H^n(X))|$.

Theorem 7.4. *Let X be an object of $D^b(\mathbb{P}^1)$, let E be a minimal bounded complex of projective representations, or of injective representations, of K_2 representing X, with arbitrary Hermitian metrics on its spaces at the two vertices, and let z_0, z_1 be charges in the upper half-plane. Then along the phase flow of E:*

(i) *the flow is the product of flows, one for each cohomological degree n, and the blocks of Ω in the summand of degree n approach $(-1)^n$ times those of the representation $H^n(X)$ with its induced metrics, exponentially in the trace norm;*

(ii) *for each n such that $H^n(X)$ is polystable, the metrics induced on $H^n(X)$, multiplied by $e^{(\varphi_n - \pi n)t}$ with $\varphi_n = \arg Z(H^n(X)) \in (0, \pi)$, converge exponentially to a harmonic metric of $H^n(X)$;*

(iii) *$\mathfrak{m}(h(t))$ converges to the mass of X for the stability condition with central charge (z_0, z_1) and heart the category of representations;*

(iv) *for each n with $H^n(X) \neq 0$ the metrics induced on $H^n(X)$ decay at the exponential rates $\phi_{n,k} - \pi n$ on the steps of the Harder–Narasimhan filtration of $H^n(X)$, as in Theorem 6.11(ii); and the metric induced on each Harder–Narasimhan factor $\operatorname{gr}_n^k[-n]$ of X, multiplied by $e^{(\phi_{n,k} - \pi n)t}$, is an asymptotic solution of the weighted gradient flow of the Kempf–Ness function of gr_n^k (Corollary 6.5), and converges exponentially to a harmonic metric if gr_n^k is polystable.*

Proof. Proposition 7.3 and Theorem 7.2. □

The hypothesis of (ii) depends on the cohomology representations. For strictly ordered charges the following representations of K_m are stable: the simple ones; those of dimension vector $(1,1)$ with a nonzero arrow; and, for $m = 2$ and $l \geq 1$, the representations L_l and $\mathsf{L}_l^\vee$. Indeed, the simple ones have no proper nonzero subrepresentation, and those of dimension vector $(1,1)$ have only $(0 \to \mathbb{C})$, of phase $\alpha_1 < \varphi$. For L_l the stability parameter has $\theta_0 = l+1$ and $\theta_1 = -l$ up to a positive factor, and a subrepresentation (V', W') with $V' \neq 0$ has $W' \supset V' + xV'$, of dimension at least $\dim V' + 1$, so that $(l+1)\dim V' - l \dim W' \leq \dim V' - l < 0$ unless $V' = \mathbb{C}[x]_{<l}$, in which case $W' = \mathbb{C}[x]_{\leq l}$; and duality preserves stability and strictly

ordered charges, the dual being regarded as a representation of K_2 as above. So, for $m = 2$ and strictly ordered charges, that is, when the phase of $\mathcal{O}(-1)[1]$ exceeds that of $\mathcal{O}$, every line bundle and every skyscraper sheaf on $\mathbb{P}^1$ is stable up to shift, whereas for $\alpha_0 \leq \alpha_1$ only $\mathcal{O}$ and $\mathcal{O}(-1)[1]$ are, among the objects of the heart (compare [Oka06]). Theorem 7.4(ii) then applies to every object whose cohomology representations are direct sums of representations of this list with the same argument of the central charge in each degree.

Remark 7.5. An example of a complex with nonzero differentials at both vertices, to which the theorems of this section do not apply, is the cone of the identity map of a representation with both spaces nonzero. When a cohomology representation is semistable and not polystable, its moment map converges to θ, but the induced metrics do not converge after any normalization by a scalar function of the time: the moment map is invariant under common rescalings, so a limit would be a zero of the shifted moment map, which does not exist (Lemma 5.3). Theorem 6.3 and Corollary 6.5 describe these metrics up to a bounded factor.

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