CONVERGENCE OF THE MINIMIZING FLOW ON KRONECKER QUIVER COMPLEXES

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Abstract. For a two-term complex $(V \to W) \to (U \to 0)$ of generalized Kronecker quiver representations, we prove exponential cancellation of its acyclic part under the polar-phase interpretation of the minimizing flow of Haiden, Katzarkov, Kontsevich and Pandit. Suppose the differential $D\colon V \to U$ is surjective, $U \neq 0$, and the cohomology $H = (\ker D \to W)$ is nonzero at both vertices. For all vertex charges in the upper half-plane and every initial Hermitian metric, the endomorphism defining the flow approaches that of H exponentially in trace norm, with the induced metric on H and extension by zero. If the source charge has larger argument than the target charge and H is polystable, the induced metric multiplied by $e^{t \arg Z(H)}$ converges exponentially to a harmonic metric, where $Z(H)$ is the central charge of the cohomology. In this chamber semistability suffices for the shifted moment map to tend to zero. The arrow-energy estimate uses the kernel of second divided differences of e^{-x} at the nodes $\ell_i, 0, \ell_j$ with $\ell_i > 0$. Its Schur product J with any positive semidefinite matrix satisfies $\operatorname{tr} J_- \leq \frac{1}{3} \operatorname{tr} J$, where J_- is the negative part. We also prove that, at fixed positive definite imaginary part, minus the polar phase is strictly monotone for the trace inner product.

1. Introduction

Haiden, Katzarkov, Kontsevich and Pandit [HKKP26] propose a notion of Kähler geometry on a triangulated category with a stability condition in the sense of Bridgeland [Bri07]. In their framework an object carries a space of metrics h and a complex-valued function $\mathcal{S}_{\mathbb{C}}$ on it with differential $\delta h \mapsto \operatorname{tr}(\Omega\, h^{-1} \delta h)$, for an endomorphism Ω whose trace is the central charge of the object. A *phase* $\arg \Omega$ is defined by a positivity condition, the *mass* is $m = \operatorname{tr} |\Omega|$, and the metrics move by the flow $h^{-1}\dot{h} = -\arg \Omega$, called the minimizing flow in [HKKP26, §4.1]. In [HKKP26, §§1, 4.1] the mass is expected to decrease along the flow, the flow to converge to a *harmonic* metric, one of constant phase, when the object admits one, and harmonic metrics to exist on polystable objects, a categorical form of the theorem of Donaldson, Uhlenbeck and Yau [Don85, UY86]. For a representation of a quiver, Ω is the central charge of each vertex shifted by the moment map, and the harmonic metrics are the zeros of the shifted moment map. For complexes of quiver representations Ω is defined through a superconnection in a graded algebra [HKKP26, §5.4], and it is asked there whether the mass decreases [HKKP26, Question 5.18].

The normality assertion in [HKKP26, Lemma 5.16], used there to define the phase by functional calculus, fails for complexes [Wue26a, Example 2.3]. Throughout this paper $\arg \Omega$

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is taken in the sense of the positivity condition of [HKKP26, §4.2], that is, as the argument of the unitary factor in the polar decomposition (Section 2). Every statement about the flow below, including the comparison with [HKKP26, Question 5.18], refers to this reading. We call the resulting flow the *phase flow*: it is the minimizing flow of [HKKP26, §4.1] with this reading of the phase.

The simplest complexes on which both a differential and the arrows of a quiver act are

$$(V \xrightarrow{(A_j)} W) \xrightarrow{(D,0)} (U \to 0), \tag{1}$$

two-term complexes of representations of the generalized Kronecker quiver with k arrows, with $D : V \to U$ surjective and $A_1, \dots, A_k : V \to W$. The cohomology is the representation $\ker D \to W$ in degree zero; it is a subcomplex, and the quotient $(V/\ker D \to 0) \to (U \to 0)$ is an acyclic complex of rank $r = \dim U$, the *acyclic part*. So (1) is quasi-isomorphic to a representation. For these complexes Ω was computed in [Wue26a], where it was shown that the mass can increase along the phase flow [Wue26a, Theorem 1.1]; that the contribution of the acyclic part to Ω nevertheless decays if the least singular value of D in the initial metrics is large [Wue26a, Theorem 1.3]; and that for one pair of charges and $\dim\ker D = \dim W$ the moment map of the cohomology representation then tends to zero if its Kempf–Ness function is bounded below [Wue26a, Theorem 1.4].

We prove that, for every initial metric, the contribution of the acyclic part decays, for all charges, and that, when the charge at the source vertex has the larger argument, the metrics induced on the cohomology converge, after a scalar normalization, to a harmonic metric of the cohomology representation whenever that representation is polystable. This removes the assumption on the initial metrics of [Wue26a, Theorems 1.3, 1.4]. The main results are Theorems 1.1 and 1.2.

1.1. **Results.** Let $h = (h_V, h_U, h_W)$ be Hermitian metrics on the three spaces of (1); adjoints, written †, are taken for the current metrics. Let z_0, z_1 be central charges in the upper half-plane, attached to the source vertex 0, which carries V and U, and to the target vertex 1, which carries W; let $n_0 = \dim\ker D$, $n_1 = \dim W$, $Z = n_0 z_0 + n_1 z_1$ and $\varphi = \arg Z \in (0, \pi)$. The pair (z_0, z_1) is the central charge of a stability condition on the bounded derived category of representations whose heart is the category of representations; the phase of a representation of dimension vector (a, b) is $\arg(az_0 + bz_1)$, which is π times the usual normalized phase. The endomorphism of [HKKP26, §5.4] is $\Omega = \Omega_V \oplus \Omega_U \oplus \Omega_W$; it is computed in (4) below. Put

$$L = DD^\dagger, \qquad \lambda = \lambda_{\min}(L), \qquad D^+ = D^\dagger(DD^\dagger)^{-1}, \qquad s = \sum_j \left\| A_j D^+ \right\|_{\mathrm{HS}}^2 .$$

For the metrics induced on $\ker D$ and W let H_j be the matrix of $A_j|_{\ker D}$ in orthonormal frames, $C_0 = \sum_j H_j^\dagger H_j$ and $C_1 = \sum_j H_j H_j^\dagger$. For the cohomology representation with these metrics, Ω is $(z_0 + C_0) \oplus (z_1 - C_1)$, a scalar standing for the corresponding multiple of the identity. We write $\|\cdot\|_1$ for the trace norm, and $(z_0 + C_0) \oplus 0$ for the endomorphism of $V = \ker D \oplus (\ker D)^\perp$ that is $z_0 + C_0$ on $\ker D$ and zero on the orthogonal complement.

Theorem 1.1 (Cancellation of the acyclic part)**.** *For all charges z_0, z_1 in the upper half-plane and every complex (1), along the phase flow $\lambda(t) \to \infty$, $s(t) \to 0$, and for $t \geq t_1$*

$$\|\Omega_V - (z_0 + C_0) \oplus 0\|_1 + \|\Omega_U\|_1 + \|\Omega_W - (z_1 - C_1)\|_1 \leq C_{\mathrm{can}}\, e^{-\frac{1}{4}(\arg z_1)(t - t_1)},$$

where C_{can} is an explicit function of the dimensions and the charges, and t_1 of these and of three numbers computed from the initial metrics. No hypothesis is made on the cohomology representation.

The charges are *ordered* if $\arg z_0 > \arg z_1$, that is, if the simple representation at the source vertex has the larger phase, and *reversed* if $\arg z_0 < \arg z_1$. For reversed charges no representation with $n_0, n_1 \geq 1$ is semistable. For ordered charges there are unique $\theta_0, \theta_1 > 0$ with $\arg(z_0 + \theta_0) = \arg(z_1 - \theta_1) = \varphi$; then $n_0\theta_0 = n_1\theta_1$, and a representation of dimension vector (n_0, n_1) is semistable, in the sense that no subrepresentation has phase larger than φ, exactly when it is semistable in the sense of King [Kin94] for the character $(-\theta_0, \theta_1)$. We call (θ_0, θ_1) the *stability parameter*. The *moment map* of the cohomology representation, shifted by the stability parameter, is $\mu = (C_0 - \theta_0,\, C_1 - \theta_1)$. Its zeros are the pairs of metrics for which the phases $\arg(z_0 + C_0)$ and $\arg(z_1 - C_1)$ at the two vertices both equal φ, that is, the harmonic metrics; they exist exactly for polystable representations [KN79, Kin94].

Theorem 1.2 (Convergence)**.** *Along the phase flow, for every initial metric:*

(i) *If the charges are ordered and the cohomology representation is semistable, then* $\mu(t) \to 0$ *and* $m(h(t)) \to |Z|$.

(ii) *If the charges are ordered and the cohomology representation is polystable, the induced metrics* $(e^{\varphi t}h_V|_{\ker D},\, e^{\varphi t}h_W)$ *converge exponentially, as Hermitian forms on the fixed vector spaces* $\ker D$ *and* W*, to a zero of the moment map, that is, to a harmonic metric of the cohomology representation; the moment map and* $m - |Z|$ *tend to zero exponentially.*

(iii) *If the charges are reversed, the arrows* H_j *tend to zero exponentially and* $m(h(t)) \to n_0|z_0| + n_1|z_1|$.

A harmonic metric of a representation is not stationary under the flow but moves by the scalar $e^{-\varphi t}$, which is why the limit is stated for the normalized metrics. The complex itself has no harmonic metric, since the phase on U is negative and the phases on V, W are positive, and its metrics do not converge even after normalization, because $\lambda \to \infty$; the limit exists for the metrics induced on the cohomology. In [HKKP26, Remark 5.15] the metrics of an object are the metrized complexes isomorphic to it in the derived category. The flow preserves the underlying complex, and Theorem 1.2 identifies its limit with a harmonic metric on another representative of the same object, the cohomology representation. The rate in (ii) is governed by the least eigenvalue of the Hessian of the Kempf–Ness function transversally to the zeros of the moment map and by the rate of Theorem 1.1. The precise statements are Theorems 5.7, 6.1, 6.6 and 6.11.

The proof of Theorem 1.1 does not use a Lyapunov function: the mass is not monotone, and the function $\mathcal{K}$ introduced before Theorem 1.5 is monotone for the charges of that theorem but not for all charges. It rests on an a priori bound for the following quantity, which is not monotone either, and on the growth of λ. Let $R(b) = (1 - e^{-b})/b$ and

$$N = \sum_j \operatorname{tr}\big(A_j R(D^\dagger D) A_j^\dagger\big),$$

the *arrow energy*; it is the term of $-\mathcal{S}_{\mathbb{C}}$, see (11), that does not contain the charges, and it bounds $\operatorname{tr} C_0$ and $(1 - e^{-\lambda})s$.

Theorem 1.3 (A priori bound)**.** *For all charges in the upper half-plane, along the phase flow*

$$\frac{dN}{dt} \leq -\rho N + n_1 M_1, \qquad \rho = \pi - \max\big\{\arg z_0, \tfrac{\pi}{3}\big\} > 0,$$

with $M_1 = \max\{\pi \max(\operatorname{Re} z_1, 0),\, 2 \operatorname{Im} z_1\}$*. Hence* $N(t) \leq \max\{N(0), n_1 M_1/\rho\}$ *for all* $t \geq 0$*, and* $N(t) \leq 2n_1 M_1/\rho$ *after an explicit time.*

The three terms of dN/dt come from the three blocks of Ω (Lemma 5.2). The block Ω_W is a normal matrix and its term is elementary. For the source block Ω_V, taking expectations in a polar-phase eigenbasis bounds its term by $\alpha_0 \operatorname{tr} K$, with K the Hermitian arrow term of that block. For the degree-one block Ω_U the required estimate is the trace inequality in (i) below. Part (ii) gives a comparison for arbitrary pairs of matrices with the same imaginary part.

Theorem 1.4 (Two matrix inequalities)**.** *The following hold.*

(i) *Let $L > 0$ and $E \geq 0$ be Hermitian matrices, and let $J_L(E)$ be the integral of $e^{-xL}Ee^{-yL}$ over the simplex $x, y \geq 0$, $x + y \leq 1$. Then the negative part of the Hermitian matrix $J_L(E)$ satisfies $\operatorname{tr} J_L(E)_- \leq \frac{1}{3} \operatorname{tr} J_L(E)$.*

(ii) *Let Ω', Ω'' be matrices with the same positive definite imaginary part and Φ', Φ'' the arguments of the unitary factors in their polar decompositions, with spectrum in $(0, \pi)$. Then $\operatorname{tr}\big((\Omega'' - \Omega')(\Phi'' - \Phi')\big) \leq 0$, with equality only if $\Omega' = \Omega''$.*

By the Hermite–Genocchi formula for divided differences, in an eigenbasis of L the matrix $J_L(E)$ is the entrywise product of E with the matrix of second divided differences of e^{-x} at the points $\ell_i, 0, \ell_k$; this matrix need not be positive semidefinite, and (i) says that the negative part of the product carries at most one third of the trace. Splitting the integration simplex at its midpoints writes the kernel as a difference of two positive semidefinite kernels. A diagonal bound on the second kernel proves (i). Statement (ii) says that minus the argument of the polar factor is a strictly monotone operator, in the sense of Minty [Min62], on the Hermitian matrices with the trace inner product; it is not monotone for the Löwner order, because the arctangent is not operator monotone on the real line (Remark 4.9). For a scalar imaginary part it is the fact that the argument of a complex number decreases in its real part. The proof uses an integral representation of anti-Löwner type [Aud11] (Section 4).

The mass is not monotone along the phase flow, also for the charges and complexes of Theorem 1.2: we give a complex with 512 arrows, stable cohomology and $r = 1$, with metrics at which the mass increases, for $z_0 = -1+\mathrm{i}$, $z_1 = 1+\mathrm{i}$ (Proposition 9.2). The largest phase can increase and the smallest phase can decrease (Proposition 9.3), whereas for representations the mass and the extreme phases are monotone, as expected in [HKKP26, §4.1] (Proposition 9.1). What converges for the complexes (1), for ordered charges and semistable cohomology, is the mass measure of [HKKP26, §4.2], recalled in Section 9 (Proposition 9.5).

For a range of charges there is a Lyapunov function. The *phase potential* is

$$\mathcal{K} = \frac{\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}})}{\sin\varphi},$$

with $\mathcal{S}_{\mathbb{C}}$ the function (11). We call $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$ the *symmetric charges*; for them and $n_0 = n_1$,

$$\mathcal{K}(h) = \log\det h_V - \log\det h_U - \log\det h_W + \operatorname{tr}\operatorname{Ein}(D^\dagger D) + N, \qquad \operatorname{Ein}(b) = \int_0^1 (1 - e^{-xb})\frac{dx}{x},$$

with the determinants of the Gram matrices in fixed reference bases.

Theorem 1.5 (Lyapunov function)**.** *Let the charges be ordered, and suppose that $\varphi \geq \frac{\pi}{3}$ or $r = 1$. Then the phase potential is strictly decreasing along the phase flow. For the symmetric charges and $n_0 = n_1$,*

$$-\frac{d\mathcal{K}}{dt} \;\geq\; \frac{\pi}{6}\operatorname{tr} J_L(DCD^\dagger) + \frac{\pi}{2}\operatorname{tr} e^{-L} > 0, \qquad C = \sum_j A_j^\dagger A_j.$$

There are ordered charges and complexes with $r = 2$, with metrics at which the phase potential increases along the flow.

The number $\frac{\pi}{3}$ is π times the constant of Theorem 1.4(i). The rate of the potential is a sum of three terms, one for each of V, W (degree zero) and U (degree one); the first two are nonnegative for elementary reasons, and the third is controlled by Theorem 1.4(i). See Theorem 7.6 and Propositions 10.1, 10.2.

Theorem 1.6 (Infimum and moduli)**.** *Let the charges be ordered, and let γ be Euler's constant.*

(i) *The potential depends on reference bases of V, U, W only through their volume elements. With volume elements on* $\ker D$ *and* W *induced from those of V, U, W through* $\det V \cong \det \ker D \otimes \det U$*, the infimum of the phase potential over all metrics of the complex is the infimum of the Kempf–Ness function (25) of its cohomology representation for the stability parameter (θ_0, θ_1), plus $\gamma\, \theta_0 r$; it is not attained. Consequently the infima for two quasi-isomorphic complexes of the class (1) with semistable cohomology and corresponding volume elements differ by $\gamma\, \theta_0 (r' - r)$.*

(ii) *On the stable locus of a holomorphic family of complexes, $\mathrm{i}\partial\bar\partial$ of the infimum is the pullback of the Kähler form of the moduli space of stable representations.*

(iii) *For semistable cohomology the potential along the flow tends to its infimum, exponentially and locally uniformly on the stable locus.*

Part (ii) follows from (i) and the classical description of the Kähler potential of a Kähler quotient (Lemma 8.4). See Theorems 8.2, 8.5 and 8.7.

1.2. **Remarks on the setting and the method.** The phase flow is not the gradient flow of the phase potential (Section 7). For a representation regarded as a complex in degree zero Ω is normal, the phase potential is the Kempf–Ness function, and its monotonicity along the flow follows from formulas displayed in [HKKP26, §§4.2, 5.1]. On a representation the flow has the same stationary points modulo scaling as the downward gradient flow of the Kempf–Ness function, and its velocity pairs negatively with the gradient, but it is not that gradient flow (Section 6).

The estimates for the polar decomposition that isolate one block of a matrix, and the differential inequality for s, are taken from [Wue26a] and recalled in Proposition 2.3. What is new in the proof of Theorem 1.1 is the control of the arrows and of λ before λ is large: Theorem 1.3 bounds the arrows, and a second differential inequality, for the trace of the exponential integral of L, forces λ to grow (Theorem 5.6). For the charges of Theorem 1.5 and semistable cohomology the Lyapunov function gives the same control with different constants (Proposition 7.10). Theorem 1.2 then follows from the convexity of the Kempf–Ness function; the relevant statements are classical [KN79, GRS21] and are proved for the present linear action in Section 6.

The following is not proved here. For semistable cohomology that is not polystable the induced metrics diverge (Proposition 6.8); the asymptotics are determined here only for the smallest such representation, regarded as a complex in degree zero, where the metrics have the form $t^{\mp 1/2}$ on a subrepresentation and the quotient (Proposition 6.9). In general the normalized induced metrics stay at bounded distance from a trajectory of the gradient flow of the Kempf–Ness function for the invariant Riemannian metric weighted by the imaginary parts of the charges [Wue26b, Theorem 6.3 and Corollary 6.5], whose asymptotics are described in [HKKP23]; whether the bounded factor converges is open beyond this example. The case $\arg z_0 = \arg z_1$ is covered by Theorems 1.1 and 1.3 but not by Theorem 1.2. Complexes

whose term of degree one sits at the target vertex, with acyclic parts at both vertices, or with terms in more than two degrees, are not treated. For cohomology that is not semistable, Theorems 1.1 and 1.3 apply, and the induced metrics follow the Harder–Narasimhan filtration of the cohomology representation [Wue26b, Theorem 6.11] (Remark 6.12).

1.3. **Organization.** Section 2 recalls the endomorphism Ω, the flow and the estimates used from [Wue26a]. Sections 3 and 4 prove Theorem 1.4. Section 5 proves Theorems 1.3 and 1.1, and Section 6 Theorem 1.2. Section 7 treats the phase potential and proves Theorem 1.5, and Section 8 proves Theorem 1.6. Section 9 treats the mass, the extreme phases and the fixed points, and Section 10 the charges for which the phase potential is not monotone, and open questions.

2. The endomorphism Ω, the flow and the function $\mathcal{S}_{\mathbb{C}}$

This section fixes notation and recalls what is used from [HKKP26] and [Wue26a]. Nothing in it is new. Throughout, $T^\dagger$ is the adjoint of a linear map for the Hermitian metrics in force (the conjugate transpose in orthonormal frames), T^* is the conjugate transpose of a matrix in fixed reference bases, and D^+ is the Moore–Penrose inverse. Inner products $\langle\cdot,\cdot\rangle$ and Hermitian forms are conjugate-linear in the first argument, and a scalar in a matrix expression denotes the corresponding multiple of the identity. For a square matrix T we write $\operatorname{Re} T = (T+T^\dagger)/2$ and $\operatorname{Im} T = (T-T^\dagger)/2\mathrm{i}$; $\|T\|_1$ is the trace norm, $\|T\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm and $\|T\|$ the operator norm; for a Hermitian matrix M, $M_\pm \geq 0$ are its positive and negative parts, $M = M_+ - M_-$, $M_+M_- = 0$. The letter φ always denotes the argument of the total central charge; ϕ with indices denotes phases of blocks and their eigenvalues. Likewise ϑ, $\varkappa$, ϱ, ς, ϖ denote objects different from those denoted by θ, κ, ρ, σ, π.

2.1. **Complexes and the endomorphism Ω.** Let V, U, W be finite-dimensional complex vector spaces, $D : V \to U$ a linear map and $A_1, \dots, A_k : V \to W$ linear maps. This is a two-term complex of representations of the generalized Kronecker quiver, with two vertices $0, 1$ and k arrows from 0 to 1:

$$\mathsf{E}: \qquad (V \xrightarrow{(A_j)} W) \xrightarrow{(D,0)} (U \to 0), \tag{2}$$

with V and W in degree 0 and U in degree 1. We assume throughout that

$$D \text{ is surjective}, \qquad n_0 := \dim\ker D \geq 1, \quad n_1 := \dim W \geq 1, \quad r := \dim U \geq 1.$$

The *cohomology representation* is $\ker D \to W$ with the arrows $A_j|_{\ker D}$, of dimension vector (n_0, n_1). It is a subcomplex of E, and the quotient $(V/\ker D \to 0) \to (U \to 0)$ is an acyclic complex, which we call the *acyclic part*; r is its rank. Fix *central charges* z_0, z_1 in the open upper half-plane, attached to the two vertices, and write

$$z_i = x_i + \mathrm{i}y_i = |z_i| e^{\mathrm{i}\alpha_i}, \qquad y_i > 0, \quad \alpha_i \in (0, \pi).$$

Let $h = (h_V, h_U, h_W)$ be Hermitian metrics. Adjoints refer to the current metrics. Put

$$B = D^\dagger D, \qquad L = DD^\dagger, \qquad C = \sum_j A_j^\dagger A_j,$$

and, for Hermitian matrices $P \geq 0$ and M,

$$R(P) = \int_0^1 e^{-xP}dx, \qquad K_P(M) = \int_0^1 e^{-(1-x)P} M e^{-xP} dx,$$
$$J_P(M) = \int_{\substack{x,y\geq 0\\ x+y\leq 1}} e^{-xP} M e^{-yP}\, dx\, dy. \tag{3}$$

In [HKKP26, §5.4] a complex of quiver representations with metrics carries an endomorphism Ω of its underlying graded vector space, defined through a superconnection in a graded algebra attached to the quiver: $\Omega = \varepsilon\,[\exp(-\sqrt{-1}F)]_f$, where $F = (A-A^*)^2 + \sqrt{-1}\sum_i f_i \otimes z_i$ is formed from the element $A = \sum_i e_i \otimes d_i + \sum_a a \otimes \rho(a)$ of the algebra of [HKKP26, Definition 5.13] tensored with the endomorphisms, with d_i the differentials and $\rho(a)$ the maps of the arrows, ε is the grading, and $[\,\cdot\,]_f$ is the component along the basis elements f_i. Apart from Ω and z_i, the symbols of this sentence are those of [HKKP26] and have this meaning only here; A^* is formed from the involution of that algebra and the adjoints for the metrics [HKKP26, (5.14)]. The differential of the function $\mathcal{S}_{\mathbb{C}}$ below is $\xi \mapsto \operatorname{tr}(\Omega\xi)$; the trace of Ω is the central charge (7), and for a representation Ω is the central charge of each vertex shifted by the moment map. For (2) it is $\Omega = \Omega_V \oplus \Omega_U \oplus \Omega_W$ with [Wue26a, Proposition 2.1]

$$\Omega_V = z_0 e^{-B} + K_B(C), \qquad \Omega_U = J_L(DCD^\dagger) - z_0 e^{-L}, \qquad \Omega_W = z_1 I - \sum_j A_j R(B) A_j^\dagger. \tag{4}$$

We abbreviate

$$K = K_B(C), \qquad J = J_L(DCD^\dagger), \qquad G = \sum_j A_j R(B) A_j^\dagger \geq 0, \tag{5}$$

three Hermitian matrices that do not depend on the charges. Hence

$$\operatorname{Im}\Omega_V = y_0 e^{-B} > 0, \qquad \operatorname{Im}(-\Omega_U) = y_0 e^{-L} > 0, \qquad \operatorname{Im}\Omega_W = y_1 I > 0, \tag{6}$$

and [HKKP26, Lemma 5.17]

$$\operatorname{tr}\Omega_V + \operatorname{tr}\Omega_U + \operatorname{tr}\Omega_W = n_0 z_0 + n_1 z_1 =: Z. \tag{7}$$

The matrix Ω_W is normal. The matrix Ω_V is in general not normal [Wue26a, Example 2.3], and neither is Ω_U, whose Hermitian and anti-Hermitian parts $J - x_0 e^{-L}$ and $-y_0 e^{-L}$ commute only if J commutes with e^{-L}.

2.2. **The phase and the flow.** For a matrix T with $\operatorname{Im} T > 0$ there is a unique Hermitian matrix Ψ with spectrum in $(0,\pi)$ such that $Te^{-\mathrm{i}\Psi} > 0$, the *polar phase* of T; it is the argument of the unitary factor in the polar decomposition $T = (TT^\dagger)^{1/2}e^{\mathrm{i}\Psi}$ [HJ13, Section 7.3]: if $e^{\mathrm{i}\Psi}v = \zeta v$, then $\langle v, (\operatorname{Im} T)v\rangle = (\operatorname{Im}\zeta)\,\langle v, |T|v\rangle$, so the eigenvalues of the unitary factor lie in the open upper half-circle (see also [Wue26a, Lemma 2.2]). Here $|T| := (TT^\dagger)^{1/2}$ is the left modulus; the unitary factor is the same in the left and right polar decompositions, and so is the trace norm (in [Bha97] the modulus is the right one, $(T^\dagger T)^{1/2}$). The phases Φ_V, Φ_W are the polar phases of Ω_V, Ω_W, and $\Phi_U + \pi$ is the polar phase of $-\Omega_U$; so

$$|\Omega_\bullet| := \Omega_\bullet e^{-\mathrm{i}\Phi_\bullet} = (\Omega_\bullet \Omega_\bullet^\dagger)^{1/2} > 0, \qquad \bullet \in \{V, U, W\}, \tag{8}$$

with $\operatorname{spec}\Phi_V, \operatorname{spec}\Phi_W \subset (0,\pi)$ and $\operatorname{spec}\Phi_U \subset (-\pi, 0)$. This is the phase defined by positivity in [HKKP26, §4.2], with the branches of [HKKP26, (5.15)]. In [HKKP26, §5.4] the phase of Ω for a complex is written with the functional calculus. For normal Ω, in particular for

representations, the two readings coincide; for complexes Ω is not normal in general [Wue26a, Example 2.3]. The flow studied here is the one defined with the polar phase:

$$h_\bullet^{-1}\dot h_\bullet = -\Phi_\bullet, \qquad \bullet \in \{V,U,W\}. \tag{9}$$

We call it the *phase flow*. It is the minimizing flow of [HKKP26, §4.1] with the phase read as above; we use a neutral name because on complexes the mass can increase along it [Wue26a, Theorem 1.1]. The *mass* is $m(h) = \sum_\bullet \operatorname{tr}|\Omega_\bullet| \geq |Z|$. The vector field is smooth and its solutions exist for all $t \geq 0$ [Wue26a, Lemma 2.4]. In orthonormal frames moving with the metrics the maps evolve by

$$\dot D = \tfrac12(D\Phi_V - \Phi_U D), \qquad \dot A_j = \tfrac12(A_j\Phi_V - \Phi_W A_j). \tag{10}$$

A *state* is a tuple (D, A_j) of matrices in orthonormal frames, up to unitary changes of the frames; equivalently, a triple of metrics on the fixed complex up to isometry, or a point of the orbit of (D, A_j) under $\mathrm{GL}(V)\times\mathrm{GL}(U)\times\mathrm{GL}(W)$ modulo the unitary groups. The phases, mass and arrow energy are invariant under these changes of frames. The primitive below also uses fixed reference volume elements.

Lemma 2.1. *Let T be a matrix with $\operatorname{Im} T > 0$ and polar phase Ψ, and let v be a unit vector with $\Psi v = \psi v$. Then $\langle v, Tv\rangle = p\,e^{\mathrm{i}\psi}$ with $p = \langle v, |T|v\rangle > 0$. Consequently, in an orthonormal eigenbasis (v_k) of Ψ with eigenvalues ψ_k and $p_k = \langle v_k, |T|v_k\rangle$, $\operatorname{tr}(|T|f(\Psi)) = \sum_k p_k f(\psi_k)$ for every real function f.*

Proof. $T = |T|e^{\mathrm{i}\Psi}$, so $\langle v, Tv\rangle = e^{\mathrm{i}\psi}\,\langle v, |T|v\rangle$. The second statement is the trace computed in the basis (v_k). □

The lemma applies to Ω_V, Ω_W and $-\Omega_U$; for the block U it gives $\langle v, \Omega_U v\rangle = p\,e^{\mathrm{i}\phi}$ for a unit eigenvector v of Φ_U with eigenvalue $\phi \in (-\pi, 0)$.

2.3. **The primitive.** Let $\mathrm{Ein}(b) = \int_0^1 (1 - e^{-xb})\,dx/x$. With respect to fixed reference bases, in which $\log\det h_\bullet$ denotes the logarithm of the Gram determinant, put

$$\mathcal{S}_{\mathbb{C}}(h) = z_0\big(\log\det h_V - \log\det h_U\big) + z_1 \log\det h_W + z_0 \operatorname{tr}\mathrm{Ein}(B) - \operatorname{tr}\big(CR(B)\big). \tag{11}$$

Then [Wue26a, Proposition 7.1], for every variation $\xi_\bullet = h_\bullet^{-1}\delta h_\bullet$ of the metrics,

$$d\mathcal{S}_{\mathbb{C}}(\xi) = \sum_\bullet \operatorname{tr}(\Omega_\bullet \xi_\bullet). \tag{12}$$

Both sides are affine in (z_0, z_1), so the terms without charges agree:

$$d\big(\operatorname{tr}(CR(B))\big)(\xi) = -\operatorname{tr}(K\xi_V) - \operatorname{tr}(J\xi_U) + \operatorname{tr}(G\xi_W). \tag{13}$$

2.4. **Charges and the stability parameter.** Put

$$\varphi = \arg Z \in (0,\pi), \quad \sigma = \sin\varphi, \quad c_0 = |z_0|\sin(\alpha_0 - \varphi), \quad c_1 = |z_1|\sin(\varphi - \alpha_1).$$

Since $\operatorname{Im}(e^{-\mathrm{i}\varphi}Z) = 0$ we have $n_0c_0 = n_1c_1$. The charges are *ordered* if $\alpha_0 > \alpha_1$ and *reversed* if $\alpha_0 < \alpha_1$; they are ordered exactly when $c_0 > 0$, or when $\alpha_1 < \varphi < \alpha_0$. For ordered charges put

$$\theta_0 = c_0/\sigma, \qquad \theta_1 = c_1/\sigma, \qquad \vartheta = n_0\theta_0 = n_1\theta_1 > 0; \tag{14}$$

then $z_0 + \theta_0$ and $z_1 - \theta_1$ have argument φ. We call the pair (θ_0, θ_1) the *stability parameter*. A representation whose dimension vector (a_0, a_1) satisfies $a_0\theta_0 = a_1\theta_1$, for instance (n_0, n_1), has phase φ. It is *semistable* if $a\theta_0 - b\theta_1 \leq 0$ for the dimension vector (a, b) of each of its subrepresentations, that is, if no subrepresentation has phase $\arg(az_0 + bz_1)$ larger

than φ; it is *stable* if the inequality is strict for proper nonzero subrepresentations, and *polystable* if it is a direct sum of stable representations of phase φ. In the convention of King [Kin94, Definition 1.1] this is stability for the character $(-\theta_0, \theta_1)$, a positive multiple of the integral character $(-n_1, n_0)$; the Kempf–Ness function (25) carries the weights $(\theta_0, -\theta_1)$ on $(\log\det h_0, \log\det h_1)$. For reversed charges no representation with $n_0, n_1 \geq 1$ is semistable, because the subrepresentation supported at the vertex 1 has phase $\alpha_1 > \varphi$.

Definition 2.2. Let (n_0, n_1, r) be given. The charges (z_0, z_1) are *admissible* if they are ordered and either $\varphi \geq \frac{\pi}{3}$ or $r = 1$.

This is the range of charges covered by Theorem 7.6; the number $\frac{\pi}{3}$ comes from the constant of Theorem 3.1.

The *symmetric charges* are $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$. If moreover $n_0 = n_1$, then $\varphi = \frac{\pi}{2}$, $\sigma = 1$, $\theta_0 = \theta_1 = 1$, and they are admissible for every r.

2.5. **The splitting adapted to** $\ker D$**.** Choose orthonormal frames adapted to $V = \ker D \oplus (\ker D)^{\perp}$ and write

$$D = [0, F], \qquad A_j = [H_j, T_j], \qquad \Phi_V = \begin{pmatrix} \Phi_{HH} & \Phi_{HA} \\ \Phi_{AH} & \Phi_{AA} \end{pmatrix}.$$

Here the indices H and A refer to $\ker D$ and to its orthogonal complement, which carries the acyclic part. So F is invertible and the H_j are the matrices of the arrows of the cohomology representation for the induced metrics. Let $D^{+} = D^{\dagger}(DD^{\dagger})^{-1}$ be the Moore–Penrose inverse of D and

$$X_j = A_j D^{+} = T_j F^{-1} : U \to W.$$

These maps vanish exactly when the arrows vanish on $(\ker D)^{\perp}$. Put

(15)
$$C_0 = \sum_j H_j^{\dagger} H_j, \quad C_1 = \sum_j H_j H_j^{\dagger}, \quad q = \operatorname{tr} C_0 = \operatorname{tr} C_1, \quad s = \sum_j \|X_j\|_{\mathrm{HS}}^2, \quad \lambda = \lambda_{\min}(L),$$

let $\ell_1 \leq \cdots \leq \ell_r$ be the eigenvalues of $L = FF^{\dagger}$, so $\lambda = \ell_1$, and

$$Y = \sum_j X_j (I - e^{-L}) X_j^{\dagger} \geq 0, \qquad G = C_1 + Y, \qquad \operatorname{tr} Y \geq (1 - e^{-\lambda})\, s. \tag{16}$$

With the choice of the moving frames made in [Wue26a, §6.1] (the frames of V are rotated so that the splitting is preserved),

$$\dot F = \tfrac{1}{2}(F\Phi_{AA} - \Phi_U F), \qquad \dot H_j = \tfrac{1}{2}(H_j \Phi_{HH} - \Phi_W H_j), \tag{17}$$

and

$$\Omega_V = \begin{pmatrix} z_0 I + C_0 & E_R \\ E_R^{\dagger} & z_0 e^{-F^{\dagger}F} + K_{F^{\dagger}F}(C_T) \end{pmatrix}, \qquad \Omega_W = z_1 I - C_1 - Y, \tag{18}$$

with $C_T = \sum_j T_j^{\dagger} T_j$ and $E_R = \sum_j H_j^{\dagger} T_j R(F^{\dagger}F)$. For the cohomology representation with its induced metrics the endomorphism Ω is

$$\Omega_H = (z_0 I + C_0) \oplus (z_1 I - C_1);$$

it is normal, with phases the Hermitian matrices $\phi_0 = \arg(z_0 I + C_0)$ and $\phi_1 = \arg(z_1 I - C_1)$, and mass $m_H = \|z_0 I + C_0\|_1 + \|z_1 I - C_1\|_1$. We write

$$\delta_0 = \Phi_{HH} - \phi_0, \qquad \delta_1 = \Phi_W - \phi_1, \qquad \mathcal{E}_V = \Omega_V - \operatorname{diag}\bigl(z_0 I + C_0,\ z_0 e^{-F^{\dagger}F}\bigr).$$

We say that *the acyclic part cancels* along a trajectory if $\lambda \to \infty$ and $\|\Omega_V - (z_0 I + C_0) \oplus 0\|_1 + \|\Omega_U\|_1 + \|\Omega_W - (z_1 I - C_1)\|_1 \to 0$. The name refers to the central charge, to which the two terms of the acyclic part contribute $z_0 r$ and $-z_0 r$; the statement is that their contributions to Ω itself tend to zero.

For ordered charges we write $\mu = (\mu_0, \mu_1) = (C_0 - \theta_0 I,\ C_1 - \theta_1 I)$ and call it the *moment map* of the cohomology representation. The pair $(-\mu_0, \mu_1)$ is the gradient of the Kempf–Ness function (Lemma 6.2), that is, up to a constant factor the moment map of $\mathrm{U}(n_0) \times \mathrm{U}(n_1)$ in the usual convention, shifted by King's character; we change the sign of the first component so that both components read $C_i - \theta_i$. A pair of metrics on $\ker D$ and W with $\mu = 0$ is a *zero of the moment map*; such pairs exist exactly when the cohomology representation is polystable [Kin94, Proposition 6.5], which extends the theorem of Kempf and Ness [KN79], and they are the harmonic metrics of [HKKP26] for a representation, since $\mu = 0$ if and only if both phases ϕ_0, ϕ_1 equal φ.

2.6. **Estimates for $A_j D^+$ and for the phases.** We use the following facts from [Wue26a], valid for all charges in the upper half-plane and all dimensions. Put $\kappa_0 = \pi/(2e)$ and $d_0 = \min(n_0, r)$.

Proposition 2.3 ([Wue26a])**.** *Along the phase flow and at every metric:*

(i) *λ is nondecreasing along the flow. By the formula for the one-sided derivative of the least eigenvalue of a differentiable family of Hermitian matrices, its right derivative is the minimum of $\lambda\big(\langle v, \Phi_{AA} v\rangle - \langle u, \Phi_U u\rangle\big)$ over the unit vectors v with $F^\dagger F v = \lambda v$, where $u = Fv/\sqrt{\lambda}$* [Wue26a, Lemma 4.3 and §6.1].

(ii) *$\|E_R\|_{\mathrm{HS}} \le \sqrt{qs/\lambda}$, $\|K_{F^\dagger F}(C_T)\|_1 \le r\kappa_0 s$, $\|J\|_1 \le rs$ and $\|Y\|_1 \le s$; hence $\|\mathcal{E}_V\|_1 \le 2\sqrt{d_0 qs/\lambda} + r\kappa_0 s$* [Wue26a, Lemma 4.2, §6.1 and §6.3].

(iii) *$\dot s \le \big(-\alpha_1 + \frac{4\pi q}{y_0 \lambda}\big) s$* [Wue26a, §6.2].

(iv) *If $\sqrt{qs/\lambda} \le y_0/8$, then $\|\delta_0\|_{\mathrm{HS}} \le \frac{4\pi}{y_0}\sqrt{qs/\lambda}$; and $\|\delta_1\|_{\mathrm{HS}} \le s/y_1$ always* [Wue26a, Lemma 3.1 and §6.2].

(v) *Let $T_\circ = |T_\circ|\, e^{\mathrm{i}\Psi_\circ}$ be an invertible matrix in polar form, with $\Psi_\circ$ Hermitian, and $T = T_\circ + M$ a matrix with $\operatorname{Im} T > 0$ and polar phase Ψ. If v is a unit vector with $|T_\circ| v = pv$ and $\Psi_\circ v = \psi v$, where $p > 0$ and $0 < \psi < \pi$, then $|\langle v, (\Psi - \psi) v\rangle| \le \pi \sqrt{\|M\|_1 / p}$* [Wue26a, Lemma 3.2].

In [Wue26a] these estimates were used when the initial value of λ is large: then the acyclic part cancels, for all charges [Wue26a, Theorem 1.3], and for the symmetric charges the moment map of the cohomology representation tends to zero if its Kempf–Ness function is bounded below [Wue26a, Theorem 1.4]. Whether the assumption on the initial value of λ can be removed was left open there.

3. A trace inequality for a Schur multiplier

For a Hermitian $r \times r$ matrix $L > 0$ and a Hermitian matrix $E \ge 0$ put

$$J_L(E) = \int_{\substack{x,y \ge 0 \\ x+y \le 1}} e^{-xL} E e^{-yL}\, dx\, dy,$$

as in (3). This Hermitian matrix is in general indefinite (Remark 3.4). Recall that $M_\pm \ge 0$ denote the positive and negative parts of a Hermitian matrix $M = M_+ - M_-$, $M_+ M_- = 0$.

Theorem 3.1 (Trace inequality)**.** *For all Hermitian $L > 0$ and $E \geq 0$,*

$$\operatorname{tr} J_L(E)_- \leq \frac{1}{3} \operatorname{tr} J_L(E).$$

In particular $\operatorname{tr} J_L(E) \geq 0$*, and* $\operatorname{tr} J_L(E)_- \leq \frac{1}{4} \operatorname{tr} J_L(E)_+$.

Diagonalize $L = \operatorname{diag}(\ell_1, \dots, \ell_r)$ with $\ell_i > 0$. Then $J_L(E)_{ik} = \mathsf{j}(\ell_i, \ell_k) E_{ik}$, where

$$\mathsf{j}(a, b) = \int_{\substack{x,y \geq 0 \\ x+y \leq 1}} e^{-xa-yb}\, dx\, dy.$$

Put $S = L^{-1} E L^{-1} \geq 0$ and $\mathsf{m}(a, b) = ab\, \mathsf{j}(a, b)$. Then

$$J_L(E) = S \circ \mathsf{M}, \qquad \mathsf{M} = [\mathsf{m}(\ell_i, \ell_k)]_{i,k}, \qquad \operatorname{tr} J_L(E) = \sum_i \mathsf{m}(\ell_i, \ell_i) S_{ii}, \tag{19}$$

where $\circ$ denotes the entrywise (Schur) product. By the Hermite–Genocchi formula $\mathsf{j}(a, b)$ is the second divided difference of e^{-x} at the points $a, 0, b$. By the recursion for divided differences it equals $-(R(a) - R(b))/(a - b)$ for $a \neq b$, with $R(x) = (1 - e^{-x})/x$, so that $J_L(E)$ is minus the Schur product of E with the Löwner matrix of R at the eigenvalues of L. The function $-R$ is increasing but, being entire and not affine, it is not operator monotone; so by Löwner's theorem [Bha97] these matrices are not all positive semidefinite. Theorem 3.1 bounds the negative part of the Schur product in trace.

Lemma 3.2. *Let $\eta(a) = 1 - e^{-a}$. For $a, b > 0$, $a \neq b$,*

$$\mathsf{m}(a, b) = \frac{b\,\eta(a) - a\,\eta(b)}{b - a}, \qquad \mathsf{m}(a, a) = 1 - (1 + a)e^{-a} > 0.$$

Proof. Integrating in y gives $\mathsf{j}(a, b) = b^{-1} \int_0^1 e^{-xa}(1 - e^{-(1-x)b})\, dx = b^{-1}[\eta(a)/a - (e^{-a} - e^{-b})/(b - a)]$. Multiply by ab and use $e^{-a} - e^{-b} = \eta(b) - \eta(a)$. The diagonal value is the limit $b \to a$, and it is positive because its derivative is ae^{-a}. □

Proof of Theorem 3.1. For $a > 0$ and $s \in I = [0, 1/2]$, put

$$F_a(s) = \int_0^s e^{-ax}\, dx, \qquad A_a = F_a(1/2), \qquad V_a(s) = e^{-a(1-s)}.$$

Splitting the triangle $x + y \leq 1$ at $x = 1/2$ and $y = 1/2$ gives

$$\mathsf{j}(a, b) = A_a A_b + \int_I (F_a V_b + V_a F_b)\, ds.$$

Write $\mathsf{j} = P - Q$, where

$$P(a, b) = A_a A_b + \tfrac{2}{3} \int_I (F_a + \tfrac{3}{4} V_a)(F_b + \tfrac{3}{4} V_b)\, ds,$$

$$Q(a, b) = \tfrac{2}{3} \int_I (F_a - \tfrac{3}{4} V_a)(F_b - \tfrac{3}{4} V_b)\, ds.$$

Both kernels are positive semidefinite. Fix a and suppress the subscript. Since $s \leq 1 - s$, we have $sV(s) \leq F(s) \leq A$. Moreover V^2 is increasing on I, so $\int_I sV^2 \geq \frac{1}{4} \int_I V^2$ by the integral Chebyshev inequality. Consequently

$$Q(a, a) = \tfrac{2}{3} \int_I F^2 - \int_I FV + \tfrac{3}{8} \int_I V^2 \leq \tfrac{1}{3} A^2 + \tfrac{1}{2} \int_I FV \leq \tfrac{1}{3} \mathsf{j}(a, a).$$

Evaluate P, Q at the eigenvalues of L. By the Schur product theorem, $E \circ P$ and $E \circ Q$ are positive semidefinite. For the negative spectral projection Π of $J = E \circ P - E \circ Q$, this gives

$$\operatorname{tr} J_- = -\operatorname{tr}(\Pi J) \le \operatorname{tr}(E \circ Q) \le \tfrac{1}{3} \sum_i E_{ii} \mathsf{j}(\ell_i, \ell_i) = \tfrac{1}{3} \operatorname{tr} J.$$

The second bound follows on substituting $\operatorname{tr} J = \operatorname{tr} J_+ - \operatorname{tr} J_-$. □

Remark 3.3. Every constant valid in Theorem 3.1 is at least $(\sqrt{2}-1)/2$. In (19) take $L = \operatorname{diag}(a, b)$ and $S = ww^\dagger$ with $w = (a^{-1}, 2^{-1/2})$. First let $b \to \infty$, then $a \to 0$. Lemma 3.2 gives a limiting matrix with diagonal entries $1/2$ and off-diagonal entries $2^{-1/2}$. Its eigenvalues are $1/2 \pm 2^{-1/2}$ and its trace is one.

Remark 3.4. For every n some matrix $[\mathsf{m}(\ell_i, \ell_j)]_{i,j}$ has at least n negative eigenvalues. It is the Gram matrix on the functions $\ell_i e^{-\ell_i x}$ of the form $\langle f, Tf \rangle$ in $L^2[0,1]$, where $(Tf)(x) = \int_0^{1-x} f(y)\, dy$. For $\omega_k = (k+1/2)\pi$, $T \cos(\omega_k x) = (-1)^k \cos(\omega_k x)/\omega_k$. The span of e^{-ax}, $a > 0$, is dense by the Stone–Weierstrass theorem, since its uniform closure contains the constants. Approximating n orthogonal negative eigenfunctions in this span gives the assertion.

4. Trace monotonicity of the polar phase

This section is independent of the rest of the paper. Let Ω be a complex $n \times n$ matrix with positive definite imaginary part $\operatorname{Im} \Omega = (\Omega - \Omega^\dagger)/2\mathrm{i}$, where † is the conjugate transpose. Then Ω is invertible, and the unitary factor W in the polar decompositions $\Omega = (\Omega\Omega^\dagger)^{1/2}\mathsf{W} = \mathsf{W}(\Omega^\dagger\Omega)^{1/2}$ has its spectrum in the open upper half of the unit circle. The *polar phase* of Ω is the Hermitian matrix Φ with spectrum in $(0, \pi)$ and $\mathsf{W} = e^{\mathrm{i}\Phi}$, and $|\Omega| = (\Omega\Omega^\dagger)^{1/2} = \Omega e^{-\mathrm{i}\Phi}$ is the left modulus. The set of matrices with a prescribed positive definite imaginary part is an affine space over the Hermitian matrices, and on it the polar phase is a real-analytic map: the unitary polar factor of an invertible matrix depends analytically on it, and Φ is $-\mathrm{i}$ times its principal logarithm.

Theorem 4.1 (Monotonicity of the polar phase)**.** *Let Ω', Ω'' be matrices with the same positive definite imaginary part, and let Φ', Φ'' be their polar phases. Then*

$$\operatorname{tr}\big((\Omega'' - \Omega')(\Phi'' - \Phi')\big) \le 0,$$

with equality only if $\Omega' = \Omega''$. More precisely, for every smooth family $\Omega(\varepsilon)$ with constant positive definite imaginary part, $\operatorname{tr}(\dot\Omega\, \dot\Phi) < 0$ wherever $\dot\Omega \neq 0$.

In other words, $\operatorname{Re}\Omega \mapsto -\Phi$ is a strictly monotone operator [Min62] on the real Hilbert space of Hermitian matrices with the trace inner product. For $n = 1$ this says that $\arg(x + \mathrm{i}y)$ decreases in x when $y > 0$, and for a scalar imaginary part it follows from that by the functional calculus. In general Ω is not normal and Φ is not a function of its eigenvalues. The derivative of the unitary polar factor solves a Sylvester equation, which is used in the perturbation theory of the polar decomposition [Bha97]; we are not aware of a statement on the sign of the derivative of its argument. The phases of a sectorial matrix in the sense of [WCKQ20] are defined through a congruence and are a different object.

Corollary 4.2. *Let $\operatorname{Im}\Omega^{(0)} > 0$, let K be Hermitian, and let $\Phi^{(0)}$ and Φ be the polar phases of $\Omega^{(0)}$ and $\Omega^{(0)} + K$. Then $\operatorname{tr}(K\Phi) \le \operatorname{tr}(K\Phi^{(0)})$. In particular, if $\Omega^{(0)} = zP$ with $\operatorname{Im} z > 0$ and $P > 0$, then $\operatorname{tr}(K\Phi) \le (\arg z) \operatorname{tr} K$.*

Proof. Theorem 4.1 with $\Omega' = \Omega^{(0)}$ and $\Omega'' = \Omega^{(0)} + K$. The polar phase of $zP = (|z|P)e^{\mathrm{i}\arg z}$ is the scalar $\arg z$. □

Corollary 4.3. *Let $\mathbb{C}^n = H \oplus H'$ be an orthogonal decomposition and*

$$\Omega = \begin{pmatrix} \Omega_{11} & K_{12} \\ K_{12}^\dagger & \Omega_{22} \end{pmatrix}, \qquad \operatorname{Im}\Omega > 0 \text{ block diagonal},$$

with polar phase $\Phi = (\Phi_{ik})$. Then $\operatorname{Re}\operatorname{tr}(K_{12}\,\Phi_{21}) \le 0$, with equality only if $K_{12} = 0$.

Proof. Let K be the Hermitian matrix with off-diagonal blocks $K_{12}, K_{12}^\dagger$ and zero diagonal blocks, $\Omega(\varepsilon) = \Omega - (1-\varepsilon)K$ and $\Sigma = \operatorname{diag}(I, -I)$. Then $\operatorname{Im}\Omega(\varepsilon) = \operatorname{Im}\Omega$, $\Sigma\Omega(\varepsilon)\Sigma = \Omega(-\varepsilon)$, and the polar phase is equivariant under unitary conjugation; so $f(\varepsilon) = \operatorname{tr}(K\Phi(\varepsilon))$ is odd, $f(0) = 0$, and $f' < 0$ for $K \neq 0$ by Theorem 4.1. Finally $f(1) = 2\operatorname{Re}\operatorname{tr}(K_{12}\Phi_{21})$. □

The rest of this section proves Theorem 4.1. The pairing $\operatorname{tr}(\dot\Omega\dot\Phi)$ is expressed through the solution of a Sylvester equation, and the resulting quadratic form is a sum of forms with matrices $(\operatorname{Im}\Omega)\circ \mathsf{L}^{(a)}$, where the matrices $\mathsf{L}^{(a)}$ are positive semidefinite by an integral representation of the kind that characterizes anti-Löwner functions [Aud11].

4.1. **The first variation of the phase.** Put

$$\tilde P = e^{-\mathrm{i}\Phi/2}\,|\Omega|\,e^{\mathrm{i}\Phi/2} > 0, \qquad \text{so that} \qquad \Omega = e^{\mathrm{i}\Phi/2}\,\tilde P\,e^{\mathrm{i}\Phi/2}.$$

We work in an orthonormal eigenbasis of Φ, $\Phi = \operatorname{diag}(\phi_a)$ with $\phi_a \in (0,\pi)$. Then $\Omega_{ab} = e^{\mathrm{i}(\phi_a+\phi_b)/2}\tilde P_{ab}$, and since $\tilde P$ is Hermitian,

$$(\operatorname{Re}\Omega)_{ab} = \tilde P_{ab}\cos\tfrac{\phi_a+\phi_b}{2}, \qquad (\operatorname{Im}\Omega)_{ab} = \tilde P_{ab}\,\varsigma_{ab}, \qquad \varsigma_{ab} := \sin\tfrac{\phi_a+\phi_b}{2} > 0. \tag{20}$$

Lemma 4.4. *Let $\Omega(\varepsilon)$ be a smooth family with $\Omega(0) = \Omega$, $\operatorname{Im}\Omega(\varepsilon) > 0$, and derivative $\dot\Omega$ at 0. There is a Hermitian matrix Θ with*

$$\tilde P\Theta + \Theta\tilde P = \frac{X - X^\dagger}{2\mathrm{i}}, \qquad X = e^{-\mathrm{i}\Phi/2}\,\dot\Omega\,e^{-\mathrm{i}\Phi/2}, \tag{21}$$

and, in the eigenbasis of Φ,

$$\dot\Phi_{ab} = \frac{\phi_a - \phi_b}{\sin\frac{\phi_a-\phi_b}{2}}\,\Theta_{ab}, \tag{22}$$

where the factor is read as 2 if $\phi_a = \phi_b$. If $\dot\Omega$ is Hermitian, the right side of (21) has entries $-\dot\Omega_{ab}\,\varsigma_{ab}$.

Proof. Let $\mathsf{U}(\varepsilon) = e^{\mathrm{i}\Phi(\varepsilon)/2}$, so that $\Omega = \mathsf{U}\tilde P\mathsf{U}$ for every ε. Since U is unitary, $\dot{\mathsf{U}} = \mathrm{i}\Upsilon\mathsf{U} = \mathsf{U}\,\mathrm{i}\Upsilon'$ with Υ Hermitian and $\Upsilon' = \mathsf{U}^{-1}\Upsilon\mathsf{U}$. Hence $\dot\Omega = \mathrm{i}\Upsilon\Omega + \mathsf{U}\dot{\tilde P}\mathsf{U} + \mathrm{i}\Omega\Upsilon'$ and

$$\mathsf{U}^{-1}\dot\Omega\,\mathsf{U}^{-1} = \dot{\tilde P} + \mathrm{i}\big(\Upsilon'\tilde P + \tilde P\Upsilon\big).$$

Apply $M \mapsto (M - M^\dagger)/2\mathrm{i}$. The Hermitian matrix $\dot{\tilde P}$ drops out and $\mathrm{i}(\Upsilon'\tilde P + \tilde P\Upsilon)$ contributes $\frac12[(\Upsilon+\Upsilon')\tilde P + \tilde P(\Upsilon+\Upsilon')]$; so (21) holds with $\Theta = \frac12(\Upsilon+\Upsilon')$. For an entire function f and a smooth family of Hermitian matrices, differentiating the Cauchy integral $f(\Phi) = \frac{1}{2\pi\mathrm{i}}\oint f(\zeta)(\zeta-\Phi)^{-1}d\zeta$ gives, in an eigenbasis of Φ, $(\frac{d}{d\varepsilon}f(\Phi))_{ab} = \dot\Phi_{ab}\,(f(\phi_a) - f(\phi_b))/(\phi_a - \phi_b)$, with the value $f'(\phi_a)$ if $\phi_a = \phi_b$ (the formula of Daleckii and Krein; see [Bha97]). For $f(x) = e^{\mathrm{i}x/2}$ and $\gamma = \phi_a - \phi_b$ this gives $\Upsilon_{ab} = -\mathrm{i}\dot{\mathsf{U}}_{ab}e^{-\mathrm{i}\phi_b/2} = \dot\Phi_{ab}\,2\gamma^{-1}\sin(\gamma/4)\,e^{\mathrm{i}\gamma/4}$ and $\Upsilon'_{ab} = e^{-\mathrm{i}\gamma/2}\Upsilon_{ab}$, hence $\Theta_{ab} = \dot\Phi_{ab}\sin(\gamma/2)/\gamma$. For Hermitian $\dot\Omega$, $(X - X^\dagger)_{ab}/2\mathrm{i} = \dot\Omega_{ab}(e^{-\mathrm{i}(\phi_a+\phi_b)/2} - e^{\mathrm{i}(\phi_a+\phi_b)/2})/2\mathrm{i} = -\dot\Omega_{ab}\,\varsigma_{ab}$. □

The equation (21) has a unique solution because $\tilde P > 0$.

Lemma 4.5. *Let*

$$\varrho_{ab} = \frac{\phi_a - \phi_b}{2(\cos\phi_b - \cos\phi_a)} \quad (\phi_a \neq \phi_b), \qquad \varrho_{ab} = \frac{1}{2\sin\phi_a} \quad (\phi_a = \phi_b),$$

a positive symmetric function of (ϕ_a, ϕ_b). *For a family with constant imaginary part,* $\operatorname{tr}(\dot\Omega\dot\Phi) = -4\,Q(\Theta)$, *where*

$$Q(\Theta) = \sum_{a,b} \varrho_{ab}\,(\tilde P\Theta + \Theta\tilde P)_{ba}\,\Theta_{ab} = \sum_a \Big\langle \theta^a, \big((\operatorname{Im}\Omega)\circ \mathsf{L}^{(a)}\big)\theta^a \Big\rangle, \qquad \mathsf{L}^{(a)}_{bd} = \frac{\varrho_{ba}+\varrho_{da}}{\varsigma_{bd}},$$

and θ^a *is the* a*-th column of* Θ.

Proof. By Lemma 4.4, $\dot\Omega_{ab} = -(\tilde P\Theta + \Theta\tilde P)_{ab}/\varsigma_{ab}$ and $\dot\Phi_{ab} = \gamma\,\Theta_{ab}/\sin(\gamma/2)$ with $\gamma = \phi_a - \phi_b$, and $\gamma/(\sin(\gamma/2)\,\varsigma_{ab}) = 2\gamma/(\cos\phi_b - \cos\phi_a) = 4\varrho_{ab}$. So $\operatorname{tr}(\dot\Omega\dot\Phi) = \sum_{a,b}\overline{\dot\Omega_{ab}}\dot\Phi_{ab} = -4Q(\Theta)$, the conjugate of an entry of a Hermitian matrix being the transposed entry. Expanding, $Q = \sum_{a,b,c}\varrho_{ab}[\tilde P_{bc}\Theta_{ca}\Theta_{ab} + \Theta_{bc}\tilde P_{ca}\Theta_{ab}]$; renaming the indices (a,b,c) of the second sum as (c,a,b) turns it into $\sum_{a,b,c}\varrho_{ca}\tilde P_{bc}\Theta_{ca}\Theta_{ab}$. Hence $Q = \sum_{a,b,c}(\varrho_{ab}+\varrho_{ca})\tilde P_{bc}\,\Theta_{ab}\Theta_{ca}$, and $\tilde P_{bc} = (\operatorname{Im}\Omega)_{bc}/\varsigma_{bc}$ by (20), while $\Theta_{ab}\Theta_{ca} = \overline{\theta^a_b}\,\theta^a_c$. □

4.2. **Positivity of the kernels.**

Lemma 4.6. *For* $x, c > 0$ *let* $g_c(x) = (1+x^2)(\arctan x - \arctan c)/(x^2-c^2)$ *for* $x \neq c$ *and* $g_c(c) = 1/(2c)$. *Then*

$$g_c(x) = \frac{2}{\pi}\int_0^\infty \frac{x}{x^2+y^2}\,w_c(y)\,dy, \qquad w_c(y) = \frac{|1-y^2|\,k_c(y)}{y^2+c^2} \geq 0,$$

where $k_c(y) = \arctan c$ *for* $0 < y < 1$ *and* $k_c(y) = \frac{\pi}{2} - \arctan c$ *for* $y > 1$.

Proof. For $x \neq c$, $(1-y^2)/((x^2+y^2)(c^2+y^2)) = p/(x^2+y^2) + p'/(c^2+y^2)$ with $p = (1+x^2)/(c^2-x^2)$ and $p' = (1+c^2)/(x^2-c^2)$. Moreover $\int_0^1 x\,dy/(x^2+y^2) = \frac{\pi}{2} - \arctan x$, $\int_1^\infty x\,dy/(x^2+y^2) = \arctan x$, $\int_0^1 x\,dy/(c^2+y^2) = \frac{x}{c}(\frac{\pi}{2} - \arctan c)$ and $\int_1^\infty x\,dy/(c^2+y^2) = \frac{x}{c}\arctan c$. With $\beta = \arctan c$ and $\beta' = \frac{\pi}{2} - \beta$,

$$\begin{aligned}\int_0^\infty \frac{x\,w_c(y)}{x^2+y^2}\,dy &= \beta\big[p(\tfrac{\pi}{2} - \arctan x) + p'\tfrac{x}{c}\beta'\big] - \beta'\big[p\arctan x + p'\tfrac{x}{c}\beta\big]\\ &= \tfrac{\pi}{2}\,p\,(\arctan c - \arctan x) = \tfrac{\pi}{2}\,g_c(x).\end{aligned}$$

Both sides are continuous in x, which gives the case $x = c$. □

Lemma 4.7. *For every* $c > 0$ *and all* $x_1, \dots, x_n > 0$ *the matrix* $\big[(g_c(x_b)+g_c(x_d))/(x_b+x_d)\big]_{b,d}$ *is positive semidefinite.*

Proof. For $f_y(x) = x/(x^2+y^2)$,

$$\frac{f_y(x)+f_y(x')}{x+x'} = \frac{xx'+y^2}{(x^2+y^2)(x'^2+y^2)} = f_y(x)f_y(x') + y^2\,\frac{1}{x^2+y^2}\,\frac{1}{x'^2+y^2},$$

a sum of two kernels of the form $\psi(x)\psi(x')$. Integrate against the positive measure $\frac{2}{\pi}w_c(y)\,dy$ of Lemma 4.6. □

Functions g on $(0,\infty)$ for which all matrices $[(g(x_b)+g(x_d))/(x_b+x_d)]$ are positive semidefinite are called anti-Löwner; Audenaert [Aud11] characterizes them by an integral representation, and Lemma 4.6 puts g_c in that form. The sufficient condition of Kwong quoted in [Aud11], operator monotonicity of g, does not apply, since g_c is not monotone: for $c = 1$ the limits at 0 and at ∞ are $\frac{\pi}{4}$, while the value at 1 is $\frac{1}{2}$.

Lemma 4.8. *For every index a the matrix $\mathsf{L}^{(a)}$ of Lemma 4.5 is positive semidefinite, with positive diagonal.*

Proof. Put $x_b = \tan(\phi_b/2) > 0$ and $c = x_a$. Then $\cos\phi_a - \cos\phi_b = 2(x_b^2 - c^2)/((1+c^2)(1+x_b^2))$ and $\phi_b - \phi_a = 2(\arctan x_b - \arctan c)$, so $\varrho_{ba} = \frac{1+c^2}{2} g_c(x_b)$, also when $\phi_b = \phi_a$; and $\varsigma_{bd} = (x_b + x_d)/(\sqrt{1+x_b^2}\sqrt{1+x_d^2})$. Hence

$$\mathsf{L}^{(a)}_{bd} = \frac{1+c^2}{2}\sqrt{1+x_b^2}\sqrt{1+x_d^2}\,\frac{g_c(x_b)+g_c(x_d)}{x_b+x_d},$$

the matrix of Lemma 4.7 conjugated by a positive diagonal matrix. The diagonal entries $2\varrho_{ba}/\varsigma_{bb}$ are positive. □

Proof of Theorem 4.1. By Lemma 4.5, $\operatorname{tr}(\dot\Omega\dot\Phi) = -4\sum_a \langle \theta^a, ((\operatorname{Im}\Omega)\circ \mathsf{L}^{(a)})\theta^a\rangle$. Let $y_{\min} > 0$ be the least eigenvalue of $\operatorname{Im}\Omega$. Since $\operatorname{Im}\Omega - y_{\min} I \geq 0$ and $\mathsf{L}^{(a)} \geq 0$, the Schur product theorem [HJ13, Section 7.5] gives $(\operatorname{Im}\Omega)\circ \mathsf{L}^{(a)} \geq y_{\min}\operatorname{diag}(\mathsf{L}^{(a)}_{bb}) > 0$ by Lemma 4.8. So $Q(\Theta) > 0$ unless $\Theta = 0$, and $\Theta = 0$ forces $\dot\Omega = 0$ by (21). The integrated statement follows along the segment $\Omega' + \varepsilon(\Omega'' - \Omega')$, on which the imaginary part is constant. □

Remark 4.9. The positivity of $\operatorname{Im}\Omega$ is needed. (The numbers of this remark are computed in floating-point arithmetic and are not used in any proof.) The polar phase is defined and analytic near any invertible matrix whose unitary polar factor has spectrum in the open upper half-circle. For $\Phi = \operatorname{diag}(\frac{1}{5}, 3)$, $\tilde P = \left(\begin{smallmatrix} 1 & 9/10 \\ 9/10 & 1 \end{smallmatrix}\right)$ and $\Omega = e^{\mathrm{i}\Phi/2}\tilde P e^{\mathrm{i}\Phi/2}$, the matrix $\operatorname{Im}\Omega$ has eigenvalues $-0.730\ldots$ and $1.0699\ldots$, and the symmetric part of the derivative of the phase on Hermitian variations has the positive eigenvalue $0.2335\ldots$ For $\operatorname{Im}\Omega > 0$ the phase is in general not the gradient of a function of $\operatorname{Re}\Omega$ for the trace inner product: at $\Omega = \left(\begin{smallmatrix} \mathrm{i} & 1 \\ 1 & 3\mathrm{i} \end{smallmatrix}\right)$ the derivative of the entry Φ_{11} with respect to $(\operatorname{Re}\Omega)_{22}$ is $0.0488\ldots$, while that of Φ_{22} with respect to $(\operatorname{Re}\Omega)_{11}$ is $0.0306\ldots$ Nor is $\operatorname{Re}\Omega \mapsto -\Phi$ monotone for the Löwner order, even for a scalar imaginary part y: there $-\Phi = \arctan(\operatorname{Re}\Omega/y) - \frac{\pi}{2}$, and the arctangent is not operator monotone on the real line, since an operator monotone function on the real line is affine. The trace norm $\operatorname{tr}|\Omega|$ is a convex function of $\operatorname{Re}\Omega$ with gradient $\cos\Phi$, so $\cos\Phi$ is a monotone map; Theorem 4.1 concerns Φ itself and does not follow from this.

5. The arrow energy and the cancellation of the acyclic part

In this section the charges are arbitrary in the upper half-plane, and no hypothesis on the cohomology representation is made. The subscripts of z_i, x_i, y_i, α_i, n_i, C_i, ϕ_i, δ_i and M_1 refer to the vertices. In λ_0 and Ξ_0 the subscript marks the value at time zero, and in N_0, t_0, t_1, t_2, κ_0, κ_1, d_0, s_1, Λ_1 and $\mathfrak{c}_1$, $\mathfrak{c}_2$, $\mathfrak{c}_3$ it is a label.

5.1. The arrow energy.

Definition 5.1. The *arrow energy* of a state is

$$N := \sum_j \operatorname{tr}\big(A_j R(D^\dagger D) A_j^\dagger\big) = \operatorname{tr}\big(CR(B)\big) = \operatorname{tr} G.$$

It is the term of the primitive (11) that does not contain the charges. For $D = 0$ it would be the sum of the squared norms of the arrows.

Lemma 5.2. *The arrow energy has the following properties.*

(a) $N = q + \operatorname{tr} Y$.

(b) $\operatorname{tr} K = \operatorname{tr}(Ce^{-B}) \geq 0$, $\operatorname{tr} J = \operatorname{tr}\big(C(R(B) - e^{-B})\big) \geq 0$, *and* $\operatorname{tr} K + \operatorname{tr} J = N$.
(c) *Along the phase flow, for all charges,*

$$\frac{dN}{dt} = \operatorname{tr}(K\Phi_V) + \operatorname{tr}(J\Phi_U) - \operatorname{tr}(G\Phi_W).$$

Proof. (a) In the adapted splitting $R(B) = I_{\ker D} \oplus R(F^\dagger F)$, so $\operatorname{tr}(CR(B)) = \operatorname{tr} C_0 + \sum_j \operatorname{tr}(T_j R(F^\dagger F) T_j^\dagger)$, and $T_j R(F^\dagger F) T_j^\dagger = X_j F R(F^\dagger F) F^\dagger X_j^\dagger = X_j(I - e^{-L})X_j^\dagger$.

(b) $\operatorname{tr} K_B(C) = \int_0^1 \operatorname{tr}(e^{-(1-x)B} C e^{-xB})\, dx = \operatorname{tr}(Ce^{-B})$. By (3), $\operatorname{tr} J_L(M) = \operatorname{tr}(M\, g(L))$ with $g(\ell) = \int_{x,y\geq 0,\, x+y\leq 1} e^{-(x+y)\ell}\, dx\, dy = \int_0^1 x e^{-x\ell} dx = (1 - e^{-\ell} - \ell e^{-\ell})/\ell^2$; for $M = DCD^\dagger$, and since $D^\dagger g(L) D = B\, g(B) = R(B) - e^{-B}$ (both sides vanish on $\ker B$), this is $\operatorname{tr}(C(R(B) - e^{-B}))$. Both traces are nonnegative because $C \geq 0$, $e^{-B} \geq 0$ and $R(b) \geq e^{-b}$ for $b \geq 0$.

(c) This is (13) at $\xi = -\Phi$. □

Lemma 5.3. *At every state and for all charges in the upper half-plane:*

(a) $\operatorname{tr}(K\Phi_V) \leq \alpha_0 \operatorname{tr} K$;
(b) $\operatorname{tr}(J\Phi_U) \leq \pi \operatorname{tr} J_- \leq \frac{\pi}{3} \operatorname{tr} J$, *and* $\operatorname{tr}(J\Phi_U) \leq 0$ *if* $r = 1$;
(c) $\operatorname{tr}(G\Phi_W) \geq \pi N - n_1 M_1$, *where* $M_1 := \max\{\pi \max(x_1, 0),\, 2y_1\}$.

Proof. (a) For each unit eigenvector v of Φ_V of phase ϕ, put $p = \langle v, |\Omega_V| v\rangle$. By Lemma 2.1, $\langle v, \Omega_V v\rangle = pe^{\mathrm{i}\phi}$. Taking imaginary parts after multiplication by $e^{-\mathrm{i}\alpha_0}$ in $\Omega_V = z_0 e^{-B} + K$ gives $\langle v, Kv\rangle = p \sin(\alpha_0 - \phi)/\sin\alpha_0$. Its product with $\phi - \alpha_0$ is nonpositive. Summation in an orthonormal eigenbasis gives the bound.

(b) Let (u_k) be an orthonormal eigenbasis of Φ_U with eigenvalues $\phi_k \in (-\pi, 0)$, and $j_k = \langle u_k, Ju_k\rangle$. Then $\operatorname{tr}(J\Phi_U) = \sum_k \phi_k j_k \leq \pi \sum_k (j_k)_-$, where $(j_k)_- = \max(-j_k, 0)$, and $\sum_k (j_k)_- \leq \sum_k \langle u_k, J_- u_k\rangle = \operatorname{tr} J_-$. By Theorem 3.1, $\operatorname{tr} J_- \leq \frac{1}{3} \operatorname{tr} J$, because $DCD^\dagger \geq 0$. For $r = 1$, $J \geq 0$ is a number and $\Phi_U < 0$.

(c) $\Omega_W = z_1 I - G$ is normal and $\Phi_W = f_1(G)$ with $f_1(x) = \arg(z_1 - x)$. We claim that $x(\pi - f_1(x)) \leq M_1$ for all $x \geq 0$. If $x \leq \max(x_1, 0)$ the left side is at most $\pi \max(x_1, 0)$. If $x > \max(x_1, 0)$, then $x - x_1 > 0$ and $\pi - f_1(x) = \arctan \frac{y_1}{x - x_1} \leq \min\{\frac{\pi}{2}, \frac{y_1}{x - x_1}\}$; for $x \leq 2x_1$ the product is at most $2x_1 \cdot \frac{\pi}{2}$, and for $x > 2\max(x_1, 0)$ we have $x - x_1 \geq x/2$ and the product is at most $2y_1$. Summing over the eigenvalues of $G \geq 0$ gives $\operatorname{tr}(G\, f_1(G)) \geq \pi \operatorname{tr} G - n_1 M_1$. □

Theorem 5.4 (A priori bound)**.** *Let*

$$\rho := \pi - \max\{\alpha_0, \tfrac{\pi}{3}\} \quad (\rho := \pi - \alpha_0 \text{ if } r = 1), \qquad \bar{N} := \frac{2n_1 M_1}{\rho}, \qquad t_0 := \frac{1}{\rho} \log^+ \frac{\rho N(0)}{n_1 M_1},$$

where $\log^+ x = \log \max\{1, x\}$ *for* $x \geq 0$. *For all charges in the upper half-plane and along every trajectory of the phase flow,*

$$\frac{dN}{dt} \leq -\rho N + n_1 M_1, \qquad N(t) \leq N(0) e^{-\rho t} + \frac{n_1 M_1}{\rho}\big(1 - e^{-\rho t}\big).$$

Hence $N(t) \leq N_0 := \max\{N(0), n_1 M_1/\rho\}$ *for all* $t \geq 0$, *and* $N(t) \leq \bar{N}$ *for* $t \geq t_0$. *At all times* $q \leq N$, $\operatorname{tr} Y \leq N$, $\operatorname{tr} J \leq N$ *and* $s \leq N/(1 - e^{-\lambda})$.

Proof. By Lemmas 5.2(c) and 5.3, $\dot{N} \leq \alpha_0 \operatorname{tr} K + \frac{\pi}{3} \operatorname{tr} J - \pi N + n_1 M_1$. Since $\operatorname{tr} K$ and $\operatorname{tr} J$ are nonnegative with sum N, the first two terms are at most $\max\{\alpha_0, \frac{\pi}{3}\} N$; for $r = 1$ the second term may be omitted. Integrate; for $t \geq t_0$ the first term of the bound is at most $n_1 M_1/\rho$. The remaining statements are Lemma 5.2(a),(b) and (16). □

The bound $\bar N$ depends only on the charges and on n_1; the initial state enters only through the time t_0. For the symmetric charges $\rho = \frac{\pi}{4}$, $M_1 = \pi$ and $\bar N = 8n_1$. Corollary 4.2 also implies Lemma 5.3(a) by comparison with $z_0 e^{-B}$. If Lemma 5.3(a) is replaced by the elementary estimates $\operatorname{tr}(K\Phi_V) \le \pi \operatorname{tr} K_+$, $\operatorname{tr} K_+ = \frac12(\|K\|_1 + \operatorname{tr} K)$ and $\|K\|_1 \le \operatorname{tr}(CR(2B)) \le N$, one obtains only $\dot N \le -\frac{\pi}{6}\operatorname{tr} J + n_1 M_1$, which gives no decay of the arrows on $\ker D$.

5.2. **Growth of the least eigenvalue of $DD^\dagger$.** Let $E_1(b) = \int_1^\infty e^{-xb}\,dx/x$ be the exponential integral. For $b > 0$,

$$\frac{e^{-b}}{1+b} \le E_1(b) \le e^{-b}\log\bigl(1+\tfrac1b\bigr), \qquad E_1'(b) = -\frac{e^{-b}}{b}\,: \tag{23}$$

the difference $E_1(b) - e^{-b}/(1+b)$ has derivative $-e^{-b}/(b(1+b)^2) < 0$ and the difference $e^{-b}\log(1+\frac1b) - E_1(b)$ has derivative $e^{-b}[\frac{1}{1+b} - \log(1+\frac1b)] < 0$, and both tend to 0 as $b \to \infty$.

Lemma 5.5. *For all charges, along the phase flow,*

$$\frac{d}{dt}\operatorname{tr} E_1(L) = -\operatorname{tr}\bigl(e^{-F^\dagger F}\Phi_{AA}\bigr) - \operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr) \le -\operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr),$$

and at every state

$$\operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr) \ge \frac{y_0\,(\operatorname{tr} e^{-L})^2}{(y_0+|x_0|)\operatorname{tr} e^{-L} + \operatorname{tr} J_+}.$$

Proof. By (10), $\dot L = D\Phi_V D^\dagger - \frac12(\Phi_U L + L\Phi_U)$, so by (23)

$$\frac{d}{dt}\operatorname{tr} E_1(L) = -\operatorname{tr}\bigl(L^{-1}e^{-L}D\Phi_V D^\dagger\bigr) + \operatorname{tr}\bigl(e^{-L}\Phi_U\bigr),$$

and $D^\dagger L^{-1}e^{-L}D = 0 \oplus e^{-F^\dagger F}$ in the adapted splitting. Both terms are nonpositive, because $\Phi_{AA} > 0 > \Phi_U$.

Let (u_k) be an orthonormal eigenbasis of Φ_U with eigenvalues $\phi_k \in (-\pi, 0)$, and put $e_k = \langle u_k, e^{-L}u_k\rangle > 0$, $j_k = \langle u_k, Ju_k\rangle$. By Lemma 2.1 applied to $-\Omega_U = z_0e^{-L} - J$, whose polar phase is $\Phi_U + \pi$, $\langle u_k, (z_0e^{-L} - J)u_k\rangle = p_k e^{\mathrm{i}(\phi_k+\pi)}$ with $p_k > 0$. Hence $\cot(\phi_k+\pi) = (x_0e_k - j_k)/(y_0e_k)$ and

$$|\phi_k| = \pi - (\phi_k+\pi) = \operatorname{arccot}\frac{j_k - x_0e_k}{y_0e_k} \ge \operatorname{arccot}\frac{(j_k)_+ + |x_0|e_k}{y_0e_k} \ge \frac{y_0e_k}{(y_0+|x_0|)e_k + (j_k)_+},$$

where $(j_k)_+ = \max(j_k, 0)$; we used that arccot is decreasing and that $\operatorname{arccot} x \ge \frac{1}{1+x}$ for $x \ge 0$ (the difference is positive at 0, tends to 0 at infinity and has derivative $-2x/((1+x^2)(1+x)^2) \le 0$). Therefore

$$\operatorname{tr}\bigl(e^{-L}|\Phi_U|\bigr) = \sum_k |\phi_k|e_k \ge \sum_k \frac{y_0e_k^2}{(y_0+|x_0|)e_k + (j_k)_+} \ge \frac{y_0\,(\sum_k e_k)^2}{(y_0+|x_0|)\sum_k e_k + \sum_k (j_k)_+}$$

by the Cauchy–Schwarz inequality in the form $\sum_k a_k^2/d_k \ge (\sum_k a_k)^2/\sum_k d_k$ for $d_k > 0$. Finally $\sum_k e_k = \operatorname{tr} e^{-L}$ and $\sum_k (j_k)_+ \le \operatorname{tr} J_+$. □

Theorem 5.6 (Growth)**.** *Let $\bar N$ and t_0 be as in Theorem 5.4, $y_0 = \operatorname{Im} z_0$, $\lambda_0 = \lambda(0) > 0$, $c_\lambda = 1/\log(1+1/\lambda_0)$, $\Xi_0 = \operatorname{tr} E_1(L)|_{t=0}$ and*

$$\kappa_1 := \frac{y_0\,c_\lambda^2}{\sqrt2\,|z_0|\,c_\lambda\,\Xi_0 + \frac43\bar N}.$$

Then for all charges and all $t > t_0$

$$E_1(\lambda(t)) \le \operatorname{tr} E_1(L(t)) \le \frac{1}{\Xi_0^{-1} + \kappa_1(t-t_0)} < \frac{1}{\kappa_1(t-t_0)}.$$

In particular $\lambda(t) \to \infty$, and $\lambda(t) \ge \Lambda$ for all $t \ge t_0 + (1+\Lambda)e^{\Lambda}/\kappa_1$, for every $\Lambda > 0$.

Proof. Put $\Xi = \operatorname{tr} E_1(L) = \sum_i E_1(\ell_i)$. By Proposition 2.3(i) all $\ell_i \ge \lambda_0$ for $t \ge 0$, so (23) gives $\Xi \le \log(1+1/\lambda_0) \operatorname{tr} e^{-L}$, that is $\operatorname{tr} e^{-L} \ge c_\lambda \Xi$. By Theorem 3.1 and Theorem 5.4, for $t \ge t_0$, $\operatorname{tr} J_+ = \operatorname{tr} J + \operatorname{tr} J_- \le \frac{4}{3} \operatorname{tr} J \le \frac{4}{3}\bar{N}$, and $y_0 + |x_0| \le \sqrt{2}|z_0|$. The function $x \mapsto x^2/(cx+d)$ is increasing on $x > 0$ for $c, d > 0$. Hence by Lemma 5.5, for $t \ge t_0$,

$$\dot{\Xi} \le -\frac{y_0 c_\lambda^2 \, \Xi^2}{\sqrt{2}|z_0| c_\lambda \, \Xi + \frac{4}{3}\bar{N}} \le -\kappa_1 \Xi^2,$$

where we used $\Xi \le \Xi_0$ in the denominator; Ξ is nonincreasing by Lemma 5.5. Integrating from t_0, $\Xi(t)^{-1} \ge \Xi(t_0)^{-1} + \kappa_1(t-t_0) \ge \Xi_0^{-1} + \kappa_1(t-t_0)$. Since $E_1(\lambda)$ is one of the nonnegative summands of Ξ, $E_1(\lambda) \le \Xi$. For the last statement, if $t \ge t_0 + (1+\Lambda)e^{\Lambda}/\kappa_1$ then $E_1(\lambda(t)) < e^{-\Lambda}/(1+\Lambda) \le E_1(\Lambda)$ by (23). □

5.3. **Cancellation.** Put

$$\Lambda_1 = \max\Big\{1, \frac{8\pi\bar{N}}{\alpha_1 y_0}\Big\}, \qquad t_1 = t_0 + \frac{(1+\Lambda_1)e^{\Lambda_1}}{\kappa_1}, \qquad s_1 = \frac{\bar{N}}{1-e^{-1}},$$

$$\mathfrak{c}_1 = 2\sqrt{d_0 \bar{N} s_1} + r\kappa_0 s_1, \qquad \mathfrak{c}_2 = \frac{\pi}{\sqrt{|z_0|}}\big(\sqrt{\mathfrak{c}_1} + \sqrt{r s_1}\big), \qquad \mathfrak{c}_3 = \max\Big\{0, 2\log\frac{2\mathfrak{c}_2}{\pi}\Big\}, \tag{24}$$

$$C_{\mathrm{can}} = \mathfrak{c}_1 + (r+1)s_1 + 2|z_0| r e^{\mathfrak{c}_3}, \qquad t_2 = t_1 + \frac{2}{\alpha_1}\log^+\frac{8\alpha_1 s_1}{\pi y_0}.$$

The numbers Λ_1, s_1, $\mathfrak{c}_1$, $\mathfrak{c}_2$, $\mathfrak{c}_3$, C_{can} depend only on the dimensions and the charges; the times t_1 and t_2 depend in addition on the three numbers $N(0)$, $\lambda(0)$ and $\operatorname{tr} E_1(L)|_{t=0}$ of the initial state.

Theorem 5.7 (Cancellation)**.** *For all charges in the upper half-plane and along every trajectory of the phase flow:*

(i) *for $t \ge t_1$: $\lambda(t) \ge \Lambda_1$, $\dot{s} \le -\frac{\alpha_1}{2}s$ and $s(t) \le s_1 e^{-\frac{\alpha_1}{2}(t-t_1)}$;*
(ii) *for $t \ge t_1$: $\lambda(t) \ge \max\{\Lambda_1, \ \frac{\alpha_1}{4}(t-t_1) - \mathfrak{c}_3\}$;*
(iii) *for $t \ge t_1$:*

$$\|\Omega_V - (z_0 I + C_0) \oplus 0\|_1 + \|\Omega_U\|_1 + \|\Omega_W - (z_1 I - C_1)\|_1 \le C_{\mathrm{can}} \, e^{-\frac{\alpha_1}{4}(t-t_1)},$$

and $|m(h(t)) - m_H(t)|$ satisfies the same bound;
(iv) *for $t \ge t_2$: $\|\delta_0\|_{\mathrm{HS}}^2 \le \frac{2\pi\alpha_1}{y_0} s$ and $\|\delta_1\|_{\mathrm{HS}} \le s/y_1$.*

So the acyclic part cancels, with $q \le \bar{N}$ from the time t_0 on, and after the time t_2 the metrics induced on the cohomology representation move by its own phases ϕ_0, ϕ_1 up to an exponentially decaying error.

Proof. (i) By Theorem 5.6 with $\Lambda = \Lambda_1$, $\lambda \ge \Lambda_1$ on $[t_1, \infty)$. By Theorem 5.4, $q \le \bar{N}$ for $t \ge t_0$, so $\frac{4\pi q}{y_0 \lambda} \le \frac{\alpha_1}{2}$ for $t \ge t_1$, and Proposition 2.3(iii) gives $\dot{s} \le -\frac{\alpha_1}{2}s$. At t_1, $\lambda \ge 1$ and $s \le \operatorname{tr} Y/(1-e^{-\lambda}) \le s_1$ by (16) and Theorem 5.4.

(ii) By Proposition 2.3(ii), Theorem 5.4 and (i), for $t \ge t_1$,

$$\|\mathcal{E}_V\|_1 \le 2\sqrt{d_0 q s/\lambda} + r\kappa_0 s \le \mathfrak{c}_1 e^{-\frac{\alpha_1}{4}(t-t_1)}, \qquad \|J\|_1 \le r s_1 e^{-\frac{\alpha_1}{2}(t-t_1)}.$$

Let v be a unit vector of $(\ker D)^\perp$ with $F^\dagger F v = \lambda v$ and $u = Fv/\sqrt{\lambda}$, so $Lu = \lambda u$. Apply Proposition 2.3(v) to $T = \Omega_V$ with $T_\circ = \operatorname{diag}(z_0 I + C_0,\ z_0 e^{-F^\dagger F})$, for which $|T_\circ| v = |z_0| e^{-\lambda} v$ and $\Psi_\circ v = \alpha_0 v$; and to $T = -\Omega_U = z_0 e^{-L} - J$ with $T_\circ = z_0 e^{-L}$ and the vector u:

$$|\langle v, (\Phi_V - \alpha_0) v\rangle| \le \pi \sqrt{\frac{\|\mathcal{E}_V\|_1\, e^{\lambda}}{|z_0|}}, \qquad |\langle u, (\Phi_U + \pi - \alpha_0) u\rangle| \le \pi \sqrt{\frac{\|J\|_1\, e^{\lambda}}{|z_0|}}.$$

By Proposition 2.3(i), the right derivative of λ satisfies

$$\dot\lambda \ge \lambda\big[\pi - \mathfrak{c}_2 e^{\lambda/2 - \frac{\alpha_1}{8}(t-t_1)}\big] \qquad (t \ge t_1).$$

Consider the line $t \mapsto \frac{\alpha_1}{4}(t - t_1) - \mathfrak{c}_3$. Where λ is at most its value we have $\mathfrak{c}_2 e^{\lambda/2 - \frac{\alpha_1}{8}(t-t_1)} \le \mathfrak{c}_2 e^{-\mathfrak{c}_3/2} \le \frac{\pi}{2}$, hence $\dot\lambda \ge \frac{\pi}{2}\lambda \ge \frac{\pi}{2} > \frac{\alpha_1}{4}$, because $\lambda \ge \Lambda_1 \ge 1$ and $\alpha_1 < \pi$. Since $\lambda(t_1) \ge 1 > -\mathfrak{c}_3$, the graph of λ cannot cross below the line.

(iii) For $t \ge t_1$: $\|\Omega_V - (z_0 I + C_0) \oplus 0\|_1 \le \|\mathcal{E}_V\|_1 + |z_0| r e^{-\lambda}$; $\|\Omega_U\|_1 \le rs + |z_0| r e^{-\lambda}$; and $\|\Omega_W - (z_1 I - C_1)\|_1 = \operatorname{tr} Y \le s$. By (ii), $e^{-\lambda} \le e^{\mathfrak{c}_3} e^{-\frac{\alpha_1}{4}(t-t_1)}$. Adding gives the bound, and the statement on the masses follows from the triangle inequality for the trace norm.

(iv) For $t \ge t_2$, (i) gives $s \le \frac{\pi y_0}{8\alpha_1}$. Then $\frac{qs}{\lambda y_0^2} \le \frac{\tilde N}{\Lambda_1}\frac{s}{y_0^2} \le \frac{\alpha_1 s}{8\pi y_0} \le \frac{1}{64}$, and Proposition 2.3(iv) gives $\|\delta_0\|_{\mathrm{HS}}^2 \le \frac{16\pi^2}{y_0^2}\frac{qs}{\lambda} \le \frac{2\pi\alpha_1}{y_0} s$. □

Remark 5.8. The time t_1 is an a priori bound; it contains e^{Λ_1} and $\log^2(1 + 1/\lambda(0))$ and exceeds by many orders of magnitude the time at which the decay is observed to begin in numerical integrations. For the symmetric charges $\alpha_1 = \frac{\pi}{4}$, so the theorem gives the rate $\frac{\pi}{8}$ for s and $\frac{\pi}{16}$ for the blocks of Ω on the acyclic part. Numerical integration of (10), which is not used in any proof and is not reproduced here, suggests that $\dot\lambda/\lambda$ approaches $\frac{\pi}{2}$, so that λ would grow exponentially.

6. Ordered and reversed charges: the limit on the cohomology

By Theorem 5.7 the long-time behaviour is that of the metrics induced on the cohomology representation, which move by its own phases up to an exponentially small error. We treat ordered charges with semistable and with polystable cohomology, and reversed charges.

For ordered charges, the *Kempf–Ness function* of the cohomology representation for the stability parameter (θ_0, θ_1) of (14) is the function of a pair (h_0, h_1) of Hermitian metrics on $\ker D$ and W

$$\mathcal{K}_{\mathrm{coh}}(h_0, h_1) = q + \theta_0 \log\det h_0 - \theta_1 \log\det h_1, \qquad q = \sum_j \operatorname{tr}\big(h_0^{-1} b_j^* h_1 b_j\big), \tag{25}$$

where b_j is the matrix of $A_j|_{\ker D}$ in fixed reference bases, b_j^* its conjugate transpose, and h_0, h_1 stand for the Gram matrices. Writing $h_i = g_i^* g_i$, one has $\mathcal{K}_{\mathrm{coh}} = \|g \cdot b\|^2 + \theta_0 \log|\det g_0|^2 - \theta_1 \log|\det g_1|^2$ with $b = (b_j)$ and $(g \cdot b)_j = g_1 b_j g_0^{-1}$; so $\mathcal{K}_{\mathrm{coh}}$ is the Kempf–Ness function of the point b on the symmetric space $\mathrm{GL}_{n_0} \times \mathrm{GL}_{n_1}/\mathrm{U}(n_0) \times \mathrm{U}(n_1)$ of pairs of metrics. It is invariant under a common rescaling of h_0, h_1 because $n_0\theta_0 = n_1\theta_1$, and its critical points are the zeros of the moment map $\mu = (C_0 - \theta_0 I,\ C_1 - \theta_1 I)$ (Lemma 6.2).

Put $\mathcal{K}_* := \inf \mathcal{K}_{\mathrm{coh}} \in [-\infty, \infty)$. A zero of the moment map exists if and only if the cohomology representation is polystable [Kin94, Proposition 6.5], and $\mathcal{K}_* > -\infty$ if and only if it is semistable [HKKP23, Theorem 5.1]. King's results are stated for integral characters. They apply here because (θ_0, θ_1) is a positive multiple $c\,(n_1, n_0)$ of an integral vector, and

the Kempf–Ness function for the stability parameter $c\,(n_1, n_0)$ at the arrows b is c times the one for (n_1, n_0) at the arrows $b/\sqrt{c}$, which have the same subrepresentations.

For the second statement we recall the argument. If a subrepresentation (V', W') of dimension vector (a_0, a_1) has $a_0\theta_0 - a_1\theta_1 > 0$, let $h_0 = 1 \oplus e^\tau$ on $V' \oplus V'^\perp$ and $h_1 = 1 \oplus e^\tau$ on $W' \oplus W'^\perp$, for reference metrics; then q stays bounded as $\tau \to \infty$, and $\mathcal{K}_{\mathrm{coh}} = q - \tau(a_0\theta_0 - a_1\theta_1) \to -\infty$, since $n_0\theta_0 = n_1\theta_1$. Conversely, for a semistable representation and an integral character, the closure of the orbit of a lift of the representation to the total space of the character does not meet the zero section [Kin94, Proposition 3.1 and Lemma 2.2], the norm function on the total space considered in [Kin94, §6] is proper on this closed set [Kin94, Lemma 6.3], hence bounded below by a positive constant there, and $\frac{1}{2}\mathcal{K}_{\mathrm{coh}}$ is its logarithm on the orbit.

We write $\|\mu\|^2 = \|\mu_0\|^2_{\mathrm{HS}} + \|\mu_1\|^2_{\mathrm{HS}}$ and $\|\delta\|^2 = \|\delta_0\|^2_{\mathrm{HS}} + \|\delta_1\|^2_{\mathrm{HS}}$; for a pair of matrices $\|\cdot\|$ denotes this Hilbert–Schmidt norm, and for a single matrix it remains the operator norm.

On a representation, regarded as a complex in degree zero, the normalized metrics move with velocity $(G_0(C_0), -G_1(C_1))$ for increasing functions G_i with $G_i(\theta_i) = 0$ (see (28)), whereas the downward gradient flow of the Kempf–Ness function has velocity $(C_0 - \theta_0, -(C_1 - \theta_1))$. So the phase flow is not that gradient flow, but it has the same stationary points and its velocity pairs negatively with the gradient $(-\mu_0, \mu_1)$.

6.1. **The moment map tends to zero.** Let the charges be ordered. Besides (24) put

$$\Gamma = \max\Big\{\frac{(|z_0| + \bar{N})^2}{y_0}, \frac{(|z_1| + \bar{N})^2}{y_1}\Big\}, \qquad c_\delta^2 = \frac{2\pi\alpha_1 s_1}{y_0} + \frac{s_1^2}{y_1^2}$$

(the formula for c_δ makes sense for all charges and is used again in Theorem 6.11). We note that

$$\theta_0,\ \theta_1 \ \le\ \vartheta \ \le\ \bar{N}. \tag{26}$$

Indeed $\theta_1 = x_1 - y_1 \cot\varphi$. If $\cot\varphi \ge 0$, then $0 < \theta_1 \le x_1 \le M_1/\pi$. If $\cot\varphi < 0$, then $\frac{\pi}{2} < \varphi < \alpha_0$, so $|\cot\varphi| = \cot(\pi - \varphi) \le (\pi - \varphi)^{-1} < (\pi - \alpha_0)^{-1} \le \rho^{-1}$ and $\theta_1 \le \max(x_1, 0) + y_1/\rho \le M_1/\pi + M_1/(2\rho)$. In both cases $\theta_1 \le 2M_1/\rho$, that is, $\vartheta = n_1\theta_1 \le \bar{N}$. Let

$$\mathfrak{h}(t) = \big(e^{\varphi t} h_V(t)|_{\ker D},\ e^{\varphi t} h_W(t)\big)$$

be the pair of induced metrics on $\ker D$ and W, multiplied by $e^{\varphi t}$. The function $\mathcal{K}_{\mathrm{coh}}$ is invariant under common rescaling, so $\mathcal{E}(t) := \mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) - \mathcal{K}_*$ is the excess of the induced pair.

Theorem 6.1 (Decay of the moment map)**.** *Let the charges be ordered and the cohomology representation semistable. Then along every trajectory, for $t \ge t_2$,*

$$\frac{d}{dt}\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) \le -\frac{1}{\Gamma}\|\mu\|^2 + \|\mu\|\,\|\delta\| \le -\frac{1}{2\Gamma}\|\mu\|^2 + \frac{\Gamma}{2}\|\delta\|^2, \qquad \|\delta(t)\| \le c_\delta\, e^{-\frac{\alpha_1}{4}(t - t_1)}, \tag{27}$$

and

$$\mathcal{E}(t) + \frac{1}{2\Gamma}\int_{t_2}^{t}\|\mu\|^2 \le \mathcal{E}^\sharp := \mathcal{E}(0) + 2\pi(N_0 + \vartheta)\,t_2 + \frac{\Gamma}{\alpha_1}c_\delta^2,$$

with $N_0 = \max\{N(0), n_1 M_1/\rho\}$ as in Theorem 5.4 and $\delta = (\delta_0, \delta_1)$, $\delta_0 = \Phi_{HH} - \phi_0$, $\delta_1 = \Phi_W - \phi_1$, as in Section 2. Moreover $\mu(t) \to 0$, $\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) \to \mathcal{K}_*$ and $m(h(t)) \to |Z|$ as $t \to \infty$.

Proof. By (9) and (17) the pair $\mathfrak{h}(t)$ has velocities $\xi_0 = \varphi - \Phi_{HH}$ and $\xi_1 = \varphi - \Phi_W$ in the frames induced by the adapted frames; that is,

$$\begin{aligned} \xi_0 &= G_0(C_0) - \delta_0, \qquad & \xi_1 &= -G_1(C_1) - \delta_1, \\ G_0(x) &= \varphi - \arg(z_0 + x), \qquad & G_1(x) &= \arg(z_1 - x) - \varphi. \end{aligned} \tag{28}$$

The function G_0 is increasing with $G_0(\theta_0) = 0$, because $\operatorname{Im}(e^{-\mathrm{i}\varphi}(z_0 + x)) = c_0 - \sigma x$, and $G_0'(x) = y_0/|z_0 + x|^2 \in [y_0/(|z_0| + \bar N)^2,\ 1/y_0]$ on $[0, \bar N]$; likewise G_1 is increasing with $G_1(\theta_1) = 0$ and $G_1'(x) = y_1/|z_1 - x|^2 \in [y_1/(|z_1| + \bar N)^2,\ 1/y_1]$. By Theorem 5.4 the spectra of C_0, C_1 lie in $[0, \bar N]$ for $t \ge t_0$, and $\theta_0, \theta_1 \le \bar N$ by (26). The first variation of (25) is $d\mathcal{K}_{\mathrm{coh}}(\xi) = -\operatorname{tr}(\xi_0\mu_0) + \operatorname{tr}(\xi_1\mu_1)$ (Lemma 6.2), so

$$\frac{d}{dt}\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) = -\operatorname{tr}\big(\mu_0 G_0(C_0)\big) - \operatorname{tr}\big(\mu_1 G_1(C_1)\big) + \operatorname{tr}(\mu_0\delta_0) - \operatorname{tr}(\mu_1\delta_1),$$

and $\operatorname{tr}(\mu_0 G_0(C_0)) = \sum (c - \theta_0) G_0(c) \ge \Gamma^{-1} \|\mu_0\|_{\mathrm{HS}}^2$ over the eigenvalues c of C_0, likewise for μ_1. This is the first inequality of (27), for all $t \ge t_0$; the bound on $\|\delta\|$ for $t \ge t_2$ is Theorem 5.7(i),(iv).

Since $\|\xi_0\|, \|\xi_1\| \le \pi$ and $\|\mu_0\|_1 \le q + n_0\theta_0$, $\|\mu_1\|_1 \le q + n_1\theta_1$, we have $|\frac{d}{dt}\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t))| \le 2\pi(N_0 + \vartheta)$ for all t, hence $\mathcal{E}(t_2) \le \mathcal{E}(0) + 2\pi(N_0 + \vartheta)t_2$. Integrating (27) from t_2 and using $\frac{\Gamma}{2}\int_{t_2}^{\infty} \|\delta\|^2 \le \frac{\Gamma}{\alpha_1} c_\delta^2$ and $\mathcal{E} \ge 0$ gives the displayed bound. By (17) and $\|\Phi_\bullet\| \le \pi$, $\|\dot\mu_0\|_{\mathrm{HS}}, \|\dot\mu_1\|_{\mathrm{HS}} \le 2\pi q \le 2\pi N_0$, so $\|\mu\|^2$ is uniformly continuous and integrable and therefore tends to zero. Then $C_0 \to \theta_0 I$ and $C_1 \to \theta_1 I$, so $m_H \to n_0|z_0 + \theta_0| + n_1|z_1 - \theta_1|$. Both $z_0 + \theta_0$ and $z_1 - \theta_1$ have argument φ, so this limit is $\operatorname{Re}\big(e^{-\mathrm{i}\varphi}(Z + n_0\theta_0 - n_1\theta_1)\big) = |Z|$, and $m \to |Z|$ by Theorem 5.7(iii).

It remains to show $\mathcal{E}(t) \to 0$. By (27) the function $\mathcal{E}(t) + \frac{\Gamma}{2}\int_t^{\infty} \|\delta\|^2$ is nonincreasing on $[t_2, \infty)$, so $\mathcal{E}(t)$ has a limit $\mathcal{E}_\infty \ge 0$. Suppose $\mathcal{E}_\infty > 0$ and choose a pair $\mathfrak{h}'$ with $\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}') - \mathcal{K}_* < \mathcal{E}_\infty/2$. By convexity of $\mathcal{K}_{\mathrm{coh}}$ along the geodesic from $\mathfrak{h}(t)$ to $\mathfrak{h}'$ in the space of pairs of metrics, with its invariant distance d (these are recalled in the next subsection; Lemma 6.2), $\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) - \mathcal{K}_{\mathrm{coh}}(\mathfrak{h}') \le \|\mu(t)\|\, d(\mathfrak{h}(t), \mathfrak{h}')$, and the left side is at least $\mathcal{E}_\infty/4$ for large t. By (28) the speed of $\mathfrak{h}(t)$ is at most $\|\mu\| / \min(y_0, y_1) + \|\delta\|$, which is square integrable, so $d(\mathfrak{h}(t), \mathfrak{h}') \le c + c'\sqrt{t}$ by the Cauchy–Schwarz inequality. Then $\|\mu(t)\|^2 \ge \mathcal{E}_\infty^2/(16(c + c'\sqrt{t})^2)$ is not integrable, a contradiction. □

Theorem 6.1 does not require a zero of the moment map to exist: for semistable cohomology that is not polystable the infimum $\mathcal{K}_*$ is not attained, and $\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) \to \mathcal{K}_*$ is the analogue for the phase flow of the corresponding fact for gradient flow lines of a Kempf–Ness function [GRS21, Theorem 4.3].

6.2. **The Kempf–Ness function on the space of metric pairs.** The convexity statements of this subsection are classical in the theory of Kempf–Ness functions; see [GRS21, Theorem 4.3], where they are proved for actions on closed Kähler manifolds. We include the short proofs for the linear action at hand. Let $\mathcal{X}$ be the space of pairs $\mathfrak{h} = (\mathfrak{h}_0, \mathfrak{h}_1)$ of Hermitian metrics on $\ker D$ and W. A tangent vector at $\mathfrak{h}$ is a pair $\xi = (\xi_0, \xi_1)$ of $\mathfrak{h}$-Hermitian endomorphisms, $\delta\mathfrak{h} = \mathfrak{h}\xi$, and $\|\xi\|^2 = \operatorname{tr}\xi_0^2 + \operatorname{tr}\xi_1^2$ defines an invariant Riemannian metric, for which $\mathcal{X}$ is complete, simply connected and nonpositively curved. Its geodesics are $u \mapsto \mathfrak{h}e^{u\xi}$, the exponential map at any point is a diffeomorphism, and closed balls are compact. Let d be the distance. In a $\mathfrak{h}$-orthonormal frame let H_j be the matrix of $A_j|_{\ker D}$, and C_0, C_1, μ_0, μ_1, q as before.

Lemma 6.2. *$d\mathcal{K}_{\mathrm{coh}}(\xi) = -\operatorname{tr}(\xi_0\mu_0) + \operatorname{tr}(\xi_1\mu_1)$, so the gradient of $\mathcal{K}_{\mathrm{coh}}$ is $(-\mu_0, \mu_1)$, of norm $\|\mu\|$. Along a geodesic,*

$$\frac{d^2}{du^2}\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}e^{u\xi}) = \sum_j \|\xi_1 H_j(u) - H_j(u)\xi_0\|_{\mathrm{HS}}^2 \ge 0, \qquad H_j(u) = e^{u\xi_1/2} H_j e^{-u\xi_0/2}.$$

Proof. In the $\mathfrak{h}$-orthonormal frame, $q(\mathfrak{h}e^{u\xi}) = \sum_j \operatorname{tr}(e^{-u\xi_0}H_j^\dagger e^{u\xi_1}H_j)$, and $\log\det$ is affine along geodesics with derivative $\operatorname{tr}\xi$. Differentiate once and twice. □

So $\mathcal{K}_{\mathrm{coh}}$ is geodesically convex and its critical points, the zeros of the moment map, are its minimizers. Let $\mathcal{Z} \subset \mathcal{X}$ be the set of zeros of the moment map. For $\mathfrak{h} \in \mathcal{X}$ let $\mathfrak{a}(\mathfrak{h})$ be the space of $\mathfrak{h}$-Hermitian pairs ξ with $\xi_1 A_j = A_j\xi_0$ on $\ker D$ for all j, the Hermitian infinitesimal automorphisms of the representation.

Lemma 6.3. *For $p \in \mathcal{Z}$, $\mathcal{Z} = \{pe^\xi : \xi \in \mathfrak{a}(p)\}$. It is a closed totally geodesic submanifold with $T_p\mathcal{Z} = \mathfrak{a}(p)$.*

Proof. If $\xi \in \mathfrak{a}(p)$, the second derivative of $\mathcal{K}_{\mathrm{coh}}$ along $u \mapsto pe^{u\xi}$ vanishes identically by Lemma 6.2, and the first derivative vanishes at $u = 0$; so $pe^\xi \in \mathcal{Z}$. Conversely, if $pe^\xi \in \mathcal{Z}$, the convex function $u \mapsto \mathcal{K}_{\mathrm{coh}}(pe^{u\xi})$ takes its minimum at $u = 0$ and $u = 1$, hence is constant on $[0,1]$, and its second derivative at 0 gives $\xi \in \mathfrak{a}(p)$. $\mathcal{Z} = \{\mu = 0\}$ is closed, and it is the image of a linear subspace under the exponential map at each of its points. □

Lemma 6.4. *For $p \in \mathcal{Z}$ put*

$$\nu_* := \min\Big\{\sum_j \|\xi_1 H_j - H_j\xi_0\|_{\mathrm{HS}}^2 :\ \xi \perp \mathfrak{a}(p),\ \|\xi\| = 1\Big\} > 0, \qquad \varrho_* := \frac{\log 2}{\sqrt{2}}.$$

Then ν_* does not depend on $p \in \mathcal{Z}$, and for every unit $\xi \perp \mathfrak{a}(p)$ and $0 \le u \le \varrho_*$, $\frac{d^2}{du^2}\mathcal{K}_{\mathrm{coh}}(pe^{u\xi}) \ge \nu_*/2$.

Proof. The quadratic form is positive on the orthogonal complement of its kernel $\mathfrak{a}(p)$. By Lemma 6.2, $\xi_1 H_j(u) - H_j(u)\xi_0 = e^{u\xi_1/2}(\xi_1 H_j - H_j\xi_0)e^{-u\xi_0/2}$ has Hilbert–Schmidt norm at least $e^{-u(\|\xi_1\|+\|\xi_0\|)/2}\,\|\xi_1 H_j - H_j\xi_0\|_{\mathrm{HS}}$, and $\|\xi_1\| + \|\xi_0\| \le \sqrt{2}$ for a unit ξ. So the second derivative is at least $e^{-\sqrt{2}u}\nu_*$. If $p' = pe^\zeta$ with $\zeta \in \mathfrak{a}(p)$, then $e^{-\zeta/2}$ is a pair of automorphisms of the representation that maps p-orthonormal frames to p'-orthonormal frames and preserves the matrices H_j; conjugation by it carries $\mathfrak{a}(p)$ and its orthogonal complement isometrically to those of p'. □

Lemma 6.5. *Let the representation be polystable, so that $\mathcal{Z} \neq \emptyset$. For $\mathfrak{h} \in \mathcal{X}$ put $\mathsf{d} = d(\mathfrak{h}, \mathcal{Z})$ and $\mathcal{E} = \mathcal{K}_{\mathrm{coh}}(\mathfrak{h}) - \mathcal{K}_*$. Then*

(i) $\mathcal{E} \le \|\mu(\mathfrak{h})\|\,\mathsf{d}$;
(ii) $\mathcal{E} \ge \frac{\nu_*}{4}\min(\mathsf{d}^2, \varrho_*\mathsf{d})$;
(iii) *if $\mathsf{d} \le \varrho_*$, then $\|\mu(\mathfrak{h})\| \ge \frac{\nu_*}{2}\mathsf{d}$ and $\mathcal{E} \le \frac{2}{\nu_*}\|\mu(\mathfrak{h})\|^2$.*

Proof. Let $p \in \mathcal{Z}$ be a nearest point and $\mathfrak{h} = pe^{\mathsf{d}\xi}$ with $\|\xi\| = 1$; then $\xi \perp \mathfrak{a}(p)$. The function $f(u) = \mathcal{K}_{\mathrm{coh}}(pe^{u\xi}) - \mathcal{K}_*$ is convex with $f(0) = f'(0) = 0$. (i) $0 = f(0) \ge f(\mathsf{d}) - \mathsf{d}f'(\mathsf{d})$ and $f'(\mathsf{d}) \le \|\mu(\mathfrak{h})\|$. (ii) $f'' \ge \nu_*/2$ on $[0, \varrho_*]$ by Lemma 6.4, so $f(u) \ge \frac{\nu_*}{4}u^2$ there, and $f(u) \ge f(\varrho_*)u/\varrho_*$ for $u \ge \varrho_*$ by convexity. (iii) $\|\mu(\mathfrak{h})\| \ge f'(\mathsf{d}) = \int_0^{\mathsf{d}} f'' \ge \frac{\nu_*}{2}\mathsf{d}$; combine with (i). □

Part (iii) is a Łojasiewicz gradient inequality with exponent $\frac{1}{2}$ for $\mathcal{K}_{\mathrm{coh}}$ near $\mathcal{Z}$.

6.3. **Convergence for polystable cohomology.**

Theorem 6.6 (Convergence to a harmonic metric)**.** *Let the charges be ordered and the cohomology representation polystable. Then along every trajectory there is a zero $\mathfrak{h}_\infty$ of the*

moment map, that is, a harmonic metric of the cohomology representation in the sense of [HKKP26]*, with*

$$d\big(\mathfrak{h}(t),\mathfrak{h}_\infty\big)\le \mathfrak{C}\,e^{-\beta t}\qquad (t\ge 0)$$

for every rate $\beta<\frac12\min\{\nu_*/(4\Gamma),\ \alpha_1/2\}$*, where the constant* $\mathfrak{C}$ *depends only on* β*, the dimensions, the charges,* $N(0)$*,* t_2*,* $\mathcal{E}(0)$ *and* ν_**. Moreover, for* $t\ge t_1$*,*

$$\|\mu(\mathfrak{h}(t))\|\le 2\bar N\, d(\mathfrak{h}(t),\mathfrak{h}_\infty),\qquad 0\le m_H(t)-|Z|\le \frac{\|\mu(\mathfrak{h}(t))\|^2}{2\min(y_0,y_1)},$$

so that the moment map of the cohomology representation and $m(h(t))-|Z|$ *tend to zero exponentially.*

The convergence is as Hermitian forms on the fixed vector spaces $\ker D$ and W: no time-dependent automorphism of the representation is applied. The limit depends on the initial metric through its position on $\mathcal{Z}$.

Proof. By Theorem 6.1, (27) holds for $t\ge t_2$ and $\mathcal{E}(t)\le\mathcal{E}^\sharp$ there.

Step 1. By Lemma 6.5(ii), $d(\mathfrak{h}(t),\mathcal{Z})\le \mathsf{d}_{\max}:=\max\{\varrho_*,4\mathcal{E}^\sharp/(\nu_*\varrho_*)\}$ for $t\ge t_2$, and by (i), $\mathcal{E}(t)\le\|\mu(t)\|\,\mathsf{d}_{\max}$.

Step 2. By Theorem 6.1, $\int_{t_2}^\infty\|\mu\|^2\le 2\Gamma\mathcal{E}^\sharp$ and $\mathcal{E}(t)\le\mathcal{E}(t')+\frac\Gamma2\int_{t'}^\infty\|\delta\|^2$ for $t\ge t'\ge t_2$. For $T>0$ there is $t'\in[t_2+T/2,t_2+T]$ with $\|\mu(t')\|^2\le 4\Gamma\mathcal{E}^\sharp/T$, hence for $t\ge t_2+T$

$$\mathcal{E}(t)\le \mathsf{d}_{\max}\sqrt{4\Gamma\mathcal{E}^\sharp/T}+\frac{\Gamma}{\alpha_1}c_\delta^2e^{-\frac{\alpha_1}{4}T}.$$

Choose T, depending only on the dimensions, the charges, $\mathcal{E}^\sharp$ and ν_*, so that the right side is at most $\frac{\nu_*}{4}\varrho_*^2$; then $d(\mathfrak{h}(t),\mathcal{Z})\le\varrho_*$ for $t\ge t_3:=t_2+T$ by Lemma 6.5(ii). For such t, Lemma 6.5(iii) and (27) give

$$\dot{\mathcal{E}}\le-\frac{\nu_*}{4\Gamma}\mathcal{E}+\frac{\Gamma}{2}c_\delta^2e^{-\frac{\alpha_1}{2}(t-t_1)},$$

so $\mathcal{E}(t)\le\mathfrak{C}_2e^{-2\beta t}$ for every $2\beta<\min\{\nu_*/(4\Gamma),\alpha_1/2\}$.

Step 3. Integrating (27) over $[t,t+1]$, $\int_t^{t+1}\|\mu\|^2\le 2\Gamma\mathcal{E}(t)+\Gamma^2\int_t^{t+1}\|\delta\|^2\le\mathfrak{C}_3e^{-2\beta t}$. By (28) and $|G_0'|\le1/y_0$, $|G_1'|\le1/y_1$, the speed of $\mathfrak{h}(t)$ is $\|\xi\|\le\|\mu\|/\min(y_0,y_1)+\|\delta\|$. Hence $\int_t^\infty\|\xi\|\le\mathfrak{C}_4e^{-\beta t}$ for $t\ge t_3$, and by completeness $\mathfrak{h}(t)$ converges to some $\mathfrak{h}_\infty$ with $d(\mathfrak{h}(t),\mathfrak{h}_\infty)\le\mathfrak{C}_4e^{-\beta t}$; $\mu(\mathfrak{h}_\infty)=0$ by continuity. On $[0,t_3]$ the speed is at most $\pi\sqrt{n_0+n_1}$, because the spectra of Φ_{HH} and Φ_W lie in $(0,\pi)$; this gives the dependence of $\mathfrak{C}$.

The moment map and the mass. Let $t\ge t_1$ and let $u\mapsto\mathfrak{h}_\infty e^{u\zeta}$, $0\le u\le1$, be the geodesic from $\mathfrak{h}_\infty$ to $\mathfrak{h}(t)$. The function q is convex along geodesics, so $q\le\max(\vartheta,q(\mathfrak{h}(t)))\le\bar N$ on it, by (26); here $q(\mathfrak{h}_\infty)=\operatorname{tr}C_0=\vartheta$. In the frames of Lemma 6.2, $\frac{d}{du}C_0=-\frac12(\zeta_0C_0+C_0\zeta_0)+\sum_j H_j^\dagger\zeta_1H_j$ and similarly for C_1, so $\big\|\frac{d}{du}\mu\big\|\le 2q\,\|\zeta\|$, and $\mu(\mathfrak{h}_\infty)=0$. For the mass, the functions $x\mapsto|z_0+x|$ and $x\mapsto|z_1-x|$ are convex with second derivatives at most $1/y_0$ and $1/y_1$, and their first-order terms at θ_0,θ_1 cancel in the sum over the eigenvalues of C_0 and C_1 because $\operatorname{tr}\mu_0=\operatorname{tr}\mu_1$. Combine with Theorem 5.7(iii). □

Remark 6.7. The rate is limited by the gap ν_* of the Hessian of $\mathcal{K}_{\mathrm{coh}}$ transversally to $\mathcal{Z}$ and by the decay of A_jD^+. When the cohomology representation is stable, $\mathfrak{a}$ consists of the scalar pairs (c,c).

6.4. **Semistable cohomology that is not polystable.** In this case Theorem 6.1 gives $\mu \to 0$, and there is no limit metric.

Proposition 6.8. *Let the charges be ordered and let the cohomology representation be semistable and not polystable. Then along every trajectory $d(\mathfrak{h}(t), \mathfrak{h}(0)) \to \infty$, and for $t \geq t_2$*

$$d\big(\mathfrak{h}(t), \mathfrak{h}(t_2)\big) \leq \frac{\sqrt{2\Gamma\mathcal{E}^\sharp}}{\min(y_0, y_1)}\sqrt{t - t_2} + \frac{4c_\delta}{\alpha_1}.$$

Proof. Suppose $d(\mathfrak{h}(t_k), \mathfrak{h}(0))$ is bounded along a sequence $t_k \to \infty$. Closed balls of $\mathcal{X}$ are compact, so a subsequence converges to a pair $\mathfrak{h}_\infty$. The moment map is continuous on $\mathcal{X}$ and $\mu(\mathfrak{h}(t)) \to 0$ by Theorem 6.1; so $\mu(\mathfrak{h}_\infty) = 0$ and the representation is polystable, a contradiction. By (28) and $0 < G_i' \leq 1/y_i$ on $[0, \infty)$, the speed of $\mathfrak{h}(t)$ is at most $\|\mu\| / \min(y_0, y_1) + \|\delta\|$. By Theorem 6.1 and the Cauchy–Schwarz inequality $\int_{t_2}^t \|\mu\| \leq (2\Gamma\mathcal{E}^\sharp)^{1/2}\sqrt{t - t_2}$, and $\int_{t_2}^\infty \|\delta\| \leq 4c_\delta/\alpha_1$ because $t_2 \geq t_1$. □

The bound $\sqrt{t}$ is not attained in the following example, the smallest one: there the distance grows like $\log t$, and the asymptotics can be determined for every initial metric. Let R be the representation of the Kronecker quiver with two arrows on $\mathbb{C}^2 \to \mathbb{C}^2$ with the arrows

$$b_1 = I, \qquad b_2 = E_{12}, \qquad E_{12}e_2 = e_1, \quad E_{12}e_1 = 0.$$

Here $n_0 = n_1 = 2$, so $\theta_0 = \theta_1 =: \theta$. A subrepresentation (V', W') has $V' = b_1 V' \subset W'$, so R is semistable; the lines $\mathbb{C}e_1 \to \mathbb{C}e_1$ form a subrepresentation of dimension vector $(1, 1)$, so it is not stable; and it is not polystable, since a direct sum of two representations of dimension vector $(1, 1)$ has arrows that are diagonal in suitable bases of the two spaces, and then $b_1^{-1}b_2 = E_{12}$ would be diagonalizable. Both $\mathbb{C}e_1 \to \mathbb{C}e_1$ and the quotient are isomorphic to the stable representation $\mathbb{C} \to \mathbb{C}$ with arrows $(1, 0)$. Put

$$\mathsf{g}_0 = G_0'(\theta) = \frac{y_0}{|z_0 + \theta|^2}, \qquad \mathsf{g}_1 = G_1'(\theta) = \frac{y_1}{|z_1 - \theta|^2}, \qquad \mathsf{k} = \frac{2\mathsf{g}_0\mathsf{g}_1}{\mathsf{g}_0 + \mathsf{g}_1}, \qquad \Lambda_\infty = (\mathsf{g}_0 + \mathsf{g}_1)\,\theta,$$

with G_0, G_1 as in (28). For a pair of metrics (h_0, h_1) on R, with Hermitian forms conjugate-linear in the first argument, let

$$p_1 = h_0(e_1, e_1), \qquad \chi_0 = \frac{h_0(e_1, e_2)}{p_1}, \qquad p_2 = h_0(e_2 - \chi_0 e_1,\, e_2 - \chi_0 e_1),$$

and define p_1', χ_1, p_2' in the same way from h_1; these are the Gram–Schmidt data of the two metrics for the flag $\mathbb{C}e_1 \subset \mathbb{C}^2$. The metrics are diagonal in the basis e_1, e_2 exactly when $\chi_0 = \chi_1 = 0$, and the automorphisms $I + xE_{12}$, $x \in \mathbb{C}$, of R shift χ_0 and χ_1 by x. Put

$$a = \frac{p_1'}{p_1}, \qquad b = \frac{p_2'}{p_2}, \qquad c = \frac{p_1'}{p_2}, \qquad \chi = \chi_1 - \chi_0.$$

Proposition 6.9 (A semistable example)**.** *Let the charges be ordered. For every initial pair of metrics on R, regarded as a complex in degree zero, along the phase flow:*

(i) *χ vanishes identically or has constant argument, it tends to zero with $|\chi(t)| = O(e^{-\Lambda' t})$ for every $\Lambda' < \Lambda_\infty$, and χ_0, χ_1 converge to a common limit χ_∞;*

(ii) $c(t) = \dfrac{1}{\mathsf{k}t} + O\Big(\dfrac{\log t}{t^2}\Big)$, $a - \theta = -\dfrac{\mathsf{g}_1}{\mathsf{g}_0 + \mathsf{g}_1}c + O(c^2)$ *and* $b - \theta = -\dfrac{\mathsf{g}_0}{\mathsf{g}_0 + \mathsf{g}_1}c + O(c^2)$;

(iii) $\lim\limits_{t\to\infty} t\,\|\mu(t)\| = \dfrac{\sqrt{2(\mathsf{g}_0^2 + \mathsf{g}_1^2)}}{2\mathsf{g}_0\mathsf{g}_1}$ *and* $\mathcal{K}_{\mathrm{coh}}(\mathfrak{h}(t)) - \mathcal{K}_* = \dfrac{1}{\mathsf{k}t} + O\Big(\dfrac{\log t}{t^2}\Big)$;

(iv) *there is a constant* $A > 0$ *with*

$$e^{\varphi t}p_1 \sim A\,t^{-1/2}, \quad e^{\varphi t}p_1' \sim \theta A\,t^{-1/2}, \quad e^{\varphi t}p_2 \sim \theta \mathsf{k} A\,t^{1/2}, \quad e^{\varphi t}p_2' \sim \theta^2 \mathsf{k} A\,t^{1/2},$$

and $d(\mathfrak{h}(t), \mathfrak{h}(0)) = \log t + O(1)$.

So for every initial metric the normalized metrics decay like $t^{-1/2}$ on the subrepresentation $\mathbb{C}e_1 \to \mathbb{C}e_1$ and grow like $t^{1/2}$ on the quotient. These rates hold in every basis $e_1,\ e_2 - xe_1$; the value $x = \chi_\infty$ is the one for which the metrics become diagonal, in the sense that $h_i(e_1, e_2 - \chi_\infty e_1)/h_i(e_1, e_1) \to 0$ for $i = 0, 1$. The exponents do not depend on the charges; the constants do, through θ, g_0 and g_1.

Proof. Frames. The vectors $e_1/\sqrt{p_1}$, $(e_2 - \chi_0 e_1)/\sqrt{p_2}$ form an h_0-orthonormal frame, and $e_1/\sqrt{p_1'}$, $(e_2 - \chi_1 e_1)/\sqrt{p_2'}$ an h_1-orthonormal frame. Since $e_2 - \chi_0 e_1 = (e_2 - \chi_1 e_1) + \chi e_1$ and $E_{12}(e_2 - \chi_0 e_1) = e_1$, the arrows have the matrices $H_1 = \left(\begin{smallmatrix} \sqrt{a} & \chi\sqrt{c} \\ 0 & \sqrt{b} \end{smallmatrix}\right)$ and $H_2 = \left(\begin{smallmatrix} 0 & \sqrt{c} \\ 0 & 0 \end{smallmatrix}\right)$, so

$$C_0 = \begin{pmatrix} a & \chi\sqrt{ac} \\ \bar\chi\sqrt{ac} & b + c + |\chi|^2 c \end{pmatrix}, \qquad C_1 = \begin{pmatrix} a + c + |\chi|^2 c & \chi\sqrt{bc} \\ \bar\chi\sqrt{bc} & b \end{pmatrix}.$$

Let $F_0(x) = \arg(z_0 + x) = \varphi - G_0(x)$ and $F_1(x) = \arg(z_1 - x) = \varphi + G_1(x)$; the matrices of the phases in these frames are $F_0(C_0)$ and $F_1(C_1)$.

Evolution of the data. For a metric h with phase Φ, $\frac{d}{dt}h(u, w) = -h(u, \Phi w)$. Writing e_1, e_2 in the frame, one finds for the data (p_1, p_2, χ_0) of h_0 and the matrix $\phi = F_0(C_0)$

$$\tfrac{d}{dt}\log p_1 = -\phi_{11}, \qquad \tfrac{d}{dt}\log p_2 = -\phi_{22}, \qquad \dot\chi_0 = -\sqrt{p_2/p_1}\;\phi_{12},$$

the second because $\det h_0 = p_1 p_2$ has logarithmic derivative $-\operatorname{tr}\phi$; and likewise for h_1 with $\phi = F_1(C_1)$. For a 2×2 Hermitian matrix C with eigenvalues $c^\pm$ and a function f, $f(C) - f[c^+, c^-]\,C$ is a scalar matrix, where $f[\,\cdot\,,\cdot\,]$ is the divided difference; so $f(C)_{12} = f[c^+, c^-]\,C_{12}$. Let $\varrho_0 = -F_0[c_0^+, c_0^-] > 0$ and $\varrho_1 = F_1[c_1^+, c_1^-] > 0$ for the eigenvalues of C_0, C_1. Since $\sqrt{p_2/p_1}\sqrt{ac} = a$ and $\sqrt{p_2'/p_1'}\sqrt{bc} = b$,

$$\dot\chi_0 = \varrho_0\, a\,\chi, \qquad \dot\chi_1 = -\varrho_1\, b\,\chi, \qquad \dot\chi = -(a\varrho_0 + b\varrho_1)\,\chi. \tag{29}$$

Limits. Theorem 6.1 applies to the flow of a representation. Indeed, let E be the orthogonal direct sum of R and a complex $(V_1 \to 0) \to (U \to 0)$ with invertible differential. It is of the form (2), its endomorphism (4) is the direct sum of those of the two summands, so the flow preserves the orthogonal splitting and restricts to the flow of each summand; the cohomology of E is R with its own metrics, and $\delta = 0$. Hence $C_0 \to \theta$ and $C_1 \to \theta$, that is, $a \to \theta$, $b \to \theta$, $c(1 + |\chi|^2) \to 0$, and $\varrho_i \to \mathsf{g}_i$. By (29), χ is a real positive multiple of $\chi(0)$, $|\chi|$ is nonincreasing, and $a\varrho_0 + b\varrho_1 \to \Lambda_\infty$; this gives the decay of χ. Then $\dot\chi_0$ and $\dot\chi_1$ are integrable, so χ_0, χ_1 converge, to the same limit. This is (i).

The reduced system. Put

$$w_1 = -G_0(a) - G_1(a + c), \qquad w_2 = -G_0(b + c) - G_1(b), \qquad w_3 = -G_0(b + c) - G_1(a + c),$$

functions of (a, b, c). If $\chi = 0$, the matrices C_0, C_1 are diagonal and $\frac{d}{dt}\log a = w_1$, $\frac{d}{dt}\log b = w_2$, $\frac{d}{dt}\log c = w_3$. In general, for a 2×2 Hermitian matrix C with eigenvalues $c^\pm$ and a smooth function f, Newton interpolation of f at the nodes C_{11}, c^+, c^- gives

$$f(C) = f(C_{11}) + f[C_{11}, c^+]\,(C - C_{11}) + f[C_{11}, c^+, c^-]\,(C - C_{11})(C - c^+),$$

and the upper left entries of $C - C_{11}$ and $(C - C_{11})(C - c^+)$ are 0 and $|C_{12}|^2$. So $|f(C)_{11} - f(C_{11})| \le \frac{1}{2}\sup|f''|\,|C_{12}|^2$, and likewise for the other diagonal entry. Here $|C_{12}|^2 \le |\chi|^2 c\,(a +$

b), and the diagonal entries of C_0, C_1 differ from those for $\chi = 0$ by $|\chi|^2 c$. Hence, with $\varepsilon(t) := |\chi(t)|^2$,

$$\frac{d}{dt}\log a = w_1 + O(\varepsilon c), \qquad \frac{d}{dt}\log b = w_2 + O(\varepsilon c), \qquad \frac{d}{dt}\log c = w_3 + O(\varepsilon c), \tag{30}$$

and $\frac{d}{dt}\log(e^{\varphi t}p_1) = G_0(a) + O(\varepsilon c)$, and similarly for p_2, p_1', p_2' with $G_0(b+c)$, $-G_1(a+c)$, $-G_1(b)$.

The fast variables. Differentiating along (30),

$$\dot w_1 = -\Lambda_a w_1 - G_1'(a+c)\, c\, w_3 + O(\varepsilon c), \qquad \dot w_2 = -\Lambda_b w_2 - G_0'(b+c)\, c\, w_3 + O(\varepsilon c),$$

with $\Lambda_a = a(G_0'(a) + G_1'(a+c))$ and $\Lambda_b = b(G_0'(b+c) + G_1'(b))$, both tending to Λ_∞. Near $(a,b,c) = (\theta,\theta,0)$ the functions (w_1, w_2, c) are real-analytic coordinates, with $w_1 = -(\mathsf{g}_0+\mathsf{g}_1)(a-\theta) - \mathsf{g}_1 c$ and $w_2 = -(\mathsf{g}_0+\mathsf{g}_1)(b-\theta) - \mathsf{g}_0 c$ to first order. Therefore, with $|w| = (w_1^2 + w_2^2)^{1/2}$,

$$a - \theta = -\frac{w_1 + \mathsf{g}_1 c}{\mathsf{g}_0 + \mathsf{g}_1} + O(|w|^2 + c^2), \qquad w_3 = -\mathsf{k}c + \frac{\mathsf{g}_1 w_1 + \mathsf{g}_0 w_2}{\mathsf{g}_0 + \mathsf{g}_1} + O(|w|^2 + c^2), \tag{31}$$

and similarly for b; the factor of c in w_3 is $-(\mathsf{g}_0+\mathsf{g}_1) + (\mathsf{g}_0^2 + \mathsf{g}_1^2)/(\mathsf{g}_0+\mathsf{g}_1) = -\mathsf{k}$.

We claim $|w| = O(c^2)$. Choose T such that for $t \ge T$: $\Lambda_a, \Lambda_b \ge \Lambda_\infty/2$, $|\frac{d}{dt}\log c| \le \Lambda_\infty/16$, $|w| \le 1$, $c \le 1$, (31) holds, and $a\varrho_0 + b\varrho_1 \ge \Lambda_\infty/2$, so that $\varepsilon(s) \le \varepsilon(T) e^{-\Lambda_\infty(s-T)}$ for $s \ge T$ by (29). Then, for constants K', K'' and after enlarging T,

$$\frac{d}{dt}|w| \le -\frac{\Lambda_\infty}{4}|w| + K'\, c^2 + K''\, \varepsilon\, c$$

in the sense of upper right derivatives. Since $c(s) \le c(t) e^{\frac{\Lambda_\infty}{16}(t-s)}$ and $e^{-\frac{\Lambda_\infty}{16}(t-T)} \le c(t)/c(T)$ for $T \le s \le t$,

$$|w(t)| \le \Big(\frac{c(t)}{c(T)}\Big)^4 |w(T)| + \frac{8K'}{\Lambda_\infty}\, c(t)^2 + K'' c(t)\, \varepsilon(T) \int_T^t e^{-\frac{3\Lambda_\infty}{16}(t-s)} e^{-\Lambda_\infty(s-T)} ds,$$

and the integral is at most $\frac{16}{13\Lambda_\infty} e^{-\frac{3\Lambda_\infty}{16}(t-T)} \le \frac{16}{13\Lambda_\infty}(c(t)/c(T))^3$. This proves the claim. By (30) and (31), $\frac{d}{dt}\log c = -\mathsf{k}c + O(c^2) + O(\varepsilon c)$, so $\frac{d}{dt}(1/c) = \mathsf{k} + O(c) + O(\varepsilon)$; then $c = O(1/t)$, $\int^t c = O(\log t)$ and $1/c = \mathsf{k}t + O(\log t)$. With (31) this is (ii).

(iii) By (ii), up to terms $O(c^2)$ and terms of order $|\chi|\sqrt{c}$, which decay exponentially, $C_0 - \theta = \frac{\mathsf{g}_1 c}{\mathsf{g}_0+\mathsf{g}_1}\operatorname{diag}(-1,1)$ and $C_1 - \theta = \frac{\mathsf{g}_0 c}{\mathsf{g}_0+\mathsf{g}_1}\operatorname{diag}(1,-1)$. Moreover $\mathcal{K}_{\mathrm{coh}} = \operatorname{tr} C_0 + \theta \log(\det h_0 / \det h_1) = a + b + c + \varepsilon c - \theta\log(ab)$, and $x - \theta\log x$ is stationary at θ, so $\mathcal{K}_{\mathrm{coh}} = 2\theta - 2\theta\log\theta + c + O(c^2) + O(\varepsilon c)$; the constant is $\mathcal{K}_*$ by Theorem 6.1.

(iv) By (30) and (ii), $\frac{d}{dt}\log(e^{\varphi t}p_1) = \mathsf{g}_0(a-\theta) + O(c^2) + O(\varepsilon c) = -\frac{\mathsf{k}}{2}c + O(c^2) + O(\varepsilon c) = -\frac{1}{2t} + O(\frac{\log t}{t^2})$; the error is integrable, so $e^{\varphi t} p_1\, t^{1/2}$ has a positive limit A. Then $p_1' = a p_1$, $p_2 = p_1'/c$ and $p_2' = b p_2$ give the other three. The metrics are $g_i^* \operatorname{diag} g_i$ with the diagonal matrices of the data and $g_i = \left(\begin{smallmatrix} 1 & \chi_i \\ 0 & 1 \end{smallmatrix}\right)$, which converge. For every pair (g_0, g_1) of invertible matrices the map $(h_0, h_1) \mapsto (g_0^* h_0 g_0,\ g_1^* h_1 g_1)$ is an isometry of the space of pairs of metrics, and the $g_i(t)$ stay in a compact set; so the distance from $\mathfrak{h}(0)$ is $\big(\sum_i \log^2(e^{\varphi t}p_i) + \sum_i \log^2(e^{\varphi t}p_i')\big)^{1/2} + O(1) = \log t + O(1)$. □

Remark 6.10. Apart from the limits $a, b \to \theta$ and $c \to 0$, the proof uses of G_0, G_1 only that they are smooth and increasing, with positive derivatives at θ, and vanish at θ. For the downward gradient flow of the Kempf–Ness function, for which these limits hold as well, the same computation applies with $G_i(x) = x - \theta$, and gives the same exponents with $\mathsf{g}_0 = \mathsf{g}_1 = \mathsf{k} = 1$; asymptotics of such gradient flows at semistable points are the subject of

[HKKP23]. For $\chi(0) \neq 0$ the rate in (i) is not attained for $\Lambda' = \Lambda_\infty$: the divided differences satisfy $\varrho_i = \mathsf{g}_i + O(c^2)$, hence $a\varrho_0 + b\varrho_1 = \Lambda_\infty - \mathsf{k}c + O(c^2)$ by (ii), and $|\chi(t)|$ is asymptotic to a constant multiple of $t\, e^{-\Lambda_\infty t}$. For general semistable cohomology the induced metrics are determined up to a bounded factor by [Wue26b, Theorem 6.3 and Corollary 6.5]; whether the factor converges is open.

6.5. **Reversed charges.**

Theorem 6.11 (Reversed charges)**.** *Let $\alpha_0 < \alpha_1$. Along every trajectory, for $t \geq t_{\mathrm{rev}} := \max\{t_2,\ t_1 + \frac{4}{\alpha_1}\log^+ \frac{2\sqrt{2}\,c_\delta}{\alpha_1-\alpha_0}\}$,*

$$q(t) \leq \bar{N}\, e^{-\frac{1}{2}(\alpha_1-\alpha_0)(t-t_{\mathrm{rev}})},$$

and $m(h(t)) \to n_0|z_0| + n_1|z_1|$ as $t \to \infty$. The phases of the induced metrics on $\ker D$ and W converge to α_0 and α_1.

Proof. By (17), $\dot{q} = \operatorname{tr}(C_0\Phi_{HH}) - \operatorname{tr}(C_1\Phi_W) = \operatorname{tr}(C_0\phi_0) - \operatorname{tr}(C_1\phi_1) + \operatorname{tr}(C_0\delta_0) - \operatorname{tr}(C_1\delta_1)$. Since $C_0, C_1 \geq 0$, $\phi_0 = \arg(z_0 I + C_0) \leq \alpha_0$ and $\phi_1 = \arg(z_1 I - C_1) \geq \alpha_1$. Hence, for all charges, $\dot{q} \leq -(\alpha_1 - \alpha_0 - \|\delta_0\| - \|\delta_1\|)\, q$. By Theorem 5.7(i),(iv), $\|\delta_0\| + \|\delta_1\| \leq \sqrt{2}\, c_\delta e^{-\frac{\alpha_1}{4}(t-t_1)}$ for $t \geq t_2$, which is at most $\frac{1}{2}(\alpha_1 - \alpha_0)$ for $t \geq t_{\mathrm{rev}}$; and $q(t_{\mathrm{rev}}) \leq \bar{N}$. Then $C_0, C_1 \to 0$, so $\phi_0 \to \alpha_0$, $\phi_1 \to \alpha_1$, $\delta \to 0$, $m_H \to n_0|z_0| + n_1|z_1|$, and $|m - m_H| \to 0$ by Theorem 5.7(iii). □

For reversed charges every representation with nonzero arrows is unstable; its Harder–Narasimhan factors are the two semisimple representations supported at one vertex each, of phases $\alpha_1 > \alpha_0$, and Theorem 6.11 says that the flow converges to their direct sum. By the triangle inequality $n_0|z_0| + n_1|z_1| > |Z|$ unless z_0, z_1 are parallel.

Remark 6.12. For ordered charges and cohomology that is not semistable, Theorem 5.7 still gives the cancellation of the acyclic part, with arrows bounded by $q \leq \bar{N}$ after the time t_0, and reduces the long-time behaviour to that of the phase flow of the cohomology representation, perturbed by the exponentially decaying δ_0, δ_1. In [HKKP26, §4.1] the asymptotics of the flow on an object without harmonic metric are expected to be controlled by a refinement of its Harder–Narasimhan filtration. For the complexes of this paper this is proved in the following form [Wue26b, Theorem 6.11 and Corollary 6.5]: the induced metrics decay at the exponential rates given by the phases of the Harder–Narasimhan filtration of the cohomology representation, on the steps of that filtration, and the metric induced on each factor is determined, up to a bounded factor, by the theory for semistable representations. The simplest case is that in which all arrows of the cohomology representation vanish: then $H_j = 0$ for all time, Ω_V is block diagonal with the block $z_0 I$ on $\ker D$, so $\Phi_{HH} = \alpha_0$ exactly, and $\Phi_W = \arg(z_1 I - Y) \to \alpha_1$.

7. The phase potential

7.1. **The rate identity.**

Definition 7.1. The *phase potential* is $\mathcal{K}(h) := \sigma^{-1}\operatorname{Im}\big(e^{-\mathrm{i}\varphi}\mathcal{S}_{\mathbb{C}}(h)\big)$, and the *dissipation* is $\mathcal{D} := \mathcal{D}_V + \mathcal{D}_U + \mathcal{D}_W$ with

$$\mathcal{D}_\bullet := \operatorname{tr}\Big(|\Omega_\bullet|\,(\Phi_\bullet - \varphi)\sin(\Phi_\bullet - \varphi)\Big).$$

For the symmetric charges and $n_0 = n_1$, $\mathcal{K} = -\operatorname{Re}\mathcal{S}_{\mathbb{C}}$, that is

$$\mathcal{K}(h) = \log\det h_V - \log\det h_U - \log\det h_W + \operatorname{tr}\operatorname{Ein}(D^\dagger D) + \sum_j \operatorname{tr}\big(A_j R(D^\dagger D) A_j^\dagger\big). \tag{32}$$

For a representation regarded as a complex in degree 0 the potential is its Kempf–Ness function for the stability parameter (θ_0, θ_1); see Proposition 7.7. The function $\mathcal{K}$ is a primitive of the real one-form $\xi \mapsto \sigma^{-1} \operatorname{Im} \sum_\bullet \operatorname{tr}(e^{-\mathrm{i}\varphi}\Omega_\bullet \xi_\bullet)$ on the space of metrics. The phase flow is not its gradient flow: the gradient of $\mathcal{K}$ for the invariant metric on the space of metrics has the components $\sigma^{-1}\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_\bullet)$, which do not differ from the phases $\Phi_\bullet$ by a constant.

Proposition 7.2 (Rate identity)**.** *Along the phase flow, for all charges in the upper half-plane,*

$$\sigma \frac{d\mathcal{K}}{dt} = -\mathcal{D}.$$

Proof. By (12) and (9), $\frac{d}{dt}\mathcal{S}_\mathbb{C} = -\sum_\bullet \operatorname{tr}(\Omega_\bullet \Phi_\bullet)$. By (7), $e^{-\mathrm{i}\varphi}\sum_\bullet \operatorname{tr}\Omega_\bullet = |Z|$ is real, so

$$\frac{d}{dt}\operatorname{Im}(e^{-\mathrm{i}\varphi}\mathcal{S}_\mathbb{C}) = -\sum_\bullet \operatorname{Im}\operatorname{tr}\big(e^{-\mathrm{i}\varphi}\Omega_\bullet(\Phi_\bullet - \varphi)\big).$$

By (8), $e^{-\mathrm{i}\varphi}\Omega_\bullet(\Phi_\bullet - \varphi) = |\Omega_\bullet|\big[(\Phi_\bullet - \varphi)\cos(\Phi_\bullet - \varphi) + \mathrm{i}(\Phi_\bullet - \varphi)\sin(\Phi_\bullet - \varphi)\big]$, and the trace of a product of two Hermitian matrices is real. □

The proof uses only (12), (9) and (7). For a representation in degree 0 the endomorphism Ω is normal and the statement follows from formulas displayed in [HKKP26, §§4.2, 5.1], although it is not stated there.

By Lemma 2.1, in an orthonormal eigenbasis (v_k) of $\Phi_\bullet$ with eigenvalues ϕ_k and $p_k = \langle v_k, |\Omega_\bullet| v_k\rangle > 0$,

$$\mathcal{D}_\bullet = \sum_k p_k(\phi_k - \varphi)\sin(\phi_k - \varphi), \qquad p_k \sin(\phi_k - \varphi) = \big\langle v_k, \operatorname{Im}\big(e^{-\mathrm{i}\varphi}\Omega_\bullet\big) v_k\big\rangle\,. \tag{33}$$

Proposition 7.3. *$\mathcal{D}_V \geq 0$ and $\mathcal{D}_W \geq 0$, with equality on a block if and only if the phase is φI there.*

Proof. For $\bullet = V, W$ the spectrum of $\Phi_\bullet$ lies in $(0, \pi)$ and $\varphi \in (0, \pi)$, so $|\phi_k - \varphi| < \pi$ and $(\phi_k - \varphi)\sin(\phi_k - \varphi) \geq 0$, with equality only for $\phi_k = \varphi$. □

The block U of degree one is different: the spectrum of $\Phi_U - \varphi$ lies in $(-\pi - \varphi, -\varphi)$, and $x \sin x < 0$ for $x \in (-\pi - \varphi, -\pi)$.

Proposition 7.4. *Let $M = \sigma J + c_0 e^{-L}$. For all charges in the upper half-plane,*

$$\mathcal{D}_U \geq \varphi \operatorname{tr} M - \pi \operatorname{tr} M_-.$$

If $c_0 \geq 0$, then

$$\mathcal{D}_U \geq \varphi c_0 \operatorname{tr} e^{-L} + \sigma\big(\varphi \operatorname{tr} J - \pi \operatorname{tr} J_-\big).$$

Proof. By (4), $\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_U) = -\sigma J - |z_0|\sin(\alpha_0 - \varphi)e^{-L} = -M$, so by (33), with $M_{kk} = \langle v_k, M v_k\rangle$, $p_k \sin(\phi_k - \varphi) = -M_{kk}$ and $\mathcal{D}_U = \sum_k (\varphi - \phi_k) M_{kk}$. If $M_{kk} \geq 0$, then $\sin(\phi_k - \varphi) \leq 0$, so $\varphi - \phi_k \in (\varphi, \pi]$ and the term is at least φM_{kk}. If $M_{kk} < 0$, then $\varphi - \phi_k \in (\pi, \pi + \varphi)$ and the term is at least $(\pi + \varphi)M_{kk}$. In both cases it is at least $\varphi M_{kk} - \pi(M_{kk})_-$. For any orthonormal basis $M_{kk} \geq -(M_-)_{kk}$, hence $\sum_k (M_{kk})_- \leq \operatorname{tr} M_-$. This is the first inequality. For the second, $\operatorname{tr} M_- = \max_{0 \leq \Pi \leq I}(-\operatorname{tr}(M\Pi))$ and $c_0 e^{-L} \geq 0$, so $\operatorname{tr} M_- \leq \sigma \operatorname{tr} J_-$. □

Remark 7.5. Fix L and the arrows, suppose J is invertible, and let $z_0 \to 0$ in the upper half-plane with z_1 fixed. Then $\Omega_U \to J$ and $\varphi \to \arg z_1$. The unitary polar factor tends to the sign of J. On the closed lower semicircle the argument extends continuously with values 0 at 1 and $-\pi$ at -1. Hence Φ_U tends to $-\pi$ times the negative spectral projection of J. The

lower bound in the first inequality of Proposition 7.4 is attained in this limit. The trace ratio in Theorem 3.1 therefore controls the degree-one contribution in this varying-charge limit.

Theorem 7.6 (Lyapunov function)**.** *Let the charges be admissible. Then at every state*

$$\mathcal{D} \geq \mathcal{D}_U \geq \varphi c_0 \operatorname{tr} e^{-L} + \sigma\big(\varphi - \tfrac{\pi}{3}\big)_+ \operatorname{tr} J > 0, \tag{34}$$

and the phase potential is strictly decreasing along the phase flow: it is a strict Lyapunov function. For the symmetric charges and $n_0 = n_1$,

$$\frac{d\mathcal{K}}{dt} = -\mathcal{D}, \qquad \mathcal{D} \geq \frac{\pi}{6} \operatorname{tr} J + \frac{\pi}{2} \operatorname{tr} e^{-L}.$$

Proof. Propositions 7.2, 7.3 and 7.4, together with Theorem 3.1: $\operatorname{tr} J_- \leq \frac{1}{3} \operatorname{tr} J$ and $\operatorname{tr} J \geq 0$, since $DCD^\dagger \geq 0$. If $r = 1$ then J is a nonnegative number, by (19) and Lemma 3.2, and $J_- = 0$. Here $L > 0$ because D is surjective, and $c_0 > 0$. □

No stability assumption on the cohomology representation and no assumption on the size of D or of the maps $A_j D^+$ enters Theorem 7.6. Some condition on the charges is needed: see Propositions 10.1 and 10.2.

7.2. **Decomposition and infimum.** In this subsection the charges are ordered. Let $H = \ker D$, with the induced metric $h_0 = h_V|_H$, and $h_1 = h_W$. Fix reference bases adapted to $H \subset V$: a basis of H followed by vectors whose classes form a basis of V/H. Let $\bar{D} : V/H \to U$ be the isomorphism induced by D and $c_D = \log |\det \bar{D}|^2$, the determinant taken in the reference bases. Let $\mathcal{K}_{\mathrm{coh}}$ be the Kempf–Ness function (25) of the cohomology representation, in the reference basis of H and that of W, and $\mathcal{K}_*$ its infimum.

Let $E_1(b) = \int_1^\infty e^{-xb}\, dx/x$ be the exponential integral and γ Euler's constant.

Proposition 7.7 (Decomposition)**.**

$$\mathcal{K} = \mathcal{K}_{\mathrm{coh}}(h_0, h_1) + \theta_0\Big(c_D + r\gamma + \sum_{i=1}^{r} E_1(\ell_i)\Big) + \operatorname{tr} Y.$$

Proof. B vanishes on H and equals $F^\dagger F$ on $H^\perp$, so $\operatorname{tr} \mathrm{Ein}(B) = \sum_i \mathrm{Ein}(\ell_i)$, and $R(B)$ is the identity on H, so $\operatorname{tr}(CR(B)) = \sum_j \operatorname{tr}(A_j R(B) A_j^\dagger) = q + \operatorname{tr} Y$, using $T_j R(F^\dagger F) T_j^\dagger = X_j(I - e^{-L})X_j^\dagger$. For the Gram matrix in the adapted reference basis, $\det h_V = \det h_0 \cdot \det h_{V/H}$ with $h_{V/H}$ the quotient metric, which is the metric of $H^\perp \cong V/H$. In the reference bases of V/H and U the endomorphism $F^\dagger F$ of $H^\perp$ has matrix $h_{V/H}^{-1}[\bar{D}]^* h_U [\bar{D}]$, so $\prod_i \ell_i = |\det \bar{D}|^2 \det h_U / \det h_{V/H}$. Hence $\log\det h_V - \log\det h_U = \log\det h_0 + c_D - \sum_i \log \ell_i$, and with $\mathrm{Ein}(b) = \gamma + \log b + E_1(b)$ for $b > 0$,

$$\mathcal{S}_{\mathbb{C}} = z_0\Big(\log\det h_0 + c_D + r\gamma + \sum_i E_1(\ell_i)\Big) + z_1 \log\det h_1 - q - \operatorname{tr} Y.$$

Finally $\operatorname{Im}(e^{-\mathrm{i}\varphi} z_0) = c_0$, $\operatorname{Im}(e^{-\mathrm{i}\varphi} z_1) = -c_1$ and $\operatorname{Im}(-e^{-\mathrm{i}\varphi}) = \sigma$. □

The identity $\log\det h_V - \log\det h_U + \log\det(F^\dagger F) = \log\det h_0 + c_D$ in this proof is the finite-dimensional comparison between the metric induced on the determinant of the cohomology and the metric of the cohomology, with torsion $\log\det(F^\dagger F)$; compare [BZ92, Theorem 1.1] for complexes of real vector spaces. The constant $r\gamma$ arises because Ein is the heat integral cut off at time 1 rather than its zeta-regularization (Remark 8.3).

Corollary 7.8. $\inf_h \mathcal{K} = \mathcal{K}_* + \theta_0(c_D + r\gamma)$*, and the infimum is not attained.*

Proof. The terms $\theta_0 \sum_i E_1(\ell_i)$ and $\operatorname{tr} Y$ are nonnegative and the first is positive, which gives "$\geq$" and non-attainment. For the converse fix a complement H' of H in V with a metric, a metric h_U^0, and take $h_V = h_0 \oplus h_{H'}$, $h_U = \epsilon^{-1} h_U^0$. As $\epsilon \to 0$ all $\ell_i \to \infty$, so $E_1(\ell_i) \to 0$ and $\operatorname{tr} Y \leq \sum_j \|T_j\|_{\mathrm{HS}}^2 / \min_i \ell_i \to 0$, while $\mathcal{K}_{\mathrm{coh}}$ depends only on (h_0, h_1). □

Lemma 7.9. *For every pair (h_0, h_1) with $q > 0$,*

$$q - \vartheta - \vartheta \log \frac{q}{\vartheta} \leq \mathcal{K}_{\mathrm{coh}}(h_0, h_1) - \mathcal{K}_*.$$

Proof. Replacing h_1 by $e^x h_1$ changes $\mathcal{K}_{\mathrm{coh}}$ to $e^x q + (\mathcal{K}_{\mathrm{coh}} - q) - \vartheta x$; at $e^x = \vartheta/q$ this is $\vartheta + \mathcal{K}_{\mathrm{coh}} - q + \vartheta \log(q/\vartheta) \geq \mathcal{K}_*$. □

The function $Q \mapsto Q - \vartheta - \vartheta \log(Q/\vartheta)$ is convex, vanishes only at $Q = \vartheta$ and tends to $+\infty$ as $Q \to 0$ and as $Q \to \infty$.

7.3. **Bounds through the potential.** Let the charges be admissible and the cohomology representation semistable. Then the Lyapunov function gives bounds that do not involve the arrow energy. Put

$$\Delta_0 := \mathcal{K}(h(0)) - \inf_h \mathcal{K} = \mathcal{K}_{\mathrm{coh}}(0) - \mathcal{K}_* + \theta_0 \sum_i E_1(\ell_i(0)) + \operatorname{tr} Y(0) \in (0, \infty), \tag{35}$$

and let $Q_\infty \geq \vartheta$ be the larger root of $Q - \vartheta - \vartheta \log(Q/\vartheta) = \Delta_0$.

Proposition 7.10 (Bounds through the potential)**.** *Let the charges be admissible and $\mathcal{K}_* > -\infty$. For all $t \geq 0$:*

(i) $0 \leq \mathcal{K}_{\mathrm{coh}}(t) - \mathcal{K}_* \leq \Delta_0$ *and* $q(t) \leq Q_\infty$;
(ii) $\theta_0 \sum_i E_1(\ell_i(t)) \leq \Delta_0$ *and* $\operatorname{tr} Y(t) \leq \Delta_0$, *hence* $s(t) \leq \Delta_0/(1 - e^{-\lambda(t)})$;
(iii) $\int_0^\infty \mathcal{D}\, dt \leq \sigma \Delta_0$;
(iv) $\int_0^\infty e^{-\lambda(t)} dt \leq \dfrac{\Delta_0}{\varphi \theta_0}$ *and* $\lambda(t) \geq \max\Big\{\lambda(0),\ \log \dfrac{\varphi \theta_0 t}{\Delta_0}\Big\}$.

In particular the conclusions of Theorem 5.7 hold with the constants (24) computed from $\bar N' := \max\{Q_\infty, \Delta_0\}$ in place of $\bar N$, and with t_1 replaced by $t_1' := \frac{\Delta_0}{\varphi\theta_0} e^{\Lambda_1'}$, where $\Lambda_1' = \max\{1, 8\pi \bar N'/(\alpha_1 y_0)\}$.

Proof. By Proposition 7.7 and Corollary 7.8, $\mathcal{K} - \inf \mathcal{K} = (\mathcal{K}_{\mathrm{coh}} - \mathcal{K}_*) + \theta_0 \sum_i E_1(\ell_i) + \operatorname{tr} Y$ is a sum of three nonnegative terms, and it is at most Δ_0 by Theorem 7.6. This gives (i), with Lemma 7.9, and (ii), by (16). (iii) is the integral of $\sigma \dot{\mathcal{K}} = -\mathcal{D}$. For (iv), (34) gives $\mathcal{D} \geq \varphi c_0 \operatorname{tr} e^{-L} \geq \varphi \sigma \theta_0 e^{-\lambda}$, and λ is nondecreasing (Proposition 2.3(i)), so $t e^{-\lambda(t)} \leq \int_0^t e^{-\lambda}$. For the last statement, by (i) and (ii) $q \leq \bar N'$ and $\operatorname{tr} Y \leq \bar N'$ for all $t \geq 0$, and by (iv) $\lambda \geq \Lambda_1'$ for $t \geq t_1'$. The proof of Theorem 5.7 uses Theorems 5.4 and 5.6 only to obtain these three facts, with $\bar N$, Λ_1, t_1 in place of $\bar N'$, Λ_1', t_1'; the rest of that proof applies unchanged. □

For the symmetric charges and $n_0 = n_1$, (iv) reads $\lambda(t) \geq \log(\pi t / 2\Delta_0)$.

8. The infimum: quasi-isomorphic complexes and moduli

The charges are ordered. This section identifies the infimum of the phase potential and shows that, up to an explicit constant, it is the same for quasi-isomorphic complexes of the class (2).

8.1. **Volume data.** A *volume datum* for a complex E is a triple $\varpi = (\varpi_V, \varpi_U, \varpi_W)$ of nonzero elements of the top exterior powers $\det V, \det U, \det W$. If these are the wedges of the reference bases, then by (11)

$$\mathcal{K}(h;\varpi) = \theta_0\big(\log\|\varpi_V\|^2_{h_V} - \log\|\varpi_U\|^2_{h_U}\big) - \theta_1 \log\|\varpi_W\|^2_{h_W} + \theta_0 \operatorname{tr}\mathrm{Ein}(B) + \operatorname{tr}\big(CR(B)\big), \tag{36}$$

and we use (36) as the definition of $\mathcal{K}$ for an arbitrary volume datum. Let $H = \ker D$. There is a canonical isomorphism

$$\iota_D : \det V \otimes (\det U)^{-1} \to \det H, \qquad (v_1\wedge\cdots\wedge v_{n_0+r})\otimes(Dv_{n_0+1}\wedge\cdots\wedge Dv_{n_0+r})^{-1} \mapsto v_1\wedge\cdots\wedge v_{n_0},$$

for any basis (v_i) of V whose first n_0 vectors span H. Put $\bar\varpi = (\iota_D(\varpi_V \otimes \varpi_U^{-1}), \varpi_W)$, a volume datum for the cohomology representation $\mathcal{H} = (H \to W)$, and

$$\mathcal{K}_{\mathrm{coh}}(\mathcal{H}; h_0, h_1; \bar\varpi) = q + \theta_0 \log\big\|\iota_D(\varpi_V \otimes \varpi_U^{-1})\big\|^2_{h_0} - \theta_1 \log\|\varpi_W\|^2_{h_1},$$
$$\mathcal{K}_*(\mathcal{H};\bar\varpi) = \inf_{h_0,h_1} \mathcal{K}_{\mathrm{coh}}.$$

Lemma 8.1. *For all metrics, with $h_0 = h_V|_H$ and $\det'$ the product of the nonzero eigenvalues,*

$$\log\|\varpi_V\|^2_{h_V} - \log\|\varpi_U\|^2_{h_U} + \log\det'(D^\dagger D) = \log\big\|\iota_D(\varpi_V \otimes \varpi_U^{-1})\big\|^2_{h_0}.$$

Proof. Take an h_V-orthonormal basis (v_i) with $v_1,\dots,v_{n_0} \in H$ and an h_U-orthonormal basis (u_i), let F be the matrix of $D|_{H^\perp}$, and write $\varpi_V = a_V v_1\wedge\cdots\wedge v_{n_0+r}$, $\varpi_U = a_U u_1\wedge\cdots\wedge u_r$. Then $\iota_D(\varpi_V\otimes\varpi_U^{-1}) = (a_V/a_U)\det F \cdot v_1\wedge\cdots\wedge v_{n_0}$, and $|\det F|^2 = \det'(D^\dagger D)$. □

This is the comparison already used in the proof of Proposition 7.7, where c_D is the logarithm of the squared modulus of ι_D in the reference bases.

Theorem 8.2 (Infimum of the potential)**.** *Let the charges be ordered. For every complex and volume datum,*

$$\mathcal{K}(h;\varpi) = \mathcal{K}_{\mathrm{coh}}(\mathcal{H}; h_0, h_1; \bar\varpi) + \theta_0\big[r\gamma + \operatorname{tr}' E_1(D^\dagger D)\big] + \operatorname{tr} Y,$$
$$\inf_h \mathcal{K}(h;\varpi) = \mathcal{K}_*(\mathcal{H};\bar\varpi) + \theta_0\gamma\, r,$$

where tr' *is the trace over the nonzero spectrum; the infimum is not attained. Consequently, let* E, E' *be two complexes of the class* (2) *with semistable cohomology and* $f = (f_V, f_U, f_W) : \mathsf{E} \to \mathsf{E}'$ *a quasi-isomorphism of complexes of representations, that is, a morphism inducing an isomorphism* $\mathcal{H} \to \mathcal{H}'$ *of the cohomology representations. If the volume data correspond,* $\bar\varpi' = f_*\bar\varpi$*, then*

$$\inf_{h'} \mathcal{K}_{\mathsf{E}'}(h';\varpi') - \inf_h \mathcal{K}_{\mathsf{E}}(h;\varpi) = \theta_0\gamma\,(r' - r).$$

Proof. The first identity is Proposition 7.7 with Lemma 8.1 and

$$\operatorname{tr}\mathrm{Ein}(D^\dagger D) = \log\det'(D^\dagger D) + r\gamma + \operatorname{tr}' E_1(D^\dagger D);$$

the infimum is Corollary 7.8. The function $\mathcal{K}_*$ is transported by isomorphisms of representations with volume data. □

We note three special cases. For an invertible triple $g = (g_V, g_U, g_W)$ and fixed ϖ, $\inf\mathcal{K}_{g\cdot\mathsf{E}} = \inf\mathcal{K}_{\mathsf{E}} - \theta_0\log|\det g_V/\det g_U|^2 + \theta_1\log|\det g_W|^2$. A shear $g_V = \left(\begin{smallmatrix} I & S \\ 0 & I\end{smallmatrix}\right)$ in a splitting $V = H\oplus H'$, which leaves D and the arrows on H unchanged and replaces the arrows T_j on H' by $T_j - A_j|_H S$, does not change the infimum. Adding an acyclic summand $H'' \xrightarrow{F''} U''$ with arbitrary arrows $H'' \to W$ adds $\theta_0[\gamma\dim U'' + \log|\det F''|^2]$, independently of those

arrows. In [Wue26a, §8.1] the shear and the removal of orthogonal acyclic summands were compared for fixed metrics on V, U, W and λ large.

Remark 8.3. The *normalized potential* $\mathcal{K} - \theta_0 \gamma\, r$ has infimum $\mathcal{K}_*(\mathcal{H}; \varpi)$, the infimum of the Kempf–Ness function of the cohomology representation, for every complex with that cohomology. The constant reflects the choice of regularization: for $b > 0$, $\log b = \int_0^1 (1 - e^{-xb}) \frac{dx}{x} - \int_1^\infty e^{-xb} \frac{dx}{x} - \gamma$, and $\mathrm{Ein}(b)$ is the first integral alone. The primitive of Ω is determined on the metrics of one complex only up to an additive constant, and the normalized potential fixes these constants compatibly.

8.2. **The potential on moduli.** Let $\mathcal{R} = \mathrm{Hom}(\mathbb{C}^{n_0}, \mathbb{C}^{n_1})^k$ be the space of representations of dimension vector (n_0, n_1), with the action of $\mathrm{G} = \mathrm{GL}_{n_0} \times \mathrm{GL}_{n_1}$, $b_j \mapsto g_1 b_j g_0^{-1}$, and $\mathrm{K} = \mathrm{U}(n_0) \times \mathrm{U}(n_1)$. Put

$$\varkappa(b) = \inf_{g \in \mathrm{G}} \Big[\|g \cdot b\|^2 + \theta_0 \log |\det g_0|^2 - \theta_1 \log |\det g_1|^2 \Big],$$

let $\mathcal{R}^{\mathrm{s}}$ be the set of stable points, $\mathcal{R}_0 = \{C_0 = \theta_0 I,\ C_1 = \theta_1 I\} \cap \mathcal{R}^{\mathrm{s}}$ the zeros of the moment map in it, and $\mathcal{M}^{\mathrm{s}} = \mathcal{R}^{\mathrm{s}}/\mathrm{G} = \mathcal{R}_0/\mathrm{K}$ the moduli space of stable representations [Kin94], with the Kähler form ω_θ obtained by reduction of $\omega_{\mathrm{flat}} = \mathrm{i}\partial\bar\partial \|b\|^2$: its pullback to $\mathcal{R}_0$ is the restriction of ω_{flat}. Let $\mathrm{pr} : \mathcal{R}^{\mathrm{s}} \to \mathcal{M}^{\mathrm{s}}$ be the projection. The stable locus may be empty, for instance for $k = 2$ and $n_0 = n_1 \geq 2$; the statements about it are then vacuous.

Lemma 8.4. *The function $\varkappa$ has the following properties.*

(a) $\varkappa(g \cdot b) = \varkappa(b) - \theta_0 \log |\det g_0|^2 + \theta_1 \log |\det g_1|^2$.
(b) *$\varkappa$ is real-analytic on $\mathcal{R}^{\mathrm{s}}$ and $\mathrm{i}\partial\bar\partial\varkappa = \mathrm{pr}^* \omega_\theta$ there.*

Proof. (a) Substitute $g'g$ in the infimum. (b) Let $b_0 \in \mathcal{R}_0$ and, for a pair $\zeta = (\zeta_0, \zeta_1)$ of Hermitian matrices, let $f(\zeta, b) = \left\| e^\zeta \cdot b \right\|^2 + 2\theta_0 \operatorname{tr} \zeta_0 - 2\theta_1 \operatorname{tr} \zeta_1$, so $\varkappa(b) = \inf_\zeta f(\zeta, b)$. Then $\partial_\zeta f(0, b_0) = 0$, and $\partial_\zeta^2 f(0, b_0)[\zeta, \zeta] = 4 \left\| \zeta \cdot b_0 \right\|^2$, with $\zeta \cdot b = (\zeta_1 b_j - b_j \zeta_0)_j$ the infinitesimal action, is positive definite modulo the scalars (c, c), on which f is constant. By the implicit function theorem there is a critical point $\zeta(b)$ of $f(\cdot, b)$, real-analytic near b_0. It is the global minimizer: $b' = e^{\zeta(b)} \cdot b$ lies in $\mathcal{R}_0$, the function $x \mapsto f(x\zeta', b')$ is convex with zero derivative at 0 for every ζ', and every element of G is $u e^{\zeta'}$ with $u \in \mathrm{K}$. For a holomorphic curve $b(w)$ with $b(0) = b_0$ and $\dot b = b'(0)$, differentiating the minimum gives $\partial_w \partial_{\bar w} \varkappa = f_{w\bar w} - f_{w\zeta} f_{\zeta\zeta}^{-1} f_{\zeta\bar w}$ with $f_{w\bar w} = \left\| \dot b \right\|^2$, $f_{w\zeta}[\zeta] = 2 \left\langle \zeta \cdot b_0, \dot b \right\rangle$ and $f_{\zeta\zeta}[\zeta, \zeta'] = 4 \operatorname{Re} \langle \zeta \cdot b_0, \zeta' \cdot b_0 \rangle$. At a point of $\mathcal{R}_0$ the K-orbit is isotropic, so the real space $\{\zeta \cdot b_0\}$ is orthogonal to its multiple by i, and a real orthonormal basis of it is a unitary basis of the tangent space $\mathfrak{g} \cdot b_0$ of the complex orbit. Hence

$$\partial_w \partial_{\bar w} \varkappa(b(w))\big|_{w=0} = \left\| \dot b \right\|^2 - \left\| \Pi_{\mathfrak{g} \cdot b_0} \dot b \right\|^2,$$

the squared length of the component of $\dot b$ orthogonal to the complex orbit, which is the reduced form evaluated on $d\,\mathrm{pr}(\dot b)$. For a general stable point: every stable orbit meets $\mathcal{R}_0$ [KN79, Kin94], and for fixed g the translation $b \mapsto g \cdot b$ is a biholomorphism of $\mathcal{R}^{\mathrm{s}}$ commuting with pr and changing $\varkappa$ by a constant, by (a); so both sides of the identity are invariant under it. □

Lemma 8.4(b) is the classical description of the Kähler potential of a Kähler quotient through the shifted ambient potential; for tori see [BG04]. We include the short proof for the reader's convenience.

Let now $\mathcal{P}$ be a complex manifold with holomorphic maps $p \mapsto D(p), A_j(p)$ (matrices in fixed bases), $D(p)$ surjective, ϖ the volume datum of the bases, $\mathsf{E}(p)$ the corresponding complexes, $\mathcal{K}_\infty(p) = \inf_h \mathcal{K}_{\mathsf{E}(p)}(h;\varpi)$, and $\mathcal{P}^{\mathrm{s}}$ the open set where the cohomology representation is stable.

Theorem 8.5 (The potential on moduli)**.** *Let the charges be ordered. On $\mathcal{P}^{\mathrm{s}}$ the function $\mathcal{K}_\infty$ is real-analytic and*

$$\mathrm{i}\partial\bar\partial\mathcal{K}_\infty = \mathrm{coh}^*\omega_\theta,$$

where $\mathrm{coh} : \mathcal{P}^{\mathrm{s}} \to \mathcal{M}^{\mathrm{s}}$ *sends p to the class of the cohomology representation of $\mathsf{E}(p)$. In particular the form is the same for all holomorphic families of complexes of the class (2) with the same family of cohomology representations, and it vanishes in every direction along which the isomorphism class of the cohomology representation is constant.*

Proof. $\ker D(p)$ is a holomorphic subbundle. In a local holomorphic frame $(e_i(p))$ of it let $b(p) \in \mathcal{R}$ be the matrices of $A_j(p)|_{\ker D(p)}$; then b is holomorphic and $\mathrm{coh} = \mathrm{pr} \circ b$. Write $\iota_{D(p)}(\varpi_V \otimes \varpi_U^{-1}) = a(p)\, e_1(p) \wedge \cdots \wedge e_{n_0}(p)$ with a holomorphic and nowhere zero. By Theorem 8.2, $\mathcal{K}_\infty(p) = \varkappa(b(p)) + \theta_0 \log|a(p)|^2 + \theta_0\gamma r$, and the last two terms are pluriharmonic. Apply Lemma 8.4(b). □

Remark 8.6. For fixed metrics on V, U, W the function $(A, D) \mapsto \mathcal{K}(h)$ is not plurisubharmonic: for the symmetric charges it has a negative Levi direction along a family of complexes with constant cohomology representation [Wue26a, Proposition 7.5]. Theorem 8.5 concerns the function after minimizing over the metrics, where the infimum is not attained; along such a family it is pluriharmonic.

8.3. **Convergence along the flow in families.** For a continuous family of initial metrics $h^{\mathrm{in}}(p)$ let $h(t;p)$ be the trajectory of the phase flow of $\mathsf{E}(p)$ starting at $h^{\mathrm{in}}(p)$, and $\mathcal{K}_t(p) = \mathcal{K}_{\mathsf{E}(p)}(h(t;p);\varpi)$.

Theorem 8.7 (Convergence in families)**.** *Let the charges be ordered.*

(i) *If the cohomology representation of $\mathsf{E}(p)$ is semistable, then $\mathcal{K}_t(p) \to \mathcal{K}_\infty(p)$ as $t \to \infty$, although the infimum is not attained and no zero of the moment map need exist. For admissible charges $\mathcal{K}_t(p)$ is strictly decreasing in t.*

(ii) *On compact subsets of $\mathcal{P}^{\mathrm{s}}$, $0 < \mathcal{K}_t - \mathcal{K}_\infty \le Ce^{-ct}$ with $C, c > 0$ depending only on the subset.*

(iii) *On $\mathcal{P}^{\mathrm{s}}$ all parameter derivatives of $\mathcal{K}_t$ converge to those of $\mathcal{K}_\infty$ in the sense of distributions; in particular $\mathrm{i}\partial\bar\partial\mathcal{K}_t \to \mathrm{coh}^*\omega_\theta$ weakly.*

Proof. (i) By Theorem 8.2, $\mathcal{K}_t - \mathcal{K}_\infty = \mathcal{E}(t) + \theta_0 \operatorname{tr}' E_1(D^\dagger D) + \operatorname{tr} Y$ with $\mathcal{E} = \mathcal{K}_{\mathrm{coh}} - \mathcal{K}_* \ge 0$ as in Section 6. The second term is at most $\theta_0 r E_1(\lambda(t))$ and the third at most $s(t)$; both tend to zero by Theorem 5.7, and $\mathcal{E}(t) \to 0$ by Theorem 6.1. Monotonicity for admissible charges is Theorem 7.6; positivity of the difference is Corollary 7.8.

(ii) For $t \ge t_1$, $\operatorname{tr} Y \le s \le s_1 e^{-\frac{\alpha_1}{2}(t-t_1)}$ and $\theta_0 \operatorname{tr}' E_1 \le \theta_0 r e^{-\lambda} \le \theta_0 r e^{\mathfrak{c}_3} e^{-\frac{\alpha_1}{4}(t-t_1)}$ by Theorem 5.7 and (23), and $\mathcal{E}(t) \le \mathfrak{C}_2 e^{-2\beta t}$ by the proof of Theorem 6.6. All constants there and in (24) are continuous functions of $N(0)$, $\lambda(0)$, $\operatorname{tr} E_1(L)|_{t=0}$, $\mathcal{E}(0)$ and ν_*, which are continuous in p for continuous initial metrics; and $\nu_*(p)$ is positive on $\mathcal{P}^{\mathrm{s}}$, being the least eigenvalue of the Hessian form of Lemma 6.4 at a zero of the moment map, which depends continuously on p modulo K by the proof of Lemma 8.4(b), on the orthogonal complement of the scalars.

(iii) Locally uniform convergence of continuous functions implies convergence of all derivatives in the sense of distributions. □

Corollary 8.8. *In Theorem 6.6 the limit $\mathfrak{h}_\infty$ depends continuously on the initial metric. Along a continuous family of complexes whose cohomology representations stay stable, with continuous initial metrics, it depends continuously on the parameter, in continuous local frames of* $\ker D$.

Proof. The constants of Theorem 6.6 are locally uniform in such families: $N(0)$, t_2 and $\mathcal{E}(0)$ are continuous, and ν_* is continuous and positive on the stable locus, as shown in the proof of Theorem 8.7(ii). So $d(\mathfrak{h}(t),\mathfrak{h}_\infty) \le \mathfrak{C}e^{-\beta t}$ uniformly, and $\mathfrak{h}(T)$ depends continuously on the data for each fixed T. □

Pointwise convergence of parameter derivatives requires regularity of the initial metrics. For example, take the constant family with symmetric charges, $D = (0,1)$ and $A_1 = (1,0)$, and let h_V, h_W initially be the identity and $h_U = e^{|\operatorname{Re} p|}$ for p in a disk. The cohomology arrow is the identity on a line and stays harmonic. The orthogonal splitting is preserved, with phases $3\pi/4$ and $-\pi/4$ on the two acyclic lines. Thus $L = e^{\pi t + |\operatorname{Re} p|}$ and $Y = 0$. Theorem 8.2 gives $\mathcal{K}_t - \mathcal{K}_\infty = E_1(e^{\pi t+|\operatorname{Re} p|})$, whose one-sided derivatives in $\operatorname{Re} p$ at zero are opposite and nonzero for every finite t. For smoothly varying initial metrics the question of pointwise convergence remains open for these complexes. Outside the stable locus the infimum need not be differentiable. For example let $k = 2$, $n_0 = n_1 = 2$ (so $\theta_0 = \theta_1$) and $b = (I_2, \left(\begin{smallmatrix} 0 & a \\ a' & 0 \end{smallmatrix}\right))$ with $a, a' \neq 0$. This representation is isomorphic to the direct sum of the two representations $(1, \pm c)$ of dimension vector $(1,1)$, $c^2 = aa'$, by a simultaneous conjugation, which does not change $\varkappa$ because $\theta_0 = \theta_1$; and on a direct sum of representations $(1, c_i)$ the infimum is attained at a diagonal zero of the moment map. Hence

$$\varkappa(b) = 2\theta_0 + 2\theta_0 \log \frac{1 + |a|\,|a'|}{\theta_0} \qquad (a, a' \neq 0),$$

which has no differentiable extension to $a = 0$, where the representation is semistable and not polystable.

9. Mass, extreme phases and fixed points

In [HKKP26, §4.1] the structures of a categorical Kähler metric are listed as axioms that "should be read more as a collection of desiderata rather than as a final definition". For the flow the text reads: "We expect the following quantities to vary monotonically under this action: $m(X,h)$ decreases, $\phi^-(X,h)$ increases, and $\phi^+(X,h)$ decreases. Moreover, the fixed points of the action on $\operatorname{Met}(X)$/rescaling are precisely the harmonic metrics." Here $\phi^\pm$ are the ends of the support of the mass measure, which is defined in [HKKP26, §4.2] by $\int f\, d\mu_{X,h} = \Omega\big(e^{-\sqrt{-1}\arg\Omega} f(\arg\Omega/\pi)\big)$; there Ω is the functional $a \mapsto \operatorname{tr}(\Omega a)$, so that the right side is $\operatorname{tr}\big(|\Omega|\, f(\arg\Omega/\pi)\big)$. We write ν_h for this measure, since μ denotes the moment map. For the endomorphism Ω of the complex (2), the polar phase $\Phi = \Phi_V \oplus \Phi_U \oplus \Phi_W$ and phases measured in radians, this is the measure

$$\nu_h = \sum_k p_k\, \delta_{\phi_k}, \qquad p_k = \langle v_k, |\Omega| v_k\rangle > 0,$$

for an orthonormal eigenbasis (v_k) of Φ with eigenvalues ϕ_k; equivalently $\int f\, d\nu_h = \operatorname{tr}(|\Omega| f(\Phi))$. Its total mass is $m(h)$, and

$$\phi_+ = \max\big(\operatorname{spec}\Phi_V \cup \operatorname{spec}\Phi_W\big), \qquad \phi_- = \min \operatorname{spec}\Phi_U.$$

A metric is harmonic if Φ is a scalar. In [HKKP26, Remark 5.15] the metrics of an object are the metrized complexes isomorphic to it in the derived category. The statements of this section concern the metrics on a given complex, which the flow preserves. For these we examine whether the mass is nonincreasing, whether ϕ_+ is nonincreasing and ϕ_- nondecreasing, and whether the fixed points modulo rescaling are the harmonic metrics. For a representation all of this holds (Proposition 9.1). For the complexes (2) the mass and both extreme phases can move in the opposite direction (Propositions 9.2 and 9.3), and there are neither fixed points nor harmonic metrics on the complex (Proposition 9.4); what converges is the mass measure (Proposition 9.5).

9.1. **Representations.** Let a representation of the generalized Kronecker quiver with metrics be given, regarded as a complex in degree zero, with dimension vector (n_0, n_1), $n_0, n_1 \geq 1$. In orthonormal frames let H_j be the matrices of the arrows, $C_0 = \sum_j H_j^\dagger H_j$ and $C_1 = \sum_j H_j H_j^\dagger$. Its endomorphism $\Omega = (z_0 + C_0) \oplus (z_1 - C_1)$ is normal, with phases $\phi_0 = \arg(z_0 + C_0)$ and $\phi_1 = \arg(z_1 - C_1)$, and the flow is $\dot{H}_j = \frac{1}{2}(H_j\phi_0 - \phi_1 H_j)$.

Proposition 9.1. *For a representation and all charges in the upper half-plane, along the phase flow:*

(i) *the mass is nonincreasing;*
(ii) *the largest phase is nonincreasing and the smallest phase is nondecreasing;*
(iii) *the fixed points on metrics modulo rescaling are the metrics with $\phi_0 = \varphi\, I$ and $\phi_1 = \varphi\, I$, that is, the harmonic metrics.*

Proof. From the flow, $\dot{C}_0 = \phi_0 C_0 - \sum_j H_j^\dagger \phi_1 H_j$ and $\dot{C}_1 = \sum_j H_j \phi_0 H_j^\dagger - \phi_1 C_1$, because ϕ_0 commutes with C_0 and ϕ_1 with C_1.

(i) This is [Wue26a, §6.2]: the derivative of $x \mapsto |z_0 + x|$ is $\cos\arg(z_0 + x)$ and that of $x \mapsto |z_1 - x|$ is $-\cos\arg(z_1 - x)$, so in eigenbases of ϕ_0 (eigenvalues a_i) and ϕ_1 (eigenvalues b_l)

$$\dot{m} = \operatorname{tr}(\cos\phi_0\, \dot{C}_0) - \operatorname{tr}(\cos\phi_1\, \dot{C}_1) = \sum_{j,i,l} |(H_j)_{li}|^2 (\cos a_i - \cos b_l)(a_i - b_l) \leq 0,$$

because the cosine decreases on $(0, \pi)$.

(ii) We have $\phi_0 = f_0(C_0)$ and $\phi_1 = f_1(C_1)$ with $f_0(x) = \arg(z_0 + x)$ decreasing and $f_1(x) = \arg(z_1 - x)$ increasing on $[0, \infty)$. Let $\bar{\phi}_0 = f_0(\lambda_{\min}(C_0))$ and $\bar{\phi}_1 = f_1(\lambda_{\max}(C_1))$ be the largest eigenvalues of ϕ_0 and ϕ_1. For a unit eigenvector v of C_0 for its least eigenvalue c, $\left\langle v, \dot{C}_0 v \right\rangle = \bar{\phi}_0 c - \sum_j \langle H_j v, \phi_1 H_j v \rangle \geq (\bar{\phi}_0 - \bar{\phi}_1)c$. The right lower derivative of $\lambda_{\min}(C_0)$ is the minimum of $\left\langle v, \dot{C}_0 v \right\rangle$ over such v; so whenever $\bar{\phi}_0 \geq \bar{\phi}_1$, the upper right derivative of $\bar{\phi}_0$ is nonpositive. For a unit eigenvector w of C_1 for its largest eigenvalue c', $\left\langle w, \dot{C}_1 w \right\rangle = \sum_j \left\langle H_j^\dagger w, \phi_0 H_j^\dagger w \right\rangle - \bar{\phi}_1 c' \leq (\bar{\phi}_0 - \bar{\phi}_1)c'$; so whenever $\bar{\phi}_1 \geq \bar{\phi}_0$, the upper right derivative of $\bar{\phi}_1$ is nonpositive. Hence $\max(\bar{\phi}_0, \bar{\phi}_1)$ is nonincreasing. The statement for the smallest phase is the same argument with $\lambda_{\max}(C_0)$ and $\lambda_{\min}(C_1)$.

(iii) At a fixed point modulo rescaling the velocities are a common scalar, $\phi_0 = c\, I$ and $\phi_1 = c\, I$ with $c \in (0, \pi)$, and $\operatorname{tr}\Omega = Z$ forces $c = \varphi$. □

9.2. **The mass.** The example of [Wue26a, Theorem 1.1] with increasing mass has charges close to the real axis and an invertible differential, and the question whether the mass decreases is asked in [HKKP26, Question 5.18]. The mass can also increase for the symmetric

charges and for complexes with stable cohomology and $r = 1$, to which Theorems 7.6 and 6.6 apply.

Proposition 9.2 (A state with increasing mass)**.** *Let $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$, $k = 512$, $V = \mathbb{C}^2$, $U = \mathbb{C}$, $W = \mathbb{C}^{512}$ with the standard metrics, $D = (0, f)$ with $f^2 = \frac{383}{100}$, and let A_j be the matrix whose only nonzero row is the j-th, equal to $(-0.1978, -0.6072)$ for 480 values of j and to $(-0.5038, 0.8392)$ for the other 32. The cohomology representation $\mathbb{C} \to \mathbb{C}^{512}$ is stable, $r = 1$, and at this state*

$$m = 749.87188\ldots, \qquad \frac{dm}{dt} = 1.77041\ldots > 0, \qquad \mathcal{D} = 48.7558\ldots > 0.$$

Proof. Since $r = 1$ and the arrows have orthogonal images, Ω is in closed form: with $v = D^\dagger$, $\ell = |v|^2$, a_j the nonzero row of A_j and $C = \sum_j a_j^\dagger a_j$,

$$\Omega_V = z_0\big(I - R(\ell)vv^\dagger\big) + C - \tfrac{1-R(\ell)}{\ell}(vv^\dagger C + Cvv^\dagger) + \tfrac{1-2R(\ell)+e^{-\ell}}{\ell^2}(v^\dagger Cv)\, vv^\dagger,$$

$\Omega_U = (v^\dagger Cv)(1 - e^{-\ell} - \ell e^{-\ell})/\ell^2 - z_0 e^{-\ell}$, and Ω_W is diagonal with entries $z_1 - |a_j|^2 + \frac{1-R(\ell)}{\ell}|a_j v|^2$. The flow preserves this form, with $\dot v = \frac12(\Phi_V - \Phi_U)v$ and $\dot a_j = \frac12(a_j\Phi_V - (\Phi_W)_{jj}a_j)$, and $\frac{d}{dt}\operatorname{tr}|T| = \operatorname{Re}\operatorname{tr}(\dot T\, e^{-\mathrm{i}\Psi})$ for a matrix T with polar phase Ψ. The 2×2 polar decomposition is explicit: $|T| = (M + \sqrt{\det M})/\sqrt{\operatorname{tr} M + 2\sqrt{\det M}}$ with $M = TT^\dagger$. The three numbers were enclosed with the ball arithmetic of Arb [Joh17] at 500 bits; the enclosures have radius below 10^{-20}.[1] Stability: every row a_j has a nonzero first entry, so the proper nonzero subrepresentations are subspaces of W, of phase $\frac{\pi}{4} < \varphi$. □

With these two rows in the proportion 15 : 1, the same formulas give $\dot m > 0$ for $k = 400$ arrows and $\dot m < 0$ for $k = 368$; both signs are enclosed by the same program. For $k \le 3$ arrows and the symmetric charges, a numerical maximization of $\dot m/m$ over the maps D, A_j from random starting points, for small dimensions, found no metric with increasing mass; this observation is not used in any proof.

9.3. **The extreme phases.**

Proposition 9.3 (Extreme phases)**.** *Let $z_0 = -1 + \mathrm{i}$, $z_1 = 1 + \mathrm{i}$, and take the standard metrics.*

(i) *For $V = \mathbb{C}^2$, $U = W = \mathbb{C}$, $D = (0, f)$ with $f^2 = \frac{1}{10}$, $A_1 = (1, 0)$ and $A_2 = (0, 10)$, the smallest phase satisfies $\frac{d}{dt}\phi_- = -0.38959518\ldots < 0$.*

(ii) *For $V = \mathbb{C}^3$, $U = \mathbb{C}$, $W = \mathbb{C}^2$, $D = (0, 0, \frac92)$,*

$$A_1 = \begin{pmatrix} 5.7 & -1.7 & 6 \\ 6 & 0 & -5.6 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 6 & 0.1 & -6 \\ -6 & -1.8 & -6 \end{pmatrix},$$

the largest eigenvalue of Φ_V is simple and exceeds the eigenvalues of Φ_W, and $\frac{d}{dt}\phi_+ = 0.0060800323\ldots > 0$.

In both cases $r = 1$ and the cohomology representation is polystable; in (i) it is stable.

Proof. (i) Here $C = \operatorname{diag}(1, 100)$, all blocks of Ω are numbers, and the flow preserves this form. With $\ell = f^2$, $x = 100/\ell$ and $\mathsf{m}(\ell) = 1 - (1 + \ell)e^{-\ell}$, the phases are $\arg(z_0 + 1)$ and $\Phi_A := \arg(z_0 + 100)$ on the two lines of V, $\Phi_W = \arg(z_1 - 1 - x(1 - e^{-\ell}))$, and $\Phi_U =$

[1] The program `prop_9_2_mass.py` and its output accompany this paper as supplementary files. The program also checks the trace identity (7), compares the derivative with a difference quotient of the enclosed mass along the velocity, and gives $\dot m < 0$ for the same state with all rows divided by 20.

$\arg(z_0e^{-\ell} - x\,\mathsf{m}(\ell)) - \pi = -\arctan\frac{1}{1+y}$ with $y = x\,\mathsf{m}(\ell)e^{\ell}$; moreover $\dot\ell = \ell(\Phi_A - \Phi_U)$ and $\dot x = x(\Phi_U - \Phi_W)$ by (10) and (17). Hence $\dot\Phi_U = \dot y/((1+y)^2+1)$ and $\dot y/y = (\Phi_U - \Phi_W) + (\ell e^{-\ell}/\mathsf{m}(\ell) + 1)\dot\ell$, an explicit expression, which was enclosed with interval arithmetic.[2] The cohomology representation is $1 : \mathbb{C} \to \mathbb{C}$ with a second arrow equal to zero.

(ii) Since $r = 1$, Ω is given by the formulas in the proof of Proposition 9.2, with $C = A_1^\dagger A_1 + A_2^\dagger A_2$ and $\Omega_W = z_1 - \sum_j A_j\big(I - \frac{1-R(\ell)}{\ell}vv^\dagger\big)A_j^\dagger$; the derivative of Ω_V along the flow is obtained by the product rule from $\dot v = \frac12(\Phi_V - \Phi_U)v$ and $\dot A_j = \frac12(A_j\Phi_V - \Phi_W A_j)$. By Lemma 4.4, the derivative of a simple eigenvalue of Φ_V with spectral projection Π is $2\operatorname{tr}(\Pi\Theta)$, where Θ solves (21) for the variation $\dot\Omega_V$. The spectra of Φ_V and Φ_W, the projection Π and the solution Θ were enclosed with interval arithmetic at 400 bits; the spectra are $\{0.0071\ldots,\ 0.1929\ldots,\ 3.13307\ldots\}$ and $\{3.12792\ldots,\ 3.12883\ldots\}$. The eigenvalue ϕ_+ carries the weight $6.09\ldots\times10^{-6}$ of the mass measure, whose total mass is $303.13\ldots$ The cohomology representation $\mathbb{C}^2 \to \mathbb{C}^2$ has the arrows H_1, H_2 given by the first two columns of A_1, A_2. The polynomial $\det(H_1 - \tau H_2) = -\frac{51}{5}\tau^2 + \frac{1053}{50}\tau + \frac{51}{5}$ has two distinct roots τ_1, τ_2; if $H_1 v_i = \tau_i H_2 v_i$, then the lines $\mathbb{C}v_i \to \mathbb{C}H_2v_i$ are two non-isomorphic stable subrepresentations of dimension vector $(1,1)$, and the representation is their direct sum. □

9.4. **Fixed points and harmonic metrics.**

Proposition 9.4. *Let $r \geq 1$. For all charges in the upper half-plane the phase flow has no fixed point and no periodic orbit on the metrics of the complex modulo rescaling, and the complex carries no harmonic metric. If $\alpha_0 \neq \alpha_1$, the phase potential has no critical point.*

Proof. The least eigenvalue λ of $DD^\dagger$ is invariant under a common rescaling of the metrics, and by Proposition 2.3(i) its right derivative is at least $\lambda\min\big(\langle v, \Phi_{AA}v\rangle - \langle u, \Phi_U u\rangle\big) > 0$ over unit vectors, since $\Phi_V > 0 > \Phi_U$. So λ is strictly increasing along every trajectory. A harmonic metric would have Φ_V and Φ_U equal to the same scalar, which is impossible for the same reason. The differential of the phase potential is $\xi \mapsto \sigma^{-1}\sum_\bullet \operatorname{tr}(\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_\bullet)\xi_\bullet)$ (Section 7), so at a critical point $\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_\bullet) = 0$ for all three blocks. But $\operatorname{tr}\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_U) = -\sigma\operatorname{tr}J - c_0\operatorname{tr}e^{-L} < 0$ if $c_0 > 0$, and $\operatorname{tr}\operatorname{Im}(e^{-\mathrm{i}\varphi}\Omega_V) = c_0\operatorname{tr}e^{-B} - \sigma\operatorname{tr}K < 0$ if $c_0 < 0$; here $\operatorname{tr}J \geq 0$ by Theorem 3.1, $\operatorname{tr}K = \operatorname{tr}(Ce^{-B}) \geq 0$, and $c_0 \neq 0$ exactly when $\alpha_0 \neq \alpha_1$. □

So on the metrics of the given complex the set of fixed points and the set of harmonic metrics coincide trivially, both being empty. In the sense of [HKKP26, Remark 5.15] the harmonic metrics of the object live on other representatives; for ordered charges and polystable cohomology Theorem 6.6 identifies the limit of the flow with a harmonic metric of the cohomology representation.

9.5. **The mass measure.** The extreme phases need not converge to φ. If the arrows vanish on $(\ker D)^\perp$, then $\Omega_V = (z_0 + C_0) \oplus z_0e^{-F^\dagger F}$ and $\Omega_U = -z_0e^{-L}$, the flow preserves this form, and the phases of the acyclic part are α_0 and $\alpha_0 - \pi$ for all time, while their weights $|z_0|e^{-\ell_i}$ in the mass measure tend to zero.

Proposition 9.5. *For every metric on the complex,*

$$m(h) - |Z| = \int \big(1 - \cos(\phi - \varphi)\big)\, d\nu_h(\phi) \geq 0.$$

Along a trajectory, $\nu_{h(t)} \to |Z|\,\delta_\varphi$ weakly if and only if $m(h(t)) \to |Z|$. In particular $\nu_{h(t)} \to |Z|\,\delta_\varphi$ for ordered charges and semistable cohomology.

[2] The program `prop_9_3_extreme_phases.py` and its output accompany this paper as supplementary files.

Proof. For a unit eigenvector v_k of Φ, $\langle v_k, \Omega v_k\rangle = p_k e^{\mathrm{i}\phi_k}$, because $\Omega = |\Omega| e^{\mathrm{i}\Phi}$. Hence $\int e^{\mathrm{i}\phi} d\nu_h = \operatorname{tr}\Omega = Z$ and $\int d\nu_h = m(h)$; take the real part of $e^{-\mathrm{i}\varphi}$ times the first identity. The phases lie in $(-\pi,\pi)$ and $\varphi \in (0,\pi)$, so on $[-\pi,\pi]$ the function $1-\cos(\phi-\varphi)$ vanishes only at $\phi=\varphi$. If $m \to |Z|$, the $\nu_{h(t)}$-measure of the complement of any neighbourhood of φ therefore tends to zero, and the total mass tends to $|Z|$; the converse follows by integrating the constant 1. The last statement is Theorem 6.1. □

10. Charges for which the potential is not monotone, and questions

The phase potential is monotone for the charges of Theorem 7.6. This section gives charges for which it is not, and states the questions left open.

10.1. **Reversed and ordered charges.**

Proposition 10.1. *If all $A_j = 0$, then for all charges in the upper half-plane*

$$\mathcal{D} = c_0\big[\, n_0(\alpha_0-\alpha_1) + \pi \operatorname{tr} e^{-L}\,\big].$$

In particular, for reversed charges one has $\mathcal{D}<0$ at every such state with $\pi \operatorname{tr} e^{-L} > n_0(\alpha_1-\alpha_0)$; such states exist if and only if $\pi r > n_0(\alpha_1-\alpha_0)$.

Proof. The matrices $\Omega_V = z_0 e^{-B}$, $-\Omega_U = z_0 e^{-L}$, $\Omega_W = z_1 I$ are normal with phases α_0, $\alpha_0-\pi$, α_1. Hence $\mathcal{D}_V = |z_0|(n_0 + \operatorname{tr} e^{-L})(\alpha_0-\varphi)\sin(\alpha_0-\varphi)$, $\mathcal{D}_U = |z_0|\operatorname{tr} e^{-L}(\pi-\alpha_0+\varphi)\sin(\alpha_0-\varphi)$ and $\mathcal{D}_W = n_1 c_1(\varphi-\alpha_1) = n_0 c_0(\varphi-\alpha_1)$. For reversed charges $c_0<0$, and $\operatorname{tr} e^{-L}$ takes all values in $(0,r)$. □

For example, for $z_0 = 1+2\mathrm{i}$, $z_1 = -3+2\mathrm{i}$, $n_0 = n_1 = r = 1$, $D = (0,\frac12)$ and standard metrics, $\mathcal{D} = -1.78927\ldots$

For ordered charges the potential can also fail to be monotone when $r \geq 2$.

Proposition 10.2. *Let $z_0 = 10^{-6}(-1+\mathrm{i})$ and $z_1 = 1+\frac{\mathrm{i}}{500}$, so that $\alpha_0 = \frac{3\pi}{4} > \alpha_1$, and let $n_0 = n_1 = 1$, $r = 2$, $k = 2$. With standard metrics on $V=\mathbb{C}^3$, $U=\mathbb{C}^2$, $W=\mathbb{C}$, let*

$$D = \begin{pmatrix} 0 & \sqrt{\ell_1} & 0 \\ 0 & 0 & \sqrt{\ell_2} \end{pmatrix}, \quad A_1 = (\tfrac15, 0, 0), \quad A_2 = (0, a_1, a_2), \qquad \ell_1 = \tfrac{1}{20},\ \ell_2 = 8,$$

$$a_1 = \sqrt{\epsilon/\ell_1}, \quad a_2 = \tfrac{7}{10}\sqrt{\epsilon\,\ell_2}, \quad \epsilon = \tfrac{3}{10}\Big(\frac{1-e^{-\ell_1}}{\ell_1^2} + \frac{49}{100}(1-e^{-\ell_2})\Big)^{-1}.$$

The cohomology representation is the nonzero map $\frac15 : \mathbb{C}\to\mathbb{C}$, which is stable, and

$$\mathcal{D} = -1.6654354\ldots \times 10^{-6} < 0, \qquad (\mathcal{D}_V, \mathcal{D}_U, \mathcal{D}_W) \approx (6.72,\ -9.09,\ 0.70)\times 10^{-6}.$$

Proof. Since $C = \operatorname{diag}(\frac{1}{25}, (a_1,a_2)^\dagger(a_1,a_2))$ has no block between $\ker D$ and its complement, all blocks of Ω are of size 1 or 2, and the polar decomposition of a 2×2 matrix is explicit, as in the proof of Proposition 9.2. The value of $\mathcal{D}$ was enclosed with interval arithmetic at 300 bits; the enclosure has radius below 10^{-30}.[3] At the same state and for the symmetric charges the same computation gives $\mathcal{D} = 3.79\ldots > 0$, in agreement with Theorem 7.6. □

[3] The program `prop_10_1_10_2_dissipation.py` and its output accompany this paper as supplementary files.

The mechanism is the following. As $z_0 \to 0$ with $z_1 = e^{\mathrm{i}\epsilon'}$ and no arrows on the cohomology, Ω_V and Ω_U tend to the Hermitian matrices $K_{F^\dagger F}(C_T)$ and J, whose phases have the eigenvalues $0, \pi$ and $0, -\pi$ respectively, and $\mathcal{D}/\sigma \to \epsilon' \operatorname{tr} Y/(1 - \operatorname{tr} Y) + \pi\big(\operatorname{tr} K_{F^\dagger F}(C_T)_- - \operatorname{tr} J_-\big)$ up to terms of order ϵ'^2 ($n_1 = 1$, $\operatorname{tr} Y < 1$). Configurations with $\operatorname{tr} J_- > \operatorname{tr} K_{F^\dagger F}(C_T)_-$ exist, and then $\mathcal{D} < 0$ for small ϵ'. By Theorems 5.7 and 6.6 the failure of monotonicity of the potential does not affect the long-time behaviour.

10.2. **Questions.** We collect the questions left open.

(1) Determine the set of charges for which $\mathcal{D} \geq 0$ at every state. It contains the admissible charges, and by Proposition 10.2 it does not contain all ordered charges when $r \geq 2$.
(2) For semistable cohomology that is not polystable, does the bounded factor between the induced metrics and the asymptotic solutions of [HKKP23] converge, as in Proposition 6.9 (Remark 6.10; [Wue26b, Corollary 6.5])?
(3) Is the mass nonincreasing along the flow for the symmetric charges and few arrows?
(4) For smoothly varying initial metrics, do the parameter derivatives of $\mathcal{K}_t$ converge pointwise on $\mathcal{P}^{\mathrm{s}}$ (Theorem 8.7)?
(5) Extend Theorems 5.7 and 6.6 to complexes with acyclic parts at both vertices or in several degrees. The block of degree one for a two-sided differential involves second divided differences of e^{-x} at ℓ_i, p_μ, ℓ_j, summed over eigenvalues p_μ of the Laplacian at the other vertex. Theorem 3.1 treats the case $p_\mu = 0$.

References

[Aud11] K. M. R. Audenaert, *A characterisation of anti-Löwner functions*, Proc. Amer. Math. Soc. **139** (2011), no. 12, 4217–4223. arXiv:1008.2943v2.

[Bha97] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics **169**, Springer, New York, 1997.

[BZ92] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205** (1992), 235 pp.

[Bri07] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[BG04] D. Burns and V. Guillemin, *Potential functions and actions of tori on Kähler manifolds*, Comm. Anal. Geom. **12** (2004), no. 1, 281–303. arXiv:math/0302350.

[Don85] S. K. Donaldson, *Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3) **50** (1985), no. 1, 1–26.

[GRS21] V. Georgoulas, J. W. Robbin and D. A. Salamon, *The moment-weight inequality and the Hilbert–Mumford criterion*, Lecture Notes in Math. **2297**, Springer, Cham, 2021. arXiv:1311.0410v2.

[HKKP23] F. Haiden, L. Katzarkov, M. Kontsevich and P. Pandit, *Semistability, modular lattices, and iterated logarithms*, J. Differential Geom. **123** (2023), no. 1, 21–66. doi:10.4310/jdg/1679503804; arXiv:1706.01073v2.

[HKKP26] F. Haiden, L. Katzarkov, M. Kontsevich and P. Pandit, *Towards categorical Kähler geometry*, preprint (2026), arXiv:2609.00978v1.

[HJ13] R. A. Horn and C. R. Johnson, *Matrix analysis*, 2nd ed., Cambridge Univ. Press, Cambridge, 2013.

[Joh17] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), no. 8, 1281–1292. doi:10.1109/TC.2017.2690633.

[KN79] G. Kempf and L. Ness, *The length of vectors in representation spaces*, in *Algebraic geometry (Copenhagen, 1978)*, Lecture Notes in Math. **732**, Springer, Berlin, 1979, 233–243.

[Kin94] A. D. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 4, 515–530. doi:10.1093/qmath/45.4.515.

[Min62] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), no. 3, 341–346. doi:10.1215/S0012-7094-62-02933-2.

[UY86] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian–Yang–Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39** (1986), no. S1, S257–S293.

[WCKQ20] D. Wang, W. Chen, S. Z. Khong and L. Qiu, *On the phases of a complex matrix*, Linear Algebra Appl. **593** (2020), 152–179. arXiv:1904.07211v2.

[Wue26a] B. J. Wuebben, *Mass increase and acyclic cancellation for phase flows on quiver complexes*, preprint (2026), doi:10.5281/zenodo.23921919.

[Wue26b] B. J. Wuebben, *Convergence of the minimizing flow on acyclic quivers*, preprint (2026), doi:10.5281/zenodo.24346923.

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