

RELATIVE METRIC ESTIMATES NEAR REDUCIBLE YANG–MILLS CONNECTIONS

BERND JOHANNES WUEBBEN

ABSTRACT. We study the natural L^2 metric near a central Yang–Mills connection on a closed surface. In the classical quadratic symplectic model, we prove a relative metric estimate with error quadratic in the harmonic deformation parameter, uniformly across all stabilizer types. We also compute the leading metric correction using the Green operators of the reference connection. The proof first eliminates the gauge directions with a fixed positive spectral gap and then minimizes over the remaining finite-dimensional orbit; it requires no lower bound on a positive orbit eigenvalue. The same argument compares every small eigenvalue of the gauge Laplacian with the infinitesimal stabilizer-action operator, including at changes of kernel dimension. As a curved finite-dimensional comparison, quaternionic projective and hyperbolic reductions admit exact orbit-metric formulas on their flat moment-map zero cone. A four-cycle is realized in both settings with independently vanishing weights. The resulting estimates distinguish uniform control of the quotient metric from estimates that depend on the smallest orbit scale.

1. INTRODUCTION

Near a reducible connection, the orbit directions of the gauge group contain vectors of arbitrarily small norm. This makes it difficult to compare quotient metrics by estimates for an inverse gauge Laplacian: the inverse becomes unbounded as the stabilizer grows. A finite-dimensional model records the small directions, but an additive error estimate for that model need not control its smallest positive eigenvalues.

For central Yang–Mills connections on a surface, the local symplectic model is classical. Atiyah and Bott identify curvature as the moment map for the gauge action [AB83]. Huebschmann constructs a stabilizer-equivariant Kuranishi chart that identifies the moduli germ with a quadratic symplectic reduction [Hue95, Hue96]. We use this chart to compare the actual L^2 quotient metric with the metric of the linear reduction. The estimates are relative and their constants do not depend on the stabilizer type or on ratios between positive orbit eigenvalues.

1.1. The surface result. Let $P \rightarrow \Sigma$ be a principal bundle with compact structure group G over a closed oriented Riemannian surface. Fix an invariant positive inner product on $\mathfrak{g}$ and a central Yang–Mills connection A_0 . Thus $F_{A_0} = \zeta \operatorname{vol}_\Sigma$ for a parallel central section ζ . We consider connections with this fixed curvature, modulo the gauge group $\mathcal{G}$. All spaces and inner products below are real. Put

$$d = d_{A_0}, \quad K = \operatorname{Stab}(A_0), \quad \mathfrak{k} = \mathcal{H}_d^0, \quad V = \mathcal{H}_d^1.$$

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Centrality gives $d^2 = 0$ on the adjoint bundle. The finite spaces $\mathfrak{k}$ and V have their L^2 inner products. Let P_j be harmonic projection in degree j , and let $Q_j = 1 - P_j$. Pairing with harmonic zero-forms identifies $\mathcal{H}_d^2$ with $\mathfrak{k}^*$. The quadratic moment map and the infinitesimal action are

$$(1) \quad \Theta(h) = \frac{1}{2}P_2[h, h], \quad T_h : \mathfrak{k} \longrightarrow V, \quad T_h \xi = [h, \xi].$$

An overall sign in the moment map depends on the convention for the gauge action and does not change its zero set. Write $Z = \Theta^{-1}(0)$, and let g_{lin} be the reduced metric on each orbit-type stratum of Z/K .

The classical chart has the form $A(h) = A_0 + a(h)$, with $a(h) = h + c(h)$ and $c(h) \in \text{im } d^*$. We recall its construction in Section 3. Its induced map on the quotient is denoted $\bar{a}$.

Theorem 1.1. *There are $\delta > 0$ and $C < \infty$ such that, for every $h \in Z \cap B_\delta(V)$, the classical chart identifies the local orbit-type strata and satisfies*

$$(2) \quad (1 - C|h|^2)g_{\text{lin}} \leq \bar{a}^*g_{L^2} \leq (1 + C|h|^2)g_{\text{lin}}.$$

The same constant applies to all strata. There is no restriction on the ratios of the nonzero eigenvalues of $T_h^*T_h$.

The leading correction is explicit. Let

$$R_2 = d^*G_2Q_2, \quad \Delta_0 = d^*d|_{\mathfrak{k}^\perp}, \quad B_h v = Q_0 \text{ad}_h^* v,$$

where G_2 is the degree-two Green operator and $\text{ad}_h \eta = [h, \eta]$. For a tangent vector v to the appropriate smooth orbit-type piece of Z , chosen L^2 -orthogonal to $\text{im } T_h$, one has

$$(3) \quad \begin{aligned} \bar{a}^*g_{L^2}([v], [v]) &= |v|^2 + \|R_2[h, v]\|^2 - \langle B_h v, \Delta_0^{-1} B_h v \rangle + \mathcal{R}_h(v), \\ |\mathcal{R}_h(v)| &\leq C|h|^3|v|^2. \end{aligned}$$

This remainder is uniform across the orbit types as well.

Here a tangent vector means a tangent vector to a smooth piece, rather than an arbitrary vector in $\ker d\Theta_h$ at a singular point. The positive correction in (3) measures the bending of the Coulomb slice. The negative correction is the part removed by nonconstant gauge transformations. These terms use only Green operators at A_0 , where the relevant complementary gap is fixed.

The same construction controls the small gauge spectrum. List eigenvalues with multiplicities, including zero eigenvalues.

Theorem 1.2. *Let $m = \dim \mathfrak{k}$ and let $0 \leq \nu_1(h) \leq \dots \leq \nu_m(h)$ be the eigenvalues of $T_h^*T_h$. For $h \in Z \cap B_\delta(V)$, the self-adjoint gauge Laplacian $d_{A(h)}^*d_{A(h)}$ has exactly m eigenvalues below a fixed positive threshold. They satisfy*

$$(4) \quad (1 - C|h|^2)\nu_j(h) \leq \lambda_j(A(h)) \leq (1 + C|h|^2)\nu_j(h), \quad 1 \leq j \leq m.$$

The next eigenvalue is bounded below uniformly, and $\ker d_{A(h)} = \ker T_h \subset \mathfrak{k}$. The estimates include parameters at which this kernel increases in dimension.

For unitary bundles, central Yang–Mills connections on a Riemann surface are Hermitian–Einstein connections. The theorems therefore compare actual gauge-theoretic metrics and operators. In higher complex dimension, a quadratic tangent model need not describe the integrable deformation locus; no such replacement is made here.

1.2. Proof mechanism and finite-dimensional comparisons. The gauge Lie algebra splits as $\mathfrak{k} \oplus \mathfrak{k}^\perp$. On $\mathfrak{k}^\perp$, the covariant derivative at A_0 has a positive gap. If u is tangent to the Coulomb graph, then

$$\langle u, d_{A(h)}\eta \rangle = \langle u, [a(h), \eta] \rangle, \quad \eta \in \mathfrak{k}^\perp.$$

The angle between the graph tangent and these gauge directions is therefore $O(|h|)$. Orthogonal projection changes squared norms by $O(|h|^2)$. Equivariance then identifies the remaining gauge action exactly with T_h . Minimizing over its image preserves the relative estimate, even if its dimension changes. A second expansion gives (3). The spectral argument uses the same Coulomb cancellation in a Schur complement.

In finite dimensions, an equivariant Darboux chart at a Kähler fixed point gives a related quadratic metric comparison simultaneously on all local central moment levels. We record this consequence of the classical construction in Section 6. It separates the elementary passage from ambient to quotient estimates from the gauge analysis needed in Theorem 1.1.

There is also an explicit curved quaternionic comparison. For a compact subgroup $K \subset \mathrm{Sp}(n)$ fixing the origin in the affine chart of $\mathbb{H}P^n$ or $\mathbb{H}H^n$, the quaternionic moment-map zero set is the same as the flat hyperkähler zero cone. If $r = |q|$ and $\varepsilon = 1$ or -1 , respectively, its orbit Gram form satisfies

$$(5) \quad G_\varepsilon(q) = (1 + \varepsilon r^2)^{-1} G_0(q).$$

Section 7 gives the exact reduced metric and its uniform bounds without a local-freeness hypothesis. Section 8 realizes the same weighted four-cycle in these curved quotients and in the surface theorem. This identifies a common operator on an explicit parameter family; it does not identify the two moduli spaces.

1.3. Relation to existing normal forms. The local symplectic reduction and its orbit-type decomposition are due to Sjamaar–Lerman [SL91]; holomorphic slice results are developed by Sjamaar [Sja95]. In the surface problem, the exact quadratic moment map and local quotient chart are Huebschmann’s results [Hue95, Theorems 2.24 and 2.32]. Our proof retains their coexact construction and estimates the full gauge quotient metric, including its leading correction. We make no claim of a new topological or symplectic local model.

Quaternionic Kähler reduction originates with Galicki–Lawson [GL88]. Swann describes its relation to hyperkähler reduction [Swa90] and the geometry of singular reductions [Swa97]. Mayrand proves a stratification and a local complex-symplectic quotient model for integrable hyperkähler actions [May22, Theorems 1.2–1.4]. Those structural results do not, by themselves, compare an arbitrary curved quotient metric with a flat one uniformly across all orbit types. Our quaternionic formulas use the explicit space-form metric.

2. RELATIVE ESTIMATES AND QUOTIENT NORMS

Let a compact group K act on a Riemannian manifold and preserve a subset Z partitioned into smooth orbit-type pieces. At z write $T_z\xi = X_\xi(z)$ for the infinitesimal action and $W_z = \mathrm{im} T_z$. On a smooth piece, the norm of a quotient tangent class is

$$(6) \quad |[v]|_g^2 = \inf_{w \in W_z} g_z(v - w, v - w).$$

This formula also applies when the stabilizer has positive dimension. It uses the image of the action, not an inverse on its Lie algebra.

Lemma 2.1. *Suppose two metrics on the same space satisfy $(1 - \eta)g_0 \leq g_1 \leq (1 + \eta)g_0$, with $0 \leq \eta < 1$. Then their orbit Gram forms and their reduced metrics satisfy the same inequalities. The assertion also holds after pullback by an equivariant diffeomorphism preserving Z .*

Proof. Evaluate the metric inequalities on $T_z\xi$ for the orbit forms. Apply them to every $v - w$ and take the infimum for the quotient forms. Equivariance identifies the vertical spaces under a diffeomorphism. $\square$

The following slightly sharper observation will be used for the metric expansion. All norms in it are Euclidean operator norms.

Lemma 2.2. *Let $H = I + E$ be a positive form on a Euclidean space, with $\|E\| \leq \eta < 1$, and let W be any subspace. If $v \perp W$, then*

$$\inf_{w \in W} H(v - w, v - w) = H(v, v) - \langle P_W E v, (I_W + P_W E|_W)^{-1} P_W E v \rangle.$$

The absolute value of the subtracted term is at most $\eta^2(1 - \eta)^{-1}|v|^2$, independently of W .

Proof. Differentiation in w gives $(I_W + P_W E|_W)w = P_W E v$. The operator on W is bounded below by $1 - \eta$. Substitution gives the formula and the bound. $\square$

2.1. Length metrics. We use the intrinsic length metric associated with the metrics on the orbit-type strata. Admissible paths are finite concatenations of paths smooth in a stratum on the interior of each segment, continuous at the endpoints, and of finite length. Endpoints may lie on smaller strata. We take the infimum of lengths and then the metric completion when needed. All local comparisons below use corresponding neighborhoods under the stated chart. Thus a coordinate ball is not implicitly identified with a ball for the reduced metric.

If relative quadratic-form bounds hold with a common constant on all strata, their square roots compare lengths of every admissible path. Taking infima gives the corresponding distance bounds, and the same bounds extend to completions. This elementary observation is the meaning of the assertions across singular strata in this paper.

An additive operator estimate cannot substitute for a relative one. For example,

$$G(r) = \text{diag}(r^2, r^8), \quad \tilde{G}(r) = G(r) + \text{diag}(0, r^4)$$

satisfy $\|\tilde{G} - G\| = r^4$, while their smallest positive eigenvalues have unbounded ratio as $r \rightarrow 0$. The estimates below retain the energy in each orbit direction.

3. THE CLASSICAL SURFACE CHART

We give the analytic details of the chart because the metric estimate depends on its particular transverse choice. The symplectic and moment-map assertions in this section are the classical surface local model [Hue95, Section 2].

3.1. The coexact equation. Fix an integer $s > 2$ and use Sobolev spaces H^s of adjoint-valued forms. The bracket is continuous in the required multiplication norms. Hodge decomposition gives

$$\Omega^1 = \mathcal{H}_d^1 \oplus \text{im } d \oplus \text{im } d^*, \quad \Omega^2 = \mathcal{H}_d^2 \oplus \text{im } d.$$

In the completed spaces these images are closed. The map

$$d : \text{im } d^* \cap H^s \Omega^1 \longrightarrow \text{im } d \cap H^{s-1} \Omega^2$$

is an isomorphism, with inverse $R_2 = d^* G_2 Q_2$.

For $h \in V$ and $c \in \text{im } d^* \cap H^s$, solve

$$(7) \quad Q_2(d(h+c) + \tfrac{1}{2}[h+c, h+c]) = 0.$$

Its derivative in c at $(0,0)$ is the preceding isomorphism. The implicit-function theorem gives a unique small solution $c(h)$, smooth in h , and K -equivariant. Its value and first derivative vanish at zero. In particular,

$$(8) \quad \|c(h)\|_{H^s} \leq C|h|^2, \quad \|Dc(h)v\|_{H^s} \leq C|h||v|.$$

Elliptic regularity applied to $c = -\frac{1}{2}R_2[h+c, h+c]$ gives smooth spatial coefficients. The same estimates hold in every fixed C^j norm after choosing the initial Sobolev index sufficiently large. We use this finite smooth family throughout; the constants may depend on that index.

Set $a(h) = h + c(h)$ and $U_h = Da(h)$. Harmonic and coexact forms are orthogonal, so

$$(9) \quad \|U_h v\|^2 = |v|^2 + \|Dc(h)v\|^2, \quad d^*a(h) = d^*U_h v = 0.$$

The homogeneous quadratic term of c and its derivative are

$$(10) \quad c_2(h) = -\tfrac{1}{2}R_2[h, h], \quad J_h v = Dc_2(h)v = -R_2[h, v].$$

Taylor's theorem in this finite parameter space gives $Dc(h) = J_h + O(|h|^2)$ in each fixed spatial norm.

3.2. The exact moment map. The affine space of connections has symplectic form $\Omega(u, v) = \int_\Sigma \langle u \wedge v \rangle$, where the angle brackets pair the adjoint-bundle factors, with the conventional sign chosen to match the curvature moment map. Harmonic forms pair trivially with coexact forms for Ω . Moreover $\text{im } d^*$ is isotropic. Indeed, on a surface it equals $\star \text{im } d$, star preserves Ω , and

$$\int_\Sigma \langle d\xi \wedge d\eta \rangle = 0$$

by integration by parts and $d^2 = 0$. It follows that

$$(11) \quad a^* \Omega = \Omega|_V.$$

The moment map of the residual K -action on the graph is $P_2(F_{A(h)} - F_{A_0})$, paired with $\mathfrak{k}$. Equivariance and (11) show that its differential equals that of the quadratic moment map Θ in (1). Both maps vanish at zero, and hence

$$(12) \quad P_2(F_{A(h)} - F_{A_0}) = \Theta(h).$$

Together with (7), this proves that $A(h)$ has curvature F_{A_0} exactly when $h \in Z$.

The gauge slice theorem identifies a small Coulomb slice modulo K with a neighborhood of $[A_0]$ in the gauge quotient. Applied to (12), this is Huebschmann's local quotient theorem [Hue95, Theorem 2.32]; its orbit-type interpretation is given in [Hue96]. We use that result for the local identification. The subsequent norm comparison concerns the quotient by the full gauge group, rather than only the metric induced on the slice.

The fixed central-curvature equation is preserved by the full gauge group, including when G is disconnected. To see this, let π_3 be projection onto the adjoint center bundle. Its flat connection d_3 is independent of the choice of connection on P , and

$$\pi_3(F_{A_0+b} - F_{A_0}) = d_3 \pi_3 b.$$

Gauge transformations commute with d_3 and, being orthogonal, with its adjoint. Thus the curvature of a gauge transform of A_0 and F_{A_0} are harmonic center-valued two-forms with exact difference. They agree by Hodge theory. Gauge covariance then proves preservation of the fixed-curvature equation.

Finally, differentiating K -equivariance gives an identity needed in both proofs:

$$(13) \quad d_{A(h)}\xi = [a(h), \xi] = U_h T_h \xi, \quad \xi \in \mathfrak{k}.$$

One may reverse both infinitesimal-action signs without changing any norm or operator in the argument.

4. THE ACTUAL QUOTIENT METRIC

Write $D_h = d_{A(h)} = d + \text{ad}_{a(h)}$. The metric on the moduli stratum is obtained by taking the infimum over all infinitesimal gauge transformations. We first take that infimum over $\mathfrak{k}^\perp$.

4.1. The complementary projection. The Poincaré estimate at A_0 gives

$$\|\eta\|_{H^1} \leq C \|d\eta\|_{L^2}, \quad \eta \in H^1 \Omega^0 \cap \mathfrak{k}^\perp.$$

Since $\|a(h)\|_{C^0} \leq C|h|$, decreasing δ gives

$$(14) \quad \|D_h \eta\| \geq (1 - C|h|) \|d\eta\| \geq c \|\eta\|_{H^1}.$$

Thus $L_h = D_h|_{\mathfrak{k}^\perp}$ has closed range in $L^2 \Omega^1$. Let Π_h be the orthogonal projection onto that range. For $u = U_h v$, Coulombness in (9) gives

$$\langle u, L_h \eta \rangle = \langle u, [a(h), \eta] \rangle.$$

Using (14), we obtain

$$(15) \quad |\langle u, L_h \eta \rangle| \leq C|h| \|u\| \|L_h \eta\|, \quad \|\Pi_h u\| \leq C|h| \|u\|.$$

Pythagoras yields the quadratic relative estimate

$$(16) \quad (1 - C|h|^2) \|U_h v\|^2 \leq \inf_{\eta \in H^1 \cap \mathfrak{k}^\perp} \|U_h v - D_h \eta\|^2 \leq \|U_h v\|^2.$$

The use of squared projection norms is essential for the quadratic error.

4.2. Proof of the relative estimate. Let v represent a tangent vector to a smooth orbit-type piece of Z . Every H^1 gauge generator splits uniquely as $\xi + \eta$, with $\xi \in \mathfrak{k}$ and $\eta \perp \mathfrak{k}$. By (13), the full quotient norm is

$$(17) \quad \bar{a}^* g_{L^2}([v], [v]) = \inf_{\xi \in \mathfrak{k}, \eta \in H^1 \cap \mathfrak{k}^\perp} \|U_h(v - T_h \xi) - D_h \eta\|^2.$$

Using smooth or higher-Sobolev gauge generators gives the same infimum by density. Elliptic regularity identifies this norm with the usual L^2 metric on the smooth moduli stratum.

Apply (16) and (9) to $v - T_h \xi$, for every ξ , before taking the infimum. This gives

$$(1 - C|h|^2) \inf_{\xi} |v - T_h \xi|^2 \leq \bar{a}^* g_{L^2}([v], [v]) \leq (1 + C|h|^2) \inf_{\xi} |v - T_h \xi|^2.$$

The outer expressions are the finite reduced norm. This proves (2). The proof uses neither an inverse of T_h nor a bound on the dimension of its image away from the origin.

4.3. The quadratic correction. The closed positive form $\|L_h\eta\|^2$ on $H^1\Omega^0 \cap \mathfrak{k}^\perp$ defines a self-adjoint operator $C_h = L_h^*L_h$. It has domain $H^2\Omega^0 \cap \mathfrak{k}^\perp$ and a uniform positive lower bound by (14). The domain assertion follows from elliptic regularity; the fixed harmonic projection has smooth finite-dimensional range.

Define a positive form H_h on all of V by

$$(18) \quad H_h(v, v) = \|U_h v\|^2 - \langle L_h^* U_h v, C_h^{-1} L_h^* U_h v \rangle.$$

This is exactly the middle expression in (16). All adjoints in this formula use the fixed L^2 metrics. Since $d^*U_h = 0$,

$$(19) \quad L_h^* U_h v = Q_0 \operatorname{ad}_{a(h)}^* U_h v = B_h v + O(|h|^2)|v|$$

in L^2 . Also

$$C_h - \Delta_0 : H^2\Omega^0 \cap \mathfrak{k}^\perp \longrightarrow L^2\Omega^0 \cap \mathfrak{k}^\perp$$

has norm $O(|h|)$, because its first-order and zeroth-order coefficients are controlled by the fixed C^1 norm of $a(h)$. The resolvent identity, or the Neumann series for $I + (C_h - \Delta_0)\Delta_0^{-1}$, gives

$$(20) \quad \|C_h^{-1} - \Delta_0^{-1}\|_{L^2 \rightarrow H^2} \leq C|h|.$$

Equations (9) and (10) give

$$\|U_h v\|^2 = |v|^2 + \|R_2[h, v]\|^2 + O(|h|^3)|v|^2.$$

Substituting this and (19)–(20) into (18) proves

$$(21) \quad H_h(v, v) = |v|^2 + \|R_2[h, v]\|^2 - \langle B_h v, \Delta_0^{-1} B_h v \rangle + O(|h|^3)|v|^2.$$

Every remainder here is a uniform operator-norm estimate on V .

For the residual quotient, choose $v \perp W_h = \operatorname{im} T_h$. By (17), the desired norm is $\inf_{w \in W_h} H_h(v - w, v - w)$. We have $H_h = I + E_h$, with $\|E_h\| \leq C|h|^2$. Lemma 2.2 shows that this final infimum differs from $H_h(v, v)$ by at most $C|h|^4|v|^2$, uniformly in W_h . Combining with (21) proves (3) and completes Theorem 1.1.

Corollary 4.1. *Let $d_{\operatorname{lin}, R}$ and $d_{L^2, R}$ be the intrinsic length metrics on $(Z \cap B_R)/K$ and its image, with the convention of Section 2. For sufficiently small R ,*

$$\sqrt{1 - CR^2} d_{\operatorname{lin}, R} \leq \bar{a}^* d_{L^2, R} \leq \sqrt{1 + CR^2} d_{\operatorname{lin}, R}.$$

The comparison extends to their metric completions.

Proof. On every stratum segment in the truncated domain, Theorem 1.1 compares speeds by the displayed square-root factors. Integrate, sum over segments, and take the infimum over corresponding paths. The two Lipschitz bounds extend the chart and its inverse to completions. $\square$

Remark 4.2. The expansion is pointwise on horizontal stratum tangent vectors. Differentiating the horizontal projection as the stabilizer changes can introduce inverse small eigenvalues. In particular, (3) is not a curvature estimate for the reduced metric.

5. ALL SMALL GAUGE EIGENVALUES

We prove Theorem 1.2. Regard $\mathcal{L}_h = D_h^* D_h$ as a nonnegative self-adjoint operator on $L^2\Omega^0$, with form domain $H^1\Omega^0$ and operator domain $H^2\Omega^0$. It has compact resolvent. Use the fixed decomposition $L^2\Omega^0 = \mathfrak{k} \oplus \mathfrak{k}^\perp$ and write, in quadratic-form blocks,

$$(22) \quad \mathcal{L}_h = \begin{pmatrix} \Gamma_h & \mathcal{B}_h^* \\ \mathcal{B}_h & C_h \end{pmatrix}.$$

The complementary operator is the one in the previous section; $C_h \geq \gamma > 0$ uniformly. Since $\mathfrak{k}$ consists of smooth parallel sections, $\mathcal{B}_h = Q_0 \mathcal{L}_h|_{\mathfrak{k}}$ is a bounded map from the finite-dimensional space $\mathfrak{k}$ to L^2 .

For $\xi \in \mathfrak{k}$, (13) and (9) show that

$$(23) \quad \|T_h \xi\|^2 \leq \langle \xi, \Gamma_h \xi \rangle \leq (1 + C|h|^2) \|T_h \xi\|^2.$$

There is also an exact cancellation. As $d\xi = 0$ and $d^*a = 0$,

$$d^*[a, \xi] = [d^*a, \xi] = 0.$$

Consequently

$$(24) \quad \mathcal{L}_h \xi = \text{ad}_a^* D_h \xi, \quad \|\mathcal{B}_h \xi\|^2 \leq C|h|^2 \langle \xi, \Gamma_h \xi \rangle.$$

The right side is the energy of the particular orbit direction, which may be much smaller than $|h|^2 |\xi|^2$.

For $0 \leq \lambda < \gamma/2$, form the finite Schur operator

$$S_h(\lambda) = \Gamma_h - \lambda I - \mathcal{B}_h^* (C_h - \lambda)^{-1} \mathcal{B}_h.$$

Equation (24) gives

$$(25) \quad (1 - C|h|^2) \Gamma_h - \lambda I \leq S_h(\lambda) \leq \Gamma_h - \lambda I.$$

These finite inequalities control the eigenvalue count of the full operator. Indeed, for $\xi \in \mathfrak{k}$ and $\eta \in H^1 \cap \mathfrak{k}^\perp$, completing the square gives

$$(26) \quad \begin{aligned} \langle \xi + \eta, (\mathcal{L}_h - \lambda)(\xi + \eta) \rangle &= \langle \xi, S_h(\lambda) \xi \rangle \\ &\quad + \|(C_h - \lambda)^{1/2} (\eta + (C_h - \lambda)^{-1} \mathcal{B}_h \xi)\|^2. \end{aligned}$$

The left side is interpreted as a closed quadratic form. The added vector belongs to H^2 , so the change of coordinates is invertible on the form domain. The number of eigenvalues of $\mathcal{L}_h$ strictly below λ is therefore the number of negative eigenvalues of $S_h(\lambda)$.

Restriction to the trial space $\mathfrak{k}$ puts the first m eigenvalues below $\|\Gamma_h\| = O(|h|^2)$. The max–min principle applied to the codimension- m subspace $\mathfrak{k}^\perp$ gives $\lambda_{m+1} \geq \gamma$. If $\gamma_j(h)$ are the ordered eigenvalues of Γ_h , including zeros, (25) and (26) imply

$$(1 - C|h|^2) \gamma_j(h) \leq \lambda_j(A(h)) \leq \gamma_j(h), \quad j \leq m.$$

For the lower bound, evaluate the inertia comparison at $\lambda = (1 - C|h|^2) \gamma_j(h)$; at most $j - 1$ eigenvalues are strictly smaller. A zero right side gives the bound by positivity. The upper bound is the ordinary min–max restriction. Repeated eigenvalues cause no change. Finally (23) compares γ_j with ν_j , proving (4).

To identify the kernel as a subspace, suppose $D_h(\xi + \eta) = 0$. Estimate (16) applies to $U_h T_h \xi = D_h \xi$, so its distance from $\text{im } L_h$ is positive unless $D_h \xi = 0$. Hence $T_h \xi = 0$ by injectivity of U_h , and (14) gives $\eta = 0$. The converse follows from (13). This proves the exact kernel assertion. If the adjoint bundle is zero, the spectral assertion is read as the absence of any spectrum; otherwise the preceding enumeration applies.

6. A QUANTITATIVE KÄHLER FIXED-POINT COMPARISON

For completeness, the finite-dimensional estimate can be obtained directly from the usual equivariant Darboux construction. Its proof is short; its role is to isolate the part of the comparison that needs no gauge-theoretic analysis.

Proposition 6.1. *Let a compact group K act by holomorphic isometries on a Kähler manifold (M, g, ω) and fix p . Set $V = T_p M$, with its flat metric g_0 and isotropy action. There is an equivariant symplectic chart $\Phi : B_\delta(V) \rightarrow M$, fixing p , such that*

$$(27) \quad (1 - C|z|^2)g_0 \leq \Phi^*g \leq (1 + C|z|^2)g_0.$$

In this chart the centered local moment map is exactly the quadratic moment map of the isotropy representation. One chart and constant therefore compare the orbit forms and reduced metrics on every level $\lambda \in (\mathfrak{k}^*)^K$ in the domain, across all orbit types.

Proof. Let $E = \exp_p$ on an invariant ball. Normal coordinates and $\nabla\omega = 0$ give

$$E^*g = g_0 + O(|z|^2), \quad E^*\omega = \omega_0 + \beta, \quad \beta = O(|z|^2), \quad D\beta = O(|z|).$$

The radial homotopy primitive

$$\alpha_z(v) = \int_0^1 t \beta_{tz}(z, v) dt$$

is invariant and satisfies $d\alpha = \beta$, $\alpha = O(|z|^3)$ and $D\alpha = O(|z|^2)$. The forms $\omega_t = \omega_0 + t\beta$ are uniformly nondegenerate on one small ball. Solve $\iota_{X_t}\omega_t = -\alpha$. Then $X_t = O(|z|^3)$ and $DX_t = O(|z|^2)$, uniformly in $t \in [0, 1]$. The flow ψ_t is defined on a smaller invariant ball for all these times. Its equation and variational equation give

$$\psi_1(z) = z + O(|z|^3), \quad D\psi_1(z) = I + O(|z|^2).$$

Thus $\Phi = E \circ \psi_1$ is equivariant and symplectic and satisfies (27).

The centered local moment map exists on this contractible neighborhood and can be normalized equivariantly under the full compact group. Equivariance and the symplectic identity show that its pullback and the quadratic moment map have identical differentials. Their values at zero agree, proving exact equality. The levels invariant under all of K , rather than only its identity component, are the levels stated in the proposition. Equivariance preserves their stabilizers. Lemma 2.1 gives the orbit and quotient estimates with the same constant. $\square$

For a fixed radius R , the associated intrinsic distance estimates have factors $\sqrt{1 \pm CR^2}$. The proposition is a quantitative consequence of equivariant Darboux and singular reduction [SL91, Sja95]; its proof uses the whole ambient metric estimate. It does not attempt to invert a moment-map derivative on a singular level. A triplet of moment maps on a quaternionic manifold requires a separate treatment, as in the explicit class below.

7. QUATERNIONIC SPACE-FORM QUOTIENTS

We now give a curved class in which the three moment equations and the quotient metric can both be computed exactly. These formulas are consequences of the classical Hopf construction and quaternionic reduction [GL88, Swa90]. They provide a useful comparison with the surface estimate, without asserting a normal form for a general quaternionic Kähler metric.

Let $K \subset \mathrm{Sp}(n)$ act on quaternionic columns by left matrix multiplication. The maps are right- $\mathbb{H}$ -linear. Extend the action to $\mathbb{H}^{n+1}$ by fixing the first coordinate, and use the affine chart $[1 : q]$. Write

$$\varepsilon = 1 \quad \text{on } \mathbb{H}P^n, \quad \varepsilon = -1 \quad \text{on } \mathbb{H}H^n, \quad s = 1 + \varepsilon|q|^2.$$

The hyperbolic chart has $|q| < 1$. Normalize both metrics to be the Euclidean metric at $q = 0$. Here $q^*v = \sum_j \bar{q}_j v_j$ is quaternion-valued.

7.1. Metric and moment section. The quaternionic Hermitian form

$$\mathcal{B}_\varepsilon(z, w) = \varepsilon \bar{z}_0 w_0 + \sum_{j=1}^n \bar{z}_j w_j$$

has $\mathcal{B}_\varepsilon(x, x) = \varepsilon$ on the local Hopf section $x(q) = (1, q)/\sqrt{s}$. For $b = q^*v$ and $a = \mathrm{Re} b$, direct differentiation gives

$$\mathcal{B}_\varepsilon(x, dx(v)) = \frac{\mathrm{Im} b}{s}, \quad \mathrm{Re} \mathcal{B}_\varepsilon(dx(v), dx(v)) = \frac{|v|^2}{s} - \frac{\varepsilon a^2}{s^2}.$$

The vertical vectors are xu , for $u \in \mathrm{Im} \mathbb{H}$, with squared norm $\varepsilon|u|^2$. Subtracting their component from $dx(v)$ gives the horizontal lift $dx(v) - \varepsilon x \mathrm{Im}(b)/s$. Its squared norm is

$$(28) \quad g_\varepsilon(v, v) = \frac{|v|^2}{s} - \frac{\varepsilon|q^*v|^2}{s^2}.$$

The principal $\mathrm{Sp}(1)$ connection is $\theta = \varepsilon \mathcal{B}_\varepsilon(x, dx)$. For $\xi \in \mathfrak{k}$, its value on the lifted fundamental field is

$$\theta(\tilde{X}_\xi) = \frac{\varepsilon q^* \xi q}{s}.$$

The connection is K -invariant. Cartan's formula expresses the covariant derivative of this section as minus contraction of the curvature. Since that curvature is a nonzero constant multiple of the quaternionic Kähler two-form triple, the quaternionic moment section in this frame is a fixed nonzero scalar multiple of $q^* \xi q/s$; see [Swa90, Chapter 3, §3.2]. The first homogeneous coordinate is fixed, so there is no constant term in the numerator. In particular the curved zero set is exactly the flat hyperkähler zero cone

$$(29) \quad Z_{\mathbb{H}} = \{q : q^* \xi q = 0 \text{ for every } \xi \in \mathfrak{k}\}$$

intersected with the chart domain. This description uses a local frame of the rank-three quaternionic bundle; it does not make the moment section a globally defined triple of real functions.

7.2. Exact comparison on every stratum. Set $T_q \xi = \xi q$, and use one fixed invariant inner product on $\mathfrak{k}$ to define orbit Gram operators. The smooth orbit-type pieces of the flat reduction carry their usual quotient metrics [May22, Theorem 1.1 and §2.5]. The same pieces and quotient norms apply to the curved zero set. Indeed, an element fixing $[1 : q]$ fixes the normalized lift $x(q)$: its possible right $\mathrm{Sp}(1)$ multiplier is forced to be 1 by the first coordinate. The stabilizer therefore acts trivially on the quaternionic rank-three bundle. Its fixed locus is the chart of a quaternionic linear subspace, and reduction on the exact stabilizer locus by $N_K(H)/H$ gives the smooth quotient pieces. This is the trivial quaternionic-isotropy case of singular quaternionic reduction [Swa97, §§3–4]; arbitrary singular quaternionic reductions need not have only quaternionic Kähler strata.

Proposition 7.1. *On $Z_{\mathbb{H}}$, the flat and curved horizontal spaces agree, and*

$$(30) \quad G_{\varepsilon}(q) = s^{-1}G_0(q).$$

On each quotient stratum, the quaternion-valued one-form $\beta_q(v) = q^*v$ descends and satisfies

$$(31) \quad g_{\varepsilon, \text{red}} = s^{-1}g_{0, \text{red}} - \varepsilon s^{-2}|\beta|^2.$$

Consequently, with $r = |q|$,

$$(32) \quad (1 + r^2)^{-2}g_{0, \text{red}} \leq g_{+, \text{red}} \leq (1 + r^2)^{-1}g_{0, \text{red}},$$

$$(33) \quad (1 - r^2)^{-1}g_{0, \text{red}} \leq g_{-, \text{red}} \leq (1 - r^2)^{-2}g_{0, \text{red}}.$$

No local-freeness assumption is required for these estimates.

Proof. The zero equation says $q^*T_q\xi = 0$. Polarization of (28) therefore gives, for every ambient vector v ,

$$g_{\varepsilon}(v, T_q\xi) = s^{-1}g_0(v, T_q\xi).$$

This proves both the Gram identity and equality of the orthogonal complements of the orbit spaces, including where their ranks change. The one-form β is invariant, because $K \subset \text{Sp}(n)$, and annihilates the vertical vectors on $Z_{\mathbb{H}}$. It thus descends on each stratum. Evaluate (28) on the common horizontal representative to obtain (31). Finally $|q^*v|^2 \leq r^2|v|^2$ gives (32) and (33). $\square$

For the intrinsic distances on corresponding truncations $r \leq R$, with the path convention of Section 2, these bounds give

$$(1 + R^2)^{-1}d_0 \leq d_+ \leq d_0, \quad d_0 \leq d_- \leq (1 - R^2)^{-1}d_0 \quad (R < 1).$$

They persist after completion. These are statements about the specified coordinate domains, not about unrestricted global distances.

7.3. Abelian charges and supports. For a torus, let Q_e be the integral charge of its e th quaternionic coordinate and use the action $(T_q\xi)_e = i(Q_e \cdot \xi)q_e$. Its flat Gram matrix is

$$(34) \quad G_0(q) = \sum_e |q_e|^2 Q_e Q_e^{\top}.$$

At a zero-moment point, every eigenvalue of the curved Gram operator is exactly s^{-1} times its flat value. This includes all zero eigenvalues and uses the same fixed inner product on the Lie algebra. A gauge-boson mass interpretation additionally requires a specified positive gauge kinetic form; (34) alone does not supply it.

Zero-moment points with independently chosen supports can be obtained by taking two copies of each charge. In a pair, set the coordinates equal to (b_e, jb_e) , with $b_e \geq 0$ real. Their moments cancel, since

$$\bar{b}_e i b_e + \overline{j b_e} i (j b_e) = b_e^2 \{i + (-j)i j\} = 0,$$

whereas their Gram contribution is $2b_e^2 Q_e Q_e^{\top}$. Thus supports of charge *pairs* may be chosen independently. Arbitrary individual-coordinate supports in an undoubled representation need not satisfy the moment equations.

8. A FOUR-CYCLE IN BOTH GEOMETRIC SETTINGS

The following example makes the independence from weak orbit scales explicit. Its matrix algebra is elementary. The geometric point is that the same matrix occurs on actual solution sets in both settings.

8.1. The common matrix. Give the edges 12, 23, 34, 41 of a four-cycle the weights a, s, a, t , where $a > 0$ and $0 \leq s, t \leq a/4$. Its Laplacian is

$$(35) \quad L(a, s, t) = \begin{pmatrix} a+t & -a & 0 & -t \\ -a & a+s & -s & 0 \\ 0 & -s & a+s & -a \\ -t & 0 & -a & a+t \end{pmatrix}.$$

The reflection (14)(23) decomposes $\mathbb{R}^4$ into two orthogonal planes. In the bases $(e_1 + e_4, e_2 + e_3)/\sqrt{2}$ and $(e_1 - e_4, e_2 - e_3)/\sqrt{2}$, its blocks are

$$\begin{pmatrix} a & -a \\ -a & a \end{pmatrix}, \quad \begin{pmatrix} a+2t & -a \\ -a & a+2s \end{pmatrix}.$$

Hence the ordered eigenvalues on $\mathbf{1}^\perp$ are

$$(36) \quad \nu_- = a + s + t - \sqrt{a^2 + (s-t)^2}, \quad 2a, \quad \nu_+ = a + s + t + \sqrt{a^2 + (s-t)^2}.$$

The inequalities $\sqrt{a^2 + x^2} \leq a + x^2/(2a)$ and $|s-t| \leq s+t$ give

$$(37) \quad \left(1 - \frac{s+t}{2a}\right)(s+t) \leq \nu_- \leq s+t.$$

Thus $\nu_- = 0$ precisely when $s = t = 0$, and $\nu_- \sim s+t$ whenever $(s+t)/a \rightarrow 0$, with no condition on s/t . The product of the three values is

$$2a\nu_-\nu_+ = 4a^2(s+t) + 8ast.$$

These identities follow from the displayed blocks.

8.2. Surface Hermitian–Einstein connections. Let Σ have genus $g \geq 2$ and area one. Choose pairwise nonisomorphic flat unitary line bundles $L_1, \dots, L_4$ whose tensor product is trivial, and fix a parallel unitary trivialization of that product. Such a choice is obtained from three generic flat lines by taking the fourth to be their inverse product. The direct sum of the flat connections is then an $SU(4)$ connection on $E = \bigoplus_i L_i$, to which the preceding theorems apply. Its infinitesimal determinant-one stabilizer is the space of diagonal matrices $i \operatorname{diag}(\xi_1, \dots, \xi_4)$ with $\sum \xi_i = 0$. With the invariant matrix inner product $-\operatorname{tr}(XY)$ and area one, this is Euclidean $\mathbf{1}^\perp$.

For $i \neq j$, the degree-zero line bundle $\operatorname{Hom}(L_j, L_i)$ is nontrivial. It has no holomorphic sections, so Riemann–Roch gives

$$\dim_{\mathbb{C}} H^{0,1}(\Sigma, \operatorname{Hom}(L_j, L_i)) = g - 1.$$

The real harmonic deformation space identifies with the underlying real space of

$$H^{0,1}(\Sigma, \operatorname{End}_0 E),$$

where $\operatorname{End}_0 E$ denotes the trace-free complex endomorphism bundle. For each edge, select one harmonic direction in each of the two ordered off-diagonal summands. Normalize its complex coordinate so that the induced real harmonic-one-form norm is the standard norm. Distinct selected summands are orthogonal.

The two summands on the edge ij have opposite torus weights $e_i - e_j$ and $e_j - e_i$. Assign them equal complex-coordinate magnitudes $\sqrt{w_{ij}/2}$, where $(w_{12}, w_{23}, w_{34}, w_{41}) = (a, s, a, t)$. Since the reference stabilizer is diagonal, the moment map is precisely the torus moment map

on this orthogonal representation. The two contributions cancel on every edge. We therefore obtain

$$(38) \quad h \in \Theta^{-1}(0), \quad |h|^2 = 2a + s + t, \quad T_h^* T_h = L(a, s, t)|_{\mathbf{1}^\perp}.$$

For small weights the classical chart gives actual flat unitary connections $A(h)$ with the specified determinant. They are Hermitian–Einstein connections on their induced holomorphic bundles. Their existence here follows from the exact moment equation and the chart, not from the matrix calculation.

Theorem 1.2 compares the three small gauge eigenvalues with (36) by factors $1 \pm C(2a + s + t)$. When $s + t > 0$, the support graph is connected and its determinant-one infinitesimal stabilizer vanishes. At $s = t = 0$ it has dimension one. The same estimate applies there, including the newly zero eigenvalue. Theorem 1.1 simultaneously compares the full quotient metrics, rather than only this restricted family of orbit operators.

8.3. Curved quaternionic quotients. Let the effective torus $T^4/U(1)_{\text{diag}}$ act on $\mathbb{H}^8$. For each edge, take two quaternionic coordinates of charge $Q_{ij} = e_i - e_j$, with values

$$\left(\sqrt{w_{ij}/2}, j\sqrt{w_{ij}/2} \right).$$

The paired-charge calculation gives

$$q \in Z_{\mathbb{H}}, \quad |q|^2 = 2a + s + t, \quad G_0(q) = L(a, s, t)|_{\mathbf{1}^\perp},$$

using the Euclidean metric on the torus Lie algebra. The points $[1 : q]$ lie in the actual zero level of the quaternionic moment section on $\mathbb{H}P^8$ and, when $2a + s + t < 1$, on $\mathbb{H}H^8$. Their curved orbit eigenvalues are exactly

$$(39) \quad \frac{\nu_-}{1 + \varepsilon(2a + s + t)}, \quad \frac{2a}{1 + \varepsilon(2a + s + t)}, \quad \frac{\nu_+}{1 + \varepsilon(2a + s + t)}.$$

Equations (38) and (39) compare actual geometric operators on a common parameter family. The two quotient spaces have different dimensions and are not identified by this construction.

For example, take $a = \rho^2$, $s = \rho^8$, and $t = \exp(-1/\rho^2)$ as $\rho \downarrow 0$. The smallest positive model eigenvalue is asymptotic to ρ^8 , whereas $r^2 = 2a + s + t \sim 2\rho^2$. An additive $O(r^4)$ error can exceed that eigenvalue. Both geometric comparisons above instead give relative error $O(\rho^2)$, regardless of the separation between s and t . Setting both weak weights to zero retains the same estimates with the enlarged kernel.

9. FURTHER QUESTIONS

The surface argument uses two special features: the adjoint connection at the reference point is flat, and the coexact graph is exactly symplectic. In higher complex dimension, integrability adds equations to the deformation problem. A useful extension would require a chart of the actual integrable locus with a uniform metric estimate; the orbit Gram matrix alone cannot provide that chart.

The quaternionic space-form calculation makes a different simplification. Its moment section and metric are explicit in one common coordinate system. For a general quaternionic Kähler metric, establishing a comparable chart across all stabilizer types remains a separate problem. In particular, the formulas above do not determine the metric of a physical compactification whose metric is not already known.

Finally, the estimates control tangent norms and the resulting local intrinsic distances. They do not control derivatives of the quotient metric near a change of orbit type. Curvature

asymptotics would require additional estimates, since the horizontal projection can vary rapidly even when the metric comparison is uniform.

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BERND JOHANNES WUEBBEN, NEW YORK, NY, wuebben@gmail.com