

UNIFORM HERMITIAN–EINSTEIN SPECTRA WITH REPEATED STABLE FACTORS

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ABSTRACT. We prove uniform relative asymptotics for all small positive Dolbeault eigenvalues on trace-free endomorphism-valued zero-forms for Hermitian–Einstein metrics on simple two-step extensions near an equal-slope Kähler polarization. The stable factors may occur with arbitrary fixed multiplicities, with distinct isomorphism types in the two layers. The finite comparison operator is the commutator Laplacian on the full complex multiplicity algebra, evaluated at an independently balanced representative of the extension. Every small eigenvalue has relative error bounded by a fixed multiple of the total balanced extension energy, and the remaining spectrum has a uniform positive lower bound. No lower bound on positive eigenvalue ratios or stability margins is imposed. The main estimate bounds the curvature residual against every multiplicity projection by its crossing energy; a degree argument then compares the actual metric with the balanced metric without inverting a degenerating finite Hessian. A fixed rank-five bundle on a product of four projective lines exhibits six small modes and arbitrarily separated positive scales.

1. INTRODUCTION

Let a holomorphic bundle become strictly semistable as a Kähler polarization approaches a wall. On the stable side its Hermitian–Einstein metric may diverge in a fixed holomorphic frame. The limiting polystable bundle has additional parallel endomorphisms, and these give rise to small eigenvalues of the endomorphism Dolbeault Laplacian. When a stable factor is repeated, the corresponding parallel endomorphisms form a matrix algebra. The small spectrum therefore contains matrix directions that are absent from a scalar graph model.

We determine the entire small spectrum for simple extensions of two polystable layers, uniformly as positive spectral scales separate. The finite operator is specified by the extension class and a quadratic moment-map equation. The principal analytical issue is to compare this independently specified balance with the actual Hermitian–Einstein metric. An ordinary implicit-function estimate for the finite equation would divide by its smallest positive eigenvalue. Such an estimate is not uniform near a face of a stability chamber. We instead use curvature identities for all multiplicity projections and a finite-dimensional degree argument.

1.1. Geometric data and normalization. Let X be a compact connected complex manifold of dimension d , and let ω_s , for s in a neighborhood of 0 in $\mathbb{R}^k$, be a smooth family of Kähler forms. Put $dV_s = \omega_s^d/d!$ and $V_s = \int_X dV_s$. Fix pairwise nonisomorphic locally free bundles F_v , of ranks r_v , which are stable and have the same slope with respect to ω_0 . The index set is the disjoint union of nonempty sets I and J . Fix positive multiplicities m_v and

Date: October 29, 2026.

2020 Mathematics Subject Classification. 53C07 (primary); 32L05, 58J50, 16G20 (secondary).

Key words and phrases. Hermitian–Einstein metrics, stability walls, Dolbeault spectrum, quiver representations, repeated stable factors.

a simple holomorphic extension

$$(1) \quad 0 \longrightarrow S = \bigoplus_{i \in I} F_i \otimes \mathbb{C}^{m_i} \longrightarrow E \longrightarrow U = \bigoplus_{j \in J} F_j \otimes \mathbb{C}^{m_j} \longrightarrow 0.$$

Simple means $H^0(X, \text{End } E) = \mathbb{C} \text{id}_E$. Repetitions are allowed within either layer; the same stable isomorphism type does not occur in both. At least one $m_v > 1$ in the applications of interest.

After shrinking the parameter neighborhood, suppose the factors have smoothly varying Hermitian–Einstein metrics $h_{v,s}$. This is the usual local continuation of the stable factor metrics; it can also be taken as an explicit hypothesis on the chosen family. Fix their scalar normalizations. Write $i\Lambda_s F_{h_{v,s}} = \kappa_v(s) \text{id}_{F_v}$ and

$$(2) \quad R_E = \sum_v r_v m_v, \quad \kappa_E(s) = \frac{1}{R_E} \sum_v r_v m_v \kappa_v(s), \quad b_v(s) = -r_v (\kappa_v(s) - \kappa_E(s)).$$

Our curvature convention is $c_1(F) = [i \text{tr } F_h / (2\pi)]$; thus

$$(3) \quad \kappa(F) = \frac{2\pi}{\text{rk}(F)V_s} \int_X c_1(F) \wedge \frac{\omega_s^{d-1}}{(d-1)!}.$$

All slope comparisons have the same sign in this normalization. The numbers b_v satisfy $\sum_v m_v b_v = 0$. They are coefficients in the dual trace pairing, so their definition already includes the factor ranks.

Choose fixed bases of $\text{Ext}^1(F_j, F_i)$ and take their current harmonic representatives $\beta_{ij,a,s}$. If $Z = (Z_{ij,a})$ is an array of m_i by m_j matrices, set

$$(4) \quad N_s(Z)_{ij} = \sum_a \beta_{ij,a,s} \otimes Z_{ij,a}, \quad \bar{\partial}_{s,Z} = \bar{\partial}_G + N_s(Z), \quad G = S \oplus U,$$

and define $q_s(Z) = V_s^{-1} \|N_s(Z)\|_{L^2}^2$ using $h_{0,s} = \bigoplus_v h_{v,s} \otimes I_{m_v}$. We retain the actual harmonic Gram matrices. Fixed cohomology coordinates need not be orthonormal for varying s . The extension E has fixed cohomology matrices Z^0 in this notation.

The group $\mathcal{G} = \prod_v \text{GL}(m_v, \mathbb{C})$ acts by $(g \cdot Z)_{ij,a} = g_i Z_{ij,a} g_j^{-1}$; its compact subgroup is $K = \prod_v \text{U}(m_v)$. Define the quadratic moment map by

$$(5) \quad \sum_v \text{tr}(Y_v \mu_s(Z)_v) = \left. \frac{d}{dt} \right|_{t=0} q_s(e^{tY/2} \cdot Z) \quad (Y_v = Y_v^*).$$

In orthonormal arrow coordinates it is the sum of incoming ZZ^* minus outgoing Z^*Z . We call Z *balanced* at s if $\mu_s(Z)_v = b_v(s) I_{m_v}$. Finite balancing below concerns the orbit $\mathcal{G} \cdot Z^0$, independently of the unknown HE metric.

A collection of subspaces $W_v \subset \mathbb{C}^{m_v}$ is a *subrepresentation* if $Z_{ij,a}^0 W_j \subset W_i$ for every arrow. It is proper and nonzero if it is neither the whole collection nor zero. The representation is $b(s)$ -stable when

$$(6) \quad \sum_v b_v(s) \dim W_v > 0 \quad \text{for every proper nonzero subrepresentation } W.$$

This is King's positive sign convention. The real-weight balancing theorem then supplies a balanced $Z \in \mathcal{G} \cdot Z^0$, unique up to K . The details and sign conversion are given in Section 2.

1.2. The finite spectrum and the main theorem. Consider the full complex vector space

$$(7) \quad \mathcal{H} = \left\{ X = (X_v) \in \bigoplus_v M_{m_v}(\mathbb{C}) : \sum_v r_v \operatorname{tr} X_v = 0 \right\}, \quad \|X\|_r^2 = \sum_v r_v \operatorname{tr}(X_v^* X_v).$$

Its complex dimension is $N = \sum_v m_v^2 - 1$. For a balanced Z , define

$$(8) \quad \mathcal{Q}_{s,Z}(X) = q_s([X, Z]), \quad [X, Z]_{ij,a} = X_i Z_{ij,a} - Z_{ij,a} X_j.$$

Let $\nu_1(s) \leq \dots \leq \nu_N(s)$ be its eigenvalues relative to $\|\cdot\|_r^2$. Simplicity makes these eigenvalues positive. Compact-unitary invariance makes them independent of the choice of balanced representative. They are defined from the geometry, extension orbit, and finite balance.

Theorem 1.1. *Fix the geometric data above. There are a neighborhood $\mathcal{U}$ of 0 and constants $C, \gamma > 0$ with the following properties. For $s \in \mathcal{U}$, the bundle E is slope-stable with respect to ω_s if and only if its multiplicity representation is $b(s)$ -stable. At such a parameter let Z be balanced and put*

$$(9) \quad \sigma(s) = \sum_{i \in I} m_i b_i(s) = q_s(Z) > 0.$$

Let $h_{\text{HE},s}$ be the HE metric whose determinant is the product determinant metric induced by the fixed factor metrics. The ordered eigenvalues, counted with complex multiplicity, of $\Delta_s = \bar{\partial}_{\text{End } E}^{*, h_{\text{HE},s}} \bar{\partial}_{\text{End } E}$ on $L^2(X, \text{End}_0 E)$ satisfy

$$(10) \quad \left| \frac{\lambda_j(s)}{\nu_j(s)} - 1 \right| \leq C\sigma(s) \quad (1 \leq j \leq N), \quad \lambda_{N+1}(s) \geq \gamma.$$

If $h_{\text{bal},s}$ is the pullback of $h_{0,s}$ through a determinant-one balancing gauge and the harmonic identification of E , then

$$(11) \quad e^{-C\sigma(s)} h_{\text{bal},s} \leq h_{\text{HE},s} \leq e^{C\sigma(s)} h_{\text{bal},s}.$$

The constants impose no lower bound on a positive stability margin, ν_1 , or any positive ratio ν_j/ν_k . No estimate of the form $|s| = O(\sigma(s))$ is required.

The harmonic identification in this statement is fixed by removing the exact part of the extension form, as in Lemma 2.1. Changing the balanced representative by K leaves the pulled-back metric unchanged; the remaining common scalar is fixed by the determinant. The constants may depend on the fixed factors, multiplicities, harmonic spaces, and smooth parameter neighborhood. They are not asserted uniform under degeneration of this background or growth of the multiplicities.

The theorem controls endomorphism-valued zero-forms. It is not a statement about all degrees of the Dolbeault complex or the entire Yang–Mills Hessian. At a nonsplit strictly semistable boundary point the original bundle need not admit an HE metric. All occurrences of $h_{\text{HE},s}$ in the theorem are on the stable side.

The example in Section 6 is a fixed rank-five bundle on $(\mathbb{P}^1)^4$ with one multiplicity-two target and three scalar sources. For genuine polarizations depending on $t, \delta > 0$, it has

$$(12) \quad \lambda_1 = \frac{5}{4}t\delta(1 + O(t) + O(\delta)), \quad \lambda_j \asymp t \quad (2 \leq j \leq 6), \quad \lambda_7 \geq \gamma.$$

The constants are uniform for independently small t and δ . In particular $\delta = e^{-1/t}$ is allowed.

1.3. Relation to earlier work and proof structure. The correspondence between stable bundles and HE metrics is due to Donaldson and Uhlenbeck–Yau; we use the standard formulation and endomorphism identities in Kobayashi [Kob87]. King [Kin94] relates subrepresentation stability to moment-map balance. Haiden–Katzarkov–Kontsevich–Pandit [HKKP23, §5.1, Theorem 5.1] give the real-weight formulation used here, including positive masses on the multiplicity algebra.

Clarke–Tipler [CT24, Propositions 3.1–3.2] construct the local elliptic reduction without an abelian restriction; their local wall theorem assumes abelian graded automorphisms. Deloche [Del25, Proposition 4.2 and Theorems 4.8–4.9] allows nonabelian graded automorphisms and proves continuity and quantitative convergence of HYM representatives under metric variation. His Proposition 4.7 obtains a zero of the geometrically deformed moment map. These results already supply nonabelian local reduction and HYM existence. They do not identify that zero, with a relative error tending to zero uniformly near weak directions, with a zero of the independently specified quadratic moment map. The distinction is essential for (10): an additive error that is small compared with the total extension energy may dominate the least positive eigenvalue. All Deloche locators refer to the published version, arXiv version 4.

The repeated-factor harmonic quiver also occurs in the flow theory of Lam [Lam26, Theorems 8.2, 8.7 and Proposition 8.9]; the finite iterated-logarithmic asymptotics are developed in [HKKP23, Theorem 5.11]. Those comparisons concern bounded metric distance in a flow problem. The present result is stationary and gives a multiplicative error tending to one. It extends the distinct-factor spectral theorem of [Wue26]; no result of that paper is needed as an unproved step in the argument below.

The new estimate is Proposition 3.2: the reduced curvature residual, paired with any orthogonal multiplicity projection, is bounded by total extension energy times the energy crossing that projection. Theorem 4.1 uses this estimate on spectral projections of a Hermitian gauge logarithm to exclude zeros on the boundary of a small ball. Brouwer degree gives the desired HE correction without a uniform inverse for the finite Hessian. Harmonicity then makes the Schur-complement coupling proportional to each vector’s own finite energy, yielding every relative eigenvalue estimate.

2. HARMONIC EXTENSIONS AND FINITE BALANCE

We first identify the algebra that enters the finite model. The use of the complete algebra, rather than one scalar per stable type, determines both the number of small eigenvalues and the correct stability condition.

Lemma 2.1. *For the extension (1), harmonic representatives of its fixed cohomology class give a presentation (4) for every small s . In this presentation*

$$(13) \quad H^0(X, \text{End } E) = \left\{ (X_v) \in \bigoplus_v M_{m_v}(\mathbb{C}) : X_i Z_{ij,a}^0 = Z_{ij,a}^0 X_j \text{ for all } i, j, a \right\}.$$

The two algebras are identified by parallel block endomorphisms. Every holomorphic isomorphism between two harmonic presentations is a parallel block gauge in $\mathcal{G}$. In particular simplicity of E is equivalent to the common centralizer being scalar.

Proof. Stable nonisomorphic bundles of a common slope have no nonzero maps between them, and each has only scalar endomorphisms; see [Kob87, Chapter V, §7]. Hence $\text{Hom}(F_v, F_w) = 0$

for $v \neq w$ and $\text{End}(F_v) = \mathbb{C} \text{Id}$. These are statements about the fixed holomorphic bundles and do not change with s .

A smooth splitting of (1) gives a closed off-diagonal form $\eta \in \Omega^{0,1}(\text{Hom}(U, S))$. Its harmonic part is $N_s(Z^0)$, and $\eta - N_s(Z^0) = \bar{\partial}C_s$ for a smooth section C_s of $\text{Hom}(U, S)$. An upper-triangular unipotent change of splitting removes this exact part. Products of two source-to-target blocks vanish, so there is no nonlinear term in this change. The same observation shows that $\bar{\partial}_{s,Z}^2 = 0$ for every coefficient array Z . Taking C_s orthogonal to the kernel gives a smooth choice in s . This gauge has determinant one.

Apply $\text{Hom}(F_i, -)$ to (1). Since $\text{Hom}(F_i, U) = 0$, one obtains $\text{Hom}(F_i, E) = \text{Hom}(F_i, S) = \mathbb{C}^{m_i}$. The evaluation image

$$\text{im} \left(\bigoplus_{i \in I} F_i \otimes \text{Hom}(F_i, E) \longrightarrow E \right) = S$$

is the displayed locally free subbundle, fiber by fiber. Every endomorphism preserves it. In a harmonic splitting an endomorphism therefore has the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Its diagonal equations make A and B parallel multiplicity blocks. Its remaining equation is

$$(14) \quad \bar{\partial}C + N_s(Z^0)B - AN_s(Z^0) = 0.$$

The last two terms are harmonic, since the multiplicity blocks are parallel. Exact and harmonic forms are orthogonal, so each term in this decomposition vanishes. Now C is holomorphic and $\text{Hom}(U, S) = 0$, whence $C = 0$. Independence of the cohomology bases gives (13). Conversely those equations plainly define a holomorphic endomorphism. The same proof for an isomorphism between two presentations gives the asserted gauge statement. $\square$

The separation of stable types across the layers is natural in this simple two-step setting. Indeed, for any extension of two polystable layers of equal slope the map

$$\text{Hom}(U, S) \longrightarrow \text{End}(E), \quad f \longmapsto \iota f \pi$$

is injective and has square-zero image. Simplicity forces this space to vanish. If a stable type occurred in both layers, it would give a nonzero map from U to S . This observation is specific to two polystable layers and is not a claim about longer filtrations.

2.1. The real domain and its dual. The Hermitian logarithms used for metric corrections form

$$(15) \quad \mathfrak{h} = \left\{ Y_v = Y_v^* : \sum_v r_v \text{tr} Y_v = 0 \right\}, \quad \mathcal{W} = \left\{ T_v = T_v^* : \sum_v \text{tr} T_v = 0 \right\}.$$

They are paired by $\langle Y, T \rangle = \sum_v \text{tr}(Y_v T_v)$. The positive definite real inner product on $\mathfrak{h}$ is $\langle X, Y \rangle_r = \sum_v r_v \text{tr}(X_v Y_v)$; division of the v -th block of T by r_v identifies $\mathcal{W}$ with $\mathfrak{h}$. Both $\mu_s(Z)$ and $b = (b_v I_{m_v})$ lie in $\mathcal{W}$. Multiplicities m_v and factor ranks r_v have different roles.

For example, at a fixed parameter orthonormalize the harmonic arrow basis in the volume-normalized inner product. Then

$$(16) \quad \mu_i(Z) = \sum_{j,a} Z_{ij,a} Z_{ij,a}^*, \quad \mu_j(Z) = - \sum_{i,a} Z_{ij,a}^* Z_{ij,a}.$$

The rank weights have not disappeared: they occur in b_v and in the inner product on the endomorphism algebra. In the original cohomology coordinates the same equations use the corresponding positive Gram matrices. Changing arrow basis commutes with $\mathcal{G}$ and preserves all subrepresentations.

Proposition 2.2. *If the fixed representation Z^0 is b -stable, its orbit contains a balanced representative. Under the scalar-centralizer hypothesis this representative is unique up to K . For every such representative*

$$(17) \quad q_s(Z) = \sum_{i \in I} m_i b_i = \sigma.$$

The stable real weights for this fixed representation form an open polyhedral cone in $\{b : \sum_v m_v b_v = 0\}$, possibly with empty intersection with the accessible parameter image.

Proof. The real-weight finite Kempf–Ness theorem [HKKP23, §5.1, Theorem 5.1] states that a representation admits a balanced metric precisely when it is polystable. Its weight convention is $\theta(W) \leq 0$; setting $\theta_v = -b_v$ gives (6) and (16). The positive masses in the group metric are r_v . Equivalently the central endomorphism parameter of that convention is $-b_v/r_v$. No second rank factor enters (6). The theorem allows real weights through its logarithmic determinant potential; no rational approximation is required. The underlying integral-character statement is King’s [Kin94, Propositions 3.1–3.2 and Theorem 6.1].

For clarity, uniqueness here can be checked directly. If two balanced points differ by $g \in \mathcal{G}$, polar decomposition and a common scalar give $g = ke^{Y/2}$, with $k \in K$ and $Y \in \mathfrak{h}$. Diagonalize Y blockwise and denote its eigenvalues by y_p . Writing w_{pq} for the total squared arrow entries from source eigenvector q to target eigenvector p , the function

$$(18) \quad f(t) = q_s(e^{tY/2} \cdot Z) - t\langle Y, b \rangle$$

satisfies

$$f''(t) = \sum_{p,q} (y_p - y_q)^2 e^{t(y_p - y_q)} w_{pq}.$$

For $Y \neq 0$ in $\mathfrak{h}$ this is positive: otherwise Y centralizes Z and must be scalar, contrary to its weighted trace zero. Balance at both endpoints would give $f'(0) = f'(1) = 0$. Thus $Y = 0$. Tracing all target equations in (16) proves (17). Finally only finitely many dimension vectors ($\dim W_v$) are possible, and each gives a linear strict inequality. These define the claimed cone. The actual polarization image need not fill that cone. $\square$

At balance the sign can also be seen without a stability convention. If Π_v projects onto a subrepresentation, then for each arrow

$$\mathrm{tr}(\Pi_i Z Z^*) - \mathrm{tr}(\Pi_j Z^* Z) = \|\Pi_i Z(1 - \Pi_j)\|_F^2 - \|(1 - \Pi_i)Z\Pi_j\|_F^2.$$

The second term is zero. Equality after summation would make both the subrepresentation and its orthogonal complement invariant; its projection would then be a nonscalar centralizer element. Thus balance with scalar centralizer implies precisely the strict positive sign.

A common complex scalar can normalize the bundle determinant of any balancing gauge to one, since it acts trivially on the arrows and its bundle determinant is its R_E -th power. Choices of scalar phase do not affect the metric. Together with the unipotent identification in Lemma 2.1, this defines the determinant convention in Theorem 1.1.

3. PROJECTED CURVATURE AND MULTIPLICITY PROJECTIONS

All estimates in this section hold on the product of a fixed small parameter neighborhood and a fixed small ball of coefficient arrays. The constants are independent of any positive finite eigenvalue. For each s consider pointwise trace-free, $h_{0,s}$ -Hermitian endomorphisms of

G . Let P_s be their L^2 projection onto the entire parallel algebra $\mathfrak{h}$, and put $Q_s = 1 - P_s$. For $u \in Q_s$ define

$$(19) \quad \widehat{K}(s, Z, u) = e^{u/2} (i\Lambda_s F_{\bar{\partial}_s, Z, h_{0,s} e^u}) e^{-u/2} - \kappa_E(s) I.$$

This is Hermitian for $h_{0,s}$. Its pointwise trace vanishes: the determinant holomorphic structure is the split determinant and $\text{tr } u = 0$, so the determinant curvature is unchanged and has constant contraction $R_E \kappa_E$.

Lemma 3.1. *The equation $Q_s \widehat{K}(s, Z, u) = 0$ has a unique small solution $u(s, Z) \in Q_s$, smooth in the finite parameters. For every fixed nonnegative integer a ,*

$$(20) \quad \|u(s, Z)\|_{C^a} \leq C_a q_s(Z).$$

It satisfies $u(s, k \cdot Z) = ku(s, Z)k^{-1}$ for $k \in K$.

Proof. At $Z = u = 0$ the linearization of (19) in the real Hermitian metric variable is

$$(21) \quad L_s = \frac{1}{2}(\Delta_{\bar{\partial}, s} + \Delta_{\partial, s}).$$

The conjugation in (19) is included in this formula; it gives the symmetric real linearization even when the split summands have different mean curvatures. The Kähler identities give (21), and

$$\langle u, L_s u \rangle_{L^2} = \frac{1}{2}(\|\bar{\partial} u\|_{L^2}^2 + \|\partial u\|_{L^2}^2).$$

Its kernel is the parallel algebra. At $s = 0$ this follows from stability and vanishing cross-Hom groups. For nearby s , a zero of the displayed sum is parallel for the same split connection; there are no additional parallel maps. Conversely every multiplicity block is parallel. The positive spectral gap of L_0 on the complement therefore persists uniformly in a small neighborhood, and so does its elliptic inverse.

Take an integer $\ell > d$, so that the relevant Sobolev spaces are algebras on the real $2d$ -dimensional manifold. The projected equation is a smooth map from $Q_s H^{\ell+2}$ to $Q_s H^\ell$. Identify these varying spaces smoothly with those at zero. The parameter-dependent implicit-function theorem applies because the derivative $Q_s L_s Q_s$ is uniformly invertible. At $Z = 0$, the split curvature is parallel; hence $u(s, 0) = 0$.

The first derivative of the contracted curvature in Z vanishes. Indeed its off-diagonal linear terms are contractions of covariant derivatives of the harmonic forms $\beta_{ij,a,s}$; the Kähler identities identify them with their vanishing Dolbeault adjoints. Thus $D_Z u(s, 0) = 0$. Taylor expansion gives the quadratic bound in Sobolev norm. Elliptic regularity gives (20) for each fixed spatial order. The group K preserves the equation, the entire parallel algebra, and its orthogonal complement. Uniqueness proves the equivariance. It is unnecessary, and generally false, that K fix every vector of the parallel algebra. $\square$

Write $\mathcal{F}_v(s, Z)$ for the volume-normalized integral of the partial trace over F_v of the v, v block of $\widehat{K}(s, Z, u(s, Z))$. Then

$$(22) \quad (P_s \widehat{K})_v = I_{F_v} \otimes \frac{\mathcal{F}_v(s, Z)}{r_v}.$$

In particular $\mathcal{F} \in \mathcal{W}$, and the full HE equation is equivalent to $\mathcal{F} = 0$ once the complement is solved.

Proposition 3.2. *There is a smooth residual $\mathcal{R}(s, Z) \in \mathcal{W}$ such that*

$$(23) \quad \mathcal{F}(s, Z) = -b(s) + \mu_s(Z) + \mathcal{R}(s, Z), \quad \|D_Z^2 \mathcal{R}(s, Z)\| \leq C q_s(Z).$$

For every tuple $\Pi = (\Pi_v)$ of orthogonal multiplicity projections,

$$(24) \quad \left| \sum_v \operatorname{tr}(\Pi_v \mathcal{R}_v(s, Z)) \right| \leq C q_s(Z) q_s([\Pi, Z]).$$

One constant works for all projections and all sufficiently small (s, Z) , including projections of different ranks at different vertices.

The last factor measures the squared norm of precisely those arrow blocks crossing between $\operatorname{im} \Pi$ and its orthogonal complement. This factor, rather than a norm of Π alone, will retain control at an arbitrarily weak finite direction.

Proof. At $Z = 0$, (22) gives $\mathcal{F} = -b$. Harmonicity kills the linear term, as in Lemma 3.1. The direct quadratic term in the Chern curvature is the incoming-minus- outgoing expression (16), with its current harmonic Gram form. The quadratic metric correction contributes $L_s u_2$, where u_2 is the quadratic Taylor term of u . Its pairing with every parallel block vanishes, because L_s is self-adjoint and annihilates those blocks. This proves the constant, linear, and quadratic parts of (23).

The unitary which equals $-I$ at all targets and I at all sources sends Z to $-Z$ and acts trivially by conjugation on the parallel algebra. Equivariance yields $\mathcal{F}(s, -Z) = \mathcal{F}(s, Z)$. Consequently the cubic term also vanishes. Joint Taylor expansion, with the background parameter restricted to a relatively compact neighborhood, gives the Hessian bound in (23). In particular it holds for each nearby s , with no comparison between $|s|$ and $q_s(Z)$.

Fix Π and decompose each arrow into its preserving and crossing parts, writing $Z = A + C$. Explicitly,

$$A_{ij,a} = \Pi_i Z_{ij,a} \Pi_j + (1 - \Pi_i) Z_{ij,a} (1 - \Pi_j), \quad C = Z - A.$$

The two parts are orthogonal in the harmonic Gram norm, and $q_s(C) = q_s([\Pi, Z])$. This remains true for nondiagonal arrow Gram matrices, since preserving and crossing matrix blocks are orthogonal for each pair of arrow indices.

At $C = 0$ the harmonic bundle is a holomorphic direct sum $G_\Pi \oplus G_{1-\Pi}$. A circle acting scalarly on G_Π and trivially on its complement fixes A . Equivariance and uniqueness of u force $u(s, A)$ to preserve this sum. Integrated Chern–Weil trace on the first summand is its degree, whence

$$(25) \quad \sum_v \operatorname{tr}(\Pi_v \mathcal{F}_v(s, A)) = \sum_v r_v \operatorname{rk}(\Pi_v) (\kappa_v - \kappa_E) = - \sum_v \operatorname{tr}(\Pi_v b_v I).$$

Conjugation by $e^{u/2}$ does not change this trace because it preserves the summand. The moment-map trace on that summand is zero: every remaining arrow begins and ends within the same summand. Thus $\langle \Pi, \mathcal{R}(s, A) \rangle = 0$.

The unitary $2\Pi - I$ fixes A and reverses C . It commutes with Π , so $\langle \Pi, \mathcal{R}(s, A + C) \rangle$ is even in C . Its first derivative at $C = 0$ vanishes. Taylor expansion in C alone, and the Hessian bound along $A + tC$, now give

$$|\langle \Pi, \mathcal{R}(s, A + C) \rangle| \leq C q_s(A + C) q_s(C).$$

Here $q_s(A + tC) \leq q_s(A + C)$ for $0 \leq t \leq 1$ by orthogonality. The finitely many rank choices of Π give compact products of Grassmannians; unitary invariance and bounded derivatives make the constant independent of Π . This proves (24). $\square$

Only the auxiliary representation A , with its crossing blocks removed, was used to form the holomorphic direct summand in the proof. The projection Π need not define a holomorphic

subbundle for the original representation Z . This is why (24) is available for every orthogonal projection, including the spectral projections used next.

4. COMPARISON WITH AN INDEPENDENTLY BALANCED METRIC

The reduced finite equation differs from quadratic balance by the residual of Proposition 3.2. Its derivative may have arbitrarily small positive eigenvalues. We obtain the required metric correction by excluding zeros on the boundary of a small ball, rather than estimating the inverse of that derivative.

Theorem 4.1. *There are constants $A, \sigma_0 > 0$ such that the following holds uniformly in the fixed background neighborhood. Let Z satisfy $\mu_s(Z) = b(s)$, let its complex centralizer be scalar, and suppose $0 < \sigma = q_s(Z) < \sigma_0$. Then some $Y \in \mathfrak{h}$ satisfies*

$$(26) \quad \|Y\|_{\text{op}} \leq A\sigma, \quad \mathcal{F}(s, e^{Y/2} \cdot Z) = 0.$$

For $Z' = e^{Y/2} \cdot Z$, the metric $h_{0,s}e^{u(s,Z')}$ is HE on $E_{s,Z'}$, and $\|u(s, Z')\|_{C^a} \leq C_a\sigma$ for every fixed a . The constants do not depend on a positive finite eigenvalue or on a ratio of such eigenvalues.

Proof. On the real space $\mathfrak{h}$ consider the homotopy into $\mathcal{W}$

$$(27) \quad H_\tau(Y) = \mu_s(e^{Y/2} \cdot Z) - b + \tau \mathcal{R}(s, e^{Y/2} \cdot Z), \quad 0 \leq \tau \leq 1.$$

Use the ball $\|Y\|_{\text{op}} < a$, where $a = A\sigma < 1$ and $\|Y\|_{\text{op}} = \max_v \|Y_v\|_{\text{op}}$. Write $M = \sum_v m_v$. Both layers are nonzero, so $M \geq 2$.

Suppose $\|Y\|_{\text{op}} = a$. Diagonalize each block of Y unitarily and sort all M eigenvalues as $y_1 \leq \dots \leq y_M$, retaining their vertices. The condition $\sum_v r_v \text{tr } Y_v = 0$, with positive weights, implies that their oscillation is at least a . An adjacent gap therefore satisfies

$$(28) \quad \delta = y_{\ell+1} - y_\ell \geq \frac{a}{M-1}.$$

Let Π be the tuple of spectral projections onto eigenvectors above this gap. It is nonzero and proper and commutes with Y .

Use orthonormal arrow coordinates at this fixed s . If p is a target eigenvector and q a source eigenvector, write w_{pq} for the sum of squared arrow entries joining them. The gauge $e^{Y/2}$ multiplies w_{pq} by $e^{y_p - y_q}$. Since $\mu_s(Z) = b$,

$$(29) \quad \begin{aligned} \langle \Pi, \mu_s(e^{Y/2} \cdot Z) - b \rangle &= \sum_{p,q} (\Pi_p - \Pi_q)(e^{y_p - y_q} - 1)w_{pq} \\ &\geq (1 - e^{-\delta})q_s([\Pi, Z]) \\ &\geq e^{-2a} \frac{a}{M-1} q_s([\Pi, Z]). \end{aligned}$$

For a high target and low source the exponential difference is positive and at least $e^\delta - 1$. For a low target and high source the projection sign is negative and its product with the exponential difference is at least $1 - e^{-\delta}$. These are all crossing entries; preserving entries contribute zero. The last inequality follows from $\delta \leq 2a$ and $1 - e^{-\delta} \geq \delta e^{-\delta}$. The crossing energy is strictly positive. Otherwise Π would commute with every arrow and be a nonscalar centralizer element.

Put $Z_Y = e^{Y/2} \cdot Z$. The gauge bounds and $[\Pi, Y] = 0$ give

$$(30) \quad q_s(Z_Y) \leq e^{2a}\sigma, \quad q_s([\Pi, Z_Y]) \leq e^{2a}q_s([\Pi, Z]).$$

Proposition 3.2 consequently yields

$$(31) \quad |\langle \Pi, \mathcal{R}(s, Z_Y) \rangle| \leq C e^{4a} \sigma q_s([\Pi, Z]).$$

Choose $A > 2C(M-1)$, and then choose σ_0 so small that $e^{6A\sigma_0} < 2$ and the entire gauge ball stays in the coefficient neighborhood. Since $a = A\sigma$, (29) strictly dominates (31). Thus H_τ has no zero on the boundary, for any $\tau \in [0, 1]$. The projection depends on Y ; only pointwise exclusion of a boundary zero is needed.

At $\tau = 0$ the unique zero is $Y = 0$. Indeed the function (18) has $f'(0) = 0$ and $f''(t) > 0$ for nonzero $Y \in \mathfrak{h}$, by the scalar-centralizer hypothesis. It follows that $\langle Y, H_0(Y) \rangle = f'(1) > 0$. The derivative at zero also satisfies

$$(32) \quad \langle Y, DH_0(0)Y \rangle = q_s([Y, Z]) > 0 \quad (Y \neq 0).$$

Under the fixed identification $T_v \mapsto T_v/r_v$ from $\mathcal{W}$ to $\mathfrak{h}$, this is a positive definite real Hessian. Its local Brouwer degree is $+1$. The operator-norm ball is a bounded open subset of a finite-dimensional real vector space; smoothness of its boundary is not required for degree. Homotopy invariance gives a zero of H_1 inside this ball.

At that zero the reduced equation and the complementary equation both vanish, so (19) is zero. This is the HE equation. Finally (30) and Lemma 3.1 give the asserted bounds for u . $\square$

The positivity in (32) is used to compute a degree at each fixed parameter. No uniform lower bound for this Hessian is used. Likewise the crossing energy in (29) may be arbitrarily small; it appears as the same positive factor in the error bound and cancels in the comparison. This gives the uniformity at weak directions.

Proposition 4.2. *After shrinking the parameter neighborhood, E is slope-stable exactly at the $b(s)$ -stable points of its representation. At these points the metric comparison (11) holds.*

Proof. Let W be a proper nonzero subrepresentation. The smooth subbundle $G_W = \bigoplus_v F_v \otimes W_v$ is preserved by $\bar{\partial}_{s, Z^0}$ and hence is an actual holomorphic subbundle of E with locally free quotient. Put $r_W = \sum_v r_v \dim W_v$. Its degree is the sum of the degrees of its factor copies, so

$$(33) \quad \kappa(G_W) - \kappa_E = -\frac{\sum_v b_v \dim W_v}{r_W}.$$

Stability of E forces the inequalities (6). This implication uses only the displayed subbundles, without asserting that they exhaust all subsheaves.

Conversely, finite stability gives a balanced Z by Proposition 2.2. Lemma 2.1 gives its scalar centralizer. Since $b(0) = 0$, shrink the background neighborhood so that every such σ is below the fixed threshold of Theorem 4.1. That theorem constructs an HE metric on $E_{s, Z'}$, and the harmonic and finite gauges pull it back to E . An HE bundle is polystable [Kob87, Chapter V]; a simple polystable bundle has only one stable summand, and therefore is stable. This argument constructs the HE metric before using stability sufficiency.

For the comparison, work first on the independently balanced presentation. With $g = e^{Y/2}$ the additional pullback of the metric has matrix $g^* e^u g$ in a split unitary frame. Scalar Hermitian-form bounds for e^u and for $g^* g = e^Y$ give

$$e^{-C\sigma} h_{0,s} \leq g^*(h_{0,s} e^u) \leq e^{C\sigma} h_{0,s}.$$

No commutation of u and Y is needed. Pullback through the same original balancing gauge and harmonic identification preserves these inequalities, even if the balancing gauge is unbounded. This proves (11). The unipotent identification and determinant-normalized finite

gauges have determinant one; $\sum_v r_v \operatorname{tr} Y_v = 0$ and $\operatorname{tr} u = 0$ preserve the fixed determinant metric. The HE metric is unique after this normalization. $\square$

No smooth selection of Y as the parameter approaches a chamber face is asserted. The proof supplies a uniformly small correction at every stable point. Bounds for derivatives of a selected gauge with respect to the parameter are a different question.

5. THE FULL SMALL DOLBEAULT SPECTRUM

We now compare the elliptic operator with the finite form. The first step holds for every sufficiently small simple coefficient array, whether balanced or not. All Hilbert spaces in this section are complex, and all endomorphisms are pointwise trace-free.

Proposition 5.1. *Let Z' be small with scalar centralizer. Use the split metric $h_{0,s}$ on $E_{s,Z'}$, and let $\nu_j(s, Z')$ be the eigenvalues of (8). There are constants $C, \gamma_1 > 0$, uniform in the fixed background neighborhood, such that the ordered Dolbeault eigenvalues $\tilde{\lambda}_j$ satisfy*

$$(34) \quad (1 - Cq_s(Z'))\nu_j(s, Z') \leq \tilde{\lambda}_j \leq \nu_j(s, Z') \quad (1 \leq j \leq N), \quad \tilde{\lambda}_{N+1} \geq \gamma_1/2.$$

In particular exactly N eigenvalues lie below a fixed positive threshold for sufficiently small Z' .

Proof. Write $D_0 = \bar{\partial}_{\operatorname{End} G}$ and $D = D_0 + \mathcal{A}$, where $\mathcal{A} = [N_s(Z'), \cdot]$. The kernel of D_0 is the full parallel space $\mathcal{H}$ of (7), by the fixed cross-Hom vanishing. Let P be complex L^2 projection onto $\mathcal{H}$, and put $Q = 1 - P$. These are the complex spectral projections, distinct from the real metric-coordinate projections of Section 3. The positive spectrum of $D_0^* D_0$ on Q has a uniform lower bound $\gamma_0 > 0$.

For every parallel $X \in \mathcal{H}$, $\mathcal{A}X$ is harmonic, because it is a constant multiplicity combination of the harmonic Hom forms. Thus

$$(35) \quad D_0^* \mathcal{A}X = 0, \quad V_s^{-1} \|DX\|_{L^2}^2 = \mathcal{Q}_{s,Z'}(X), \quad V_s^{-1} \|X\|_{L^2}^2 = \|X\|_r^2.$$

Let $q = q_s(Z')$. Since the harmonic coefficient space is fixed and finite-dimensional, its C^0 and L^2 norms are uniformly equivalent. Consequently $\|\mathcal{A}\|_{L^2 \rightarrow L^2(0,1)} \leq C\sqrt{q}$. For $y \in QH^1$ this gives

$$\|Dy\|_{L^2} \geq (\sqrt{\gamma_0} - C\sqrt{q})\|y\|_{L^2}.$$

The complementary compression $C_0 = QD^*DQ$, understood initially as a closed quadratic form, therefore satisfies $C_0 \geq \gamma_1 I$ for some fixed $\gamma_1 > 0$ and all sufficiently small q .

Set $G_0 = PD^*DP$ and $B = QD^*DP$. Harmonicity in (35) gives, as equalities of smooth sections,

$$(36) \quad BX = Q\mathcal{A}^*AX, \quad \|BX\|_{L^2}^2 \leq Cq\|\mathcal{A}X\|_{L^2}^2 = Cq\langle X, G_0X \rangle_{L^2}.$$

The right side is each vector's own finite energy. There is no replacement by $q\|X\|^2$ and no division by ν_1 .

We specify the form domains to justify the spectral comparison. The form domain of D^*D is $H^1(\operatorname{End}_0 G)$ and its operator domain is $H^2(\operatorname{End}_0 G)$. Removing the finite-dimensional smooth space $\mathcal{H}$ preserves these Sobolev spaces; the form and operator domains of C_0 are QH^1 and QH^2 . In particular B maps the finite space into L^2 , and $(C_0 - \lambda)^{-1}$ maps QL^2 into QH^2 for $0 \leq \lambda \leq \gamma_1/2$.

For such λ define the finite Schur complement

$$(37) \quad S_\lambda = G_0 - \lambda I - B^*(C_0 - \lambda)^{-1}B.$$

Equation (36) gives the Hermitian-form bounds

$$(38) \quad (1 - Cq)G_0 - \lambda I \leq S_\lambda \leq G_0 - \lambda I.$$

Completion of the square on $\mathcal{H} \oplus QH^1$ gives

$$(39) \quad \begin{aligned} \mathfrak{q}_{D^*D-\lambda}(X+y) &= \langle S_\lambda X, X \rangle \\ &\quad + \left\| (C_0 - \lambda)^{1/2} (y + (C_0 - \lambda)^{-1} BX) \right\|^2. \end{aligned}$$

The triangular change of variables here is a bijection of the form domain, since its correction lies in QH^2 . The negative index of $D^*D - \lambda$ thus equals the negative index of S_λ .

The norm of G_0 is $O(q)$. At $\lambda = \gamma_1/2$ the finite Schur complement is strictly negative, whereas the complementary operator is positive. Exactly N eigenvalues therefore lie below this threshold and the threshold is not an eigenvalue. For the indexed lower bound put $\alpha = 1 - Cq > 0$. If $\lambda < \alpha\nu_j$, the form $\alpha G_0 - \lambda I$ has at most $j - 1$ negative directions. The same is true of S_λ by (38); (39) then implies $\tilde{\lambda}_j \geq \alpha\nu_j$ by passage to the limit. For the upper bound apply ordinary min-max to the first j finite eigenvectors inside $\mathcal{H}$. These arguments also apply when eigenvalues coincide. They establish (34) with complex multiplicities. $\square$

Lemma 5.2. *Let $g = e^{Y/2}$ for $Y \in \mathfrak{h}$, put $a = \|Y\|_{\text{op}}$, and let $Z' = g \cdot Z$. For all $1 \leq j \leq N$,*

$$(40) \quad e^{-4a}\nu_j(s, Z) \leq \nu_j(s, Z') \leq e^{4a}\nu_j(s, Z).$$

This comparison holds for arbitrary positive harmonic Gram matrices.

Proof. Conjugation $T_g X = (g_v X_v g_v^{-1})_v$ preserves the full complex space $\mathcal{H}$, since it preserves each block trace. The bounds $\|g_v^{\pm 1}\|_{\text{op}} \leq e^{a/2}$ imply

$$(41) \quad e^{-2a}\|X\|_r^2 \leq \|T_g X\|_r^2 \leq e^{2a}\|X\|_r^2.$$

The exact intertwining identity

$$[T_g X, g \cdot Z]_{ij,a} = g_i [X, Z]_{ij,a} g_j^{-1}$$

gives the same factors for the squared energy. For a nondiagonal Gram matrix, apply its positive square root to the arrow index. Left and right multiplicity operations commute with this change because every arrow in the block has the same source and target. Thus the same energy bounds hold without changing constants. Rayleigh quotients have factors $e^{\pm 4a}$, and generalized min-max through the invertible complex map T_g proves the lemma. $\square$

The map T_g need not preserve Hermitian matrices. Its use on $\mathcal{H}$ is one reason to retain the full complex operator throughout the argument. At exact central balance there is an additional adjoint symmetry, discussed in Section 7, but no such symmetry is needed during this comparison.

Proof of Theorem 1.1. Proposition 4.2 proves the stability equivalence and metric statement. At a stable parameter choose the independently balanced Z . Theorem 4.1 gives $Z' = e^{Y/2} \cdot Z$ with $\|Y\|_{\text{op}} \leq A\sigma$ and $\|u\|_{C^0} \leq C\sigma$. Moreover $q_s(Z') \leq e^{2A\sigma}\sigma$. Proposition 5.1 and Lemma 5.2 therefore compare the split-metric small eigenvalues at Z' to $\nu_j(s, Z)$ with a common factor $1 + O(\sigma)$.

It remains to pass to the actual metric on the same holomorphic bundle $E_{s,Z'}$. If two bundle metrics satisfy $e^{-c}h_0 \leq h_1 \leq e^c h_0$, their induced squared norms on $\text{End } G$ lie between e^{-2c} and e^{2c} times each other. The same is true on $\text{End } G$ -valued $(0, 1)$ -forms, since the base metric is fixed in this comparison. The differential D is unchanged. Hence every Rayleigh

quotient has comparison factors $e^{\pm 4c}$ on the common H^1 form domain. Apply this with $h_1 = h_{0,s}e^u$ and $c = C\sigma$.

The trace-free spaces in this argument are invariant for both Laplacians. Indeed $\text{End } G = \mathbb{C}I \oplus \text{End}_0 G$ is orthogonal for every induced endomorphism metric, $D(fI) = (\partial f)I$, and $\text{tr}(D\xi) = \bar{\partial} \text{tr } \xi$. This orthogonal splitting also holds on one-forms. Both D and its adjoint preserve the corresponding summands. Min-max therefore applies to the same trace-free domain for each metric.

The additional factor is $1 + O(\sigma)$, so it proves (10) for $1 \leq j \leq N$. It also leaves a fixed positive fraction of the complementary bound in Proposition 5.1. Finally holomorphic pullback, with the metric pulled back at the same time, is unitary on the relevant L^2 spaces and intertwines the Dolbeault operators. It identifies these eigenvalues with those on the original E , regardless of the size of the original balancing gauge. Shrinking $\mathcal{U}$ absorbs all uniform exponential factors into the stated constants. $\square$

6. A FIXED RANK-FIVE BUNDLE WITH SEPARATED SPECTRAL SCALES

We give a geometric example in which the stable region has three independent polarization parameters. Its full finite model has six complex modes, and one can tend to zero at an arbitrary positive rate relative to the other five.

Let $X = (\mathbb{P}^1)^4$. Write ω_ℓ for the pullbacks of Fubini–Study forms normalized by $\int_{\mathbb{P}^1} \omega_\ell = 1$, and put $\omega_h = \sum_{\ell=1}^4 h_\ell \omega_\ell$, with $h_\ell > 0$. Consider

$$(42) \quad F_0 = \mathcal{O}_X, \quad F_j = \mathcal{O}_X(2e_j - 2e_4) \quad (1 \leq j \leq 3),$$

where e_ℓ denotes degree one on the ℓ -th factor, and an extension

$$(43) \quad 0 \longrightarrow \mathcal{O}_X \otimes \mathbb{C}^2 \longrightarrow E \longrightarrow F_1 \oplus F_2 \oplus F_3 \longrightarrow 0.$$

All four stable types are line bundles. They are distinct and have degree zero at $h = (1, 1, 1, 1)$. Product invariant Hermitian metrics are HE for every product polarization. The normalization (3) gives

$$(44) \quad V_h = h_1 h_2 h_3 h_4, \quad \kappa_0 = 0, \quad \kappa_j = 4\pi(h_j^{-1} - h_4^{-1}), \quad \kappa_E = \frac{1}{5} \sum_{j=1}^3 \kappa_j.$$

Indeed $iF_{F_j} = 4\pi(\omega_j - \omega_4)$ and $\Lambda_{\omega_h} \omega_\ell = h_\ell^{-1}$.

The Künneth formula and line-bundle cohomology on $\mathbb{P}^1$ give

$$(45) \quad \text{Ext}^1(F_j, F_0) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(2)) \cong \mathbb{C}^3.$$

The nonconstant factors are j and 4; the others contribute $H^0(\mathcal{O}) = \mathbb{C}$. Fix cohomology bases orthonormal at the reference polarization in the volume-normalized inner product. For every h , the current harmonic Gram matrix is $c_j(h)I_3$ with $c_j(h) > 0$ smooth. To see this, all the chosen metrics and forms are invariant under $\text{SU}(2)^4$. The only nontrivial action on (45) is the irreducible three-dimensional representation on the fourth-factor degree-two sections. Schur's lemma makes the invariant harmonic inner product a scalar multiple of the reference one. The cohomology coefficients below remain fixed as the harmonic representatives vary.

For small real $a = (a_1, a_2, a_3)$ let $A = a_1 + a_2 + a_3$ and choose the actual polarization chart

$$(46) \quad h_4 = 1, \quad h_j = \left(1 + \frac{A/2 + a_j}{4\pi}\right)^{-1} \quad (1 \leq j \leq 3).$$

It is defined for all sufficiently small signed a and has positive coefficients. Substitution into (44) gives

$$(47) \quad \kappa_E = A/2, \quad \kappa_j - \kappa_E = a_j, \quad b_0 = A/2, \quad b_j = -a_j, \quad \sigma = A.$$

The target moment is $(A/2)I_2$, so its trace is A . Thus the three finite demands are independent geometric parameters.

6.1. The extension and its complete endomorphism algebra. Define the fixed extension class by the following matrices, whose three columns multiply the basis elements in (45):

$$(48) \quad \begin{aligned} Z_1^0 &= \begin{pmatrix} \sqrt{3/2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}, & Z_2^0 &= \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \end{pmatrix}, \\ Z_3^0 &= \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1 & 0 \end{pmatrix}. \end{aligned}$$

An extension of locally free sheaves by locally free sheaves is locally free. These classes therefore define one fixed rank-five holomorphic bundle, rather than merely a finite representation. The covariances $B_j = Z_j^0(Z_j^0)^*$ are

$$(49) \quad B_1 = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}.$$

They have determinants $3/4, 3/4, 1/2$, respectively, and

$$(50) \quad B_1 + B_2 + B_3 = 3I_2, \quad [B_1, B_2] = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \neq 0.$$

In particular the interactions are not simultaneously diagonal.

If $XZ_j^0 = Z_j^0x_j$ for $X \in M_2(\mathbb{C})$ and scalar x_j , full row rank of each Z_j^0 implies $X = x_jI_2$. All x_j coincide. Lemma 2.1 proves that E is simple. The full trace-free complex space is

$$(51) \quad \mathcal{H} = \{(X, x_1, x_2, x_3) : \text{tr } X + x_1 + x_2 + x_3 = 0\}, \quad \|(X, x)\|^2 = \|X\|_F^2 + \sum_j |x_j|^2.$$

It has dimension six. The multiplicity two of the target is already included in its matrix trace; it is not an additional factor rank.

Proposition 6.1. *The local stable region of the fixed bundle (43) in the chart (46) is exactly $a_1, a_2, a_3 > 0$. For these parameters its balanced covariances are*

$$(52) \quad C_j = a_j \frac{TB_jT}{\text{tr}(TB_jT)}, \quad \sum_{j=1}^3 C_j = \frac{A}{2}I_2,$$

where T is positive Hermitian and $\det T = 1$. The target shape T is unique, uniformly bounded together with its inverse, and smooth in the normalized weights $w_j = a_j/A$ on the entire closed simplex.

Proof. If a subrepresentation contains any source, its target contains the span of that source's three images, which is all of $\mathbb{C}^2$. The proper nonzero subrepresentations therefore have either a target line and no sources, or the full target with a proper subset S of sources. Their demands are $A/2$ and $\sum_{j \notin S} a_j$, respectively. They are all positive when $a_j > 0$. Conversely, taking the full target and every source except j gives demand a_j , so finite stability requires all three to be positive. Theorem 1.1 identifies this with the actual local stable region.

The scalar harmonic Gram factor $c_j(h)$ can be absorbed into the positive gauge at source j . After a target gauge T , choose its source multiplier to have square $c_j(h) \operatorname{tr}(TB_jT)/a_j$. This produces (52) in the same extension orbit. A common scalar on all blocks normalizes the entire bundle determinant; $\det T = 1$ fixes only the target shape.

To prove existence and uniformity directly, put $H = T^2$ and minimize

$$(53) \quad f_w(H) = \sum_j w_j \log \operatorname{tr}(HB_j), \quad H = H^* > 0, \quad \det H = 1.$$

All B_j are positive definite. With $c = 1 - 1/\sqrt{2} > 0$, one has $B_j \geq cI$ and $f_w(I) = \log 2$. Therefore

$$f_w(H) \geq \log(c \operatorname{tr} H), \quad \operatorname{tr} H \leq \frac{2}{c} = 4 + 2\sqrt{2} \quad \text{at a minimizer.}$$

This is a uniform coercivity bound for all weights in the closed simplex. The determinant condition bounds both eigenvalues above and away from zero on the indicated sublevel set, proving existence and compactness of the minimizers.

For a determinant-one geodesic $H(t) = H_0^{1/2} e^{tY} H_0^{1/2}$ with $Y = Y^*$ and $\operatorname{tr} Y = 0$, each summand of (53) is $\log \sum_\ell q_{j\ell} e^{ty_\ell}$ in an eigenbasis for Y . Every $q_{j\ell}$ is strictly positive, because $B_j > 0$. The second derivative is the variance of the y_ℓ under positive normalized weights. It is strictly positive when $Y \neq 0$. A convex combination of these variances remains positive even if some $w_j = 0$. Hence f_w is strictly geodesically convex and the minimizer is unique.

At $H = T^2$ its first variation in direction TYT is

$$\operatorname{tr} \left(Y \sum_j w_j \frac{TB_jT}{\operatorname{tr}(TB_jT)} \right).$$

It vanishes for every trace-free Hermitian Y exactly when the matrix in parentheses is $I_2/2$, proving (52). On the compact family of minimizers the Hessian has a uniform positive lower bound on unit tangent vectors. The finite-dimensional implicit-function theorem gives a smooth extension near each point of the closed simplex in its affine weight hyperplane. Uniqueness patches these extensions. This proves the asserted smoothness, including at corners. $\square$

At a boundary weight the corresponding limiting arrow disappears. The source gauge relating it to the original nonzero class is no longer invertible there. Boundedness of T is not boundedness of the full metric, and the proposition asserts no HE metric on the original boundary bundle.

6.2. Six finite modes. For the covariances (52), the finite form is

$$(54) \quad \mathcal{Q}(X, x) = \sum_{j=1}^3 \operatorname{tr}((X - x_j I) C_j (X - x_j I)^*) \quad \text{on (51).}$$

At $a_1 = a_2 = a_3 = 2t$ one has $T = I$ and $C_j = tB_j$.

Proposition 6.2. *At these equal demands the six finite eigenvalues, divided by t , are*

$$(55) \quad 3, \quad 5, \quad \frac{5 - \sqrt{7}}{2}, \quad \frac{5 + \sqrt{7}}{2}, \quad \frac{5 - \sqrt{3}}{2}, \quad \frac{5 + \sqrt{3}}{2}.$$

The scalar-target compression instead has only three positive eigenvalues, $2t, 2t, 5t$.

Proof. It suffices to take $t = 1$. In $M_2(\mathbb{C})$ use the orthonormal basis

$$I_2/\sqrt{2}, \quad \sigma_z/\sqrt{2}, \quad \sigma_x/\sqrt{2}, \quad \sigma_y/\sqrt{2},$$

where the last three are the Pauli matrices. Their complex span is the full matrix algebra. On the complete seven-dimensional space before removing the identity, (54) has matrix

$$(56) \quad L = \begin{pmatrix} 3I_4 & -K \\ -K^* & 2I_3 \end{pmatrix}, \quad K = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of K^*K are $6, 3/2, 1/2$. Each squared singular value s^2 gives the quadratic factor $(\lambda - 3)(\lambda - 2) - s^2$, and the remaining target direction has eigenvalue 3. Thus

$$\det(\lambda I_7 - L) = \lambda(\lambda - 5)(\lambda - 3)(\lambda^2 - 5\lambda + 9/2)(\lambda^2 - 5\lambda + 11/2).$$

The zero is exactly the scalar identity. Its orthogonal complement is (51), which gives (55). Restricting the target to its scalar part replaces K by its first row and gives $\lambda(\lambda - 5)(\lambda - 2)^2$. $\square$

6.3. Arbitrary separation on genuine polarizations. Set $a = (t\delta, t, t)$, with $t, \delta > 0$. The target solution depends only on $w = (\delta, 1, 1)/(2 + \delta)$, so the form $\mathcal{Q}/t$ is smooth in δ down to zero.

Theorem 6.3. *There are $t_0, \delta_0, c, C, \gamma > 0$ such that, for independently chosen $0 < t < t_0$ and $0 < \delta < \delta_0$, the HE spectrum of the fixed bundle (43) at (46) satisfies*

$$(57) \quad \lambda_1 = \frac{5}{4}t\delta(1 + O(\delta) + O(t)), \quad ct \leq \lambda_j \leq Ct \quad (2 \leq j \leq 6), \quad \lambda_7 \geq \gamma.$$

All constants are independent of the ratio between δ and t .

Proof. At $\delta = 0$ the target and sources 2, 3 form a simple representation, since each of these sources has full image span. The absent first source adds one centralizer direction. On (51) the kernel is therefore exactly the line

$$(58) \quad (X, x_1, x_2, x_3) = (\alpha I_2, -4\alpha, \alpha, \alpha).$$

Its squared norm is $20|\alpha|^2$, and the remaining five eigenvalues of $\mathcal{Q}/t$ are positive. On this fixed vector the strong-source terms vanish for every target shape. Since $\text{tr } C_1 = t\delta$, the weak-source term is exactly $25t\delta|\alpha|^2$. Smooth perturbation of the simple zero eigenvalue gives

$$(59) \quad \nu_1 = \frac{5}{4}t\delta(1 + O(\delta)), \quad ct \leq \nu_j \leq Ct \quad (2 \leq j \leq 6).$$

The derivative $5/4$ is the ratio $25/20$ of energy derivative to mass. The finite form scales exactly with t , so these constants are uniform in $t > 0$.

All a_j are positive, and $\sigma = t(2 + \delta)$. Theorem 1.1 applies with relative error $O(t)$ and a fixed complementary gap. Combining it with (59) proves (57). $\square$

For every fixed $p > 0$, the choice $\delta = t^p$ yields a small eigenvalue of order t^{p+1} while the other five have order t . The choice $\delta = e^{-1/t}$ gives a faster separation. Both are paths of actual Kähler polarizations of the same fixed bundle.

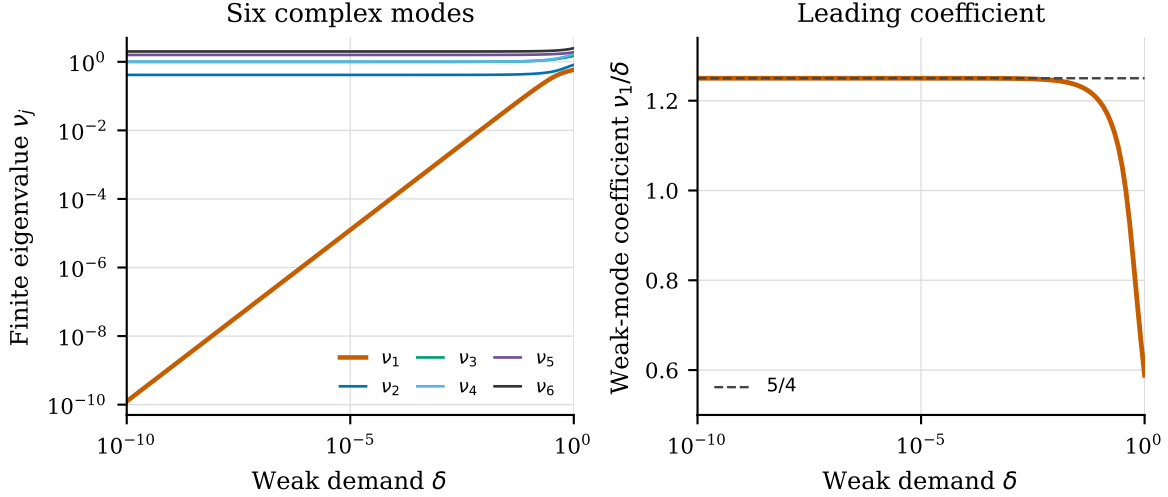


FIGURE 1. Finite commutator eigenvalues for demands $(\delta, 1, 1)$ (equivalently $t = 1$ in the homogeneous finite model). The first mode tends to zero while the other five stay positive; the second panel displays its coefficient tending to $5/4$. These are numerical evaluations of (54), not computed eigenvalues of the HE differential operator. Theorem 6.3 supplies the relative comparison for small geometric t .

7. SCOPE AND REPRODUCIBILITY

The theorem concerns a fixed compact Kähler background, fixed locally free factors, two nonzero layers, and fixed multiplicities. Its uniformity includes approaches to faces through the simple stable region; it does not include the nonsimple endpoint itself. Higher extensions have composable arrows and nonlinear integrability relations, so the exact harmonic presentation and parity argument used here do not apply without further work. Singular graded sheaves and actual Yang–Mills flow estimates require additional analysis.

7.1. Complex endomorphisms and adjoint symmetry. Let L_Z be the finite operator associated with $\mathcal{Q}_{s,Z}$ and the rank mass. In orthonormal arrow coordinates, direct expansion gives

$$(60) \quad r_v((L_Z(X^*))_v - (L_Z X)_v^*) = [X_v^*, \mu_s(Z)_v].$$

At central balance the right side vanishes. Hence the Hermitian real fixed space of adjoint complexifies to the full operator, with the same eigenvalue multiplicities. This agrees with the centralizer identity of [HKKP23, §5.2, Proposition 5.3]. Thus the six-mode example is a failure of scalar-vertex compression, not a failure of correctly complexifying the Hermitian operator at central balance.

Off balance a Hermitian restriction can lose even the correct kernel. Take one target and one source, each of dimension two, with arrows I_2 and $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The identity arrow forces the two centralizer blocks to agree; commutation with J then gives $X = Y = aI + bJ$. On the full trace-free complex space the kernel is the line $X = Y = bJ$. Its Hermitian part is zero. This representation is not centrally balanced: its moments are $\text{diag}(2, 1)$ and $\text{diag}(-1, -2)$. It illustrates why the intermediate gauge comparison in Lemma 5.2 is performed on the complex space. Likewise a single identity arrow has connected support but all simultaneous matrix

endomorphisms in its centralizer. Connectedness alone never replaces the scalar-centralizer condition.

7.2. Finite calculations. The accompanying scripts use the explicit matrices printed in Section 6. Exact rational computations verify the covariances, full centralizer dimension, positive trace-free Gram form, characteristic polynomials, and the off-balance kernel comparison above. A separate implementation forms the raw complex commutator matrix in a Pauli basis and verifies the same six-mode polynomial and the coefficient $25/20$. These calculations check finite algebra; the proofs of HE existence and relative elliptic estimates are those in Sections 3–5.

For the plotted family we solve the two real target balance equations. This is valid for these real matrices: uniqueness of the positive Hermitian minimizer and invariance under complex conjugation imply that it is real symmetric. Small eigenvalues are evaluated as the squares of singular values of the commutator map restricted to a mass-orthonormal trace-free basis. This avoids forming a rounded Gram matrix before computing its weakest eigenvalue. The numerical coefficient at $\delta = 10^{-10}$ is approximately 1.24999999996, consistent with the proved limit $5/4$.

The source distribution contains the calculation scripts, generated data, figure source, dependency versions, and commands for rebuilding the figure and this PDF. No partial differential equation is solved numerically in these computations. The numerical output is an illustration and reproducibility check of the finite model, not an additional hypothesis in the theorems.

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