HERMITIAN–EINSTEIN METRICS AND MODULI OF THREE-STEP EXTENSIONS WITH REPEATED FACTORS

BERND JOHANNES WUEBBEN

Abstract. We construct Hermitian–Einstein metrics and obtain relative asymptotics for all six small eigenvalues of the Dolbeault Laplacian on trace-free endomorphism-valued zero-forms in a rank-five deformation problem with a repeated line factor on the product of three projective lines. The estimates are uniform as positive spectral scales separate or vanish. At nearby product polarizations, the metric construction applies to every sufficiently small simple integrable deformation satisfying the quadratic moment-map equation at the centered slope level. Its logarithmic error relative to the split metric is quadratic in the harmonic coefficients. The key estimate uses harmonic orthogonality to eliminate cubic curvature terms and bounds the remaining error against each multiplicity projection by the energy mixing its image and kernel. This gives relative spectral comparison with the matrix commutator form without a lower bound on its positive eigenvalues. On smooth pieces of the full local Hermitian–Einstein quotient, we likewise compare the harmonic L^2 metric with the coefficient quotient metric. At a reducible locus where one copy of the repeated line splits off, the induced metric agrees with that of the remaining stable rank-four summand.

1. Introduction

A repeated stable factor changes the local geometry of an extension. Its scalar rescalings are replaced by a matrix group, and the contractions associated with different extension classes need not commute. In a three-step filtration there is a second difficulty: successive extensions compose, so integrability imposes relations and requires a nonlinear correction. We study these two features together in a complete deformation problem on a product of projective lines.

The central estimate controls curvature against every projection in the multiplicity space. Its crossing energy is the squared norm of the coefficient components mixing the projection's image and kernel. The estimate remains effective when this is much smaller than the total squared coefficient norm. This leads to actual HE metrics and relative estimates for all small trace-free endomorphism eigenvalues. The same complete model describes the local moduli geometry where a repeated line splits off. We then construct a curve of genuinely changing bundles that reaches this part in finite length.

1.1. The example and its complete deformation space.

On $X = (\mathbb{P}^1)^3$ let

$$F_1 = \mathcal{O}(-2,-2,4), \quad F_2 = \mathcal{O}(1,-2,1), \quad F_3 = \mathcal{O}(-2,1,1), \quad F_4 = \mathcal{O},$$

and use the smooth rank-five bundle $E_0 = F_1 \oplus (F_2 \otimes \mathbb{C}^2) \oplus F_3 \oplus F_4$. Give it the product Hermitian metric h_0. Block ij denotes $\mathrm{Hom}(F_j, F_i)$ with its indicated multiplicities. The

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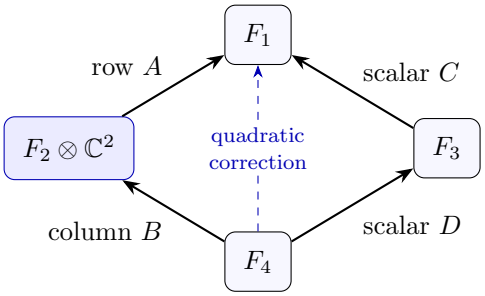
four adjacent blocks $12, 13, 24, 34$ form the diagram in Figure 1. The path through $F_2 \otimes \mathbb{C}^2$ multiplies a row by a column; the path through F_3 is scalar.



Harmonic relation: $A_0B_0 + A_1B_1 = C_0D_0 + C_1D_1$

FIGURE 1. The two composable paths in the adjacent-layer subfamily. The repeated middle factor retains a full two-by-two matrix action. The central correction enforces integrability after the nonzero harmonic path classes cancel. The complete deformation space also contains the other blocks listed in Lemma 2.1.

The full harmonic deformation space V has dimension 119, including reverse arrows and directions outside these four blocks. Its complete obstruction has five components. For the nearby product polarizations (6), let K_p be the split Hodge homotopy and $P_{2,p}$ the degree-two harmonic projection. The convergent equations

$$a_p(x) = \iota x - K_p(a_p(x)^2), \qquad \kappa_p(x) = P_{2,p}(a_p(x)^2)$$

give the full integrability condition $\kappa_p(x) = 0$. The effective complex group is $G = (\mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1)/\mathbb{C}^*$, with compact form K. Its trace-free parallel algebra $\mathfrak{g}$ has complex dimension six. Theorem 2.2 identifies every holomorphic map between small reconstructed bundles with a constant harmonic intertwiner. Thus all matrix directions, rather than only vertex scalars, enter both the spectrum and the quotient.

Write $q_p(x) = \|\iota x\|_p^2$, where the squared norm is divided by spatial volume. On the adjacent subspace the only quadratic correction lies in block 14, and all higher products vanish. The two path obstruction classes can be nonzero and opposite, while their sum has a nonzero coexact primitive. The family in Section 3 demonstrates this directly. It also has a boundary kernel generated by a matrix that is not Hermitian for h_0. Restricting the finite calculation to Hermitian matrix directions would miss that complex holomorphic kernel.

1.2. **Curvature control and the actual small spectrum.** Let $b(p)$ be the centered slope differences, let μ_p be the quadratic matrix moment map, and let m_p be the remaining curvature equation after solving on the complement of all parallel Hermitian endomorphisms. We prove

$$m_p(x) = -b(p) + \mu_p(x) + R_p(x), \qquad |\langle \Pi, R_p(x)\rangle| \le C q_p(x) q_p([\Pi, x])$$

for every orthogonal multiplicity projection Π, including arbitrary projections in $\mathbb{C}^2$. The estimate holds on the entire ambient ball. Its proof has two parts. Harmonic orthogonality removes the cubic curvature terms. At a projection-preserving array, a partial Chern–Weil identity makes the projected remainder zero, and a reflection makes its first crossing derivative zero. Taylor's formula then gives the crossing-energy factor.

For an integrable simple point with $\mu_p(x) = b(p)$ and small $q = q_p(x)$, Theorem 4.5 constructs a fixed-determinant HE metric satisfying

$$e^{-Cq}h_0 \le H \le e^{Cq}h_0. \tag{1}$$

The finite correction uses a degree argument and the same crossing factor in its positive moment increment and residual error. It does not divide by the weakest positive orbit eigenvalue.

Let $\nu_1 \le \cdots \le \nu_6$ be the eigenvalues of $\xi \mapsto \|[\xi, \iota x]\|_p^2$ on the full complex algebra $\mathfrak{g}$. For the actual trace-free endomorphism zero-form Dolbeault eigenvalues, Lemma 5.1 and (1) give

$$\left|\log \frac{\lambda_j}{\nu_j}\right| \le Cq \quad (\nu_j > 0,\ j \le 6), \qquad \lambda_j = 0 \Longleftrightarrow \nu_j = 0, \qquad \lambda_7 \ge \gamma > 0. \tag{2}$$

All constants are independent of positive eigenvalue ratios. For the balanced family in Section 5, $q \asymp t^2$ and $\lambda_1 = \frac{5}{4}t^2\varepsilon^2(1 + O(t^2 + \varepsilon^2))$, while the other five small eigenvalues are comparable to t^2. At $\varepsilon = 0$ the weak mode is exactly zero and a line splits off. We construct that nonsimple endpoint metric separately.

1.3. **The quotient and a finite-length boundary curve.** The corrected locus $\{\kappa_p = 0, m_p = 0, |x| < r\}/K$ describes an open neighborhood of actual HE classes, with uniform coverage in fixed connection norms. Section 6 proves compact-orbit uniqueness, identifies the effective stabilizers, and constructs smooth locally closed complex pieces. On these pieces Theorem 6.3 gives

$$(1 - Cr^2)g_{\mathrm{fin},p} \le g_{\mathrm{HE},p} \le (1 + Cr^2)g_{\mathrm{fin},p}, \qquad g_{\mathrm{fin},p}([v]) = \inf_{\xi\in\mathfrak{g}} \|v - C_x\xi\|_p^2. \tag{3}$$

The constant is uniform as the orbit map C_x and the obstruction derivative lose rank. The exact tangent kernel is the complex orbit tangent. Tangents transported to the fixed subspace of the limiting stabilizer have the specified limits in (67). At fixed polarization, intrinsic lengths separate points and compare on the same local path domains.

On $p_2 = 7p_1/5$, the piece where one copy of F_2 splits off is the active stable part of the full rank-four problem of [Wue26a]. We construct the change between the two Hodge presentations and prove that their actual active tangent metrics agree exactly. Volume normalization introduces no rank factor.

Finally, the unrotated balanced family has constant interior bundle type. Section 7 rotates a cubic into an orthogonal polynomial direction while preserving all moment and integrability identities. The resulting bundles are pairwise nonisomorphic, and their actual bundle tangent has norm comparable to t. Including the parameter derivative $\partial_p a_p$ and the assigned parameter metric $|dp|^2$, we obtain

$$ct\varepsilon \le \mathrm{Length}_{\mathrm{act}}(\gamma_t|_{(0,\varepsilon]}) \le Ct\varepsilon.$$

The constants hold for independent small t, ε. The curve converges to the rank-four-plus-line HE class. This is a tail-length estimate in the stated varying-polarization space; its lower bound is not an ambient shortest-distance bound.

1.4. **Relation to earlier work.** The qualitative local deformation theory near a fixed HE bundle, including automorphisms and the polystability correspondence, is developed in [BuS20]. Complementary elliptic reduction and adapted symplectic forms for varying backgrounds appear in [CT24, Del25]. We use these mechanisms with the complete matrix kernel and verify the uniform product-domain estimates. Our adapted form tames the complex structure; the local complex pieces are constructed directly.

The two-step result [Wue26c] already treats repeated stable factors, the estimate for every multiplicity projection, and the finite degree correction. Its bipartite geometry makes the harmonic presentation integrable without composable relations and supplies a parity cancellation. Here we retain the full nonlinear reconstruction, replace that cancellation by harmonic orthogonality, and treat a nonsimple endpoint. The finite correction argument is restated and proved in the form required by this directed matrix model.

The order of relative quotient minimization originates in [Wue26b]. We use the general analytic formulation and transverse linear lemma of [Wue26a], after verifying every hypothesis. The full matrix quotient, its polar normalization, its rank-four geometric overlap, and the parameter-inclusive moving curve are proved here. All objects remain locally free on a fixed compact threefold; the results do not assert a matrix flow theorem or a description of non-locally-free limits.

2. The complete matrix deformation model

We first construct the local deformation problem in which the examples will lie. Keeping the entire deformation space is necessary both for isomorphism statements and for the quotient metric. The four composable blocks used later form a proper subspace of this model.

2.1. **Geometry and norms.** Let $X = (\mathbb{P}^1)^3$, and let ω_i be the product Fubini–Study forms, normalized to have integral one on their factors. Set

$$F_1 = \mathcal{O}(-2,-2,4), \quad F_2 = \mathcal{O}(1,-2,1), \quad F_3 = \mathcal{O}(-2,1,1), \quad F_4 = \mathcal{O}, \tag{4}$$

$$E_0 = F_1 \oplus (F_2 \otimes \mathbb{C}^2) \oplus F_3 \oplus F_4, \qquad (m_1,m_2,m_3,m_4) = (1,2,1,1). \tag{5}$$

The product Hermitian metric h_0 uses the standard metric on $\mathbb{C}^2$. Its connection is A_0 and its Dolbeault operator is d; the induced operator on endomorphism forms has the same notation. Block ij maps type j to type i. For a small real parameter $p = (p_1, p_2)$, put

$$\omega_p = (1+p_1)^{-1}\omega_1 + (1+p_2)^{-1}\omega_2 + \omega_3. \tag{6}$$

All Hermitian L^2 products are divided by $\mathrm{Vol}_p = ((1+p_1)(1+p_2))^{-1}$ and are linear in their second argument. Their real parts are the associated real metrics. No norm is divided by the bundle rank. The normalized volume measure is fixed. Sobolev norms H^s, with $s \geq 6$, use ω_0, h_0; nearby norms are uniformly equivalent. We fix the product determinant metric on $\det E_0 = \mathcal{O}(-2,-5,7)$. Every eigenvalue denoted λ_j is that of the zero-form Dolbeault Laplacian $\bar\partial^*_{\mathrm{End}_0}\bar\partial_{\mathrm{End}_0}$ on complex trace-free endomorphisms, without a factor two. Eigenvalues are ordered with complex multiplicity, including zeros.

The contracted line curvatures and their rank-weighted average are

$$\begin{aligned} &\kappa_1 = -4\pi(p_1+p_2), && \kappa_2 = 2\pi(p_1-2p_2), \\ &\kappa_3 = 2\pi(-2p_1+p_2), && \kappa_4 = 0, && \kappa_E = -\tfrac{4\pi}{5}p_1 - 2\pi p_2. \end{aligned} \tag{7}$$

Write $b_i = \kappa_E - \kappa_i$, interpreting the second component as a scalar matrix on $\mathbb{C}^2$. Thus

$$b(p) = \big(\tfrac{16\pi}{5}p_1 + 2\pi p_2, -\tfrac{14\pi}{5}p_1 + 2\pi p_2, \tfrac{16\pi}{5}p_1 - 4\pi p_2, -\tfrac{4\pi}{5}p_1 - 2\pi p_2\big). \tag{8}$$

In particular $b_1 + 2b_2 + b_3 + b_4 = 0$ and $3b_1 + 2b_2 + 5b_4 = 0$. The latter identity will determine which balanced arrays arise from the actual polarizations (6).

Lemma 2.1. *The split trace-free cohomology dimensions in degrees zero through three are* $(6, 119, 5, 0)$. *The full degree-one space* V *has the following blocks:*

ij	12	13	21	31	23	32	24	34	41
$\dim H^1(F_iF_j^{-1})$	8	8	8	8	8	8	4	4	27
$\dim V_{ij}$	16	8	16	8	16	16	8	4	27.

The only degree-two block is 14, *naturally identified with* $S_4 = H^0(\mathbb{P}^1_3, \mathcal{O}(4))$ *after choosing the two negative-factor harmonic generators. The parallel trace-free algebra is*

$$\mathfrak{g} = \{(z_1, M, z_3, z_4) : z_1 + \operatorname{tr} M + z_3 + z_4 = 0\}, \qquad |\xi|^2 = |z_1|^2 + \operatorname{tr}(M^*M) + |z_3|^2 + |z_4|^2. \tag{9}$$

Here M *is an arbitrary complex two-by-two matrix.*

Proof. The six line blocks $12, 13, 21, 31, 23, 32$ have degrees obtained by permuting $(-3, 0, 3)$, with one negative entry. Projective-line cohomology and the Künneth formula give dimension $2 \cdot 1 \cdot 4 = 8$ in degree one. Blocks $24, 34$ have degrees $(1, -2, 1), (-2, 1, 1)$ and dimension four. Block 41 has degrees $(2, 2, -4)$ and dimension $3 \cdot 3 \cdot 3 = 27$. Block 14 has degrees $(-2, -2, 4)$ and degree-two dimension five. Blocks $42, 43$ have a degree -1 factor and vanish. There are no off-diagonal degree-zero or degree-three groups. Tensoring with the multiplicity Hom spaces gives the table. The diagonal blocks contribute $1 + 4 + 1 + 1$ constants and no higher cohomology. Removing common scalars gives six constants and (9). These are the standard projective-space and product computations of [Stacks, Tags 01XT and 0BED]. □

The effective constant groups are

$$G = (\mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1)/\mathbb{C}^*, \qquad K = (\mathrm{U}_1 \times \mathrm{U}_2 \times \mathrm{U}_1 \times \mathrm{U}_1)/S^1.$$

They act by $(gx)_{ij} = g_i x_{ij} g_j^{-1}$. Determinant-one lifts have an ineffective fifth-root center. Let $\mathfrak{g}_{\mathbb{R}}$ denote the Hermitian part of (9), and write $C_x\xi = [\xi, x]$ for the infinitesimal complex action.

The split harmonic spaces, as spaces of differential forms, are fixed as p varies. Indeed the product Laplacian is a sum of commuting nonnegative factor Laplacians with positive coefficients $1 + p_1, 1 + p_2, 1$. Its kernel is their common kernel. If $\iota : V \to \Omega^{0,1}(\operatorname{End}_0 E_0)$ is inclusion, then

$$\begin{aligned} q_p(x) := \|\iota x\|_p^2 &= (1 + p_1)(\|x_{12}\|_0^2 + \|x_{32}\|_0^2 + \|x_{34}\|_0^2) \\ &\quad + (1 + p_2)(\|x_{13}\|_0^2 + \|x_{23}\|_0^2 + \|x_{24}\|_0^2) + \|x_{21}\|_0^2 + \|x_{31}\|_0^2 + \|x_{41}\|_0^2. \end{aligned} \tag{10}$$

The three groups have their $(0, 1)$ component on the first, second and third factors, respectively. Put $|x|^2 = q_0(x)$.

Denote harmonic projections and Green operators by $P_{j,p}, G_{j,p}$, and the Hodge homotopy by $K_p = d_p^* G_p$. They have uniform bounds between fixed Sobolev spaces and depend smoothly on p. To see this, use the fixed kernels and the positive complementary gap at $p = 0$; uniform ellipticity preserves the gap and gives uniform elliptic inverse estimates. Differentiating the inverse identity gives the parameter derivatives. The zero-form projection P_0 is independent of p. The degree-one projection is also independent of p as an operator: in each Hom block only the indicated differential component pairs with a harmonic form, and both its pairing and the harmonic Gram matrix are multiplied by the same constant in (10).

2.2. **Convergent reconstruction and all isomorphisms.**

Theorem 2.2. *On a fixed product neighborhood $|p| < p_0$, $|x| < r_0$, there are maps*

$$a_p(x) = \iota x - K_p(a_p(x)^2), \qquad \kappa_p(x) = P_{2,p}(a_p(x)^2), \tag{11}$$

holomorphic in x and smooth in p. Their values are spatially smooth, and

$$\|a_p(x) - \iota x\|_{H^s} \le C|x|^2, \qquad \|D_x a_p(x)v - \iota v\|_{H^s} \le C|x||v|, \tag{12}$$
$$d_p^* a_p = 0, \qquad P_1 a_p = x.$$

The complete equation $\kappa_p(x) = 0$ is equivalent to integrability. Every sufficiently small integrable perturbation is complex-gauge equivalent to such a member, with uniform local gauge bounds.

Every holomorphic bundle map between two sufficiently small integrable members at the same p is a constant block matrix g_0 satisfying

$$y_{ij} g_{0,j} = g_{0,i} x_{ij} \quad \text{in every harmonic coordinate.} \tag{13}$$

Conversely every such matrix is a holomorphic map. Thus isomorphisms are exactly the G orbit relation with both coordinates in the ball. No bound on the isomorphism or its inverse is required.

Proof. Sobolev multiplication is bounded for $s \ge 6$ on this real sixfold. The map $a \mapsto \iota x - K_p(a^2)$ is a contraction on $\|a\|_{H^s} \le 2\|\iota\||x|$ for one sufficiently small radius. Its homogeneous series is

$$a_1 = \iota x, \qquad a_n = -K_p \sum_{j=1}^{n-1} a_j \wedge a_{n-j} \quad (n \ge 2).$$

A Catalan majorant proves convergence uniformly in p. All matrix products have their indicated order. Since $\operatorname{tr}(a \wedge a) = 0$, the reconstruction is trace-free. The Hodge identities give $K_p a = 0$, $P_1 a = x$, $d_p^* a = 0$. Differentiating the equation gives

$$J_{p,x} v = \iota v - K_p(J_{p,x} v \wedge a_p + a_p \wedge J_{p,x} v), \qquad J_{p,x} = D_x a_p. \tag{14}$$

The linear inverse has norm at most two after decreasing the radius. This proves the derivative bound, and repeated differentiation proves the fixed-order parameter estimates. The order-minus-one homotopy and elliptic bootstrap give spatial smoothness.

For $c = da + a^2$, the Hodge and Bianchi identities give

$$c = \iota_2 \kappa_p(x) + K_p(c \wedge a - a \wedge c). \tag{15}$$

The last operator is small in H^{s-1}. Hence $\kappa_p = 0$ implies $c = 0$, and the converse follows by harmonic projection.

For a small perturbation b, solve $d_p^*(e^v \cdot (d+b) - d) = 0$ with $v \in QH^{s+1}$, where Q is the complement of all parallel complex zero-forms. The derivative in v is $-d_p^* d : QH^{s+1} \to QH^{s-1}$, a uniformly invertible elliptic operator. The analytic implicit-function theorem gives a local gauge with uniform bounds and first derivatives. For an integrable Coulomb representative b', Hodge decomposition gives $b' = \iota P_1 b' - K_p((b')^2)$, so contraction uniqueness identifies it with $a_p(P_1 b')$.

Every constant block matrix commutes with the split differential, projections and homotopy. Termwise conjugation therefore proves equivariance of each a_n. For a fixed g, summing the two convergent series proves $a_p(gx) = g a_p(x) g^{-1}$ whenever both coordinates are in the ball. This argument does not require the entire g-orbit of the ball to stay in the ball.

Finally decompose a holomorphic map as $g = g_0+g_1$, where g_0 is the full parallel projection, including scalars. Its equation is

$$dg_1 + a_p(y)g_1 - g_1a_p(x) + a_p(y)g_0 - g_0a_p(x) = 0.$$

Applying d_p^* kills the last two terms. Pairing with g_1 and using the fixed Poincaré inequality gives

$$\|dg_1\|_p^2 \le C(\|a_p(x)\|_\infty + \|a_p(y)\|_\infty)\|dg_1\|_p\|g_1\|_p, \qquad \|g_1\|_p \le C\|dg_1\|_p.$$

Smallness forces $g_1 = 0$, independently of $\|g_0\|$. Harmonic projection gives (13). Conversely that equation intertwines every term of the ordered reconstruction series, proving the full morphism assertion. This is the co-closed-slice mechanism of [BuS20, Proposition 4.5 and Corollary 4.6], with the full multiplicity algebra retained. □

The trace-free holomorphic endomorphisms are therefore $\ker C_x \subset \mathfrak{g}$. At a nonpolystable point this complex algebra need not be closed under the reference Hermitian adjoint. Section 3 gives an explicit instance.

2.3. **The adjacent-layer equations.** Put $S_j = H^0(\mathbb{P}^1_3, \mathcal{O}(j))$. On each of the first two factors choose $e_0, e_1 \in H^0(\mathcal{O}(1))$, a nonzero $\theta \in H^1(\mathcal{O}(-2))$, and harmonic $\eta^0, \eta^1 \in H^1(\mathcal{O}(-3))$ with $P(\eta^j e_m) = \delta_{jm}\theta$. Serre duality supplies such bases. The 36-dimensional adjacent-layer subspace has entries

$$\begin{aligned}
x_{12} &= \eta_1^0 A_0 + \eta_1^1 A_1, && A_j \in S_3 \otimes M_{1,2},\\
x_{13} &= \eta_2^0 C_0 + \eta_2^1 C_1, && C_j \in S_3,\\
x_{24} &= (e_{0,1}B_0 + e_{1,1}B_1)\theta_2, && B_j \in S_1 \otimes M_{2,1},\\
x_{34} &= \theta_1(e_{0,2}D_0 + e_{1,2}D_1), && D_j \in S_1.
\end{aligned} \tag{16}$$

Only block 14 occurs in the square. Its further products with these four blocks vanish on both sides. Consequently the restriction of the complete model is exactly

$$a_p(x) = \iota x - K_p((\iota x)^2), \tag{17}$$

$$\kappa_p(x) = (\theta_1 \wedge \theta_2)(A_0B_0 + A_1B_1 - C_0D_0 - C_1D_1). \tag{18}$$

The minus sign orders the differential components by projective-line factor. The products A_jB_j are ordered row–column products, hence scalar quartics. Their five coefficients are the complete equations on this subspace. The full 119-coordinate recursion can have higher terms.

3. A composable family at the split metric

We begin with a family whose weak eigenvalue can be computed explicitly. Its bundles are semistable at the equal-slope polarization and have no Hermitian–Einstein metric there. The operator in this section uses the fixed split metric h_0. Section 5 constructs a different, exactly balanced family and proves the corresponding assertion for its actual HE metrics.

3.1. **Two nonzero compositions and their correction.** On the first two projective lines choose identical nonzero data $\tau \in H^0(\mathcal{O}(1))$, $\theta \in H^1(\mathcal{O}(-2))$, and a harmonic $\eta \in H^1(\mathcal{O}(-3))$ with $P(\tau\eta) = \theta$. Multiplication by τ on this degree-one cohomology is surjective, dual to the injective map $H^0(\mathcal{O}) \to H^0(\mathcal{O}(1))$. Put $\rho = \tau\eta$. On the third factor use homogeneous coordinates $(s : z)$ and set $P = s^3$, $Q = z^3$, $\tau_3 = s$. The cubics are orthogonal and have equal norm. Put $c = (1, 1)^T$.

For $0 \le \varepsilon \le \varepsilon_0 < 1/2$, let A_ε have precisely the four harmonic blocks

$$\begin{aligned} &(A_\varepsilon)_{12} = \eta_1(Pe_1^T + \varepsilon Q e_2^T), && (A_\varepsilon)_{13} = \eta_2(P + \varepsilon Q), \\ &(A_\varepsilon)_{24} = \tau_1\theta_2\tau_3 c, && (A_\varepsilon)_{34} = \theta_1\tau_2\tau_3. \end{aligned} \tag{19}$$

Suppressed tensor factors are constant sections. Its square is the block-14 two-form

$$W_\varepsilon = (\rho_1 \wedge \theta_2 - \theta_1 \wedge \rho_2)\tau_3(P + \varepsilon Q). \tag{20}$$

It is closed and has zero harmonic projection. With the reference homotopy K_0, set

$$\Gamma_\varepsilon = -K_0 W_\varepsilon, \qquad N_{t,\varepsilon} = tA_\varepsilon + t^2\Gamma_\varepsilon E_{14}, \qquad 0 < t < t_0. \tag{21}$$

The notation E_{14} here records the block of the scalar form.

Proposition 3.1. *The operators $d + N_{t,\varepsilon}$ are integrable and co-closed and are the complete reconstructions of $x = tA_\varepsilon$. Both ordered composition classes are nonzero and opposite, and $\Gamma_\varepsilon \neq 0$. Every fixed spatial norm of N is $O(t)$, uniformly in the stated parameter rectangle.*

Proof. Product Hodge theory gives $P_2 W_\varepsilon = 0$ from $P\rho = \theta$. The Hodge identity gives $d\Gamma_\varepsilon = -W_\varepsilon$, $d^*\Gamma_\varepsilon = 0$, and $P_1\Gamma_\varepsilon = 0$. Every product of block 14 with the active blocks vanishes. Thus $dN + N^2 = 0$ and $N = \iota x - K_0(N^2)$ exactly.

The form θ is nowhere zero as a smooth section of $\Lambda^{0,1} \otimes \mathcal{O}(-2)$: its Serre-dual section is a nonzero constant. The form ρ vanishes at the zero of τ and has nonzero harmonic projection. Hence ρ, θ are linearly independent, so $W_\varepsilon \neq 0$ and its primitive is nonzero. Each path has class, up to sign, $\theta_1 \wedge \theta_2\tau_3(P + \varepsilon Q) \neq 0$. The forms depend affinely on ε, and the fixed Green operator has bounded action in every fixed smooth order; this proves the norm assertion. □

Replacing only the scalar route by $P + \varepsilon Q + \delta R$, with nonzero $R \in S_3$, leaves harmonic curvature $-t^2\delta\theta_1 \wedge \theta_2\tau_3 R$. No choice of a central block-14 primitive can remove this class with the four harmonic arrows fixed.

3.2. **Noncommutation, stabilizers and bundle type.** Up to positive norm constants, the multiplicity contractions in the row and column data are

$$U_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad [U_\varepsilon, V] = (1 - \varepsilon^2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{22}$$

They cannot be simultaneously diagonalized by a unitary multiplicity basis. In particular the scalar vertex graph omits relevant directions.

Proposition 3.2. *For $t > 0$ and $\varepsilon > 0$, the full holomorphic endomorphism algebra is scalar. At $\varepsilon = 0$ it is*

$$z_1 = z_3 = z_4 = s_0, \qquad M = \begin{pmatrix} s_0 & 0 \\ s_0 - n & n \end{pmatrix}, \qquad s_0, n \in \mathbb{C}. \tag{23}$$

The bundle then splits as a rank-four three-step extension and one copy of F_2, with effective automorphism group $\mathbb{C}^*$. The trace-free kernel is generated by

$$\Xi = (1, M_0, 1, 1), \qquad M_0 = \begin{pmatrix} 1 & 0 \\ 5 & -4 \end{pmatrix}, \qquad |\Xi|^2 = 45. \tag{24}$$

All these bundles, for $t > 0$, are semistable but not polystable at ω_0. They have no HE metric at that polarization.

Proof. By Theorem 2.2, it suffices to compute constant intertwiners. At positive ε, the two independent cubic coefficients in block 12 force both rows of M to equal those of $z_1 I_2$. The other scalar arrows force $z_3 = z_4 = z_1$. At zero, only the first row is fixed; the column equation $Mc = s_0 c$ gives (23). Its trace is $4s_0 + n$, giving (24). In the multiplicity basis (c, e_2) the centralizer is diagonal. The column uses the first basis vector, and the row annihilates the second, which splits off the stated F_2. The active rank-four part retains both paths and their correction.

The three-step filtration has slope-zero polystable quotients and refines to a filtration by the four line types. Hence it is semistable and has graded object E_0. If it were polystable, it would be isomorphic to that graded object. Its endomorphism algebra would then have dimension seven, whereas the computed dimensions are one and two. Thus it is not polystable. The HE-to-polystable implication [Kob87, Theorem 5.8.3] rules out an HE metric. □

For fixed ε, changing $t > 0$ stays on one holomorphic orbit: the determinant-one gauge of weights $(1, 0, 0, -1)$ on the four types scales every adjacent block by t and the central one by t^2. Different ε give different bundles over the fixed base, since the ratio of the P, Q coefficients in scalar block 13 is invariant. These are distinct holomorphic isomorphism classes with the same polystable graded object. The composable route $4 \to 2 \to 1$ also shows that this is not a two-layer presentation with each of these fixed stable types assigned to a single layer.

3.3. **The weak eigenvalue.** Let the fixed squared harmonic norms be

$$a = \|\eta_1 P\|_0^2 = \|\eta_1 Q\|_0^2 = \|\eta_2 P\|_0^2 = \|\eta_2 Q\|_0^2, \qquad b = \|\tau_1\theta_2\tau_3\|_0^2 = \|\theta_1\tau_2\tau_3\|_0^2.$$

Orthogonality of the cubics and blocks makes the full finite form $t^2 Q_\varepsilon$, where, on (9),

$$\begin{aligned} Q_\varepsilon(\xi) = a\big(&\|z_1 e_1^T - e_1^T M\|^2 + \varepsilon^2\|z_1 e_2^T - e_2^T M\|^2 + (1+\varepsilon^2)|z_1 - z_3|^2\big) \\ &+ b\big(\|Mc - cz_4\|^2 + |z_3 - z_4|^2\big). \end{aligned} \tag{25}$$

Let $\mu_1 \le \cdots \le \mu_6$ be its eigenvalues with the complex inner product (9), counting zeros.

Lemma 3.3. *For a sufficiently small fixed ε_0,*

$$\mu_1(\varepsilon) = \tfrac{10a}{9}\varepsilon^2 + O(\varepsilon^4), \qquad c \le \mu_j(\varepsilon) \le C \quad (2 \le j \le 6).$$

Proof. At zero the kernel is $\mathbb{C}\Xi$ and its complement has a positive gap. The form depends affinely on $e = \varepsilon^2$. On the unit kernel vector the coefficient of e is

$$\frac{a}{45}\|e_2^T - e_2^T M_0\|^2 = \frac{a}{45}\|(-5, 5)\|^2 = \frac{10a}{9}.$$

With respect to $\mathbb{C}\Xi \oplus \Xi^\perp$, the operator is

$$\begin{pmatrix} e(10a/9) & eB^* \\ eB & L_\perp + eC \end{pmatrix}.$$

Min–max isolates one small eigenvalue while keeping the other five in a fixed positive interval. The scalar Schur equation for that eigenvalue λ is $\lambda = e(10a/9) - e^2 B^*(L_\perp + eC - \lambda)^{-1} B$. The inverse is uniformly bounded, proving the expansion. □

Corollary 3.4. *Let $\lambda_j(t, \varepsilon)$ be the eigenvalues, with complex multiplicity and counting zeros, of the trace-free endomorphism zero-form Dolbeault Laplacian for $d + N_{t,\varepsilon}$, h_0, ω_0. The*

operator is $\bar\partial^*\bar\partial$ without a factor two. Uniformly for independent $0<t<t_0$, $0<\varepsilon\le\varepsilon_0$,

$$\lambda_1=\tfrac{10a}{9}t^2\varepsilon^2(1+O(\varepsilon^2+t^2)),\tag{26}$$
$$ct^2\le\lambda_j\le Ct^2\quad(2\le j\le 6),\qquad \lambda_7\ge\gamma>0,$$
$$\left|\log\frac{\lambda_j}{t^2\mu_j(\varepsilon)}\right|\le Ct^2\quad(1\le j\le 6).$$

At $\varepsilon=0$ there is exactly one trace-free zero mode; the five positive-mode estimates and the seventh gap persist.

Proof. Proposition 3.1 gives an integrable, co-closed reconstruction with $\|N\|_\infty\le Ct$ and $q_0(x)=t^2\{2a(1+\varepsilon^2)+3b\}$. The relative spectral lemma proved in Section 5.1 compares all six eigenvalues to the full finite form with relative error $O(t^2)$, including exact kernels and a seventh gap. Combine it with Lemma 3.3. No inverse of μ_1 is used, so there is no condition on the ratio of t and ε. □

4. Multiplicity projections and the HE correction

Quadratic balance determines a useful reference metric, but does not solve the nonlinear HE equation. The correction must be controlled even when the finite matrix action has very small positive eigenvalues. The required estimate pairs the curvature error with every orthogonal projection in every multiplicity space. The proof uses harmonic orthogonality to eliminate cubic terms; it applies to the complete deformation space, which contains reverse arrows and higher products.

4.1. **The ambient complementary equation.** Let P_0 project the trace-free Hermitian zero-forms onto $\mathfrak{g}_{\mathbb{R}}$, and put $Q_0=1-P_0$. For $u\in Q_0H^{s+1}$, form the connection compatible with $d+a_p(x)$ and h_0e^u, and transport it to h_0 by $e^{u/2}$. Denote that unitary connection by $\Phi(p,x,u)$ and its centered contracted curvature by

$$\mathcal{M}(p,x,u)=i\Lambda_pF_{\Phi(p,x,u)}-\kappa_E(p)I.\tag{27}$$

The connection is defined off the integrable locus as well. Its fixed determinant has constant mean curvature, so $\mathcal{M}$ is pointwise trace-free.

Lemma 4.1. *On one uniform product neighborhood there is a unique small solution $u(p,x)\in Q_0H^{s+1}$ of*

$$Q_0\mathcal{M}(p,x,u(p,x))=0,\qquad \|u(p,x)\|_{H^{s+1}}\le C|x|^2,\qquad \|D_xu(p,x)v\|_{H^{s+1}}\le C|x||v|.\tag{28}$$

It is smooth in p,x, equivariant under K, and has uniform fixed-order spatial and parameter estimates on smaller neighborhoods.

Proof. At $x=u=0$, $\mathcal{M}=-b(p)$ is parallel. The derivative in the real Hermitian metric variable, including the unitary transport, is

$$L_p=\tfrac12(\Delta_{\bar\partial,p}+\Delta_{\partial,p}),\qquad \langle u,L_pu\rangle_p=\tfrac12(\|\bar\partial u\|_p^2+\|\partial u\|_p^2).\tag{29}$$

Indeed the curvature derivative of h_0e^u is the contracted $\bar\partial\partial$ term, and differentiating the transport adds half its curvature commutator; the Kähler identities give the symmetric average in (29). The kernel on Hermitian fields is the full parallel algebra. The uniform split gap and ellipticity give an inverse $Q_0H^{s-1}\to Q_0H^{s+1}$. The parameter implicit-function theorem therefore gives a smooth solution with $u(p,0)=0$.

The linear curvature variation in x vanishes by harmonicity and the Kähler identities. Hence $D_xu(p,0)=0$, proving the estimates by Taylor's formula. Elliptic regularity and

differentiated equations give the higher bounds. Uniqueness proves K-equivariance. This is the complementary reduction of [CT24, Section 3] and [Del25, Lemma 4.1 and Proposition 4.2], on a fixed Sobolev complement containing the entire matrix kernel. □

Set $H_x = h_0 e^{u(p,x)}$, $\Phi_p(x) = \Phi(p, x, u(p,x))$ and $m_p(x) = P_0\mathcal{M}(p, x, u(p,x))$. At an integrable point, $m_p(x) = 0$ is exactly the remaining HE equation. Define the quadratic moment map by the real trace pairing

$$\langle \xi, \mu_p(x)\rangle = \langle [\xi, \iota x], \iota x\rangle_p = \left.\frac{d}{dt}\right|_{t=0} q_p(e^{t\xi/2}\cdot x), \qquad \xi \in \mathfrak{g}_{\mathbb{R}}. \tag{30}$$

It is incoming xx^* minus outgoing x^*x, with the current harmonic Gram forms. Both m_p and μ_p are in the full six-dimensional Hermitian algebra. Write

$$m_p(x) = -b(p) + \mu_p(x) + R_p(x). \tag{31}$$

4.2. **Quartic order and every projection.**

Lemma 4.2. *On the entire ambient ball,*

$$|R_p(x)| \le C|x|^4, \qquad \|D_x R_p(x)\| \le C|x|^3, \qquad \|D_x^2 R_p(x)\| \le C|x|^2.$$

The constants are uniform in p.

Proof. Let $\alpha = \Phi_p(x)^{0,1} - d$. The transformation rule gives

$$\alpha = e^{u/2} a_p e^{-u/2} - (de^{u/2})e^{-u/2} = \iota x + B + O(|x|^3), \qquad B = a_p - \iota x - \tfrac{1}{2}du.$$

Here $B = O(|x|^2)$ and is orthogonal to every current harmonic one-form: its first summand is coexact and its second exact. The linear curvature term, integrated against a parallel Hermitian matrix, vanishes by integration by parts. The exact remaining identity is

$$\langle \xi, m_p(x)\rangle = -\langle \xi, b(p)\rangle + \langle [\xi, \alpha], \alpha\rangle_p. \tag{32}$$

Parallel left and right multiplication preserve the harmonic subspace. Consequently both cubic cross terms between ιx and B vanish. The remaining difference from (30) is $O(|x|^4)$. The argument holds for every ambient x, so all Taylor coefficients of R_p of degree below four vanish. Its smoothness and uniform fourth-derivative bound give the two derivative estimates. □

An orthogonal multiplicity projection means a tuple $\Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4)$ with $\Pi_i^2 = \Pi_i = \Pi_i^*$ on $\mathbb{C}^{m_i}$. It need not be diagonal in the fixed basis of $\mathbb{C}^2$. Since R_p has total trace zero, its trace pairing with Π is defined without imposing a trace condition on Π.

Proposition 4.3. *For every such projection and every sufficiently small ambient array,*

$$|\langle \Pi, R_p(x)\rangle| \le C q_p(x) q_p([\Pi, x]). \tag{33}$$

This includes nonintegrable arrays and all ranks and directions of the multiplicity projections.

Proof. Decompose $x = A + C$ into the preserving and crossing blocks of Π:

$$A_{ij} = \Pi_i x_{ij} \Pi_j + (1 - \Pi_i) x_{ij} (1 - \Pi_j), \qquad C = x - A.$$

These parts are orthogonal for every harmonic Gram form. Thus $q_p(A + sC) \le q_p(x)$ for $0 \le s \le 1$ and $q_p(C) = q_p([\Pi, x])$.

At $C = 0$, the circle generated by Π fixes A. Equivariance of reconstruction and uniqueness in Lemma 4.1 make the compatible connection preserve $E_\Pi = \bigoplus_i F_i \otimes \operatorname{im}\Pi_i$ and its complement. The integrated trace of its curvature on E_Π is topological:

$$\operatorname{Vol}_p^{-1} \int_X \operatorname{tr}\big(\Pi i \Lambda_p F\big) dV_p = \sum_i \operatorname{rk}(\Pi_i)\kappa_i.$$

This Chern–Weil identity applies to any compatible connection on the preserved smooth bundle, including a nonintegrable one. Also $[\Pi, A] = 0$, so $\langle \Pi, \mu_p(A)\rangle = 0$. It follows that $\langle \Pi, R_p(A)\rangle = 0$.

The unitary reflection $2\Pi - I$ fixes A and sends C to $-C$. Compact equivariance makes the scalar function $C \mapsto \langle \Pi, R_p(A+C)\rangle$ even, so its first derivative at zero vanishes. Taylor's integral formula and Lemma 4.2 now give

$$|\langle \Pi, R_p(A + C)\rangle| \le C q_p(x) \|C\|_p^2.$$

All projections have norm at most $\sqrt{5}$ and range over a finite union of compact Grassmannians. The same constant therefore works for all of them, proving (33). □

The projection method and the finite degree argument below come from the two-step repeated-factor analysis in [Wue26c]. The new analytic point here is the quartic cancellation on the complete composable model: a two-step sign symmetry is unavailable in the presence of its reverse arrows. The preserving arrays in the proof were not required to satisfy the nonlinear relations.

4.3. **The finite correction without a spectral lower bound.** We give the finite argument in the normalization needed here. Fix a matrix representation with total multiplicity $M \ge 2$ and positive Hermitian arrow Gram forms, allowing mixing only among coordinates with the same ordered endpoints. Let q be its squared norm and μ its moment map as in (30). The Hermitian domain and target both have total trace zero.

Lemma 4.4. *Suppose $\mu(Z) = b$, the complex centralizer of Z is the common scalars, and $\sigma = q(Z) > 0$. Let R be continuous at all $e^{Y/2}\cdot Z$ with $Y = Y^*$, $\operatorname{tr} Y = 0$, $\|Y\|_{\mathrm{op}} \le a_0$. Assume there that $|\langle \Pi, R(W)\rangle| \le C q(W) q([\Pi, W])$ for every multiplicity projection. For sufficiently small σ there is Y with $\|Y\|_{\mathrm{op}} \le A\sigma$ such that*

$$\mu(e^{Y/2} \cdot Z) - b + R(e^{Y/2} \cdot Z) = 0.$$

The constants depend on C, M, a_0 and the fixed norm conventions, uniformly over a relatively compact set of positive arrow Gram forms.

Proof. Consider $H_s(Y) = \mu(e^{Y/2}\cdot Z) - b + sR(e^{Y/2}\cdot Z)$, $0 \le s \le 1$, on $\|Y\|_{\mathrm{op}} < a = A\sigma$. Take $a < \min(1, a_0)$. On its boundary diagonalize every block of Y and sort all M eigenvalues, keeping their vertex labels. Trace zero implies an oscillation at least a, so an adjacent gap δ is at least $a/(M-1)$. Let Π project onto the eigenvectors above that gap.

Whiten arrow coordinates for each ordered pair of vertices. This commutes with the vertex action. Let w_{ij} be the sum of squared entries from source eigenvector j to target eigenvector i. The positive gauge multiplies these weights by $e^{y_i - y_j}$. Using balance at Z gives

$$\begin{aligned} \langle \Pi, \mu(e^{Y/2}\cdot Z) - b\rangle &= \sum_{i,j} (\Pi_i - \Pi_j)(e^{y_i - y_j} - 1) w_{ij} \\ &\ge (1 - e^{-\delta}) q([\Pi, Z]) \ge e^{-2a} \frac{a}{M-1} q([\Pi, Z]). \end{aligned} \tag{34}$$

Both crossing orientations contribute positively. The crossing energy is positive because Π is nonscalar and the centralizer is scalar. Writing $W = e^{Y/2} \cdot Z$, the gauge bounds give $q(W) \le e^{2a}\sigma$ and $q([\Pi, W]) \le e^{2a} q([\Pi, Z])$. Hence the residual pairing is at most $Ce^{4a}\sigma q([\Pi, Z])$. Choose $A > 2C(M-1)$ and then σ small enough that $e^{6A\sigma} < 2$. The positive term in (34) strictly dominates the error. No H_s has a boundary zero. The projection is chosen pointwise; its continuity is unnecessary.

To compute the degree at $s = 0$, fix nonzero Y and set $f(t) = q(e^{tY/2} \cdot Z) - t\langle b, Y\rangle$. Then $f'(0) = 0$ and

$$f''(t) = \sum_{i,j} (y_i - y_j)^2 e^{t(y_i - y_j)} w_{ij} > 0.$$

Equality would make Y a scalar centralizer element, and trace zero would force $Y = 0$. Thus $\langle Y, H_0(Y)\rangle = f'(1) > 0$. The radial homotopy from H_0 to the identity has no boundary zero, so its Brouwer degree is $+1$. Homotopy invariance through H_s gives a zero of H_1 inside the ball. □

Theorem 4.5. *There are uniform $p_0, r, C > 0$ such that the following holds. Suppose $|p| < p_0$, $\kappa_p(x) = 0$, $\ker C_x = 0$ in the trace-free complex algebra, $\mu_p(x) = b(p)$, and $0 < q_p(x) < r^2$. Then $d + a_p(x)$ has a fixed-determinant HE metric H satisfying*

$$e^{-Cq_p(x)} h_0 \le H \le e^{Cq_p(x)} h_0. \tag{35}$$

The bundle is stable. The constant is independent of all positive singular values of C_x and their ratios.

Proof. Choose a fixed positive-gauge radius a_0. Decrease r so that every $e^{Y/2}x$ in that gauge ball remains in the ambient domain of Proposition 4.3. Its bound verifies every hypothesis of Lemma 4.4, with $M = 5$. We obtain $\|Y\|_{\mathrm{op}} \le Cq_p(x)$ and $x' = e^{Y/2} \cdot x$ with $m_p(x') = 0$. Exact equivariance preserves integrability. The complementary and finite curvature equations now both vanish, so $h_0 e^{u(p,x')}$ is HE on $d + a_p(x')$.

Pull back by the constant holomorphic isomorphism $g = e^{Y/2}$. In an h_0-unitary frame the resulting metric has matrix $g^* e^{u(p,x')} g$. Since $q_p(x') \le e^{2\|Y\|_{\mathrm{op}}} q_p(x)$, Lemma 4.1 and Hermitian-form order give (35). This uses no commutation between u and Y. Both are trace-free, so the determinant metric is preserved. An HE bundle is polystable; its scalar endomorphism algebra permits only one stable summand, proving stability. □

The theorem supplies a pointwise small correction. It neither asserts a uniform derivative for a chosen balancing gauge nor characterizes every stable orbit by a small quadratically balanced representative. The full local HE quotient is constructed in Section 6 from the corrected equation $m_p = 0$ itself. Reverse arrows allow cancellation in μ_p, so small $b(p)$ alone does not control $q_p(x)$.

5. Actual HE metrics and all six small modes

The full matrix algebra gives six small complex eigenvalues. We first compare them to the finite commutator form, retaining exact zeros. We then construct a balanced family for which one positive eigenvalue vanishes at a nonsimple endpoint. The endpoint metric is obtained by a separate correction on its active summand.

5.1. **Relative spectral reduction.** For a complete integrable reconstruction $N = a_p(x)$ define

$$\mathcal{Q}_{p,x}(\xi) = \|[\xi, \iota x]\|_p^2, \qquad 0 \le \nu_1(p,x) \le \cdots \le \nu_6(p,x), \quad \xi \in \mathfrak{g}, \tag{36}$$

using the full complex norm (9).

Lemma 5.1. *There are uniform $r, C, \gamma > 0$ such that, when $\kappa_p(x) = 0$ and $q = q_p(x) < r^2$, the zero-form Dolbeault eigenvalues on $\operatorname{End}_0 E_0$ for $d + N$ with metric h_0 satisfy*

$$\lambda_j = 0 \iff \nu_j = 0, \qquad \left|\log \frac{\lambda_j}{\nu_j}\right| \le Cq \quad (\nu_j > 0,\ 1 \le j \le 6), \qquad \lambda_7 \ge \gamma. \tag{37}$$

The same statement holds for any bundle metric H with $e^{-C_0 q} h_0 \le H \le e^{C_0 q} h_0$, with constants also depending on C_0. The split slopes need not be equal.

Proof. Parallel complex endomorphisms commute with the split Hodge operators. Commuting (11) with ξ gives

$$\|[N - \iota x, \xi]\|_p \le C\|N\|_\infty \|[N, \xi]\|_p \le C\sqrt{q}\, \|[N, \xi]\|_p.$$

The harmonic part and the coexact remainder are orthogonal. Thus, with $q_N(\xi) = \|[N, \xi]\|_p^2$,

$$\mathcal{Q}_{p,x} \le q_N \le (1 - Cq)^{-1} \mathcal{Q}_{p,x}, \qquad \ker q_N = \ker \mathcal{Q}_{p,x}. \tag{38}$$

Let $D_N = d + [N, \cdot]$ and let η be the L^2 operator norm of multiplication by $[N, \cdot]$; then $\eta \le C\sqrt{q}$. If γ_0 is the uniform split zero-form gap, put $\gamma_N = (\sqrt{\gamma_0} - \eta)^2$. On the complement $\mathfrak{g}^\perp \subset L^2(\operatorname{End}_0 E_0)$, $\|D_N v\|_p^2 \ge \gamma_N \|v\|_p^2$. Co-closedness implies $\langle [N, \xi], dv \rangle_p = 0$ for every parallel ξ. Expanding the square and applying Young's inequality therefore yields

$$\|D_N(\xi + v)\|_p^2 \ge (1 - 2\eta^2/\gamma_N) q_N(\xi) + \tfrac{1}{2}\gamma_N \|v\|_p^2. \tag{39}$$

For example, bound the cross term by $2\eta\sqrt{q_N(\xi)}\|v\|_p$ and absorb half the complementary lower bound. The trial space $\mathfrak{g}$ gives the upper bounds by the eigenvalues of q_N. The block diagonal lower form in (39) gives the matching bounds for all six eigenvalues, after shrinking q so their upper bounds lie below $\gamma_N/2$. The codimension-six complementary inequality gives the seventh gap. Combining with (38) proves (37). Its ingredients are parallelness, co-closedness and the split gap, all valid at unequal slopes.

Finally the Hermitian-form inequality for H changes both numerator and denominator in each endomorphism Rayleigh quotient by factors $e^{\pm 2C_0 q}$. Min–max changes every positive eigenvalue by at most $e^{\pm 4C_0 q}$ and preserves the gap. The kernel is holomorphic and independent of the metric, giving the zero assertion as well. □

This is the relative Kuranishi spectral argument applied to the full parallel algebra. We included its proof to verify the unequal-slope case and the precise zero convention. For the metrics constructed in Theorem 4.5, the lemma gives all six HE eigenvalues with error $O(q_p(x))$, independently of their positive ratios.

5.2. **An explicitly balanced family.** Use the forms η, θ, τ from Section 3. Rescale the orthogonal coordinate cubics P, Q together and rescale τ_3 separately so that

$$\|\eta_1 P\|_0^2 = \|\eta_1 Q\|_0^2 = \|\eta_2 P\|_0^2 = \|\eta_2 Q\|_0^2 = \|\tau_1\theta_2\tau_3\|_0^2 = \|\theta_1\tau_2\tau_3\|_0^2 = 1. \tag{40}$$

These two rescalings preserve $P(\tau\eta) = \theta$ and the composition identities. For independent $0 < t < t_0$ and $0 \le \varepsilon \le \varepsilon_0 < 1/2$, set

$$e = \varepsilon^2, \qquad z(e) = \frac{4e - 2 + \sqrt{64 + 44e + 16e^2}}{10}, \qquad T = \frac{t^2}{18\pi}, \tag{41}$$

$$A = e + 1 + z, \qquad B = -2e + 7(1 + z), \qquad \beta = (1 - TB)^{-1},$$

$$p_2 = \beta - 1, \qquad p_1 = 5T\beta A, \qquad \alpha = 1 + p_1, \tag{42}$$

$$D = \frac{\beta}{\alpha} z, \qquad L = \sqrt{\frac{\beta}{\alpha}}\,\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+e} & \varepsilon \\ \sqrt{1+e} & -\varepsilon \end{pmatrix}. \tag{43}$$

Here z is a positive scalar function, and A, B are scalar auxiliary functions. The homogeneous third-factor coordinate no longer appears in these formulas. We have

$$z(0) = \tfrac{3}{5}, \qquad 5z^2 + (2 - 4e)z - 3 - 3e = 0. \tag{44}$$

For a fixed small t_0, the denominator $1 - TB$ exceeds $1/2$; α, β, D are positive, $|p| = O(t^2)$, and $\alpha, \beta = 1 + O(t^2)$ uniformly on the whole rectangle.

Let ℓ_P, ℓ_Q be the rows of L and put

$$C_{t,\varepsilon} = \frac{P\ell_P e_1 + Q\ell_Q e_1}{\sqrt{D}} = \sqrt{\frac{1+e}{2z}}(P + Q).$$

Define $x = t\mathcal{A}_{t,\varepsilon}$ by

$$\begin{aligned} &\mathcal{A}_{12} = \eta_1(P\ell_P + Q\ell_Q), && \mathcal{A}_{13} = \eta_2 C_{t,\varepsilon}, \\ &\mathcal{A}_{24} = \tau_1\theta_2\tau_3 e_1, && \mathcal{A}_{34} = \theta_1\tau_2\tau_3\sqrt{D}, \end{aligned} \tag{45}$$

with every other harmonic block zero.

Proposition 5.2. *The tuple (45) satisfies the complete integrability equation and $\mu_p(x) = b(p)$ at its stated polarization. Its reconstruction is*

$$N = t\mathcal{A}_{t,\varepsilon} - t^2 K_p W_{t,\varepsilon} E_{14}, \tag{46}$$

where

$$W_{t,\varepsilon} = (\rho_1 \wedge \theta_2 - \theta_1 \wedge \rho_2)\tau_3\sqrt{\frac{\beta(1+e)}{2\alpha}}(P + Q) \neq 0. \tag{47}$$

Its full centralizer is scalar for $\varepsilon > 0$. At zero it is

$$z_1 = z_3 = z_4 = s_0, \qquad M = \operatorname{diag}(s_0, n). \tag{48}$$

The energy is

$$q_p(x) = t^2\beta\left(2 + 2e + \frac{1+e}{z} + z\right) \asymp t^2. \tag{49}$$

Proof. The defining identity for $C_{t,\varepsilon}$ makes the two ordered path products have opposite nonzero harmonic classes. Their sum is (47), a nonzero exact form by the same ρ, θ argument as before. All further products with block 14 vanish, proving (46) and exact integrability.

Since $L^*L = (\beta/\alpha)\operatorname{diag}(1 + e, e)$, the moment of the unscaled tuple is

$$\mu_p(\mathcal{A}) = \beta\left(1 + 2e + \frac{1+e}{z},\ -eI_2,\ z - \frac{1+e}{z},\ -1 - z\right). \tag{50}$$

In particular the incoming and outgoing multiplicity contractions differ by the scalar $-\beta e I_2$. The weighted total trace is zero, and (44) gives the second relation $3\mu_1 + 2\mu_2 + 5\mu_4 = 0$. Directly from (42), using $B - 7A = -9e$ and $2A + B = 9(1 + z)$,

$$b_2(p) = -t^2\beta e, \qquad b_4(p) = -t^2\beta(1 + z).$$

The two linear relations determine the remaining entries, proving $\mu_p(x) = b(p)$. Summing the four squared block norms gives (49).

At positive ε the two rows of L are independent, and the independent cubic coefficients force $M = z_1 I_2$ in any constant intertwiner. The other arrows force all scalar entries to agree. At zero the rows and the column use only e_1, giving (48). Theorem 2.2 identifies these with the full holomorphic endomorphisms. □

The aggregate moment is scalar at the repeated vertex, but its individual contractions do not commute. For $U_P = \ell_P^* \ell_P$ and $V = e_1 e_1^*$,

$$[U_P, V] = \frac{\beta}{2\alpha} \begin{pmatrix} 0 & -\varepsilon\sqrt{1+e} \\ \varepsilon\sqrt{1+e} & 0 \end{pmatrix} \neq 0 \quad (\varepsilon > 0). \tag{51}$$

This is compatible with exact matrix balance. The family is used here for calibration and spectral degeneration; its interior isomorphism type will be discussed in Section 7.2.

5.3. **The nonsimple endpoint and its weak mode.**

Theorem 5.3. *Every tuple (45) has a fixed-determinant HE metric with*

$$e^{-Ct^2} h_0 \leq H_{t,\varepsilon} \leq e^{Ct^2} h_0. \tag{52}$$

For $\varepsilon > 0$ the bundle is stable. At $\varepsilon = 0$ it is a stable rank-four bundle plus the equal-slope line $F_2 \otimes \mathbb{C}e_2$; the metric on this line is its fixed product metric.

The full finite form is $t^2\beta Q_\varepsilon$, where

$$\begin{aligned} Q_\varepsilon(\xi) = (1+e)\|z_1 e_1^T - e_1^T M\|^2 + e\|z_1 e_2^T - e_2^T M\|^2 + \frac{1+e}{z}|z_1 - z_3|^2 \\ + \|Me_1 - e_1 z_4\|^2 + z|z_3 - z_4|^2. \end{aligned} \tag{53}$$

Let $\theta_1 \leq \cdots \leq \theta_6$ be the ordered eigenvalues of Q_ε on (9). For the actual HE Dolbeault eigenvalues on trace-free endomorphism zero-forms, uniformly for independent positive t, ε,

$$\begin{gathered} \lambda_1(t,\varepsilon) = \tfrac{5}{4} t^2\varepsilon^2(1 + O(\varepsilon^2 + t^2)), \\ ct^2 \leq \lambda_j \leq Ct^2 \quad (2 \leq j \leq 6), \qquad \lambda_7 \geq \gamma > 0, \\ \left|\log \frac{\lambda_j}{t^2\beta\theta_j(\varepsilon)}\right| \leq Ct^2 \quad (1 \leq j \leq 6). \end{gathered} \tag{54}$$

At zero, $\theta_1 = \lambda_1 = 0$ exactly, and the remaining estimates continue to hold.

Proof. At positive ε, Proposition 5.2 verifies Theorem 4.5, giving the metric and stability. At zero, keep the decomposition into the active four-dimensional summand and $F_2 \otimes \mathbb{C}e_2$. Compact equivariance makes the reconstruction and complementary correction preserve it. The unused line projection Π has zero crossing energy, so (33) gives $\langle \Pi, R \rangle = 0$. Its quadratic moment is zero and $b_2 = 0$ at $e = 0$. Off-diagonal multiplicity entries of the reduced curvature vanish by the circle preserving the decomposition. Thus the unused reduced equation vanishes exactly.

Restrict the positive gauge to trace-zero real diagonals on the active rank-four summand, with zero on the unused line. The active representation is connected, has scalar centralizer,

and is quadratically balanced. Its residual inherits (33) by extending every active projection by zero. Lemma 4.4 supplies an active correction of order t^2. This proves the full HE equation and (52) at the nonsimple point separately. The logarithm of the unused line metric is a harmonic function with zero average, since $P_0 u = 0$; it therefore vanishes. The active part is simple and HE and hence stable, with the same mean curvature.

Orthogonality of P, Q gives (53) by summing the row and column commutators. At $e = 0$ its trace-free kernel is generated by

$$\Xi = (1, \operatorname{diag}(1, -4), 1, 1), \qquad |\Xi|^2 = 20. \tag{55}$$

The other five directions have a positive gap. Every term of (53) except the second annihilates Ξ for all e; the second has value $25e$. The form is smooth in e. The same fixed-complement Schur argument as in Lemma 3.3 therefore gives

$$\theta_1(\varepsilon) = \tfrac{25}{20}\varepsilon^2 + O(\varepsilon^4), \qquad c \le \theta_j \le C \quad (2 \le j \le 6).$$

Apply Lemma 5.1 to the constructed HE metrics, including the separate boundary metric and its exact kernel. Finally $\beta = 1 + O(t^2)$ proves (54). □

At fixed positive t, the endpoint polarization is not $p = 0$. Its active rank-four summand and unused line have equal slope there. This differs from the equal-slope split background of Section 3, where the nonsplit bundles have no HE metric. Both spectral statements allow, for example, $\varepsilon = e^{-1/t}$; their leading constants and reference metrics are different.

6. The full HE quotient and its rank-four part

We now use the complete corrected equation, rather than quadratic balance, to describe nearby HE classes. Two matrix-specific issues enter. An isomorphism need not be close to a unitary matrix, even when both metrics are close to the reference metric. Also stabilizers must be treated up to compact conjugacy. The first issue is resolved by two successive polar decompositions; the second by local complex products and compact tubular neighborhoods.

6.1. Coverage and compact-orbit uniqueness.

For small $|p|$ and r put

$$\mathcal{B}_p(r) = \{x : |x| < r,\ \kappa_p(x) = 0,\ m_p(x) = 0\}, \qquad \mathcal{N}_p(r) = \mathcal{B}_p(r)/K. \tag{56}$$

At each such point $H_x = h_0 e^{u(p,x)}$ is HE and $\Phi_p(x)$ is its isometric unitary presentation on h_0.

Proposition 6.1. *Uniform constants $p_0, r_0, c, C > 0$ can be chosen so that, for $|p| < p_0$ and $r \le r_0$, the map Φ_p identifies $\mathcal{N}_p(r)$ homeomorphically with an open subset of the nearby fixed-determinant HE unitary-gauge quotient. Its image contains every class admitting a representative with $\|A - A_0\|_{H^s} < cr$, and every image class has a representative with $\|A - A_0\|_{H^s} < Cr$. Two points of $\mathcal{B}_p(r)$ give isomorphic bundles if and only if they are in the same K orbit.*

Proof. The reconstruction and complementary bounds give $\|\Phi_p(x) - A_0\|_{H^s} \le C|x|$. Conversely, apply the uniform complex Coulomb construction to the Dolbeault part of a nearby HE connection. It gives a raw coefficient x_{raw} and transported metric H_{raw}, both within $C\|A - A_0\|_{H^s}$ of their reference values. A small Hermitian constant Y uniquely solves

$$P_0 \log(e^{-Y}(h_0^{-1} H_{\mathrm{raw}})e^{-Y}) = 0. \tag{57}$$

Its derivative in Y at the reference pair is $-2\,\mathrm{id}$ on all of $\mathfrak{g}_{\mathbb{R}}$. The finite implicit-function theorem is uniform on one fixed neighborhood. The constant gauge e^Y preserves the holomorphic slice and sends the raw metric to a small metric with logarithm in Q_0. Uniqueness

in Lemma 4.1 identifies it with H_x, and the original HE equation gives $\kappa_p(x) = m_p(x) = 0$. This proves the two connection radii.

To prove uniqueness, let g be an isomorphism between balanced points x, y. Theorem 2.2 makes it a constant block matrix; multiply it by a common scalar to arrange determinant one. The Hom connection has zero contracted curvature because the HE constants agree. Thus g is parallel by [Kob87, Proposition 4.1.7]. Take its actual-metric adjoint $g^\dagger$ and set $P = (g^\dagger g)^{1/2}$. This positive endomorphism is parallel, holomorphic and therefore a constant intertwiner fixing x. It need not be h_0-Hermitian. The matrix $q = gP^{-1}$ is a constant holomorphic isometry from H_x to H_y, with determinant one. The metric bounds imply

$$e^{-Cr^2} I \le q^* q \le e^{Cr^2} I.$$

Now take the separate reference polar decomposition $q = ke^Y$, $k \in K$, $Y \in \mathfrak{g}_{\mathbb{R}}$, $|Y| \le Cr^2$. After removing k by compact equivariance, the isometry identity is

$$e^{u'_y} = e^{-Y} e^{u_x} e^{-Y}, \qquad P_0 u_x = P_0 u'_y = 0.$$

For this fixed $u_x = O(r^2)$ the derivative of the left side of (57) as a logarithmic map is $-2\,\mathrm{id} + O(r^2)$ throughout $|Y| \le Cr^2$. Its value at zero is zero; integration along the segment tY shows that it has no other zero there. Hence $Y = 0$ and $y = kx$. Conversely compact equivariance supplies an isometry for every K orbit. No condition number of the original g has been estimated.

The inverse construction is continuous on nearby connection representatives. On a reconstructed $\Phi_p(x)$ its first gauge is exactly $e^{-u/2}$, whose logarithm belongs to the complementary gauge space; it returns $d + a_p(x)$. The normalization then has $Y = 0$ and returns x. Compact-orbit uniqueness makes the inverse well defined on gauge classes. The condition that the returned coefficient lie in the open ball has open preimage; gauge saturation is open. This proves the homeomorphism and openness assertions. □

6.2. **Stabilizers and smooth complex pieces.** On the affine space of h_0-unitary connections use the closed form $\Omega_{\mathrm{aff}}(U, V) = 2\,\mathrm{Im}\langle U^{0,1}, V^{0,1}\rangle_p$ and set $\Omega_p = \Phi_p^* \Omega_{\mathrm{aff}}$. For a Hermitian constant ξ, integration by parts gives

$$d\langle m_p, \xi\rangle_x(v) = \Omega_{p,x}(v, iC_x\xi). \tag{58}$$

Indeed the infinitesimal unitary action on the connection is $-d_{\Phi_p(x)}(i\xi)$; the centered scalar has zero pairing with ξ. Cyclicity of trace gives the same identity for scalar and matrix blocks. By (12) and (28),

$$\Omega_{p,x}(v, iv) \ge c\|v\|_p^2, \qquad |\Omega_{p,x}(v, w)| \le C\|v\|_p\|w\|_p. \tag{59}$$

At zero the first expression is $2\|v\|_p^2$. Thus Ω_p is symplectic and tames the coefficient complex structure. It need not be of type $(1,1)$; this distinction is also present in the adapted constructions of [CT24, Section 3] and [Del25, Section 4.1].

Lemma 6.2. *At $x \in \mathcal{B}_p(r)$, the full constant endomorphism algebra is closed under the reference adjoint. The effective complex stabilizer is the complexification of $H = K_x$, and both effective stabilizers are connected.*

Proof. Write a complex trace-free stabilizer element as $\xi + i\eta$ with ξ, η Hermitian. Then $C_x\xi = -iC_x\eta$. Moment equivariance at $m_p(x) = 0$ gives $Dm_p(iC_x\eta) = 0$. Pairing with ξ in (58) and putting $v = C_x\eta$ gives $0 = \Omega_p(iv, v) = -\Omega_p(v, iv)$. Taming forces $v = 0$ and then $C_x\xi = 0$. The infinitesimal stabilizer is therefore the complexification of its Hermitian part. Adding scalars makes the full intertwiner algebra a unital finite-dimensional $*$-algebra.

Functional calculus in this algebra keeps the reference polar factors of any invertible element in the algebra. A positive factor is an exponential of a Hermitian element, and a unitary factor has a Hermitian logarithm obtained by spectral interpolation on its finite spectrum. These paths prove connectedness of the full unit and unitary groups. Dividing by their common scalars preserves connectedness and gives the claimed effective groups. Determinant-one lifts may still have a finite ineffective center; connectedness is asserted for the effective stabilizers used in (56). □

We describe the smooth pieces and their complex tangent spaces directly. Fix $x \in \mathcal{B}_p(r)$ and $H = K_x$. Choose an Ad_H- and adjoint-invariant complement $\mathfrak{a}$ to $\mathfrak{h} = \mathrm{Lie}(H)_{\mathbb{C}}$ in $\mathfrak{g}$, and an H-invariant complex complement N to $\operatorname{im} C_x$ in V. The local holomorphic map

$$(\eta, z) \longmapsto e^{\eta} \cdot (x + z), \qquad \eta \in \mathfrak{a}, \quad z \in N \tag{60}$$

has invertible derivative. Its local inverse is H-equivariant, so on fixed sets it identifies $\mathfrak{a}^H \times N^H$ with a neighborhood of x in V^H. Set $\Sigma = x + N^H$.

On V^H the components of m_p paired with $\mathfrak{h}_{\mathbb{R}}$ are constant by (58), and vanish at x. Equivariance also makes m_p commute with H. Its remaining target is therefore the Hermitian part of $\mathfrak{a}^H$. For z near x in Σ, solve $m_p(e^Y z) = 0$ in that real space. At x,

$$\langle Dm_p(C_x Y), Y\rangle = \Omega_p(C_x Y, iC_x Y) > 0 \quad (Y \neq 0).$$

The derivative has equal-dimensional domain and target and is injective, hence invertible. The ordinary real implicit-function theorem gives a local balancing section $Y(z)$. Its radius may depend on x. Because Y commutes with H, the section remains in V^H.

After shrinking, the infinitesimal stabilizer in V^H is exactly $\mathfrak{h}$: it contains that algebra and its dimension is minimal at x. Every point just constructed balances by a gauge commuting with H, so Lemma 6.2 excludes extra components. The stabilizer of the unbalanced slice point is consequently $H_{\mathbb{C}}$ as well. Intersect Σ with the complete locus $\kappa_p = 0$. Take its reduced analytic space, refine by the determinantal rank loci of $D\kappa_p$, and take regular loci and successive singular loci. The connected regular parts are smooth locally closed complex pieces. Reduction here specifies ordinary geometric tangents; the complete obstruction ideal is unchanged in the deformation problem.

These pieces describe the full local quotient of the indicated orbit type. For this assertion use a compact equivariant tubular neighborhood of $K\cdot x$ in the unitary representation V. The nearest-orbit projection is equivariant, so a nearby stabilizer lies in a conjugate of H. A point of the same compact stabilizer type can be aligned to H; connectedness and equal dimension turn this containment into equality. A local section of $K \to K/H$ makes the alignment near x. This is the ordinary compact smooth tubular construction; see [Pal61, Proposition 2.2.2 and Remark 2.2.3]. Once aligned, (60) and the fixed-space balancing apply. Compact identifications between representatives tending to x have subsequences tending into H. Since H acts trivially on N^H, the local product and balancing uniqueness remove any remaining local ambiguity. Proposition 6.1 identifies the balanced images with the corresponding quotient pieces.

Repeating at each orbit type and analytic regular part covers the local quotient by these smooth complex pieces. Their transitions come from the holomorphic products (60), after constant compact conjugation of stabilizers, and are holomorphic on the reduced pieces. Balancing itself is real smooth; its derivative contributes a complex orbit vector, so the induced quotient tangent is complex linear. No assertion of a Whitney stratification is needed.

6.3. **Relative tangent norms and intrinsic lengths.** For $x \in \mathcal{B}_p(r)$, set $D_x = d + [a_p(x), \cdot]$ and define

$$g_{\mathrm{fin},p,x}([v]) = \inf_{\xi \in \mathfrak{g}} \|v - C_x\xi\|_p^2, \tag{61}$$

$$g_{\mathrm{HE},p,x}([v]) = \inf_{\zeta \in W^{1,2}(\mathrm{End}_0\, E_0)} \|J_{p,x}v - D_x\zeta\|_{H_x,p}^2. \tag{62}$$

For $v \in \ker D\kappa_{p,x}$ the second expression is the volume-normalized harmonic Kodaira–Spencer squared norm. It is Vol_p^{-1} times the Weil–Petersson convention of [BiS16, Section 4.1]. A real unitary-connection variation has twice the squared norm of its Hermitian $(0,1)$ part. All lengths here use the real part of (62).

We recall precisely the analytic input from [Wue26a, Theorem 8.1]. For a fixed parallel complex algebra and its fixed zero-form complement, suppose the split Poincaré estimate is uniform, the harmonic space and normalized zero-form inner product are fixed, and spatial norms are uniformly equivalent. Suppose the full equivariant reconstruction satisfies

$$Jv = \iota v + B_x v, \qquad \langle \iota w, B_x v\rangle_p = 0, \qquad \|B_x v\|_p \le C|x|\|v\|_p,$$

as well as $D_x\xi = -JC_x\xi$ and $D_xJv = 0$ on the full integrable tangent kernel. Smooth compact-equivariant metrics with $\|\log(h_0^{-1}H_x)\|_\infty \le Cq_p(x)$ then give a smooth positive Hermitian form $\mathcal{B}_{p,x}$ on all of V, satisfying

$$(1 - Cq_p(x))\|v\|_p^2 \le \mathcal{B}_{p,x}(v,v) \le (1 + Cq_p(x))\|v\|_p^2, \tag{63}$$

$$g_{\mathrm{HE},p,x}([v]) = \inf_{\xi \in \mathfrak{g}} \mathcal{B}_{p,x}(v - C_x\xi, v - C_x\xi). \tag{64}$$

The construction first projects off $D_x(QW^{1,2})$, whose weak normal operator has a uniform inverse. The projection of Jv onto that range is $O(|x|)\|v\|_p$, so removing it changes the squared norm by $O(q_p(x))$. One applies (63) to every $v - C_x\xi$ before taking the finite infimum. The argument permits nonabelian groups and every ambient v; the cohomological interpretation uses the integrable tangent condition.

Theorem 6.3. *For uniform $p_0, r_0, C > 0$ with $Cr_0^2 < 1/2$, on every smooth complex piece of $\mathcal{N}_p(r)$ constructed above,*

$$(1 - Cr^2)g_{\mathrm{fin},p} \le g_{\mathrm{HE},p} \le (1 + Cr^2)g_{\mathrm{fin},p}. \tag{65}$$

The estimate also holds on $\ker D\kappa_{p,x}/\operatorname{im} C_x$, and the exact Kodaira–Spencer kernel is $\operatorname{im} C_x$. The constants require no lower bound for positive singular values of C_x or $D\kappa_{p,x}$. At fixed p, the two intrinsic distances on the same local class of piecewise smooth paths compare by $\sqrt{1 \pm Cr^2}$; they separate points on each finite-distance component and have canonically equivalent metric completions.

Proof. We verify the recalled hypotheses on the full matrix model. The normalized measure, zero-form metric and parallel algebra are fixed by Section 2; the complementary Poincaré estimate is uniform. Theorem 2.2 gives complete reconstruction, equivariance and the derivative bounds. Its derivative correction is current-p coexact, hence exactly orthogonal to harmonics. Differentiating (15) at an integrable point proves $D_xJv = 0$ on $\ker D\kappa$. Lemma 4.1 supplies the smooth compact-equivariant metric and its quadratic bound. Thus (63)–(64) hold uniformly. Taking the infima proves (65). The positive ambient lower bound makes its zero set exactly $\operatorname{im} C_x$; Hodge theory identifies this with the cohomology kernel. On the declared complex pieces these are precisely the removed orbit tangents.

For separation, take a real tangent v to the balanced locus and subtract its orthogonal projection onto the compact orbit tangent $T_x = iC_x\mathfrak{g}_{\mathbb{R}}$, obtaining $w \perp T_x$. It remains

balanced infinitesimally. For Hermitian η, ξ put $e = w - C_x\eta - iC_x\xi$. Moment equivariance at $m_p = 0$ gives $\Omega_p(iC_x\xi, iC_x\eta) = 0$. Equations (58)–(59) then imply

$$c\|C_x\eta\|_p^2 \le \Omega_p(C_x\eta, iC_x\eta) = -\Omega_p(e, iC_x\eta) \le C\|e\|_p\|C_x\eta\|_p.$$

Thus $\|C_x\eta\|_p \le C\|e\|_p$. Projecting $w = e + C_x\eta + iC_x\xi$ onto T_x also gives $\|iC_x\xi\|_p \le C\|e\|_p$. Consequently

$$\inf_{\zeta\in\mathfrak{g}} \|v - C_x\zeta\|_p \ge c \inf_{\xi\in\mathfrak{g}_{\mathbb{R}}} \|v - iC_x\xi\|_p. \tag{66}$$

The right side is the speed in the Euclidean compact quotient. Its path length bounds below $\min_{k\in K}\|x - ky\|_p$: choose horizontal compact lifts on constant-rank smooth subintervals and integrate ordinary speed, then use subdivision and continuous endpoints. The local orbit inverse needed to choose such a lift does not enter (66). Compactness of K makes this bound positive for distinct orbits.

Use continuous piecewise C^1 paths in the specified pieces, with local balanced lifts, on one fixed local domain. Integrating the square roots of (65) and taking infima over the same paths proves the distance comparison. Equivalent Cauchy sequences give the stated completion correspondence. These are local intrinsic distances on their finite-distance components. □

6.4. **Transported limiting tangents.** There is a precise continuity statement at changes of stabilizer. Let $(p_n, x_n) \to (p_*, x_*)$ be points of the corrected locus and let $H = K_{x_*}$. Choose an H-invariant complex complement N to $\operatorname{im} C_{x_*}$. The local product (60) gives

$$x_n = g_n \cdot z_n, \qquad g_n = e^{\eta_n} \longrightarrow 1, \qquad z_n \in x_* + N, \quad z_n \longrightarrow x_*.$$

For lifted fibre tangents $v_n \in \ker D\kappa_{p_n,x_n}$, differentiate this product at fixed p_n and let w_n be the derivative of its z coordinate. Suppose

$$w_n \longrightarrow w_* \in N^H, \quad w_* \text{ is tangent to the limiting reduced complex piece.} \tag{67}$$

Then both quotient norms converge to the corresponding norm of w_* at (p_*, x_*).

Here is the verification, including the metric transport. Pull the coefficient forms back by the linear action of g_n; the resulting positive forms converge to the H-invariant form at p_*. The derivative of g_n contributes only an orbit vector. For the actual forms pull the metric H_{p_n,x_n} back to the bundle at z_n, calling it H'_n. It converges in fixed spatial norms to H_{p_*,x_*}, although its logarithm need not be Q_0-normalized. Let R'_n project off $D_{p_n,z_n}(QW^{1,2})$ in this metric. The fixed complementary inverse persists, so

$$\mathcal{B}'_n(v, w) = \langle R'_n J_{p_n,z_n} v, R'_n J_{p_n,z_n} w\rangle_{H'_n,p_n}$$

is positive near the limit and converges to the H-invariant limiting ambient form. Its finite orbit quotient is exactly the transported actual norm, by splitting all gauges into $\mathfrak{g} \oplus Q$.

Apply [Wue26a, Lemma 8.2] to each of these two sequences of forms. Its elementary mechanism is as follows. First remove the nonzero limiting orbit block $C_{z_n}\mathfrak{a}$ by its continuous orthogonal projection P_n. Since N complements the limiting orbit, $(1-P_*)|_N$ is injective with a positive least singular value. The remaining vectors $C_{z_n}\mathfrak{h}$ lie in the moving H-representation subspace of N. After $1 - P_n$ projection, their unit directions stay within $o(1)$ of the moving subspace of V. The limiting vector $(1 - P_*)w_*$ is H-fixed, hence orthogonal to that moving subspace in the limiting invariant form. Its additional projection therefore tends to zero. This proves the norm convergence without dividing by any disappearing orbit singular value. Polarization gives convergence of the corresponding Hermitian pairings.

The condition (67) concerns tangents to the limiting piece. It is not a statement about arbitrary coarse Zariski tangent vectors or arbitrary untransported vectors. The variation of p_n here compares fibrewise norms; parameter velocities enter separately in Section 7.

On overlaps of smooth pieces the local products give smooth families of holomorphic isomorphisms. If these are not isometries, remove their actual-metric positive polar factor as in Proposition 6.1. Matrix functional calculus makes the resulting isometries smooth. Differentiating an isometry g between two reconstructions gives

$$J_2(DT_x v) = g(J_1 v)g^{-1} - D_2(\dot{g}g^{-1}). \tag{68}$$

The second term is exact and conjugation preserves norms, so $T^* g_{\mathrm{HE},2} = g_{\mathrm{HE},1}$. The finite forms compare by $1+O(q_1+q_2)$ using (65) on each side. This also identifies the exact tangent kernels and the local length forms on the overlap.

6.5. **An actual rank-four overlap.** Fix $\mathbb{C}e_1 \subset \mathbb{C}^2$ and its orthogonal complement, and write

$$E_A = F_1 \oplus (F_2 \otimes \mathbb{C}e_1) \oplus F_3 \oplus F_4, \qquad E_0 = E_A \oplus (F_2 \otimes \mathbb{C}e_2).$$

Let H be the relative-phase circle of these two summands. Its fixed harmonic space is the full 83-dimensional rank-four deformation space of E_A. All mixing blocks have nonzero H-weight, and $H^1(\mathcal{O}_X) = 0$ leaves no deformation of the isolated line. Reconstruction preserves this decomposition and is the full active reconstruction plus that line.

An H-fixed HE point requires equal slopes of the line and the whole bundle, which is exactly

$$b_2(p) = 0, \qquad p_2 = \tfrac{7}{5} p_1. \tag{69}$$

Conversely, on this locus a small active HE bundle with its prescribed determinant, direct summed with the fixed line metric, is rank-five HE. The active determinant is $\mathcal{O}(-3,-3,6)$. For an H-fixed corrected point, $u(p,x)$ is block diagonal. Its line logarithm is harmonic and has zero average, so is zero. Pointwise trace-freeness then makes the active logarithm trace-free, with zero averages in its four parallel diagonal blocks. Conversely an active correction with exactly this normalization, extended by zero, is the full correction by uniqueness of Lemma 4.1.

For $p_1 > 0$ set $l = 6\pi p_1$ and interchange the first two projective-line factors, relabeling F_2, F_3 accordingly. The new parameters and active slope differences are

$$(p_1', p_2') = \left(\frac{7l}{30\pi}, \frac{l}{6\pi} \right), \qquad b_A' = l(1, -2/5, 0, -3/5). \tag{70}$$

These are exactly the rank-four geometry of [Wue26a, Sections 6 and 9]. The involution preserves h_0 and normalized spatial integrals and permutes all harmonic blocks and their norm weights.

That paper uses the reference Hodge contraction, whereas we use K_p. For an active integrable $a_p(x)$ apply the uniform reference complex Coulomb construction to put it in the reference reconstruction. Its inverse acts only on the fixed split zero-form complement. The resulting coefficient map and small gauge are holomorphic in x, smooth in p, and uniformly controlled. Transport the active HE metric by that gauge and impose the small constant diagonal normalization (57). Uniqueness identifies it with the corrected rank-four metric. The reverse current-p Coulomb construction gives the inverse quotient germ. Shrinking the domains puts both coefficient presentations in their respective small balls. The balanced coordinate map is real smooth on each piece, while its induced quotient map is holomorphic. Its gauge derivative is the exact term in (68).

This overlap is nonempty: take $\varepsilon = 0$ in Theorem 5.3. Its active bundle is stable and its remaining summand is the isolated line. For each fixed small $t > 0$, restrict to the open active stable locus. Equal-slope stability and the rank difference make every Hom map between the active part and the line vanish. The stabilizer is therefore precisely the relative-phase circle. Compact conjugation removes a nearby choice of unused multiplicity line. The local H-type quotient is consequently the open active stable part of the full rank-four quotient.

Finally the endomorphism complex splits orthogonally into fixed and moving H-weights. In the fixed part, $\mathrm{End}_0(E_A)$ is orthogonal to the relative scalar line, and the differential preserves both. For an active trace-free tangent, minimization over all rank-five gauges therefore reduces exactly to minimization over $\mathrm{End}_0(E_A)$. There is no rank factor because both norms use only spatial-volume normalization. If T denotes the actual transition just constructed,

$$T^* g_{\mathrm{HE},4} = g_{\mathrm{HE},5}|_{\mathrm{active}}, \tag{71}$$

$$(1 - C(q_4 + q_5))g_{\mathrm{fin},5}|_{\mathrm{active}} \le T^* g_{\mathrm{fin},4} \le (1 + C(q_4 + q_5))g_{\mathrm{fin},5}|_{\mathrm{active}}. \tag{72}$$

This is a geometric overlap at the stated positive-parameter boundary points. The preceding transported-tangent result specifies the limiting directions from the matrix side that recover these rank-four forms.

7. A nonconstant curve of finite length

The balanced family in Section 5 has a changing polarization and spectrum, but its positive-parameter bundles are isomorphic. We identify that limitation explicitly, then rotate one cubic into a new polynomial direction. The rotation preserves exact balance and produces a nonzero bundle deformation. Since the polarization still varies, its length requires a metric that includes the parameter velocity.

7.1. **The parameter derivative and the total tangent form.** Put $J = D_x a_p$ and $S = \partial_p a_p$, where S acts on a real parameter velocity $b \in \mathbb{R}^2$. The harmonic projection P_1 is independent of p, as shown in Section 2. Differentiating $P_1 a_p = x$ at fixed x gives $\langle \iota w, Sb\rangle_p = 0$. Differentiating the full recursion gives

$$Sb = -(\partial_p K_p)[b](a_p^2) - K_p(a_p \wedge Sb + Sb \wedge a_p).$$

The last operator is $O(|x|)$ in each fixed spatial norm. The uniform parameter derivatives of K_p therefore imply

$$\|Sb\|_{H^s} \le C q_p(x)|b|, \qquad \langle \iota w, Sb\rangle_p = 0. \tag{73}$$

This uses the full reconstruction, not just its quadratic restriction.

On the complete total integrable locus, a real first-order tangent (b, v) satisfies

$$D_x \kappa_p(x)v + \partial_p \kappa_p(x)b = 0. \tag{74}$$

Differentiating (15) makes $Jv + Sb$ closed for D_x. The complex structure of X is fixed; varying p changes the slice and its norms, but not the underlying meaning of this bundle deformation class.

Assign the Euclidean parameter metric $|dp|^2 = dp_1^2 + dp_2^2$. This is specified data, rather than a canonical metric on Kähler classes. At an actual HE metric H with $e^{-Cq}h_0 \le H \le e^{Cq}h_0$,

define

$$G_{\mathrm{fin}}(b,[v]) = |b|^2 + \inf_{\xi\in\mathfrak{g}} \|v - C_x\xi\|_p^2, \tag{75}$$

$$G_{\mathrm{act}}(b,[v]) = |b|^2 + \inf_{\zeta\in W^{1,2}(\mathrm{End}_0\, E_0)} \|Jv + Sb - D_x\zeta\|_{H,p}^2. \tag{76}$$

The second term is the harmonic norm of the actual bundle variation. It uses the value of H, not its derivative with respect to p.

Proposition 7.1. *On one uniform neighborhood and for all tangents* (74),

$$(1 - Cq_p(x))G_{\mathrm{fin}} \le G_{\mathrm{act}} \le (1 + Cq_p(x))G_{\mathrm{fin}}. \tag{77}$$

The exact kernel before quotienting is $\{(0, C_x\xi) : \xi \in \mathfrak{g}\}$. *On specified smooth geometric pieces this defines a positive real quotient metric, compatible with holomorphic isometric changes of presentation at the same parameter* p.

Proof. Write $Jw = \iota w + B_x w$. The harmonic cross terms of both $B_x w$ and Sb vanish, while $\|B_x w\|_p \le C\sqrt{q}\|w\|_p$ and $\|Sb\|_p \le Cq|b|$. Thus

$$\|Jw + Sb\|_p^2 = \|w\|_p^2 + \|B_x w + Sb\|_p^2, \quad \|B_x w + Sb\|_p^2 \le Cq\|w\|_p^2 + Cq^2|b|^2.$$

As in the recalled relative estimate, projection of Jw onto $D_x(QW^{1,2})$ has norm at most $C\sqrt{q}\|w\|_p$. The extra contribution of Sb is at most $Cq|b|$. Removing this projection loses at most $Cq\|w\|_p^2 + Cq^2|b|^2$. Changing the norm from h_0 to H costs a relative $O(q)$. After adding $|b|^2$, we have a relative $1 \pm Cq$ estimate for

$$|b|^2 + \inf_{\eta\in QW^{1,2}} \|Jw + Sb - D_x\eta\|_{H,p}^2 \quad \text{against} \quad |b|^2 + \|w\|_p^2.$$

Apply it to every ambient $w = v - C_x\xi$ and minimize over the full complex algebra. Exact equivariance identifies the resulting expression with (76), proving (77). Its zero condition is precisely the stated kernel.

For an isometric transition $g(p,x)$, differentiate along (b,v). The transformation law is

$$\delta a_2 = g\delta a_1 g^{-1} - D_2(\dot{g}g^{-1}),$$

where the dot includes both parameter and coefficient derivatives. The exact term disappears under harmonic minimization, and conjugation preserves norms. The parameter term is the same, so the actual total forms agree. Under a change of parameter coordinates, $|dp|^2$ is transported as a tensor. Applying (77) on both sides gives a relative $1+O(q_1+q_2)$ comparison of the finite forms. □

For a common class of piecewise C^1 paths in these pieces, the square-root estimate compares lengths and the resulting length pseudodistances. Their completions agree after the common zero-distance identification. The fixed-p separation theorem is Theorem 6.3; no global separation assertion across all varying-p fibres is needed for the curve below.

7.2. **The unrotated family is isotrivial.** Suppress the fixed form factors in (45) and write

$$x_{12} = t\{a(P+Q)e_1^T + b(P-Q)e_2^T\}, \quad x_{13} = tc(P+Q), \quad x_{24} = te_1, \quad x_{34} = td,$$

where

$$a = \sqrt{\beta/\alpha}\sqrt{1+e}/\sqrt{2}, \quad b = \sqrt{\beta/\alpha}\,\varepsilon/\sqrt{2}, \quad c = \sqrt{(1+e)/(2z)}, \quad d = \sqrt{(\beta/\alpha)z}.$$

These satisfy $a = cd$. At positive ε, the diagonal gauge

$$g_4 = 1, \quad g_3 = (td)^{-1}, \quad g_1 = (t^2 a)^{-1}, \quad M = \mathrm{diag}(t^{-1}, b/(ta)) \tag{78}$$

sends the four coefficients to $((P+Q)e_1^T+(P-Q)e_2^T, P+Q, e_1, 1)$. A common scalar normalizes the determinant without affecting its action. The normalized tuple is only used for the algebraic orbit calculation; composing two such gauges maps the original small endpoints directly to one another.

The current Hodge primitive also changes with p. On these four adjacent blocks, replacing K_p by K_0 changes only the central primitive. The difference is closed and has zero cohomology class, because $H^1(F_1F_4^{-1})=0$. It is therefore $d\phi$ in block 14. A gauge $1\pm\phi E_{14}$, with sign fixed by the gauge convention, removes it exactly; all further products with the active blocks vanish. Reference reconstruction is equivariant. Thus (78) identifies the actual bundles throughout the positive-parameter interior. The gauges are smooth on closed positive-ε intervals, and the bundle Kodaira–Spencer class there is zero.

7.3. **A cubic rotation with a nonzero bundle tangent.** Choose a third cubic R, proportional to s^2z, orthogonal to P, Q and with their common norm. For $0\le\varepsilon\le\varepsilon_0$ put

$$P_\varepsilon=\cos\varepsilon\,P+\sin\varepsilon\,R,\qquad \dot P_\varepsilon=-\sin\varepsilon\,P+\cos\varepsilon\,R. \tag{79}$$

Then P_ε, Q remain orthogonal with the original norms; $\dot P_\varepsilon$ is orthogonal to both and has that norm. Keep all scalar functions and the polarization in (41)–(43), and replace P by P_ε in (45). Explicitly,

$$\begin{aligned} x_{12}&=t\eta_1(P_\varepsilon\ell_P+Q\ell_Q), & x_{13}&=t\eta_2\sqrt{\frac{1+e}{2z}}(P_\varepsilon+Q),\\ x_{24}&=t\tau_1\theta_2\tau_3e_1, & x_{34}&=t\theta_1\tau_2\tau_3\sqrt{D}. \end{aligned} \tag{80}$$

Isomorphisms throughout are over the identity of the fixed base X; we do not quotient by automorphisms of X.

Theorem 7.2. *For one independent rectangle $0<t<t_0$, $0\le\varepsilon\le\varepsilon_0$, the tuples (80) are completely integrable and quadratically balanced at their stated polarizations. They have HE metrics satisfying (52). For fixed $t>0$, their positive-ε bundles are stable and pairwise nonisomorphic. The actual bundle Kodaira–Spencer norm along ε is between ct and Ct.*

Let γ_t be the resulting bundle and polarization curve, with the total metric (76). Its tail satisfies

$$ct\varepsilon\le \mathrm{Length}_{\mathrm{act}}(\gamma_t|_{(0,\varepsilon]})\le Ct\varepsilon. \tag{81}$$

The constants impose no relation between t and ε. The curve converges in the local HE quotient topology to the stable rank-four bundle plus the equal-slope line of Theorem 5.3. Its intrinsic arc-length completion adds this endpoint. On a larger path domain admitting this endpoint curve, its distance to the endpoint is at most $Ct\varepsilon$; the lower bound in (81) concerns the displayed curve's own length.

Proof. We prove the construction, the nonzero tangent and length estimate, and finally the geometric convergence.

The two ordered products in (80) have opposite nonzero harmonic classes. Their sum is

$$W=(\rho_1\wedge\theta_2-\theta_1\wedge\rho_2)\tau_3\sqrt{\frac{\beta(1+e)}{2\alpha}}(P_\varepsilon+Q),$$

which is nonzero and exact. Its coexact primitive is the complete quadratic correction; every further product with block 14 vanishes. Only the unchanged pairwise inner products of P_ε, Q enter the moment. Hence (50) and its proof still give $\mu_p(x)=b(p)$ and $q_p(x)\asymp t^2$. At positive ε, independent polynomials and independent rows force the full centralizer to be

scalar. Theorem 4.5 constructs the HE metric and proves stability. At zero the tuple is exactly the previously constructed nonsimple endpoint, with its separate active-summand correction.

At every positive ε, simplicity makes the fixed-determinant HE metric unique. The elliptic implicit-function theorem on trace-free Hermitian metrics gives local smooth dependence: its linearization at an HE metric has kernel precisely the parallel trace-free Hermitian endomorphisms, which vanish here. Its inverse is bounded at each fixed interior point. Thus the metric is smooth along the interior curve; no uniform endpoint inverse is required.

To distinguish bundles, replace the central current primitive by its reference primitive as in the preceding subsection. Every isomorphism is then a constant block intertwiner. The cubic span of block 12 is exactly $\operatorname{span}(P_\varepsilon, Q)$ for $\varepsilon > 0$, invariant under every invertible multiplicity matrix and outer scalar. These planes are different for different $\varepsilon \in (0, \varepsilon_0]$: their projection onto $\operatorname{span}(P, R)$ is the line $\mathbb{C}(\cos\varepsilon\, P + \sin\varepsilon\, R)$. The interior bundles are therefore pairwise nonisomorphic. The endpoint also differs from them by its larger endomorphism algebra.

Fix t and put $b = p'(\varepsilon)$, $v = x'(\varepsilon)$. Smoothness of the scalar functions in $e = \varepsilon^2$ gives

$$|b| \le Ct^2\varepsilon, \qquad \|v\|_p \le Ct. \tag{82}$$

Every block-12 coefficient of $C_x\xi$ lies in $\operatorname{span}(P_\varepsilon, Q)$. The component $t\eta_1\dot{P}_\varepsilon\ell_P$ of v_{12} is orthogonal to that entire subspace for every matrix gauge direction. Its squared norm is

$$t^2\alpha\|\ell_P\|^2 = t^2\beta(1+2e)/2 \ge ct^2. \tag{83}$$

Consequently $ct^2 \le \inf_{\xi\in\mathfrak{g}} \|v - C_x\xi\|_p^2 \le Ct^2$.

The bundle part alone is nonzero. The ambient relative estimate (63) applies pointwise to the calibrated metric in (52), whether or not its logarithm is Q_0-normalized. Its pointwise proof needs only that metric bound. Minkowski's inequality for the quotient seminorm and (73) give

$$\begin{aligned} \inf_\zeta \|Jv + Sb - D_x\zeta\|_{H,p} &\ge \inf_\zeta \|Jv - D_x\zeta\|_{H,p} - \|Sb\|_{H,p} \\ &\ge \sqrt{1 - Ct^2}\, \inf_\xi \|v - C_x\xi\|_p - Ct^2|b| \ge ct. \end{aligned} \tag{84}$$

The estimate for Jv is valid on ambient vectors even though v alone need not satisfy fixed-p linearized integrability. The combined $Jv + Sb$ is closed by (74). The upper bound follows from (82). This proves the nonzero actual Kodaira–Spencer assertion.

Proposition 7.1 now gives $ct \le \sqrt{G_{\mathrm{act}}(b, [v])} \le Ct$. For any $0 < \varepsilon_1 < \varepsilon_2 \le \varepsilon_0$, integration gives

$$ct(\varepsilon_2 - \varepsilon_1) \le \operatorname{Length}_{\mathrm{act}}(\gamma_t|_{[\varepsilon_1,\varepsilon_2]}) \le Ct(\varepsilon_2 - \varepsilon_1).$$

Taking the lower endpoint to zero proves (81). The embedded parameter interval has a positive arc-length metric and one added endpoint in its completion. Its length lower bound does not constrain shortcuts through other deformation directions.

For the geometric endpoint, put each positive-ε tuple in the corrected locus using a calibration gauge $\widehat{x}_\varepsilon = e^{Y_\varepsilon/2}x_\varepsilon$, $|Y_\varepsilon| \le Ct^2$. Pointwise choices suffice. Along any sequence $\varepsilon_n \downarrow 0$, a subsequence of these bounded finite-dimensional matrices converges. Smoothness of reconstruction and u makes the corresponding corrected pairs converge to a balanced presentation of $e^{Y_0/2}x(t, 0)$. Choose the radii with a common interior margin by reducing t_0. Every such limit has the same holomorphic endpoint type. Proposition 6.1 identifies all these limiting presentations with one compact orbit. The subsequence criterion and compactness of K prove convergence of the full curve in the coefficient quotient topology, hence in the local HE quotient topology, including the parameter.

Smooth local corrected lifts exist in the interior by the just proved metric smoothness and normalization (57). Admitting curves smooth on the open interval with finite improper length therefore attaches the geometric endpoint as an admissible terminal orbit. The upper bound in (81) bounds its ambient distance in any such declared path domain. □

At the endpoint $p_2(t,0) = 7p_1(t,0)/5$. Thus its fixed-stabilizer piece is the actual rank-four overlap of Section 6.5, and the active harmonic metric agrees there by (71). The transverse ε-velocity of the curve is not asserted to meet the fixed-H limiting-tangent condition (67). The finite tail length and geometric endpoint required only the speed bounds and compact-orbit convergence proved above.

8. Finite calculations and the boundary geometry

The two examples use the same six-dimensional parallel space but have different limiting kernels. The following data collect the exact finite computations used in their proofs:

metric and family	multiplicity part of Ξ	$\lvert\Xi\rvert^2$	$\frac{d}{de}Q_e(\Xi)\Big\rvert_{e=0}$
h_0, (19)	$\begin{pmatrix} 1 & 0 \\ 5 & -4 \end{pmatrix}$	45	$50a$
$H_{t,\varepsilon}$, (45)	$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}$	20	25

In both rows the other three scalar entries of Ξ are one. The ratios of the last two columns are $10a/9$ and $5/4$. The first family is not HE at its stated polarization. The second finite form acquires the scale factor $t^2\beta$, and the actual HE comparison has relative error $O(t^2)$.

For a concrete matrix representation of either finite form, use coordinates

$$(z_1, M_{11}, M_{12}, M_{21}, M_{22}, z_3), \qquad z_4 = -z_1 - M_{11} - M_{22} - z_3.$$

The inner-product matrix is $I + vv^*$, where $v = (1,1,0,0,1,1)^T$. The expressions (25) and (53) then specify every entry of the positive matrix whose generalized eigenvalues are being used. At the boundary its rank is five, and at every positive parameter its rank is six. If one deletes both off-diagonal multiplicity coordinates in the first family, the remaining four-dimensional restriction has rank four even at zero: that restriction loses the true holomorphic kernel.

The balanced identities can also be checked without radicals. Eliminate e from (44):

$$e = \frac{5z^2 + 2z - 3}{4z + 3}.$$

Rational z near $3/5$ gives exact rational data for the slope and moment identities. The accompanying calculation verifies those identities, the kernel dimensions, and the two first-variation coefficients. These finite checks supplement the printed arguments; they do not replace the analytic estimates.

Figure 3 distinguishes the two sources of motion. The polynomial direction $\dot{P}_\varepsilon$ remains perpendicular to every finite orbit direction, whereas $p'(\varepsilon) = O(t^2\varepsilon)$. This is why the nonzero bundle tangent persists in the parameter-inclusive metric. The unrotated family has zero interior bundle tangent despite its varying HE spectrum. The distinction is visible algebraically before invoking any estimate on the actual metric.

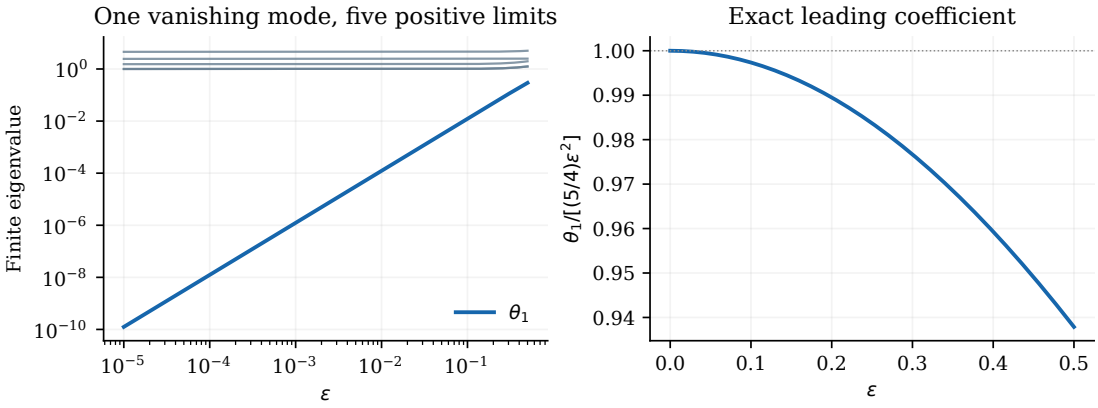


FIGURE 2. Eigenvalues of the explicit finite form (53), before the factor $t^2\beta$. Left: all six eigenvalues for positive ε, with the weak mode emphasized. Right: its ratio to $(5/4)\varepsilon^2$. At $\varepsilon = 0$ that eigenvalue is exactly zero and the other five remain positive. Passage to the trace-free HE spectrum is the relative estimate in Theorem 5.3, not a numerical solution of the HE equation.

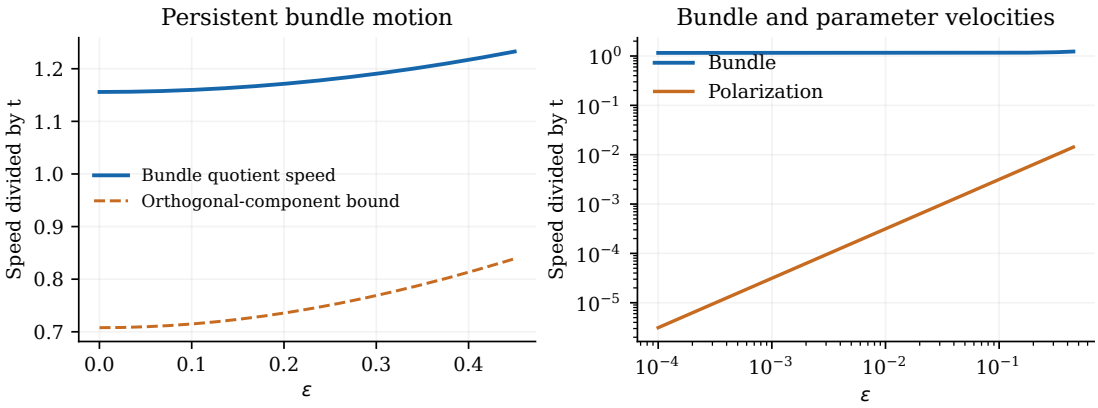


FIGURE 3. The cubic-rotation curve at $t = 0.1$, computed in the finite model. Left: the coefficient quotient speed divided by t, with the explicit lower bound from (83). Right: the polarization speed divided by t and the bundle quotient speed. The latter stays positive as the parameter speed tends to zero. The actual bundle and total metric bounds follow from (84) and Proposition 7.1.

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