

SINGULAR GRADED LIMITS AND METRIC GROWTH UNDER HERMITIAN–YANG–MILLS FLOW

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ABSTRACT. We construct a semistable toric tangent bundle for which the weight-filtration description of Hermitian–Yang–Mills metric growth cannot extend to singular graded sheaves with a uniformly bounded remainder. The bundle lies on a smooth projective surface, and its Jordan–Hölder graded sheaf is singular at one point. For every smooth initial metric with fixed Hermitian–Einstein determinant, the induced squared norm on its unique saturated equal-slope line subsheaf is $o(T^{-1/2})$ as the flow time T tends to infinity, uniformly on compact subsets away from that point. The rank-polarized iterated weight filtration of Haiden–Katzarkov–Kontsevich–Pandit instead assigns the scale $T^{-1/2}$, with no further nonzero weights. A splitting-independent local residue forces divergent normalized extension energy, which an exact dissipation identity converts into the strict decay estimate. Approaching the wall from the stable side, we also identify the Hermitian–Einstein limit with the reflexive graded bundle and locate its multiplicity-one curvature concentration at the same point.

1. INTRODUCTION

The Hermitian–Yang–Mills flow on a semistable holomorphic bundle approaches its polystable graded object at the level of connections, allowing curvature concentration where the graded sheaf is singular. The behavior of the Hermitian metric contains further information. Even when the limiting connection is understood, the metric can diverge along the complex gauge orbit, and the rates of this divergence distinguish extensions with the same graded factors.

Haiden–Katzarkov–Kontsevich–Pandit describe these rates for semistable quiver representations and for bundles on compact Riemann surfaces by an iterated weight filtration [HKKP23, HKKP18]. Lam extends this description to semistable bundles with locally free Jordan–Hölder graded object [Lam26, Theorems 1.1, 8.2]. In that setting the weights give an asymptotic model with bounded error. The extension to singular graded sheaves is explicitly anticipated in [Lam26, Introduction and Remark 8.3], using the quotient category by sheaves supported in complex codimension at least two. A description away from a codimension-two set is also anticipated in [HKKP18, Introduction].

We give a smooth rank-two bundle on a compact Kähler surface for which two precise bounded-remainder versions of that singular extension fail, even on compact subsets away from the singular point. The categorical weight filtration has only two weights and no later

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iterated term. Nevertheless the logarithm of the induced line metric has an additional unbounded negative discrepancy from the prescribed model. A local residue at the singular point forces this discrepancy through a global holomorphic-section identity.

1.1. The flow theorem. Let X be the toric blowup of $\mathbb{F}_1$ with cyclic rays

$$(1, 0), (1, 1), (0, 1), (-1, 0), (1, -1).$$

Write S, f for the pullback section and fibre classes and E for the exceptional divisor. Its tangent bundle $V = T_X$ admits the saturated sequence

$$(1) \quad 0 \longrightarrow L \xrightarrow{\iota} V \longrightarrow \mathcal{I}_p M \longrightarrow 0, \quad L = \mathcal{O}_X(2S + f - E), \quad M = \mathcal{O}_X(2f),$$

where p is the fixed point of the cone $((1, 1), (0, 1))$. The ample class $\alpha_0 = 4S + 7f - 2E$ gives both factors degree 8. The bundle is strictly semistable, and its graded sheaf is $L \oplus \mathcal{I}_p M$. Its geometry is recalled in full in Section 2.

Fix any Kähler form $\omega \in \alpha_0$, and Hermitian–Einstein metrics h_L, h_M on the two line bundles. Put $h_{\det} = h_L h_M$, $v_0 = \int_X \omega^2/2 = 18$ and $\kappa = 8\pi/9$. For a smooth Hermitian initial metric H_{in} with determinant $h_{\det}$, consider the actual metric flow

$$H^{-1} \partial_T H = -2(i\Lambda_\omega F_H - \kappa I), \quad H(0) = H_{\text{in}}.$$

The ratio $u(T) = \iota^* H(T)/h_L$ is a globally smooth nonnegative function, positive on $U = X \setminus \{p\}$ and zero at p . Set $Z(T) = \int_X u(T) \omega^2/2$.

Theorem 1.1. *For every initial metric just specified and every compact $K \subset U$,*

$$(2) \quad \frac{u(T)}{Z(T)} \longrightarrow \frac{1}{18} \quad \text{in } C^\infty(K),$$

and

$$(3) \quad \frac{-Z'(T)}{Z(T)^3} \longrightarrow +\infty, \quad \sqrt{T} Z(T) \longrightarrow 0.$$

In particular $u(T) = o(T^{-1/2})$ uniformly on K . If K is nonempty,

$$(4) \quad \sup_K \left(\log u(T) + \frac{1}{2} \log T \right) \longrightarrow -\infty.$$

The constants and convergence times may depend on the initial metric.

In the slope-8 part of $\text{Coh}(X)/\text{Coh}_0(X)$, the full subobject lattice of V is $0 < L < V$. Its rank-polarized iterated weights are $-1/2$ on L and $+1/2$ on M , with all further weights zero (Section 3). These are squared-metric weights. Thus Theorem 1.1 detects a discrepancy on exactly the line to which the categorical model assigns the scale $T^{-1/2}$.

To state the bounded-remainder consequence precisely, choose a fixed smooth splitting $\Psi : L \oplus M \rightarrow V|_U$ restricting to ι on L , and let

$$G_T = \Psi_*(T^{-1/2} h_L \oplus T^{1/2} h_M), \quad G_0 = \Psi_*(h_L \oplus h_M).$$

Let A_T be the positive G_0 -self-adjoint endomorphism representing $H(T)$, and P the G_0 -orthogonal projection onto $\iota(L)$.

Corollary 1.2. *On no nonempty compact subset $K \subset U$ can either of the following hold with a constant independent of T :*

$$(5) \quad C_K^{-1} G_T \leq H(T) \leq C_K G_T,$$

$$(6) \quad \left\| \log A_T - \frac{1}{2} \log T (I - 2P) \right\|_{G_0} \leq C_K.$$

The first assertion follows by restricting the metric order to L . For the second, spectral Jensen gives $\langle e, A_T e \rangle \geq \exp\langle e, \log A_T e \rangle$ for a G_0 -unit vector in L ; both proposed bounds would force a positive multiple of $T^{-1/2}$ as a lower bound for u . The full argument is in Section 6. The two matrix formulations are distinguished because ordinary matrix logarithms and relative metric order need not agree in a noncommuting problem.

The anticipated extension to singular graded sheaves does not specify a topology for the bounded remainder. Corollary 1.2 tests the two formulations displayed here; it does not attribute either to a numbered singular-case conjecture. The locally free graded theorem is unaffected. Nor does (4) determine the limit of $\log u / \log T$: a leading coefficient $-1/2$ with an additional unbounded, slower term remains possible.

1.2. Why a point singularity affects the metric away from it. Locally at p , the inclusion and quotient are $(z_1, -z_2)^t$ and (z_2, z_1) . On a punctured ball a smooth splitting has a Dolbeault extension form β . Its residue is independent of the splitting and nonzero. It follows that, in fixed smooth graded metrics,

$$(7) \quad \int_{\rho < r < R} |\beta|_0^2 dV \geq c(\rho^{-2} - R^{-2}).$$

The same lower bound holds for the orthogonal splitting selected by $H(T)$, despite its time dependence.

The global norm of the inclusion satisfies the exact identity

$$-Z'(T) = 2 \int_X |D'_{H(T) \otimes h_L^{-1}} \iota|^2 dV.$$

Fixing the determinant gives a cubic metric factor in the normal component of this derivative. On every annulus,

$$\frac{-Z'(T)}{Z(T)^3} \geq 2 \int_{\rho < r < R} (u/Z)^3 |\beta_T|_0^2 dV.$$

The normalized section has a uniform scalar bound. The known connection limit of Sibley–Wentworth [SW15], together with Hartogs extension, then gives $u/Z \rightarrow 1/v_0$ off p . First let $T \rightarrow \infty$ with ρ fixed, and then let $\rho \rightarrow 0$. The residue makes $-Z'/Z^3$ tend to infinity, from which the strict little-oh decay follows by integration. The proof requires no estimate for the shrinking size of a region of curvature concentration.

1.3. A complementary stationary limit. The same example has a stable side. Set $\alpha_t = 4S + (7+t)f - 2E$ for $-1 < t < 0$ and choose a smooth Kähler family $\omega_t \in \alpha_t$ through $\omega_0 = \omega$. Let D_t be the actual Hermitian–Einstein connections on V , presented on one fixed Hermitian bundle with smoothly varying determinant, and let F_t^0 denote their trace-free curvature. At the wall put $D_* = D_L \oplus D_M$.

Theorem 1.3. *As $t \uparrow 0$, the ideal unitary limit of D_t is $(L \oplus M, D_*)$ together with the point p of multiplicity one. Every sequence has a subsequence converging smoothly modulo unitary gauge on compact subsets of $X \setminus \{p\}$. The full curvature measures, and likewise the trace-free curvature measures, converge with the atom $8\pi^2\delta_p$. In particular,*

$$(8) \quad |F_t^0|_t^2 dV_t \rightharpoonup |F_*^0|_0^2 dV + 8\pi^2\delta_p, \quad \int_X |F_t^0|_t^2 dV_t = 26\pi^2, \quad \int_X |F_*^0|_0^2 dV = 18\pi^2.$$

If Π_t is the orthogonal projection onto the image of L , then

$$(9) \quad \|\bar{\partial}_{D_t} \Pi_t\|_{L^2(dV_t)}^2 = -2\pi t, \quad \|i\Lambda_0 F_{D_t} - \kappa I\|_{L^2(dV)} = O(|t|).$$

The sheaf $L \oplus \mathcal{I}_p M$ also occurs as the special fibre of a flat family whose nonzero fibres are isomorphic to V .

Projectively anti-self-dual compactness supplies a reflexive bundle and an integral point cycle. To identify them, normalize the original holomorphic inclusion to have integral squared norm one. A scalar bound retains its mass, and the limiting map is nonzero and parallel. This forces the limiting bundle to be $L \oplus M$. Its second Chern number leaves one point of concentration; the fact that every original inclusion vanishes at p forces that point to be p itself.

The stationary proof also verifies the strengthened hypotheses in the Daskalopoulos–Wentworth correction [DW07] and reconciles the corresponding gauges with the smooth compactness limit. The point cycle is necessary: $c_2(V) = 5$ whereas $c_2(L \oplus M) = 4$, so there is no global smooth bundle isomorphism between them.

1.4. Relation to existing results and organization. The toric construction and its stability wall are taken from [Wue26], and are recalled to make the local singularity explicit. The convergence of the actual flow to the reflexive graded connection, including its singular set, is the theorem of Sibley–Wentworth in its corrected form. Our flow result adds a quantitative obstruction for the metric: its restriction to the canonical line is strictly smaller than the categorical two-weight scale. The new use of the residue is uniform in the changing splitting, which is what allows the connection limit to be combined with the global dissipation identity.

Related normalized induced-metric estimates in [LZZ18, Theorem 3.5] assume the strict slope gap imposed in that paper’s Section 3. The present equal-slope extension has no such gap. We use its standard flow normalization and mean-curvature estimate, not that separate induced-metric theorem.

Sections 2 and 3 give the example and its complete weight calculation. Section 4 proves the local residue estimate and its metric weight. Sections 5–6 prove Theorem 1.1 and Corollary 1.2. Section 7 proves Theorem 1.3, including its exact projection identity and the curvature measure. The final section identifies the remaining quantitative question.

2. A TORIC SURFACE WITH A SINGULAR GRADED QUOTIENT

We first describe the bundle and its semistability wall. This construction appeared in [Wue26, Example 4.3]; the details below fix the local maps and normalizations used in both analytic arguments.

2.1. The surface and the polarization. Let X be the smooth complete toric surface with rays

$$(10) \quad v = (1, 0), \quad w = (1, 1), \quad b = (0, 1), \quad -v = (-1, 0), \quad b' = (1, -1)$$

in cyclic order. Consecutive pairs span its maximal cones. Their determinants are one, so X is smooth. Removing the ray w gives the fan of the first Hirzebruch surface $\mathbb{F}_1$; thus X is its blowup at the fixed point of the cone (v, b) . We use S and f for the pullbacks of the negative section and fibre classes, and E for the exceptional divisor. The intersection pairing is

$$(11) \quad S^2 = -1, \quad Sf = 1, \quad f^2 = 0, \quad E^2 = -1, \quad SE = fE = 0.$$

The class of the polarization will be denoted by α_t , leaving H for a Hermitian metric:

$$(12) \quad \alpha_t = 4S + (7 + t)f - 2E.$$

Here t is a real parameter. Flow time will always be denoted by T .

In the ray order $(v, -v, b, b', w)$ a divisor representing α_t has coefficients $(4, 0, 0, 7 + t, 2)$. Its polygon is

$$-4 \leq x \leq 0, \quad 0 \leq y \leq x + 7 + t, \quad x + y \geq -2.$$

For $t > -1$ the vertices are

$$(-4, 2), \quad (-2, 0), \quad (0, 0), \quad (0, 7 + t), \quad (-4, 3 + t).$$

Every edge has positive lattice length, and the normal fan is the fan of X . Thus α_t is ample. The degrees of the invariant divisors are respectively

$$(13) \quad (1 + t, 7 + t, 2, 4, 2).$$

The same calculation shows that this path leaves the ample cone at $t = -1$.

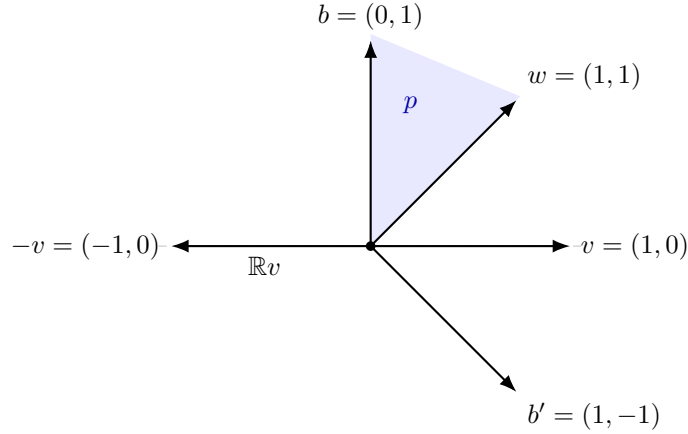


FIGURE 1. The fan of X . The added ray w subdivides the cone (v, b) to give the blowup. The shaded cone (w, b) corresponds to the point p where the graded quotient is singular. The line $\mathbb{R}v$ determines the saturated tangent subsheaf.

2.2. The saturated tangent line. For a toric tangent bundle, a vector subspace W of the torus Lie algebra defines an equivariant saturated subsheaf whose first Chern class is the sum of the invariant divisors with ray in W . The toric stability criterion tests these subsheaves; see [Kly90, HNS22]. In dimension two it suffices to consider lines. Lines containing no ray have degree zero. Among the ray lines, $\mathbb{C}v$ contains both v and $-v$, whereas each of the other three contains only one ray.

Consequently their degrees are $8 + 2t$, 2 , 4 and 2 . The tangent bundle $V = T_X$ has slope

$$(14) \quad \mu_t(V) = 8 + t.$$

It is slope stable for $-1 < t < 0$, strictly semistable at $t = 0$, and unstable for $t > 0$. At $t = 0$ the unique equality among these tests is $\mathbb{C}v$.

The saturated rank-one subsheaf for this line is

$$(15) \quad L = \mathcal{O}_X(2S + f - E), \quad M = \mathcal{O}_X(2f).$$

To see its class directly, the divisors of v and $-v$ have classes $S - E$ and $S + f$. Their sum is $c_1(L)$. A saturated rank-one subsheaf of a locally free sheaf on a smooth surface is reflexive and hence a line bundle. The determinant of V is $\mathcal{O}_X(2S + 3f - E) = L \otimes M$.

Write p for the fixed point of (w, b) . In the corresponding affine chart, $v = w - b$, so the invariant vector field has expression $z_1 \partial_{z_1} - z_2 \partial_{z_2}$. The saturated inclusion and quotient are therefore

$$(16) \quad 0 \longrightarrow \mathcal{O} \xrightarrow{\iota=(z_1, -z_2)^t} \mathcal{O}^2 \xrightarrow{q=(z_2, z_1)} (z_1, z_2) \longrightarrow 0.$$

The kernel assertion follows, for example, by comparing coefficients in $z_2 a + z_1 b = 0$: the regular sequence z_1, z_2 implies $(a, b) = c(z_1, -z_2)$. The quotient is the maximal ideal of the origin. Every other maximal cone contains a ray in $\mathbb{R}v$. Saturation there removes the divisorial zero of the invariant vector field, leaving a nowhere-vanishing local generator. Thus p is the sole zero of ι . Globally we obtain

$$(17) \quad 0 \longrightarrow L \xrightarrow{\iota} V \longrightarrow \mathcal{I}_p M \longrightarrow 0.$$

We also use q for the composite map $V \rightarrow \mathcal{I}_p M \hookrightarrow M$. In particular, q_p is the zero map on fibres.

Both factors in (17) have slope 8 at the wall and are stable, being torsion-free of rank one. Hence the Jordan–Hölder graded sheaf and its reflexive hull are

$$(18) \quad G = L \oplus \mathcal{I}_p M, \quad G^{**} = L \oplus M, \quad G^{**}/G \simeq \mathbb{C}_p \otimes M_p.$$

The distinction between a saturated subsheaf and a subbundle matters here: L is locally free, but its inclusion into V is not a subbundle inclusion at p .

2.3. Degrees, volume and curvature conventions. All intersections below follow from (11):

$$(19) \quad \begin{aligned} L^2 &= -1, & M^2 &= 0, & LM &= 4, & c_1(V)^2 &= 7, & c_2(V) &= 5, \\ \deg_t L &= 8 + 2t, & \deg_t M &= 8, & \alpha_t^2 &= 36 + 8t. \end{aligned}$$

Here $c_2(T_X) = 5$ is also the number of two-dimensional cones in the fan. In particular $c_2(L \oplus M) = 4$: passing to the reflexive graded bundle loses one unit of second Chern class.

Fix a Kähler form $\omega = \omega_0$ representing α_0 . The volume form is $dV = \omega^2/2$ and

$$(20) \quad v_0 = \text{Vol}(X, \omega) = 18, \quad \kappa = \frac{2\pi \deg_0 L}{v_0} = \frac{8\pi}{9}.$$

Degrees mean $\int_X c_1(\cdot) \wedge \omega$, without division by volume. Choose Hermitian–Einstein line metrics h_L, h_M and put $h_{\det} = h_L h_M$. Their mean curvatures are both κ . Constant rescaling of either line metric changes neither line connection.

On skew-Hermitian endomorphisms we use the inner product $-\text{tr}(AB)$, and on Hermitian endomorphisms we use $\text{tr}(AB)$. Form norms are the ordinary norms of the Kähler metric, and all integral norms use the unnormalized volume. These conventions fix both the factor 2 in the metric flow and the energy $8\pi^2$ of an integral point defect. The explicitly Euclidean coefficient norm used in the local residue calculation will be specified separately.

3. THE CATEGORICAL WEIGHTS AT THE WALL

The singularity disappears from the graded factors when sheaves supported at finitely many points are set equal to zero. The extension itself does not split in that category. We determine its full subobject lattice, including nonequivariant subobjects, before calculating its weights.

Let $\text{Coh}_0(X)$ be the Serre subcategory of zero-dimensional coherent sheaves. We work in the abelian category $\mathcal{C}$ of slope-semistable objects of slope 8 in $\text{Coh}(X)/\text{Coh}_0(X)$. Equivalently, one considers semistable torsion-free sheaves of that slope, identifying modifications

at finitely many points and taking kernels and cokernels in the quotient category. Divisorial torsion is not set equal to zero. The finite-slope condition excludes it from $\mathcal{C}$. Rank is a positive additive function on nonzero objects of $\mathcal{C}$.

This is the category appropriate to the rank-polarized weight filtration in the singular surface case; compare [Lam26, Remark 8.3] and [HKKP18, §4.1, equation (4.3)]. Formula (17) becomes an extension of M by L in $\mathcal{C}$.

Lemma 3.1. *The subobject lattice of V in $\mathcal{C}$ is the three-element chain $0 < L < V$. Its simple constituents are the distinct rank-one objects L and M , and the extension does not split in $\mathcal{C}$.*

Proof. A rank-one torsion-free sheaf on a smooth surface has a line bundle as its double dual. It differs from that line bundle only at finitely many points, so they define the same object in $\mathcal{C}$. A same-rank subobject of the same degree can differ from it only by a zero-dimensional quotient: a nonempty effective divisor has positive degree against an ample class. It follows that a rank-one object of $\mathcal{C}$ is simple.

We shall also need the fact that a morphism between vector bundles in $\text{Coh}(X)/\text{Coh}_0(X)$ is represented by an ordinary global bundle map. Indeed, after modifying its source at finitely many points, such a morphism is an ordinary map off that finite set; a vector-bundle target has no zero-dimensional subsheaf. In local frames its coefficients extend across the missing points by Hartogs' theorem, and the extensions agree by uniqueness. This proves the assertion, including uniqueness of the representative.

A nonzero map between distinct line bundles of equal degree cannot exist. Its zero divisor would be effective of degree zero and hence empty, which would make the map an isomorphism. Thus $\text{Hom}(L, M) = \text{Hom}(M, L) = 0$.

If the extension split in $\mathcal{C}$, its right inverse would be a map $s : M \rightarrow V$. By the preceding Hartogs argument it would extend to a holomorphic bundle map on X . The identity $qs = \text{id}_M$, valid away from finitely many points, would then hold everywhere. At p this contradicts $q_p = 0$.

Finally, any proper nonzero subobject A of V has rank one. If $A \neq L$, simplicity gives $A \cap L = 0$, and the composite $A \rightarrow V \rightarrow M$ is a nonzero map between simple objects. It is an isomorphism and would split the extension. This is impossible, so L is the only such subobject. $\square$

Proposition 3.2. *The rank-polarized iterated weight filtration of V assigns weight $-\frac{1}{2}$ to L and $+\frac{1}{2}$ to M . All subsequent iterated weights vanish.*

Proof. The polarized chain in Lemma 3.1 is the subobject lattice of the two-vertex representation with one nonzero arrow $M \rightarrow L$ and unit masses at the vertices. In the construction of Haiden–Katzarkov–Kontsevich–Pandit, the weight filtration depends only on this polarized lattice [HKKP23, §§3.1, 3.3, 4.3]. Here it minimizes

$$w_L^2 + w_M^2 \quad \text{subject to} \quad w_M - w_L \geq 1.$$

The minimum occurs at $w_L = -1/2$, $w_M = 1/2$.

To check that there is no further iterated contribution, one must also perform the next lattice construction. Its elements are pairs (A, B) with $A \in [0, L]$ and $B \in [L, V]$ such that the interval $[A, B]$ is complemented. Here an interval is complemented if each of its elements has a lattice complement; for subobjects this is the condition of semisimplicity. The two entries record the choices in the two successive weight intervals. The derived lattice therefore has three elements,

$$(0, L), \quad (L, L), \quad (L, V).$$

The pair $(0, V)$ is excluded because the interval $[0, V]$ is not complemented: the middle element L has no complement. With the charges measured from the least element, the charge of (A, B) is

$$(1 - i/2) \operatorname{rk} A + (1 + i/2)(\operatorname{rk} B - \operatorname{rk} L).$$

Thus its real part records rank, and its imaginary part records rank weighted by the first filtration weights. The three charges are

$$0, \quad 1 - \frac{i}{2}, \quad 2.$$

Phase is the argument of a nonzero charge. Among nonzero elements, only the greatest element has phase zero. Retaining the least element by convention therefore leaves a two-element complemented lattice. Its weight filtration is trivial, and so are all subsequent iterations. This is the iteration specified in [HKKP23, §§4.4–4.5]. $\square$

The normalization in Proposition 3.2 is the normalization for squared metric norms. In the two-vertex solution of [HKKP23, Example 5.12, equations (5.104)–(5.106)], the source metric grows as $T^{1/2}$ and the target metric decays as $T^{-1/2}$. Taking a square root would instead describe the norm of a frame, and would change these exponents by a factor of two.

There is a useful analytic interpretation of the same calculation. Suppose the orthogonal line metric were $a(T)h_L$, with determinant fixed. The quotient metric would be $a(T)^{-1}h_M$, and the squared norm of an extension form, measured in these metrics, would acquire a factor $a(T)^2$. If its energy relative to the fixed line metrics were a positive finite constant c , the line-norm dissipation would have cubic size $ca(T)^3$. The scalar equation $-a' = ca^3$ gives $a(T) \asymp T^{-1/2}$. This explains the scale appearing in the weight calculation. In our example every fixed-metric extension form has infinite energy near p . The next section proves that assertion in a form uniform over all splittings, which will allow its use for the moving orthogonal splitting of the actual flow.

4. THE LOCAL EXTENSION FORM AND ITS RESIDUE

The local sequence (16) splits smoothly on a punctured ball. Its Dolbeault extension form depends on that splitting, but one of its integrals does not. This invariant integral is the analytic obstruction we shall use.

4.1. An explicit splitting. Work on $B_R \subset \mathbb{C}^2$ in the coordinates of (16), and put $r^2 = |z_1|^2 + |z_2|^2$. A smooth right inverse of q on $B_R \setminus \{0\}$ is

$$(21) \quad \tau_0 = \frac{1}{r^2} \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \end{pmatrix}.$$

The two-column frame (ι, τ_0) has determinant one. Since $q\tau_0 = 1$ and q is holomorphic, $\bar{\partial}\tau_0$ lies in the line spanned by ι . Direct differentiation gives

$$(22) \quad \bar{\partial}\tau_0 = \iota\beta_0, \quad \beta_0 = \frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{r^4}, \quad \bar{\partial}\beta_0 = 0.$$

For instance the $d\bar{z}_1$ coefficient of $\bar{\partial}\tau_0$ is $(-z_1\bar{z}_2, |z_2|^2)^t/r^4$, which is $\iota(-\bar{z}_2/r^4)$. The other coefficient is obtained in the same way.

For this calculation only, give $d\bar{z}_1, d\bar{z}_2$ unit coefficient norm, and use Lebesgue volume on $\mathbb{R}^4$. Then

$$(23) \quad |\beta_0|^2 = r^{-6}, \quad \int_{\rho < r < R} |\beta_0|^2 dV_{\text{Euc}} = \pi^2(\rho^{-2} - R^{-2}).$$

The divergence is therefore stronger than logarithmic. Merely choosing a different splitting might conceivably change this conclusion. The residue rules out that possibility.

Lemma 4.1. *For every smooth right inverse τ of q on $B_R \setminus \{0\}$, write $\bar{\partial}\tau = \iota\beta_\tau$. There is $c > 0$, independent of τ and of $0 < \rho < R$, such that*

$$(24) \quad \int_{\rho < r < R} |\beta_\tau|^2 dV_{\text{Euc}} \geq c(\rho^{-2} - R^{-2}).$$

The conclusion also holds with fixed smooth base and line metrics that extend positively to a neighborhood of the closed ball, with a different constant c .

Proof. On the punctured ball any two right inverses differ by a multiple of ι . Thus $\tau = \tau_0 + \iota f$ for a smooth function f , and

$$(25) \quad \beta_\tau = \beta_0 + \bar{\partial}f.$$

Let S_r have the outward boundary orientation. The form $dz_1 \wedge dz_2 \wedge \beta_0$ restricts to twice r^{-3} times the oriented Euclidean area form. One can check this at $(r, 0)$ and use $U(2)$ invariance; the determinant factor of $dz_1 \wedge dz_2$ is cancelled by the conjugate determinant factor in β_0 . Since $\text{area}(S_r) = 2\pi^2 r^3$, we obtain

$$(26) \quad \int_{S_r} dz_1 \wedge dz_2 \wedge \beta_0 = 4\pi^2.$$

For completeness, at $(r, 0)$ the form is $r^{-3} dz_1 \wedge dz_2 \wedge d\bar{z}_2$; on the positively oriented tangent frame in the y_1, x_2, y_2 directions it equals $2r^{-3}$.

The change in the integrand under (25) is $d(f dz_1 \wedge dz_2)$. Its integral over the sphere is zero. Hence (26) holds with β_τ in place of β_0 . Restriction to S_r and Cauchy–Schwarz now give

$$4\pi^2 \leq C \int_{S_r} |\beta_\tau| dA \leq C(2\pi^2 r^3)^{1/2} \left(\int_{S_r} |\beta_\tau|^2 dA \right)^{1/2}.$$

Thus the spherical integral is bounded below by $c_1 r^{-3}$. Integration in r proves (24). Fixed smooth positive metrics on the closed ball are uniformly equivalent to these Euclidean norms and volume, proving the final assertion. $\square$

In particular there is no smooth splitting whose extension form is square-integrable near the origin in fixed smooth graded metrics. The uniformity over splittings is essential: the orthogonal splitting selected by the evolving metric will vary with time, and we have no uniform bound on its derivatives near the singular point.

4.2. The metric factor in the extension energy. Fix the metrics h_L, h_M from Section 2. For a smooth Hermitian metric H on V with determinant h_{det} , define

$$(27) \quad u_H = \frac{\iota^* H}{h_L}.$$

This is a smooth nonnegative function on X , vanishing precisely at p . On $U = X \setminus \{p\}$ use the H -orthogonal right inverse τ_H of q . The induced line and quotient metrics in this splitting are

$$(28) \quad u_H h_L, \quad u_H^{-1} h_M.$$

Indeed their product is the fixed determinant metric, and the frame (ι, τ_H) is compatible with the determinant identification.

Write $|\beta_H|_0$ for the norm using the fixed line metric $h_L h_M^{-1}$ and the fixed base form ω . From (28) the norm using the actual induced metrics satisfies

$$(29) \quad |\beta_H|_H^2 = u_H^2 |\beta_H|_0^2.$$

Let Π_H be the orthogonal projection onto $\iota(L)$ on U . In the orthogonal splitting, $\bar{\partial}\Pi_H$ has the off-diagonal block β_H up to sign, so $|\bar{\partial}\Pi_H|^2 = |\beta_H|_H^2$.

Regard ι as a holomorphic section of $V \otimes L^{-1}$. Its normal covariant derivative is obtained by applying the adjoint $(1, 0)$ off-diagonal block to that section. The section itself has squared norm u_H . The tangential and normal components of the derivative are orthogonal. Consequently

$$(30) \quad |D'_{H \otimes h_L^{-1}} \iota|^2 \geq u_H |\beta_H|_H^2 = u_H^3 |\beta_H|_0^2.$$

There are three factors of u_H : two from the Hom metric and one from the norm of the section. This cubic weight is what connects the residue to the scalar decay rate.

For each fixed smooth H , the local resolution implies $u_H \asymp r^2$ near p , with constants depending on H . Thus the fixed-metric divergence of β_H does not mean that the actual weighted quantities in (29) or (30) are infinite at a finite time. The obstruction arises after normalization and passage to the large-time limit.

5. THE LINE INCLUSION ALONG THE METRIC FLOW

Let $H(T)$ solve

$$(31) \quad H^{-1} \partial_T H = -2\Phi_H, \quad \Phi_H = i\Lambda_\omega F_H - \kappa I, \quad H(0) = H_{\text{in}},$$

where H_{in} is any smooth positive Hermitian metric with $\det H_{\text{in}} = h_{\text{det}}$. Long-time existence is the standard Hermitian–Yang–Mills heat-flow theorem [Don85]. The scalar determinant flow starts at its Hermitian–Einstein metric; uniqueness for that scalar equation gives $\det H(T) = h_{\text{det}}$ for every $T \geq 0$.

We use the intrinsic functions

$$(32) \quad u(T, x) = \frac{\iota^* H(T)}{h_L}(x), \quad Z(T) = \int_X u(T, x) dV.$$

Although the orthogonal splitting is singular at p , the section ι is globally smooth and holomorphic. In particular $Z(T) > 0$ for every finite T , and its derivative can be computed on the entire compact surface without introducing a puncture.

5.1. Dissipation and a scalar bound.

Lemma 5.1. *For every $T \geq 0$,*

$$(33) \quad Z'(T) = -2 \int_X |D'_{H(T) \otimes h_L^{-1}} \iota|^2 dV \leq 0.$$

There is a constant C , depending on H_{in} and the fixed geometry but not on T , such that

$$(34) \quad \sup_X u(T, \cdot) \leq CZ(T).$$

Proof. The curvature contraction on $V \otimes L^{-1}$ is Φ_H , because $i\Lambda F_{h_L} = \kappa$. Differentiating the norm of the fixed section along (31) gives

$$\partial_T u = -2 \langle \Phi_H \iota, \iota \rangle.$$

For a holomorphic section the integrated Kähler Bochner identity is

$$\int_X \langle i\Lambda F_{V \otimes L^{-1}} \iota, \iota \rangle dV = \int_X |D'_{H \otimes h_L^{-1}} \iota|^2 dV.$$

The section and the bundle metric are smooth at p , so integration introduces no singular boundary term. This proves (33).

With the real Laplacian $\Delta = 2g^{j\bar{k}} \partial_j \partial_{\bar{k}}$, the pointwise Bochner identity reads

$$\Delta u = 2|D'_{H \otimes h_L^{-1}} \iota|^2 - 2\langle \Phi_H \iota, \iota \rangle.$$

The mean-curvature maximum principle along (31) bounds $\sup_X |\Phi_H(T)|_{H(T)}$ by its initial value. Its scalar heat inequality follows by differentiating the contracted curvature along the metric flow; in the present normalization $(\partial_T - \Delta)|\Phi_H|^2 \leq 0$. The conventions and this inequality also appear in [LZZ18, equations (1.5)–(1.6), (3.78)]. It follows that $\Delta u \geq -C_0 u$, with C_0 independent of T . The scalar mean-value estimate, or Moser iteration on the fixed compact surface, now gives $\|u\|_{L^\infty} \leq C\|u\|_{L^1}$. This is (34). $\square$

Only the mean curvature is bounded in this argument. The full curvature may concentrate at p , and the metric need not remain uniformly comparable to a fixed smooth reference metric. The scalar estimate remains uniform because its potential is measured in the current bundle metric and the base geometry is fixed.

5.2. The connection limit and normalized sections. We use the corrected form of Sibley–Wentworth’s convergence theorem [SW15, Theorem 1.1 and Remark 1.2]. In the present case it has the following content. After transporting the metric flow to a fixed Hermitian bundle, the resulting Yang–Mills connections converge modulo unitary gauge smoothly on compact subsets of $U = X \setminus \{p\}$ to the Hermitian–Einstein connection on $G^{**} = L \oplus M$. The algebraic singular set is $\{p\}$, with multiplicity one. For the argument below, the subsequential formulation along any sequence $T_j \rightarrow \infty$ suffices.

The hypotheses are satisfied: X is compact Kähler, V is a holomorphic vector bundle, the initial metric is smooth, and (18) is its graded sheaf at the fixed wall polarization. Metric flow and connection flow are related by time-dependent complex isometries and unitary gauge transformations. Those identifications preserve the scalar norms of sections used here. In the chosen line metrics the limiting connection is

$$(35) \quad D_* = D_L \oplus D_M.$$

We use normalized holomorphic inclusions to pass from this connection convergence to information about the evolving metric.

Proposition 5.2. *For every compact $K \subset U$,*

$$(36) \quad \frac{u(T, \cdot)}{Z(T)} \longrightarrow \frac{1}{v_0} \quad \text{in } C^\infty(K) \quad (T \rightarrow \infty).$$

Proof. Transport $(V, H(T))$ isometrically to a fixed Hermitian bundle and normalize the transported inclusion by $Z(T)^{-1/2}$. Denote the resulting section of $V_T \otimes L^{-1}$ by j_T . It is holomorphic for the transported Dolbeault operator and has global squared integral norm one. Lemma 5.1 gives the pointwise bound

$$(37) \quad |j_T|^2 = \frac{u(T)}{Z(T)} \leq C.$$

Choose any sequence $T_j \rightarrow \infty$. The preceding connection theorem provides gauges in which the connections converge smoothly on compact subsets of U . Interior elliptic estimates for the holomorphic section equation, together with (37), give bounds for all derivatives of the sections on smaller compact sets. Passing to a diagonal subsequence, we obtain a holomorphic limit

$$j_\infty \in H^0(U, (L \oplus M) \otimes L^{-1}).$$

In holomorphic frames near p , its coefficients extend across the puncture by Hartogs' theorem. It is therefore a global section of $\mathcal{O}_X \oplus (M \otimes L^{-1})$.

The second summand has no nonzero holomorphic section. Indeed it is a nontrivial line bundle of degree zero; any nonzero section would define an effective divisor of degree zero and hence trivialize it. It follows that j_∞ lies in the first summand and is constant there. We still have to show that this constant is nonzero and determine its norm. Smooth convergence off p alone would not exclude concentration of all the section's mass at p .

The pointwise bound does exclude that possibility. On any coordinate ball $B_r(p)$,

$$\int_{B_r(p)} |j_{T_j}|^2 dV \leq C \text{Vol}(B_r(p)),$$

uniformly in j . Pass to the limit on $X \setminus B_r(p)$ and then let $r \downarrow 0$. Since the global integral is one, the limiting section also has integral norm one. Its constant squared norm is therefore $1/v_0$.

Finally, $|j_T|^2 = u/Z$ is invariant under every isometry and unitary gauge used above. Every sequence of times has a subsequence with the same scalar C^∞ limit on each compact subset. A subsequence contradiction proves the full convergence (36). $\square$

The order of the preceding operations is useful to keep in mind. Connection convergence gives a limiting holomorphic section away from the singular set. The global scalar estimate retains its mass. Hartogs extension and the equal-slope condition then identify it. No estimate for the speed of connection convergence was needed.

6. FAILURE OF THE BOUNDED METRIC ASYMPTOTICS

We now combine the local residue with the global normalization. Their combination forces an unbounded coefficient in the cubic dissipation law, and this is stronger than a fixed multiple of the $T^{-1/2}$ decay predicted by the two-weight model.

Proposition 6.1. *Along (31),*

$$(38) \quad \frac{-Z'(T)}{Z(T)^3} \rightarrow +\infty, \quad \sqrt{T} Z(T) \rightarrow 0.$$

Consequently, for every nonempty compact $K \subset U$,

$$(39) \quad \sup_{x \in K} \sqrt{T} u(T, x) \rightarrow 0, \quad \sup_{x \in K} \left(\log u(T, x) + \frac{1}{2} \log T \right) \rightarrow -\infty.$$

Proof. Choose a coordinate ball whose closure lies in the chart of (16). At each finite time, take the $H(T)$ -orthogonal right inverse of q on the punctured ball, and denote its extension form by β_T . In the fixed smooth line and base metrics, Lemma 4.1 gives

$$(40) \quad \int_{\rho < r < R} |\beta_T|_0^2 dV \geq c(\rho^{-2} - R^{-2})$$

for all T and all $0 < \rho < R$, with the same $c > 0$. Since the determinant is fixed, (30) applies at every time. Together with (33), it gives

$$(41) \quad \frac{-Z'(T)}{Z(T)^3} \geq 2 \int_{\rho < r < R} \left(\frac{u(T, x)}{Z(T)} \right)^3 |\beta_T|_0^2 dV.$$

Fix $\rho > 0$. Proposition 5.2 implies $u(T)/Z(T) \geq 1/(2v_0)$ on this closed annulus for all sufficiently large T . Thus

$$(42) \quad \liminf_{T \rightarrow \infty} \frac{-Z'(T)}{Z(T)^3} \geq \frac{c}{4v_0^3}(\rho^{-2} - R^{-2}).$$

The left-hand side is independent of ρ . Letting $\rho \downarrow 0$ proves that it is $+\infty$.

It follows that

$$\frac{d}{dT} Z(T)^{-2} = -\frac{2Z'(T)}{Z(T)^3} \longrightarrow +\infty.$$

For any $A > 0$, this derivative is at least A after some time T_A . Integration yields

$$Z(T)^{-2} \geq Z(T_A)^{-2} + A(T - T_A) \quad (T \geq T_A).$$

After division by T and passage to the limit, $\liminf Z(T)^{-2}/T \geq A$. As A is arbitrary, $\sqrt{T} Z(T) \rightarrow 0$. The uniform convergence of u/Z on K proves the first part of (39). Its logarithm proves the second, since $u(T, x) > 0$ on K . $\square$

The argument first lets time tend to infinity on a fixed annulus and only then decreases its inner radius. It gives no uniform entry time as that radius shrinks. In particular it neither selects a time-dependent annulus nor estimates a bubble radius. This limit order is enough for the strict little-oh conclusion.

6.1. Two precise bounded-remainder assertions. The weights in Proposition 3.2 lead to the following candidate metric on the punctured surface. Choose any fixed smooth filtration-compatible splitting

$$\Psi : L \oplus M \longrightarrow V|_U, \quad \Psi|_L = \iota,$$

and define, for $T \geq 1$,

$$(43) \quad G_T = \Psi_*(T^{-1/2}h_L \oplus T^{1/2}h_M).$$

One possible bounded-remainder statement is bounded relative metric order on compact subsets:

$$(44) \quad C_K^{-1}G_T \leq H(T) \leq C_K G_T \quad \text{on } K,$$

with C_K independent of T . The order is the order of positive Hermitian forms on each fibre.

A second statement uses the ordinary logarithm of a positive matrix. Let $G_0 = \Psi_*(h_L \oplus h_M)$, write A_T for the positive G_0 -self-adjoint endomorphism representing $H(T)$, and let P be the G_0 -orthogonal projection onto $\iota(L)$. The proposed assertion is

$$(45) \quad \left\| \log A_T - \frac{1}{2} \log T (I - 2P) \right\|_{G_0} \leq C_K \quad \text{on } K.$$

Here the norm is the operator norm and the logarithm is defined by spectral calculus. These two assertions should not be identified in a noncommuting matrix problem.

Corollary 6.2. *Neither (44) nor (45) holds for any nonempty compact $K \subset U$, for any fixed splitting Ψ as above.*

Proof. Restricting the lower inequality in (44) to L would give $u(T, x) \geq C_K^{-1} T^{-1/2}$, contradicting (39).

For (45), fix a G_0 -unit vector e in $\iota(L)$. The elementary spectral Jensen inequality gives

$$(46) \quad u(T, x) = \langle e, A_T e \rangle_{G_0} \geq \exp \langle e, (\log A_T) e \rangle_{G_0}.$$

Indeed, diagonalize A_T and apply the scalar concavity of $\log$ to the eigenvalues with weights given by the squared coefficients of e . Since $Pe = e$, (45) implies that the last expression is at least $e^{-C_K T^{-1/2}}$, again a contradiction. $\square$

Thus the failure is detected by restriction to the canonical line. It cannot be removed by altering the fixed smooth splitting or by passing from metric order to the specified ordinary matrix-log formulation. The argument does not infer either of those matrix assertions from the other; it tests a necessary scalar inequality for each.

7. STATIONARY DEGENERATION AND THE CURVATURE MEASURE

The same line inclusion identifies the limit of Hermitian–Einstein connections as the polarization approaches the wall from the stable side. Here the parameter is $t \uparrow 0$, not flow time. We prove the compactness and localization statement independently of the heat-flow convergence used in Section 5.

Retain the smooth Kähler family $\omega_t \in \alpha_t$ chosen in the introduction, with $\omega_0 = \omega$, and restrict to $-t_0 \leq t \leq 0$ for some $t_0 \in (0, 1)$. Such families exist: if η is a smooth closed real $(1, 1)$ -form representing f , then $\omega + t\eta$ is Kähler for sufficiently small $|t|$. Linearity in t is unnecessary below; we use smooth dependence, the prescribed cohomology classes, and $\Lambda_t - \Lambda_0 = O(t)$ in smooth operator norms. Write $dV_t = \omega_t^2/2$ and Λ_t for contraction with ω_t . Then

$$(47) \quad \text{Vol}_t = 18 + 4t, \quad \kappa_t = \frac{2\pi(8 + t)}{18 + 4t}.$$

Let $h_{\det, t}$ be smoothly varying Hermitian–Einstein metrics on $\det V$, chosen to agree with $h_{\det}$ at zero. Such a choice follows from the scalar Poisson equation: normalize the logarithmic correction to have zero ω -mean, and use the smooth inverse of the Laplacian on the complement of its constants. The mean-zero solvability condition is precisely the degree identity.

For $t < 0$ the bundle V is stable. By the Donaldson–Uhlenbeck–Yau theorem [Don85, UY86], it has a Hermitian–Einstein metric K_t with determinant $h_{\det, t}$. Fix a smooth Hermitian metric h on the underlying bundle with determinant $h_{\det}$. Transport the Chern connection of K_t through the positive isometry $(V, K_t) \rightarrow (V, h)$ and denote the resulting unitary connection by D_t . It satisfies

$$(48) \quad F_{D_t}^{0,2} = 0, \quad i\Lambda_t F_{D_t} = \kappa_t I.$$

Its Dolbeault operator lies in the complex gauge orbit of the fixed holomorphic bundle V . Write d_t for its determinant connection, $f_t = \text{tr } F_{D_t}$ for the determinant curvature, and

$$(49) \quad F_t^0 = F_{D_t} - \frac{1}{2} f_t I$$

for the trace-free curvature. The superscript 0 in F_t^0 denotes the trace-free part, not a form type. The connections d_t and their curvatures vary smoothly through $t = 0$; no such assertion is made for K_t .

7.1. The flat sheaf degeneration. The algebraic degeneration is explicit. Write s for the coordinate on $\mathbb{A}^1$ and $\text{pr} : X \times \mathbb{A}^1 \rightarrow X$ for the projection. Define

$$(50) \quad \mathcal{V} = \text{coker} [\text{pr}^* L \longrightarrow \text{pr}^* V \oplus \text{pr}^* L, \quad \ell \longmapsto (\iota\ell, -s\ell)].$$

There is an exact sequence

$$0 \longrightarrow \text{pr}^* L \longrightarrow \mathcal{V} \longrightarrow \text{pr}^*(\mathcal{I}_p M) \longrightarrow 0.$$

For the injection, if $(0, \ell)$ lies in the image of the defining map, injectivity of ι forces its preimage and ℓ to vanish. The quotient assertion follows from (17). Both end terms are flat over $\mathbb{A}^1$, hence so is $\mathcal{V}$, and the sequence remains exact on fibres. For $s \neq 0$ the map $[(e, \ell)] \mapsto e + s^{-1}\iota\ell$ identifies the fibre with V . At $s = 0$ the fibre is $L \oplus \mathcal{I}_p M$.

Locally at $(p, 0)$ the family has the presentation

$$\mathcal{O} \xrightarrow{(z_1, -z_2, -s)^t} \mathcal{O}^3 \longrightarrow \mathcal{V} \longrightarrow 0.$$

The loss of local freeness in the special fibre is visible in this matrix. Its double dual has length-one quotient at p , as in (18). One may pair the real stable-side parameters by $s = -t$, but flatness and the analytic limit below are distinct assertions.

7.2. Quantitative curvature estimates.

Proposition 7.1. *The trace-free curvature has constant total energy*

$$(51) \quad \int_X |F_t^0|_t^2 dV_t = 26\pi^2.$$

For a constant C independent of $t \in [-t_0, 0)$,

$$(52) \quad \|i\Lambda_0 F_{D_t} - \kappa I\|_{L^2(dV)} \leq C|t|.$$

Moreover, with $\text{HYM}_0(D) = \int_X |i\Lambda_0 F_D|^2 dV$,

$$(53) \quad \text{HYM}_0(D_t) - 2\kappa^2 v_0 = \|i\Lambda_0 F_{D_t} - \kappa I\|_{L^2(dV)}^2 \leq C^2 t^2.$$

Proof. By (48), F_t^0 is a primitive $(1, 1)$ -form and is therefore anti-self-dual on the Kähler surface. With the trace norm fixed above, Chern–Weil gives

$$\int_X |F_t^0|_t^2 dV_t = 8\pi^2 \left(c_2(V) - \frac{1}{4}c_1(V)^2 \right) = 8\pi^2 \left(5 - \frac{7}{4} \right) = 26\pi^2.$$

The determinant curvature is smooth, so the full curvature has uniformly bounded integral squared norm as well. More explicitly, the surface energy identity gives

$$(54) \quad \text{YM}_t(D_t) = \text{HYM}_t(D_t) + 4\pi^2(2c_2(V) - c_1(V)^2) = \frac{8\pi^2(8+t)^2}{18+4t} + 12\pi^2.$$

The contraction operators satisfy $\|\Lambda_0 - \Lambda_t\| \leq C_1|t|$ pointwise. Base norms and volumes are uniformly equivalent, and $|\kappa_t - \kappa| \leq C_2|t|$. Thus

$$i\Lambda_0 F_{D_t} - \kappa I = i(\Lambda_0 - \Lambda_t)F_{D_t} + (\kappa_t - \kappa)I$$

has integral squared norm bounded by $C^2 t^2$. All bundle norms in this calculation use the single metric h . Finally, putting $e_t = i\Lambda_0 F_{D_t} - \kappa I$, the degree identity implies

$$\int_X \text{tr}(e_t) dV = 2\pi \deg_0 V - 2\kappa v_0 = 0.$$

The cross term in the square of $\kappa I + e_t$ therefore integrates to zero, proving (53). $\square$

There is also an exact estimate for the line projection. Let Π_t be the K_t -orthogonal projection onto $\iota(L)$ on U , transported to the D_t presentation.

Proposition 7.2. *For each fixed $t < 0$, Π_t extends across p as a $W^{1,2}$ projection and*

$$(55) \quad \|\bar{\partial}_{D_t}\Pi_t\|_{L^2(dV_t)}^2 = -2\pi t.$$

Proof. Fix $t < 0$ and work first in the original smooth holomorphic presentation. With $z = (z_1, -z_2)^t$, the projection is $z(z^*K_t z)^{-1}z^*K_t$. Positivity and smoothness of this fixed metric give $|\Pi_t| \leq C_t$ and $|d\Pi_t| \leq C_t/r$. The derivative is square-integrable in real dimension four. Extending the bounded projection across the point creates no distributional derivative supported there: the boundary term on a sphere of radius r is $O_t(r^3)$.

The induced line metric has squared frame length $z^*K_t z \asymp r^2$. Its logarithmic derivative is $O_t(r^{-1})$ and its curvature is locally integrable. Compare it to a smooth metric on L and apply Stokes on $X \setminus B_r(p)$. The curvature difference integrates to a boundary term of size $O_t(r^2)$, which tends to zero. Thus the integral of the induced first Chern form is the ordinary degree of L .

In the orthogonal splitting the second-fundamental-form identity is

$$\mathrm{tr}(\Pi_t i\Lambda_t F_{D_t}) = i\Lambda_t F_{L,\mathrm{ind}} + |\bar{\partial}_{D_t}\Pi_t|^2.$$

Its coefficient can be checked by writing the connection off-diagonal blocks as $(-B^*, B)$, with B of type $(1, 0)$; $\bar{\partial}\Pi_t$ has the block B^* , of the same norm as B . Integrating gives

$$\|\bar{\partial}_{D_t}\Pi_t\|_2^2 = \kappa_t \mathrm{Vol}_t - 2\pi \deg_t L = 2\pi[(8+t) - (8+2t)] = -2\pi t.$$

No point correction appears in a first Chern degree. The local constants were allowed to depend on t ; they justify integration at each parameter, after which the identity itself is exact. $\square$

7.3. Compactness for the changing Kähler form. We apply projectively anti-self-dual compactness in real dimension four. The precise version needed permits base metrics in a compact smooth family, fixed determinant and bounded trace-free energy; it yields a smooth extended limit and a positive integral zero-cycle, with energy $8\pi^2$ per unit multiplicity. See [FL99, §2.3]; local uniform regularity is given in its Theorem 4.17 and Remark 4.18. Removability with determinant fixed is the unperturbed, zero-spinor case of [FL98, Theorem 4.10].

The determinant in our presentation varies smoothly. Put $\theta_t = d_t - d_0$ and replace D_t by

$$\widehat{D}_t = D_t - \tfrac{1}{2}\theta_t I.$$

This fixes the determinant and leaves the projective connection and F_t^0 unchanged. The term subtracted tends smoothly to zero, so restoring it after compactness changes neither the point cycle nor the local convergence.

For clarity, the critical integral squared-curvature bound is used through anti-self-dual small-energy regularity. Uniform constants are available because the chosen family ω_t , $t \in [-t_0, 0]$, is compact and smooth. On a sufficiently small coordinate ball one has a bound of the form

$$(56) \quad \sup_{B_{r/4}} |F_t^0|_t \leq Cr^{-2} \left(\int_{B_r} |F_t^0|_t^2 dV_t \right)^{1/2}$$

when the integral is below a fixed threshold. At a point with no atom in the weak limiting measure, choose a small ball with boundary of zero limiting measure and sufficiently small limiting energy. The same smallness holds eventually along the sequence, and (56) and

Coulomb-gauge estimates give all derivative bounds on smaller balls. The smooth determinant adds only a uniformly bounded central term.

Thus for every sequence $t_j \uparrow 0$ there is a subsequence with smooth convergence modulo unitary gauge away from a finite set Σ . Finite-energy removal extends the limiting projective connection with its determinant to a smooth rank-two bundle V_∞ . Restoring the determinant term and passing to the limit in (48) gives

$$(57) \quad F_{D_\infty}^{0,2} = 0, \quad i\Lambda_0 F_{D_\infty} = \kappa I.$$

The compactness theorem also gives

$$(58) \quad \begin{aligned} |F_{t_j}^0|_{t_j}^2 dV_{t_j} &\rightharpoonup |F_{D_\infty}^0|_0^2 dV + 8\pi^2 \sum_{q \in \Sigma} m_q \delta_q, \\ c_2(V_\infty) &= 5 - \sum_{q \in \Sigma} m_q, \quad m_q \in \mathbb{Z}_{>0}. \end{aligned}$$

The local multiplicities are integral even though the total projective charge $c_2(V) - c_1(V)^2/4 = 13/4$ is not an integer. The local clutching bundle on a four-sphere has an $SU(2)$ lift, and its second Chern number is integral; this is also the normalization in [FL98, equations (2.32)–(2.33), Lemma 4.21]. The identities (58) do not yet identify either V_∞ or the points of Σ .

7.4. The normalized inclusion locates the point. Let $g_t : (V, K_t) \rightarrow (V, h)$ be the positive isometry defining D_t . Choose $a_t > 0$ so that

$$(59) \quad s_t = a_t g_t \iota \in H^0(X, \text{Hom}(L, V_t)), \quad \int_X |s_t|^2 dV_t = 1.$$

The mean curvature of this Hom bundle is

$$i\Lambda_t F_{\text{Hom}(L, V_t)} = (\kappa_t - i\Lambda_t F_L)I.$$

It is uniformly bounded, since F_L is smooth and the base forms vary smoothly. The scalar Bochner inequality and a mean-value estimate with uniform base Sobolev constants give

$$(60) \quad \|s_t\|_{L^\infty} \leq C \|s_t\|_{L^2(dV_t)} = C.$$

This is the same normalization mechanism as in Proposition 5.2, but the connection compactness input is now (58).

In its smooth compactness gauges, interior estimates for the holomorphic section equation give a locally smooth limit $s_\infty : L \rightarrow V_\infty$ on $X \setminus \Sigma$. The uniform bound (60) prevents its norm-one mass from disappearing into finitely many shrinking balls. Hence the limit is nonzero. Its coefficients in holomorphic frames extend across Σ by Hartogs. On the extended Hom bundle the mean curvature is zero, by (57) and $i\Lambda_0 F_L = \kappa$. The integrated Bochner identity makes s_∞ parallel. Its image is therefore a parallel copy of L , and the determinant identifies its parallel orthogonal complement with M . We conclude that

$$(61) \quad (V_\infty, D_\infty) \simeq (L \oplus M, D_*).$$

Since $c_2(L \oplus M) = 4$, equation (58) implies $\sum m_q = 1$. There is exactly one concentration point, of multiplicity one.

To locate it, recall that $s_t(p) = 0$ for every t . If the unique point q of Σ were different from p , the sections would converge smoothly near p . Their limit would satisfy $s_\infty(p) = 0$, which contradicts its nonzero parallelism. Therefore $\Sigma = \{p\}$. This identifies the point by the zero of the original inclusion, rather than by the total energy alone.

Chern–Weil on the limit gives

$$\int_X |F_*^0|_0^2 dV = 8\pi^2 \left(4 - \frac{7}{4}\right) = 18\pi^2.$$

Together with (58) this proves

$$(62) \quad |F_t^0|_t^2 dV_t \rightharpoonup |F_*^0|_0^2 dV + 8\pi^2 \delta_p.$$

The identity initially holds along the extracted subsequence. Each fixed holomorphic line has a unique Hermitian–Einstein connection, since the ratio of two such metrics has harmonic logarithm and is constant. Thus every sequence admits a subsequence with the same ideal unitary limit. A subsequence contradiction proves the full convergence of gauge classes in the Uhlenbeck topology and the full weak convergence in (62). This does not require a continuous choice of gauges as t varies.

Finally $\text{tr } F_{D_t}$ converges smoothly, and

$$|F_{D_t}|^2 = |F_t^0|^2 + \frac{1}{2} |\text{tr } F_{D_t}|^2.$$

The full curvature measures therefore have the same point atom. There is no global smooth complex bundle isomorphism $V \simeq L \oplus M$, since their second Chern classes differ. Nor can global gauges on V make connection coefficients and their first spatial derivatives uniformly bounded relative to one smooth reference connection: that would bound curvature pointwise and exclude the atom in (62). These conclusions complete the proof of the stationary theorem.

7.5. Compatibility with the corrected minimizing-sequence theorem. There is a second useful verification of the reflexive limit. The original minimizing-sequence statement of Daskalopoulos–Wentworth must be used with its correction [DW04, DW07]. It requires minimization of HYM_α for $\alpha = 2$ and for a set of exponents in $[1, \infty)$ having an accumulation point. The single quadratic excess in (53) is not, by itself, the corrected hypothesis.

In fact (52) gives all these conditions. Define

$$\Phi_\alpha(A) = \text{tr } |A|^\alpha, \quad \|A\|_{L^\alpha(S^\alpha)} = \left(\int_X \text{tr } |A|^\alpha dV \right)^{1/\alpha}.$$

For each fixed $1 \leq \alpha \leq 2$, finite volume and the finite-dimensional Schatten norm inequalities imply

$$\|i\Lambda_0 F_{D_t} - \kappa I\|_{L^\alpha(S^\alpha)} = O(|t|).$$

The reverse triangle inequality for this norm yields

$$(63) \quad \int_X \Phi_\alpha(i\Lambda_0 F_{D_t}) dV \longrightarrow 2v_0 |\kappa|^\alpha.$$

These are the type minima for the equal wall slopes. Taking the accumulation set $(1, 2)$ verifies the correction. To match the source’s volume convention 2π , scale ω by $c = (2\pi/18)^{1/2}$. Contraction scales by c^{-1} and volume by c^2 , so both each functional and its type minimum scale by $c^{2-\alpha}$. All connections lie in the same complex gauge orbit. The corrected theorem therefore identifies a strong L_{loc}^2 subsequential limit with $(L \oplus \mathcal{I}_p M)^{**} = L \oplus M$.

One can compare its gauges with the preceding smooth compactness gauges without asserting an automatic improvement of weak convergence. On a common subsequence away from the finite exceptional sets, let $A_j \rightarrow A$ smoothly and $B_j \rightarrow B$ strongly in L_{loc}^2 be the two presentations. The relative unitary maps obey

$$du_j = u_j A_j - B_j u_j.$$

Unitarity bounds u_j in L^∞ , and this equation bounds them in $W_{\text{loc}}^{1,2}$. Rellich compactness gives a strongly local L^2 and almost-everywhere limit u , unitary almost everywhere. The equation passes to the limit: for the terms involving B_j , combine strong L^2 convergence with boundedness of u_j and dominated convergence for the fixed limit B . Thus $du = uA - Bu$. As A and B are smooth, this first-order equation bootstraps u to a smooth unitary intertwiner. Its holomorphic equation and that of its inverse extend across the finite set by Hartogs, yielding the same global connection isomorphism. This confirms the compatibility of the two limit identifications; the location and measure of the point defect were supplied by the preceding argument.

8. THE REMAINING ASYMPTOTIC QUESTION

Theorem 1.1 establishes an unbounded negative discrepancy from $-\frac{1}{2} \log T$ on the canonical line. It does not identify its size. The distinction matters: both a change in the leading power and a slower unbounded correction would be consistent with the proved little-oh estimate.

Question 8.1. For the bundle and flow of Theorem 1.1, what is the asymptotic behavior of $Z(T)$ beyond $Z(T) = o(T^{-1/2})$? Is the limit of $\log Z(T)/\log T$ equal to $-1/2$?

The proof indicates the missing estimate. At present the residue gives a lower bound for dissipation on each fixed annulus, and the normalized inclusion becomes uniform there after an annulus-dependent time. A sharper description would need quantitative control relating these times to the inner radius, together with a matching upper estimate for the relevant weighted extension energy. The stationary identity $\|\bar{\partial}\Pi_t\|_2^2 = -2\pi t$ concerns a different family and does not supply that relation to flow time.

There is also a distinction between identifying the limiting connection and describing the metric along the original holomorphic presentation. The connection limit retains the reflexive bundle and the point cycle. The metric asymptotics must account for the analytic effect of the singular extension as well. In this example that effect is visible in the residue, although the quotient-category lattice has only the two ordinary weights. Determining how such local extension data should enter a corrected singular asymptotic description remains open.

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