

SMALL DOLBEAULT EIGENVALUES NEAR STABILITY WALLS

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ABSTRACT. We determine the small spectrum of the Dolbeault Laplacian on trace-free endomorphisms of Hermitian–Einstein bundles near intersections of slope-stability walls. On a compact Kähler manifold of dimension at least two, consider an extension of $\bigoplus_{j=1}^k F_j$ by F_0 , where the factors are pairwise nonisomorphic and stable of equal slope at the limiting Kähler class, and every extension component is nonzero. Write $r_i = \text{rk } F_i$ and let ϵ_j be the Einstein constant of F_j minus that of the extension. In the chamber $\epsilon_j > 0$, exactly k eigenvalues tend to zero. They agree, to relative error $O(\sum_j \epsilon_j)$, with the eigenvalues of $\text{diag}(\epsilon_1, \dots, \epsilon_k) + r_0^{-1}vv^*$, where $v_j = \sqrt{r_j \epsilon_j}$. The estimate is uniform in all positive ratios of the slope gaps. Thus the ranks and slopes determine the leading spectrum even when the gaps vanish at arbitrarily different rates.

For two factors we obtain $\lambda_1 = \delta - D\delta^2 + o(\delta^2)$, where $\delta > 0$ is the quotient Einstein constant minus the subbundle Einstein constant and $D \geq 0$ is given by the Green operator of the graded connection. After rescaling the extension and making unitary identifications with the graded limit, we determine the normalized first eigensection through order δ in every spatial C^m norm. Toric tangent bundles provide examples with arbitrarily many small eigenvalues whose orders of vanishing can be prescribed independently.

1. INTRODUCTION

When a stable bundle approaches a strictly semistable polarization, its Hermitian–Einstein connection can converge, after gauge, to the connection on its graded object. The additional endomorphisms of the graded bundle then appear as small eigenvalues of the endomorphism Dolbeault Laplacian. We ask how those eigenvalues depend on the distances to the stability walls, especially when several distances tend to zero at different rates.

Let X be a compact connected Kähler manifold of complex dimension $n \geq 2$. For a Kähler form ω we use

$$\deg_\omega F = \int_X c_1(F) \frac{\omega^{n-1}}{(n-1)!}, \quad \mu_\omega(F) = \frac{\deg_\omega F}{\text{rk } F}, \quad V_\omega = \int_X \frac{\omega^n}{n!}.$$

The Einstein constant is $\kappa_\omega(F) = 2\pi\mu_\omega(F)/V_\omega$. Our operator is $\Delta = \bar{\partial}_{\text{End}}^* \bar{\partial}_{\text{End}}$ on complex trace-free endomorphisms. Eigenvalues are counted with complex multiplicity. A scalar multiple of a bundle metric induces the same endomorphism metric, so the spectrum of a stable bundle with its Hermitian–Einstein metric is well defined once ω is fixed.

Fix an extension of holomorphic vector bundles

$$(1) \quad 0 \longrightarrow F_0 \longrightarrow E \longrightarrow \bigoplus_{j=1}^k F_j \longrightarrow 0.$$

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Assume that the F_i are pairwise nonisomorphic, stable of the same slope at ω_0 , and that every component of the extension class in $\text{Ext}^1(F_j, F_0)$ is nonzero. Write $r_i = \text{rk } F_i$ and $r = \sum_{i=0}^k r_i$. The extension graph has one central vertex and k leaves; we call (1) a star extension.

Let ω_s be a smooth family of Kähler forms, with s in a neighborhood of zero in $\mathbb{R}^d$. Put

$$(2) \quad \epsilon_j(s) = \kappa_{\omega_s}(F_j) - \kappa_{\omega_s}(E), \quad e(s) = \sum_{j=1}^k \epsilon_j(s).$$

These are slope surpluses of the quotient factors. Stability requires $\epsilon_j > 0$: the kernel of $E \rightarrow F_j$ must have smaller slope than E .

Theorem 1.1. *Under the preceding assumptions, for s sufficiently close to zero, E is stable with respect to ω_s if and only if $\epsilon_j(s) > 0$ for every j . In this region let $\lambda_1(s) \leq \lambda_2(s) \leq \dots$ be the endomorphism Dolbeault eigenvalues of its Hermitian–Einstein metric. Let $\nu_1(s) \leq \dots \leq \nu_k(s)$ be the eigenvalues of*

$$(3) \quad S(\epsilon) = \text{diag}(\epsilon_1, \dots, \epsilon_k) + \frac{1}{r_0} vv^*, \quad v_j = \sqrt{r_j \epsilon_j}.$$

There are constants $C, \gamma > 0$ such that

$$(4) \quad \left| \frac{\lambda_j(s)}{\nu_j(s)} - 1 \right| \leq C e(s) \quad (1 \leq j \leq k), \quad \lambda_{k+1}(s) \geq \gamma.$$

The constants are independent of the positive ratios ϵ_i/ϵ_j . No transversality of the parameter family to the walls is required.

The uniformity in (4) is a relative assertion for each small eigenvalue. An additive error controlled by the largest gap would not determine an eigenvalue associated with a much smaller gap. Here one may, for example, let one gap be exponentially small in the reciprocal of another. The constants may depend on the fixed bundles, extension, parameter family and number of factors.

The matrix in (3) is the weighted star-graph Laplacian written on the complement of its constant mode. Its Rayleigh quotient is

$$(5) \quad \frac{\sum_{j=1}^k r_j \epsilon_j |x_j - x_0|^2}{\sum_{i=0}^k r_i |x_i|^2}, \quad \sum_{i=0}^k r_i x_i = 0.$$

When all surpluses equal ϵ , its eigenvalues are ϵ with multiplicity $k - 1$ and $(r/r_0)\epsilon$ once. The harmonic norms of the extension classes do not occur in this expression.

The proof first solves the Hermitian–Einstein equation orthogonally to the block-scalar endomorphisms. If t_j scales the j th harmonic extension and $w_j = t_j^2$, the remaining equations have the form

$$V_s r_j \epsilon_j - w_j b_j(s, w) = 0, \quad b_j(s, 0) = \|\beta_{j,s}\|^2 > 0.$$

The factor w_j in its own equation is essential. On the face $t_j = 0$ the j th summand splits off, and its integrated curvature is fixed by its degree. Equivariance and smoothness therefore give exact divisibility by w_j , which yields a relative estimate for every extension weight without dividing by the smallest slope gap. A relative Schur estimate identifies the harmonic-family spectrum with the graph spectrum. A pointwise comparison of the corrected and split metrics then transfers the estimate to the actual Hermitian–Einstein metric.

For two factors write $\delta = \kappa_{\omega_s}(F_1) - \kappa_{\omega_s}(F_0)$. Theorem 4.1 refines the leading result to

$$\lambda_1 = \delta - D\delta^2 + o(\delta^2), \quad D \geq 0,$$

and gives a smooth eigensection expansion. The subbundle projection has Rayleigh quotient exactly δ at an HE metric. This identity fixes the first term without an unknown metric correction and makes the second term a Green-operator expression at the graded limit.

Relation to previous work. The Hermitian–Einstein correspondence supplies the metrics on the stable factors and identifies simple HE bundles with stable bundles [UY86, Kob87]. Clarke–Tipler [CT24] prove local chamber decomposition and convergence of HYM connections to the graded connection for simple bundles with locally free graded object and abelian graded automorphism group. Their Proposition 3.2 gives a parameter-dependent complementary implicit-function construction; their Theorem 1.2 gives convergence along paths in a stable chamber. The bundles considered here satisfy those hypotheses. Deloche [Del25] extends this continuity theory to balanced metrics without the abelian automorphism hypothesis. His Theorem 4 also gives a norm bound for the difference from the graded connection. We use a specialized construction to determine the small spectrum quantitatively.

Sektnan–Tipler [ST24] construct HYM connections on pullback bundles in adiabatic classes. Section 5.2 develops scaled harmonic extensions and balances their curvature against slope defects. Proposition 50 (Proposition 5.27 of the preprint) supplies a lower bound for the linearized operator; these are substantial precedents for the construction used here. Our estimates concern all small eigenvalues near an interior Kähler class and are relative uniformly in the independent positive wall distances.

The earlier spectral paper [Vue26a] gives relative graph comparison for harmonic families and quantitative transfer along HYM flow. Its two-factor harmonic family has expansion $ct^2 - dt^4 + O(t^6)$. The coefficient in our HE expansion is exactly $D = d/c^2$ after normalization. The new assertion concerns its occurrence for the actual equilibrium metrics, with the slope gap as parameter. For several factors, the principal result is the cancellation of extension norms from the leading HE spectrum, together with the uniform relative error across arbitrarily separated wall distances. The proof does not take a joint limit in polarization and flow time.

The toric companion [Vue26b] supplies tangent-bundle extensions of the form (1), including examples with arbitrarily many independent slope coordinates. Section 5 states the geometric inputs and applies the analytic theorem. The analytic proof does not use a finite toric census.

2. THE HERMITIAN–EINSTEIN EQUATION NEAR THE GRADED BUNDLE

We retain the hypotheses of Theorem 1.1. Norm estimates in this section are uniform after shrinking the parameter neighborhood. They do not compare $|s|$ with any positive slope surplus.

2.1. Factor metrics and harmonic extensions. Choose smoothly varying HE metrics $h_{i,s}$ on the fixed holomorphic bundles F_i , with scalar normalizations. Here is the local analytic justification. First choose normalized HE determinant metrics, using the scalar Laplace equation. With those determinants fixed, the linearization of the trace-free HE equation at $s = 0$ is the endomorphism Dolbeault Laplacian on trace-free Hermitian sections. It is invertible because F_i is stable. For an integer $\ell > n$, the equation is a smooth map from $H^{\ell+2}$ to H^ℓ . The implicit-function theorem and elliptic regularity give the metrics for small s . Their existence and simplicity also show that the factors remain stable.

Let $\beta_{j,s}$ be the harmonic representative of the fixed nonzero class in $H^{0,1}(\text{Hom}(F_j, F_0))$. These representatives vary smoothly. Indeed the holomorphic Hom bundle is fixed, so the cohomology dimension is constant; the spectral projection onto the kernel of its smoothly varying elliptic Laplacian is smooth. Elliptic regularity gives smoothness in every spatial

norm. Write

$$B_j(s) = \|\beta_{j,s}\|_s^2 > 0, \quad h_{\text{split},s} = \bigoplus_{i=0}^k h_{i,s}.$$

Each B_j has a uniform positive lower bound.

On the fixed smooth direct sum define

$$(6) \quad \bar{\partial}_{s,t} = \bar{\partial}_0 + \sum_{j=1}^k t_j \beta_{j,s}, \quad t \in \mathbb{R}^k,$$

where $\beta_{j,s}$ occupies the block $F_j \rightarrow F_0$. The operator is integrable in every complex dimension: $\bar{\partial}_0 \beta_{j,s} = 0$, and every product of two off-diagonal entries vanishes. When all $t_j \neq 0$, independent constant scalings of the quotient factors identify this extension with (1). Changes of harmonic representative are implemented by upper-triangular complex gauges. Its determinant holomorphic structure is independent of t .

2.2. The complementary equation. Let $\mathcal{H}$ be the real space of constant block-scalar Hermitian endomorphisms with total trace zero. It has dimension k . Let P_s be its L_s^2 projection and $Q_s = I - P_s$, acting on pointwise trace-free Hermitian sections. We seek a metric $h = h_{\text{split},s} e^u$, where u is a trace-free Hermitian endomorphism satisfying $P_s u = 0$. Express its mean curvature in the fixed Hermitian structure:

$$(7) \quad \widehat{K}(s, t, u) = e^{u/2} (i \Lambda_{\omega_s} F_{\bar{\partial}_{s,t}, h_{\text{split},s} e^u}) e^{-u/2}.$$

Because $\text{tr } u = 0$, its trace is the constant $r \kappa_{\omega_s}(E)$.

Lemma 2.1. *For small (s, t) there is a unique small normalized $u(s, t)$ solving*

$$Q_s(\widehat{K}(s, t, u) - \kappa_{\omega_s}(E)I) = 0.$$

It is smooth in the parameters and spatial variables, and for every m

$$(8) \quad \|u(s, t)\|_{C^m} \leq C_m |t|^2.$$

The construction is equivariant under independent constant unitary scalings of the factors.

Proof. Use the real Hermitian spaces $H^{\ell+2} \rightarrow H^\ell$, $\ell > n$. Block-diagonal positive changes of metric identify the Hermitian bundles for different s , preserve trace and block scalars, and are bounded with bounded inverses in every fixed Sobolev norm. The projections Q_s depend smoothly on s . After those identifications, $Q_s|_{\mathcal{H}^\perp}$ is an isomorphism onto $\mathcal{H}^\perp$ for small s . These maps give fixed Banach spaces for the parameter-dependent equation.

At $t = u = 0$, the split curvature $K_{0,s}$ is block scalar, with entries $\kappa_i(s) I_{F_i}$. Differentiation of the un-conjugated mean curvature in the convention $h = h_{\text{split},s} e^u$ gives $\Delta_{\partial,s} u$. Differentiating the conjugation in (7) adds $[u, K_{0,s}]/2$. The Kähler identity on endomorphism-valued zero-forms is $\Delta_{\partial} - \Delta_{\bar{\partial}} = [K_{0,s}, \cdot]$. Thus the real Hermitian linearization is

$$(9) \quad L_s = \Delta_{\bar{\partial},s} + \frac{1}{2}[K_{0,s}, \cdot] = \frac{1}{2}(\Delta_{\bar{\partial},s} + \Delta_{\partial,s}).$$

It is self-adjoint and nonnegative and annihilates $\mathcal{H}$. At $s = 0$ all factor Einstein constants agree, so $L_0 = \Delta_{\bar{\partial},0}$. Its trace-free Hermitian kernel is exactly $\mathcal{H}$: each factor has only scalar endomorphisms, and the Hom groups between distinct stable factors of equal slope vanish.

Consequently $Q_0 L_0$ is an isomorphism from the complementary $H^{\ell+2}$ space to H^ℓ . In fixed spaces, $Q_s L_s$ converges in this operator norm to $Q_0 L_0$. The Neumann-series criterion gives uniformly bounded inverses. The implicit-function theorem now produces $u(s, t)$, and uniqueness gives $u(s, 0) = 0$.

The derivative of the split mean curvature in t_j has upper off-diagonal entry $-\bar{\partial}_{\text{Hom}}^* \beta_{j,s}$ and lower entry its Hermitian adjoint. Both vanish by harmonicity for every s . Differentiating the complementary equation therefore gives $u_{t_j}(s, 0) = 0$, proving (8). Applying the IFT at higher Sobolev indices and using uniqueness, or bootstrapping the equation, gives each C^m estimate. All equations, projections and normalizations are invariant under the indicated unitary scalings; uniqueness proves equivariance. $\square$

2.3. Scalar equations on the coordinate faces. Set $p_j = I_{F_j} - (r_j/r)I$ for $1 \leq j \leq k$. These form a basis of $\mathcal{H}$. The remaining curvature equations are

$$f_j(s, t) = \langle p_j, \widehat{K}(s, t, u(s, t)) - \kappa_{\omega_s}(E)I \rangle_s = 0.$$

Changing the sign of any t_i is induced by a unitary sign on F_i , and fixes every p_j . Thus every f_j is separately even in the t_i .

We use the following elementary smoothness fact, jointly with the parameters. A smooth function $f(s, t)$ that is separately even in $t_1, \dots, t_k$ can be written $F(s, t_1^2, \dots, t_k^2)$ with F smooth on the closed positive orthant and smoothly extendible across its faces. In one variable, Taylor expansion has only even powers, and its differentiated remainders give derivatives of every order in $w = t^2$ up to $w = 0$. A smooth extension from the half-line then gives the assertion; apply this successively in each variable, retaining the other variables and s as parameters. If such an F vanishes on $w_j = 0$, the formula

$$F(s, w) = w_j \int_0^1 \partial_{w_j} F(s, w_1, \dots, \theta w_j, \dots, w_k) d\theta$$

gives smooth divisibility on the orthant. We write $w_j = t_j^2$ below.

Proposition 2.2. *The scalar equations have the form*

$$(10) \quad f_j(s, w) = V_s r_j \epsilon_j(s) - w_j b_j(s, w), \quad b_j(s, w) = B_j(s) + O\left(\sum_i w_i\right).$$

The functions b_j are smooth jointly in the parameters and orthant variables, and the estimates are uniform near the origin.

Proof. On the face $t_j = 0$, F_j splits holomorphically from the rest. Its circle of scalar unitary transformations fixes the Dolbeault operator, the background metric and the complementary equation. Lemma 2.1 implies that $u(s, t)$ commutes with this circle. The corrected metric therefore also splits off F_j . Conjugation by $e^{u/2}$ preserves the trace of that block. Since the total residual is pointwise trace-free, Chern–Weil gives exactly

$$(11) \quad f_j(s, t)|_{t_j=0} = 2\pi \deg_{\omega_s}(F_j) - V_s r_j \kappa_{\omega_s}(E) = V_s r_j \epsilon_j(s).$$

This equality is independent of all other extension parameters. The smoothness fact just stated gives the factorization in (10).

It remains to determine $b_j(s, 0)$. At the split metric the quadratic curvature term of the entry $t_j \beta_{j,s}$ is positive on F_0 and negative on F_j . Its pairing with p_j is $-t_j^2 \|\beta_{j,s}\|^2$. The diagonal terms from other edges pair to zero, as do their off-diagonal cross terms. The quadratic metric correction contributes $L_s u_2$, whose pairing with p_j vanishes because L_s is self-adjoint and $L_s p_j = 0$. Hence $b_j(s, 0) = B_j(s)$. Smoothness yields the remainder uniformly in s . $\square$

Proposition 2.3. *For small s with every $\epsilon_j(s) > 0$, there is a positive solution $w(s)$ of the scalar equations, and*

$$(12) \quad w_j(s) B_j(s) = V_s r_j \epsilon_j(s) (1 + O(e(s))) \quad (1 \leq j \leq k).$$

The estimates are uniform in the positive gap ratios. The corresponding corrected metric is an HE metric on E , and E is stable.

Proof. Extend the b_j smoothly to a full neighborhood of $w = 0$, preserving positivity after shrinking it. Introduce independent real targets η_j and solve

$$w_j b_j(s, w) = V_s r_j \eta_j.$$

The w -Jacobian at $w = \eta = 0$ is $\text{diag}(B_j(s))$, uniformly invertible. The parameter IFT gives $w(s, \eta) = O(|\eta|)$, with $w(s, 0) = 0$. For positive targets the equations themselves force each $w_j > 0$, so the solution lies where the extension agrees with the original functions. Set $\eta_j = \epsilon_j(s)$. Then

$$\frac{w_j B_j(s)}{V_s r_j \epsilon_j(s)} = \frac{B_j(s)}{b_j(s, w)} = 1 + O(e(s)).$$

No estimate divides an error by $\min_j \epsilon_j$.

Together with the complementary equation, vanishing of the scalar pairings against a basis of $\mathcal{H}$ gives the full HE equation. Take $t_j = \sqrt{w_j}$; all are nonzero, so the holomorphic bundle is E . To see simplicity, an endomorphism of E preserves F_0 because $\text{Hom}(F_0, F_j) = 0$. Its actions on F_0 and on each quotient F_j are scalar. Compatibility with each nonzero extension class forces all these scalars to agree. The possible remaining map from the quotient to F_0 is zero because $\text{Hom}(F_j, F_0) = 0$. Thus an HE metric makes E stable. Necessity of the positive surpluses follows from the kernels of $E \rightarrow F_j$, as in the introduction. $\square$

3. RELATIVE SPECTRAL ESTIMATES

We first use the metric $h_{\text{split},s}$ on the holomorphic bundle defined by (6). The comparison below is the harmonic-family argument of [Wue26a]; we give its proof to keep track of uniformity in both the polarization and the extension weights.

3.1. The harmonic family. Write D_0 for the split Dolbeault operator on trace-free endomorphisms and

$$D_t = D_0 + A_t, \quad A_t x = \left[\sum_j t_j \beta_{j,s}, x \right].$$

The complex space $\mathcal{H}_{\mathbb{C}} = \ker D_0$ consists of constant block-scalar endomorphisms. Write them as $x = \text{diag}(x_i I_{F_i})$, where $\sum_i r_i x_i = 0$. Let P be the L_s^2 projection onto this space and $Q = I - P$. The spectral gap of $D_0^* D_0$ on its complement is uniformly positive for small s . Since the harmonic forms vary smoothly, $a_t = \|A_t\|_{L^2 \rightarrow L^2} \leq C|t|$.

For $x \in \mathcal{H}_{\mathbb{C}}$, harmonicity gives $D_0^* A_t x = 0$. Distinct matrix entries are orthogonal, so

$$(13) \quad G_t(x) := \|D_t x\|^2 = \sum_{j=1}^k w_j B_j(s) |x_j - x_0|^2, \quad \|x\|^2 = V_s \sum_{i=0}^k r_i |x_i|^2.$$

No pointwise orthogonality among the forms themselves is required. Let $g_1(t) \leq \dots \leq g_k(t)$ be the eigenvalues of this quadratic form with the displayed mass inner product.

Lemma 3.1. *For small (s, t) with all $t_j \neq 0$, the first k eigenvalues of $\Delta_t = D_t^* D_t$ satisfy*

$$(14) \quad (1 - C|t|^2) g_j(t) \leq \lambda_j(h_{\text{split},s}, \bar{\partial}_{s,t}) \leq g_j(t).$$

All remaining eigenvalues have a uniform positive lower bound. The constant does not depend on any ratio of the nonzero t_j .

Proof. For $y \perp \mathcal{H}_\mathbb{C}$, $\|D_t y\| \geq \|D_0 y\| - a_t \|y\|$. Hence the compression $C_t = Q\Delta_t Q$ is bounded below by a constant $\gamma > 0$ for small parameters. For $x \in \mathcal{H}_\mathbb{C}$ the harmonic cancellation gives $\Delta_t x = A_t^* A_t x$. Its coupling to the complement, $F_t = Q\Delta_t P$, therefore satisfies

$$\|F_t x\|^2 \leq a_t^2 G_t(x).$$

For $0 \leq \lambda < \gamma/2$ this implies the quadratic-form bound

$$(15) \quad 0 \leq F_t^*(C_t - \lambda)^{-1} F_t \leq \frac{2a_t^2}{\gamma} G_t.$$

The k trial directions in $\mathcal{H}_\mathbb{C}$ give k eigenvalues $O(|t|^2)$ by min–max. The complementary subspace has codimension k , so the next eigenvalue is at least γ . Below $\gamma/2$, the Schur complement of $\Delta_t - \lambda$ is the finite operator

$$G_t - \lambda I - F_t^*(C_t - \lambda)^{-1} F_t$$

on $\mathcal{H}_\mathbb{C}$. Because $C_t - \lambda$ is positive, the number of negative eigenvalues of $\Delta_t - \lambda$ equals that of this Schur complement. Comparison with $(1 - 2a_t^2/\gamma)G_t - \lambda I$ and $G_t - \lambda I$ brackets every ordered eigenvalue and proves (14). The argument uses neither a lower bound on the smallest graph eigenvalue nor a separation between graph eigenvalues. $\square$

3.2. The actual metric. For the solution in Proposition 2.3, Lemma 2.1 gives the pointwise metric comparison

$$e^{-C|w|} h_{\text{split},s} \leq h_{\text{HE},s} \leq e^{C|w|} h_{\text{split},s}.$$

If $e^{-a}h_0 \leq h_1 \leq e^a h_0$, the induced squared norms on endomorphism-valued zero-forms and one-forms lie between e^{-2a} and e^{2a} times the original squared norms. The Dolbeault operator and base Kähler form are the same on both sides. The Rayleigh quotient, followed by min–max, therefore gives

$$(16) \quad e^{-4a} \lambda_j(h_0) \leq \lambda_j(h_1) \leq e^{4a} \lambda_j(h_0).$$

These inequalities hold on the same algebraically trace-free section space. They use no derivatives of the metric comparison.

Since $|w| = O(e(s))$, equations (12), (14) and (16) show that the first k HE eigenvalues agree, to relative error $O(e(s))$, with the graph form whose edge weights are $V_s r_j \epsilon_j$ and masses $V_s r_i$. Every weight comparison is relative, so it is also a relative comparison of the entire graph quadratic form. Min–max preserves that comparison for each ordered eigenvalue. The common volume cancels, leaving (5). The complementary spectral gap persists under the same metric comparison.

To obtain (3), set $y_j = x_j - x_0$. Eliminating x_0 by the trace constraint gives mass matrix and energy matrix

$$M = \text{diag}(r_1, \dots, r_k) - \frac{1}{r} (r_i r_j)_{i,j=1}^k, \quad D = \text{diag}(r_1 \epsilon_1, \dots, r_k \epsilon_k).$$

Direct multiplication gives

$$M^{-1} = \text{diag}(r_1^{-1}, \dots, r_k^{-1}) + \frac{1}{r_0} \mathbf{1}\mathbf{1}^*.$$

The generalized eigenvalues of (D, M) are those of $D^{1/2} M^{-1} D^{1/2}$, which is $S(\epsilon)$. This proves Theorem 1.1.

3.3. Consequences for separated gaps. The determinant lemma and the trace give

$$(17) \quad \prod_{j=1}^k \nu_j = \frac{r}{r_0} \prod_{j=1}^k \epsilon_j, \quad \sum_{j=1}^k \nu_j = \sum_{j=1}^k \left(1 + \frac{r_j}{r_0}\right) \epsilon_j.$$

Also, by Cauchy–Schwarz,

$$\text{diag}(\epsilon) \leq S(\epsilon) \leq \frac{r}{r_0} \text{diag}(\epsilon).$$

If $\epsilon_{(1)} \leq \dots \leq \epsilon_{(k)}$ are the ordered surpluses, then

$$(18) \quad \epsilon_{(j)} \leq \nu_j \leq \frac{r}{r_0} \epsilon_{(j)}.$$

Thus distinct orders of vanishing in the slope surpluses give the same ordered vanishing orders in the small HE spectrum. The matrix S determines the constants as well, including repeated scales.

For example, with two leaves and $\epsilon_1/\epsilon_2 \rightarrow 0$, the larger eigenvalue has leading term $(1 + r_2/r_0)\epsilon_2$. The determinant identity then gives

$$\nu_1 \sim \frac{r_0 + r_1 + r_2}{r_0 + r_2} \epsilon_1.$$

The larger edge effectively adds the second leaf's rank to the core mass before the smaller eigenvalue is formed. This is the elementary two-edge case of the graph contraction principle in [Wue26a].

4. THE SECOND-ORDER TERM FOR TWO FACTORS

Consider a nonsplit extension $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ with F, G nonisomorphic and stable of equal slope at ω_0 . Write $a = \text{rk } F$, $b = \text{rk } G$ and $r = a + b$. In the stable side put

$$\delta_s = \kappa_{\omega_s}(G) - \kappa_{\omega_s}(F) > 0, \quad N_s = V_s \frac{ab}{r}.$$

Theorem 1.1 gives $\lambda_1 \sim \delta_s$. We now identify the next coefficient for the actual HE metrics.

4.1. An exact identity at an HE metric. Use the orthogonal smooth splitting determined by any HE metric on E . Its Dolbeault operator has the form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_G \end{pmatrix}.$$

The diagonal metrics here are induced metrics; they need not be HE. Let P_F be the orthogonal projection onto F , and set $q = P_F - (a/r)I$. Then $\|q\|^2 = N_s$. The subbundle degree formula gives

$$(19) \quad \|\beta\|^2 = 2\pi a(\mu(E) - \mu(F)) = N_s \delta_s, \quad \frac{\langle q, \Delta q \rangle}{\|q\|^2} = \delta_s,$$

because $\bar{\partial}_{\text{End}} q = -\beta$. This Rayleigh upper bound is a consequence of the classical degree identity.

The off-diagonal HE equation and the Kähler identity give $\bar{\partial}_{\text{Hom}}^* \beta = 0$ for the induced connections. In a unitary coframe write $\beta = \sum_{\nu} \beta_{\nu} \bar{\theta}^{\nu}$ and put

$$(20) \quad C_s(\beta) = \text{diag} \left(\sum_{\nu} \beta_{\nu} \beta_{\nu}^*, - \sum_{\nu} \beta_{\nu}^* \beta_{\nu} \right).$$

Block multiplication in $\bar{\partial}_{\text{End}}^* \bar{\partial}_{\text{End}} q$ gives

$$(21) \quad \Delta q = C_s(\beta).$$

The term linear in β is $-\bar{\partial}_{\text{Hom}}^* \beta$ and vanishes; the remaining quadratic term is (20).

Let Q project onto $q^\perp$ in complex trace-free endomorphisms, and write $z = C_s(\beta) - \delta_s q$. Suppose the compression $A = Q\Delta Q$ has spectrum at least $\gamma > \delta_s$. For the unique small eigenvector, take its q component to be one. The complementary equation gives $v = -(A - \lambda)^{-1}z$. Pairing the remaining equation with q yields

$$(22) \quad \lambda = \delta_s - \frac{\langle z, (A - \lambda)^{-1}z \rangle}{N_s}.$$

In particular

$$\delta_s - \frac{\|z\|^2}{N_s(\gamma - \delta_s)} \leq \lambda \leq \delta_s.$$

4.2. The normalized extension limit. Let β_s be the harmonic representative for the split factor HE metrics, and $B_s = \|\beta_s\|^2$. The scalar equation in Section 2, now paired with q , reads

$$-N_s \delta_s + B_s t^2 + O(t^4) = 0.$$

The coefficient B_s is exact: the quadratic metric correction is in the image of L_s and pairs trivially with its parallel kernel vector q . Parity makes the scalar equation even, and its derivative in $w = t^2$ is $B_s + O(w) > 0$. Consequently

$$(23) \quad t_s^2 = \frac{N_s}{B_s} \delta_s + O(\delta_s^2).$$

These estimates are uniform in s , including flat approaches to $\delta = 0$.

We need an orthogonal identification that preserves the subbundle. The involution $J = \text{diag}(-I_F, I_G)$ sends t to $-t$; uniqueness gives $u(s, -t) = Ju(s, t)J$. Together with $u_t(s, 0) = 0$, this makes the diagonal entries even of order $O(t^2)$ and the off-diagonal entries odd of order $O(t^3)$. Write

$$e^u = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad T = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}.$$

Then

$$T^* e^u T = \text{diag}(A, D - B^* A^{-1} B).$$

Thus $S = T \text{diag}(A^{-1/2}, (D - B^* A^{-1} B)^{-1/2})$ is an isometry from the split metric to the corrected metric and preserves F . Its diagonal correction is $O(t^2)$ and its upper entry is $O(t^3)$ in every C^m norm. Including the derivative of S in $S^{-1} \bar{\partial}_{s,t} S$, the upper Dolbeault entry is therefore

$$\beta_{\text{HE},s,t} = t\beta_s + O_{C^m}(t^3).$$

The estimates at one higher spatial derivative justify this formula in C^m . The diagonal connections converge smoothly to the split wall connections. With (23) we obtain

$$(24) \quad \frac{\beta_{\text{HE},s}}{\sqrt{\delta_s}} \longrightarrow \beta_* := \sqrt{\frac{N_0}{B_0}} \beta_0 \quad \text{in every } C^m, \quad \|\beta_*\|^2 = N_0.$$

This convergence takes place in the rescaled presentation of the extension. On the original fixed E , the isomorphism to the scaled extension multiplies F by t ; the pulled-back metric on F is therefore $t^2 h_F$ to leading order. Smooth convergence to the split metric is not asserted in that original presentation.

4.3. The Green coefficient and eigensection. Let Δ_0 be the endomorphism Dolbeault Laplacian at the split wall metric, and G_0 its inverse on the complement of q_0 in trace-free endomorphisms. Define

$$(25) \quad z_0 = C_0(\beta_*) - q_0, \quad D = \frac{\langle z_0, G_0 z_0 \rangle}{N_0}.$$

The vector z_0 is orthogonal to q_0 , since $\langle q_0, C_0(\beta_*) \rangle = \|\beta_*\|^2 = N_0$.

Theorem 4.1. *As $s \rightarrow 0$ through $\delta_s > 0$,*

$$(26) \quad \lambda_1(s) = \delta_s - D\delta_s^2 + o(\delta_s^2).$$

Here $D \geq 0$, with equality precisely when $C_0(\beta_*) = q_0$ pointwise. After smooth unitary identifications with the wall spaces, the normalized first eigensection, with positive q_s component, is

$$(27) \quad \psi_s = \frac{q_s}{\sqrt{N_s}} - \delta_s \frac{G_0 z_0}{\sqrt{N_0}} + o_{C^m}(\delta_s) \quad \text{for every } m.$$

All other trace-free eigenvalues remain uniformly positive.

Proof. Use the orthogonal presentation above and an L^2 isometry U_s incorporating the varying volume density. We use q_s, Q_s, Δ_s for the transported endomorphism objects. Nonlinear expressions must be formed before this transport: in particular the transported $C_s(\beta)$ means $U_s C_s(\beta)$, not C_0 applied to a transported form. Density multiplication does not commute with the quadratic operation C .

All differential coefficients converge smoothly to those of Δ_0 . The latter has trace-free kernel $\mathbb{C}q_0$. Its complementary gap persists for $A_s = Q_s \Delta_s Q_s$. Indeed, a sequence of normalized complementary sections with energy tending to zero would have bounded H^1 norm by the first-order elliptic estimate, a strongly convergent L^2 subsequence, and a nonzero limit in $\ker \Delta_0 \cap q_0^\perp$, a contradiction. Min-max and (19) now give exactly one eigenvalue $\lambda_s < \gamma/2$ for small s , and positivity follows from simplicity of E .

Equations (21) and (24) imply

$$z_s = Q_s \Delta_s q_s = U_s C_s(\beta_{\text{HE},s}) - \delta_s q_s, \quad z_s / \delta_s \longrightarrow z_0$$

in every Sobolev norm. After smooth fixed-complement identifications, uniform elliptic estimates give

$$\|v\|_{H^{j+2}} \leq C_j (\|(A_s - \lambda_s)v\|_{H^j} + \|v\|_{L^2}).$$

The complementary gap absorbs the last term. The projections have finite rank with smooth kernels and are uniformly bounded on every Sobolev space. Hence

$$\|(A_s - \lambda_s)^{-1}\|_{H^j \rightarrow H^{j+2}} \leq C_j.$$

The resolvent identity and coefficient convergence from H^{j+2} to H^j show that these inverses converge to G_0 in the stated operator norm. Substituting into (22) proves (26).

Before normalization the small eigensection is $q_s + v_s$, where

$$v_s = -(A_s - \lambda_s)^{-1} z_s = -\delta_s G_0 z_0 + o_{H^{j+2}}(\delta_s).$$

Its squared norm is $N_s + \|v_s\|^2$. Normalization therefore changes the q_s coefficient only by $O(\delta_s^2)$. Sobolev embedding, with arbitrary j , gives (27); the phase is fixed by $\langle q_s, \psi_s \rangle_s > 0$. Positivity of G_0 proves the assertion about D and its zero set. $\square$

The coefficient is unchanged by nonzero scalar rescaling of the extension representative, after the induced unitary identifications. It depends on the wall geometry and the projective extension class. Ranks and Chern classes alone do not determine it. To compare with [Wue26a], write $c = B_0/N_0$ and $z = C_0(\beta_0) - cq_0$. Its fixed harmonic-family coefficient is $d = \langle z, G_0 z \rangle / N_0$. Since $z_0 = z/c$, we have exactly $D = d/c^2$. Theorem 4.1 identifies this coefficient for the actual HE metrics and the exact slope-gap parameter.

5. TORIC TANGENT BUNDLES

The geometric inputs in this section are proved in the companion [Wue26b]. We state them explicitly to distinguish the geometry needed for Theorem 1.1 from the analytic conclusion.

For $k \geq 1$ let

$$X_k = \mathbb{P}_{(\mathbb{P}^1)^{2k-1}}(\mathcal{O} \oplus \mathcal{O}(1^k, -1^{k-1})),$$

using quotient projectivization. The twisting line bundle has degree 1 on the first k base factors and degree -1 on the remaining $k-1$. Let $\pi : X_k \rightarrow (\mathbb{P}^1)^k$ be the projection onto the first k base factors, with components π_j . This is a smooth toric Fano variety of dimension and Picard number $2k$. Put $D = -K_{X_k}$ and $h_j = c_1(\pi_j^* \mathcal{O}_{\mathbb{P}^1}(1))$. The tangent bundle fits into

$$(28) \quad 0 \longrightarrow F_0 := T_{X_k/(\mathbb{P}^1)^k} \longrightarrow T_{X_k} \longrightarrow \bigoplus_{j=1}^k \pi_j^* \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0.$$

At D , F_0 is stable of rank k , the quotient line bundles are stable of the same slope, and all factors are pairwise nonisomorphic. Every component of the extension class is nonzero. The functions

$$z_j(H) = \left(2h_j - \frac{D}{2k}\right) H^{2k-1}$$

extend to independent real analytic local coordinates at D . The local stable locus is $z_j > 0$ for every j .

Choose smooth Kähler representatives of the nearby ample classes. The slopes in [Wue26b] are intersection numbers without a factorial denominator. In the present normalization,

$$(29) \quad \epsilon_j(H) = \frac{2\pi z_j(H)}{(2k-1)!V_H} = \frac{4\pi k z_j(H)}{H^{2k}}.$$

At the wall the differential of this map is a common positive multiple of dz_j , so the ϵ_j are also independent local coordinates after completion by k further coordinates.

Corollary 5.1. *For the HE tangent bundle of X_k in its nearby stable chamber, there are exactly k small trace-free Dolbeault eigenvalues. To relative error $O(\sum_j \epsilon_j)$ they are the eigenvalues of*

$$\text{diag}(\epsilon_1, \dots, \epsilon_k) + \frac{1}{k} vv^*, \quad v_j = \sqrt{\epsilon_j}.$$

Along a path with every $\epsilon_j = \epsilon$, the leading spectrum is ϵ with multiplicity $k-1$ and 2ϵ once. Positive paths with arbitrary distinct orders of vanishing of the ϵ_j are realized by genuine nearby polarizations.

Proof. Apply Theorem 1.1 with $r_0 = k$ and $r_j = 1$. For equal surpluses the rank-one perturbation has eigenvalue ϵ on the complement of $(1, \dots, 1)$ and 2ϵ on its span. Independent local coordinates realize the indicated positive paths while staying in the ample cone. $\square$

Example 5.2. For X_2 , write the positive surpluses as ϵ_1, ϵ_2 . The two comparison eigenvalues are

$$\nu_{\pm} = \frac{3}{4}(\epsilon_1 + \epsilon_2) \pm \frac{1}{4}\sqrt{9(\epsilon_1 - \epsilon_2)^2 + 4\epsilon_1\epsilon_2}.$$

If $\epsilon_1/\epsilon_2 \rightarrow 0$, the actual HE eigenvalues satisfy $\lambda_1 \sim \frac{4}{3}\epsilon_1$ and $\lambda_2 \sim \frac{3}{2}\epsilon_2$. The relative theorem applies equally when $\epsilon_1 = e^{-1/\epsilon_2^2}$.

Example 5.3. For X_3 choose $\epsilon_1 = s^6$, $\epsilon_2 = s^4$ and $\epsilon_3 = s^2$, $s > 0$. The characteristic polynomial of the comparison matrix is

$$\lambda^3 - \frac{4}{3}(\epsilon_1 + \epsilon_2 + \epsilon_3)\lambda^2 + \frac{5}{3}(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3)\lambda - 2\epsilon_1\epsilon_2\epsilon_3.$$

Its largest root is asymptotic to $\frac{4}{3}s^2$. The second elementary symmetric function then gives the middle root $\frac{5}{4}s^4$, and the determinant gives the smallest root $\frac{6}{5}s^6$. Consequently the actual tangent-bundle spectrum satisfies

$$\lambda_1 \sim \frac{6}{5}s^6, \quad \lambda_2 \sim \frac{5}{4}s^4, \quad \lambda_3 \sim \frac{4}{3}s^2.$$

These are changes of polarization on one fixed six-dimensional variety.

Theorem 4.1 and equation (16) of [Wue26b] supply further examples from the strictly semistable smooth toric Fano census in dimensions three through six, overlapping this family at X_2 and X_3 . All 31 indecomposable cases give star extensions of distinct locally free stable factors, with nonzero components. Twenty-six have two factors and hence also satisfy Theorem 4.1. The five cases with more factors have ranks

$$(2, 1, 1), (3, 1, 1), (4, 1, 1), (4, 1, 1), (3, 1, 1, 1).$$

In dimension n , for a quotient of rank r_j with first Chern class C_j , the general conversion is

$$\epsilon_j(H) = \frac{2\pi}{r_j(n-1)!V_H} \left(C_j - \frac{r_j}{n}c_1(T_X) \right) H^{n-1}.$$

Computing the two-factor coefficient D would additionally require the wall Green operator and normalized harmonic extension; the toric degree data do not provide those analytic quantities.

6. SCOPE AND FURTHER QUESTIONS

The star structure is responsible for the disappearance of harmonic norms from the leading spectrum. It gives one positive scaling variable for each independent scalar equation, and a split summand on every coordinate face. For general two-step extension graphs with cycles, the number of edge weights can exceed the number of independent moment equations; the ranks and slope surpluses then need not determine all leading weights. Repeated stable factors also replace block scalars by noncommutative endomorphism algebras. Neither setting is included in Theorem 1.1.

The estimates concern equilibrium metrics for varying Kähler forms. Relating them to the long-time HYM flow as the polarization also approaches a wall requires a uniform two-parameter analysis. The inverse-time behavior on the wall and the positive equilibrium gap inside the chamber do not by themselves justify interchanging those limits.

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