SPIN STRUCTURES AND LOCAL GROMOV–WITTEN INVARIANTS OF CURVES

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Abstract. We compute the local Gromov–Witten invariants of a curve of genus at least two with positive maximal flat symplectic normal holonomy, in every covering degree and source genus. The orthogonal bundle of the harmonic reduction determines which partitions occur. When it is orientable, the series agrees with the antidiagonal local-curve series of Bryan and Pandharipande. When it is nonorientable, only self-conjugate partitions contribute. The determinant sign on the embedded curve multiplies the degree variable in this sum. The restriction on partitions comes from the change of orientation at the relative contacts when two spin structures are glued across a nonseparating node. On the closure of the locus of triple covers with four simple branch points over a genus-two curve, we compute the intersection matrix of the ten classes measuring the difference between the real spin Euler class and the corresponding complex Chern class, one for each even spin structure.

1. The local contribution of a curve

How much does the normal operator of an embedded curve tell us about the contribution of its multiple covers? Even if every cover is isolated in the normal directions, counting the unramified covers with their determinant signs gives only the first possible source genus in each degree. Higher genera allow branching, nodes and contracted components. We compute their combined contribution for a curve with positive maximal flat symplectic normal holonomy. Already in degree two, changing one of two identical spin structures can make the entire disconnected contribution vanish. The reason appears when the target is cut at a nonseparating node. Reversing one spin gluing tensors the boundary character with the sign representation. It exchanges a Young diagram with its transpose, so only self-conjugate diagrams contribute when the two ends are glued back together.

Let C be a smooth projective complex curve of genus $g \geq 2$, and let $\rho : \pi_1(C) \to \mathrm{Sp}(4, \mathbb{R})$ be the holonomy of a flat symplectic four-plane bundle. We use the convention that ρ is *positive maximal* if its compatible complex rank-two bundle has degree $2g - 2$. In particular the total space has first Chern number zero on its zero section. The harmonic reduction gives a holomorphic bundle E and a nondegenerate symmetric pairing $\gamma : E \otimes E \to K_C$. Choose a theta characteristic a, that is, a holomorphic square root of K_C. Untwisting the pairing gives the complex orthogonal bundle $F = E \otimes a^{-1}$, the underlying bundle of the Cayley correspondence of Bradlow, García-Prada and Gothen [BGG12]. Its first Stiefel–Whitney class $w_1(F)$ measures whether its transition functions can be chosen in $\mathrm{SO}(2, \mathbb{C})$. In that case its two isotropic lines give $F = M \oplus M^{-1}$; write $s = |\deg M|$. For $w_1(F) \neq 0$ monodromy

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interchanges the isotropic directions, so an antidiagonal action scaling the two lines separately need not exist.

A Hermitian metric turns the symmetric pairing into an antilinear map $R_\gamma : E \to \Lambda^{0,1}C \otimes E$ by taking the metric dual and conjugating the form factor. The positive normal operator is $D = \bar\partial_E + R_\gamma$. In a split model $E = a_1 \oplus a_2$ with $a_i^2 = K_C$, the pairing is the sum of the two square maps and a product metric gives two independent real spin operators. For $w_1(F) \neq 0$, every unmarked deformation type has a representative of this form with $a_1 \neq a_2$. Conversely, the direct sum of the two uniformizing spin lifts realizes each such pair. Write ϵ for the determinant sign of D on C, with orientation continued from the complex Dolbeault operator. On such a representative $\epsilon = (-1)^{h^0(a_1)+h^0(a_2)}$. Isolation, reduction to these representatives and invariance under normal rescaling are proved in [Wue26j, Lemma 14.1, Propositions 14.2 and 15.4, and Appendices D–E].

Let $\overline{\mathcal{M}}_k(C, m)$ be the proper stack of connected unmarked stable maps of arithmetic genus k and degree $m > 0$ to this fixed C. Over a map f, the antilinear term of D_f uses df, and sections on the branches of a node must agree. The kernel is zero on every nonconstant component by the spin energy identity. Matching then kills the constants on every contracted subcurve. Its oriented real cokernel V_D consequently has rank $4(k - 1 - m(g - 1))$, the real virtual dimension of the stack. The ordinary connected local invariant and its disconnected series are

$$L_{m,k} = \int_{[\overline{\mathcal{M}}_k(C,m)]^{\mathrm{vir}}} e(V_D), \qquad Z_D(Q, v) = \exp\left(\sum_{m\geq 1,\, k} L_{m,k} Q^m v^{2k-2} \right). \tag{1}$$

The sum starts at $k = 1 + m(g - 1)$ by Riemann–Hurwitz on the normalization. Every connected component counted by Z_D has positive degree. At the least genus each connected cover contributes its determinant sign divided by its deck order. Giving each cover the positive reciprocal of its deck order instead would recover the ordinary unramified Hurwitz number.

Bryan and Pandharipande computed the residue invariant of a direct sum of two holomorphic line bundles with opposite torus weights [BP08, Corollaries 7.2–7.3]. Their answer is a sum over all partitions of the covering degree. Our first formula identifies the real normal Euler integral with their residue invariant in the orientable case. For a nonorientable F, the sewing of the normal operators changes that sum.

For a box u in a Young diagram λ, write $h(u)$ for one plus the number of boxes strictly to its right or below it in its row or column. Put $a(\lambda) = \sum_{(i,j)\in\lambda}(j - i)$, the total content. With $t = e^{iv}$, let $X_j = 2 - t^j - t^{-j}$ and $b_0 = g - 1$. Thus $X_j = 4\sin^2(jv/2)$.

Theorem 1.1. *For the positive maximal normal operator above, in every degree and source genus the disconnected fixed-target series is*

$$Z_D(Q, v) = \begin{cases} \displaystyle\sum_{\lambda} (-1)^{s|\lambda|} Q^{|\lambda|} t^{sa(\lambda)} \prod_{u\in\lambda} X_{h(u)}^{b_0}, & w_1(F) = 0, \\ \displaystyle\sum_{\lambda=\lambda^\top} (\epsilon Q)^{|\lambda|} \prod_{u\in\lambda} X_{h(u)}^{b_0}, & w_1(F) \neq 0. \end{cases}$$

The empty diagram contributes one. The same series gives the local invariants of a marked horizontal section of a flat symplectic four-plane germ with this derivative holonomy and the split horizontal and compatible vertical almost-complex structures.

The obstruction to the second comparison occurs at the seam of a single handle. Pinch a nonseparating circle on C, and choose the two spin structures to agree on the normalization

and to differ in the identification of the two end fibers. For a relative map, the contact orders make a partition μ of m. The mixed spin Euler integral acquires the sign $\epsilon^m(-1)^{m-\ell(\mu)}$ relative to the square for a repeated spin. The second factor is the sign of a permutation of cycle type μ. In the character basis, tensoring by sign interchanges a partition and its transpose. Gluing the handle takes a trace. A partition contributes to that trace exactly when it is fixed by transposition. The orientation calculation has to include even contacts and contracted components to give this sign on the full relative virtual class.

Self-conjugate diagrams also occur in the local Real Gromov–Witten theory of Georgieva and Ionel [GI21, Theorem 1.1 and Corollary 9.14]. Their domain and target carry antiholomorphic involutions and the maps intertwine them. For their canonically oriented Real Calabi–Yau series $R_G(q,u)$ on a connected symmetric curve of genus G, the formal expression in the second line is precisely $R_{2g-1}((-1)^{g-1}\epsilon Q, iv)$. Indeed the crosscap sign in their formula is raised to the even power $2g-2$, and each hyperbolic sine acquires a factor of i. Our moduli in (1) are ordinary complex stable maps; the orientation change comes from the *normal spin matching* over one node. In particular, taking the logarithm here selects connected ordinary domains. In the Real theory the logarithm also counts pairs of components exchanged by the involution [GI21, (2.16)].

For a concrete consequence, put $x = X_1$. A balanced genus-j factor is the disconnected antidiagonal local series of a curve of genus j with both normal lines of degree $j-1$; denote it by $B_j(Q,v)$. Its logarithm starts with Qx^{j-1}. The *elementary multiplicities* $n_{d,j}$ of a compactly realized isolated germ are the integers in the factorization $Z_D = \prod_{d,j} B_j(Q^d, v)^{n_{d,j}}$, as justified by [Wue26j, Proposition 3.1]. The balanced local factor is the elementary curve of Ionel and Parker under the comparison of [Wue26b, §3]. These multiplicities record a genus of the elementary curve and differ from the integrals at a specified source genus.

Corollary 1.2. *For two distinct spin structures of the same parity on a genus-two target, the connected degree-three series is* $28x^3/3 - 6x^4 + x^5$. *Its nonzero degree-three elementary multiplicities in increasing genus are*

$$(n_{3,4}, n_{3,5}, n_{3,6}, n_{3,7}) = (-64, 64, -20, 2).$$

Proof. There is no self-conjugate partition of two. In degree three there is just $(2,1)$, with hooks $3,1,1$. Thus the logarithm gives $X_3x^2 + x^3/3$, with $X_3 = 9x - 6x^2 + x^3$. The only proper divisor of three is one. Subtracting the degree-three term of the balanced genus-two logarithm, $220x^3/3 - 70x^4 + 21x^5 - 2x^6$, gives the four stated integers. □

The degree-three answer is the same for any two distinct even spin structures on a given genus-two curve. It gives the integral of a product, summed over all stable maps. To recover the individual classes one must locate them on the moduli space. We identify the cokernel of a single spin operator with a real form of the residue space at the ramification points. This gives a signed divisor formula on the surface of covers with two simple branch points. On the four-dimensional closure of the cubic covers with four simple branch points the individual spin corrections have square 6 and mixed pairing $-1/3$ for any two of the ten distinct even spins (Theorem 7.3). Their common support is detected by the difference of two squared Szegő kernels. These intersections explain what the partition sum forgets about an individual square root of K_C.

2. Relative degeneration and the sign of a handle

We prove Theorem 1.1. Deformation of the metric and the normal rescaling theorem in [Wue26j, Proposition 15.4 and Appendices D–E] identify the fixed-target integral for any of

the stated germs with its value at a harmonic linear representative. At each fixed degree and source genus this continuation takes place over the compact stable-map stack. The vertical area identity on positive normalizations, followed by matching on contracted components, keeps the normal family injective.

2.1. **The orientable normal bundle.** In the splitting $E = L_+ \oplus L_-$, the two degrees are $b_0 + s$ and $b_0 - s$. Choose an orthogonal product metric. Let π and f denote the universal source and universal map over $\overline{\mathcal{M}}_k(C, m)$, and put $W_\pm = -R\pi_* f^* L_\pm$, $n = k - 1 - mb_0$. The antilinear term of D exchanges the lines; the family from Dolbeault to the positive operator is complex-linear in the complex structure $(i, -i)$. At its positive end the kernel is zero. At its Dolbeault end the orientation change on the second summand is $(-1)^{n+ms}$, since this is the parity of its Dolbeault index. Stabilization by forms supported away from the nodes continues the family index, including the complex equations at the nodes, across jumps in the individual cohomology bundles. Thus

$$e(V_D) = (-1)^{n+ms} c_{2n}(W_+ + \overline{W}_-). \tag{2}$$

The right side is the cohomological degree $2n$ term (in complex degrees) of the equivariant Euler product for weights $+1, -1$. All other terms integrate to zero. Bryan–Pandharipande's antidiagonal formula [BP08, Corollary 7.2] now gives the first line of the theorem. Exchanging the weight convention transposes all diagrams and preserves the sum.

2.2. **Two smoothings of the spin pair.** Pinch a nonseparating circle on C; denote the normalization of the central target by B and its ordered inverse images of the node by p, q. For each ϵ we need one smoothing of a pair of spins whose ratio has nontrivial holonomy around the loop of the dual graph. The unmarked component classification in [Wue26j, Appendix E] then applies to every pair of that sign. For $\epsilon = 1$ use the exceptional spin at an index-two node. The two spin structures restrict to the same theta characteristic on B and differ by a sign in sewing at q. Over an exceptional rational component the spin has degree one, so its sections interpolate the two end values uniquely. Both extensions thus have the same parity. For $\epsilon = -1$, take the two locally free extensions of $A^2 = K_B(p + q)$ at an ordinary node. Riemann–Roch gives $h^0(A) - h^0(A(-p - q)) = 1$; the line of end values is isotropic for the sum of squared residues, by the residue theorem. Hence exactly one of the two isotropic clutch lines admits the extra section. The two parities differ. The roots smooth after a local base change, by smoothness of the relative Picard space at a line and étaleness of squaring; parity is preserved in spin smoothing [Cor89, §6]. In both cases their ratio has trivial inertia and its central double torsor is connected by the nontrivial clutch. This persists under smoothing. See Calculation D1 of [Wue26g] for the square frames. Use $d = 2$ for the exceptional spin and $d = 1$ for the locally free spin.

After $\delta = z^2$, resolve the local smoothing $UV = \delta$ before taking the stabilizer quotient. The two central components are the d-th root of B at p, q and the two-ended rational component $[\mathbb{P}^1/\mu_d]$ with its rotation action. Both have coarse covering degree m, by intersection with the exceptional component in the surface. We use the simultaneous relative expansions at p, q of [AF16, §§4.4–4.7]. In a frame whose square is the logarithmic differential, take the antilinear term proportional to $U\, d\bar{U} - V\, d\bar{V}$ on the neck. Its coefficient relative to the conjugate logarithmic differential is $|U|^2 + |V|^2$ on a smooth fiber and can be joined by positive interpolation to the metric off the neck. Pullback by contraction makes it zero on the inserted components.

Stokes' identity on a main normalization gives nonnegative interior energy and opposite end residues weighted by the coarse contact orders; orbifold integration cancels the uniformizer

index. The square of a holomorphic solution on an inserted component is constant. Conservation of contact order thus cancels the end residues on the glued source. The solution vanishes on each nonconstant main component, then on all components by matching. In particular contracted subcurves of arbitrarily high genus do not create a kernel. Stabilization with forms off the nodes gives an oriented cokernel bundle on the plumbing charts. The antilinear term is the pullback of a bounded ambient coefficient on each half-neck, so the nodal estimates of [BS24, Appendix B] apply with this term. The family Euler product restricts under virtual Gysin to the paired Euler insertion on the central splitting. The expanded family and virtual restriction are described in Calculations D6–D7 of [Wue26g].

2.3. **Eliminating the two-ended component.** On a connected exceptional relative source of arithmetic genus j let S denote the pulled-back sign line. Its space of constants is a line or zero according as its double torsor is trivial or connected; for $d = 1$ it is always a line. These alternatives are locally constant on the relative stack. The normalization sequence gives the two real cokernels a common complex quotient $A = R^1\pi_*S$. Mumford's Hodge relation on the coarse double torsor, divided by the same relation on its invariant summand, gives $c(A)c(A^\vee) = 1$. In positive rank $c_{\mathrm{top}}(A)^2 = 0$, so the paired insertion vanishes. In rank zero eliminate the surjective exceptional Dolbeault equations. The remaining equations involve the main operator, its evaluations at the ends and one constant for each trivial-torsor vertex. Pass to the finite étale cover framing the invariant end lines by square-compatible frames. The residual choices are finite signs, and on every such sector the insertion is pulled back from the main relative space. Finite transfer thus reduces the exceptional integral to its virtual class paired against 1.

The log tangent of the two-ended target is trivial. A connected positive-degree exceptional source with a ends meets both colors, so its virtual dimension $2j - 2 + a$ is nonnegative. Integration against 1 vanishes in positive dimension. Only genus zero with two ends survives. Stability and Riemann–Hurwitz then leave the power map of degree k, branched only at the two ends; unbranched rubber levels give the same relative map. The two profiles agree, say $\mu = (k_1, \dots, k_r)$. The following multiplicity allows repeated parts as well as even contacts.

Lemma 2.1. *Suppose the surviving maps over the two-ended component are power cylinders with coarse contact profile $\mu = (k_i)$ and the insertion is pulled back from the main relative factor. For one prescribed matching of the separately labelled ends the weight is $\prod_i k_i$. Against the main integral with both lists unlabelled it is $z_\mu = |\operatorname{Aut}\mu| \prod_i k_i$. This holds for target index $d = 1$ and $d = 2$.*

Proof. At one flag of coarse order k put $a = (d, k)$, $f = d/a$ and $c = k/a$. The primitive representable source index is f; in a uniformizer the contact has order c. The inverse-band matching diagonal of the rigidified evaluation sector $B(\mu_d/\mu_f)$ has degree a: its fiber is the torsor of isomorphisms between two given evaluations. Choose one auxiliary twist R divisible by every c. The target index becomes dR, the source index dR/k, and the map of rigidified sectors to the primitive sector is a μ_c-gerbe. Comparison of the diagonals contributes c per flag. The single target gluing gerbe of degree $1/R$ and the central multiplicity R cancel on the common expansion base. Thus the flag factor is $ac = k$ [AF16, §§5.5–5.10, Lemma 5.15 and Proposition 5.18].

With $N = dR$, the cylinder is the index-k rotation quotient $[\mathbb{P}^1/\mu_{N/k}] \to [\mathbb{P}^1/\mu_N]$. The simply connected atlas gives a unique connected cover for this subgroup. Its log differential is an isomorphism, and its deck group has order k. The two matching factors and its virtual degree give $k^2/k = k$. In particular for $d = 2$ and k even the two relative spin clutches each

carry weight $k/2$: the deck action changes both primitive signs together, and the two resulting orbits have stabilizer of order $k/2$. The pullback assumption gives the same insertion on these two orbits.

Labelling equal cylinders introduces the divisor $A = |\operatorname{Aut}\mu|$ once, simultaneously on both ends. The unlabelled main integral divides the separately labelled integral by A^2. Their ratio gives $A\prod_i k_i$, as asserted. □

2.4. **The orientation at a contact.** In the primitive chart the sign line has an invariant matching fiber at order k_i exactly when $d \mid k_i$. Write $I = \{i : d \mid k_i\}$. On a surviving cylinder in I, elimination of its constant at the first end leaves the complex equation $q_i - \sigma_i p_i = 0$, where p_i, q_i are the end values in square frames and $\sigma_i = \pm 1$. Changing the target clutch negates each σ_i. On the cut main source Stokes' boundary inequality, after a common phase rotation of the frames, is

$$\sum_{i\in I} k_i \operatorname{Re}(p_i^2 - q_i^2) \geq 0$$

with equality only for the zero solution of the main equations. For $0 \leq u \leq 1$ impose instead

$$\operatorname{Re} p_i - u\sigma_i \operatorname{Re} q_i = 0, \qquad \operatorname{Im} q_i - u\sigma_i \operatorname{Im} p_i = 0.$$

Each term in the boundary sum then equals $k_i(u^2-1)((\operatorname{Re} q_i)^2+(\operatorname{Im} p_i)^2) \leq 0$. The augmented operators remain injective throughout. Their endpoint at $u = 0$ is independent of the clutch. At $u = 1$ the transformation from the original complex row to the new two real rows has determinant $-\sigma_i$. Thus each changed invariant matching fiber reverses the orientation once. For $d = 2$ the number of even parts has parity $m - r$; for $d = 1$ all r parts contribute. The mixed insertion equals the repeated square multiplied by

$$(-1)^{|I|} = \epsilon^m(-1)^{m-r}. \tag{3}$$

Before this comparison one must keep the two spin matchings at an even contact separate. After it the sign depends only on μ, so finite transfer preserves the formula.

2.5. **The repeated square and the trace.** Let H denote the spin pulled back to the closed source after the cylinders have been attached, and $W_H = -R\pi_*H$. The square of the Euler class of an oriented real $2n$-plane bundle is its top Pontryagin class. Continuation to Dolbeault gives

$$e(V_H)^2 = (-1)^n c_{2n}(W_H + \overline{W}_H).$$

We compare this square with an ordinary index on the coarse source. Choose a locally free spin L on the ordinary nodal target. Pull it back to the twisted target. Its square agrees with that of H in logarithmic frames, so $F' = HL^{-1}$ on the target is a sign line. There $[H] - [L] = [F'] - 1$ in the perfect K-group: express L by divisors off the node and tensor the resulting point sheaves by F'. Pull this identity back along the universal map before pushing it down to the parameter space. The zeroth cohomologies of the sign and trivial lines have finite monodromy; the positive even Chern characters of their first cohomologies vanish by the Hodge relation on the coarse double torsor and its invariant summand. Conjugation cancels the odd Chern characters. Consequently $c_{2n}(W_H + \overline{W}_H) = c_{2n}(W_L + \overline{W}_L)$ rationally, including on the twisted sectors. On normalizing the r power cylinders the two complex rows at each cylinder cancel its single constant, leaving $W_L = W_B + \mathcal{O}^r$ after framing. Here W_B is the ordinary relative index on the main coarse source. It descends with the relative virtual class by [ACW17, Theorem 2.1] and [AMW14, Theorem 1.1.2].

For equivariant weight s_0, the inverse antidiagonal pairing on a profile of length r is $z_\mu(-s_0^2)^r$; the required degree of the equivariant Euler product of W_B contributes $(-1)^{n-r}s_0^{-2r}c_{2n}(W_B +$

$\overline{W}_B$). Their product is z_μ times the repeated square. The factor z_μ agrees with Lemma 2.1 on unlabelled relative maps. This establishes the normalization and the orientation for the full contraction, including both clutch types. The arithmetic genus of the disconnected cut source, defined by $1-\chi(\mathcal{O})$, is $k-r$. The relative shift contributes $2r$ to its genus exponent, restoring $2k-2$.

Let A_m be the balanced two-point operator of genus $g-1$ and levels (b_0,b_0) in the antidiagonal relative theory. The degree-m coefficient of Z_D is now $\epsilon^m \operatorname{Tr}(\Omega A_m)$, where $\Omega e_\mu = (-1)^{m-\ell(\mu)} e_\mu$ in the relative basis indexed by contact profiles. In the idempotent indexed by λ, the coefficient of e_μ is proportional to $\dim(\lambda)\chi^\lambda(\mu)$, with the remaining factor depending only on m, μ, s_0. Tensoring the character with sign therefore gives $\Omega v_\lambda = v_{\lambda^\top}$, with unit coefficient. The eigenvalue of A_m is $\prod_{u\in\lambda} X_{h(u)}^{b_0}$ by the handle and level eigenvalues in [BP08, §7]. Taking the trace proves the second line of Theorem 1.1.

3. The largest genus in the elementary factorization

The source-genus expansion of a fixed positive-degree term of (1) generally has infinitely many coefficients. Its elementary multiplicities have finite support. The last nonzero genus can be read from the two extreme diagrams in the antidiagonal formula. Write $n_{m,j}$ for the multiplicities defined after Theorem 1.1, using the compact realization and fixed-target comparison of [Wue26j, Proposition 3.1 and its proof].

Theorem 3.1. *Suppose the hypotheses of Theorem 1.1 hold. If $w_1(F)=0$ and $s>0$, the largest j with $n_{m,j}\neq 0$ is $1+D_m$ in each degree $m\geq 1$, where*

$$D_m = mb_0 + (b_0+s)\binom{m}{2}, \qquad n_{m,1+D_m} = (-1)^{m(b_0+s)+D_m}.$$

For $w_1(F)=0$ and $s=0$, the only elementary multiplicity is $n_{1,g}=1$. For $w_1(F)\neq 0$, one has $n_{1,g}=\epsilon$; for $m\geq 2$ the largest genus is $1+D_m$, with

$$D_m = b_0\binom{m+1}{2}, \qquad n_{m,1+D_m} = -2\epsilon(-1)^{mb_0+D_m}.$$

Proof. Use Laurent degree in t, so that X_i has degree i. The sum of the hooks of $\lambda\vdash m$ is m plus the number of pairs of boxes in a common row or column. It is at most $\binom{m+1}{2}$, with equality only for a single row or column. The total content is at most $\binom{m}{2}$, attained only by the row. For $s>0$ the row alone therefore contributes the top term of degree D_m, with coefficient $(-1)^{m(b_0+s)}$. When $s=0$, Theorem 1.1 gives $Z_D=B_g$.

For $w_1(F)\neq 0$ and $m>1$ neither extreme diagram is self-conjugate. Thus the numerator in degree m, and hence the degree-m term in $\log Z_D$, has Laurent degree strictly less than D_m: in passing to the logarithm, products from smaller degrees are strictly below the bound by strict superadditivity. For the same reason the top term in the balanced genus-g logarithm is contributed by the row and column alone, with coefficient $2(-1)^{mb_0}$. It enters the elementary subtraction for $d=1$ with multiplicity ϵ.

It remains to eliminate proper divisors $m=de$ with $d,e>1$. The degree-e coefficient in $\log B_j$ has Laurent degree at most $(j-1)\binom{e+1}{2}$. Induct on m. In the nonorientable case the smaller-degree contributions are strictly below D_m because $\binom{d+1}{2}\binom{e+1}{2} < \binom{de+1}{2}$. In the orientable case the corresponding difference is

$$D_m - D_d\binom{e+1}{2} = \frac{de(e-1)}{4}\big((b_0+s)(d+1)-2b_0\big) > 0.$$

For $m > 1$, the degree-one balanced logarithm in the orientable subtraction also falls below D_m, since $s > 0$. What remains in degree m is $\sum_j n_{m,j} X_1^{j-1}$. Since the top Laurent coefficient of X_1^D is $(-1)^D$, the two leading coefficients give the asserted signs. □

The orientable formula also recovers s at degree two. The two diagrams are transposes, so the disconnected coefficient is $2\cos(sv)(X_1X_2)^{b_0}$; its first term after the leading one distinguishes every $s \geq 0$. At zero mass the bound for $s > 0$ is attained by the cyclic spectral curve $y^m = \sigma$ in $L_+ = aM$, taking the labelling with $\deg M = s$. For $m \geq 2$ choose a section σ of L_+^m with simple zeros ($\deg L_+^m \geq 2g$); total ramification over these zeros gives genus $1 + D_m$. For a nonorientable genus-two target with positive sign, expansion of Corollary 1.2 in v, in place of X_1, gives the connected invariants $28/3, -25/3, 589/180, -61/81$ in source genera four through seven.

4. The Euler class of a single spin operator

A single spin line yields a real Euler class before pairing the two normal directions. This class retains local collision data which disappear from the closed partition function. We compute it first on the finite-map component and then include the stable maps with a contracted elliptic tail. Let C be a smooth closed complex curve of genus $g \geq 2$, and let θ be a theta characteristic on C, a holomorphic line with an isomorphism $\theta^2 \simeq K_C$. If $f : Y \to C$ is a holomorphic cover, its pullback spin operator on $L = f^*\theta$ is the real Cauchy–Riemann operator obtained by adding the metric conjugate of $df : L \to K_Y L^{-1}$ to $\bar\partial_L$. Lee and Parker [LP12] constructed its real obstruction bundle on the space of stable maps. Its restriction to unramified covers has rank zero, and the orientation of a regular cover is its spin parity. Ramification produces a bundle of positive rank. We determine its Euler class on the first Hurwitz spaces on which this occurs.

Write $H_m = 1 + m(g-1)$. In degree $m \geq 2$ and source genus $H_m + 1$, the connected unmarked finite maps form a compact smooth stack surface $\mathcal{F}_m$. Its general point is a cover with two simple ramification points. For a connected boundary divisor B of nodal covers put $s_\theta(B) = (-1)^{h^0(\widetilde{f}^*\theta)}$, where $\widetilde{f}$ is the normalization of a member. For a divisor of covers with a single triple ramification point p, put $s_\theta(B) = (-1)^{h^0(f^*\theta(p))}$. Both are parities of theta characteristics. Our divisor formula is

$$e(\operatorname{coker} D_\theta) = -\frac{1}{4}\sum_B s_\theta(B)[B] \quad \text{in } H^2(\mathcal{F}_m;\mathbb{Q}). \tag{4}$$

Here B runs over the reduced divisors of the two indicated types. At two simple ramification points over the same value the obstruction has rational Euler class zero. The equality holds before integrating against another class.

4.1. The spin obstruction and the residue pairing.

Let $f : Y \to C$ be a finite map of smooth compact curves and fix an isomorphism $\theta^2 \simeq K_C$. Put $L = f^*\theta$ and $P = K_Y L^{-1}$. The differential of f gives a holomorphic map $\sigma : L \to P$ whose divisor is the ramification divisor R. Write $\deg R = 2r$. A Hermitian metric on L gives a conjugate-linear identification $c : P \to \Lambda^{0,1}T^*Y \otimes L$. The real spin operator and the complex space of ramification residues are

$$D_\theta = \bar\partial_L + c\sigma, \qquad \mathcal{E}_f = H^0(Y, P/\sigma L). \tag{5}$$

The identity $\|D_\theta u\|^2 = \|\bar\partial u\|^2 + \|c\sigma u\|^2$ follows from Stokes' theorem applied to σu^2. In particular D_θ is injective and its cokernel V_θ has real rank $2r$. We orient it by continuing

the complex determinant orientation along $\bar\partial_L + tc\sigma$, $0 \le t \le 1$, with the dual determinant convention for a cokernel [LP12, §12].

Theorem 4.1. *Over a family of finite maps with smooth domains and fixed total ramification $2r$, the complexification of V_θ identifies with the complex orthogonal bundle $\mathcal{E}$ whose form is*

$$b(v, v') = \sum_{p \in R_{\mathrm{red}}} \operatorname{Res}_p \frac{vv'}{\sigma}. \tag{6}$$

More precisely, up to the universal normalization of the Serre pairing, V_θ identifies with a positive real form of $\mathcal{E}$. At each map the image $S = H^0(P)/\sigma H^0(L)$ is a maximal isotropic subspace. Projection from V_θ to $\mathcal{E}/S = S^*$ is a real isomorphism of orientation sign $(-1)^{h^0(L)}$.

Proof. Resolve $[L \xrightarrow{\sigma} P]$ by its Dolbeault complex. In degrees zero, one and two the terms are $\Omega^0(L)$, $\Omega^{0,1}(L) \oplus \Omega^0(P)$ and $\Omega^{0,1}(P)$. The differentials, written in local coefficients, are

$$d_0 u = (\bar\partial u, \sigma u), \qquad d_1(\beta, v) = \bar\partial_P v - \sigma\beta.$$

The sheaf map is injective and its cokernel has finite support. Hence the middle cohomology is $\mathcal{E}_f$ and all other cohomology vanishes. On the middle term take the symmetric bilinear pairing

$$B\big((\beta, v), (\beta', v')\big) = \int_Y (\beta v' + \beta' v),$$

with the one-form factors in a fixed order. Integration by parts identifies the B-annihilator of $\operatorname{im} d_0$ with $\ker d_1$. On cohomology B is (6) up to a constant. Indeed, lift local representatives of v at the zeros of σ by a cutoff to a smooth section $\widetilde{v}$ of P. A closed middle cochain representing v is then $(\bar\partial\widetilde{v}/\sigma, \widetilde{v})$. Pairing two such lifts and applying Stokes off disks about the zeros gives the sum of the boundary integrals of vv'/σ. At a zero of order d, the local form on $\mathbb{C}[z]/(z^d)$ takes the coefficient of z^{d-1} in the product, after multiplication by a unit. It is nondegenerate, and distinct support points are orthogonal.

The conjugate-linear involution

$$J(\beta, v) = (c(v), c^{-1}(\beta))$$

makes $B(\,\cdot\,, J\,\cdot\,)$ a positive Hermitian L^2 metric after a constant normalization. In a holomorphic frame of squared norm h, J sends the local coefficients (β, v) to $(h^{-1}\overline{v}, h\overline{\beta})$. The harmonic middle space is the Hermitian orthogonal complement of $\operatorname{im} d_0 + J \operatorname{im} d_0$. In particular it is J-invariant. Projection to the first factor identifies the J-fixed part of the middle term with $\Omega^{0,1}(L)$, and sends $(1 + J)d_0 u/2$ to $D_\theta u/2$. A fixed vector real orthogonal to all these projections is Hermitian orthogonal to $\operatorname{im} d_0$, by testing both u and iu. It is then also orthogonal to $J \operatorname{im} d_0$. Thus the real cokernel identifies with the fixed harmonic space. Elliptic stabilization gives the identification in families since the harmonic dimension is constant. Equivalently the orthogonal structure group of $\mathcal{E}$ reduces from $\mathrm{O}(2r, \mathbb{C})$ to $\mathrm{O}(2r, \mathbb{R})$.

The residue theorem makes S isotropic, and Riemann–Roch gives $\dim_{\mathbb{C}} S = r$. The long exact sequence identifies $\mathcal{E}_f/S$ with the annihilator of $\sigma H^0(L)$ in $H^1(L) = H^0(P)^*$. On a harmonic class it is the map $(\beta, v) \mapsto [\beta]$. The image of the zeroth-order term $c\sigma$ on the Dolbeault kernel is the Hermitian dual of $\sigma H^0(L)$ in $H^1(L)$. Consequently the map from the fixed harmonic space to S^* is the cokernel projection of Lee and Parker [LP12, Lemma 12.1 and Proposition 12.2]. Its sign is $(-1)^{h^0(L)}$, from conjugating the complex kernel before taking its determinant. This last comparison is at a single map and uses no constant-rank assumption on the Dolbeault cohomology of nearby maps. □

Write $\kappa = c_1(K_C)$ and use rational cohomology. On a stratum of fixed ramification profile we may pass to a finite cover ordering the ramification points. Denote by ev_p evaluation at a marked ramification point p. The following formula concerns the restriction to this stratum.

Theorem 4.2. *If a ramification index is even, $e(V_\theta) = 0$. If all indices are odd, write $R = 2D$, $A = L(D)$, and put $I = H^0(Y, L(D)/L)$. Then*

$$e(V_\theta) = (-1)^{h^0(A)} c_r(I). \tag{7}$$

For a fixed target an index $2s+1 \geq 5$ also gives zero. If the ramification consists of r points of index three, the restriction to the ordered stratum is

$$e(V_\theta) = (-1)^{h^0(A)} 6^{-r} \prod_{p \in R_{\mathrm{red}}} \mathrm{ev}_p^* \kappa. \tag{8}$$

Proof. At an even index the corresponding orthogonal residue summand has odd complex rank. A sum of complex orthogonal bundles admits a positive real form obtained by taking a positive real form in each summand. Any two such reductions are isomorphic, since the space of reductions in each fiber is contractible. After orienting the summands on a finite cover, the Euler class thus has the Euler class of an odd-rank oriented real bundle as a factor. The latter vanishes over $\mathbb{Q}$. Transfer gives the assertion on the original stratum.

For odd indices $A^2 = K_Y$ by Riemann–Hurwitz. Multiplication by σ identifies P with $L(2D)$. In this description I is a maximal isotropic subbundle of $\mathcal{E}$, since the local meromorphic representatives in I have product with no residue in (6). The two maximal isotropics have

$$k = \dim(S \cap I) = h^0(A) - h^0(L).$$

This is the residue construction for a theta characteristic [Mum71, §1]. The line A is the theta characteristic of the oddly ramified cover used in spin Hurwitz theory [LP13, §1]. For two maximal isotropics meeting in dimension k, the real projections from a positive real form to their duals induce orientations which differ by $(-1)^{r-k}$. To see this, move the pair through maximal isotropics with intersection of dimension k to a pair in hyperbolic coordinate form. In the $r-k$ exchanged coordinates the projections differ by complex conjugation. Applying Theorem 4.1 and $c_r(I^*) = (-1)^r c_r(I)$ gives the sign $(-1)^{h^0(L)+r-k+r} = (-1)^{h^0(A)}$. The bundle I exists over the entire profile stratum. The parity of the theta characteristic A is locally constant, so the pointwise orientation comparison proves (7) across jumps of $h^0(L)$.

At an index $2s+1$ point with image x, the filtration by pole order on I has graded lines $\theta_x \otimes (T_pY)^{\otimes j}$, $1 \leq j \leq s$. The leading nonzero jet of f identifies $(T_pY)^{\otimes(2s+1)}$ with T_xC. The first Chern classes of the graded lines are therefore

$$\left(\frac{1}{2} - \frac{j}{2s+1}\right) \mathrm{ev}_p^* \kappa.$$

When $s \geq 2$, two such factors already have product zero since $\kappa^2 = 0$ on C. For $s = 1$ the factor is $\mathrm{ev}_p^* \kappa/6$, giving (8). □

On the stratum with $r \geq 2$ distinct triple critical values, the product in (8) itself vanishes in ordinary cohomology. It pulls back from the top cohomology of the noncompact ordered configuration space of r points on C. A prescribed coincidence of two values gives zero by $\kappa^2 = 0$. The class of a family meeting the boundary can nevertheless have a contribution supported there.

Corollary 4.3. *On a stratum of finite nodal maps with fixed dual graph, label the components, the nodes, and the two preimages of each node on a finite cover. Let $V_{\theta,i}$ be the oriented*

cokernel on the i-th normalized component and let ev_j *send a map to the image* x_j *of its* j*-th node. Then the restriction of the nodal Euler class is*

$$\prod_i e(V_{\theta,i}) \prod_j \mathrm{ev}_j^* \, c_1(\theta).$$

An unramified component contributes its index-zero determinant sign.

Proof. In the $W^{1,p}$ domain with $p > 2$, the nodal domain is the kernel of the surjection taking the differences of values over the nodes, with target $\bigoplus_j \theta_{x_j}$. The spin operators on the normalized components are injective. The exact sequence of kernels and cokernels therefore gives

$$0 \longrightarrow \bigoplus_j \theta_{x_j} \longrightarrow V_{\theta,\mathrm{nod}} \longrightarrow \bigoplus_i V_{\theta,i} \longrightarrow 0.$$

The determinant isomorphism gives the matching lines their complex orientations, by continuation from the complex-linear normalization sequence [Zin11]. All ranks in the sequence are even, so reordering the factors preserves the product orientation. Interchange of the two branches at a node tensors its matching line with a flat sign line and leaves its rational first Chern class unchanged. □

5. Collision divisors and elliptic tails

Fix a smooth compact curve C of genus $g \geq 2$, put $H_m = 1+m(g-1)$, and let $m \geq 2$. Inside the stack of connected stable maps $\overline{\mathcal{M}}_{H_m+1}(C, m)$, let $\mathcal{F}_m = \mathcal{F}_m(C)$ be the locus of maps finite over C. These maps are flat, since the source is a nodal curve and its structure sheaf is torsion-free over the smooth target. The locus is open by properness of the universal curve and openness of quasi-finiteness. Its complement is a union of open and closed elliptic-tail loci, as proved in Section 5.1. Thus $\mathcal{F}_m$ is proper. Bryan and Pandharipande introduced this surface and its branch morphism in their calculation of local invariants of curves [BP01, §§1.3.2, 4].

For a finite map, Riemann–Hurwitz on the normalization and the genus formula for a nodal curve give

$$2 = \deg R + 2\,\#\{\text{nodes}\},$$

where R is the ramification divisor on the normalization. Thus a nodal map has one node and its normalization is étale. At a smooth ramification point of index b the local presentation $w = z^b$ identifies the relative cotangent complex with the module $\mathcal{O}/(z^{b-1})$ in degree zero. At the node there are coordinates $uv = 0$, $w = u+v$, since both branches are unramified. Writing its local algebra over the target as $\mathcal{O}_C[u]/(u^2 - wu)$ gives the relative cotangent complex $\mathcal{O}/(2u-w)$ in degree zero. Here $2u-w = u-v$ is a nonzerodivisor and the quotient has length two. In each case the module has projective dimension one. Its sheaf $\mathcal{H}om$ to $\mathcal{O}$ vanishes, and its sheaf $\mathcal{E}xt^1$ has the same finite length as the module. The local-to-global sequence for deformations over C consequently gives tangent dimension two and zero obstruction. This proves smoothness of the stack and identifies its virtual class with its fundamental class.

The branch morphism $\beta : \mathcal{F}_m \to C^{(2)}$ sends a smooth cover to its branch divisor and a nodal one to twice the image of the node. Let $\Delta = \{2x : x \in C\}$. The connected components of the reduced preimage of Δ are of three kinds.

- N_i consists of nodal covers. The two points above the node belong to a possibly disconnected étale normalization.
- T_j consists of smooth covers having a single ramification point of index three.

- U_k consists of smooth covers having two distinct simple ramification points over the same point of C.

For any such component B, write $a_B : B \to C$ for its branch value and $d_B = \int_B a_B^*[\mathrm{pt}]$. This is a stack degree. For a fixed branch value it sums $1/|\operatorname{Aut}_C(f)|$ over the corresponding isomorphism classes. Indeed, the monodromy data over the complement of the branch value are discrete and finite in fixed degree, and move uniquely up to isomorphism in a disk of branch values. In the nodal case the data consist of an étale cover and an unordered pair of distinct sheets to be identified, with connected quotient. Automorphisms interchanging the branches are included.

In a versal deformation of a nodal map the local model is $uv = t,\ w = u + v + v_0$. Its two critical values have squared difference $16t$. At a point of T_j the model is $w = z^3 + uz + v$, for which the squared difference is a nonzero constant times u^3. At a point of U_k the two critical values v_1, v_2 are independent parameters on an ordering cover; their squared difference is $(v_1 - v_2)^2$. The local deformation calculation above gives all the parameters in these models. In particular the reduced divisors are smooth and pairwise disjoint, and

$$\beta^*\Delta = \sum_i N_i + 3\sum_j T_j + 2\sum_k U_k. \tag{9}$$

The multiplicities are orders on stack charts. Since the pullback of Δ to $C \times C$ is twice the diagonal, $\Delta^2 = 4 - 4g$. Restricting (9) to a component of multiplicity ℓ_B gives

$$B^2 = \frac{4-4g}{\ell_B}\, d_B. \tag{10}$$

These are the branch intersections used in [BP01, §4.3].

For a theta characteristic θ define the following signs on the reduced divisors. On N_i, let $\widetilde{f}$ be the normalization of the cover. On T_j, write p for the ramification point. Put

$$s_\theta(N_i) = (-1)^{h^0(\widetilde{f}^*\theta)}, \qquad s_\theta(T_j) = (-1)^{h^0(f^*\theta(p))}. \tag{11}$$

In each expression the line is a theta characteristic on a smooth curve, and its parity is locally constant. On a disconnected normalization h^0 is the sum over its components.

Theorem 5.1. *In $H^2(\mathcal{F}_m;\mathbb{Q})$ the Euler class of the oriented spin obstruction plane is*

$$e(V_\theta) = -\frac{1}{4}\left(\sum_i s_\theta(N_i)[N_i] + \sum_j s_\theta(T_j)[T_j]\right), \tag{12}$$

where brackets denote the classes of the reduced divisors. For $D = D_\theta \oplus D_\eta$, the finite-map contribution is

$$F_m(D) = -\frac{g-1}{4}\left(\sum_i d_{N_i} s_\theta(N_i)s_\eta(N_i) + \frac{1}{3}\sum_j d_{T_j} s_\theta(T_j)s_\eta(T_j)\right). \tag{13}$$

Proof. The cokernels over finite nodal maps extend the oriented plane on the smooth locus by the normal obstruction bundle construction [LP12, Theorem 8.1 and Lemma 11.1]. Off the N_i and T_j there are two distinct simple ramification points. By Theorem 4.1 their residue spaces are orthogonal lines. Ordering the points and choosing a unit vector for each complex quadratic form gives a finite cover trivializing the orthogonal bundle on this entire open set, including the U_k. Uniqueness up to homotopy of its positive real form and transfer imply that

the rational Euler class restricts to zero there. By the Thom isomorphism for the disjoint smooth divisors, it is therefore a linear combination of the classes $[N_i]$ and $[T_j]$.

Set $\kappa = c_1(K_C)$. On N_i the normalization sequence of Corollary 4.3 gives $e(V_\theta)|_{N_i} = s_\theta(N_i)a_{N_i}^*\kappa/2$. On T_j Theorem 4.2 gives $e(V_\theta)|_{T_j} = s_\theta(T_j)a_{T_j}^*\kappa/6$. Integrating over a component B and dividing by its nonzero self-intersection (10) determines its coefficient. It is $-s_\theta(B)/4$ in both cases. Finally the oriented cokernel for D is the direct sum of the two spin obstruction planes. Integrating the product of their Euler classes and applying (10) gives (13). □

5.1. **Contracted elliptic tails.** Let $N \to C$ be a complex vector bundle of rank two and degree $2g-2$, and D a real Cauchy–Riemann operator on N. Assume that the pullback operator on every smooth connected cover of C is injective. On an étale cover the index is zero, so the operator is invertible. Orient all normal determinants by continuation from complex-linear operators, and denote the determinant sign of an étale cover f by $\sigma_D(f)$. The signed stack degree of the connected étale covers of degree m is

$$c_m(D) = \sum_{[f:Y\to C]} \frac{\sigma_D(f)}{|\operatorname{Aut}_C(f)|}.$$

Write $L_{m,H_m+1}(D)$ for the absolute normal Euler number on $\overline{\mathcal{M}}_{H_m+1}(C,m)$ and $F_m(D)$ for its integral over the finite-map locus. Put $F_1(D)=0$, since a finite degree-one cover is an isomorphism. When D is the normal operator of a super-rigid embedded curve, the fixed-target comparison identifies this absolute Euler pairing with the local Gromov–Witten degree [Zin11, Theorem 1.2(2)–(3)]. For $m \geq 2$ the relatively oriented Euler number of [BS24, Definition 6.2 and Remark 6.4] differs from this absolute number by the determinant sign of the operator on C.

Proposition 5.2. *The complement of the finite-map locus is a disjoint union of open and closed components indexed by the connected degree-m étale covers $f : Y \to C$. Each component has its ordinary fundamental class as virtual class. Their contributions give*

$$L_{m,H_m+1}(D) = F_m(D) - \frac{m(g-1)}{12}\, c_m(D). \tag{14}$$

Proof. Consider a stable map, and let s be the number of nonconstant components on the normalization of its domain. Write R for the sum of their ramification divisors, g_0 for the sum of the genera of the normalized contracted components, and Γ for the connected dual graph. Riemann–Hurwitz and the arithmetic genus give

$$2 = s + \frac{\deg R}{2} + g_0 + b_1(\Gamma).$$

If $s=2$, both positive-degree components are étale and the graph is a tree. A contracted rational vertex must have valence at least three, which is impossible in a tree with at most two leaves. If $s=1$ and $\deg R = 2$, the same argument excludes a contracted subtree. If $s=1$ and $\deg R = 0$, the nonconstant component has smooth normalization Y of genus H_m, and $g_0 + b_1(\Gamma) = 1$. When $g_0 = 1$ the tree consists of Y joined directly to a smooth genus-one component. When $b_1(\Gamma) = 1$, every contracted vertex has valence at least three. The identity $\sum_v (\mathrm{val}(v)-2) = 0$ leaves just a self-node on Y, or one contracted rational vertex with a self-loop joined to Y by one edge. Thus every map with a contracted component consists of a smooth connected étale cover with a one-pointed stable curve of genus one attached.

For fixed f, gluing at $x \in Y$ defines a proper morphism from

$$\mathcal{G}_f = \big[(Y \times \overline{\mathcal{M}}_{1,1})/\operatorname{Aut}_C(f)\big] \longrightarrow \overline{\mathcal{M}}_{H_m+1}(C, m).$$

The deck group acts on Y, and the pointed tail retains its stack stabilizers. To compute the deformation theory at $Y \cup_{x=y} E$, first preserve the attaching node. The cover is rigid because $df : T_Y \to f^*T_C$ is an isomorphism. Its marked point moves freely. The tail deforms as a pointed stable curve with constant image in C. Varying this image makes the matching equation at the node transverse. The remaining obstruction is $T_{f(x)}C \otimes H^1(E, \mathcal{O}_E)$. The boundary map from the smoothing line is

$$T_xY \otimes T_yE \xrightarrow{df_x \otimes \delta_y} T_{f(x)}C \otimes H^1(E, \mathcal{O}_E), \tag{15}$$

up to the choice of sign for the connecting homomorphism. Indeed, in the plumbing $uv = t$, the linear term on the cover gives the principal part $df_x(t/v)$ on the tail. Thus δ_y is the connecting homomorphism for $0 \to \mathcal{O}_E \to \mathcal{O}_E(y) \to T_yE \to 0$. The dualizing sheaf is trivial for both a smooth elliptic curve and an irreducible one-nodal genus-one curve. Serre duality and Riemann–Roch give $H^1(\mathcal{O}_E(y)) = 0$ and $H^0(\mathcal{O}_E(y)) = \mathbb{C}$. Hence (15) is an isomorphism on all of $\mathcal{G}_f$.

The full stable-map theory is therefore unobstructed of complex dimension two along $\mathcal{G}_f$, and its tangent space consists of the attachment and pointed-tail deformations. These are precisely the tangents to the source of the gluing morphism. Gluing identifies the stabilizers and is injective on geometric points, since the positive-degree component and the attaching node are intrinsic. It is thus a proper representable étale open immersion, and its image is closed as well. This proves the asserted component description and the fundamental-class claim. It also gives the properness used in Section 5. An alternative description by lifting to the maximal étale cover appears in [BP01, §2.2].

It remains to orient the normal obstruction. On Y the operator f^*D is invertible. On the constant tail the normal operator is $\bar\partial$ with coefficients in $N_{f(x)}$. Its kernel consists of the constants, which evaluate isomorphically onto the matching fiber. The normalization exact sequence at the attaching node consequently identifies the full normal cokernel with

$$H^1(E, \mathcal{O}_E) \otimes N_{f(x)}.$$

This uses the arithmetic H^1 also for a nodal tail. In families the first factor is the dual of the Hodge line $\mathbb{E}$. The normalization isomorphism of determinants gives the constants and the matching fiber their complex orientations; their determinants cancel. Continuing from the complex-linear normalization sequence leaves the factor $\sigma_D(f)$ from the invertible cover operator [Zin11, §2], [BS24, Appendix B]. Thus the full normal orientation is the complex orientation of $\mathbb{E}^\vee \otimes f^*N$ multiplied by $\sigma_D(f)$.

Write $\lambda = c_1(\mathbb{E})$. On the product of the two curves

$$c_2(\mathbb{E}^\vee \otimes f^*N) = -\lambda\, c_1(f^*N).$$

The genus-one discriminant has a simple zero at the nodal boundary and weight twelve, so 12λ is the boundary class on $\overline{\mathcal{M}}_{1,1}$. The boundary point has stabilizer of order two, whence $\int \lambda = 1/24$. Finally $\int_Y c_1(f^*N) = m(2g-2)$. Dividing by $|\operatorname{Aut}_C(f)|$ and summing proves (14). □

We record the implication for the exponents of the elementary factors. Let $n_{d,a}$ be the coefficients in the expansion of the connected local series in the connected elementary series $G_a(q^d, t)$, with $a \geq H_d$. Here G_a is the connected Gromov–Witten series of the elementary genus-a curve, the logarithm of its normalized partition function. A connected degree-j,

genus-h contribution has monomial $q^j t^{2h-2}$. The same counts are attached to the balanced universal factors in Section 6.1. Write $C_j(a)$ for the unsigned connected étale degree-j count of a genus-a curve, and $F_j^{\mathrm{elem}}(a)$ for the finite-map Euler number of its doubled spin operator at first excess. All étale signs of this operator are positive. The elementary series begins in degree j at genus $1+j(a-1)$, and its degree-one leading coefficient is one [IP18, Propositions 3.2–3.3], [DIW26, Proposition 3.46]. At genus H_m, comparison therefore gives

$$c_m(D) = \sum_{d|m} n_{d,H_d} C_{m/d}(H_d).$$

At genus H_m+1, a proper divisor $d \mid m$ can still contribute only the factor of genus H_d, since increasing that genus by one raises the least source genus by m/d. For the equal-degree factor of genus H_m the finite term is $F_1^{\mathrm{elem}}(H_m) = 0$. In each divisor term the tail multiplier in (14) is the same, since $(m/d)(H_d-1) = m(g-1)$. The least-genus identity cancels their sum against the tail contribution on the left and yields

$$n_{m,H_m+1} = F_m(D) - \sum_{\substack{d|m \\ d<m}} n_{d,H_d}\, F_{m/d}^{\mathrm{elem}}(H_d). \tag{16}$$

6. Signed Hurwitz numbers and their elementary multiplicities

Let a, b be the theta characteristics of the two normal lines, $w = ba^{-1} \in H^1(C;\mathbb{Z}/2)$ their difference, and $\epsilon = (-1)^{h^0(a)+h^0(b)}$. Put $H_m = 1+m(g-1)$ and $c_m = L_{m,H_m}$. For a cell u of a partition λ, let $h(u)$ denote its hook length and $c(u)$ its column minus its row. Write $H_\lambda = \prod_{u\in\lambda} h(u)$. Let $\mathcal{P}_w$ consist of all partitions if $w = 0$ and all self-conjugate partitions if $w \neq 0$, including the empty partition in either case. Set

$$Z_w(y) = \sum_{\lambda\in\mathcal{P}_w} H_\lambda^{2g-2} y^{|\lambda|}.$$

Theorem 6.1. *The connected local counts at the first two possible source genera satisfy* $c_m = \epsilon^m [y^m] \log Z_w(y)$ *and*

$$L_{m,H_m+1} = -\frac{\epsilon^m (g-1)}{12} [y^m] \frac{\displaystyle\sum_{\lambda\in\mathcal{P}_w} H_\lambda^{2g-2} \left(\sum_{u\in\lambda} h(u)^2\right) y^{|\lambda|}}{Z_w(y)}. \tag{17}$$

For equal spin structures the right side is the first correction in the balanced hook formula. The restriction to self-conjugate partitions comes from a sign on one handle of the target.

6.0.1. *The spin sign of a collision.* At a nodal cover let $f : Y \to C$ be its étale normalization, including all components. At a smooth cover with one triple ramification point let $f : Y \to C$ be the cover itself. Write R for the ramification divisor of f, so $R = 0$ in the first case and $R = 2r$ in the second. The two signs in the boundary intersection are the parities of

$$f^*a \otimes \mathcal{O}_Y(R/2), \qquad f^*b \otimes \mathcal{O}_Y(R/2).$$

For an odd branch these are the spin structures of spin Hurwitz theory [EOP08]. Their product has a simpler expression.

Lemma 6.2. *Let* $\alpha \in H^1(C;\mathbb{Z}/2)$ *be the sign character of the sheet permutations of* f*. Then the product of the two spin signs equals*

$$\epsilon^m (-1)^{\langle w\smile\alpha,[C]\rangle}. \tag{18}$$

Proof. The bundle $E = f_*\mathcal{O}_Y(R/2)$ has rank m and carries the orthogonal form $\mathrm{Tr}(uv)$. Around $x = z^3$ its ramified summand has basis $z^{-1}, 1, z$ over $\mathbb{C}\{x\}$. The matrix of the trace form in this basis has antidiagonal entries 3 and all other entries zero. The remaining sheets have the ordinary permutation form. Thus the pairing extends nondegenerately over the branch value. On its complement the determinant character of E is the permutation sign. A 3-cycle is even, so this character extends across the missing point and $w_1(E) = \alpha$.

For an orthogonal bundle of rank m over a spin curve the mod-two index formula is

$$h^0(a \otimes E) \equiv m h^0(a) + q_a(w_1(E)) + \langle w_2(E), [C] \rangle \pmod 2,$$

where $q_a(v) = h^0(a \otimes v) - h^0(a) \pmod 2$ [Ati71, BS18]. By the projection formula the left side is the parity on Y. Adding this identity for a and b cancels $w_2(E)$. Since $q_b(v) - q_a(v) = \langle w \smile v, [C] \rangle$, the result follows. The same calculation applies to a disconnected source by direct sum. □

6.0.2. *One signed handle and one marked collision.* We remove the common factor ϵ^m in degree m until the end of the calculation. For $w \neq 0$ choose a simple nonseparating circle representing its Poincaré dual as the first generator in a surface presentation, disjoint from the branch value. The remaining sign in (18) is $\mathrm{sgn}(A_1)$ for its sheet permutation A_1. Take $\chi = \mathrm{sgn}$ in this case and $\chi = 1$ for $w = 0$. For $G = S_m$ put

$$K_\chi = \sum_{A,B \in G} \chi(A)\, ABA^{-1}B^{-1} \in \mathbb{C}[G].$$

On an irreducible representation V of dimension d and character ψ, this central element has scalar

$$\frac{|G|}{d^2} \sum_{A \in G} \chi(A) |\psi(A)|^2 = \left(\frac{|G|}{d} \right)^2 \mathbf{1}_{V \otimes \chi \simeq V}.$$

Indeed the sum over B of the conjugates of A^{-1} acts by $|G|\overline{\psi(A)}/d$, and the last equality is character orthogonality. The other $g-1$ handles contribute K_1.

The identity coefficient of $K_\chi K_1^{g-1}$ counts the labelled étale covers with this sign. It equals the regular trace divided by $|G|$. Relabelling sheets divides by $|G|$ once more, giving the inverse automorphism weights. The summand of V is therefore $(|G|/d)^{2g-2}$ when $V \otimes \chi \simeq V$. For $V = V_\lambda$ the hook-length formula gives $|G|/d = H_\lambda$, and tensoring with sign transposes λ. The resulting disconnected series is Z_w. Decomposition into connected components gives the formula for c_m, as in [Wue26j, Appendix F].

Denote by N_m the signed mass over one fixed target point of étale covers with two distinct sheets identified to give a connected nodal cover. Denote by T_m the signed mass of connected covers with a single 3-cycle over that point. Both masses still omit ϵ^m. Before imposing connectivity the first mark has $\binom{m}{2}$ choices on each labelled fiber. After gluing, the marked component is unique, and all its companions are unmarked étale covers. Removing their generating function gives

$$\sum_{m \geq 1} N_m y^m = \frac{y^2 Z_w''(y)}{2 Z_w(y)}. \tag{19}$$

In particular this includes a node joining two distinct components of the normalization. For the triple mark the handle product is multiplied by the class sum of all 3-cycles. The

Jucys–Murphy elements $J_j = \sum_{i<j}(ij)$ satisfy

$$\sum_j J_j^2 = \binom{m}{2} 1 + \sum_{\tau \text{ a 3-cycle}} \tau.$$

Indeed the nondiagonal terms of J_j^2 list the 3-cycles with largest letter j, each once. In the Young basis the J_j act by the contents of the cells containing j [Juc74]. Removing the unmarked components as above now gives

$$\sum_{m\ge 1} T_m y^m = \frac{1}{Z_w(y)} \sum_{\lambda\in\mathcal{P}_w} H_\lambda^{2g-2} \left(\sum_{u\in\lambda} c(u)^2 - \binom{|\lambda|}{2} \right) y^{|\lambda|}. \tag{20}$$

Proof of Theorem 6.1. The two collision intersections give the finite contribution $-\epsilon^m(g-1)(3N_m + T_m)/12$. The contracted elliptic tails add $-m(g-1)c_m/12$. Since $y(\log Z_w)' = yZ_w'/Z_w$, the three terms over the common denominator Z_w have total insertion on a partition of size m equal to

$$3\binom{m}{2} + \sum_{u\in\lambda} c(u)^2 - \binom{m}{2} + m = m^2 + \sum_{u\in\lambda} c(u)^2 = \sum_{u\in\lambda} h(u)^2.$$

The last equality is the hook-content square identity [Men25]. Restoring ϵ^m proves (17). □

6.0.3. *Degree three.* Write $s = 2g-2$. In degree three the partitions $(3), (2,1), (1^3)$ have hook products $6, 3, 6$, respectively. Only $(2,1)$ is self-conjugate. Equations (19) and (20), including the lower coefficients in the division by Z_w, give

	N_3	T_3
$w = 0$	$6\cdot 6^s + 3\cdot 3^s - 2\cdot 2^s$	$(4^g - 1)3^s$
$w \neq 0$	3^{2g-1}	-3^s.

Thus the finite contribution for equal spins is $-(g-1)(22\cdot 6^s + 8\cdot 3^s - 6\cdot 2^s)/12$, and for distinct spins it is $-2\epsilon(g-1)3^s/3$. At this prime degree the proper divisor one contributes ϵ times the finite degree of the balanced genus-g core. The degree-one finite contribution at positive excess is zero. Cancellation of the elliptic tails therefore gives, for $a \neq b$,

$$n_{3,H_3+1} = \frac{\epsilon(g-1)}{6}\left(11\cdot 6^{\,2g-2} - 3\cdot 2^{\,2g-2}\right). \tag{21}$$

For $g = 2$ the ordinary collision masses are 235 and 135, with finite contribution -70. For distinct spins they are 27 and -9 before restoring ϵ, with finite contribution -6ϵ. Their difference gives $n_{3,5} = 64\epsilon$.

6.1. **Elementary exponents and unramified quotients.** Let $F_h(q)$ be the balanced antidiagonal local factor of genus h in the rational universal algebra of complex one-cycles of virtual dimension zero [Par25, Wue26b]. We use the convention that F_h has constant term one. Its image in the algebra of almost-complex cycle families is the elementary cluster of genus h of Ionel and Parker [IP18, Wue26b]. For the isolated spin germ, the characteristic insertions at the compact cycles $m[C]$ have a unique balanced lift [Wue26j, Proposition 3.1]

$$\widetilde{P} = \prod_{d\ge 1,\ h\ge H_d} F_h(q^d)^{n_{d,h}}. \tag{22}$$

The exponents are integral and have finite support in h at fixed d. This uses geometric decomposition after compact realization and the multiplicity-one detection theorem of [Wue26b]. Along the fixed-target cluster decompositions, connected Gromov–Witten degrees add on

disjoint targets. Thus the same exponents give the expansion of the connected series in the elementary logarithms. The use of fixed-target degrees in this expansion relies on that geometric comparison. In particular no evaluation of arbitrary analytic parameter families is required here.

Put $n_d = n_{d,H_d}$. In this notation the tail cancellation of (16) subtracts the equal-spin finite-map integrals on unramified covers. We next express the subtraction in terms of the individual maps to C.

6.1.1. *A count using the maximal unramified quotient.* Fix a target point x. Let $\mathcal{B}_m(x)$ denote the isomorphism classes of the finite maps over the nodal and triple divisors with collision value x. Such an object has one distinguished node or one triple ramification point. Write $k(f) = 1$ in the nodal case and $k(f) = 3$ in the triple case, and write $\tau(f)$ for the product of the two spin parities in (4). All automorphisms in this section fix the target. Put $\Gamma = \pi_1(C)$ and regard $H(f) = f_*\pi_1(Y)$ as a conjugacy class of subgroups. This is the factorization by the image of the fundamental group used by Bryan and Pandharipande [BP01, §2.2, Proposition 2.3]. It uses the connected nodal space Y itself when f has a node. The connected cover of C corresponding to $H(f)$ is finite: f lifts to it, and the lift is holomorphic locally on every component. A nonconstant holomorphic map from a compact component to a connected Riemann surface forces that surface to be compact. Composition of degrees shows that $[\Gamma : H(f)]$ divides m. This cover is the maximal unramified quotient of f, in the sense that every factorization through an unramified connected cover factors through it after making compatible choices of base points. For finite-index $H \leq \Gamma$ write $\sigma(H)$ for the determinant sign of the spin pair on its unramified cover.

Theorem 6.3. *For every $m \geq 1$,*

$$n_{m,H_m+1} = \frac{g-1}{4} \sum_{f \in \mathcal{B}_m(x)} \frac{\sigma(H(f)) - \tau(f)}{k(f)\,|\operatorname{Aut}_C(f)|}.$$

In particular the difference of the exponents for two spin pairs is obtained by taking the corresponding difference in these boundary and unramified signs on the same covers.

Proof. By the orbit formula for the determinant signs [Wue26j, Theorem 16.1], there are integers b_K indexed by conjugacy classes of finite-index subgroups such that

$$\sigma(H) = \sum_{[K]} b_K |(\Gamma/K)^H|, \qquad n_d = \sum_{[\Gamma:K]=d} b_K.$$

In the second sum each conjugacy class is used once. The first formula is the fixed-point count, or mark, of a unit in the completed Burnside ring; only subgroups of index dividing $[\Gamma : H]$ contribute. For a fixed cover $z : Z \to C$ corresponding to K of index d, the number of lifts of f to Z is $|(\Gamma/K)^{H(f)}|$. The groupoid of such lifts over an object f has mass this number divided by $|\operatorname{Aut}_C(f)|$, by orbit-stabilizer. Here Z is a fixed target, so its deck group does not divide this number. Counting the same lifted maps on Z allows the collision value to be any of the d preimages of x. The mass at each such value is the same, by continuation along the connected curve Z. An unramified change of target preserves the node or the triple point and hence the weight $1/k(f)$. All boundary signs in the equal-spin elementary pair on Z are positive. By Theorem 5.1 and $H_d - 1 = d(g-1)$ we obtain

$$F^{\mathrm{elem}}_{m/d}(H_d) = -\frac{g-1}{4} \sum_{f \in \mathcal{B}_m(x)} \frac{|(\Gamma/K)^{H(f)}|}{k(f)\,|\operatorname{Aut}_C(f)|}$$

whenever $d \mid m$. For $d \nmid m$ the fixed sets are empty. Multiply by b_K, sum, and apply (16) and the signed boundary formula for D. □

Corollary 6.4. *Let $a \neq b$ and let p be an odd prime. The signs of n_{p,H_p} and n_{p,H_p+1} are $-\epsilon$ and ϵ, respectively, and*

$$|n_{p,H_p+1}| \; > \frac{g-1}{4}\binom{p}{2}\,|n_{p,H_p}|.$$

For each g there is a path of tame structures on a closed symplectic six-torus for which the isolated positive-sign core at its two ends has different local elementary exponents at both these source genera for every odd prime p.

Proof. Every collision in prime degree has $H(f) = \Gamma$. Otherwise its maximal unramified quotient would have degree p and the lift degree one. A finite flat degree-one cover of a smooth curve is an isomorphism, inconsistent with either collision. Thus its unramified sign in Theorem 6.3 is ϵ. The boundary sign is ϵ^p times the permutation sign around a loop Poincaré dual to the difference ba^{-1}, by the parity formula in Section 6. Consequently each negative permutation sign contributes with sign ϵ. Let U_- be the inverse-deck mass of connected unramified covers with that negative sign. Each one supplies $\binom{p}{2}$ unordered pairs in the fiber at x whose identification forms a node of the same character. Their total mass is $\binom{p}{2}U_-$. In least genus $n_{p,H_p} = -2\epsilon U_-$ by the unramified formula. Also $U_- > 0$: choose surface generators with the chosen loop first, send it to a transposition, send the generator of another handle to a p-cycle, and send all remaining generators to the identity. The source is connected and the surface relation holds. These nodes alone give the weak inequality. There is also a nodal cover with disconnected normalization of degrees 1 and $p-1$. Choose a connected cyclic $(p-1)$-cover whose sign on the chosen loop is negative by sending the loop to a full cycle. Join one of its sheets to the degree-one sheet at x. The resulting map has the same negative permutation sign, giving a further positive mass after multiplication by ϵ. This proves the strict inequality. The path and its primitive core class are supplied by the compact spin realization of [Wue26j, §15]. Its ends are the equal spin germ and a distinct-spin germ of the same positive sign. The equal-spin germ has only the genus-g elementary factor in degree one. The displayed comparison concerns the endpoints; it requires no assignment of a branched count to a single parameter of the path. □

One may express the result as a comparison of integral universal products. Let z be an indeterminate and let $Q_h(z) \in \mathbb{Z}[z]$ be the polynomial with $Q_h(4\sin^2(t/2)) = 4\sin^2(ht/2)$. It is integral by the cosine recurrence, with $Q_h(0) = 0$ and $Q_1(z) = z$. With $\mathcal{P}_w$ the partition set of §6 for the spin difference w, put

$$\mathcal{Z}(q,z) = \sum_{\lambda \in \mathcal{P}_w} (\epsilon q)^{|\lambda|} \prod_{s \in \lambda} Q_{h(s)}(z)^{g-1}.$$

This has a unique formal expansion as an integral product of ordinary balanced hook series of genera h evaluated at q^d. Indeed, in degree d subtract the already determined factors of smaller degree by multiplying by their inverses. The resulting coefficient is an integral polynomial in z divisible by $z^{d(g-1)}$. The degree-one terms of the balanced factors are the distinct monomials z^{h-1}, so its coefficients give the required integer exponents for this degree. The divisibility is preserved under products and inversion, and each degree requires only finitely many genera. Using these exponents as in (22) gives a balanced element in the completed universal algebra.

Corollary 6.5. *This element agrees with $\widetilde{P}$ modulo the subgroup of balanced products generated, with degree completion, by the $F_h(q^d)$ with $h \geq H_d + 2$.*

Proof. Substitute $q = yt^{-2(g-1)}$ and $z = 4\sin^2(t/2)$ in the formal product. There are no negative powers of t after this normalization. The constant and quadratic coefficients in t in its connected logarithm are the unramified connected counts and (17): a box of hook length k contributes a relative quadratic term $-(g-1)k^2/12$. These are exactly the corresponding numerical coefficients of the geometric product. Triangularity in covering degree and then in source genus, as in (16), identifies its integer exponents in both genera. Applying those exponents in the universal factors proves the claim. Thus the two lowest genera can also be recovered directly from the collision and tail geometry, independently of the all-genus comparison in Theorem 1.1. □

7. Individual spin classes and their intersections

On covers with two simple branch points the residue description determines a class for each spin structure. Its character decomposition gives a lattice indexed by the two-torsion of the Jacobian. On covers with four simple branch points there is a further distinction: the Euler integral of two spins has contributions on several components of the stable-map stack. We isolate the component of finite smooth covers and compute the ten individual intersection pairings on its closure in the genus-two cubic case.

7.1. **The lattice of spin Euler classes.** The individual Euler classes on the Hurwitz surface contain information beyond their pairwise integrals. Fix a spin line θ_0, put $G = H^1(C;\mathbb{Z}/2)$, and use the flat sign line to write $\theta_0 z$ for its twist by $z \in G$. On the complement of a node's image use the permutation action of the normalization. At a triple point its local monodromy is even. In either case the permutation sign on the complement of the collision extends to a class $\alpha \in G$. For a boundary divisor B in (4) write this class as α_B. It is constant on B: locally the covering data away from a disk about the moving point do not vary, and removing and filling the point induces an isomorphism on the extended sign characters. Define

$$D_\alpha = \sum_{B:\alpha_B=\alpha} s_{\theta_0}(B)[B], \quad \Lambda = \sum_{\alpha\in G} \mathbb{Z} D_\alpha, \qquad \Lambda_{\mathrm{spin}} = \sum_{z\in G} \mathbb{Z}\cdot 4e(V_{\theta_0 z}),$$

as lattices in $H^2(\mathcal{F}_m;\mathbb{Q})$. Here V_θ is the oriented spin cokernel.

Theorem 7.1. *For every $m \geq 2$, the D_α are independent, the spin Euler classes span the same rational space, and*

$$\Lambda/\Lambda_{\mathrm{spin}} \simeq \bigoplus_{j=1}^{2g} (\mathbb{Z}/2^j\mathbb{Z})^{\binom{2g}{j}}.$$

The least positive integer taking any one of the $e(V_\theta)$ into Λ is four.

Proof. Every α occurs already on a nodal divisor. For nonzero α, choose surface generators with value one on a_1 alone. If m is even, send a_1 to an m-cycle. If m is odd, send a_1 to a transposition and a_2 to an m-cycle. The other chosen generators are trivial, so this gives a connected unramified cover with sign α. For $\alpha = 0$ and odd m an m-cycle suffices. For even $m \geq 4$, assign an $(m-1)$-cycle and a 3-cycle involving one of its letters and the last sheet so that the generated action is transitive; both permutations are even. In each instance glue a pair of points in a fiber. For $m = 2$ and $\alpha = 0$ take two trivial sheets and glue their points over the chosen value. Distinct reduced boundary divisors are disjoint with strictly negative self-intersection. The D_α therefore have mutually orthogonal nonzero squares.

Set $q_{\theta_0}(z) = h^0(\theta_0 z) - h^0(\theta_0) \bmod 2$. The ratio of the two boundary parities, by the orthogonal trace-bundle calculation in Section 6, gives

$$-4e(V_{\theta_0 z}) = (-1)^{mq_{\theta_0}(z)} \sum_{\alpha \in G} (-1)^{\langle z \smile \alpha, [C] \rangle} D_\alpha.$$

The common row signs do not change the generated lattice. In dual binary bases for the nondegenerate cup product the character matrix is the $2g$-fold tensor power of the matrix with rows $(1,1)$ and $(1,-1)$. Integral row and column operations diagonalize this two-by-two matrix with entries $1, 2$. Tensoring those unimodular operations gives the Smith factors 2^j with the stated multiplicities. Every coefficient of a single $e(V_\theta)$ in the independent D_α is $\pm 1/4$, proving the last assertion. □

The theorem compares two lattices generated by geometric classes in rational cohomology. It does not require that the individual boundary divisors span the integral cohomology of the stack.

7.2. **Four simple branch points.** Fix two theta characteristics a, b on C. Write c_d for the signed connected étale degree and F_d for the finite-map Euler integral on the first Hurwitz surface. Let M_d be the closure of the smooth connected covers with four distinct simple branch points, with the sum of its reduced four-dimensional component cycles. Write V_a, V_b for the real spin cokernels and put $I_d = \int_{[M_d]} e(V_a)e(V_b)$.

Proposition 7.2. *For every $d \geq 2$, putting $t = d(g-1)$, one has*

$$L_{d,H_d+2} = I_d - \frac{t}{12} F_d + \frac{t(5t-1)}{1440} c_d.$$

For a genus-two target and distinct even spins this gives $I_2 = 0$, $L_{2,5} = -1/160$, $n_{2,5} = 0$ and $I_3 = 3/2$. For a repeated even spin on the same target, $I_3 = 161/6$.

Proof. First remove the open and closed summands consisting of maps of degree one over a connected étale target cover of full degree d; its lift is taken modulo its deck group. Their degree-one series contributes $[v^{2t+4}]x^t c_d = t(5t-1)c_d/1440$. Covers factoring through a smaller étale target remain. Normalize a map in the complement. The sum of half the ramification degree, the genera of contracted normalizations and the excess from the dual graph is two. The remaining positive-dimensional contracted strata have a pointed elliptic tail on a two-branch finite core, an elliptic component bridging two sheets, or a rational tripod. The first family has a proper parametrization by the universal pointed first Hurwitz surface times $\overline{\mathcal{M}}_{1,1}$, including the inserted bridge when the mark meets a node. The degree on the generic image is one.

The other strata lie on M_d: attaching a degree-two bubble by the pencil $|p+q|$ smooths an elliptic bridge with attaching pair p, q. For the tripod attach the rational function $z + z^{-1} + (z-1)^{-1}$ at its three simple poles. It has four simple branch values, since the binary discriminant in the target coordinate is $w^4 - 2w^3 - 5w^2 + 6w - 23$, of discriminant -5619712. At a direct node glue rational bubble covers of the matching contact orders and smooth the balanced equations after base change. In each case the stabilized target remains C. The remaining support is the union of M_d and the elliptic-tail image, both pure of dimension four. A smooth simply branched cover has generic virtual multiplicity one, and at a generic tail the derivative at the unramified attaching point identifies the smoothing line with $T_{f(p)}C \otimes H^1(E, \mathcal{O}_E)$. The generic virtual multiplicity there is also one. A cycle of the top dimension is determined by these multiplicities. On the tail the matching constants

cancel, leaving $\mathbb{E}^\vee \otimes \mathrm{ev}^*(a \oplus b)$ with top class $-\lambda\,\mathrm{ev}^*\,c_1(K_C)$. Its integral along the two new parameters is $-t/12$. This proves the formula; Calculation B3 of [Wue26g] gives the full normalization along the lower-dimensional strata. Substitution of the degree-two and cubic coefficients of Theorem 1.1 and the signed finite sums of Sections 5–6 gives the stated values; for the repeated spin one uses the orientable balanced series. See also Calculations B1–B4 of [Wue26g]. □

For genus two, distinct even spins give the entire connected double series

$$L_{2,k} = (-1)^k \frac{2^{2k-2} - 4}{(2k-2)!}, \qquad k \geq 3.$$

Indeed the disconnected coefficient of Q^2 vanishes and its connected coefficient is $-x^2/2$. Subtracting the balanced factor in degree one leaves no elementary double exponent at genus five or higher. A separate geometric bound for reduced multisections is available. If the normal equations are independent line equations of degrees at most $d_{\max}$, and the reduced graph is generically one-to-one without contracted components, the ramification degree plus twice the number of identified nodes is at most $m(m-1)d_{\max}$. On the normalized ordered-pair cover the difference of the two solutions satisfies a real line equation with positive zero orders. Its zeros at ramification and at the ordered pairs identified by a node give the inequality by integrating the first Chern class. Pairs on which one component difference is identically zero are counted using the other line. In the triangular case one first separates distinct scalar projections and then applies the line bound in each equal-projection group. The projected-jet variation, the prime-degree consequence and the restriction at composite towers are detailed in Calculations A1–A10 of [Wue26g]. In particular for noneffective spins and prime degree p the elementary genus is at most $1 + (g-1)\binom{p+1}{2}$.

7.3. **The ten even spins in genus two.** On $M_3 \subset \overline{\mathcal{M}}_6(C, 3)$ put $W_a = -R\pi_* f^* a$ and $\Delta_a = (c_2(W_a) - e(V_a))/2$.

Theorem 7.3. *For every smooth genus-two target the ten even spins have* $\int_{[M_3]} \Delta_a^2 = 6$ *and* $\int_{[M_3]} \Delta_a \Delta_b = -1/3$ *for* $a \neq b$. *The ten classes span a positive-definite space of rank ten.*

Proof. Let ι be the hyperelliptic involution of C. For distinct even spins a, b, the two residue-one Szegő kernels have the same polar part along the diagonal. Their squared difference restricts there to a quadratic differential Q_{ab}. The common section locus on the genus-five finite Hurwitz surface is its zero scheme. To see the condition, consider the trace-free part of $f_*\mathcal{O}_Y$. The two pulled-back sections identify it with an elementary enlargement of $a^{-1} \oplus b^{-1}$ at a point p. In square frames at p, the normalized meromorphic sections of the two spins have expansions $z^{-1} + c_a z + \cdots$ and $z^{-1} + c_b z + \cdots$. Expressing the cubic multiplication in the enlarged frame leaves one regularity condition, $c_a = c_b$, or $Q_{ab}(p) = 0$. This calculation works over local Artin bases. The resulting cover is unique, smooth, with one triple ramification point over $\iota(p)$; its monodromy is S_3, so it has no deck factor. The full cubic multiplication and the boundary classification are in Calculation C1 and Calculation C2, §§2–3 of [Wue26g]. On the four-cycle M_3 the common support consists of the disjoint curves $E_p \simeq \overline{\mathcal{M}}_{1,1}$ obtained by attaching an elliptic tail at that triple point.

Suppose that p is a simple zero of Q_{ab}. At the triple point the map deforms as $w^3 + tw + v$; let $w = x$ mark the attachment and let τ smooth the tail node. The versal equation is $\tau B = 0$, where $B|_{\tau=0} = t + 3x^2$ up to a unit, by the principal-part obstruction to smoothing. Thus $B = 0$ is a smooth chart of the main component and $\tau = 0$ cuts its tail divisor. On this

divisor the ramification marking satisfies $t = -3x^2$. Each supported correction Δ_θ is the class of its kernel surface with coefficient one, by complex-to-real determinant continuation in Calculation C2, §5 of [Wue26g]. On the finite surface the two kernel divisors meet with length one at this simple zero. Pullback under the degree-two ramification marking makes their intersection inside the tail divisor equal to $2[E_p]$. Its normal line in the main component is the node-smoothing line, of first Chern class $-\lambda$ on E_p. The self-intersection formula for that divisor therefore gives

$$\mathrm{Contr}_{E_p}(\Delta_a\Delta_b) = -2\int_{\overline{\mathcal{M}}_{1,1}} \lambda = -\frac{1}{12}.$$

The stack Hodge degree includes the pointed-tail automorphisms. Since K_C^2 has degree four, four simple zeros give total mixed degree $-1/3$.

Put $S_a = \int \Delta_a^2$. The equal-spin integral is $161/6$ from Proposition 7.2; the distinct-spin integral is $3/2$. The Chern class of the complex index is independent of a rationally, since spin differences are torsion. Subtracting one half the square of the difference of the two real Euler classes from one half the sum of their squares gives

$$\frac{3}{2} = \frac{161}{6} - 2(S_a + S_b) + 4\int \Delta_a\Delta_b.$$

It follows that $S_a + S_b = 12$ whenever Q_{ab} has four simple zeros.

Write the six hyperelliptic branch values as e_i. Each spin is an unordered partition into two triples. Given two distinct partitions choose sides sharing two elements with monic root polynomial P; the opposite sides share two with root polynomial R. Call the exchanged values c, d. The hyperelliptic Szegő formula [DS21, Appendix C, (C.3)–(C.4)] gives $Q_{ab} = (c-d)(P'R - PR')(dx/y)^2/4$. The binary quadratic $P'R - PR'$ has discriminant $4\prod_{P(a_i)=R(b_j)=0}(a_i - b_j) \neq 0$. Only c, d can be exceptional branch roots. At a fixed c this happens precisely when $u_{a_1} + u_{a_2} = u_{b_1} + u_{b_2}$ for $u_e = 1/(c-e)$. Of five distinct u_e there are at most three such disjoint-pair equalities. Indeed project to distinct reals and let their successive gaps be $A, B, C', D > 0$. The five possible equations are $A = C'$, $A = C' + D$, $A = D$, $A + B = D$, $B = D$; at most three coexist. The graph on ten spins with the four-simple-zero pairs as edges has at least $45 - 18 = 27$ edges. At a fixed spin and branch value at most one omitted element realizes the equality, so every vertex has degree at least three. A disconnected graph with this minimum degree would have at most 21 edges, a bipartite graph at most 25. Connectedness and an odd cycle force $S_a = 6$ throughout. The displayed equation now gives $-1/3$ for every distinct pair. The Gram matrix has eigenvalues 3 and $19/3$, the latter with multiplicity nine. □

The local contribution at a zero of order j is $-j/12$, including a nonsimple zero. Indeed the supported Euler products on the different closed elliptic-tail curves add by excision. At each simple zero the local calculation in the proof applies irrespective of the other zeros. One double zero must therefore contribute $-1/6$ by the global mixed pairing and the two simple contributions. With two double zeros send their underlying branch values to $0, \infty$. They are the ramification points of the map P/R; its deck involution is $z \mapsto -z$. Thus the common pairs are $\{\pm u\}$, $\{\pm v\}$. Writing $k = uv$, the hyperelliptic curve has equation $y^2 = z(z^2 - u^2)(z^2 - v^2)$ and automorphism $(z, y) \mapsto (k/z, k^{3/2}y/z^3)$. It exchanges the two supports and preserves each spin because it sends its triple to its complement. Choose square-compatible lifts and invariant metrics by averaging over the finite extension by signs. The spin orientation continued from the complex determinant and the supported Euler product are then preserved. Hence the two contributions agree, each being $-1/6$.

On $y^2 = \prod_{j=1}^{6}(x-j)$, the two exceptional pairs, indexing a spin by its triple containing 1, are $126, 135$ and $135, 156$. The smooth-source kernel pairing equals 5 for every even spin: the ramification-tail term equals 1 and the complete square is 6. Calculations C3–C4 of [Wue26g] give the characteristic-degree calculation on the singular spectral incidence and its separate tail calculation. Identifying the two compactifications at cusps and tacnodes remains open.

8. Triple ramification and multisections of a spin line

The triple-ramification divisors give a test for the appearance of actual curves. A multisection here is the image of a curve in the total space of a line or vector bundle whose projection to the base is finite and surjective. We call its parametrization simple if it has generic degree one onto its image.

Theorem 8.1. *Let C be any smooth projective curve of genus $g \geq 2$, and let $\theta^2 = K_C$. On every connected component of the space of connected degree-m covers of C with exactly one triple ramification point and no other ramification, there is a cover $f : Y \to C$ such that*

$$h^0(Y, f^*\theta) > h^0(C, \theta).$$

For every $m \geq 3$ the holomorphic total space of θ consequently contains simple multisections of degree m with smooth normalization of genus $H_m + 1$ and a single triple ramification point over C. Their supports can be chosen arbitrarily close to the zero section.

Proof. Let T be such a connected component. It is a proper curve finite in coarse space over C by Section 5. We may pass to a connected finite level cover so that the family $\pi : \mathcal{Y} \to T$ is a smooth projective family over a smooth curve. For example, a symplectic level structure of level at least three on the first homology of the smooth sources rigidifies their automorphisms. Write $f : \mathcal{Y} \to C$ also for the evaluation map, $p : T \to \mathcal{Y}$ for the ramification section, $E = p(T)$, and $K = c_1(f^*K_C)$. The number $K \cdot E$ is strictly positive since $T \to C$ moves the branch value. Relative Riemann–Hurwitz and the first nonzero jet along p give

$$c_1(\omega_\pi) = K + 2E, \qquad K^2 = 0, \qquad E^2 = -K \cdot E/3.$$

Put $L = f^*\theta$. Grothendieck–Riemann–Roch for this smooth family gives the degree of its negative determinant of cohomology as

$$\begin{aligned} \deg c_1(-R\pi_* L) &= -\int_{\mathcal{Y}} \left(\tfrac{1}{2}c_1(L)^2 - \tfrac{1}{2}c_1(L)c_1(\omega_\pi) + \tfrac{1}{12}c_1(\omega_\pi)^2\right) \\ &= \tfrac{1}{6}K \cdot E - \tfrac{1}{3}E^2 = \tfrac{5}{18}K \cdot E. \end{aligned}$$

Suppose that every section on every fiber is a pullback. By cohomology and base change $\pi_* L$ is then the constant bundle with fiber $H^0(C, \theta)$. Relative duality and Riemann–Roch show that $S = \pi_*(\omega_\pi L^{-1})/(df)\pi_* L$ is a line bundle, with $\deg S^* = \deg c_1(-R\pi_* L)$. By the residue pairing S is isotropic in the rank-two orthogonal residue bundle. That bundle has the global isotropic line $I = \pi_*(L(E)/L)$ of degree $K \cdot E/6$. The two isotropic directions in an orthogonal plane are distinct; once one is chosen, the other is a line paired perfectly with it. Connectedness therefore forces $\deg S^* = \pm K \cdot E/6$. Both values contradict the degree $5K \cdot E/18$. This argument uses only the algebraic isotropics and includes effective θ.

For each $m \geq 3$ choose standard surface generators. On the first handle use $(12), (23)$, and on the second use $(1\,2\,\ldots\,m)$ and the identity, with the remaining generators trivial. The commutator product is a 3-cycle, and these permutations generate S_m. Filling the puncture gives a connected cover with the required ramification. Its Hurwitz component consists of covers with the same full monodromy. Take the additional section just proved to exist. Its

graph (f, s) is simple: the normalization of its image gives an intermediate cover, whereas the natural action of S_m on m sheets is primitive. A normalized image of degree one over C would make s a pullback, and the other possibility makes the graph generically injective. The graphs of positive scalar multiples of s have different maximum fiber norms and converge to the zero section. Their normalizations have the stated genus by Riemann–Hurwitz. □

Theorem 8.2. *Let a, b be distinct non-effective theta characteristics on C. Identify their underlying smooth complex lines, and take any continuous path of real Cauchy–Riemann operators on this line between their spin operators. Suppose the operators on C are invertible throughout. In every degree $m \geq 3$ there is an interior parameter at which a smooth cover $f : Y \to C$ with full symmetric monodromy and a single triple ramification point has nonzero pullback kernel. Its kernel sections give simple almost-complex multisections accumulating at the zero section. Such paths and pairs exist in every base genus $g \geq 2$. A single split path on a closed symplectic six-torus realizes these multisections in all degrees $m \geq 3$.*

Proof. Choose the first generator in the preceding permutation construction to represent the Poincaré dual of the nonzero difference ba^{-1} modulo two. Its monodromy is odd. On the corresponding compact triple divisor T, the ratio of the two induced spin parities is -1 by Section 6, since both embedded spin lines have even parity. By Theorem 4.2 the oriented obstruction planes at the two endpoints have opposite nonzero Euler degrees on T. If every pullback along the path were injective there, the constant-rank smooth-domain Fredholm family would give an oriented plane bundle over $T \times [0, 1]$. Its two Euler degrees would be equal. This proves the existence of a nonzero section. It cannot descend since the base operator has zero kernel. In using primitivity for its graph, normalization of a simple image is the usual factorization for a nonconstant pseudoholomorphic map. The projection from the normalization to C is holomorphic. A degree-one projection is an isomorphism, and the resulting graph over C solves the base equation. This proves simplicity just as in the holomorphic case. The positive real multiples again give distinct supports. The assertion permits the same parameter to serve more than one degree.

To construct a path, decrease the zeroth-order term of the spin operator on a to zero. Every strictly positive amplitude is injective by the spin norm identity, and $\bar{\partial}_a$ is invertible by non-effectivity and index zero. Join the complex operators by a path in the complement of the theta divisor of $\mathrm{Pic}^{g-1} C$, which is connected. A family of representatives along the interval identifies the underlying smooth lines. Reverse the amplitude deformation at b. For existence in each genus take a hyperelliptic curve $y^2 = \prod_{i=1}^{2g+2}(x - e_i)$, with ramification points W_i and divisor H over infinity. For $|A| = g + 1$ the line $\mathcal{O}_C(\sum_{i \in A} W_i - H)$ squares to K_C and has no sections. Indeed, an invariant meromorphic function with at most these simple poles and vanishing over infinity is zero. An anti-invariant one is $yP(x)/\prod_{i \in A}(x - e_i)$ for a polynomial P, and vanishing over infinity again forces zero. Replacing one element of A changes the line, since two distinct points on C are not linearly equivalent. Finally keep one spin summand fixed and put the path in the other summand of the normal bundle of degree $2g - 2$. The graph equation defines a path of almost-complex germs with the prescribed projections. The compact realization of [Wue26j, Proposition 3.1 and §15] extends the family from a fixed small tube to tame structures on a symplectic six-torus. For each kernel section a sufficiently small multiple lies in this tube, giving a curve in the class $m[C]$ with the specified normalization. □

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