

HERMITIAN–EINSTEIN METRICS AND SMALL SPECTRA FOR THREE-STEP EXTENSIONS

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ABSTRACT. We prove uniform relative asymptotics for every small Dolbeault eigenvalue on trace-free endomorphism-valued zero-forms in a class of three-step Hermitian–Einstein extensions, allowing arbitrary separation of positive spectral scales. The graded factors are distinct line bundles of equal slope at a reference polarization on a compact Kähler manifold. Adjacent extension classes vary in fixed cohomology subspaces; the remaining terms are quadratic coexact Hodge primitives of their compositions, with no independent first-to-third-layer parameters. Assume integrability and a connected extension graph. At nearby polarizations, suppose diagonal rescaling balances the squared norms of the adjacent classes against the slope differences, with sufficiently small sum of rescaled squared norms. The rescaled split metric then approximates the Hermitian–Einstein metric with the same determinant, with logarithmic error bounded by a uniform constant times that sum. Every small eigenvalue has the same relative accuracy with respect to the corresponding eigenvalue of the graph Laplacian weighted by these squared norms; the complementary spectrum has a uniform positive gap. The proof controls curvature traces by the energy crossing each vertex cut. A family on the product of three projective lines has a spectral scale undetected by its polarization parameters. At each fixed bundle in this family, both singular values of the intrinsic metric response to the two polarization parameters agree, with the same relative accuracy, with those of the differentiated finite balancing equation.

1. INTRODUCTION

Near a stability wall, a Hermitian–Einstein connection approaches the connection on its graded object. The small endomorphism eigenvalues record how the graded factors remain joined. When some extension directions are much weaker than others, convergence of the connection does not by itself determine each small eigenvalue: an error small compared with the total extension energy may exceed the weakest eigenvalue. The same difficulty appears when one asks how the Hermitian–Einstein metric responds to a change of polarization.

For extensions of two polystable bundles, a finite weighted graph can retain every small spectral scale. We consider the next case, where extension arrows compose. Their compositions can be nonzero differential forms, and their cohomology classes can impose genuine relations on the extension parameters. The nonlinear terms required by integrability must be included in both the metric construction and the spectral comparison. We prove relative estimates for a specified three-step class, uniformly in its extension directions and in all positive scale ratios.

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1.1. A first example and the general statement. The simplest example is a fixed rank-three bundle on $(\mathbb{P}^1)^3$ with graded lines

$$\mathcal{O}(-1, -1, 2), \quad \mathcal{O}(1, -2, 1), \quad \mathcal{O}.$$

Two nonzero adjacent harmonic extensions have a nonzero but exact composition form. A quadratic endpoint term makes their matrix integrable. In two actual polarization coordinates $k_1, k_2 > 0$, the finite model is the path with edge weights k_1, k_2 , whose positive eigenvalues are

$$\nu_{1,2} = k_1 + k_2 \mp \sqrt{k_1^2 - k_1 k_2 + k_2^2}.$$

Section 2 constructs the bundle, proves that it intrinsically requires three polystable layers at the reference slope, and specifies its balanced metric independently of the unknown HE metric. The actual small eigenvalues satisfy $\lambda_j/\nu_j = 1 + O(k_1 + k_2)$, and the rest of the spectrum stays separated from zero. This remains true if one parameter is exponentially smaller than the other.

The general construction starts with distinct equal-slope line factors F_i , divided into three ordered layers. For each pair in adjacent layers choose a finite-dimensional space of harmonic extension classes. The endpoint correction is the negative coexact Green primitive of the sum of their two-arrow products. There are no independent linear endpoint parameters. Hodge decomposition makes integrability exactly equivalent to the vanishing of the cup-product obstruction. We call this the canonical three-step family; its complete definition is in Section 3.

For each nonzero adjacent block draw an edge from source to target. Write n_e for its initial weight, the squared harmonic norm divided by the base volume. Let D be the incidence matrix, with column $e_i - e_j$ for $j \rightarrow i$, and let $b_i = \kappa_E - \kappa_i$ be the centered HE constants of the factors. A positive diagonal gauge is chosen by the finite equation

$$Dw = b, \quad w_e = n_e e^{x_i - x_j}, \quad \sum_i x_i = 0.$$

If the support is connected and admits a positive flow, this gauge exists uniquely. Put $\sigma = \sum_e w_e$ and let $h_{\text{bal}} = g_0^* h_0$, where $g_0 = \text{diag}(e^{x_i/2})$. Let $\nu_1 \leq \dots \leq \nu_{m-1}$ be the graph eigenvalues on trace-free diagonal constants with unit vertex masses.

Theorem 3.3 constructs the fixed-determinant HE metric and proves

$$(1) \quad e^{-C\sigma} h_{\text{bal}} \leq H \leq e^{C\sigma} h_{\text{bal}}, \quad \left| \frac{\lambda_j}{\nu_j} - 1 \right| \leq C\sigma, \quad \lambda_m \geq \gamma > 0.$$

Here λ_j are the actual Dolbeault eigenvalues on complex trace-free endomorphism-valued zero-forms, counted with complex multiplicity. The constants depend only on the fixed coefficient spaces and a compact neighborhood of background metrics. There is no restriction on positive edge ratios, on eigenvalue ratios, or on the rank of the obstruction derivative. The positive-flow condition is also exactly the local stability criterion for this family. At disconnected small coefficients a separate auxiliary statement identifies the kernel as the diagonal constants on each graph component.

1.2. Why the estimates remain relative. After solving the elliptic equation complementary to constant diagonal endomorphisms, the remaining curvature has the form

$$\mathcal{F} = -b + \mu + R.$$

Harmonicity removes the first-order terms, and the three-layer sign symmetry removes the cubic jet. A total estimate $R = O(\sigma^2)$ follows, but is too weak for (1). The useful estimate is instead

$$(2) \quad \left| \sum_{i \in S} R_i \right| \leq Cq \sum_{e \text{ crossing } S} n_e \quad \text{for every vertex cut } S,$$

where q is the current total linear-arrow energy. Removing all independent coefficients crossing a cut also removes every quadratic path with separated endpoints. The resulting unitary connection splits, and its integrated partial curvature trace is fixed by Chern–Weil. This remains valid even when the cut leaves the obstruction locus. The resulting exact vanishing and cut-sign symmetry imply (2) by Taylor expansion.

A finite degree argument now corrects the balanced gauge by $O(\sigma)$. On a boundary sphere, sort its diagonal values and select a cut across a largest adjacent gap. The finite moment increment across that cut is positive and dominates the residual by the same cut energy. That energy may be arbitrarily small, but it appears on both sides. There is no need to divide an absolute remainder by a weak eigenvalue.

For the spectrum, each quadratic endpoint commutator is estimated along the two edges of its path. Its energy is at most Cq times the linear graph energy, separately for every diagonal direction. Co-closedness removes the first coupling to the complementary elliptic space, whose spectral gap is fixed. Min–max then gives every relative eigenvalue estimate, including repeated eigenvalues. Sections 4 and 5 provide the full arguments.

1.3. A geometric family and metric response. The four lines

$$\mathcal{O}(-2, -2, 4), \quad \mathcal{O}(1, -2, 1), \quad \mathcal{O}(-2, 1, 1), \quad \mathcal{O}$$

on $(\mathbb{P}^1)^3$ give a nonlinear diamond with 24 adjacent-layer coordinates and five quadratic relations. Its endpoint primitive is nonzero. Section 6 proves that full-support torus orbits are exactly bundle isomorphism classes over the fixed base and constructs a local 16-dimensional family of nonisomorphic bundles. The theorem is uniform on that family, at points where the obstruction derivative loses rank, and on every connected support satisfying its own stable inequalities.

The finite weights depend on an orbit invariant $\rho = n_{12}n_{24}/(n_{13}n_{34})$ in addition to the polarization. An explicit family has $\rho = 1 + |M|^2$. Even with the two polarization parameters constrained by $k = 3l/5$, its smallest eigenvalue divided by the total energy tends to zero as $|M| \rightarrow \infty$. Thus a uniform spectral description must retain extension-orbit data.

We also determine how the actual HE metric changes with these two polarization parameters while the bundle is fixed. Its intrinsic tangent is $(H^{-1}\partial_v H)_0$, measured in the current normalized L^2 norm. The finite prediction is the derivative of the balancing equation,

$$\mathcal{R}_0 = L(w)^{-1}(B_0 - D \operatorname{diag}(w)A_s),$$

where B_0 differentiates the centered demands and A_s differentiates the logarithmic harmonic norms. Both matrices are printed in Section 8. Theorem 9.1 compares the two singular values of this operator with the two singular values of the actual metric response, each to relative error $O(\sigma)$.

A metric inequality cannot simply be differentiated to prove this result. We differentiate the cut estimates, obtain a directional inverse bound, and retain the exact intrinsic metric velocity and the changing fixed-bundle identification. The unbounded family then has two

actual response scales $1/\delta$ and $1/l$, where $\delta/l \sim \sqrt{3/(5\rho)}$. Section 10 computes their leading constants and the two polarization directions that distinguish them. The third graph eigenvalue does not create a third singular value for a two-dimensional input.

1.4. Relation to earlier work. Clarke–Tipler prove local chamber and HYM convergence results for a fixed semistable bundle with abelian graded automorphisms, and construct the complementary reduction on an ambient coefficient ball [CT24, Proposition 1.1, Theorem 1.2 and Propositions 3.1–3.2]. Their setting already includes successive extensions. Their Remark 1.4 distinguishes compact families from the problem of uniformity over all small graded deformations. Delloque develops complementary reduction and quantitative convergence under metric variation with a broader graded automorphism group [Del25, Propositions 4.2, 4.7 and Theorems 4.8–4.9]. The present result concerns the stated canonical three-step family. Its additional conclusions are relative control of every small eigenvalue and the explicit metric response, uniform through unrestricted positive scale separation and variation of the adjacent extension directions.

The cut-relative correction and energy-relative spectral argument have two-step predecessors in [Wue26b, Wue26c]. The intrinsic differentiation and complementary inverse were used for two-step response in [Wue26a]. We retain these mechanisms and prove the new nonlinear geometric and differentiated cut estimates needed here. The scalar distinct-line theorem does not subsume the matrix multiplicities in [Wue26c]. Nor does the 24-coordinate diamond exhaust the full graded deformation space. Quotient metrics, stabilizer-changing derivatives and actual flow require separate arguments.

2. A CHAIN AND A FIRST OBSTRUCTED EXAMPLE

We begin with a fixed rank-three bundle whose two small eigenvalues have closed formulas. It displays the nonlinear term needed for integrability without an obstruction equation on its parameters. A tensor-product diamond then separates those two issues: its integrability equation is nontrivial, but it needs no quadratic primitive. Section 6 will combine both phenomena on a family of genuinely different bundles.

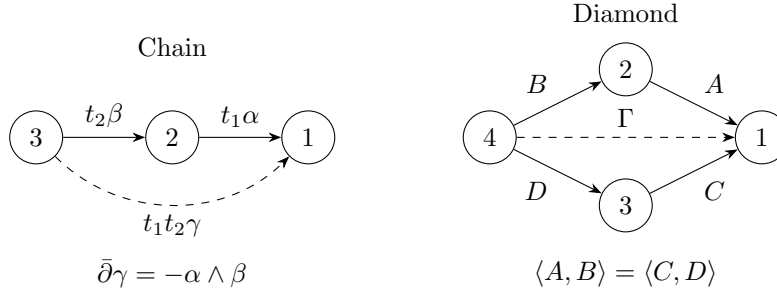


FIGURE 1. Two three-step constructions. Solid arrows are the independent harmonic extension classes; dashed arrows are the quadratic corrections required by integrability. The chain of Section 2 has an exact composition and no parameter relation. The nonlinear diamond of Section 6 has both a nonzero correction and a cup-product relation. The graph for the leading spectral model contains only the solid edges.

2.1. A chain on three projective lines. On $X = (\mathbb{P}^1)^3$, normalize the Fubini–Study forms by $\int_{\mathbb{P}^1} \omega_i = 1$. Give all product line bundles their fixed product metrics, with $iF_{\mathcal{O}(d)} = 2\pi \sum_i d_i \omega_i$. Set

$$F_1 = \mathcal{O}(-1, -1, 2), \quad F_2 = \mathcal{O}(1, -2, 1), \quad F_3 = \mathcal{O}.$$

The three lines have slope zero at $\omega_1 + \omega_2 + \omega_3$. Their selected adjacent Hom bundles and endpoint bundle are

$$A = \mathcal{O}(-2, 1, 1), \quad B = \mathcal{O}(1, -2, 1), \quad C = \mathcal{O}(-1, -1, 2).$$

Projective-line cohomology and Künneth give

$$h^1(A) = h^1(B) = 4, \quad H^q(C) = 0 \quad \text{for every } q.$$

Every off-diagonal Hom has zero H^0 , because its degree vector has a negative entry. We use the standard formulas recorded in [Stacks, Tags 01XT and 0BED].

Choose nonzero harmonic $\mathcal{O}(-2)$ -valued $(0, 1)$ -forms η_1, η_2 on the first two factors and nonzero holomorphic sections s_i of $\mathcal{O}(1)$, with another such section s'_3 on the third. Put

$$\alpha = \eta_1 \otimes s_2 \otimes s_3, \quad \beta = s_1 \otimes \eta_2 \otimes s'_3.$$

They remain harmonic for $\omega_h = \sum_i h_i \omega_i$: the product Laplacian is the sum of factor Laplacians multiplied by h_i^{-1} . Their composition is a nonzero differential form. On the product of open sets where the factors are nonzero, its two differential legs are independent. Its cohomology class vanishes because $H^2(C) = 0$.

Let D_C be the Dolbeault operator on C , and let $G_{2,h}$ be its Green operator on $(0, 2)$ -forms, zero on harmonic forms. Put

$$(3) \quad \gamma_h = -D_{C,h}^* G_{2,h}(\alpha \wedge \beta), \quad D_C \gamma_h = -\alpha \wedge \beta, \quad D_{C,h}^* \gamma_h = 0.$$

In particular $\gamma_h \neq 0$. Write D_0 for the split Dolbeault operator on $F_1 \oplus F_2 \oplus F_3$. The family

$$(4) \quad N_h(t) = \begin{pmatrix} 0 & t_1 \alpha & t_1 t_2 \gamma_h \\ 0 & 0 & t_2 \beta \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\partial}_{h,t} = D_0 + N_h(t)$$

is integrable for every $(t_1, t_2) \in \mathbb{C}^2$. Equation (3) cancels its sole possible $(0, 2)$ curvature entry. It is also co-closed. The parameter space selects one direction in each adjacent H^1 ; it is not the full graded deformation space. For example the reverse group $H^1(\text{Hom}(F_1, F_3))$ has dimension four.

Under a diagonal gauge, t_1, t_2 acquire ratios $g_1/g_2, g_2/g_3$; their product has exactly the endpoint ratio g_1/g_3 . Thus every nonzero pair gives the same holomorphic isomorphism class for fixed h . The bundle can also be held fixed as h varies. The form $\gamma_h - \gamma_1$ is closed and $H^1(C) = 0$, so choose smoothly f_h with $D_C f_h = \gamma_h - \gamma_1$, for example by the reference Green inverse. With convention $D' = g D g^{-1}$, the gauge

$$\Phi_h = I - f_h E_{13}$$

sends the unit-parameter reference bundle E to its h -presentation. Conjugation contributes $+D_C f_h$ in that entry; every other product with this endpoint matrix vanishes. The gauge has determinant one.

2.2. Why three layers are intrinsic. Let E_{12} be the nonsplit α extension of F_2 by F_1 . The class β lifts uniquely to an extension of F_3 by E_{12} : $H^2(C) = 0$ gives existence, and $H^1(C) = 0$ gives uniqueness. Its image β is nonzero. This gives $0 \subset F_1 \subset E_{12} \subset E$, agreeing with (4).

At the reference polarization the bundle is semistable. The connecting maps of these sequences and the off-diagonal Hom vanishings imply

$$\mathrm{Hom}(F_1, E) = \mathbb{C}, \quad \mathrm{Hom}(F_2, E) = \mathrm{Hom}(F_3, E) = 0.$$

For instance the connecting map for F_2 sends its identity to the nonzero class α , and the connecting map for F_3 has nonzero image β .

A stable locally free bundle of slope zero mapping to E maps, after factoring through earlier terms, nontrivially to one line quotient. A stable source of rank greater than one cannot have a rank-one quotient of nonpositive slope. A nonzero map between equal-slope lines has no zero divisor and is an isomorphism. Thus every stable summand of a slope-zero polystable subbundle must be one of the F_i . The displayed Hom groups force its image to be the single copy of F_1 . But E/F_1 is the nonsplit β extension and is not polystable. Therefore E cannot be expressed as an extension of two polystable bundles of the reference slope. The third layer is not an artifact of the chosen matrix.

2.3. The two eigenvalues. Fix $h_3 = 1$ and set

$$h_1 = (1 + k_1/(2\pi))^{-1}, \quad h_2 = (1 + k_2/(2\pi))^{-1}.$$

For small signed k_i these are genuine Kähler forms. Since $\kappa_{\mathcal{O}(d)} = 2\pi \sum_i d_i/h_i$, the centered demands are

$$b = (k_1, k_2 - k_1, -k_2).$$

Let $c_1(h) = \|\alpha\|_2^2 = c_1(1)/h_1$ and $c_2(h) = \|\beta\|_2^2 = c_2(1)/h_2$, with volume-normalized norms. For $k_1, k_2 > 0$ the independently balanced amplitudes are $t_i^0 = \sqrt{k_i/c_i(h)}$. The positive determinant-one gauge is

$$g(t) = \mathrm{diag}(t_1^{2/3}t_2^{1/3}, t_1^{-1/3}t_2^{1/3}, t_1^{-1/3}t_2^{-2/3}).$$

Define $h_{\mathrm{bal}} = (g(t^0)\Phi_h)^*h_0$ on the fixed E . Its finite operator is

$$(5) \quad L(k) = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda(\lambda^2 - 2(k_1 + k_2)\lambda + 3k_1k_2)$. Thus, putting $\sigma = k_1 + k_2$, its two positive eigenvalues are

$$(6) \quad \nu_{1,2} = \sigma \mp \sqrt{k_1^2 - k_1k_2 + k_2^2}.$$

Theorem 3.3 gives the actual metric estimate (15), $\lambda_j/\nu_j = 1 + O(\sigma)$ for $j = 1, 2$, and $\lambda_3 \geq \gamma$. Its hypotheses are precisely the three singleton layers, the two chosen harmonic lines and primitive (3). The positive flow condition is $k_1, k_2 > 0$. Necessity is also visible in the subbundles F_1 and E_{12} . There is no restriction on k_1/k_2 , including exponentially separated approaches to the wall.

In this example the exact diagonal commutator energy has the additional term

$$\|\gamma_h\|_2^2 |t_1 t_2|^2 |\xi_1 - \xi_3|^2.$$

The inequality $|\xi_1 - \xi_3|^2 \leq 2|\xi_1 - \xi_2|^2 + 2|\xi_2 - \xi_3|^2$ makes it a relative $O(\sigma)$ correction to the path form. This elementary calculation is the model for the general spectral proof in Section 5.

2.4. An obstruction that comes from a tensor product. On $(\mathbb{P}^1)^4$ put $U = \mathcal{O}(-1, 1)$, $V = \mathcal{O}(1, -1)$ on the first two factors, and use U', V' with the same degrees on the last two. Take

$$(F_1, F_2, F_3, F_4) = (U \boxtimes U', V \boxtimes U', U \boxtimes V', V \boxtimes V').$$

Choose nonzero harmonic forms α in $H^1(\mathcal{O}(-2, 2, 0, 0))$ and β in $H^1(\mathcal{O}(0, 0, -2, 2))$. Each group has dimension three. Their product is a nonzero harmonic class in $H^2(\mathcal{O}(-2, 2, -2, 2))$, of dimension nine; the endpoint H^1 vanishes. The four blocks

$$N_{12} = x\alpha, \quad N_{13} = y\beta, \quad N_{24} = v\beta, \quad N_{34} = w\alpha$$

satisfy

$$(7) \quad (D_0 + N)^2 = (xv - yw)\alpha \wedge \beta E_{14}.$$

Thus $xv = yw$ is a genuine obstruction equation. No endpoint form can repair a nonzero value, because its cohomology class is nonzero. The canonical quadratic primitive is zero here: the products are already harmonic. At unit coefficients the bundle is $E_A \boxtimes E_B$, the tensor product of the two nonsplit rank-two extensions.

At the reference slope its maximal polystable subbundle is F_1 . Indeed Künneth gives $\text{Hom}(F_1, E) = \mathbb{C}$ and $\text{Hom}(F_i, E) = 0$ for $i > 1$; the preceding stable-source argument applies to its line filtration. Its quotient by F_1 has a nonsplit extension of F_4 by $F_2 \oplus F_3$. It therefore also needs three polystable layers. This geometric fact does not remove its tensor-product origin. The full graded cohomology dimensions are $(h^0, h^1, h^2) = (4, 24, 36)$; the four selected scalar parameters are only a subfamily.

Set $h_2 = h_4 = 1$, $h_1 = (1 + a/(2\pi))^{-1}$ and $h_3 = (1 + b/(2\pi))^{-1}$. The centered demands are $(a + b, -a + b, a - b, -a - b)$. For $a, b > 0$, choosing $s = \sqrt{a/\|\alpha\|_2^2}$ and $t = \sqrt{b/\|\beta\|_2^2}$ gives balanced coefficients (s, t, t, s) and weights (a, b, b, a) . The gauge is

$$g_0 = \text{diag}(\sqrt{st}, \sqrt{t/s}, \sqrt{s/t}, 1/\sqrt{st}).$$

The positive graph eigenvalues, in increasing order, are $2\min(a, b), 2\max(a, b), 2(a + b)$. The total arrow energy in the normalization of Theorem 3.3 is $2(a + b)$; its error is therefore $O(a + b)$. The theorem gives the actual HE metric and all three relative spectral estimates, and stability holds exactly for $a, b > 0$. These conclusions can also be understood from the two factor problems.

This example makes a cut obstruction particularly visible. Removing the crossing arrows for $S = \{1, 2, 4\}$ sets $y = w = 0$ but leaves xv . The residual in (7) is then nonzero and nonexact. The general proof uses the auxiliary unitary connection at this cut, retaining the topological partial trace even though integrability fails.

3. CANONICAL THREE-STEP EXTENSIONS AND FINITE BALANCE

We now specify the class for which the comparison is uniform. The linear extension classes join adjacent layers. Their compositions determine a quadratic correction between the first and third layers. There are no independent linear parameters between those endpoint layers.

3.1. Geometric data and the obstruction. Let X be a fixed compact connected Kähler manifold of complex dimension n . Let $F_1, \dots, F_m$, $m \geq 2$, be pairwise nonisomorphic holomorphic line bundles of equal slope at a reference form ω_0 . We assume

$$(8) \quad H^0(X, \text{Hom}(F_j, F_i)) = 0 \quad (i \neq j).$$

For distinct line bundles of equal slope this follows from stability. Partition the indices into layers I_0, I_1, I_2 . The allowed arrows $j \rightarrow i$ have $j \in I_{r+1}$ and $i \in I_r$. For each such ordered pair choose a subspace

$$V_{ij} \subset H^1(X, \text{Hom}(F_j, F_i)), \quad V = \bigoplus_{ij} V_{ij}.$$

All directions with the same endpoints are included in one space V_{ij} . A zero space contributes no arrow.

Let ω_s be a smooth family of Kähler forms in a neighborhood of zero with compact closure in its parameter domain. Choose smoothly varying HE metrics $h_{i,s}$ on the lines, and write $h_{0,s}$ for their orthogonal sum. Their mean curvatures and the centered demands are

$$(9) \quad \kappa_i(s) = \frac{2\pi \deg_s F_i}{\text{Vol}_s X}, \quad \kappa_E(s) = \frac{1}{m} \sum_i \kappa_i(s), \quad b_i(s) = \kappa_E(s) - \kappa_i(s).$$

Here $\deg_s F = \int_X c_1(F) \wedge \omega_s^{n-1} / (n-1)!$ and $\text{Vol}_s X = \int_X \omega_s^n / n!$. All L^2 inner products in this paper are divided by $\text{Vol}_s X$. A constant diagonal endomorphism therefore has squared norm $\sum_i |\xi_i|^2$.

Denote the split Dolbeault operator by D_0 and its Chern connection by $\nabla_{0,s}$. Let $H_s(z_{ij})$ be the harmonic representative of $z_{ij} \in V_{ij}$. The harmonic inclusions and their Gram matrices vary smoothly with s , and these Gram matrices are uniformly positive on the chosen parameter closure. The linear block form $L_s(z)$ has entries $H_s(z_{ij})$. For $i \in I_0$, $k \in I_2$, define

$$(10) \quad Q_{ik,s}(z) = -D_{ik,s}^* G_{ik,2,s} \sum_{j \in I_1} H_s(z_{ij}) \wedge H_s(z_{jk}), \quad N_s(z) = L_s(z) + Q_s(z),$$

where D_{ik} acts on $\text{Hom}(F_k, F_i)$ and $G_{ik,2,s}$ is its Green operator on $(0,2)$ -forms, zero on harmonic forms. Every other entry of Q_s is zero. The cohomological obstruction is

$$(11) \quad \Omega_{ik}(z) = \sum_j z_{ij} \smile z_{jk} \in H^2(X, \text{Hom}(F_k, F_i)), \quad Z = \Omega^{-1}(0).$$

The cup product in this formula is independent of s .

Lemma 3.1. *On the whole coefficient space V ,*

$$(12) \quad (D_0 + N_s(z))^2 = (H_s \Omega_{ik}(z))_{ik}, \quad D_{0,s}^* N_s(z) = 0.$$

Consequently $D_0 + N_s(z)$ is integrable exactly when $z \in Z$. The map Q_s is quadratic with uniform bilinear bounds in every fixed spatial norm. A constant diagonal gauge g acts by g_i/g_j on z_{ij} and satisfies $N_s(g \cdot z) = g N_s(z) g^{-1}$.

Proof. The wedge products in (10) are closed. Hodge decomposition gives their negative, plus their harmonic projection, after applying D_{ik} to $Q_{ik,s}$. These are exactly the entries of $D_0 N_s + N_s \wedge N_s$: there is no product involving a quadratic endpoint block because there are only three layers. Harmonicity gives co-closedness for the linear terms, and $(D_{ik,s}^*)^2 = 0$ gives it for the quadratic terms. The cohomology dimensions are fixed, so the positive spectra on the harmonic complements stay uniformly separated from zero. Smooth elliptic parameter

dependence gives the asserted Green bounds. Finally each product along $k \rightarrow j \rightarrow i$ has character $(g_i/g_j)(g_j/g_k) = g_i/g_k$, proving equivariance. $\square$

For $z \in Z$, write $E_{s,z}$ for this holomorphic bundle. The definition allows singularities of Z and changes in the rank of $D\Omega$. It does not assert that $E_{s,z}$ is a fixed holomorphic bundle as s varies. The examples in Sections 2 and 6 supply such identifications separately. The word canonical here refers precisely to the Green operator choice (10).

3.2. The graph determined by finite balance. The support graph $\Gamma(z)$ has one edge $e : j \rightarrow i$ for each nonzero block z_{ij} . Set

$$n_e(s, z) = \|H_s(z_e)\|_2^2, \quad q_s(z) = \sum_e n_e(s, z), \quad \mu_s(z) = D(n_e(s, z))_e,$$

where D is the incidence matrix with column $e_i - e_j$ for the arrow $j \rightarrow i$. Thus an edge contributes positively to its target and negatively to its source. A *positive flow for b* means a vector $w_e > 0$ on exactly the supported edges with $Dw = b$. This is a finite system of linear equations and strict inequalities.

Lemma 3.2. *If $\Gamma(z)$ is connected and b admits a positive flow, there is a unique positive diagonal gauge g_0 of determinant one for which $z^0 = g_0 \cdot z$ satisfies $\mu_s(z^0) = b$. It is determined entirely by n_e and b .*

Proof. On $\mathcal{K} = \{x \in \mathbb{R}^m : \sum_i x_i = 0\}$ consider

$$f(x) = \sum_e n_e e^{(D^t x)_e} - \langle b, x \rangle.$$

Choose any positive flow w^* for b . Then f is the sum of $n_e e^{t_e} - w_e^* t_e$ with $t = D^t x$. Each scalar term is bounded below and tends to infinity as t_e tends to either end of the real line. Connectedness makes D^t injective on $\mathcal{K}$; hence f is coercive. Its Hessian is positive definite on $\mathcal{K}$. Its unique minimizer x^0 solves $D(n_e e^{D^t x^0}) = b$. Take $g_0 = \text{diag}(e^{x_i^0/2})$. $\square$

Put $a_i = 1 - r$ for $i \in I_r$. Each arrow has $a_i - a_j = 1$, so every balanced representative satisfies

$$(13) \quad w_e = n_e e^{(D^t x^0)_e}, \quad \sum_e w_e = \langle a, b(s) \rangle =: \sigma(s).$$

Since $b(0) = 0$, this total energy tends to zero with s . Individual weights can approach zero at arbitrarily different rates. Define

$$(14) \quad h_{\text{bal}} = g_0^* h_{0,s}, \quad q_w(\xi) = \sum_{e:j \rightarrow i} w_e |\xi_i - \xi_j|^2, \quad \sum_i \xi_i = 0.$$

The first expression is a metric on $E_{s,z}$, defined before an HE metric has been found. The second is a quadratic form on complex diagonal constants, with mass $\sum_i |\xi_i|^2$. Write its eigenvalues as $0 < \nu_1 \leq \dots \leq \nu_{m-1}$.

3.3. The uniform theorem.

Theorem 3.3. *For the data above there are $\varepsilon, C, \gamma > 0$ such that the following assertions hold for all s sufficiently close to zero and all $z \in Z$ with connected support admitting a positive flow for $b(s)$. If $0 < \sigma(s) < \varepsilon$, then $E_{s,z}$ is stable and has a unique HE metric $H_{s,z}$ with determinant $\prod_i h_{i,s}$. Moreover,*

$$(15) \quad e^{-C\sigma} h_{\text{bal}} \leq H_{s,z} \leq e^{C\sigma} h_{\text{bal}},$$

and the eigenvalues of the actual endomorphism Dolbeault Laplacian $\Delta_{H,s} = \bar{\partial}_E^{*H,s} \bar{\partial}_E$ on $\text{End}_0 E$ obey

$$(16) \quad \left| \frac{\lambda_j}{\nu_j} - 1 \right| \leq C\sigma \quad (1 \leq j \leq m-1), \quad \lambda_m \geq \gamma.$$

Eigenvalues are counted with complex multiplicity. The constants depend only on the fixed factors, coefficient spaces and compact background family. They are independent of the input amplitudes and their directions, all positive edge and spectral ratios, and the rank of $D\Omega$.

In this neighborhood, stability is equivalent to the positive-flow condition when the support is connected. A disconnected support never gives a stable bundle.

The operator in the theorem has form domain H^1 and operator domain H^2 . Its kernel on $\text{End}_0 E$ is zero; on all endomorphisms the kernel consists of the scalar constants. The estimates include unbounded families of extension orbits: the balancing gauge brings each orbit into one small coefficient ball, and its own norm is not restricted.

We also need a statement at small coefficients without integrability. Define the elliptic zero-form operator

$$(17) \quad \Delta_{s,z}^{\text{aux}} = (D_0 + [N_s(z), \cdot])^* (D_0 + [N_s(z), \cdot])$$

using $h_{0,s}$ and ω_s .

Proposition 3.4. *For every sufficiently small $z \in V$, the first $m-1$ eigenvalues of (17), including zeros, compare with those of the linear support graph by factors $1 \pm Cq_s(z)$. Its m th eigenvalue is at least γ . Its kernel consists exactly of diagonal constants on the connected components of $\Gamma(z)$, with total trace zero.*

This assertion concerns a zero-form operator even when $z \notin Z$; there is no Dolbeault complex at such a coefficient. At a change of support, an HE assertion requires the positive-flow condition for that new support. We prove the metric estimate next, then the spectral and stability statements.

4. CURVATURE ESTIMATES ACROSS EVERY CUT

A total remainder estimate of order σ^2 is insufficient when the smallest graph eigenvalue is much smaller than σ^2 . We instead estimate each partial curvature trace by the energy crossing the same vertex cut. This retains the weak scale before solving the finite equation.

4.1. The complementary equation on the whole coefficient ball. For a pointwise trace-free $h_{0,s}$ -Hermitian endomorphism u , put $H = h_{0,s}e^u$. The semiconnection $D_0 + N_s(z)$ has a unique H -unitary connection with that $(0,1)$ part. Its $(1,0)$ part is

$$(18) \quad \nabla_{s,z,u}^{1,0} = \nabla_{0,s}^{1,0} + e^{-u}(\nabla_{0,s}^{1,0}e^u) - e^{-u}N_s(z)^*e^u.$$

Metric compatibility proves this formula without integrability. Its trace is the fixed split determinant connection. Since contraction kills the $(0,2)$ and $(2,0)$ curvature parts,

$$\widehat{K}(s, z, u) = e^{u/2} i \Lambda_s F_{s,z,u} e^{-u/2} - \kappa_E(s) I$$

is $h_{0,s}$ -Hermitian and pointwise trace-free on the entire coefficient ball. At an integrable coefficient it is the centered Chern mean curvature.

Let Π_s be normalized L^2 projection onto the real constant diagonal trace-free sections, and let $\Pi_s^\perp = I - \Pi_s$. At $(z, u) = (0, 0)$, $\widehat{K} = -b$ lies in the diagonal space. Its real metric derivative is

$$(19) \quad \mathcal{L}_s u = \Delta_{\partial, s} u + \frac{1}{2}[u, K_{0, s}] = \frac{1}{2}(\Delta_{\partial, s} + \Delta_{\bar{\partial}, s})u, \quad K_{0, s} = \text{diag}(\kappa_i(s)).$$

The second identity is the endomorphism Kähler identity. This self-adjoint nonnegative operator has precisely the parallel diagonal Hermitian sections as its kernel. Indeed its energy is half the sum of the two derivative energies, and (8) excludes other parallel sections. It has a uniform positive gap on the complement.

For parameter dependence, identify the varying Hermitian section spaces with the reference space by smooth positive metric isometries. The projections are then a smooth finite-rank family. Restricted to the reference complement, $\Pi_s^\perp$ is a bounded isomorphism onto the current complement for s near zero. Thus the Sobolev implicit-function theorem has fixed domains, for example $H^{\ell+2} \rightarrow H^\ell$ with an integer $\ell > n$. The uniformly invertible complementary derivative gives a unique small solution

$$(20) \quad \Pi_s^\perp \widehat{K}(s, z, u_s(z)) = 0, \quad \Pi_s u_s(z) = 0.$$

It exists on one ambient coefficient ball. Harmonicity of the linear arrows makes its first coefficient derivatives vanish. Elliptic regularity and smooth parameter dependence give, at each fixed spatial order,

$$(21) \quad \|u_s(z)\|_{C^r} \leq C_r \|z\|^2.$$

All coefficient derivatives here and below are real derivatives. Constant diagonal unitaries preserve the construction, so uniqueness makes u_s equivariant under their action.

This complementary reduction is a standard part of local HYM deformation theory; see [CT24, Propositions 3.1–3.2] and [Del25, Lemma 4.1 and Proposition 4.2]. We require its ambient domain because a cut in an obstructed family can leave the integrable locus.

4.2. The partial-trace estimate. Identify the remaining diagonal curvature with a zero-sum vector and write

$$(22) \quad \mathcal{F}_s(z) = \Pi_s \widehat{K}(s, z, u_s(z)) = -b(s) + \mu_s(z) + R_s(z).$$

For a vertex subset S , let $R_S = \sum_{i \in S} R_i$ and let $q_{\partial S, s}(z)$ be the sum of the squared harmonic norms of the linear blocks with one endpoint in S and one outside.

Proposition 4.1. *On the whole ambient coefficient ball,*

$$(23) \quad |R_S(s, z)| \leq C q_s(z) q_{\partial S, s}(z)$$

for every vertex subset S , with the same constant.

Proof. First consider the coefficient jets. The constant and linear terms in R_s vanish. The projected quadratic curvature from the linear blocks is exactly μ_s . A term linear in the quadratic block $Q_s(z)$ is off diagonal, so its diagonal projection vanishes. The quadratic metric term projects to zero by self-adjointness of $\mathcal{L}_s$ and its diagonal kernel. Thus the quadratic jet of R_s also vanishes.

The diagonal unitary J with value $(-1)^r$ on layer I_r satisfies $N_s(-z) = J N_s(z) J^{-1}$. It changes the sign of the linear blocks and preserves the quadratic ones. Since J acts trivially on the diagonal target, $\mathcal{F}_s$ is even in z . The cubic jet vanishes. Taylor's theorem therefore gives

$$(24) \quad \|D_z^j R_s(z)\| \leq C_j \|z\|^{4-j} \quad (0 \leq j \leq 3).$$

In particular $D_z^2 R_s(z) = O(\|z\|^2)$.

For a fixed cut write $z = (p, r)$, where r collects all crossing linear coefficients. At $r = 0$, every linear block preserves the smooth summand $E_S = \bigoplus_{i \in S} F_i$ and its complement. A two-arrow path with separated endpoints contains a crossing arrow, so its quadratic term vanishes as well. All remaining quadratic terms are internal. The semiconnection is consequently block diagonal, even if one or both of its summands are nonintegrable.

The scalar unitary circle on E_S fixes this coefficient point. Equivariance and uniqueness in (20) force $u_s(p, 0)$ to be block diagonal. The compatible unitary connection splits. For its restriction ∇_S , Chern–Weil gives

$$(25) \quad \frac{1}{\text{Vol}_s X} \int_X \text{tr}(i\Lambda_s F_{\nabla_S}) \frac{\omega_s^n}{n!} = \frac{2\pi \deg_s(E_S)}{\text{Vol}_s X} = \sum_{i \in S} \kappa_i(s).$$

This is a trace identity for smooth unitary connections; it uses $d\omega_s = 0$ and does not require integrability. Block diagonal conjugation by $e^{u/2}$ preserves the partial trace. Hence $\mathcal{F}_S(s, p, 0) = -b_S(s)$ and $\mu_S(s, p, 0) = 0$, giving $R_S(s, p, 0) = 0$.

The diagonal unitary equal to -1 on S and $+1$ outside sends r to $-r$. It acts on each quadratic path by the correct endpoint character. Its action on the diagonal target is trivial, so $\partial_r R_S(s, p, 0) = 0$. Taylor integration now gives

$$R_S(s, p, r) = \int_0^1 (1-t) D_r^2 R_S(s, p, tr)[r, r] dt, \quad |R_S(s, p, r)| \leq C(\|p\|^2 + \|r\|^2)\|r\|^2.$$

Uniform equivalence of coefficient norms and harmonic norms proves (23). $\square$

The proof removes the independent crossing coefficients and reconstructs all their quadratic products. It never deletes arbitrary entries from a fixed integrable matrix. Even this reconstructed cut may be nonintegrable; the unitary trace identity is the reason the argument still applies.

4.3. Correction without a lower bound for the weak eigenvalues. Take the balanced point z^0 from Lemma 3.2. It lies in Z and has total energy σ . For $Y \in \mathcal{K}$, put $z(Y) = e^{Y/2} \cdot z^0$. Equivariance preserves Z along this whole orbit. Consider

$$H_\tau(Y) = \mu_s(z(Y)) - b + \tau R_s(z(Y)), \quad 0 \leq \tau \leq 1.$$

On the boundary $\|Y\|_\infty = r = A\sigma$, order the m diagonal values of Y . Their oscillation is at least r because their sum is zero. One adjacent gap is at least $r/(m-1)$. Let S be the vertices above this gap. Every crossing edge contributes positively to the moment increment paired with 1_S , in either orientation. Since $|Y_i - Y_j| \leq 2r$,

$$(26) \quad \langle 1_S, \mu_s(z(Y)) - b \rangle \geq e^{-2r} \frac{r}{m-1} q_{\partial S, s}(z^0).$$

Proposition 4.1 bounds the competing term by $Ce^{4r}\sigma q_{\partial S, s}(z^0)$. Connectedness makes this cut energy positive. Choose $A > 2C(m-1)$, then ε so small that $e^{6A\varepsilon} < 2$. The positive increment strictly dominates the remainder. Thus H_τ has no boundary zero for any τ . The entire gauge ball stays in the original coefficient ball, since its coefficients grow by at most e^r .

The unperturbed map has positive radial pairing at every nonzero Y :

$$\langle Y, H_0(Y) \rangle = \sum_e w_e (e^{Y_i - Y_j} - 1)(Y_i - Y_j) > 0.$$

Connectedness gives strict positivity. Its straight homotopy to the identity has no boundary zero, so its Brouwer degree is +1. Homotopy invariance gives a zero of H_1 with $\|Y\|_\infty \leq C\sigma$. The choice of cut may jump as Y varies; only the pointwise exclusion of boundary zeros used it. No weak eigenvalue was inverted.

Both curvature projections vanish at this zero. Because it is integrable, the compatible connection is the Chern connection of an HE metric. Pulling $h_{0,s}e^{u_s(z(Y))}$ back by $e^{Y/2}g_0$ gives a metric on $E_{s,z}$. Its determinant is the specified product metric. Both u and Y are $O(\sigma)$, and therefore this pullback satisfies (15). Pullback by g_0 preserves inequalities between Hermitian forms exactly, however large its condition number may be.

5. EVERY SMALL EIGENVALUE AND THE STABILITY CRITERION

The quadratic correction for integrability also changes the finite commutator energy. Its contribution is small relative to each graph energy, including an arbitrarily weak one. This is the second relative estimate needed for Theorem 3.3.

5.1. The nonlinear commutator energy. Let K be the complex space of constant trace-free diagonal endomorphisms. By (8), it is the kernel of the split endomorphism Dolbeault operator on zero-forms, and $\dim_{\mathbb{C}} K = m - 1$. The positive gap on $K^\perp$ is uniformly $\gamma_0 > 0$. Write $A_z = [N_s(z), \cdot]$ and let η be its L^2 multiplication norm. On the small coefficient ball, $\eta \leq C\|z\|$.

For $\xi \in K$, co-closedness gives $D_0^*A_z\xi = 0$. Linear and quadratic blocks have different layer endpoints, so their matrix entries are orthogonal. Consequently

$$(27) \quad \|A_z\xi\|_2^2 = q_{L_s(z)}(\xi) + \sum_{i \in I_0, k \in I_2} \|Q_{ik,s}(z)\|_2^2 |\xi_i - \xi_k|^2,$$

where $q_{L_s(z)}$ is the linear graph form with current edge norms. The bilinear Green bounds and the finite number of paths give

$$\|Q_{ik,s}(z)\|_2^2 \leq C \sum_j \|z_{ij}\|^2 \|z_{jk}\|^2.$$

For each path use $|\xi_i - \xi_k|^2 \leq 2|\xi_i - \xi_j|^2 + 2|\xi_j - \xi_k|^2$, and bound the unused coefficient factor by the total energy. This proves

$$(28) \quad q_{L_s(z)}(\xi) \leq \|A_z\xi\|_2^2 \leq (1 + Cq_s(z))q_{L_s(z)}(\xi).$$

There is no division by an arrow norm or an obstruction product. Quadratic paths cannot join different components of the linear graph, so both forms have exactly the componentwise constants as their kernel.

5.2. The full elliptic operator. For $\zeta \in K^\perp \cap H^1$,

$$\|(D_0 + A_z)\zeta\|_2 \geq (\sqrt{\gamma_0} - \eta)\|\zeta\|_2.$$

Set $\gamma_z = (\sqrt{\gamma_0} - \eta)^2$, bounded below by a positive constant on a smaller coefficient ball. In the energy of $\xi + \zeta$, co-closedness removes $\langle A_z\xi, D_0\zeta \rangle$. Young's inequality for the remaining cross term yields

$$(29) \quad \|(D_0 + A_z)(\xi + \zeta)\|_2^2 \geq \left(1 - \frac{2\eta^2}{\gamma_z}\right) \|A_z\xi\|_2^2 + \frac{\gamma_z}{2} \|\zeta\|_2^2.$$

For example, absorb $2\eta\|A_z\xi\|_2\|\zeta\|_2$ using $\gamma_z\|\zeta\|_2^2/2$. Density extends the inequality to the whole H^1 form domain.

The trial space K gives an upper bound for each of the first $m-1$ eigenvalues. The block diagonal lower form in (29) gives the corresponding relative lower bounds. Its finite eigenvalues are $O(q_s(z))$, hence below the complementary threshold on the small ball. The codimension- $(m-1)$ subspace $K^\perp$ gives a fixed lower bound for the m th eigenvalue. Combining with (28) proves the eigenvalue assertions of Proposition 3.4, including their multiplicities.

If the full energy is zero, (29) first forces $\zeta = 0$ and then $A_z \xi = 0$. Equation (28) identifies ξ with graph constants. Conversely such constants commute with every surviving linear and quadratic block. This proves the exact kernel assertion, including disconnected and nonintegrable coefficients.

At the corrected coefficient from Section 4, each edge weight is $e^{Y_i - Y_j} w_e = e^{O(\sigma)} w_e$. Finite min-max therefore compares its graph eigenvalues with ν_j relatively. The auxiliary comparison adds another relative $O(\sigma)$ error. Replacing $h_{0,s}$ by $h_{0,s} e^u$ multiplies both the zero-form masses and one-form energies on endomorphisms by $e^{O(\sigma)}$; the differential and trace-free form domain are unchanged. Finally simultaneous pullback of the holomorphic structure and metric is an isometry. These steps prove (16) for the actual HE metric.

5.3. Why the finite condition is exactly stability. The auxiliary kernel statement implies simplicity for a small integrable coefficient with connected support. Every coefficient can be moved to an arbitrarily small one by the diagonal layer gauge: after subtracting a common scalar from the layer heights, its action multiplies linear arrows by t and quadratic blocks by t^2 . Thus connected integrable members are simple, however large their original coefficients.

The constructed HE metric makes the bundle polystable. Simplicity excludes a nontrivial orthogonal holomorphic direct sum, and hence the bundle is stable. The standard parallel-Hom argument gives uniqueness after fixing the determinant; see [Kob87, Theorem 5.8.3 and Proposition 4.1.7]. This proves sufficiency without assuming that a list of visible subbundles exhausts all possible destabilizing subsheaves.

For necessity, call S a vertex ideal if no supported arrow leaves S . Its smooth summand E_S is preserved by every linear arrow and quadratic path, and is therefore a holomorphic subbundle at integrable coefficients. Its mean curvature relative to that of E is $-b(S)/|S|$. Stability requires

$$(30) \quad b(S) > 0 \quad \text{for every proper nonempty vertex ideal } S.$$

These inequalities are equivalent to a positive flow when the support is connected. To see this directly, let $\mathcal{C} = D\mathbb{R}_{\geq 0}^{\text{edges}}$ in the zero-sum vertex hyperplane. Connectedness makes D surjective onto that hyperplane, so $\mathcal{C}$ is full dimensional. Its dual consists of tuples satisfying $x_i \geq x_j$ on each arrow $j \rightarrow i$, modulo constants. Sorting such a tuple writes it, modulo a constant, as a nonnegative combination of indicators of proper vertex ideals. Condition (30) therefore puts b in the interior of $\mathcal{C}$ by finite-dimensional separation. Conversely an interior point pairs positively with each nonzero dual indicator.

Finally,

$$\text{int } \mathcal{C} = D\mathbb{R}_{> 0}^{\text{edges}}.$$

One inclusion follows because a surjective linear map is open. For the other, subtract a sufficiently small multiple of $D\mathbf{1}$ from an interior point, represent the result by nonnegative edge coefficients, and add that multiple to each coefficient. This proves the equivalence. If the support is disconnected, all quadratic paths remain internal to its components; the bundle is a nontrivial holomorphic direct sum and cannot be stable. Theorem 3.3 is proved.

Only compactness of the background family entered the constants. The coefficient estimates took place on one ball, and balancing brought every stable connected orbit there. In particular, this argument does not need compactness of a set of extension orbits or regularity of the obstruction locus. Its restriction is the explicit three-layer construction, including the absence of independent endpoint parameters.

6. A NONLINEAR DIAMOND ON THREE PROJECTIVE LINES

The next family has both a nonzero obstruction map and a nonzero quadratic correction. Its four adjacent extension spaces are taken in full, rather than restricted to one direction per edge. This variation produces a spectral scale that is invisible in the two polarization parameters alone.

6.1. Twenty-four coefficients and five relations. On $X = (\mathbb{P}^1)^3$, use product metrics and normalized Fubini–Study forms as before, and set

$$(31) \quad F_1 = \mathcal{O}(-2, -2, 4), \quad F_2 = \mathcal{O}(1, -2, 1), \quad F_3 = \mathcal{O}(-2, 1, 1), \quad F_4 = \mathcal{O}.$$

The layers are $\{1\}, \{2, 3\}, \{4\}$. These distinct lines have slope zero at the product polarization, and every off-diagonal Hom has zero H^0 . Write $U_i = H^0(\mathbb{P}_i^1, \mathcal{O}(1))$ and $S_d = H^0(\mathbb{P}_3^1, \mathcal{O}(d))$. Fix identically normalized harmonic $\theta_i \in H^1(\mathbb{P}_i^1, \mathcal{O}(-2))$ for $i = 1, 2$. Serre duality identifies $H^1(\mathcal{O}(-3))$ with U_i^* so that $H_i(\eta u) = \langle \eta, u \rangle \theta_i$. The adjacent spaces are

$$(32) \quad V_{12} = U_1^* \otimes S_3, \quad V_{13} = U_2^* \otimes S_3, \quad V_{24} = U_1 \otimes S_1, \quad V_{34} = U_2 \otimes S_1.$$

The missing factors are 1 or θ_i : V_{24} carries θ_2 and V_{34} carries θ_1 . The dimensions are 8, 8, 4, 4. The endpoint bundle is $C_{14} = \mathcal{O}(-2, -2, 4)$, with

$$H^0(C_{14}) = H^1(C_{14}) = 0, \quad H^2(C_{14}) = \mathbb{C}(\theta_1 \wedge \theta_2) \otimes S_4.$$

Thus the endpoint obstruction space has dimension five.

Choose bases of the two U_i and the dual bases. Write e_0, e_1 for each chosen basis and η^0, η^1 for its Serre-dual harmonic representatives; tensor positions specify the projective-line factor. Write $z = (A, C, B, D)$ in edge order 12, 13, 24, 34, with $A = (a_0, a_1), C = (c_0, c_1)$ pairs of cubics and $B = (b_0, b_1), D = (d_0, d_1)$ pairs of linear polynomials. Ordering differential legs by factor 1 then factor 2 gives

$$(33) \quad \Omega(z) = a_0 b_0 + a_1 b_1 - c_0 d_0 - c_1 d_1 \in S_4.$$

The minus sign comes from exchanging those legs. The zero locus Z has five explicit quadratic equations in 24 variables. The active-relation locus has the additional condition

$$(34) \quad \langle A, B \rangle = \langle C, D \rangle \neq 0.$$

This is open in Z and forces full support. No positive lower bound for the common product will be needed. The full graded End cohomology has $h^1 = 83$; the present 24-dimensional coefficient space is exactly the adjacent-layer part, not the full deformation space.

6.2. The primitive and the fixed holomorphic bundle. Take $\omega_h = h_1 \omega_1 + h_2 \omega_2 + \omega_3$ near the product form. The four linear harmonic spaces are fixed under this change. The product harmonic projections are also fixed: in a fixed factor degree, scaling the product forms multiplies the corresponding inner product by a scalar. Define

$$(35) \quad \Gamma_h(z) = -D_{14,h}^* G_{14,2,h}(A \wedge B + C \wedge D).$$

Then

$$(36) \quad \begin{aligned} D_{14}\Gamma_h(z) &= -A \wedge B - C \wedge D + (\theta_1 \wedge \theta_2) \otimes \Omega(z), \\ D_{14,h}^*\Gamma_h(z) &= 0. \end{aligned}$$

Set the four adjacent entries of N_h equal to A, C, B, D , its $(1, 4)$ entry equal to $\Gamma_h(z)$, and every other entry equal to zero. There are no higher products, and hence

$$(37) \quad (D_0 + N_h(z))^2 = (\theta_1 \wedge \theta_2) \otimes \Omega(z) E_{14}.$$

This identity holds on the whole ambient space.

The difference $\Gamma_h(z) - \Gamma_1(z)$ is closed even off Z , by (36). Since $H^1(C_{14}) = 0$, put

$$(38) \quad f_h(z) = D_{14,1}^* G_{14,1,1}(\Gamma_h(z) - \Gamma_1(z)), \quad \Phi_h(z) = I - f_h(z)E_{14}.$$

Here the Green operator acts on endpoint one-forms at the reference polarization. Thus $D_{14}f_h = \Gamma_h - \Gamma_1$, and the convention $D' = gDg^{-1}$ gives

$$\Phi_h(z)(D_0 + N_1(z))\Phi_h(z)^{-1} = D_0 + N_h(z).$$

Every product with the endpoint matrix vanishes. The gauge has determinant one, is quadratic in z , and has endpoint character g_1/g_4 . Let E_z denote the fixed holomorphic bundle at $h = (1, 1)$ for $z \in Z$. Formula (38) identifies it with its varying h -presentation.

6.3. Isomorphism classes on full support. The effective torus $T = (\mathbb{C}^*)^4/\mathbb{C}^*$ acts by target/source ratios. Every term in Γ_h has character g_1/g_4 , so this action gives actual holomorphic isomorphisms. The determinant-one gauge $\text{diag}(t, 1, 1, t^{-1})$ multiplies linear arrows by t and the endpoint term by t^2 . Every orbit therefore has arbitrarily small representatives.

Proposition 6.1. *For $z, z' \in Z$ with all four blocks nonzero, E_z and $E_{z'}$ are holomorphically isomorphic over the fixed base X if and only if z' is in the T -orbit of z .*

Proof. Only necessity remains. The bundles are semistable at the reference form, with a filtration whose line quotients are F_1, F_2, F_3, F_4 . The connecting maps and cross-Hom vanishings give $\text{Hom}(F_1, E_z) = \mathbb{C}$ and $\text{Hom}(F_i, E_z) = 0$ for $i > 1$: the first two connecting classes are the nonzero A, C , and the top extension has nonzero image (B, D) . The stable-source argument of Section 2 shows that the maximal equal-slope polystable subbundle is precisely F_1 . In E_z/F_1 , the corresponding subbundle is $F_2 \oplus F_3$. Its preimage is the subbundle E_{123} supported on the first three vertices. Thus any isomorphism preserves this three-layer filtration.

The induced maps on its four distinct graded lines are nonzero scalars g_i . Applying them to the extension classes of E_{123} and of E_z/F_1 gives

$$A' = (g_1/g_2)A, \quad C' = (g_1/g_3)C, \quad B' = (g_2/g_4)B, \quad D' = (g_3/g_4)D.$$

These are exactly the torus ratios. No automorphism of the base is allowed in the equivalence under consideration. $\square$

This also shows that the full-support bundles cannot be extensions of two polystable reference-slope bundles: the maximal such subbundle is F_1 , and its quotient is not polystable. The four degree vectors in (31) fail each equality $F_i + F_j = F_k + F_l$ for a partition into two pairs. They therefore do not have the four-line tensor pattern of Section 2.4. This observation only excludes that pattern, not arbitrary abstract tensor factorizations by other bundles.

Proposition 6.2. *The family contains a local 16-dimensional complex family of pairwise nonisomorphic bundles on which both the common obstruction product and the quadratic correction are nonzero.*

Proof. With homogeneous coordinates s_3, t_3 on the third projective line, take

$$(39) \quad A = C = (s_3^3, t_3^3), \quad B = D = (s_3, t_3).$$

The common product is $s_3^4 + t_3^4 \neq 0$. The derivative of Ω in A alone is $(\delta a_0, \delta a_1) \mapsto s_3 \delta a_0 + t_3 \delta a_1$, which is onto S_4 . Thus Z is smooth of dimension $24 - 5 = 19$ near this point. The effective three-dimensional torus acts freely on full support. Fixing one nonzero coefficient on each of the edges 12, 13, 24 uses three independent ratios and defines a local holomorphic slice of dimension 16. Proposition 6.1 identifies distinct nearby points of that slice with distinct bundle classes.

To see that the primitive is nonzero, restrict the third coordinate to $t_3 = 0$, $s_3 \neq 0$. The composition form is a nonzero scalar multiple of

$$(e_0 \eta^0)_1 \otimes \theta_2 - \theta_1 \otimes (e_0 \eta^0)_2.$$

The harmonic representative θ of $H^1(\mathcal{O}(-2))$ is nowhere zero as a section of its smooth form bundle; it is Serre dual to a nonzero constant. The form $e_0 \eta^0$ vanishes at the zero of e_0 but has nonzero cohomology, so it is not proportional to θ . The displayed tensor is nonzero. Hence $D\Gamma \neq 0$ and $\Gamma \neq 0$. The same holds near the chosen point. $\square$

At $A = C = (s_3^3, 0)$, $B = D = (s_3, 0)$, the derivative of Ω has image $s_3 S_3$ and rank four. The analytic theorem requires neither constant rank nor a uniform inverse for this derivative. This rank calculation alone makes no claim about the local scheme structure there.

6.4. The scalar diamond inside the family. The last specialization can be described by fixed forms. Choose a nonzero section τ of $\mathcal{O}(1)$ and a harmonic η in $H^1(\mathcal{O}(-3))$ with $H(\tau\eta) = \theta$. Such a choice exists because multiplication $H^1(\mathcal{O}(-3)) \rightarrow H^1(\mathcal{O}(-2))$ is surjective, dual to the injection $H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(1))$. Choose the same data on the first two factors, and a section τ_3 on the third. The four forms are

$$\alpha_{12} = \eta_1 \otimes 1 \otimes \tau_3^3, \quad \alpha_{13} = 1 \otimes \eta_2 \otimes \tau_3^3, \quad \beta_{24} = \tau_1 \otimes \theta_2 \otimes \tau_3, \quad \beta_{34} = \theta_1 \otimes \tau_2 \otimes \tau_3.$$

Their two products w_1, w_2 have harmonic parts $\chi, -\chi$, where $\chi = \theta_1 \wedge \theta_2 \otimes \tau_3^4 \neq 0$. Put $\gamma_{i,h} = -D_{14,h}^* G_{14,2,h} w_i$. The scalar entries $(x\alpha_{12}, y\alpha_{13}, v\beta_{24}, w\beta_{34})$ have endpoint correction $xv\gamma_{1,h} + yw\gamma_{2,h}$ and curvature $(xv - yw)\chi E_{14}$. The sum $w_1 + w_2$ is nonzero by the same antisymmetrized tensor argument, so the correction at unit coefficients is nonzero. All nonzero scalar members with $xv = yw$ are one bundle:

$$g = (xv^2 w)^{-1/4} \text{diag}(xv, v, w, 1)$$

has the required four ratios and determinant one, with a consistent fourth root. The positive gauge is unambiguous for positive coefficients.

At the cut $S = \{1, 2, 4\}$, setting $y = w = 0$ leaves $xv\chi E_{14}$. Even reconstructing the quadratic primitive cannot restore integrability. This is precisely a cut on which the ambient partial-trace argument of Section 4 is needed.

7. BALANCE, SUPPORT STRATA, AND AN ADDITIONAL WEAK SCALE

7.1. Polarization and the orbit invariant. Use independent small parameters k, l and put

$$(40) \quad h_1 = \left(1 + \frac{k+5l}{24\pi}\right)^{-1}, \quad h_2 = \left(1 + \frac{5k+l}{24\pi}\right)^{-1}, \quad \sigma = k+l.$$

This is an invertible chart in two actual Kähler directions. The four line mean curvatures follow by contracting their degrees. Their average is $-3\sigma/8$, and the centered demand is

$$(41) \quad b = \frac{1}{8}(5\sigma, 3k - 5l, -5k + 3l, -3\sigma).$$

The normalized volume measure $dV_h/(h_1 h_2) = \omega_1 \omega_2 \omega_3$ is fixed. The harmonic norms have the exact dependence

$$(42) \quad n_{12} = \frac{n_{12}(1, z)}{h_1}, \quad n_{13} = \frac{n_{13}(1, z)}{h_2}, \quad n_{24} = \frac{n_{24}(1, z)}{h_2}, \quad n_{34} = \frac{n_{34}(1, z)}{h_1}.$$

On full support the positive quantity

$$(43) \quad \rho(z) = \frac{n_{12} n_{24}}{n_{13} n_{34}}$$

is unchanged by diagonal complex gauge and by h . Equality of the two cup products in (34) does not force $\rho = 1$: that equality concerns cohomology products rather than products of norms.

Set $P = 5\sigma/8$ and $C_0 = (3k - 5l)/8$. Solving the four linear balance equations gives all candidate weights in the form

$$(44) \quad (w_{12}, w_{13}, w_{24}, w_{34}) = (a, P - a, a + C_0, l - a).$$

For $k, l > 0$ the interval

$$(45) \quad \max(0, -C_0) < a < \min(P, l)$$

is nonempty and is exactly the positivity interval. The weights belong to the given orbit precisely when

$$(46) \quad \frac{a(a + C_0)}{(P - a)(l - a)} = \rho.$$

Its left side increases from zero to infinity on (45), since its logarithmic derivative is $1/a + 1/(a + C_0) + 1/(P - a) + 1/(l - a) > 0$. Thus the balanced weights are unique. Equivalently a is the unique root in that interval of

$$(1 - \rho)a^2 + [C_0 + \rho(P + l)]a - \rho Pl = 0.$$

This equation remains valid at $\rho = 1$. The weights always sum to σ .

Let $r_e = \sqrt{w_e/n_e}$. Equation (46) implies $r_{12}r_{24} = r_{13}r_{34}$, and the explicit positive gauge is

$$(47) \quad g_0 = (r_{12}r_{24}^2r_{34})^{-1/4} \text{diag}(r_{12}r_{24}, r_{24}, r_{34}, 1).$$

It has determinant one and the prescribed edge ratios. The reference metric on the fixed holomorphic bundle E_z is

$$(48) \quad h_{\text{bal}, h, z} = (g_0 \Phi_h(z))^* h_0.$$

Its determinant is the fixed product metric on $\mathcal{O}(-3, -3, 6)$. In particular the determinant bundle is not trivial; its contraction is $4\kappa_E$, as required.

TABLE 1. Balanced weights and the strict stable region for a three-edge support. The edge order is 12, 13, 24, 34 and $\sigma = k + l$.

Missing block	Weights	Strict region
12	$(0, 5\sigma/8, (3k - 5l)/8, l)$	$l > 0, 3k > 5l$
13	$(5\sigma/8, 0, k, (3l - 5k)/8)$	$k > 0, 3l > 5k$
24	$((5l - 3k)/8, k, 0, 3\sigma/8)$	$k > 0, 5l > 3k$
34	$(l, (5k - 3l)/8, 3\sigma/8, 0)$	$l > 0, 5k > 3l$

7.2. All connected supports. A connected subgraph of the diamond has either four edges or exactly three. In the latter case, setting its missing weight equal to zero in (44) gives the table below. Each row applies only to integrable coefficients with exactly that support.

On a tree, positive edge ratios determine a positive diagonal gauge modulo a scalar: propagate ratios from one vertex and then fix the determinant. The total energy is again σ . Quadratic paths forced to vanish by a missing edge remain zero under this gauge. With this tree gauge g_0 , define $h_{\text{bal},h,z} = (g_0\Phi_h(z))^*h_0$ as on full support.

The necessity of the regions is geometric. The subbundles on $\{1, 2\}$ and $\{1, 3\}$ always have relative mean curvatures $-k/2$ and $-l/2$. If 12 is missing, $\{2\}$ is an additional invariant subbundle and requires $b_2 > 0$; if 13 is missing, $\{3\}$ requires $b_3 > 0$. If 24 is missing, the invariant subset $\{1, 3, 4\}$ requires $-b_2 > 0$; if 34 is missing, $\{1, 2, 4\}$ requires $-b_3 > 0$. These are exactly the inequalities in Table 1. Sufficiency follows from Theorem 3.3, so no further destabilizing subsheaf reduces the listed regions. On full support the region is $k, l > 0$; disconnected supports give direct sums and are never stable.

Corollary 7.1. *For every integrable connected member of (33) and all small (k, l) in its strict region, its fixed-determinant HE metric on E_z satisfies*

$$e^{-C\sigma}h_{\text{bal},h,z} \leq H_{h,z} \leq e^{C\sigma}h_{\text{bal},h,z}, \quad \left| \frac{\lambda_j}{\nu_j} - 1 \right| \leq C\sigma \quad (j = 1, 2, 3), \quad \lambda_4 \geq \gamma.$$

Here ν_j are the eigenvalues of the graph with weights (44) or the relevant row of Table 1, and unit vertex masses. The constants are uniform across all coefficient directions, connected supports and positive spectral ratios.

Proof. The fixed factors, harmonic spaces, endpoint Green primitive, co-closedness and obstruction are exactly the data of Section 3. There is no independent endpoint coefficient. Their uniform bounds follow from product Hodge theory on a compact positive h -neighborhood. The positive balances above have total energy σ , so Theorem 3.3 applies. Pullback by $\Phi_h(z)$ preserves the determinant, metric inequalities and spectrum, and places the conclusion on the fixed bundle E_z . All estimates arose on one ambient ball, including its zero-block strata; no lower bound for the input norms or compactness of the orbit space is used. $\square$

7.3. The explicit cubic at a fixed orbit. For the scalar specialization in Section 6, symmetry between the first two factors gives $\rho = 1$. Thus $a = 5l/8$ and

$$(49) \quad w = \frac{1}{8}(5l, 5k, 3k, 3l).$$

The characteristic polynomial on all four vertices is

$$(50) \quad \lambda \left[\lambda^3 - 2\sigma\lambda^2 + \left(\frac{15}{16}(k^2 + l^2) + 3kl \right) \lambda - \frac{15}{16}kl\sigma \right].$$

For verification, the graph matrix is

$$\frac{1}{8} \begin{pmatrix} 5\sigma & -5l & -5k & 0 \\ -5l & 5l+3k & 0 & -3k \\ -5k & 0 & 5k+3l & -3l \\ 0 & -3k & -3l & 3\sigma \end{pmatrix}.$$

Expanding its determinant gives (50). Comparing edge by edge with the product square gives, in increasing order,

$$\frac{3}{8}(2\min(k, l), 2\max(k, l), 2\sigma)_j \leq \nu_j \leq \frac{5}{8}(2\min(k, l), 2\max(k, l), 2\sigma)_j.$$

The weakest scale here is controlled by $\min(k, l)$. The next family shows why this conclusion is not uniform in the extension orbit.

7.4. Comparable polarizations and an arbitrarily weaker mode. Choose identical orthonormal bases on U_1, U_2 and their Serre-dual harmonic bases. For $M \in \mathbb{C}$ set

$$(51) \quad A = (s_3^3, Ms_3^3), \quad C = (s_3^3, 0), \quad B = D = (s_3, 0).$$

Both products equal $s_3^4 \neq 0$. All blocks are nonzero, and the projective direction of A changes with M , so Proposition 6.1 gives pairwise nonisomorphic bundles. Equal dual norms and orthogonality give

$$(52) \quad \rho = 1 + |M|^2.$$

The quadratic primitive never vanishes. Otherwise the composition identity would make $(e_0\eta^0)_2$ proportional to θ_2 ; the added first-factor term $(e_0\eta^1)_1 \otimes \theta_2$ cannot remove their independent second-factor components. The zero of $e_0\eta^0$ contradicts that proportionality. Interchanging the two paths similarly realizes arbitrarily small positive ρ .

Keep $k = 3l/5$ and put $\delta = l - a$. The balanced weights become

$$(53) \quad w = (l - \delta, \delta, 3l/5 - \delta, \delta), \quad \rho\delta^2 = (l - \delta)(3l/5 - \delta), \quad 0 < \delta < 3l/5.$$

As $\rho \rightarrow \infty$, $\delta/l \rightarrow 0$ and $\sqrt{\rho}\delta/l \rightarrow \sqrt{3/5}$. The Rayleigh quotient of $\xi = (-1/3, -1/3, 1, -1/3)$ is $8\delta/3$. Hence

$$0 < \nu_1 \leq \frac{8}{3}\delta, \quad \frac{\nu_1}{\sigma} \rightarrow 0.$$

The two polarization parameters remain comparable. For arbitrary joint limits $l \rightarrow 0$ and $|M| \rightarrow \infty$, with no relation between their rates, Corollary 7.1 still gives $\lambda_1/\nu_1 \rightarrow 1$ and $\lambda_1/\sigma \rightarrow 0$. The extra scale belongs to genuine bundle variation, not a large presentation of one orbit.

8. DIFFERENTIATING THE FINITE BALANCE

For the rest of the paper, fix a full-support $z \in Z$ in the diamond family and vary only $s = (k, l)$. Thus the holomorphic bundle E_z is fixed. We use the Euclidean norm on the two-dimensional input space. The output of the finite response is a trace-free diagonal potential, with its Euclidean norm on $\mathcal{K} \subset \mathbb{R}^4$.

8.1. The two-dimensional input. In edge order 12, 13, 24, 34, let

$$(54) \quad D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad B_0 = \frac{1}{8} \begin{pmatrix} 5 & 5 \\ 3 & -5 \\ -5 & 3 \\ -3 & -3 \end{pmatrix}.$$

Then $b = B_0 s$. At fixed z , the matrix of logarithmic harmonic-norm derivatives has rows

$$(55) \quad A_s = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \\ \alpha_1 \end{pmatrix}, \quad \alpha_1 = \frac{(1, 5)}{24\pi + k + 5l}, \quad \alpha_2 = \frac{(5, 1)}{24\pi + 5k + l}.$$

This is the derivative of (42), not an additional approximation.

Write $g_0 = e^{x^0/2}$ for the balancing gauge and $L = D \operatorname{diag}(w) D^\dagger$ on $\mathcal{K}$. Differentiating $D(n_e e^{(D^\dagger x^0)_e}) = b$ gives

$$(56) \quad dx^0 = \mathcal{R}_0 ds, \quad \mathcal{R}_0(s, z) = L^{-1}(B_0 - D \operatorname{diag}(w) A_s).$$

Everything on the right is specified by the input geometry and finite balance. Let $r_1 \geq r_2$ be its singular values. There are two, because there are two polarization directions, whereas L has three positive eigenvalues. These are singular values of a map with two-dimensional domain, rather than the three reciprocal eigenvalues of L .

8.2. An inverse estimate that retains every cut. For a connected weighted graph on m vertices and a zero-sum vector f , write $f(S) = \sum_{i \in S} f_i$ and $q_{\partial S} = \sum_{e \text{ crossing } S} w_e$.

Lemma 8.1. *On the zero-sum vertex space,*

$$(57) \quad \|L^{-1}f\|_\infty \leq (m-1) \max_{\emptyset \neq S \neq \{1, \dots, m\}} \frac{|f(S)|}{q_{\partial S}}.$$

Consequently, if $|(EX)(S)| \leq \eta q_{\partial S} \|X\|_\infty$ for every S, X , then $\|L^{-1}E\|_{\infty \rightarrow \infty} \leq (m-1)\eta$. If L' has the same support and $|w'_e - w_e| \leq \eta w_e$, then

$$(58) \quad \|L^{-1}(L' - L)\|_{\infty \rightarrow \infty} \leq 2(m-1)\eta.$$

Proof. Put $v = L^{-1}f$ and sort its values. Its oscillation is at least $\|v\|_\infty$, so one adjacent gap is at least $\|v\|_\infty/(m-1)$. For the vertices above that gap,

$$f(S) = \sum_{e \text{ crossing } S} w_e (v_{\text{inside}} - v_{\text{outside}}) \geq q_{\partial S} \frac{\|v\|_\infty}{m-1}.$$

This proves the first assertion; connectedness makes its denominators positive. Apply it to EX for the second. For the third, the cut sum is $\sum_{\partial S} (w'_e - w_e)(X_{\text{inside}} - X_{\text{outside}})$, whose absolute value is at most $2\eta q_{\partial S} \|X\|_\infty$. $\square$

In fixed dimension all norm conversions are harmless. In particular, (55) and the first estimate give

$$(59) \quad \|L^{-1}D \operatorname{diag}(w) A_s v\| \leq C \|v\|.$$

Each cut sum of its forcing is bounded by $C q_{\partial S} \|v\|$. These conclusions use graph structure, not merely relative comparison of arbitrary positive matrices.

8.3. Differentiated curvature estimates. The ambient construction supplies the smooth residual $\mathcal{F}(s, z) = -b(s) + \mu_s(z) + R(s, z)$. For every s , the coefficient jets of R through degree three vanish. Smooth parameter dependence in the Sobolev implicit-function theorem therefore gives

$$(60) \quad \|\partial_s D_z^2 R(s, z)\| \leq C\|z\|^2, \quad \|D_z^3 R(s, z)\| \leq C\|z\|.$$

The first estimate follows by differentiating the identically zero jets with respect to s . The same ambient ball works for these mixed derivatives. For a cut $z = (p, r)$, its exact Taylor formula is

$$R_S(s, p, r) = \int_0^1 (1-t) D_r^2 R_S(s, p, tr)[r, r] dt.$$

A diagonal potential variation X has coefficient velocity $\dot{z}_e = (X_i - X_j)z_e/2$. Hence $\|\dot{p}\| \leq C\|X\|_\infty\|p\|$ and the analogous bound holds for r . Differentiating the Taylor formula in either s at fixed coefficients or x along the diagonal orbit gives

$$(61) \quad |(\partial_s R_S)[v]| \leq Cq_s(z)q_{\partial S, s}(z)\|v\|,$$

$$(62) \quad |(\partial_x R_S)[X]| \leq Cq_s(z)q_{\partial S, s}(z)\|X\|_\infty.$$

For the second estimate, the derivative of $D_r^2 R$ contributes $O(\|z\|)\|\dot{z}\|\|r\|^2$, and the two derivatives of r contribute $O(\|z\|^2)\|r\|\|\dot{r}\|$. All are of the stated size. The first estimate uses (60). Uniform Gram equivalence handles the changing harmonic norms. The estimates hold off the obstruction locus as well and use no inverse of $D\Omega$.

8.4. The derivative at a corrected zero. At an HE zero from Section 4, write $x = x^0 + Y$, $\|Y\|_\infty \leq C\sigma$, and $\zeta = e^{x/2} \cdot z$. Its edge energies are $\tilde{w}_e = n_e(s, z)e^{(D^t x)_e} = (1 + O(\sigma))w_e$. The reduced equation is

$$(63) \quad F(s, x) = -b(s) + D\tilde{w} + R(s, \zeta) = 0.$$

Its derivative is $F_x = L_{\tilde{w}} + R_x$. Lemma 8.1, (58) and (62) give the factorization

$$(64) \quad F_x = L(I + \mathcal{E}), \quad \mathcal{E} = L^{-1}(F_x - L), \quad \|\mathcal{E}\| \leq C\sigma.$$

The order of the factors matters. This proves local invertibility even when L is arbitrarily ill-conditioned. An ordinary implicit-function theorem supplies the smooth local zero branch that we differentiate.

At fixed x , the derivative of $\tilde{w}$ is $\text{diag}(\tilde{w})A_s$. Define

$$d_s[v] = D \text{diag}(\tilde{w} - w)A_s v + (\partial_s R)(s, \zeta)[v].$$

Its every-cut bound is $|d_s[v](S)| \leq C\sigma q_{\partial S}(w)\|v\|$. Thus the inverse has already been controlled when we write

$$(65) \quad \|L^{-1}d_s[v]\| \leq C\sigma\|v\|, \quad dx[v] = (I + \mathcal{E})^{-1}(\mathcal{R}_0 v - L^{-1}d_s[v]).$$

The forcing has a uniform lower bound because

$$(66) \quad B_0^\dagger B_0 = \frac{1}{16} \begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix}, \quad \|L\| \leq 2\sigma, \quad \|D \text{diag}(w)A_s\| \leq C\sigma.$$

For small σ , the smallest singular value of $B_0 - D \text{diag}(w)A_s$ is at least $1/2$, and hence

$$(67) \quad \|\mathcal{R}_0 v\| \geq \frac{1}{4\sigma}\|v\|.$$

Combining (64)–(67) proves

$$(68) \quad \|dx[v] - \mathcal{R}_0 v\| \leq C\sigma\|\mathcal{R}_0 v\| \quad (v \in \mathbb{R}^2).$$

This directional relative estimate is uniform in z , ρ , k/l and every positive spectral ratio. It is the extra information needed to control the two response singular values.

9. THE INTRINSIC HERMITIAN–EINSTEIN RESPONSE

Let $H(s; z)$ be the fixed-determinant HE metric on the fixed bundle E_z . Its intrinsic response to a polarization velocity $v \in \mathbb{R}^2$ is

$$(69) \quad \mathcal{D}_{s,z}v = (H^{-1}\partial_v H)_0.$$

The subscript denotes pointwise trace-free part; in our normalization the determinant is fixed and the trace already vanishes. This is a Hermitian endomorphism for the current metric H . For a response V , its squared normalized L^2 norm is $\text{Vol}_h^{-1} \int_X \text{tr}(V^2) dV_h$. It is the tangent to the space of positive Hermitian forms, rather than a derivative of a matrix logarithm in a fixed frame.

Theorem 9.1. *For every full-support $z \in Z$, the metric $H(s; z)$ is smooth in $s = (k, l)$ on $k, l > 0$, $\sigma = k + l$ sufficiently small. Its intrinsic response has real rank two. If $d_1 \geq d_2 > 0$ are its singular values in the stated input and output norms and $r_1 \geq r_2 > 0$ are those of (56), then*

$$(70) \quad \left| \frac{d_j(s, z)}{r_j(s, z)} - 1 \right| \leq C\sigma, \quad j = 1, 2.$$

The constants are uniform in all full-support integrable extension directions, all $\rho > 0$, and all positive edge and spectral ratios. The derivative keeps z , and therefore the holomorphic bundle, fixed.

We prove a stronger comparison of every input direction after an explicitly specified isometric transport. Both the fixed-bundle identification and the complementary elliptic inverse must be included in that calculation.

9.1. Metric velocities and the changing presentation. At a corrected zero use $x = x^0 + Y$ and $\zeta = e^{x/2} \cdot z$. The exact metric from the construction is

$$(71) \quad H = (g_x \Phi_h(z))^* (h_0 e^{u(s, \zeta)}), \quad g_x = e^{x/2}.$$

At every fixed spatial order the ambient implicit function gives

$$(72) \quad \|u\|_{C^r} \leq C_r \sigma, \quad \|D_\zeta u\|_{C^r} \leq C_r \sqrt{\sigma}, \quad \|\partial_s u\|_{C^r} \leq C_r \sigma.$$

The last derivative fixes ζ : the zero and first coefficient jets of u vanish at every s , and their parameter derivatives vanish too.

Although g_x may be unbounded, it does not create an unscaled derivative of Φ_h . The quadratic endpoint character in (38) gives, at fixed x ,

$$(73) \quad K_v := g_x(\partial_v \Phi_h) \Phi_h^{-1} g_x^{-1} = -(\partial_v f_h)(\zeta) E_{14}, \quad \|K_v\|_{C^r} \leq C_r \sigma \|v\|.$$

Products of two endpoint matrices are zero, so no additional product term occurs. This estimate is uniform even for a large original presentation z .

The pointwise isometry from (E_z, H) to the fixed split Hermitian bundle is

$$U = e^{u/2} g_x \Phi_h(z).$$

We use it at the current parameter, without differentiating the norm used to define the response. For $\text{ad}_u(V) = [u, V]$, put

$$(74) \quad C_u = \cosh(\text{ad}_u/2), \quad S_u = \frac{\sinh(\text{ad}_u/2)}{\text{ad}_u/2},$$

with analytic values at zero. These even operators preserve Hermitian endomorphisms and satisfy $C_u - I, S_u - I = O(\sigma^2)$ in fixed spatial norms. Differentiating (71) in a diagonal potential direction X at fixed s gives the centered velocity

$$(75) \quad V_X = C_u X + S_u u_x[X], \quad u_x[X] = D_{\zeta} u \left[\left(\frac{1}{2} (X_i - X_j) \zeta_{ij} \right)_{ij} \right].$$

The two derivatives of g_x each supply a factor $1/2$; their sum gives coefficient one on X . Equations (72) and (75) imply

$$(76) \quad \|u_x[X]\|_{C^r} \leq C_r \sigma \|X\|, \quad \|V_X - X\|_2 \leq C \sigma \|X\|.$$

At fixed x , the background velocity is

$$(77) \quad W_v = S_u(\partial_s u)[v] + Z_v + Z_v^*, \quad Z_v = e^{u/2} K_v e^{-u/2}.$$

Thus $\|W_v\|_2 \leq C \sigma \|v\|$. The product metric h_0 and the normalized base volume measure are fixed; there is no missing diagonal background-metric term. Formula (77) includes all derivatives of the fixed-bundle identification.

9.2. The exact differentiated equation and the complement. For variations of a metric on a fixed holomorphic bundle, the derivative of Chern mean curvature in intrinsic direction V is $\Delta_{\partial} V$. At an HE metric the endomorphism Kähler identity gives $\Delta_{\partial} = \Delta_{\bar{\partial}}$. Differentiating the HE equation therefore yields

$$(78) \quad \Delta_{H,h} \mathcal{D}_{s,z} v = \kappa'_E[v] I - i(\partial_v \Lambda_h) F_H.$$

The right side is pointwise trace-free: the determinant connection is fixed and its contracted curvature has trace $4\kappa_E(s)$. A varying conjugation introduces no extra term after transport by U , because its commutator with the scalar HE mean curvature vanishes.

In the real Hilbert space of h_0 -Hermitian trace-free sections, let P be projection onto constant diagonals $\mathcal{K}$ and let $Q = I - P$. Denote the operator transported by U by Δ . At HE it preserves this real space, since adjunction exchanges the two Dolbeault Laplacians and their difference is the zero curvature commutator. Write

$$(79) \quad \Delta = \begin{pmatrix} G & \mathcal{B}^* \\ \mathcal{B} & C \end{pmatrix}, \quad S = G - \mathcal{B}^* C^{-1} \mathcal{B}, \quad JX = X - C^{-1} \mathcal{B} X.$$

The complementary block has $C \geq \gamma I$ uniformly. In fact $N_h(\zeta) = O(\sqrt{\sigma})$ and $u = O(\sigma)$ in the necessary spatial norms. In the isometric realization the $(0,1)$ operator differs from the split operator by multiplication of norm $O(\sqrt{\sigma})$. The split gap on Q and the triangle inequality give the assertion. Uniform elliptic estimates also give

$$\|C^{-1} f\|_{H^{a+2}} \leq C_a \|f\|_{H^a}.$$

Its form and operator domains are QH^1 and QH^2 ; P is a fixed smooth finite-rank projection.

Stability gives positivity of S on $\mathcal{K}$. Block elimination proves

$$(80) \quad Q\Delta J = 0, \quad P\Delta J = S, \quad \Delta^{-1} = JS^{-1}J^* + QC^{-1}Q.$$

Only C^{-1} is bounded uniformly in ordinary norm. The finite inverse S^{-1} retains the degenerating directions explicitly.

There is an exact relation with the reduced equation. The complementary curvature vanishes throughout the ambient family. Differentiating that identity at an HE zero gives $Q\Delta V_X = 0$. Therefore

$$(81) \quad V_X = JTX, \quad T = PV_X, \quad F_x = ST.$$

The projection of u_x is zero. Since $C_u - I$ and $S_u - I$ are $O(\sigma^2)$, we have $T = I + O(\sigma^2)$. Equations (76) and (81) also give $J - I = O(\sigma)$ and $J^*J = I + O(\sigma^2)$; the latter uses orthogonality of the diagonal and complementary spaces. Thus the elliptic elimination preserves the metric velocity to the required relative order.

9.3. Smoothness and the relative comparison. Equation (64) is invertible at every constructed zero, so ordinary local implicit-function theory gives smooth metric branches near every stable interior parameter. The fixed-determinant HE metric on the simple fixed bundle is unique. These branches agree on overlaps and define the actual smooth $H(s; z)$ throughout the indicated chamber. Their local parameter radii need not be uniform as an edge weight tends to zero.

For $X = dx[v]$, the exact transported response is

$$U(\mathcal{D}_{s,z}v)U^{-1} = V_X + W_v.$$

Combine (68), (67), (76) and (77). They give the directional estimate

$$(82) \quad \|U(\mathcal{D}_{s,z}v)U^{-1} - \text{diag}(\mathcal{R}_0v)\|_2 \leq C\sigma\|\mathcal{R}_0v\| \quad (v \in \mathbb{R}^2).$$

The isometry U preserves the norm, and a constant diagonal has its Euclidean norm. Hence, as quadratic forms on the input space,

$$(1 - C\sigma)^2\mathcal{R}_0^*\mathcal{R}_0 \leq \mathcal{D}_{s,z}^*\mathcal{D}_{s,z} \leq (1 + C\sigma)^2\mathcal{R}_0^*\mathcal{R}_0.$$

This proves rank two and (70). No smoothness of Z was required, because the derivative varies s at fixed z .

The intrinsic differentiation and the complementary inverse are the same standard mechanisms used for two-step response in [Wue26a]. The additional estimates here are the differentiated every-cut bounds and the quadratic fixed-bundle term (73). They make the comparison uniform on this two-dimensional input even when a third graph mode is present and the positive scales separate without restriction.

10. TWO SCALES OF POLARIZATION RESPONSE

The unbounded family (51) gives an explicit application of Theorem 9.1. Keep $k = 3l/5$ and write $\tau = \delta/l$. Equation (53) becomes

$$(83) \quad \rho\tau^2 = (1 - \tau)(3/5 - \tau), \quad 0 < \tau < 3/5, \quad \sqrt{\rho}\tau \longrightarrow \sqrt{3/5}.$$

Let

$$\bar{L}(\tau) = D \text{diag}(1 - \tau, \tau, 3/5 - \tau, \tau) D^t, \quad A_0(\tau) = \bar{L}(\tau)^{-1} B_0$$

on the zero-sum space. The harmonic-norm contribution in (56) is uniformly bounded by (59). Thus

$$(84) \quad \mathcal{R}_0 = l^{-1}A_0(\tau) + O(1)$$

uniformly in both l and τ .

10.1. Exact finite calculation. Solving $\bar{L}A_0 = B_0$ with zero column sums gives

$$(85) \quad A_0(\tau) = \frac{1}{64\tau(4\tau-3)} \begin{pmatrix} 5(8\tau+1)(4\tau-3) & (40\tau-3)(4\tau-3) \\ -5(8\tau+3) & 3(8\tau+3) \\ 5(9-8\tau) & 3(8\tau-9) \\ 5(-32\tau^2+36\tau-3) & -160\tau^2+84\tau+9 \end{pmatrix}.$$

Multiplication verifies both its defining equation and the zero column sums. Its Gram matrix has trace and determinant

$$(86) \quad \text{tr}(A_0^\dagger A_0) = \frac{12800\tau^4 - 19840\tau^3 + 8904\tau^2 - 816\tau + 459}{512\tau^2(4\tau-3)^2},$$

$$(87) \quad \det(A_0^\dagger A_0) = \frac{25(64\tau^2 - 48\tau + 27)}{512\tau^2(4\tau-3)^2}.$$

As $\tau \rightarrow 0$, the larger squared singular value is $51/(512\tau^2)(1 + O(\tau))$, obtained from the trace. Dividing the determinant by that eigenvalue gives the smaller squared singular value $25/17(1 + O(\tau))$. Combining with (84) yields

$$(88) \quad r_1 = \frac{\sqrt{102}}{32\delta}(1 + O(\tau + l)), \quad r_2 = \frac{5}{\sqrt{17}l}(1 + O(\tau + l)).$$

The bounded perturbation contributes relative $O(l)$ even to the smaller singular value. No relation between l and τ is required.

The two input directions can be seen without diagonalizing the Gram matrix. Define unit vectors

$$v_w = \frac{(-5, 3)}{\sqrt{34}}, \quad v_h = \frac{(3, 5)}{\sqrt{34}}, \quad u_w = \frac{(-1, -1, 3, -1)}{\sqrt{12}}.$$

The exact identities are

$$(89) \quad \lim_{\tau \rightarrow 0} \tau A_0(\tau) = \frac{1}{64} \begin{pmatrix} 5 & -3 \\ 5 & -3 \\ -15 & 9 \\ 5 & -3 \end{pmatrix}, \quad A_0(\tau)v_h = \frac{(5, 0, 0, -5)}{\sqrt{34}}.$$

The residue maps v_w to $(\sqrt{102}/32)u_w$ and annihilates v_h . Thus v_h avoids the additional $1/\delta$ scale for every τ , while v_w excites it. This is the distinction between the two-dimensional polarization response and the full inverse spectrum.

10.2. The actual metric consequence. Theorem 9.1 transfers (88) to the actual intrinsic metric derivative:

Corollary 10.1. *Along arbitrary joint limits $l \rightarrow 0$ and $|M| \rightarrow \infty$ in (51), with $k = 3l/5$,*

$$(90) \quad d_1 = \frac{\sqrt{102}}{32\delta}(1 + O(\tau + l)), \quad d_2 = \frac{5}{\sqrt{17}l}(1 + O(\tau + l)).$$

Equivalently,

$$\frac{\sigma d_1}{\sqrt{\rho}} \longrightarrow \frac{\sqrt{170}}{20}, \quad \sigma d_2 \longrightarrow \frac{8}{\sqrt{17}}.$$

In the isometric realization U of Section 9,

$$(91) \quad U(\mathcal{D}v_h)U^{-1} = \frac{\text{diag}(5, 0, 0, -5)}{l\sqrt{34}} + O(1),$$

$$(92) \quad \delta U(\mathcal{D}v_w)U^{-1} \longrightarrow \frac{\sqrt{102}}{32} \text{diag}(u_w) \quad \text{in } L^2.$$

Proof. The singular-value assertion follows from (70) and (88), since $\sigma = 8l/5$. The normalized limits use $\sqrt{\rho}\delta/l \rightarrow \sqrt{3/5}$. For v_h , the exact second identity in (89) and the bounded harmonic-norm term give $\mathcal{R}_0 v_h = (5, 0, 0, -5)/(l\sqrt{34}) + O(1)$. The directional error (82) is $O(\sigma/l) = O(1)$. For v_w , multiply the finite residue calculation by δ . Its remainder is $O(\tau)$, the bounded term contributes $O(\delta)$, and (82) contributes $O(\sigma)$. All tend to zero without a relation between the two limiting rates. $\square$

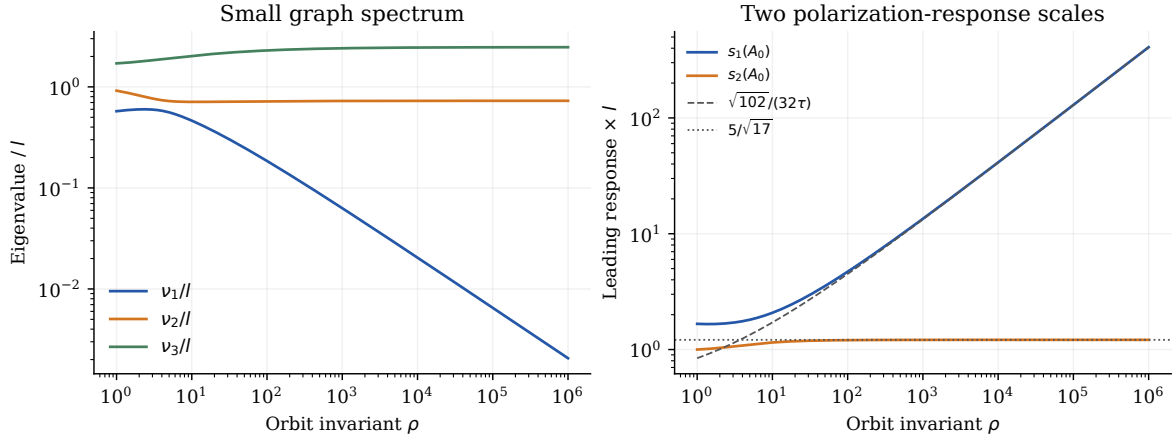


FIGURE 2. Finite models along $k/l = 3/5$. Left: the three graph eigenvalues divided by l for the weights in (53). Right: the two singular values of $A_0(\tau)$, equal to l times the leading polarization-response singular values. Here ρ and τ are related by (83). The dashed and dotted lines are the proved leading constants in (88). These are finite-model calculations, not numerical solutions of the HE equation.

The additional weak scale is therefore visible both in the graph spectrum and in one direction of the intrinsic metric derivative. The other polarization direction retains scale l^{-1} , as the exact identity in (89) explains.

REFERENCES

- [CT24] A. Clarke and C. Tipler, *Semi-stability and local wall-crossing for Hermitian Yang–Mills connections*, J. Symplectic Geom. **22** (2024), no. 3, 673–694. Theorem and proposition locators refer to [arXiv:2304.05245v1](https://arxiv.org/abs/2304.05245v1).
- [Del25] R. Delloque, *Continuity of HYM connections with respect to metric variations*, J. London Math. Soc. (2) **112** (2025), no. 1, e70219. doi:10.1112/jlms.70219. Locators refer to the published-version text [arXiv:2403.16814v4](https://arxiv.org/abs/2403.16814v4).
- [Kob87] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, Princeton, NJ, 1987. [Mathematical Society of Japan edition](https://www.msj.or.jp/en/publications/).

- [Stacks] The Stacks Project Authors, *The Stacks Project*, [Tag 01XT \(cohomology of projective space\)](#) and [Tag 0BED \(Künneth formula\)](#), accessed October 31, 2026.
- [Wue26a] B. J. Wuebben, *Metric response and Hermitian Yang–Mills flow for two-step extensions*, preprint, October 27, 2026. [doi:10.5281/zenodo.23717742](https://doi.org/10.5281/zenodo.23717742).
- [Wue26b] B. J. Wuebben, *Uniform Hermitian–Einstein spectra for bipartite two-step extensions*, preprint, October 27, 2026. [doi:10.5281/zenodo.23717660](https://doi.org/10.5281/zenodo.23717660).
- [Wue26c] B. J. Wuebben, *Uniform Hermitian–Einstein spectra with repeated stable factors*, preprint, October 29, 2026. [doi:10.5281/zenodo.23780241](https://doi.org/10.5281/zenodo.23780241).