

LOCAL STABILITY CHAMBERS FOR TORIC TANGENT BUNDLES

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ABSTRACT. We classify the anticanonical slope stability of the tangent bundles of $X_{p,q} = \mathbb{P}_{(\mathbb{P}^1)^{p+q}}(\mathcal{O} \oplus \mathcal{O}(1^p, -1^q))$, where p, q are nonnegative integers and $p + q \geq 1$. The tangent bundle is stable when $p = q$, strictly semistable when $|p - q| = 1$, and unstable otherwise. For $p = k$ and $q = k - 1$, the germ of the semistable polarization locus at the anticanonical class is real analytically equivalent to $[0, \infty)^k \times \mathbb{R}^k$, with stable locus $(0, \infty)^k \times \mathbb{R}^k$. In particular, the number of independent local chamber boundaries is unbounded.

For a smooth projective toric variety whose tangent bundle is semistable at a polarization H_0 , we describe the local chamber at H_0 by the order ideals of a finite poset determined by the equal-slope ray subspaces. If each such subspace is spanned by the rays it contains in every maximal cone, nearby stable polarizations exist precisely when the variety is not a nontrivial toric product. Applied to the existing anticanonical census in dimensions three through six, this criterion gives nearby stability in all 31 indecomposable strictly semistable cases, whose local chambers have one, two, or three independent boundaries. We determine locally free, pairwise nonisomorphic Jordan–Hölder factors both for these cases and for the family $X_{k,k-1}$.

1. INTRODUCTION

Slope stability depends on the polarization. A tangent bundle that is strictly semistable at the anticanonical class can become stable after a small change of that class, whereas a direct-sum decomposition prevents stability for every polarization. We ask how this distinction is reflected in the local geometry of the stable locus on a smooth toric variety.

Let X be smooth and projective of dimension n . For an ample real divisor class H and a torsion-free sheaf E , our convention is

$$\mu_H(E) = \frac{c_1(E)H^{n-1}}{\operatorname{rk} E}.$$

The stable and semistable loci below are subsets of $\operatorname{Amp}(X)_{\mathbb{R}}$. We use quotient projectivization throughout: $\mathbb{P}(E)$ parametrizes one-dimensional quotients of E .

Our first result concerns a family of $\mathbb{P}^1$ -bundles. In $\mathcal{O}(1^p, -1^q)$ the notation means p entries equal to 1 and q entries equal to -1 .

Theorem 1.1. *For nonnegative integers p, q with $p + q \geq 1$, put*

$$X_{p,q} = \mathbb{P}_{(\mathbb{P}^1)^{p+q}}(\mathcal{O} \oplus \mathcal{O}(1^p, -1^q)).$$

These varieties are smooth Fano, and $T_{X_{p,q}}$ is, with respect to $-K_{X_{p,q}}$,

$$\begin{cases} \text{stable,} & p = q, \\ \text{strictly semistable,} & |p - q| = 1, \\ \text{unstable,} & |p - q| \geq 2. \end{cases}$$

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For $X_k = X_{k,k-1}$, of dimension and Picard number $2k$, the germ of the semistable locus at $-K_{X_k}$ is real analytically equivalent to

$$[0, \infty)^k \times \mathbb{R}^k,$$

and the stable locus corresponds to $(0, \infty)^k \times \mathbb{R}^k$. There is an exact sequence of vector bundles

$$(1) \quad 0 \longrightarrow T_{X_k/(\mathbb{P}^1)^k} \longrightarrow T_{X_k} \longrightarrow \bigoplus_{i=1}^k \pi_i^* \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0,$$

where π_i is projection onto the i th positive-twist factor. The relative tangent bundle is stable of rank k at $-K_{X_k}$. All displayed factors have the same slope and are pairwise nonisomorphic, and every component of the extension class is nonzero.

The k boundary hypersurfaces in this statement are the irredundant boundaries of the semistable chamber. There are $2^k - 1$ proper nonzero equal-slope subspaces. For $k \geq 2$, their equality hypersurfaces include hypersurfaces defined by sums of several boundary coordinates, so the entire equality arrangement is larger than the k transverse chamber boundaries.

The proof reduces all slopes to integrals on $[-1, 1]$. The anticanonical polytope has cross-sectional volume $(2+t)^p(2-t)^q$. The sign of a single odd moment separates the stable, semistable, and unstable cases. At $p = q + 1$, all equality subspaces contain the vertical direction and every negative-twist direction. This identifies both the chamber and the relative tangent factor in (1).

The family also illustrates a more general description. The equal-slope ray subspaces of a semistable toric tangent bundle form a finite distributive lattice. Its associated poset determines the active slope inequalities. The essential geometric issue is whether changes of polarization vary the resulting slope gaps independently. In Theorem 3.3, a condition on each maximal cone implies this independence: a hypothetical principal relation among the block divisors is successively annihilated by positive relations among rays. The local chamber then has the form of an order-ideal cone, and its stable part is nonempty exactly when the variety is not a toric product.

Finally, Section 4 applies this criterion to the existing anticanonical census in dimensions three through six. Among its 479 strictly semistable cases, 31 are indecomposable as toric products. Their degrees and complete equal-slope subspaces are reconstructed here from the ray configurations. All satisfy the geometric condition, and their local chambers are orthants with one, two, or three boundary coordinates. The other 448 varieties are products and have no stable tangent polarization. The imported census supplies the exhaustive list; the new finite calculation determines its local chamber geometry.

Relation to previous work. The starting point is Klyachko's description of equivariant bundles [Kly90] and the toric slope criterion in the form of Hering–Nill–Süß [HNS22]. The reduction to finitely many ray subspaces and the polyhedral description in curve classes are already established there. The pullback to the ample cone is nonlinear, and ample-cone stability regions need not be globally convex or polyhedral. Our assertions concern their local analytic form.

Reynolds studies tangent stability on Fano Bott manifolds in Chapter 4 of [Rey23]. His Lemmas 4.2.3–4.2.4 compute the vertical facets of $\mathbb{P}^1$ -bundles over products of projective lines, and Proposition 4.2.6 gives a bound on their volumes. This is the ambient construction used in Theorem 1.1. Our calculation also treats all nonvertical ray subspaces and determines the complete equality structure in every dimension. The Hirzebruch-surface case $k = 1$ is classical;

see [BDGP21, Section 6]. The Bott-tower constructions in [DDK20, Section 6] concern other equivariant rank-two bundles.

Napame–Tipler [NT24] give stability criteria for pullbacks along toric fibrations and applications to stable perturbations. Clarke–Tipler [CT24] describe local chambers and convergence of Hermitian Yang–Mills connections for simple bundles whose graded object is locally free with abelian automorphism group. The locally free, pairwise nonisomorphic factors obtained here give the required graded objects. The tangent bundles of X_k and the 31 indecomposable census examples are simple because they are stable at a nearby polarization. Thus these examples satisfy the geometric hypotheses of [CT24]. The distributive-lattice argument itself is the standard tight-set construction for submodular polyhedra; see [Fuj84].

2. PROJECTIVE BUNDLES AND THE NUMBER OF CHAMBER BOUNDARIES

2.1. The fan and anticanonical degrees. Write $d = p + q$, $n = d + 1$, and let h_i be the hyperplane classes on the base. Set $a_i = 1$ for $i \leq p$, $a_i = -1$ otherwise, $A = \sum_i a_i h_i$, $S = \sum_i h_i$, and $\xi = c_1(\mathcal{O}_{X_{p,q}}(1))$. Then

$$(2) \quad h_i^2 = 0, \quad \xi^2 = A\xi, \quad D := -K_{X_{p,q}} = 2\xi + 2S - A.$$

In a lattice basis $e_1, \dots, e_d, v$, the fan rays are

$$(3) \quad v, -v, \quad e_i, -e_i + a_i v \quad (1 \leq i \leq d).$$

Every maximal cone chooses one ray from each pair, including the vertical pair. These are the projective-bundle cones of [CLS11, Section 7.3]. The divisor relations give $D_{-v} = \xi$, $D_v = \xi - A$, and both divisors of the i th base pair have class h_i .

The anticanonical polytope in the dual coordinates $(x_1, \dots, x_d, t)$ is

$$(4) \quad -1 \leq t \leq 1, \quad -1 \leq x_i \leq 1 + a_i t.$$

At $t = \pm 1$, all interval lengths $2 + a_i t$ are positive. Its normal cones are therefore precisely the cones just described, and their primitive generators are lattice bases. In particular D is ample.

For the integral slope calculation through the next subsection, divide divisor degrees by $(n - 1)!$. Thus facet volumes use Lebesgue measure in the affine facet lattice. Put

$$(5) \quad \begin{aligned} w(t) &= (2 + t)^p (2 - t)^q, \\ M &= \int_{-1}^1 w(t) dt, \\ I_+ &= \int_{-1}^1 \frac{w(t)}{2 + t} dt \quad (p > 0), \\ I_- &= \int_{-1}^1 \frac{w(t)}{2 - t} dt \quad (q > 0). \end{aligned}$$

The two vertical facets have degrees $3^p, 3^q$. Each ray of a positive pair has degree I_+ , and each ray of a negative pair has degree I_- . For a sloping facet $x_i = 1 + a_i t$, the other coordinates and t are integral coordinates for its affine lattice; no additional normal-length factor occurs. The full polytope has volume M , so

$$(6) \quad \frac{D^n}{(n - 1)!} = nM = 3^p + 3^q + 2pI_+ + 2qI_-, \quad \frac{\mu_D(T_X)}{(n - 1)!} = M.$$

Terms for an absent sign are omitted. The vertical formulas agree with [Rey23, Lemmas 4.2.3–4.2.4].

2.2. The slope test. For a subspace $W \subset N_{\mathbb{C}}$, the saturated induced equivariant subsheaf $\mathcal{F}_W \subset T_X$ has rank $\dim W$ and

$$(7) \quad c_1(\mathcal{F}_W) = \sum_{u_\rho \in W} D_\rho.$$

It suffices to test slope inequalities on subspaces spanned by rays [HNS22, Proposition 1.2 and Remark 2.6]. For the configuration (3), these subspaces have a particularly simple form. If $v \notin W$, at most one ray from each base pair belongs to W . Their images in the base lattice are independent, so their number is $\dim W$. If $v \in W$, either both rays of a base pair belong to W or neither does; hence

$$(8) \quad W = \text{span}_{\mathbb{C}}(v, e_i : i \in T)$$

for a subset $T \subseteq \{1, \dots, d\}$.

Interchanging p, q reverses the vertical coordinate, so suppose $p \geq q$. If $p = q \geq 1$, symmetry gives $I_+ = I_- = I$. We have $M > I$ because $2 + t > 1$ in the interior, whereas

$$M - 2I = \int_{-1}^1 t(2-t)(4-t^2)^{p-1} dt = - \int_{-1}^1 t^2(4-t^2)^{p-1} dt < 0.$$

A subspace not containing v thus has normalized slope $I < M$. For a proper subspace containing v , let J be the nonempty set of omitted base indices. By (6), its normalized slope margin is

$$\frac{c_1(\mathcal{F}_W)D^{n-1}}{(n-1)!} - (\dim W)M = |J|(M - 2I) < 0.$$

This proves stability in the balanced case.

If $p \geq q + 2$, write $h = p - q - 1 \geq 1$. Then

$$M - 2I_+ = \int_{-1}^1 t(2+t)^h(4-t^2)^q dt > 0,$$

since pairing t and $-t$ gives the positive integrand $t((2+t)^h - (2-t)^h)(4-t^2)^q$ for $0 < t < 1$. Take W to contain v and every base direction except one positive direction. Its rank is $n - 1$ and its normalized slope margin is $M - 2I_+ > 0$. Thus the tangent bundle is unstable.

It remains to treat $p = q + 1$. Put $k = p$ and $q = k - 1$. Define

$$I_q = \int_{-1}^1 (4-t^2)^q dt,$$

and, for $q \geq 1$, put

$$J_q = \int_{-1}^1 (4+t^2)(4-t^2)^{q-1} dt.$$

These are I_+ and I_- , respectively, and $M = 2I_q$. Integrating the derivative of $t(4-t^2)^q$ gives

$$(9) \quad qJ_q = (q+1)I_q - 2 \cdot 3^q \quad (q \geq 1).$$

For $q \geq 1$ we have $I_q > 2 \cdot 3^q$, and it follows that

$$(10) \quad I_q < J_q < 2I_q \quad (q \geq 1).$$

Thus every subspace not containing v has slope strictly less than M . For $q \geq 1$, a subspace of the form (8), with a positive and b negative base directions, has rank $1 + a + b$ and margin

$$(11) \quad 4 \cdot 3^q + 2aI_q + 2bJ_q - 2(1+a+b)I_q = 2(b-q)(J_q - I_q).$$

For $q \geq 1$ this is nonpositive, with equality exactly when $b = q$. For $q = 0$, there are no negative directions and $I_0 = 2$; the margin is directly $4 + 2aI_0 - 2(1+a)I_0 = 0$.

Consequently the proper nonzero equal-slope subspaces are precisely

$$(12) \quad V_0 + \text{span}_{\mathbb{C}}(e_i : i \in T), \quad V_0 = \text{span}_{\mathbb{C}}(v, e_{k+1}, \dots, e_{2k-1}), \quad T \subsetneq \{1, \dots, k\}.$$

This proves the trichotomy and exhausts all equalities.

2.3. Local coordinates and stable factors. We now return to ordinary intersection degrees, without the factorial normalization used in the integral calculation. The space V_0 in (12) corresponds to the relative tangent bundle $F = T_{X_k/(\mathbb{P}^1)^k}$. Indeed the projection to the positive base factors is locally a product with fiber

$$\mathbb{P}_{(\mathbb{P}^1)^{k-1}}(\mathcal{O} \oplus \mathcal{O}(-1, \dots, -1)).$$

Its relative tangent sequence is (1). Write $C_0 = c_1(F) = D - 2 \sum_{i=1}^k h_i$ and $C_i = 2h_i$ for $i \geq 1$. These classes are independent: C_0 has nonzero ξ coefficient in the basis $\xi, h_1, \dots, h_{2k-1}$.

For a nearby polarization set

$$(13) \quad z_i(H) = \left(2h_i - \frac{D}{2k}\right) H^{2k-1}, \quad 1 \leq i \leq k.$$

The centered divisor classes in this expression have rank k . The derivative of $H \mapsto H^{n-1}$ at an ample class is an isomorphism on numerical divisor and curve spaces, by the Hodge-index pairing [GRT16, Theorem 6.4 and Corollary 6.5]. Thus the z_i extend to real analytic local coordinates at D .

Only the inequalities in (12) can be active sufficiently near D , since the finite list of other inequalities is strict there. For the subspace indexed by T , its slope margin is $-\sum_{i \notin T} z_i(H)$. Hence semistability is exactly $z_i \geq 0$ for every i , and stability exactly $z_i > 0$ for every i . This proves the chamber statement of Theorem 1.1.

The subbundle F is stable at D . A subsheaf of greater slope would contradict semistability of T_X ; an equivariant equal-slope proper subsheaf would contradict the minimality of V_0 in (12). The toric criterion tests ordinary stability, so this suffices. The quotient factors are line bundles of the same slope. Their first Chern classes, together with C_0 , are independent, which excludes isomorphic factors. Finally a zero component of the extension class would split off its quotient line bundle. A nontrivial direct sum cannot be stable for any polarization, whereas the local chamber contains stable classes. Every extension component is therefore nonzero, completing the proof of Theorem 1.1.

Example 2.1. For $k = 1$, X_k is the Hirzebruch surface $\mathbb{F}_1$. If S is its negative section and f a fiber, the first Chern classes of the two stable factors are $2S + f$ and $2f$. For $H = aS + bf$ with $b > a > 0$, the relevant inequality is $2b - 3a < 0$. Thus the stable side near $-K = 2S + 3f$ is $a < b < 3a/2$, in agreement with the classical surface calculation. For $k = 2, 3$, the anticanonical degrees are 352 and 38976, respectively; the proper equality subspaces number 3 and 7.

3. EQUAL-SLOPE SUBSPACES AND LOCAL CHAMBERS

Let $X = X_{\Sigma}$ be a smooth projective toric variety of dimension $n \geq 2$. Write $R = \Sigma(1)$, let u_{ρ} be the primitive ray generators, and put $D = \sum_{\rho \in R} D_{\rho} = c_1(T_X)$. Fix an ample real class H_0 such that T_X is semistable, and set

$$\alpha_0 = H_0^{n-1}, \quad d_{\rho} = D_{\rho} \alpha_0 > 0, \quad \mu = D \alpha_0 / n > 0.$$

Positivity holds because each invariant divisor is nonzero and effective. For a subspace $V \subset N_{\mathbb{C}}$ write $R_V = \{\rho : u_{\rho} \in V\}$ and $d(V) = \sum_{\rho \in R_V} d_{\rho}$.

We use the toric criterion (7) also for real ample classes. To justify this extension, specialize an arbitrary subsheaf to a torus-fixed point in the closure of its orbit in the projective Quot scheme. Flat specialization preserves rank and numerical first Chern class, and saturation only increases degree against an ample real class. Thus any ordinary destabilizing or equal-slope subsheaf gives an equivariant one with at least that slope. This is the same equivariant reduction underlying [HNS22, Proposition 2.3].

Let $\mathcal{L}$ consist of $0, N_{\mathbb{C}}$ and the subspaces with $d(V) = \mu \dim V$. Each such subspace is ray-spanned: otherwise its contained-ray span has the same positive degree and smaller rank, contradicting semistability. Thus $\mathcal{L}$ is finite.

Lemma 3.1. *The set $\mathcal{L}$ is a distributive lattice under sum and intersection. For $V, W \in \mathcal{L}$,*

$$R_{V+W} = R_V \cup R_W, \quad R_{V \cap W} = R_V \cap R_W.$$

Proof. Ray membership and semistability give

$$d(V) + d(W) \leq d(V + W) + d(V \cap W) \leq \mu(\dim(V + W) + \dim(V \cap W)) = d(V) + d(W).$$

Both inequalities are equalities. The two resulting subspaces have the required slope, and positivity of all d_ρ excludes any extra ray in $V + W$. Since equality subspaces are ray-spanned, $V \mapsto R_V$ is injective. Its image is closed under union and intersection, which proves distributivity. $\square$

This is the standard tight-set argument for a submodular rank function [Fuj84]. In fact $(d_\rho/\mu)_{\rho \in R}$ belongs to the base polytope of the ray matroid, and its tight sets are exactly the sets R_V above.

Partition R into blocks B_p according to identical membership in all equality sets. Define $p \leq q$ if every equality set containing B_q also contains B_p , and denote this finite poset by P . The equality sets are precisely unions of blocks over order ideals of P . Indeed, intersection of all equality sets containing B_q gives its principal ideal; unions of these give every ideal. For an ideal I , denote its subspace by V_I .

Put $s = |P|$ and $C_p = \sum_{\rho \in B_p} D_\rho$. There are positive integers r_p such that

$$(14) \quad \dim V_I = \sum_{p \in I} r_p, \quad \sum_p r_p = n, \quad C_p \alpha_0 = \mu r_p.$$

To see this, choose a linear extension of P . The rank increment when B_p is added to an initial ideal equals $\sum_{\rho \in B_p} d_\rho/\mu$. It is positive and integral, and the formula shows independence of the chosen linear extension.

Definition 3.2. In $Z = \{z \in \mathbb{R}^s : \sum_p z_p = 0\}$ define

$$\mathcal{C}(P) = \left\{ z : \sum_{p \in I} z_p \leq 0 \text{ for every nonempty proper order ideal } I \right\}.$$

Let $\mathcal{C}^\circ(P)$ denote the set where all displayed inequalities are strict. The superscript denotes this strict locus; it does not assert that the cone is full-dimensional.

We impose the following condition on equal-slope subspaces:

$$(15) \quad V = \text{span}_{\mathbb{C}}\{u_\rho : \rho \in \sigma(1), u_\rho \in V\} \quad \text{for every } V \in \mathcal{L} \text{ and every maximal cone } \sigma.$$

Because the rays of a maximal smooth cone form a basis, this says that exactly $\dim V$ of them lie in V .

Theorem 3.3. *Under (15), the germs of the semistable and stable polarization loci at H_0 are real analytically equivalent to*

$$\mathcal{C}(P) \times \mathbb{R}^{\rho(X)-s+1}, \quad \mathcal{C}^\circ(P) \times \mathbb{R}^{\rho(X)-s+1},$$

respectively. Stable polarizations occur arbitrarily close to H_0 if and only if X is not a nontrivial toric product. In that case rational ample stable classes occur arbitrarily close to H_0 .

Lemma 3.4. *If a ray subspace V satisfies (15), its induced sheaf $\mathcal{F}_V$ is a vector subbundle of T_X . The cones of Σ contained in $V_{\mathbb{R}}$ form a complete fan in $V_{\mathbb{R}}$.*

Proof. On each smooth affine chart, T_X splits into the coordinate weight lines of its ray basis. The condition makes V a sum of a subset of these lines, so the induced inclusion is locally a direct summand. For completeness, take $v \in V_{\mathbb{R}}$ and a maximal cone containing it. In that ray basis, all coordinates outside V vanish. Thus v lies in the face generated by the rays in V . These faces cover $V_{\mathbb{R}}$. $\square$

Lemma 3.5. *Under (15), the numerical classes C_p are linearly independent.*

Proof. Numerical and linear equivalence agree for divisors on a smooth complete toric variety, and the invariant divisor sequence identifies $N^1(X)_{\mathbb{R}}$ with $\mathbb{R}^R/M_{\mathbb{R}}$ [CLS11, Propositions 6.3.15 and 6.4.1]. Consequently a relation $\sum_p a_p C_p = 0$ is represented by a character $m \in M_{\mathbb{R}}$ satisfying $\langle m, u_\rho \rangle = a_p$ on B_p .

Order the blocks by a linear extension $p_1, \dots, p_s$, and let I_j be the initial ideal. By Lemma 3.4, its rays positively span V_{I_j} . They therefore admit a relation $\sum_{\rho \in R_{V_{I_j}}} b_\rho u_\rho = 0$ with every $b_\rho > 0$: sum nonnegative expressions for the negatives of all generators. Inductively suppose $a_{p_i} = 0$ for $i < j$. Pairing this positive relation with m yields

$$a_{p_j} \sum_{\rho \in B_{p_j}} b_\rho = 0,$$

and hence $a_{p_j} = 0$. All coefficients vanish. $\square$

Proof of Theorem 3.3. For a curve class α near α_0 , set

$$z_p(\alpha) = \left(C_p - \frac{r_p}{n} D \right) \alpha.$$

These coordinates sum to zero and vanish at α_0 . Lemma 3.5 implies that their linear map onto Z has rank $s - 1$. The Hodge-index argument used in Section 2 shows that $H \mapsto z(H^{n-1})$ is a submersion at H_0 . Complete its components to real analytic local coordinates. All initially strict inequalities for ray-spanned subspaces remain strict in a sufficiently small neighborhood. The remaining inequalities are exactly those defining $\mathcal{C}(P)$, by (14).

The strict cone is nonempty exactly when the undirected Hasse diagram of P is connected. If it is disconnected, a connected component and its complement are both ideals, whose coordinate sums cannot both be negative. If it is connected, orient each covering relation upward and sum $e_q - e_p$ over the edges $p < q$. For every nonempty proper ideal, no edge enters and at least one leaves, so its coordinate sum is negative. For a singleton poset the condition is vacuous.

It remains to interpret connectedness. Disconnectedness gives two complementary equality ideals. Their ray spans have ranks summing to n , and every ray belongs to one of them, so they split $N_{\mathbb{R}}$. A maximal-cone lattice basis shows that this is an integral lattice splitting. Every cone splits into its component faces. Completeness forces every combination of component cones to occur: take interior points in the two cones and a cone containing their sum; the

fan intersection property makes their sum a face of that cone. Thus the fan is a product fan. Conversely, a toric product splits T_X into equivariant summands. Semistability and the weighted-average identity force all summands to have slope μ . Their complementary ray sets are equality ideals, which disconnect P .

Finally, strict inequalities define an open subset. Rational ample classes are dense there. A nontrivial direct sum cannot be stable at any polarization, since one summand has slope at least the total weighted average. $\square$

Remark 3.6. Only rank $s - 1$ of the centered divisor classes is needed for the coordinate argument. Condition (15) is a sufficient geometric condition that implies the stronger independence of all C_p . We do not assert that every semistable toric tangent bundle satisfies it.

3.1. The graded bundle.

Proposition 3.7. *Under (15), a maximal chain of equality subspaces gives a Jordan–Hölder filtration by vector subbundles. Its stable quotients have ranks r_p and first Chern classes C_p , and are pairwise nonisomorphic. In particular the graded bundle is locally free with automorphism group $(\mathbb{C}^*)^s$.*

Proof. Local freeness follows from Lemma 3.4; additivity gives the ranks and Chern classes. Write $A \subset B$ for successive terms. A saturated equivariant subsheaf $G \subset B/A$ lifts to a subsheaf $\tilde{G} \subset T_X$. It is saturated, since

$$0 \longrightarrow (B/A)/G \longrightarrow T_X/\tilde{G} \longrightarrow T_X/B \longrightarrow 0$$

has torsion-free end terms. If G had greater slope, so would $\tilde{G}$, contrary to semistability. If it had equal slope, its subspace would lie strictly between two successive terms of the maximal equality chain. This is impossible. The toric criterion therefore gives ordinary stability. Independence of C_p excludes isomorphic quotients. Stable bundles of equal slope have only scalar endomorphisms and no nonzero morphisms between distinct factors. $\square$

4. THE STRICTLY SEMISTABLE CENSUS IN DIMENSIONS THREE THROUGH SIX

The anticanonical census of [Wue26] contains 479 strictly semistable cases in dimensions three through six. Its full scope is imported here; the present calculation reconstructs the local data of the indecomposable cases. The identifiers below refer to its smooth Fano polytope lists, with primitive fan generators as vertices.

Theorem 4.1. *Among the strictly semistable cases of the dimension three through six census, the 31 varieties that are not nontrivial toric products admit stable polarizations arbitrarily close to the anticanonical class. Each satisfies (15). Its equality poset has one minimal element and k incomparable elements above it, with counts*

n	$k = 1$	$k = 2$	$k = 3$	indecomposable	product
3	2	0	0	2	5
4	2	1	0	3	21
5	7	1	0	8	74
6	15	2	1	18	348

The local semistable germ is $[0, \infty)^k \times \mathbb{R}^{\rho(X)-k}$, with stable locus its strict orthant. The other 448 cases have no stable tangent polarization.

Proof. For each selected indecomposable configuration, the calculation gives the following complete description. There is a minimal equal-slope ray block B_0 , of rank r_0 , and upper blocks

$B_1, \dots, B_k$. The nonzero proper equality sets are exactly B_0 together with any proper subset of the upper blocks. Their ranks are additive, and every equality subspace meets every maximal cone in exactly its rank many ray generators. The ray matrices, degree vectors, and block partitions are printed in Appendix A; the summary appears in Table 1. Theorem 3.3 applies.

For this poset, the ideal containing all blocks except B_j gives $z_j \geq 0$. These k inequalities imply all the other ideal inequalities, and their strict versions do the same for stability. Thus the local cone is an orthant. The remaining selected ray configurations split into nontrivial direct sums. Smoothness and completeness convert these into toric products, as in the proof of Theorem 3.3. Their tangent bundles cannot be stable. $\square$

The number k need not equal the rank of the quotient by the minimal subbundle. For example, F.6D.6547 has block ranks $(3, 3)$ and a single chamber boundary. The multiple-boundary cases have ranks $(2, 1, 1)$, $(3, 1, 1)$, $(4, 1, 1)$, $(4, 1, 1)$, and $(3, 1, 1, 1)$. The first and last are X_2 and X_3 from Theorem 1.1. In the coordinates of Appendix A, the images of the ordered family basis $(e_1, \dots, e_{2k-1}, v)$ are

$$(-e_1, -e_3, -e_2, e_4) \quad (k=2), \quad (-e_1, -e_2, -e_5, -e_3, -e_4, e_6) \quad (k=3).$$

These signed permutations identify the full fans, not only their ray sets.

In all 31 cases, let F be the stable minimal subbundle. The images of the upper blocks in $N_{\mathbb{C}}/V_0$ span a direct sum: they span the quotient and their ranks add to its dimension. Each projected ray filtration belongs to one summand, so the quotient bundle splits. We obtain

$$(16) \quad 0 \longrightarrow F \longrightarrow T_X \longrightarrow Q_1 \oplus \dots \oplus Q_k \longrightarrow 0,$$

with stable locally free factors of the same slope, pairwise nonisomorphic by Proposition 3.7. Each extension component is nonzero, by nearby stability and the splitting argument used for (1). Thus 26 cases give nonsplit extensions of two stable bundles; the remaining five give three or four stable factors. The product cases need not be polystable: a product can have a factor with strictly semistable nonpolystable tangent bundle.

Table 1: The indecomposable strictly semistable cases. The last column gives the block ranks.

Identifier	n	ρ	k	$\mu_{-K}(T_X)$	Ranks
F.3D.0003	3	5	1	12	(2, 1)
F.3D.0013	3	2	1	18	(2, 1)
F.4D.0072	4	4	2	88	(2, 1, 1)
F.4D.0084	4	4	1	88	(3, 1)
F.4D.0114	4	2	1	128	(3, 1)
F.5D.0245	5	7	1	460	(4, 1)
F.5D.0320	5	6	1	498	(4, 1)
F.5D.0391	5	6	1	498	(4, 1)
F.5D.0409	5	5	1	652	(4, 1)
F.5D.0496	5	7	2	536	(3, 1, 1)
F.5D.0536	5	7	1	536	(4, 1)
F.5D.0569	5	7	1	536	(4, 1)
F.5D.0841	5	2	1	1250	(4, 1)
F.6D.1739	6	4	1	7224	(5, 1)
F.6D.2022	6	4	1	7224	(5, 1)
F.6D.3068	6	6	1	5832	(5, 1)

Identifier	n	ρ	k	$\mu_{-K}(T_X)$	Ranks
F.6D.3115	6	6	1	5848	(5, 1)
F.6D.3127	6	6	1	5848	(5, 1)
F.6D.3996	6	5	1	7404	(5, 1)
F.6D.4188	6	6	1	6376	(5, 1)
F.6D.4954	6	5	1	7452	(5, 1)
F.6D.5485	6	8	2	5160	(4, 1, 1)
F.6D.5541	6	6	3	6496	(3, 1, 1, 1)
F.6D.5880	6	8	1	5160	(5, 1)
F.6D.5962	6	6	1	6496	(5, 1)
F.6D.5965	6	5	2	7236	(4, 1, 1)
F.6D.6078	6	8	1	5160	(5, 1)
F.6D.6547	6	4	1	$\frac{25984}{3}$	(3, 3)
F.6D.6614	6	5	1	7236	(5, 1)
F.6D.7472	6	4	1	9024	(5, 1)
F.6D.7551	6	2	1	15552	(5, 1)

4.1. Exact computation and its scope. The fan of a smooth Fano polytope consists of cones over its faces. For the 31 configurations we determine every facet by its rational supporting equation and check that its vertices form an integral basis. Every other ray lies strictly below its supporting hyperplane, and every ridge occurs in two facets. This nonempty collection is closed under facet adjacency and hence is the entire boundary. An independent calculation enumerates all subsets of n rays and recovers the same facets directly.

For degrees we use equivariant localization. If $m_{\sigma,i}$ is the character basis dual to the rays of a maximal cone and $w_{\sigma,i} = \langle m_{\sigma,i}, \eta \rangle$ for a generic integral η , then

$$D^n = \sum_{\sigma} \frac{(\sum_i w_{\sigma,i})^n}{\prod_i w_{\sigma,i}}, \quad D_{\rho} D^{n-1} = \sum_{\sigma \ni \rho} \frac{w_{\sigma,\rho} (\sum_i w_{\sigma,i})^{n-1}}{\prod_i w_{\sigma,i}}.$$

The same integers are independently obtained by triangulating the corresponding facets of the polar polytope and summing their lattice volumes. Ray flats are enumerated by adjoining a ray and taking linear closure; every flat has a ray basis, which proves completeness of this procedure. Independent enumeration by intersections of ray-spanned hyperplanes gives the same 4,890 proper flats. Every slope inequality and every equality is then tested exactly.

The product partitions of all 479 selected cases are obtained from the connected components of their linear ray matroids. Fundamental circuits relative to one basis and exhaustive minimal dependent subsets give identical partitions. The complete census and its exclusion of other strictly semistable cases remain inputs from [Wue26]; they are not consequences of checking only the 31 indecomposable records. The data and exact verifier are available in the public repository <https://github.com/bwuebben/toric-tangent-stability>. Its `paper3/paper/reproducibility/` directory contains the 31 complete records, the rays of all 479 selected cases, and an exact verifier requiring only Python's standard library. Its `README.md` gives the command `python3 verify.py`, the data conventions, and the optional comparison with the full source files. This verifier reconstructs facets, degrees, flats, and product partitions; selection of the 479 cases from the complete census remains an imported input.

APPENDIX A. RAY CONFIGURATIONS, DEGREES, AND EQUALITY BLOCKS

For each record, the columns of U are the primitive ray generators in their given order, numbered from 0. The vector d lists $D_\rho(-K_X)^{n-1}$. The blocks $B_0; B_1; \dots; B_k$ determine every proper equality subspace as described in the proof of Theorem 4.1. The fan is recovered by taking cones over the faces of $\text{conv}(U)$, so the displayed data determine all inputs to the 31 indecomposable computations.

F.3D.0003.

$$U = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}, \quad \mu = 12.$$

$$d = (6, 3, 2, 6, 3, 5, 5, 6).$$

$$B_0 = \{1, 2, 4, 5, 6, 7\}; \quad B_1 = \{0, 3\}.$$

F.3D.0013.

$$U = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad \mu = 18.$$

$$d = (9, 15, 6, 9, 15).$$

$$B_0 = \{1, 2, 4\}; \quad B_1 = \{0, 3\}.$$

F.4D.0072.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{pmatrix}, \quad \mu = 88.$$

$$d = (44, 44, 52, 54, 44, 44, 18, 52).$$

$$B_0 = \{2, 3, 6, 7\}; \quad B_1 = \{0, 1\}; \quad B_2 = \{4, 5\}.$$

F.4D.0084.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mu = 88.$$

$$d = (27, 68, 44, 45, 50, 24, 44, 50).$$

$$B_0 = \{0, 1, 3, 4, 5, 7\}; \quad B_1 = \{2, 6\}.$$

F.4D.0114.

$$U = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 128.$$

$$d = (112, 112, 112, 48, 64, 64).$$

$$B_0 = \{0, 1, 2, 3\}; \quad B_1 = \{4, 5\}.$$

F.5D.0245.

$$U = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad \mu = 460.$$

$$d = (161, 207, 230, 161, 207, 92, 161, 207, 276, 207, 230, 161).$$

$$B_0 = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 11\}; \quad B_1 = \{2, 10\}.$$

F.5D.0320.

$$U = \begin{pmatrix} 0 & -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \mu = 498.$$

$$d = (177, 279, 260, 260, 249, 249, 104, 270, 177, 279, 186).$$

$$B_0 = \{0, 1, 2, 3, 6, 7, 8, 9, 10\}; \quad B_1 = \{4, 5\}.$$

F.5D.0391.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mu = 498.$$

$$d = (229, 166, 310, 249, 166, 229, 229, 249, 310, 229, 124).$$

$$B_0 = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}; \quad B_1 = \{3, 7\}.$$

F.5D.0409.

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \mu = 652.$$

$$d = (342, 378, 326, 342, 336, 378, 276, 326, 112, 444).$$

$$B_0 = \{0, 1, 3, 4, 5, 6, 8, 9\}; \quad B_1 = \{2, 7\}.$$

F.5D.0496.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}, \quad \mu = 536.$$

$$d = (268, 268, 134, 308, 218, 218, 268, 134, 72, 216, 308, 268).$$

$$B_0 = \{2, 3, 4, 5, 7, 8, 9, 10\}; \quad B_1 = \{0, 1\}; \quad B_2 = \{6, 11\}.$$

F.5D.0536.

$$U = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}, \quad \mu = 536.$$

$$d = (197, 298, 180, 268, 298, 155, 268, 144, 197, 155, 412, 108).$$

$$B_0 = \{0, 1, 2, 4, 5, 7, 8, 9, 10, 11\}; \quad B_1 = \{3, 6\}.$$

F.5D.0569.

$$U = \begin{pmatrix} -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \quad \mu = 536.$$

$$d = (88, 268, 268, 264, 278, 278, 298, 108, 180, 130, 222, 298).$$

$$B_0 = \{0, 3, 4, 5, 6, 7, 8, 9, 10, 11\}; \quad B_1 = \{1, 2\}.$$

F.5D.0841.

$$U = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 1250.$$

$$d = (1125, 1125, 1125, 1125, 500, 625, 625).$$

$$B_0 = \{0, 1, 2, 3, 4\}; \quad B_1 = \{5, 6\}.$$

F.6D.1739.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu = 7224.$$

$$d = (6048, 4992, 750, 6048, 3612, 2250, 4992, 3612, 4992, 6048).$$

$$B_0 = \{0, 1, 2, 3, 5, 6, 8, 9\}; \quad B_1 = \{4, 7\}.$$

F.6D.2022.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu = 7224.$$

$$d = (6372, 5872, 2000, 3612, 6372, 3612, 1000, 5872, 5872, 2760).$$

$$B_0 = \{0, 1, 2, 4, 6, 7, 8, 9\}; \quad B_1 = \{3, 5\}.$$

F.6D.3068.

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix}, \quad \mu = 5832.$$

$$d = (2447, 3813, 2800, 3287, 2973, 3585, 1960, 2916, 1195, 2916, 3287, 3813).$$

$$B_0 = \{0, 1, 2, 3, 4, 5, 6, 8, 10, 11\}; \quad B_1 = \{7, 9\}.$$

F.6D.3115.

$$U = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 5848.$$

$$d = (2773, 4058, 4764, 2067, 2924, 2773, 1840, 2067, 4058, 2924, 2067, 2773).$$

$$B_0 = \{0, 1, 2, 3, 5, 6, 7, 8, 10, 11\}; \quad B_1 = \{4, 9\}.$$

F.6D.3127.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mu = 5848.$$

$$d = (1210, 2924, 2924, 3630, 3932, 3428, 3428, 2672, 2168, 2672, 2168, 3932).$$

$$B_0 = \{0, 3, 4, 5, 6, 7, 8, 9, 10, 11\}; \quad B_1 = \{1, 2\}.$$

F.6D.3996.

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix}, \quad \mu = 7404.$$

$$d = (2504, 5304, 5896, 2800, 5304, 3096, 5896, 4665, 3702, 3702, 1555).$$

$$B_0 = \{0, 1, 2, 3, 4, 5, 6, 7, 10\}; \quad B_1 = \{8, 9\}.$$

F.6D.4188.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 6376.$$

$$d = (1871, 2020, 4660, 2640, 4529, 3188, 2509, 3891, 1275, 3825, 4660, 3188).$$

$$B_0 = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\}; \quad B_1 = \{5, 11\}.$$

F.6D.4954.

$$U = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \mu = 7452.$$

$$d = (6453, 4680, 3726, 3887, 3726, 1560, 5847, 4493, 3887, 1960, 4493).$$

$$B_0 = \{0, 1, 3, 5, 6, 7, 8, 9, 10\}; \quad B_1 = \{2, 4\}.$$

F.6D.5485.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 5160.$$

$$d = (1850, 2880, 2340, 1170, 2580, 1850, 2880, 2580, 2580, 1170, 2880, 740, 2580, 2880).$$

$$B_0 = \{0, 1, 2, 3, 5, 6, 9, 10, 11, 13\}; \quad B_1 = \{4, 8\}; \quad B_2 = \{7, 12\}.$$

F.6D.5541.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mu = 6496.$$

$$d = (3248, 3792, 3248, 3792, 3248, 3240, 3248, 1080, 3248, 3248, 3792, 3792).$$

$$B_0 = \{1, 3, 5, 7, 10, 11\}; \quad B_1 = \{0, 9\}; \quad B_2 = \{2, 4\}; \quad B_3 = \{6, 8\}.$$

F.6D.5880.

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 5160.$$

$$d = (1110, 2580, 2830, 2830, 1950, 2830, 2830, 1390, 2580, 1665, 1665, 1365, 1365, 3970).$$

$$B_0 = \{0, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13\}; \quad B_1 = \{1, 8\}.$$

F.6D.5962.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mu = 6496.$$

$$d = (3384, 3384, 3248, 2700, 3656, 3656, 1760, 1620, 5008, 3656, 3656, 3248).$$

$$B_0 = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10\}; \quad B_1 = \{2, 11\}.$$

F.6D.5965.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \mu = 7236.$$

$$d = (4950, 4266, 3618, 3618, 3618, 4266, 1980, 4266, 4950, 4266, 3618).$$

$$B_0 = \{0, 1, 5, 6, 7, 8, 9\}; \quad B_1 = \{2, 3\}; \quad B_2 = \{4, 10\}.$$

F.6D.6078.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & -1 \end{pmatrix}, \quad \mu = 5160.$$

$$d = (2980, 2680, 2680, 1110, 2580, 2580, 1110, 1950, 2680, 780, 2220, 2680, 1950, 2980).$$

$$B_0 = \{0, 1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13\}; \quad B_1 = \{4, 5\}.$$

F.6D.6547.

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu = \frac{25984}{3}.$$

$$d = (5056, 6496, 6496, 5056, 5056, 6496, 5056, 1440, 4320, 6496).$$

$$B_0 = \{0, 3, 4, 6, 7, 8\}; \quad B_1 = \{1, 2, 5, 9\}.$$

F.6D.6614.

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}, \quad \mu = 7236.$$

$$d = (4455, 4455, 2025, 2970, 3618, 4158, 4158, 4158, 5643, 3618, 4158).$$

$$B_0 = \{0, 1, 2, 3, 5, 6, 7, 8, 10\}; \quad B_1 = \{4, 9\}.$$

F.6D.7472.

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mu = 9024.$$

$$d = (7632, 6352, 4480, 4512, 7632, 4512, 3200, 6352, 6352, 3120).$$

$$B_0 = \{0, 1, 2, 4, 6, 7, 8, 9\}; \quad B_1 = \{3, 5\}.$$

F.6D.7551.

$$U = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu = 15552.$$

$$d = (14256, 14256, 14256, 14256, 14256, 6480, 7776, 7776).$$

$$B_0 = \{0, 1, 2, 3, 4, 5\}; \quad B_1 = \{6, 7\}.$$

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