TRIPLE CURVES AND UNBOUNDED DENOMINATORS IN THREEFOLD CURVE COUNTING

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Abstract. We exhibit a smooth projective Calabi–Yau threefold on which the exponents of the scalar local factors in the universal series of effective curves have unbounded denominators, even among homology classes of divisibility three. These factors are defined by scaling both normal directions to a curve with the same weight. For a curve of odd genus $2k+1$, the scalar class of its tripled zero section has exact denominator 3^k in the integral cycle algebra in Chern number and virtual dimension zero. We also determine the denominator in even genus. From even genus four onward it is smaller than the least product of weights needed to isolate the triple by single-fiber evaluations, even allowing the normal bundle to vary. A deformation separates the moving curves. Counting their embedded images in the cycle space and their contributions in every source genus on the same family determines the difference of the universal classes.

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1. Introduction

On a single smooth projective Calabi–Yau threefold, the scalar local factors in the universal series of effective curves can occur with exponents of arbitrarily large denominator. This already happens in classes of divisibility three. To see what is being factored, assign to each effective integral homology class the unit of its compact space of cycles. Form the generating series of these units. A choice of local model for the multiples of an embedded curve gives a factor for each class and genus. Scaling the two normal directions with opposite weights in a balanced bundle gives integral exponents [Wue26b, Wue26e]. Scaling both directions with weight one gives the scalar factors introduced by Pardon [Par25]. The same spaces of cycles enter both products.

The compact example of [Wue26a, Theorem 1.1] concerns a threefold containing a Hirzebruch surface. A primitive class on the surface has genus-three multiplicity -14. At three times this class the genus-ten scalar exponent has fractional part $2/3$. There are two questions here. What denominator does the local class of a tripled curve carry? Can a single compact target realize increasingly large denominators once the covering degree has been fixed?

For the local question let C be a smooth projective curve of genus g and N a rank-two normal bundle of degree $2g-2$. The tripled zero section is the effective cycle $3[C]$ in the total space of N. An evaluation at a point of C applies a fiber covector to the three sheets of a cubic cycle and takes a homogeneous symmetric polynomial in the three values. Its degree is its weight under scalar dilation. A tuple with compact common zero locus gives an integral supported Thom class. Dividing it by the product of the weights gives Pardon's scalar class $x_{g,3}$ [Par25, Definition 4.7]. Its denominator is the least positive integer that takes it into $H_{\mathbb{Z}}$, the part of the integral algebra of semi-Fano cycles in first Chern number and virtual dimension zero. Here semi-Fano means that each irreducible component has nonnegative first Chern number. We use the integral generators of [Wue26e, Theorem 1.1], obtained by giving opposite weights to two normal lines of degree $g-1$ whose product is K_C.

Theorem 1.1. *In genera zero and one the scalar and opposite-weight classes coincide in every degree. For $g \geq 2$ the denominator of the scalar triple is*

$$\operatorname{den}(x_{g,3}) = \begin{cases} 3^{k}, & g = 2k+1, \\ 3^{\max\{0,\, k-2-v_3(2k+1)\}}, & g = 2k+2, \end{cases}$$

where v_3 is the valuation at three. For a fixed curve of genus $g \geq 2$, the least product of weights, over all rank-two bundles of degree $2g-2$ and all tuples of homogeneous single-fiber evaluations with compact common zero locus on cubic cycles, is $3^{\lfloor (g-1)/2 \rfloor}$. A general stable bundle with determinant K_C attains the minimum.

For example, on a genus-four curve the best choice of normal bundle allows a single evaluation of weight three. The resulting supported class is divisible by three in $H_{\mathbb{Z}}$, so the normalized class is integral. In genus five the minimum is nine and the class itself has denominator nine. Computing a minimum weight product thus gives a sharp geometric bound on the normalization, while computing the class denominator requires the divisibility of the numerator as well. For use below we write s_{g-1} for the exponent of three in $\operatorname{den}(x_{g,3})$ and put $s_0 = 0$.

Let $e_{\beta,h}(Y)$ denote the exponent of the scalar genus-h factor at an effective class β. On our compact threefold we find infinitely many of these individual exponents to which the odd-genus formula applies exactly.

Theorem 1.2. *There is a smooth projective Calabi–Yau hypersurface Y containing a first Hirzebruch surface with negative section s and ruling f. For each $a = 12j + 3$, $j \geq 0$, put $\alpha_a = as + (a+1)f$ and $k = (a(a-1)/2 - 1)/2$. The class α_a is primitive and $e_{3\alpha_a, 9k+1}(Y)$ has exact denominator 3^k. The first of these exponents, in genus ten, has fractional part $2/3$.*

The local Gromov–Witten theory of Bryan and Pandharipande [BP08] computes numerical residues for two normal weights. A numerical residue alone does not identify a class in a universal cycle algebra. To make the comparison, we vary a balanced sum of two normal lines in a family of algebraic targets and impose conditions on the characteristic polynomials of the sheets. At prime covering degree the nonconstant coefficients in the ratio of the two weights can be computed on embedded images, after deformation at each prescribed bound on source genus. Intersection on the cycle space counts each of these images with its sign. On the same family, stable maps with that image include maps with contracted components, whose degree-one series is $(2\sin(u/2))^{2h-2}$ for an image of genus h. Theorem 2.1 consequently identifies the difference of the scalar and opposite-weight logarithms in prime degree. For triples, the lower-degree terms are integral, so the cubic trace in §3 determines the denominator of the full class. A stable-pair calculation on the Hirzebruch surface singles out a primitive genus with multiplicity prime to three. Its contribution survives in the scalar exponent of Theorem 1.2. The general formula in §4 sums over all effective integral roots of the triple class.

The genus-ten example distinguishes three measurements of the triple. For a genus-three core the scalar-minus-opposite-weight cubic logarithm has coefficient $4720/3$ at $(2\sin(u/2))^{18}$. Changing variables and taking the coefficient of u^{18} instead gives the fixed source-genus-ten residue $23593/5$. It is the former coefficient that multiplies -14 in the change of the compact exponents. A different comparison uses the point class of an isolated triple in a nonlinear neighborhood of the curve. For a general stable normal bundle of canonical determinant there are eight one-parameter families of embedded cyclic triples in genus three. A general quadratic term obstructs all eight, and the isolated triple exceeds the scalar class by $8x_{10,1}/3$, where $x_{10,1}$ is the class of a simple genus-ten curve (Theorems 9.1 and 9.2). This describes the moving covers behind the failure of triple rigidity studied in [BP06].

In composite degree the characteristic polynomials can be proper powers. We prove the comparison outside the weights where these powers occur and compute the correction from doubled images. This determines the scalar-to-opposite-weight conversion in degree four for every core genus and in degree six for a genus-two core (§12). Appendices C and D treat individual split cubic weights and the contribution of cuspidal cubics over a genus-five curve.

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2. Changing the normal weights

Let C be a smooth connected projective curve of genus $g \geq 2$, and let $L \oplus M$ be a balanced bundle, meaning $LM = K_C$ and $\deg L = \deg M = g - 1$. Write H for the rational Chern-number-zero, virtual-dimension-zero summand of Pardon's semi-Fano universal cycle algebra [Par25, Definition 1.1 and Theorem 1.2] and $x_{h,1} \in H$ for its multiplicity-one scalar generator in genus h. For two nonzero integral fiber weights with ratio z, let $X_{g,m}(z)$ denote the normalized class of the multiple zero section of covering degree m. Section 5 constructs

its Laurent polynomial in z. We use the logarithm for disjoint union and write

$$P_{m,j} = [z^j q^m] \log\Big(1 + \sum_{l>0} X_{g,l}(z) q^l\Big).$$

The core genus g is fixed in this notation. Let $N_{k,m,j}$ be the coefficient of z^j in the connected genus-k, degree-m local Gromov–Witten residue of $L \oplus M$, with the two independent fiber weights as in [BP08]. Set $n_s = (g-1)\binom{s}{2}$ and $w = (2\sin(u/2))^2$.

Theorem 2.1. *Let d be the largest proper divisor of $m \geq 2$. If $n_d < |j| \leq n_m$, then*

$$P_{m,j} = \sum_{h\geq 0} a_{h,j} x_{h,1}, \qquad \sum_{h\geq 0} a_{h,j} w^{h-1} = \sum_{k\geq 0} N_{k,m,j} u^{2k-2}.$$

The left sum has finite support. Its coefficients vanish for $h < 1 + m(g-1) + |j|$. For each genus bound K, the construction in Theorem 6.1 gives $a_{h,j}$ for $h \leq K$ as the signed number of constrained embedded curves of genus h divided by $\prod_{i=2}^{m} i^{2(i-1)(g-1)}$.

For a prime p one has $d = 1$ and $n_d = 0$. For the degree-p logarithm $P_p(z) = \sum_j P_{p,j} z^j$, symmetry gives

$$P_p(1) - P_p(-1) = 4 \sum_{\substack{j>0 \\ j \text{ odd}}} P_{p,j}.$$

This determines the connected scalar–antidiagonal discrepancy at p. At composite m the boundary of the stated range has a geometric meaning. If $d > 1$, a linear system of spectral polynomials at $j = n_d$ necessarily meets the locus of powers of degree-d polynomials. Division of cycles then detects a nonzero lower-degree primitive. We give the intersection calculation in §5.

Let X be a smooth projective complex threefold with $K_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. For an effective integral class β, let $U_{X,\beta} \in H$ be the class of its compact cycle space. Write F_h^{sc} and F_h^{ad} for the scalar and balanced antidiagonal universal local series of a genus-h core, both with constant term one. The exponents $e_{\beta,h}$ and $c_{\beta,h}$ are defined by

$$\sum_{\beta} U_{X,\beta} Q^\beta = \prod_{\beta\neq 0,h} F_h^{\mathrm{sc}}(Q^\beta)^{e_{\beta,h}} = \prod_{\beta\neq 0,h} F_h^{\mathrm{ad}}(Q^\beta)^{c_{\beta,h}}$$

in the effective integral class monoid completed by ample degree. The $e_{\beta,h}$ are rational, and $c_{\beta,h}$ are the integral elementary-cluster coefficients by [Wue26b, Theorem 1.1]. If the image of α in the free quotient of $H_2(X;\mathbb{Z})$ is primitive, write $D_{g,p}(w)$ for the difference of the scalar and antidiagonal connected degree-p local residues of a core of genus g. Theorem 2.1 gives the coefficient identity

$$\sum_h (c_{p\alpha,h} - e_{p\alpha,h}) w^{h-1} = \sum_{p\gamma = p\alpha} \sum_{g\geq 2} c_{\gamma,g} D_{g,p}(w). \tag{1}$$

Indeed the only covering degrees at $p\alpha$ are one and p, and at every effective integral root γ the degree-one generators of both products agree. For core genera zero and one the two local classes agree in all degrees. Finite support at each class makes (1) a polynomial identity after taking the difference.

The first possible change is in genus $p + 2$. The use of the rational basis $x_{h,1}$ in this argument can also be made before assigning any numerical invariant to a complex family: proper pullback to connected maps is injective on the multiplicity-one span when all source genera are used [Wue26b, Sections 5–7]. For integral denominators we use the stronger result

that the $a_{h,d} := X_{h,d}(-1)$ freely generate $H_{\mathbb{Z}}$, with $a_{h,1} = x_{h,1}$. This is the integral basis theorem of [Wue26e, Theorem 1.1]. In particular the intersection of this integral algebra with the rational span of the $x_{h,1}$ is their free integral span: they are distinct degree-one variables in that polynomial presentation.

The proof of Theorem 2.1 occupies Sections 5–7. The argument needs a family of targets to give the same count in two different theories. For a prescribed bound on source genus we vary the target and impose the spectral coefficient conditions until the constrained connected images are regular embedded curves. The cohomological insertion counts their signs. The relative virtual class on maps also sees all contracted components, and each embedded image of genus h contributes the entire degree-one series w^{h-1}. This construction uses an algebraic relative obstruction theory on the chosen family and proper genus detection on the cycle spaces.

3. The cubic trace and its exact denominator

For the numerical calculation let F_g and E_g be the disconnected degree-generating series at scalar and opposite weights. Set $D_n = D_{n+1,3}$ and $n = g - 1$. The transition to the integral lattice will use the full row of coefficients of D_n, including coefficients in genera greater than the genus of the base.

3.1. **The quadratic and cubic corrections.** Write $a_{g,m} = X_{g,m}(-1)$ and $x_{g,m} = X_{g,m}(1)$. With $s = 2\sin(u/2)$ the scalar and antidiagonal degree-two residues are, respectively,

$$s^{4n}\big((4-2s)^n + (4+2s)^n\big), \qquad 2s^{4n}(4-s^2)^n, \qquad n = g-1$$

by [BP08, §8]. The linear terms of the two series agree. Thus the difference $J_n = [q^2](\log F_{n+1} - \log E_{n+1})$ is

$$J_n(w) = 2\sum_{l=1}^{n} 4^{n-l}\left(\binom{n}{2l} - (-1)^l\binom{n}{l}\right) w^{2n+l}. \tag{2}$$

In particular $J_1 = 2w^3$, $J_2 = 24w^5 - 2w^6$. The first formula and Theorem 2.1 give the universal identity

$$a_{2,2} = x_{2,2} - 2x_{4,1}.$$

At this genus the only nonconstant modes of $X_{2,2}(z)$ have weights ± 1 and are equal. Each is $x_{4,1}/2$, as in the geometric double-cover calculation of [Wue26b].

At degree three let $w^3\lambda_i$ be the three eigenvalues of the balanced scalar handle operator. Its characteristic equation after division by w^3 in the eigenvalue variable is

$$X^3 - AX^2 + \tfrac{1}{3}BX - \tfrac{1}{3}C = 0, \tag{3}$$

where, with $L_0 = w^2 + 54w - 243$,

$$A = w^2 + 50w + 81, \quad B = 4(w-6)L_0, \quad C = 4(w-4)(w-3)^2L_0.$$

This is the scalar specialization of the partition multiplication and cap operators of [BP08, §6 and Appendix], derived in Appendix B. The antidiagonal eigenvalues are U, U, V, where $U = w^3(4-w)(3-w)^2$ and $V = w^3(3-w)^2$ by the hook-length formula. Writing $T_n = \sum_i \lambda_i^n$ gives

$$D_n = w^{3n}T_n - 2U^n - V^n - w^n J_n. \tag{4}$$

Here $[q]F_{n+1} = [q]E_{n+1} = w^n$. The final term removes the contribution of a degree-one and a degree-two connected component. For example $T_1 = A$ and $T_2 = A^2 - 2B/3$. Substitution gives

$$\begin{aligned} D_1 &= 120w^4 - 20w^5 + 2w^6, \\ D_2 &= 14448w^7 - 1136w^8 + \tfrac{4720}{3}w^9 - 332w^{10} + 40w^{11} - 2w^{12}. \end{aligned} \tag{5}$$

Let $\iota(w^{h-1}) = x_{h,1}$, extended linearly. Recovering the disconnected universal generators from their logarithms gives

$$a_{3,3} = x_{3,3} - \iota(D_2) + x_{3,1}(-24x_{6,1} + 2x_{7,1}).$$

Indeed the quadratic difference here is $-\iota(J_2)$, and the linear generators agree. In particular the coefficient of $x_{10,1}$ in the scalar-minus-antidiagonal cubic logarithm is $4720/3$. The corresponding coefficient in the disconnected scalar residue before the logarithm and the change of weights is $[w^9]w^6T_2 = 292/3$.

The two positive odd weights contributing to $[w^9]D_2$ can also be distinguished. Write b_1, b_3 for their respective coefficients at $x_{10,1}$. All higher odd weights vanish at this genus by Theorem 2.1. Let $B_d(z)$ denote the balanced handle at weights $(z, 1)$ in degree d, with the normalizations in Appendix B. The genus-three connected coefficient is

$$\operatorname{tr}(B_3(z)^2) - w^2 \operatorname{tr}(B_2(z)^2) + w^6/3.$$

For clarity, its values at the two additional weight ratios needed to separate the modes are

z	2	-2
$[w^9]$	549	$-19891/6.$

They follow by substituting $(t_1, t_2) = (\pm 2, 1)$ in the multiplication, metric and cap in Appendix B and reducing with $r^2 - (2 - w)r + 1 = 0$, $r = e^{iu}$. Equality of the opposite weights and subtraction at $z = 1, -1$ and $z = 2, -2$ give

$$4(b_1 + b_3) = 4720/3, \qquad 5b_1 + \tfrac{65}{4}b_3 = 23185/6.$$

Thus $b_1 = 674/3$ and $b_3 = 506/3$ in the universal algebra.

3.2. **The valuation of the cubic trace.** Write v_3 for the usual valuation on rational numbers, with $v_3(3) = 1$. The common denominator of a finite list of rational numbers is its least positive integral multiplier making the entire list integral.

Theorem 3.1. *The common denominator of the coefficients of D_n is 3^{s_n}, where*

$$s_{2k} = k, \qquad s_{2k+1} = \max\{0, k - 2 - v_3(2k + 1)\}.$$

For $s_n > 0$ the polynomial $3^{s_n}D_n$ has the following reduction in $\mathbb{F}_3[w]$:

$$\begin{aligned} n = 2k: &\qquad 2(-1)^k w^{9k}, \\ n = 2k + 1: &\qquad 2(-1)^k \frac{2k + 1}{3^{v_3(2k+1)}} w^{9k+2}(w - 1). \end{aligned}$$

Proof. Equation (3) gives a recurrence over $\mathbb{Z}[1/3][w]$ for T_n, with $T_0 = 3$, $T_1 = A$ and $T_2 = A^2 - 2B/3$. In (4) all terms other than $w^{3n}T_n$ are integral polynomials. Work in the completion of $\mathbb{Q}(w)$ for the Gauss valuation at three, with residue field $\mathbb{F}_3(w)$. Its valuation on a polynomial is the least valuation of a coefficient. The polynomial

$$P(X) = 3X^3 - 3AX^2 + BX - C$$

reduces to a linear polynomial with derivative $\bar{B} = w^3$. Hensel's lemma gives a unit root z_0 with $\bar{z}_0 = \bar{C}/\bar{B} = \bar{A}$. For the two remaining roots let $S = A - z_0$ be their sum and $T = C/(3z_0)$ their product. Since

$$P(A) = AB - C = 72L_0(3w^2 - 14w - 25),$$

the divided difference $(P(A) - P(z_0))/(A - z_0)$ reduces to $\bar{B}$, and hence

$$v_3(S) = 2, \quad \overline{S/9} = 2(w-1)/w, \qquad v_3(T) = -1, \quad \overline{3T} = w^3.$$

The remaining roots have equal valuation $-1/2$ in a splitting field, since their sum has positive valuation and their product has valuation -1. Their ratio reduces to -1. For $k > 0$ their $2k$th power sum divided by $(-T)^k$ therefore reduces to 2. Adding the integral term z_0^{2k} and multiplying by $3^k w^{6k}$ proves the even case. Directly $D_0 = 0$.

For $n = 2k + 1$ the power sum of the remaining pair is

$$\sum_{l=0}^{k} \frac{n}{2l+1} \binom{k+l}{2l} (-T)^{k-l} S^{2l+1}.$$

This is the coefficient of t^n in $-t\frac{d}{dt}\log(1 - St + Tt^2)$. The $l = 0$ term has valuation $v_3(n) - k + 2$. Each subsequent term has valuation greater by at least $5l - v_3(2l+1) > 0$. If the first valuation is nonnegative, the entire trace is integral at three. Otherwise clearing its denominator leaves only the first term in the reduction. Inserting the reductions of $S/9$ and $3T$ and the factor $w^{3(2k+1)}$ gives the second formula. □

Proposition 3.2. *Only primes at most m divide* $\operatorname{den}(x_{g,m})$. *For a prime p and $g \geq 2$ the p-part of* $\operatorname{den}(x_{g,p})$ *is the p-part of the common denominator of the coefficients of $D_{g,p}(w)$.*

Proof. In the balanced split bundle use scalar weights $(1, 1)$. The elementary symmetric functions of the d projected fiber coordinates on each of the two normal lines are sections of their tensor powers of orders at most d. Their evaluations at finitely many points detect every coefficient, and simultaneous vanishing means that the cycle is the d-fold zero section. The supported integral Thom class is divided by a product of weights from 1 through d in Pardon's construction [Par25, Definition 4.7]. Hence $x_{g,d}$ is integral at every prime $p > d$.

In the difference of the degree-p logarithms, the only term with denominator p from the logarithm uses p copies of the common linear coefficient. It cancels. Every term involving between two and $p - 1$ factors belongs to $H_{(p)} = H_{\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$: the denominator from the logarithm is a unit at p and the indices of its factors are smaller than p. Also $a_{g,p}$ is integral. Theorem 2.1 gives $x_{g,p} - \iota(D_{g,p}) \in H_{(p)}$. Because the $x_{h,1}$ belong to the polynomial basis, clearing the denominator at p on their span is equivalent to clearing it on their coefficients. □

Proof of the denominator formula in Theorem 1.1. For genera zero and one the local models of Section 2 have empty separating tuple: the two lines of $\mathcal{O}(-1)^{\oplus 2}$ have no nonzero sections after any positive-degree cover, and in genus one one can take a nontorsion degree-zero line and its inverse. A section of a pulled-back degree-zero line would trivialize it and taking the norm would make a positive power of the chosen line trivial. Thus the whole compact cycle space in each multiple is the point, independent of weights. For $g \geq 2$ take $n = g - 1$. Theorem 2.1 at degree two and equation (2) give $x_{g,2} - a_{g,2} = \iota(J_n)$ with $J_n \in \mathbb{Z}[w]$. Thus every scalar double is integral. Expanding the cubic logarithms gives

$$x_{g,3} = a_{g,3} + x_{g,1}\iota(J_n) + \iota(D_{g,3}).$$

The first two terms on the right are integral, so the denominator over $\mathbb{Z}$ is exactly the common coefficient denominator of $D_{g,3}$. Its value is 3^{s_n} by the valuation in Theorem 3.1. In particular no factor of two survives. □

Theorem 3.1 also identifies the entire leading residue: in $3^{s_n}x_{n+1,3} \bmod 3H_{\mathbb{Z}}$, replace each w^{h-1} in that theorem by $x_{h,1}$ whenever $s_n > 0$. For $g = 2, 3, 4, 5, 6, 7, 8$ the resulting denominators are $1, 3, 1, 9, 1, 27, 3$.

For arbitrary nonzero normal weights (r, s) in the genus-two quadratic case, the symmetry and the Laurent bound give $X_{2,2}(r/s) = Y + (r/s + s/r)Z$. The difference at equal and opposite weights is $2x_{4,1}$, whence $X_{2,2}(r/s) = x_{2,2} + (r - s)^2 x_{4,1}/(2rs)$.

For comparison, at a genus-three core, the scalar-minus-opposite-weight connected logarithm has coefficient $4720/3$ at $(2\sin(u/2))^{18}$. At fixed arithmetic source genus ten its coefficient at u^{18} is $23593/5$. Finally, for a general stable normal bundle a neighborhood with a general second jet isolates the triple and its point class exceeds the scalar class by $(8/3)x_{10,1}$. Here eight counts cyclic covers and three is the weight of their amplitude.

4. Scalar exponents on a compact threefold

Corollary 4.1. *Let the image of α in $H_2(X;\mathbb{Z})/\mathrm{tors}$ be primitive, where X is a smooth projective Calabi–Yau threefold. For each g let $C_g = \sum_{3\gamma=3\alpha} c_{\gamma,g}$, summing over the effective integral roots. The common denominator of the scalar product exponents $e_{3\alpha,h}$ is 3^M, where*

$$M = \max\big(\{0\} \cup \{s_n - v_3(C_{n+1}) : n \geq 0,\ C_{n+1} \neq 0\}\big).$$

In particular all these exponents are integral if and only if $3^{s_n} \mid C_{n+1}$ for every $n \geq 0$.

Proof. By (1), the polynomial of differences at this class is $\sum_{n\geq 0} C_{n+1}D_n$. There are finitely many occupied genera, and $c_{3\alpha,h}$ is an integer for each h by [Wue26b, Theorem 1.1]. For $M > 0$ scale by 3^M and reduce modulo three. Each maximizing index contributes a nonzero unit times the corresponding reduced polynomial of Theorem 3.1. Their monomial supports, $\{9k\}$ or $\{9k + 2, 9k + 3\}$, are disjoint for distinct indices. This proves equality of the denominator; the sum over torsion-distinct roots of a single genus has already been taken in C_g. For $M = 0$ all rows in the sum are integral. □

In particular (5) gives

$$e_{3\alpha,10} = c_{3\alpha,10} - \tfrac{4720}{3}C_3. \tag{6}$$

Indeed D_1 has degree six, and for $n \geq 3$ the smallest possible power of w is $3n + 1$ by Theorem 2.1.

Let Y be the smooth anticanonical threefold in the toric resolution of X_9 whose fan is specified in Appendix A. Its second homology is torsion-free and it contains a divisor $F \simeq \mathbb{F}_1$ with normal line K_F. Let s, f be the negative section and a ruling fiber. For $a = 12j + 3$, $j \geq 0$, set

$$\alpha_a = as + (a + 1)f, \quad G = a(a - 1)/2, \quad k = (G - 1)/2.$$

We use the same notation for curve classes on F and their images in Y. Two integral nef divisors on Y recover the coefficients of s, f by intersection. Their difference takes value one on α_a. The four other nef degrees of (30) vanish on both s and f. Their common zero face confines every effective curve of class α_a, and every effective summand, to F. Thus α_a has primitive free image and it is the sole integral root at $3\alpha_a$.

These are the classes in Theorem 1.2.

4.1. **The highest primitive genus.** Put $D = as + (a+1)f$. In the presentation $F = \mathrm{Bl}_p \mathbb{P}^2$ one has $D = (a+1)H - E$. Its arithmetic genus is G, and

$$N = h^0(F, \mathcal{O}_F(D)) = \binom{a+3}{2} - 1 = \tfrac{1}{2}(a+1)(a+4).$$

We show that the compact elementary exponents satisfy

$$c_{\alpha_a,h} = 0 \quad (h > G), \qquad c_{\alpha_a,G} = (-1)^{N-1}N. \tag{7}$$

First consider stable pairs on $Z = \mathrm{Tot}(K_F)$. All positive compact curves are supported on F: affine fibers contain no complete curves, and on a normalization projecting nontrivially to F the tautological section of the negative canonical line vanishes. For an effective decomposition $D = \sum D_i$, with at least two nonzero classes $D_i = a_i s + b_i f$, $a_i, b_i \geq 0$, we claim

$$\sum_i \chi(\mathcal{O}_{D_i}) - \chi(\mathcal{O}_D) = \sum_{i<j} D_i D_j = \sum_i b_i(a - a_i) - \sum_{i<j} a_i a_j > 0. \tag{8}$$

These numbers depend only on the divisor classes by Riemann–Roch. Choose $A = \max a_i$ and $t = a - A$. For $t > 0$ the first sum in the last expression is at least $(a+1)t$, and the second is at most $At + t(t-1)/2$. Their difference is at least $t(t+3)/2$. For $t = 0$ some positive multiple of a fiber lies in a different summand, giving an intersection of at least a.

For a pure curve $W \subset Z$ filter $\mathcal{O}_W$ by powers of the ideal of F. The graded quotients are of the form $K_F^{-i} \otimes \mathcal{O}_F/J_i$. The divisorial hull of J_i is $\mathcal{O}_F(-D_i)$, and its quotient by J_i has finite length l_i. The sum of the divisor classes $[D_i]$ is $[W]$. The Euler characteristic of the ith layer is $\chi(\mathcal{O}_{D_i}) - iK_F D_i + l_i$, with both corrections nonnegative. By (8), two nonzero divisor layers in class D give $\chi(\mathcal{O}_W) > 1 - G$. If there is only one, it is the zeroth layer, since a nonzero higher layer along a curve requires a nonzero zeroth layer along that curve. Then $I_F\mathcal{O}_W$ has finite support and vanishes by purity. Such a curve is an effective Cartier divisor in $|D|$. The section of a stable pair has image $\mathcal{O}_W$ and finite-length cokernel. It follows that the minimum pair Euler characteristic is $1 - G$ and all pairs at the minimum are quotient pairs in $|D|$. For a product of pairs in a nontrivial additive splitting of D, apply (8) to their combined divisor layers. Its total Euler characteristic is strictly greater than $1 - G$.

The minimum pair space is $\mathbb{P}^{N-1}$ as a scheme. Indeed surjectivity of the section in a family follows from Nakayama's lemma, identifying it with the corresponding Hilbert functor. For every $W \in |D|$, the normal sequence for $W \subset F \subset Z$ has terms $\mathcal{O}_W(D)$, $N_{W/Z}$ and $K_F|_W$. The last has no sections since $H^0(F, K_F) = 0$ and $H^1(F, K_F - D) = 0$. The latter follows from duality and Kodaira vanishing, as $D - K_F$ is ample. Thus $h^0(N_{W/Z}) = h^0(\mathcal{O}_W(D)) = N - 1$ by rationality of F. The closed immersion of $|D|$ exhausts the closed points, so the local dimension and the embedding dimension of the Hilbert space both equal $N - 1$ there. Its local rings are regular, proving the scheme assertion. The symmetric pair obstruction theory on this smooth proper scheme has degree $(-1)^{N-1}N$ [PT10]. The strict product inequality implies that the coefficient of the formal logarithm of the stable-pair series at Q^D has the same lowest term, $(-1)^{N-1}Nz^{1-G}$ in the pair variable z.

To identify this term with a compact elementary exponent, deform Y to the normal line of F. Blow up $Y \times \mathbb{A}^1$ at $F \times 0$ and remove the strict transform of $Y \times 0$. The resulting smooth family has fibers Y off zero and Z over zero and contains a Cartier divisor $F \times \mathbb{A}^1$. In each positive face class all connected maps have image in that divisor by the nef degrees on Y and negativity on Z. This is true as a statement about families of maps: on every geometric source the Cartier line restricts to the pullback of K_F. Sections vanish on every positive component by negativity, then on all contracted components by connectedness. In a two-term

presentation of the derived direct image the first differential is therefore fiberwise injective and locally split injective. The canonical section vanishes on the universal domain, including over nonreduced bases. Hence the relative moduli is the product of the proper map stack of F and the parameter line. Deformation invariance of its ordinary relative stable-map class gives equality of the connected GW degrees of Y and Z in each face class.

The fixed-target toric correspondence [PP14, Theorem 1] applies to Z. The pair spaces in these classes are proper: in the projective line completion of K_F their geometric supports are on the zero section, and a flat specialization of a pair supported there remains set-theoretically on that section. For instance a power of its ideal annihilating the generic pair sheaf annihilates the flat family over a discrete valuation ring. Thus the equivariant degrees have their ordinary specialization. Since K_Z is trivial there is no prefactor in the empty-insertion correspondence under $z = -e^{iu}$. Taking the formal logarithm through class D uses only the finitely many positive summands on this face, with disconnected maps nonconstant on each connected component. The result identifies the connected fixed-target GW series for Y in α_a with the rational function of z whose lowest term was computed above.

On the compact projective Y, the algebraic and symplectic GW degrees agree [Sie99]. At a class with primitive free image the finite elementary-cluster expansion of [IP18, DIW26] is $\sum_h c_{\alpha_a,h} w^{h-1}$, with these same compact exponents by [Wue26b, Theorem 1.1]. Since $w = (1+z)^2/z$, this is a rational function of z. Equality of the expansions at $z = -1$ gives equality of rational functions. At $z = 0$ the summand of largest genus h has lowest Laurent power $1-h$ with coefficient $c_{\alpha_a,h}$. Comparison with the lowest pair term proves (7).

Proof of Theorem 1.2. For $a = 12j+3$ the integer $G = 2k+1$ is odd and $N \equiv 2 \pmod 3$. In (1) the top occupied row is D_{2k}. Theorem 3.1 shows that its denominator is 3^k and that $3^k D_{2k}$ reduces to $2(-1)^k w^{9k}$. Every lower row has denominator dividing 3^{k-1}. The coefficient multiplying the top row is a unit at three by (7), while the compact elementary exponents are integral. This proves the exact denominator at output genus $9k+1$. For $a = 3$ one has $G = 3$, $N = 14$ and $c_{\alpha_3,3} = -14$. Equation (6) gives $e_{3\alpha_3,10} = c_{3\alpha_3,10} + 66080/3$. For $a = 15$ the corresponding genus is 469 and the denominator is 3^{52}. □

4.2. **The first possible fractional exponents.** If the free image of a class is primitive its scalar and opposite-weight exponents agree. At free divisibility two, (1) and the integral coefficients of (2) imply integrality in all genera, summing over every integral double root. There is also a uniform statement in small genus. For all effective β on a smooth projective Calabi–Yau threefold as above,

$$e_{\beta,h} = c_{\beta,h} \quad (0 \le h \le 3), \qquad e_{\beta,4} = c_{\beta,4} - 2 \sum_{2\gamma=\beta} c_{\gamma,2}.$$

Indeed pull the logarithms of the two compact products back to connected maps in source genus at most four. A positive-degree map to a core of genus $g \ge 2$ in projection degree d has source genus at least $1 + d(g-1)$ by normalization and Riemann–Hurwitz. The zero and one core genera agree in every degree. At the remaining small degrees Theorem 2.1 and (2) leave precisely the term $2x_{4,1}$ from $(g,d) = (2,2)$ through source genus four. This assertion also holds for the equivariant cycle insertions: their supported pullbacks are over the zero-section multiple, hence empty below that same genus. Induct on ample degree, reading source genera in increasing order by the proper detection of [Wue26b, Sections 5–6]. Proper multiples of a genus-four change can first be detected in genus seven. This gives the displayed identities.

One may also try to deduce integrality from the scalar stable-pair series. If all scalar local pair coefficients were integral Laurent series in the pair variable z, division of a compact

integral pair series by already determined lower factors would leave an integral coefficient $\sum_g e_{\beta,g} z^{1-g}(1+z)^{2g-2}$ in the next class. The largest occupied genus has a unique lowest Laurent term with unit coefficient; descending induction would give integral exponents. The local hypothesis fails. In the genus-three scalar cubic the disconnected trace in (38), under $w = (1+z)^2/z$, has coefficient $20 + 292/3 = 352/3$ at z^{-9}. Here one uses the fixed-target equivariant local GW/pairs correspondence [Par25, Section 4.2]; the lowest compact proper pair count used in Theorem 1.2 is integral.

5. Fiber weights and spectral polynomials

5.1. **The class over a product of projective spaces.** Pardon's algebra H is obtained from compactly supported cohomology on spaces of cycles in families of threefolds. A closed oriented real parameter space of dimension b, together with a compactly supported b-cocycle on its relative cycle space, gives a class of relative degree zero. A cocycle over an interval of such families equates its two restrictions if its support stays compact. One may also restrict the target to an open neighborhood containing that support. Multiplication uses the disjoint union of targets. We use the complex families of [Par25, §3] and their map Φ to the corresponding algebra of almost-complex families [Wue26b, §2]. The latter allows smooth oriented parameter chains.

Take the two lines general in their Picard varieties subject to $LM = K_C$. Thus $H^0(C,L) = H^0(C,M) = 0$ and $L^2 \neq K_C$. For $A = L, M$, the vector space $H^0(C, A^i)$ has dimension $(i-1)(g-1)$ for $i \geq 2$, by Riemann–Roch and Serre duality. A compact curve in the rank-two bundle projects finitely onto C, since a fiber is affine. With multiplicities included, a cycle of covering degree m gives on each axis a monic polynomial of degree m. Its coefficient of order i is a holomorphic section of A^i. In families these coefficients are continuous functions of the image cycle. Near a transverse base fiber the local sheets give elementary symmetric functions. Their extension at a branch value is the characteristic polynomial of multiplication in the finite divisor on the line axis. Equivalently one can push the cycle to the line and use the coordinates on its space of monic spectral divisors [Wue26c, §2].

Fix a basis of the dual of each coefficient space using linear combinations of evaluations at finitely many points of C. Write $F \subset C$ for their union. These are n_m tests on each axis. They all vanish precisely on $m[C]$. Let P, Q denote the hyperplane lines on the factors of $B_i = \mathbb{P}^i \times \mathbb{P}^{2n_m - i}$. Twist L, M by P, Q respectively. Each test now takes values in P^s or Q^s, where s is its order. Put

$$W_{g,m} = \prod_{s=2}^{m} s^{2(s-1)(g-1)}.$$

On the relative cycle space take the pullback of the Thom class of the direct sum of all test lines and divide by $W_{g,m}$. We denote its image in H by Y_i. Its cohomological degree equals $\dim_{\mathbb{R}} B_i$, and its support is the compact family of core cycles. The choice of a general splitting computes the same normalized one-circle classes as any splitting of the specified degrees. Along a path in the Picard variety use a fixed sufficiently large number of fiber evaluations and frames on small intervals. They separate all coefficients throughout the path. Appending a redundant test multiplies the insertion by its normalized Euler class and is exactly the hyperplane stabilization of the Borel construction [Wue26c, §2].

Lemma 5.1. *For nonzero integral fiber weights r, s the corresponding class is*

$$X_{g,m}(r/s) = \sum_{i=0}^{2n_m} (r/s)^{i-n_m} Y_i.$$

It is invariant under inversion of the weight ratio. The logarithmic coefficient $P_{m,j}$ is primitive for the coproduct induced by addition of cycles. Pullback ρ_e under multiplication of cycles by e sends $P_{m,j}$ to $P_{m/e,j}$ if $e \mid m$, and to zero otherwise.

Proof. Over two large projective spaces form the same supported tuple. Pull it back by the pair of degree-$|r|, |s|$ Veronese maps from $\mathbb{P}^{2n_m}$, conjugating the corresponding source when a weight is negative. On that oriented real-analytic base the resulting lines are the lines of Chern classes rH, sH, where H is the hyperplane class. The extra scalar needed for the one-circle normalization is $r^{-n_m} s^{-n_m}$. The fundamental class of the graph of the pair maps to the sum of the coordinate linear products $[\mathbb{P}^i \times \mathbb{P}^{2n_m - i}]$ with coefficients $r^i s^{2n_m - i}$, as is seen by pairing with the two hyperplane classes. Pairing a base chain with the supported test cocycle commutes with its boundary: the cocycle restrictions on common faces agree in the shifted cochain complex for the dualizing sheaf of the base [Par25, §2.4 and Remark 3.6]. These chain relations can be taken real analytic on each simplex. Indeed embed the product of projective spaces as Hermitian projectors, approximate a subdivided bounding chain by affine interpolation of vertices followed by the analytic tubular retraction, and use the straight interpolation and retraction on the boundary prism. The support over each simplex is the compact core family. This proves the polynomial formula. Swapping the lines and moving the splitting along the Picard variety gives its symmetry. For each pair of weights the cycle addition and multiplication identities for the normalized tests are those of [Wue26c, §3]; their proofs use only the orders and the nonvanishing of the weights. Taking the logarithm and comparing coefficients as Laurent polynomials gives the asserted operations. □

The degree in z of the degree-l logarithm is at most n_l by superadditivity of n_l. If $|j| > n_d$ for the largest proper divisor d of m, the primitive $P_{m,j}$ is therefore killed by every proper division. By [Wue26b, Lemma 7.1] it is a finite linear combination of $x_{h,1}$. This use of the primitive-division criterion is independent of any evaluation by numerical invariants.

We recall the geometric detection that will determine the coefficients. On the space of connected stable maps of a fixed arithmetic genus k to fibers of a smooth family, the map to the image cycle is proper over compactly supported insertions [Wue26b, §5]. Pullback of the insertion, with the oriented parameter chain, gives a class T_k in the corresponding colimit of map-space cohomology groups. The construction annihilates products of positive-degree classes represented on disjoint targets. In an acyclic balanced normal bundle the only degree-one image is its zero section, so $T_k(x_{h,1}) = 0$ if $k < h$. At $k = h$ the only map to this cycle is the regular embedding, and $T_h(x_{h,1}) \neq 0$ by the fixed-genus detection theorem of [Wue26b, §§5–6]. We shall use only these properties of T_k. They require no numerical functional on a family of singular maps.

5.2. **The algebraic pairing.** Let $\mathcal{F} = \overline{\mathcal{M}}_k(C, m)$ be the proper stack of connected unmarked stable maps, with universal source π and map f. Write $V_A = R\pi_* f^* A$ for $A = L, M$ and $a = k - 1 - m(g-1)$. Riemann–Hurwitz on the positive components gives $a \geq 0$ if $\mathcal{F}$ is nonempty. Both complexes have virtual rank $-a$, and $\mathcal{F}$ has virtual complex dimension $2a$. Expanding the two inverse moving Euler classes in the fiber characters gives

$$N_{k,m,j} = \int_{[\mathcal{F}]^{\mathrm{vir}}} c_{a-j}(-V_L) c_{a+j}(-V_M), \qquad |j| \leq a, \tag{9}$$

and $N_{k,m,j} = 0$ outside this range [BP08, §2]. The ranks refer to Euler characteristics and are unchanged by jumps of the spaces of sections.

The pairing of the normalized tuple over $B_{n_m + j}$ with the fiberwise algebraic stable-map virtual class is precisely $N_{k,m,j}$. To verify it, first use the compactification $\mathbb{P}_C(\mathcal{O} \oplus L \oplus M)$

over a point. The tuple with two independent torus characters x, y is supported on maps into the core and extends by zero to the projective stable-map space. The only fixed component meeting this support is $\mathcal{F}$: invariant finite multisets in the affine fiber lie at its origin. The fixed obstruction theory there is the one for maps to C. The moving theory is $V_L \otimes x \oplus V_M \otimes y$, where the characters also denote their weight lines. Virtual localization [GP99] identifies the equivariant pushforward with

$$x^{n_m} y^{n_m} \sum_l N_{k,m,l} (x/y)^l.$$

The monomial comes from the normalized Euler class of the tests on the fixed locus. The pushforward is an ordinary equivariant polynomial because the support is proper. In the associated projective bundle over B_{n_m+j} the fiberwise obstruction theory is the descent of the fixed-target theory. Keeping the residual two-torus action thus substitutes $x + c_1(P), y + c_1(Q)$ in this polynomial. Integrating over the base at $x = y = 0$ gives the claimed coefficient, including when $a > n_m$. This calculation uses the stable-map theory for the family of threefolds over B_{n_m+j}. Maps to the total projective variety considered as a single target have a different obstruction theory.

5.3. **Avoidance of multiples.** Assume $j > 0$ by symmetry. Keep the L tests equal to zero and translate the M tests to homogeneous polynomials of their respective orders on $\mathbb{P}^{n_m - j}$. Along the linear homotopy the image cycle is confined to the M axis. The prescribed coefficients determine one relative monic divisor in each fiber, with multiplicities. Its graph over the parameter base and the interval is compact.

Let $e > 1$ divide m and put $s = m/e$. The eth powers of monic spectral polynomials of degree s form a closed subvariety of the affine coefficient space of codimension $n_m - n_s$. Indeed the first s coefficients of a power successively recover those of its monic root by division by e. The remaining coefficients are polynomials in the recovered coefficients, of the required orders under scaling. This argument works also for polynomials whose divisors are reducible. A section whose values are general homogeneous polynomials avoids the union of the power loci if $j > n_d$: evaluation is surjective at every point of the projective base and its dimension is less than every such codimension. A neighborhood of the compact translated insertion then misses all cycle multiples, since a convergent sequence of multiples has a convergent subsequence of roots by compactness of bounded cycles. In particular this is possible for every nonzero Laurent exponent in the stated range when m is prime. A divisor in the permitted neighborhood can still have individual repeated components.

At the first excluded positive weight for composite m, take $s = d$, $e = m/d$ and $j = n_d$. The equations on the remaining coefficients, of orders $i > d$, are sections of $\mathcal{O}(i)$ with $(i-1)(g-1)$ copies for each i. Their top Chern number on $\mathbb{P}^{n_m - n_d}$ is

$$\prod_{i=d+1}^{m} i^{(i-1)(g-1)} > 0.$$

Thus the entire power locus cannot be avoided by choosing other homogeneous values. In fact $\rho_e P_{m,n_d} = P_{d,n_d} \neq 0$: the latter has nonzero connected minimum-genus detection at its outermost weight, as we verify next.

Lemma 5.2. *For $s \geq 2$ the coefficients at the two outermost weights of $P_s(z) = \sum_j P_{s,j} z^j$ are both*

$$\frac{x_{H_s,1}}{A_s}, \qquad H_s = 1 + (g-1)\binom{s+1}{2}, \qquad A_s = \prod_{i=2}^{s} i^{(i-1)(g-1)}.$$

Proof. At $j = n_s$ the second projective factor is a point. Translate its M-tuple to a smooth integral spectral divisor D, for example $v^s = b$ with b having a simple zero and all zeros simple. For general M such a b exists: the Abel map for degree $s(g-1) \geq g$ is dominant and the reduced divisors are dense. Eisenstein's criterion gives integrality; adjunction gives genus H_s. The other tests still vanish, so the unique supported cycle is D with zero L field. For the projection $f : D \to C$ the monic equation gives $f_*\mathcal{O}_D = \bigoplus_{i=0}^{s-1} M^{-i}$. Consequently $R = f^*L$ has $H^0(D, R) = 0$. In the surface the normal line $J = f^*M^s$ satisfies $JR = K_D$, so $H^1(J) = 0$. Coefficient variation gives all $H^0(J)$ of dimension n_s. Their tests cut these surface deformations with degree one. On the smooth spectral locus these are all the curves: the M projection is birational to a smooth divisor and the lift is a section of R, which vanishes. This describes its topology as well, by the family of monic divisors and the continuous inverse given by their coefficients. The normalized L tests there give the ordinary Euler class $c_1(\mathcal{O}(1))^{n_s}$. Base Gysin replaces $\mathbb{P}^{2n_s}$ by a linear $\mathbb{P}^{n_s}$. What remains is the point class of the fixed smooth divisor in the coefficient space, divided by A_s.

We identify that point class in the universal algebra. Choose a basis $c_1, \ldots, c_{n_s}$ of $H^1(D, R)$ and a class $\delta \in H^1(D, RJ^{-1})$ inducing an isomorphism $H^0(J) \to H^1(R)$ by multiplication. Indeed $H^1(RJ^{-1})$ is dual to $H^0(J^2)$. Choose n_s points whose evaluations in fixed frames form a basis of $H^0(J)^*$, and take the sum of evaluations in the squared frames. The pairing induced by this functional is nondegenerate. On a cover of C by two affine opens represent the c_i and δ by cocycles on their preimages in D. The monic basis lifts them to polynomials in v on the overlap with the appropriate line twists. Write $P_*(v)$ for the fixed monic equation. In the L line change the transition by adding the translation

$$t\Big(\sum_{i=1}^{n_s} \sigma_i \widetilde{c_i}(v) + \sigma_0 P_*(v)\widetilde{\delta}(v)\Big), \qquad 0 \leq t \leq 1,$$

where the σ_i are the homogeneous coordinates on $\mathbb{P}^{n_s}$. The point test is still the equation of the fixed M spectral divisor. On D the translation splits for every parameter at $t = 0$, and for $t > 0$ precisely at $[1 : 0 : \cdots : 0]$, with unique lift. Thus the point test has compact support throughout this deformation. Near the final point the lifting obstruction is a holomorphic section of $H^1(R) \otimes \mathcal{O}(1)$, using reduction modulo the variable monic polynomial to identify the cohomology in families. Its derivative in the projective coordinates is the basis (c_i) and its derivative in the divisor coordinates is multiplication by δ, up to a common sign. Both are isomorphisms. Its zero scheme is a graph over either set of coordinates. The lifts vary holomorphically on this graph by cohomology and base change and $H^0(R) = 0$. The divisor point test on the graph is therefore a positive parameter point class. In that fixed target the normal sequence of the lifted curve has subline R, quotient J, and connecting isomorphism on $H^0(J)$ just computed. Its normal bundle is acyclic. Deformation to the normal bundle identifies the point class with $x_{H_s,1}$. Finally every product in the logarithmic subtraction has strictly smaller weight than n_s. This proves the formula, including its normalization. □

6. The common algebraic family for the residue comparison

Let C be a smooth projective curve of genus at least two and B a smooth projective variety. We start with

$$X_0 = \operatorname{Tot}\big((L \boxtimes P) \oplus (M \boxtimes Q)\big) \longrightarrow C \times B, \qquad L \otimes M = K_C,$$

where P, Q are line bundles on B. Fix a degree $m > 0$ over C. By the support of a cycle we mean the union of its distinct image curves. If $A = \sum_i e_i A_i$ is a cycle, its reduced cycle is $\sum_i A_i$. A *proper multiple* is a cycle eA' with $e > 1$ and A' effective. Necessarily $e \mid m$.

Let τ be a tuple of the coefficient evaluations of §5, translated by a section of its test bundle $T \to B$. For this section it suffices that the evaluations be linear combinations of symmetric functions of the intersection with finitely many framed fibers over $F \subset C$, that the translation be smooth, and that $\mathrm{rk}_{\mathbb{C}} T = \dim_{\mathbb{C}} B$. We allow the supported Thom class to be localized in an open set of the degree-m cycle space, provided its zero set there is compact. All motions below refer to that localized class. They preserve any scalar factor in its normalization.

Theorem 6.1. *Suppose the compact zero set of τ contains no proper multiple. For every K there is a smooth quasi-projective variety S smooth over B, an algebraic section $i : B \to S$, and a smooth family of Zariski locally trivial fibrations $X \to C \times S$ with fiber $\mathbb{A}^2$ restricting along i to X_0. There is a homotopy of the parameter section and the tests, with compact joint zero set in the chosen cycle neighborhoods, with the following properties.*

1. *At its endpoint, the zeros with connected support and sum of normalization genera at most K are finitely many smooth embedded curves of multiplicity one. The relative space of embedded curves is smooth of dimension $\dim S$ there. The target variations surject onto the normal obstruction space there. The relative space meets the parameter graph transversely, and the tests have transverse zero on this intersection.*
2. *On the single algebraic family X/S, the products of the test Thom class with the normal Thom classes of the parameter graphs have equal compactly supported classes at the two ends. At the initial end, capping the relative stable-map virtual class with the graph class restricts it to the relative stable-map class of X_0/B.*

In particular every connected stable map of arithmetic genus at most K on the endpoint zero set has one of the indicated image cycles.

The tests in this statement may have zeros of disconnected support. The compact support in the second assertion includes such zeros. We impose connectedness only in determining the image of a connected stable map. We first describe the target deformations used in the proof.

6.1. **Deformations at finitely many fibers.** Consider a family already constructed over a smooth quasi-projective $S \to B$. Off a finite subset $P_0 \subset C$, disjoint from F, keep its identification with the pullback of X_0. Initially $S = B$ and P_0 is empty. Fix $p \notin F \cup P_0$. On a Zariski neighborhood U of p choose frames for L, M and a uniformizer t regular there with its only zero at p. Shrink U to avoid the old punctures. In these frames the fiber variable y takes values in the pullback of the plane bundle $E = P \oplus Q$ over B.

Let b, r be positive integers. On $U \setminus \{p\}$ use the change of coordinates

$$y \longmapsto y + (\iota_\ell \Omega) \sum_{a=1}^{b} \sum_{v=0}^{r-1} c_{av} t^{-a} \ell(y)^v, \qquad \ell \in E^\vee, \quad \Omega \in \bigwedge^2 E, \quad c_{av} \in \mathbb{C}. \tag{10}$$

All the displayed parameters are coordinates on a vector bundle over S. Since $\ell(\iota_\ell \Omega) = 0$, this automorphism preserves $\ell(y)$, has inverse obtained by negating the c_{av}, and has Jacobian determinant one in the fiber. At p we glue a new copy of the original affine chart by a composition of two such automorphisms with independent parameters. For a finite set of new punctures use the old target on their complement. Each new chart is chosen to miss every other puncture. On the intersection of two new charts use their transitions through the old chart. This gives an algebraic family $X' \to C \times S'$ with S' the total space of the new parameter bundle. It restricts to the old family whenever all $c_{av} = 0$, including at the algebraic

zero section $S \to S'$. Off the union of the old and new punctures the original identification is preserved. Since the inserted determinants are one, the relative tangent determinant over C in each threefold is still pulled back from $L \otimes M$. Every compact curve in it has threefold Chern number zero. The affine-plane fibers contain no compact curves.

Here is the infinitesimal effect. Let $h_i : D_i \to X_s$ be the smooth normalizations of the distinct positive components of a cycle, with finite projections $f_i : D_i \to C$. Set $D = \coprod_i D_i$, $f = \coprod_i f_i$, and let $V|_{D_i} = h_i^* T_{X_s/C}$. Over a general p the covers are unramified and all $r = \sum_i \deg f_i$ sheet values in the affine plane are distinct. Indeed the normalized maps are birational onto different curves. The number r counts the sheets of the reduced cycle. Choose two independent forms ℓ which each separate their values at p, and use a nonzero Ω. These choices exist since only finitely many lines in $E^\vee$ are forbidden.

Lemma 6.2. *Let Z be a finite subscheme of the smooth domains, disjoint from $f^{-1}(p)$. For sufficiently large b, variations of the coefficients c_{av} in the two factors of (10), at $c = 0$, span the obstruction space $H^1(D, V(-Z))$ for deforming the graphs with fixed covers and prescribed restriction to Z.*

Proof. Since f^*p has positive degree on each component, we can arrange

$$H^1(D, V(bf^*p)(-Z)) = 0.$$

The connecting morphism from principal parts of order b over p then surjects onto $H^1(D, V(-Z))$. On each sheet, multiplication by t^b identifies these principal parts with two copies of $R_b = \mathbb{C}[t]/(t^b)$. Write $y_1(t), \dots, y_r(t)$ for the formal sheet values. For either ℓ the matrix $(\ell(y_j(t))^v)_{1 \le j \le r,\ 0 \le v < r}$ has Vandermonde determinant a unit in R_b. For each v the coefficients $\sum_{a=1}^{b} c_{av} t^{b-a}$ run through R_b. Thus one shear prescribes arbitrary principal parts on every sheet in the direction $\iota_\ell \Omega$. The two directions span the plane. Differentiation of the transition on the punctured overlap gives exactly the resulting connecting class, up to the choice of sign for the Čech differential. Prescribing the graph on Z replaces V by $V(-Z)$ because the difference of two local lifts vanishes on Z. □

We will need this lemma on a compact set of smooth normalization tuples, simultaneously for Z of lengths zero, one and two. The corresponding relative symmetric powers of the disjoint smooth source curves are proper over the tuple space. For each tuple one can take three good punctures, so that at least one avoids the base images of any such Z. At a given puncture b can be chosen uniformly over these subschemes by relative Serre vanishing for the ideal of the universal divisor. Surjectivity of the graph deformation problem with the jets imposed is open by the rank condition on a two-term resolution of the sections of $V(-Z)$. Its differential augmented by the parameter variations is surjective. This remains true where the dimensions of its kernel or of $H^1(D, V(-Z))$ change. Compactness gives finitely many choices of directions on neighborhoods of the tuples and their marked subschemes. Distinct choices of ℓ, Ω at the same Borel parameter can be placed in separate factors of the parameter bundle. Extend each choice locally over B and then smoothly by a cutoff. With the c_{av} still zero, moving the parameter section to these reference values leaves the target family unchanged. Coincident choices of new punctures can also be accommodated by composing their shears in the same transition. The linearized variations add at $c = 0$.

6.2. **Specialization of the weighted images.** We give the compactness argument for the tuples to which the lemma will be applied. Each $X' \to C \times S'$ is affine of finite type by Zariski descent, and $C \times S'$ is quasi-projective. Hence X' is quasi-projective, and admits a projective closure over S'. In each threefold fiber the projection to C has contractible fibers. Thus on

degree-m curves any chosen relative polarization has fixed degree locally in the parameter. On compact sets of parameters these degrees are bounded. The degree of a positive summand is bounded by that of the whole cycle for the polarization.

Choose a relatively compact open neighborhood of the original compact zero set, with closure in the given open locus of cycles. Such neighborhoods can be taken in the fixed-degree relative Chow space of a projective closure, on the open set of cycles with support in the affine target. They can also be obtained from a compact bound on supports and masses [DIW26, Theorem 2.11]. For $e \mid m$ the image of multiplication by e is closed in this locus. Along a convergent sequence of multiples the roots have the same supports and their masses are divided by e. A subsequential limit of the roots is therefore a root of the limiting cycle. Choose the closure disjoint from these images for all $e > 1$. Its boundary misses the zero set. Whenever the parameter family is enlarged, take such a neighborhood of the whole current compact zero set in the enlarged relative cycle space, with closure on the old graph inside the preceding neighborhood. The compact boundary misses the simultaneous zero of the old graph and tests. It continues to miss this zero for all sufficiently small motions of both. The supported Thom product in the new neighborhood thus restricts to the previous product by excision. Applied successively, this construction keeps compact support also for zeros whose normalizations exceed the genus cutoff.

For a cycle $A = \sum_{i=1}^{r} e_i A_i$ write d_i for the projection degree and h_i for the normalization genus of A_i. Set

$$\lambda(A) = \left(\sum_i d_i,\ \sum_i h_i,\ -r \right)$$

with the lexicographic order. The first entry uses the reduced cycle. Only finitely many values occur for $\sum_i h_i \le K$.

Lemma 6.3. *For a convergent sequence in the chosen neighborhood with fixed λ, the limiting value of λ is at most that of the sequence, and the sum of normalization genera cannot increase. If λ stays equal, then, after relabeling and taking a subsequence, the normalization maps converge to the smooth normalizations of the limiting cycle, with their original weights. Connected supports have connected limit.*

Proof. Keep the degrees, genera and weights of the individual labels constant along a subsequence. Stable-map compactness applied to their unweighted normalization maps gives a stable limit of each label. One can take these limits in the projective closure. The bounds on the component degrees give properness there, and the limiting total support lies in the affine open. The weighted sum of their image cycles is the given limit. Write A'_j for its distinct image curves and b_{ij} for the sum of the covering degrees onto A'_j in the ith stable limit. With $d'_j = \deg(A'_j/C)$ one has

$$d_i = \sum_j b_{ij} d'_j, \qquad e'_j = \sum_i e_i b_{ij}.$$

Every column sum of (b_{ij}) is positive. It follows that the reduced degree drops unless every column sum is one. Each A'_j has normalization genus at least two since it covers C. By Riemann–Hurwitz any component covering it has at least its genus. The genus formula for the nodal sources now bounds the sum of the distinct normalization genera by $\sum_i h_i$.

At equal reduced degree every positive component maps with degree one onto a different image, and each old label has at least one positive component. Thus the number of distinct images cannot decrease. If the entire triple is unchanged, each label has only one positive

component, with its full genus. The dual graph is a tree and all contracted vertices are rational. Any nontrivial such tree has a contracted leaf, contrary to stability without markings. The limiting maps are therefore smooth and birational onto their images, and retain the corresponding weights. Finally, a Hausdorff limit of nonempty compact connected sets is connected. □

6.3. **Separation through a fixed genus.**

Proof of Theorem 6.1. Consider the current constrained cycles of connected support whose normalization genera sum to at most K. Suppose the values of λ below a given value have been treated, with finitely many transverse multiplicity-one embedded zeros retained. Around each of these the embedded smooth maps form an open set of the full cycle space [DIW26, Proposition 2.21]. After any one enlargement of the family the implicit function theorem preserves this isolated zero with its sign under sufficiently small motions of the graph and tests. Take smaller disjoint neighborhoods of the retained zeros.

On the complement, take the least remaining value of λ. The normalization tuples with this value over the current constrained set form a compact set up to finite relabeling. Indeed their total cycles have compact closure as above. Lemma 6.3 places the limit in the same bounded connected locus. Minimality forces equality of the triples and hence convergence in the smooth normalization charts. The same argument along sequences of perturbations tending to the current graph has two consequences. An earlier absent value cannot occur on this complement under sufficiently small perturbations. Every new zero of the present value must have its tuple in the chosen neighborhoods of the compact original tuples. If there is no remaining value, the induction ends.

Enlarge the family using Lemma 6.2 on this compact set. The space of ordered smooth graph maps over the enlarged parameter S' is smooth near the tuples. To see this, first deform their finite covers of C in the smooth Hurwitz stacks. For a smooth finite source cover f_i, the cokernel of $T_{D_i} \to f_i^* T_C$ has finite length and the kernel vanishes. After these cover deformations and the parameter in B have been prescribed, the remaining obstruction for the graphs lies in $\bigoplus_i H^1(D_i, V_i)$, spanned by the parameter variations. Thus the graph space submerses onto the product of the Hurwitz stacks and B. It has complex dimension $\dim S'$, since the unmarked threefold index of each component is its Chern number zero and the target parameters occur only once. Each graph has trivial stabilizer by birationality. This calculation only counts distinct normalized components. Their weights enter the tests through the cycle sum.

Mark a critical point of a cover. This marking is finite over its Hurwitz stack, while nonimmersion requires the two vertical first derivatives of the graph to vanish. Mark instead two distinct source points over the same base value, on the same or on different components. This adds at most one complex dimension over the covers. Coincidence of the two images then imposes two vertical equations. Take smooth strata of these marked spaces over the Hurwitz stacks and pull back by the submersion just proved. The fixed-cover jet variations of Lemma 6.2 are transverse to the equations on every such stratum at a zero. In the second calculation a common change of target coordinates cancels in the variation of the difference of two equal images. In the first calculation it preserves the vanishing of the derivative at a critical point. Hence both bad loci have dimension strictly less than $\dim S'$. The length-two subschemes in the lemma include the limits of two marks approaching one another.

Move the parameter graph by finitely many smooth sections spanning its vertical normal directions along the compact base B. Move the tests independently by smooth sections spanning T there. The graph condition and the tests together have real codimension $2 \dim S'$,

and the auxiliary directions span their joint normal bundle even on the marked strata. Sard's theorem on countably many smooth charts gives arbitrarily small choices transverse on the unmarked spaces and missing all the bad strata. We can impose transversality to the parameter graph alone at the same time. A connected constrained tuple of the present type now has one component, since distinct embedded images in the tuple are disjoint. Its weight is one, since the collar excludes proper multiples. The surviving zeros of this type are finite by Lemma 6.3, compactness, and transversality in the smooth charts. They are isolated in the full cycle space on the parameter graph with the tests imposed, again by embedded-cycle openness. Add them to the retained list. An enlargement preserves regularity of the parameterized graph equations at the old zeros, since the old parameter directions are still present at its zero section. Openness then gives the same regularity at their continuations. At an embedded curve the obstruction for the fixed target is H^1 of its normal bundle. Since the parameterized graph problem has zero obstruction, its target variations surject onto this group. There are only finitely many values of λ to treat.

Take S to be the final parameter scheme. Its algebraic zero sections embed every preceding family in this last family. Concatenating the smooth parameter paths gives the claimed homotopy. The reference values of ℓ, Ω can be moved before each small perturbation inside $c = 0$, where the target and tests are constant. The compact joint support there is just transported along the reference path. The successive collars and excision give a homotopy of the joint Thom product, relative to the complement of its compact support. A connected stable map of genus at most K has connected support, and its arithmetic genus bounds the sum of the genera of the distinct image normalizations by Riemann–Hurwitz. It consequently maps to one of the retained embedded curves with degree one.

Finally, maps to fibers of the smooth open X/S form an open substack of the relative stable-map stack for a projective closure. The obstruction theory on this open, relative to the prestable source and to S, is dual to the derived sections of the pullback of $T_{X/S}$. It gives the vertical virtual class in Borel–Moore homology after taking cycle classes. The universal domain and this obstruction theory commute with base change at the regular algebraic section i. Its refined Gysin pullback gives the relative virtual class of the original family [BF97, §7]. The normal Thom class computes the same Gysin map on cycle classes. For each connected source genus the inverse image of the compact supporting set on cycles is compact by properness of the image-cycle map [Wue26b]. This proves the second assertion and completes the proof. □

7. Virtual degree and the universal class on the same family

Let $X \to S$ be the smooth algebraic family of threefolds obtained in Theorem 6.1. At a transverse solution the image is a smooth embedded curve $D \subset X_s$ of multiplicity one. Its normal bundle N_D has degree $2h - 2$, where h is the genus of D. The deformations over S constructed in §6 surject onto the obstruction space $H^1(D, N_D)$. Indeed, the deformations with the projection cover fixed already span the vertical obstruction space, and the cover itself is unobstructed. The quotient by deformations of the cover gives the obstruction space of the embedded map. We first compute the contribution from all connected maps with image cycle $[D]$.

7.1. **The relative virtual class.** Let $\mathcal{H}$ be an open neighborhood of $[D]$ in the relative Hilbert scheme of smooth connected proper curves in X/S. It can be taken inside the Hilbert scheme of a projective completion by restricting to curves contained in X. Shrink $\mathcal{H}$ so that the normal obstruction map remains surjective. Write $p : \mathcal{C} \to \mathcal{H}$ for the universal curve and N for its relative normal bundle in $X \times_S \mathcal{H}$. Let $\mathcal{M}_k(X/S)$ be the stack of connected

unmarked stable maps of arithmetic genus k to fibers of $X \to S$. We use its vertical stable-map obstruction theory over the stack of prestable curves times S, with the fundamental class of the smooth parameter space. This gives a virtual class on the quasi-projective target open of §6.

Proposition 7.1. *The scheme $\mathcal{H}$ is smooth of dimension* $\dim S$. *For $k \geq h$, let*

$$q : \mathcal{D} = \overline{\mathcal{M}}_k(\mathcal{C}/\mathcal{H}, 1) \longrightarrow \mathcal{H}$$

be the proper stack of degree-one maps to its fibers. Composition with the universal embedding identifies $\mathcal{D}$ with the open part of $\mathcal{M}_k(X/S)$ over the multiplicity-one images parametrized by $\mathcal{H}$. The virtual class of this open part, pushed forward by q, is $b_{k,h}[\mathcal{H}]$, where

$$b_{k,h} = [u^{2k-2}]\big(2\sin(u/2)\big)^{2h-2}. \tag{11}$$

For $k < h$ the inverse image of this image locus is empty.

Proof. For a smooth embedding the tangent complex relative to S is Rp_*N. This follows by canceling the tangent of the source in the deformation complex of the map. The absolute obstruction at $[D]$ is the cokernel of the map $T_sS \to H^1(D, N_D)$. Its vanishing gives smoothness, and $\chi(D, N_D) = 0$ gives the asserted dimension. In particular the derived space of smooth embedded maps over the varying target is classical here. Its absolute tangent is $T_{\mathcal{H}}$.

Write $\pi : \mathcal{F} \to \mathcal{D}$ for the universal source, $a : \mathcal{F} \to \mathcal{C}$ for the universal map, and $\iota : \mathcal{D} \to \mathcal{M}_k(X/S)$ for composition. The cone of the induced morphism of virtual tangent complexes is

$$\mathrm{Cone}\big(q^*Rp_*N \longrightarrow R\pi_*a^*N\big). \tag{12}$$

To see the arrow, work first over the stack of prestable sources. With the target curve fixed the vertical tangent is $R\pi_*a^*T_{\mathcal{C}/\mathcal{H}}$, and with the threefold fixed it is $R\pi_*a^*(T_{X/S}|_{\mathcal{C}})$. Their quotient is $R\pi_*a^*N$. Allowing the embedded curve to move adds the triangle for $T_{\mathcal{H}} \to T_S|_{\mathcal{H}}$, whose fiber is Rp_*N. A normal displacement of the embedded curve acts on the composite by pullback under a. The two tangent triangles therefore give (12). Adding the common motion of the prestable source leaves the cone unchanged. Composition with the universal embedding is a morphism of the derived mapping stacks. Upon truncation it induces the maps of stable-map obstruction theories, including their maps to the intrinsic cotangent complexes [STV15, §§1, 2.2]. The construction is relative to the parameter scheme and local in the target along the proper source.

At a geometric point $a : F \to D$ of $\mathcal{D}$, the normalization of the unique positive component maps isomorphically to D. It has no self-node, since two distinct points on the normalization have distinct images. Each connected contracted subcurve attaches to this component at just one point, for the same reason. Functions on a contracted connected fiber are constant and take the value at the attachment. Hence $a_*\mathcal{O}_F = \mathcal{O}_D$ and $R^1a_*\mathcal{O}_F$ is a sheaf of finite length $k - h$, by the Euler characteristics. In particular $k \geq h$. The unit $\mathcal{O}_D \to Ra_*\mathcal{O}_F$ and the projection formula identify the fiber of (12) with

$$H^0(D, N_D \otimes R^1a_*\mathcal{O}_F)[-1].$$

Both terms in (12) are perfect and commute with derived base change along the flat universal curves. A perfect complex whose geometric fibers are concentrated in one degree with constant rank is a vector bundle in that degree. Thus (12) is $W[-1]$ for a bundle W of rank $2(k-h)$ on $\mathcal{D}$.

The relative tangent complex of the derived composition morphism restricts to $W[-2]$ on the classical truncation. For an ordinary square-zero extension the relative infinitesimal

automorphism, tangent and obstruction groups in degrees $-1, 0, 1$ consequently vanish. The classical morphism ι is formally étale and locally of finite presentation, hence étale. Each image determines its point of $\mathcal{H}$ uniquely, and the stabilizers on both sides consist of the same automorphisms of the map from F. Thus ι is a representable étale monomorphism, and identifies $\mathcal{D}$ with the asserted open substack.

On this common classical stack the vector bundle stack of the obstruction theory of $\mathcal{D}$ embeds in the one for $\mathcal{M}_k(X/S)$ with quotient W. By compatibility the embedding of the intrinsic normal cone for the latter theory factors through its embedding for the former. The zero-section Gysin maps therefore give [BF97, §§4–5]

$$\iota^*[\mathcal{M}_k(X/S)]^{\mathrm{vir}} = c_{2(k-h)}(W) \cap [\mathcal{D}]^{\mathrm{vir}}. \tag{13}$$

Here the class on the right uses the smooth family of target curves over $\mathcal{H}$. The equality holds on the stacks, including the stabilizers of all contracted subcurves.

The right side of (13) has dimension $\dim \mathcal{H}$. Its proper pushforward is a multiple of $[\mathcal{H}]$ on each connected component. To compute the multiple, intersect with a point $[D] \in \mathcal{H}$. Refined base change gives the usual virtual class of $\overline{\mathcal{M}}_k(D, 1)$ [BF97, Proposition 7.2]. On this fiber the class of $R\Gamma(D, N_D)$ in K-theory vanishes because it is a constant class of rank zero. Consequently the coefficient is

$$\int_{[\overline{\mathcal{M}}_k(D,1)]^{\mathrm{vir}}} c_{2(k-h)}(-R\pi_* a^* N_D).$$

In rational cohomology Grothendieck–Riemann–Roch on the universal source shows that this expression depends only on the rank and degree of N_D. We may use $A \oplus K_D A^{-1}$ for a degree-$(h-1)$ line A outside the theta divisor. Both summands are acyclic, and a section of their pullback to F is determined by its restriction to the positive component. The integral is thus the local degree-one contribution computed by Pandharipande [Pan99, Theorem 2]. Multiplying his series in the excess genus by u^{2h-2} gives (11). □

7.2. **The local class in families of cycles.** Let $\sigma : B \to S$ be the transverse smooth section of Theorem 6.1. Give its normal bundle the orientation of the vertical tangent of $S \to B$, and give the tests their complex orientations. At a solution with image D, let $\epsilon(D)$ be the local intersection sign on $\mathcal{H}$ of the parameter graph and the tests. The total real codimension is $2 \dim_{\mathbb{C}} \mathcal{H}$. The parameter graph is transverse before the tests are imposed, so its inverse image in $\mathcal{H}$ is the regular space of embedded curves in the family over B. Its transverse-fiber-product orientation agrees with the determinant orientation of the parametrized map equation, by the tangent exact sequence for the graph. Interchanging the normal to the graph and the tests gives no sign, as their real ranks are even. Proposition 7.1 and the projection formula now give $\epsilon(D) b_{k,h}$ for the unnormalized algebraic contribution at this solution.

Lemma 7.2. *In the smooth family colimit the local unnormalized test class at this solution equals* $\epsilon(D)\Phi(x_{h,1})$.

Proof. Embedded connected curves of multiplicity one form an open subset of the space of cycles, with their embedded-map topology [DIW26, Proposition 2.21]. Thus the transverse solution can be excised in cycle space. To identify its class we first deform the complex structure of the marked target germ so that the curve becomes regular in that target, keeping the curve and its test values fixed.

Write $V = H^0(D, N_D)$ and $U = H^0(D, K_D \otimes N_D^\vee)$, both of dimension l by Riemann–Roch. If $l > 0$, choose points $p_1, \dots, p_l$ on unramified sheets over distinct points of the base curve

away from the test fibers. Choose covectors λ_i and vectors v_i in the respective normal fibers so that both maps

$$V \longrightarrow \mathbb{C}^l, \quad s \longmapsto (\lambda_i s(p_i))_i, \qquad U \longrightarrow \mathbb{C}^l, \quad \eta \longmapsto (\eta(p_i)(v_i)/dx)_i$$

are isomorphisms. Here x is a local coordinate on the base. Inductively, nonzero sections in the two remaining evaluation kernels are simultaneously nonzero on some unramified sheet. At such a point choose the vector and covector to reduce each kernel dimension by one. Extend these data in local frames. Multiply the endomorphisms $v_i\lambda_i$ by smooth $(0,1)$-forms supported in small disks about p_i, normalized to pair to one with dx, and denote their sum by A. Under Serre duality the matrix of $J : V \to H^1(D, N_D)$, $s \mapsto [As]$, tends to the product of the two evaluation matrices, one transposed, as the disks shrink. Hence it is invertible for a fixed choice of the disks. Splitting off the invertible complement of kernel and cokernel in $\bar\partial_{N_D}$ gives the matrix $tJ + O(t^2)$ on these finite-dimensional spaces for $\bar\partial_{N_D} + tA$. This operator is invertible for all sufficiently small $t > 0$.

Over the chosen base disks take disjoint holomorphic submersion charts about the sheets. Use the base coordinate x and vertical coordinates y in which the marked sheet is $y = y_i(x)$. Over the support of the perturbation a sufficiently small target tube meets only these charts. Write $A = A_i(x)d\bar x$ on the chosen sheet and use zero on the other sheets. Replace its local graph equation by

$$\partial_{\bar x} y = -tA_i(x)(y - y_i(x)).$$

The coefficient vanishes near the boundary of the base disk, so the change glues on the tube. Its holomorphic dependence on y makes the horizontal $(0,1)$ vector field commute with the two vertical $(0,1)$ coordinate fields. Hence the resulting almost complex structure is integrable. Along the marked sheet the added vector field vanishes and its normal linearization is the prescribed change of Dolbeault operator. Extend the change smoothly over a parameter neighborhood in B, with a parameter cutoff. All framed test fibers are unchanged. For $l = 0$ the fixed target was already regular.

At $t = 0$ the differential of the parametrized map equation together with the tests is invertible in an unparametrized slice. The implicit function theorem therefore isolates the retained solution on a small closed interval of t. By the open embedding into the relative cycle space its local Thom class extends by zero over the interval, with the same sign. At the regular endpoint the local family of curves projects diffeomorphically to B, preserving orientation since the fixed-fiber normal operator is an invertible complex operator. The test class is now the base Gysin image of the unit of the isolated curve, multiplied by $\epsilon(D)$.

Deform the holomorphic embedding to its normal bundle. The curve stays regular along this deformation, with the same normal Dolbeault operator up to isomorphism at the positive parameters. The implicit function theorem and embedded-cycle openness isolate its compact section along the full interval. At the normal endpoint a compact degree-one cycle is a section, since the vector-space fibers have no compact positive curves. Acyclicity makes the zero section the only such cycle. Its unit is the empty-tuple representative of the multiplicity-one class of genus h and Chern number zero [Par25, Definition 4.7]. The two local deformations and the base Gysin map identify the asserted class [Wue26b, §§2–3]. □

7.3. **Extraction by source genus.**

Proof of Theorem 2.1. Fix K and use Theorem 6.1 for all connected source genera through K. For each $h \le K$ let $a_h^{(K)}$ be the sum of $\epsilon(D)$ over the finitely many connected embedded solutions of genus h, divided by $W_{g,m}$. The parameter graph and the tests form a single compactly supported intersection class on the algebraic family, isotopic to the graph and tests

at the original algebraic section. Hence the pairing computed in §5 is the sum of the local pairings in Proposition 7.1. For every $k \le K$ this gives

$$N_{k,m,j} = \sum_{h \le k} a_h^{(K)} b_{k,h}. \tag{14}$$

Here and below a smaller source genus cannot map to a multiplicity-one image of larger genus.

In the smooth family over the moving graph the same test classes give a compactly supported homotopy on cycles. For fixed $k \le K$ the image-cycle map from connected genus-k stable maps is proper [Wue26b, §5]. Outside neighborhoods of the finitely many embedded solutions through genus K, the compact constrained set misses its closed image. The pullback of that part of the test class is therefore zero. Over each retained solution Lemma 7.2 applies before this pullback, including when the inverse image contains contracted components. Finally, the product terms subtracted in the logarithm require positive images in disjoint targets and have zero connected pullback. Thus

$$T_k(P_{m,j}) = \sum_{h \le k} a_h^{(K)} T_k(x_{h,1}). \tag{15}$$

In the range of weights of the theorem, the primitive-division lemma gives a finite expansion $P_{m,j} = \sum_h a_{h,j} x_{h,1}$ in the complex cycle algebra, as in §5. Since $T_k(x_{h,1}) = 0$ for $h > k$ and $T_k(x_{k,1}) \neq 0$ [Wue26b, §§5–6], induction on k in (15) identifies $a_{h,j} = a_h^{(K)}$ for $h \le K$. Equation (14) therefore holds with the coefficients of the complex universal class for every k, since K was arbitrary. Substitution of (11) gives the asserted sine expansion. The fixed-genus residue of §5 vanishes for $k < 1 + m(g-1) + |j|$, and $b_{h,h} = 1$. The same induction in (14) gives the stated lower bound on the genus of a nonzero coefficient. □

8. Cubic evaluations on a general normal bundle

Let D be a smooth projective curve of genus $g \ge 2$, and put $k = \lfloor (g-1)/2 \rfloor$. For a rank-two bundle N over D, a point $p \in D$ and a covector $\ell \in N_p^\vee$, let $F_{p,\ell}$ be the product of the three values of ℓ on the fiber of an effective degree-three cycle in $\mathrm{Tot}_D N$, counted with multiplicity. More generally, Pardon allows any homogeneous symmetric polynomial of positive degree in these values [Par25, (4.15)]. Its degree is its weight under scalar multiplication on N.

Theorem 8.1. *For a general stable N with determinant K_D, the common zero of k general point products is $3[D]$. For every rank-two N of degree $2g-2$, any tuple of the above evaluations with compact common zero set has at least k weights divisible by three. In particular the least possible product of weights is 3^k, and $3^k x_{g,3}$ is integral.*

Proof. Write $K = K_D$. The stable fixed-determinant moduli space M is irreducible of dimension $3g-3$. On a dense open its bundles are 2-rigid [BP06, Theorem 2.2 and §3]: the pullback to each stable map of degree two has no nonzero section. Thus $H^0(N) = 0$, and for every line subbundle $S \subset N$ one has $H^0(S^2) = 0$. Indeed a rational square root of a regular section is regular, and the normalization of the spectral double of a nonsquare carries a nonzero section of the pullback of N. There are no compact curves in a fiber of N. Consequently any cubic cycle distinct from $3[D]$ is integral with multiplicity one. Its normalization gives a smooth connected triple cover $f : F \to D$. The normal coordinate on F gives a nonzero map $T \to N$, where T is the Tschirnhausen bundle defined by the trace splitting $f_*\mathcal{O}_F = \mathcal{O}_D \oplus T^\vee$.

We first exclude a generically injective map $T \to N$. Write $b = \deg T$. The discriminant divisor Δ has degree $2b$ and satisfies $\mathcal{O}(\Delta) = (\det T)^2$ [Mir85, §§3–4]. An injection $T \to N$

gives an effective determinant divisor E of degree $n = 2g - 2 - b$, with $\Delta + 2E \in |K^2|$. There are finitely many covers for fixed Δ, by monodromy on the punctured curve. For fixed T, E the upper modifications have at most n parameters, by successive elementary modifications of the dual at the specified points. Stratify the divisors by the multiplicities at their t distinct support points. Unless these points lie on a canonical divisor, the Abel map for $\Delta + 2E$ has differential of rank g, dual to weighted evaluation of canonical forms. Thus the dimension, including modifications, is at most $t - g + n \le 2b + 2n - g = 3g - 4$. On a canonical divisor the support has only finitely many choices once the divisor is given, yielding the bound $g - 1 + n = 3g - 3 - b$. This suffices for $b > 0$. If $b = 0$ the cover is étale. For a noncyclic triple, $\det T$ is its nontrivial sign character, so $E \in |K(\det T)^{-1}|$ has $g - 2$ parameters and the total is at most $3g - 4$. For a cyclic triple T splits as two lines of order three. These estimates involve finite-type Hurwitz and modification spaces since $0 \le b \le 2g - 2$.

For a saturated subline S of degree $d \le g - 2$ the projective space of extensions by S with quotient KS^{-1} has dimension $3g - 4 - 2d$, by Riemann–Roch for S^2K^{-1}. Stability excludes split extensions and larger degrees. Letting S vary adds g parameters, so a general N has no subline of degree $d > k$. A map from a prescribed degree-zero line R would saturate in $R(A)$, with $a = \deg A \ge 0$. The divisor and extension have at most $a + 3g - 4 - 2a < \dim M$ parameters. This excludes all lines of order dividing three and finishes the rank-two case.

For rank one, let S be the saturation of the image. The normal coordinate in f^*S has trace zero and generates the cubic field since the degree is prime. Its characteristic equation reduces to $y^3 = s$, with $s \in H^0(S^3)$ nonzero, since its coefficient in $H^0(S^2)$ vanishes. Hence $1 \le d \le k$, degree zero having been excluded. Conversely each such s defines an integral cubic, since a rational cube root would give a section of S. Its generic fiber recovers S by saturation and then s by the equation.

The cubics modulo dilation thus form a projective algebraic set $Z \subset \mathbb{P}H^0(D, \operatorname{Sym}^3 N)$. Indeed its affine cone consists of the tensors whose value at every point lies in the closed cone of pure cubes. A nonzero such tensor determines a rational line direction by the inverse Veronese map. In a frame adapted to its saturation S, the tensor belongs to S^3 across its zeros as well. To bound $\dim Z$, the pairs $(S, \operatorname{div} s)$ of degree d are parametrized by the finite étale pullback of $\operatorname{Sym}^{3d} D$ under cubing on Picard varieties. The projective extension family over these pairs has dimension $3d + 3g - 4 - 2d$. The stable part maps to M. Over general N the fibers have dimension at most $d - 1$ and parametrize the degree-d locus in Z, since inclusions up to scalar specify the subbundle. It follows that $\dim Z \le k - 1$, with Z empty for $k = 0$.

On this cone the product $F_{p,\ell}$ is the linear form $q \mapsto \ell^{\otimes 3}(q(p))$. For fixed $[q] \in Z$ its zeros in the surface $\mathbb{P}_D(N^\vee)$ form a proper closed subset. The incidence of k simultaneous zeros above Z therefore has dimension at most $k - 1 + k < 2k$. Its projection is closed by projectivity of Z, so general products cut out $3[D]$. Their joint map is proper. Indeed its norm is bounded below on the unit sphere of the coefficient cone. Bounded coefficients bound the cubic roots in a compact disk bundle. In $\mathbb{P}_D(\mathcal{O} \oplus N)$ all these cycles have the degree of $3[D]$ by dilation, and their projective Chow limits remain in the disk. Over the Borel base $\mathbb{P}^k$ their joint integral Thom class represents $3^k x_{g,3}$ by [Par25, Definition 4.7 and (4.16)]. The empty tuple over a point applies when $k = 0$.

For the lower bound suppose $k \ge 1$, take any N of degree $2g - 2$, and prescribe an effective divisor P of degree $k - 1$. We claim that some $S \in A = \operatorname{Pic}^k D$ admits both a nonzero map $S \to N$ and a nonzero section of $S^3(-P)$. Let $p : D \times A \to A$ be the projection, $\mathcal{P}$ the normalized Poincaré bundle and θ the principal polarization. For $V = -Rp_*(N \otimes \mathcal{P}^{-1})$,

Grothendieck–Riemann–Roch gives

$$\operatorname{ch}(V) = 2k + 2\theta, \qquad c_j(V) = (2\theta)^j/j!.$$

Indeed $\operatorname{ch}(N)\operatorname{td}(D) = 2$, and the square of the mixed Poincaré class pushes forward to -2θ. On a closed locus where all maps $S \to N$ vanish, the differential in a two-term presentation is fiberwise injective. Its cokernel represents V there by a rank-$2k$ bundle, so $c_{2k+1}(V)$ restricts to zero.

If $g = 2k + 1$, the integral of this class on A is 2^g, and $\deg S^3(-P) = g$ gives a section by Riemann–Roch. If $g = 2k + 2$, let T be the inverse image of the theta divisor $W_{g-1}(D)$ under $S \mapsto S^3(-P)$. Translation and cubing give $[T] = 9\theta$, whence $\int_T c_{2k+1}(V) = 9g\,2^{g-1} > 0$. Some $S \in T$ therefore admits a map, and $S^3(-P)$ has a section by definition. In either case push the resulting spectral cubic into N by $S \to N$. It is a compact effective cycle whose generic fiber lies off the zero section, even if the line map has zeros. Over each point of P the entire cubic lies on the zero section.

The three values of any fiber covector on this cubic are $z, \zeta z, \zeta^2 z$, where ζ is a primitive cube root of unity. A homogeneous symmetric polynomial of weight prime to three vanishes on them, since multiplication by ζ permutes them. If fewer than k weights are divisible by three, include their points in P, completing it to degree $k - 1$. Every evaluation then vanishes on the entire dilation orbit of this cubic. At a base point where the three normal coordinates are nonzero some point product is unbounded on the orbit, contradicting compactness of the common zero set. □

Corollary 8.2. *Fix $1 \le d \le k$ and $d - 1$ distinct points with sum P. For general stable N of determinant K the number of pairs $(S, [s])$ with $S \subset N$ a degree-d subline and $0 \neq s \in H^0(S^3(-P))$ is*

$$2^{2d+1} 9^{g-2d-1} \binom{g}{2d+1}.$$

Proof. Put $r = 2d + 1$. The pairs (S, B) with $S^3(-P) = \mathcal{O}(B)$, $B \in \operatorname{Sym}^r D$, form a smooth finite étale pullback W of that symmetric power. Its fundamental class pushes to $9^{g-r}\theta^{g-r}/(g - r)!$ on $\operatorname{Pic}^d D$ by the Poincaré formula [Deb95, Introduction]. The preceding index calculation with d in place of k gives $\int_W c_r(V)$ equal to the displayed number. Positivity forces a nonzero map from some S. For general N each such map is saturated, since a saturation $S(A)$ with $a = \deg A > 0$ has at most $r + a + (3g - 4 - 2d - 2a) = 3g - 3 - a$ parameters including the extension. The saturated extension incidence is smooth of dimension $3g - 3$, so its general fibers over M are finite and reduced by generic smoothness. Thus every kernel at a rank drop has dimension one, since its projectivization parametrizes the maps up to scalar. An invertible minor in the two-term presentation identifies the projective kernel incidence locally with the rank-drop scheme. Its codimension is r and the Thom–Porteous formula gives its length as $\int_W c_r(V)$. □

Corollary 8.3. *For every smooth curve D of genus $g \ge 3$ and every holomorphic rank-two bundle N of degree $2g - 2$, there is a connected smooth degree-three cover f with $H^0(f^*N) \neq 0$ and genus at most $3g + 1$. If $H^0(N) = 0$, the corresponding cyclic images in the normal bundle are integral and their nonzero dilates approach $3[D]$.*

Proof. On $\operatorname{Pic}^1 D$ use the degree-one Picard complex in the preceding proof. The inverse image W of the effective locus W_3 by cubing has $\int_W c_3(V) = 8 \cdot 9^{g-3}\binom{g}{3} > 0$. If the kernel were everywhere zero, the rank-two actual cokernel bundle on W would have zero third Chern class. A line S with both $S \to N$ nonzero and a nonzero section s of S^3 therefore exists,

without any saturation or generality assumption. If s is a meromorphic cube its cube root is holomorphic and gives a section of N, which pulls back to any connected étale cyclic triple. Otherwise normalize the cyclic cubic $y^3 = s$. At most three zeros have order nonzero modulo three and each contributes two to ramification. Its genus is at most $3g - 2 + 3$. The generic injection from S to N embeds the generic fiber multiset, and dilation then gives the claimed cycle limit. □

9. The isolated triple in genera three and four

Let D be a smooth projective curve of genus h and $K = K_D$.

Theorem 9.1. *Suppose $h = 3$ or 4, and let N be a general stable bundle of rank two with determinant K. The smooth connected degree-three covers $f : F \to D$ for which $H^0(F, f^*N) \neq 0$, up to isomorphism over D, are in bijection with the degree-one subbundles $S \subset N$ such that $H^0(D, S^3) \neq 0$. There are eight if $h = 3$ and 288 if $h = 4$. Each cover is cyclic, has source genus $3h + 1$, and satisfies $h^0(F, f^*N) = 1$.*

The effective degree-three cycles in $\operatorname{Tot}_D N$ are $3[D]$ and the smooth embedded curves

$$y^3 = as, \qquad a \in \mathbb{C}^*, \qquad y \in \operatorname{Tot}_D S,$$

where s is a generator of $H^0(D, S^3)$ for one of these subbundles. The resulting parameter lines meet only at $a = 0$, and scalar multiplication by λ acts on them by $a \mapsto \lambda^3 a$. For $h = 2$ a general N with the same determinant condition is 3-rigid and its only effective degree-three cycle is $3[D]$.

Proof. The spectral description in the proof of Theorem 8.1 leaves only degree-one lines, since $\lfloor (h-1)/2 \rfloor = 1$ for $h = 3, 4$. Corollary 8.2 gives the two counts. On the finite étale pullback of $\operatorname{Sym}^3 D$, the loci of multiple zeros and of nonunique effective divisors in a complete linear system have dimension at most two. Indeed three general points impose independent conditions on differentials. Adding the extensions of KS^{-1} by S leaves dimension at most $3h - 4$ over these loci. Thus $h^0(S^3) = 1$ and the divisor is reduced. Riemann–Hurwitz gives $3h + 1$. Each point of this finite incidence has a unique subbundle map up to a constant. The quotient of its Tschirnhausen bundle $S \oplus S^2$ by the unique degree-two subline recovers S, and its inclusion in N is unique up to scalar. Thus distinct sublines give distinct covers over D. To identify the normal sections on a given cover, write $f_*\mathcal{O}_F = \mathcal{O}_D \oplus S^{-1} \oplus S^{-2}$. The last summand tensored by N has no sections because N has no subline of degree two or more. The middle summand gives the unique inclusion. Finally at $h = 2$ the cubic space is a point. A nonzero section on a connected stable map would be nonzero on some normalized nonconstant component since sections on attached complete contracted curves are constant. The vanishing just proved on smooth covers through degree three excludes this possibility. □

Put $r_3 = 8$, $r_4 = 288$ and $H = 3h + 1$. An integral insertion on an open and closed compact subset of a cycle space is the extension by zero of its unit. Products use disjoint targets. In genus h let $x_{h,d}$ be the rational universal class associated to degree d in the normal bundle with scalar action on both directions [Par25, Definition 4.7]. For $d = 1$ this is the integral class of an isolated unobstructed embedded curve. If D is embedded in a smooth quasi-projective threefold X and $d[D]$ is isolated in a tube about D, denote its integral unit by J_d. The local degree here is the coefficient of $[D]$ under retraction of the tube to D. Excision identifies the unit in every smaller tube.

In source arithmetic genus k, let $G_k(X, 3D)$ be the ordinary connected Gromov–Witten degree on the maps with this image cycle, once it is isolated. For comparison let $S^{\mathrm{sc}}_{h,3}(u)$

be the connected degree-three residue of $\operatorname{Tot} N$ for scalar multiplication on the fibers, with u^{2k-2} recording the source genus. It can be evaluated on the proper stack of stable maps to D by equivariant localization [BP08]. The distinguished rational universal class $x_{h,3}$ is the coefficient before taking a logarithm, whereas the numerical series uses connected domains throughout.

Theorem 9.2. *Let $h = 3$ or 4, let N be general with determinant K_D, and let the embedding $D \subset X$ in a smooth quasi-projective threefold have normal bundle N and a second jet whose obstruction at each of the finitely many dilation lines is nonzero, as defined in Section 10. Then $3[D]$ is isolated in a neighborhood of D. Such neighborhoods exist in smooth algebraic threefolds. In the rational complex universal algebra,*

$$J_3 = x_{h,3} + \frac{r_h}{3} x_{H,1}. \tag{16}$$

In the fixed-target Gromov–Witten theory,

$$\sum_k G_k(X, 3D) u^{2k-2} = S^{\mathrm{sc}}_{h,3}(u) + \frac{r_h}{3} (2\sin(u/2))^{2H-2}. \tag{17}$$

In degrees one and two one has $J_d = x_{h,d}$.

10. Isolation of the tripled curve

Let D have genus $h = 3$ or 4, and let N be a general stable bundle of rank two with determinant K_D. Write $Y = \operatorname{Tot}_D N$. By Theorem 9.1, the degree-three cycles in Y away from $3[D]$ are smooth embedded curves in finitely many orbits of fiber dilation. Each orbit comes from a line subbundle $S \subset N$ of degree one. For a generator $s \in H^0(D, S^3)$ its members are the curves

$$F_a = \{y^3 = as\} \subset \operatorname{Tot}_D S, \qquad a \in \mathbb{C}^\times.$$

The three zeros of s are simple, and $H^0(S) = H^0(S^2) = 0$. We first compute the obstruction to deforming F_a when the threefold varies.

10.1. **The normal bundle of a cubic.** It suffices to take $a = 1$. Put $F = F_1$, write $f : F \to D$ for the projection and $Q = K_D S^{-1}$, and denote the class of the extension of Q by S defining N by $e \in H^1(SQ^{-1})$. The equation of F in $\operatorname{Tot}_D S$ gives the normal sequence

$$0 \longrightarrow f^*S^3 \longrightarrow N_{F/Y} \longrightarrow f^*Q \longrightarrow 0. \tag{18}$$

Its connecting map on global sections is cup product with $3y^2 f^*e$, up to sign. Indeed, local splittings of $N \to Q$ lift a Q-displacement to vertical vector fields whose differences are the corresponding S-displacements given by e. The derivative of the cubic equation on these differences is multiplication by $3y^2$.

Lemma 10.1. *For all the cubics of Theorem 9.1, after shrinking the dense open of normal bundles if necessary, the tangent $H^0(F, N_{F/Y})$ is generated by varying a, and projection in (18) induces an isomorphism*

$$H^1(F, N_{F/Y}) \xrightarrow{\sim} H^1(F, f^*Q) = H^1(D, QS^{-2}) \cong H^0(D, S^3)^\vee.$$

Proof. Use $f_*\mathcal{O}_F = \bigoplus_{i=0}^{2} S^{-i}$, with the ith summand represented by y^i in a local frame. The three spaces $H^0(QS^{-i})$ have dimensions $h-2, h-3, h-3$ by Riemann–Roch. The first maps isomorphically to $H^1(S)$ in the y^2 summand of $H^1(f^*S^3)$. This is three times the connecting map for N, which is acyclic by degree-two rigidity [BP06, Theorem 2.2]. In genus three these are the only nonzero blocks.

In genus four take generators q_1 and q_2 of $H^0(QS^{-1})$ and $H^0(QS^{-2})$. Their blocks map to $H^1(S^3)$ and $H^1(S^2)$, respectively. Since $y^3 = s$, both Serre pairings are, up to the common factor and sign, $\langle e, sq_1q_2\rangle$. The product is a nonzero section of Q^2, the Serre dual of the extension space. Its annihilator is a hyperplane in each projective space $\mathbb{P}H^1(SQ^{-1})$ of dimension six. The possible S with a cubic section range over a three-dimensional locus in $\operatorname{Pic}^1 D$. These exceptional extensions therefore have image of dimension at most eight in the nine-dimensional moduli space of bundles with fixed determinant.

The connecting map in (18) is consequently an isomorphism in both genera. The remaining tangent space is $H^0(f^*S^3) = H^0(S^3)$, spanned by the derivative of F_a. Serre duality gives the asserted obstruction space, since its other two summands are dual to $H^0(S)$ and $H^0(S^2)$. □

10.2. **The second infinitesimal neighborhood.** Let $D \subset X$ be an embedding in a smooth quasi-projective threefold, with an identification $N_{D/X} = N$. Write $\mathcal{X} \to \mathbb{A}^1$ for its deformation to the normal cone, obtained by blowing up $D \times 0$ in $X \times \mathbb{A}^1$ and removing the proper transform of $X \times 0$. The central fiber is Y and every noncentral fiber identifies with X. For an embedded cubic $F \subset Y$, let

$$\kappa_F(X) \in H^1(F, N_{F/Y}) \tag{19}$$

be the restriction of the Kodaira–Spencer class of this deformation at zero, projected to the normal bundle of F. This is the obstruction to lifting F over the tangent vector ∂_t of the parameter line.

The class depends only on the second infinitesimal neighborhood defined by $\mathcal{I}_D^3$. Indeed, in adapted coordinates (z, n) along D, write a change of charts as

$$z' = h_0(z) + A(z)n + O(n^2), \qquad n' = G(z)n + B(z)(n, n) + O(n^3).$$

In normal-cone coordinates replace n by tn and divide n' by t. The derivative at zero has horizontal component $A(z)n$ and vertical component $B(z)(n, n)$. Projecting this cocycle to $N_{F/Y}$ gives (19). The definition thus applies to embeddings without a retraction onto D. Dilation lifts to the normal-cone family, acting on its parameter by the inverse character. Nonvanishing on one member of a dilation orbit is equivalent to nonvanishing on every member.

Proposition 10.2. *For $h = 3, 4$, suppose that $\kappa_F(X) \neq 0$ for one cubic in each orbit of Theorem 9.1. Then $3[D]$ is isolated among all compact effective cycles of local degree three in a tube about D in X. This condition on the second infinitesimal neighborhood is Zariski open in algebraic families with identified D and N. It is attained by smooth quasi-projective threefolds with normal bundle N and determinant-preserving changes of the fiber coordinates.*

Proof. In a family with fixed D and N, the restricted Kodaira–Spencer classes in (19) vary algebraically in the fixed obstruction lines. There are only finitely many conditions. We exhibit an algebraic family where each vanishing condition cuts out a proper hyperplane.

Start with Y. Fix a point $p \in D$, a frame of N on an affine neighborhood U, and a local parameter u regular on U with no other zero there. Choose a covector ℓ on the framed plane which is nonzero on every one of the finitely many lines S_p, and a nonzero $v_\ell \in \ker \ell$. Reglue the plane bundle over U to its restriction over $D \setminus \{p\}$ by composing the transition on the overlap with

$$n \longmapsto n + v_\ell\, \ell(n)^2 \sum_{j=1}^{m} c_j u^{-j}. \tag{20}$$

This polynomial automorphism has determinant one and its inverse negates the coefficients. It fixes the zero section to first order. The resulting smooth threefold is affine of finite type

over D, hence quasi-projective. Its projection to D has the same vertical tangent determinant as the original bundle, since the added transition has determinant one.

For a given S the restriction of $v_\ell \ell^2$ to S^2, followed by projection to Q, generates QS^{-2} at p. The Kodaira–Spencer class on the cubic therefore projects to the principal-parts boundary in $H^1(QS^{-2})$, in the y^2 summand of $H^1(f^*Q)$. For sufficiently large m, these boundaries span that space by $H^1(QS^{-2}(mp)) = 0$. Multiplying the principal parts by the local generator preserves their span. Choose m large enough for all the S. By Lemma 10.1 each κ_F is then a nonzero linear functional of the c_j, with values in a one-dimensional space. Arbitrarily small choices outside the union of their kernels satisfy the hypothesis.

Now fix an X satisfying this hypothesis. At a nonzero cubic F the tangent sequence for the relative Hilbert scheme of $\mathcal{X}/\mathbb{A}^1$ is exact through

$$0 \longrightarrow H^0(N_{F/Y}) \longrightarrow T_{[F]}\,\mathrm{Hilb} \longrightarrow T_0\mathbb{A}^1 \xrightarrow{\kappa_F} H^1(N_{F/Y}).$$

Indeed, local lifts of the embedding over a first-order parameter differ on overlaps by precisely the normal cocycle (19). The Hilbert tangent has dimension one. The actual orbit F_a has dimension one in the central fiber and its derivative generates that tangent. Equality of the local dimension and embedding dimension gives a regular local ring. Thus the relative Hilbert scheme near $[F]$ consists entirely of the central orbit. In particular this conclusion concerns every deformation of the embedded curve.

Suppose extra cycles converged to $3[D]$ in X. In a smooth tubular identification let $t_i > 0$ be the ratio of the maximum normal distance of the ith cycle to a fixed small radius. Then $t_i \to 0$. Regard the cycles in the fibers $\mathcal{X}_{t_i}$, using the divided normal coordinates. They lie in a common closed disk and reach its prescribed radius. The smooth structures after this rescaling converge on a larger disk to the complex structure of Y. A closed Kähler form on the central disk tames these structures for large i. Its integral on every cycle is three times its integral on D by retraction to the zero section. It follows that the masses are uniformly bounded. Cycle compactness, including Hausdorff convergence of support, gives a limit of degree three meeting the prescribed positive radius [DIW26, Theorem 2.11 and Remark 2.13]. Theorem 9.1 identifies the limit with a single smooth cubic of multiplicity one.

Openness of the embedded locus and its topology in the cycle space give smooth embedded nearby curves of the same genus [DIW26, Proposition 2.21]. To pass to the relative Hilbert scheme, use a projective completion of the normal-cone family. The ample degree of the cycles is locally constant, so the Hilbert polynomial of the nearby embedded curves agrees with that of F. Any flat Hilbert limit has support F and generic multiplicity one by convergence of supports and cycles. The kernel of its structure sheaf onto $\mathcal{O}_F$ has finite length, hence vanishes by equality of Hilbert polynomials. The resulting Hilbert limit $[F]$ contradicts the central-orbit description above. □

11. Comparison with the scalar local curve

Let $D \subset X$ satisfy Proposition 10.2, with $h = g(D)$ equal to three or four. Write $N = N_{D/X}$, and put

$$H = 3h + 1, \qquad r_3 = 8, \qquad r_4 = 288.$$

Theorem 9.1 gives r_h families of embedded genus-H curves in the total space of N. Write Φ for the map from complex to smooth families of threefolds in the universal cycle algebra. We compare the scalar insertion at $3[D]$ in this bundle with the compact open class J_3 of the isolated cycle in X. Write $G_k(X, 3D)$ for the local virtual degree of connected unmarked stable maps of arithmetic genus k with image cycle $3[D]$. It includes maps with contracted

subcurves. The inverse image of the isolated cycle is open and closed in the degree-three stable-map space over a sufficiently small tube. It is proper by fixed-genus compactness. The ordinary algebraic stable-map virtual class restricts to it, using a projective completion and the open set of maps with image in X.

Proposition 11.1. *Let $w = (2\sin(u/2))^2$, and let $S^{\mathrm{sc}}_{h,3}(u)$ be the connected scalar residue in degree three. In the smooth cycle algebra, and in the fixed-target connected Gromov–Witten theory, respectively, one has*

$$\Phi(J_3) = \Phi(x_{h,3}) + \frac{r_h}{3}\Phi(x_{H,1}), \tag{21}$$

$$\sum_k G_k(X, 3D)u^{2k-2} = S^{\mathrm{sc}}_{h,3}(u) + \frac{r_h}{3}w^{H-1}. \tag{22}$$

Proof. The scalar insertion. Choose $p \in D$ outside the branch divisors of the r_h families and a linear form ℓ on N_p nonzero on each of their defining sublines. For a degree-three cycle let F be the product of the three values of ℓ over p, with multiplicities. On a family $y^3 = as_i$ of Theorem 9.1, this product is $c_i a$ with $c_i \neq 0$. Thus its unique zero is $3[D]$. Under scalar dilation F has weight three.

Take $B = \mathbb{P}^1$ and $L = \mathcal{O}_B(1)$. In the diagonal Borel quotient a function of weight one is a section of L. Indeed a pair (v, z) defines the functional on $\mathbb{C}v$ taking v to z, which is unchanged under simultaneous scaling. The function F therefore gives a section of L^3 on the relative cycle space. The pullback of its Thom class divided by three, over the oriented base B, represents $x_{h,3}$ by the scalar construction [Par25, §4.3].

We identify its algebraic virtual pairing before deforming the insertion. Compactify the normal bundle by $\mathbb{P}_D(\mathcal{O} \oplus N)$ and extend the supported class by zero. The only fixed maps meeting its support are maps to D. On their proper connected moduli stack $\overline{\mathcal{M}}_k(D, 3)$, with universal curve π and map f, the moving virtual normal complex is $R\pi_* f^* N$ with scalar character one. If η is the equivariant parameter, the normalized test restricts to $3\eta/3 = \eta$. The virtual localization formula [GP99] gives the equivariant pushforward

$$\eta\,[u^{2k-2}]S^{\mathrm{sc}}_{h,3}(u).$$

Here the inverse moving Euler class is exactly the fixed-target residue of [BP08, §2.2]. It applies to N since its K-class on D is determined by rank and determinant and its scalar character is common to both normal directions. Passing to the associated bundle over B substitutes $c_1(L)$ for η, and $\int_B c_1(L) = 1$. This computes the pairing of the relative algebraic virtual class with the test in source genus k: on a trivialization of the frame torsor the universal source, universal map and their relative obstruction complex are pulled back from the fixed-target equivariant data, and these identifications descend with the scalar character.

The deformation. Form the deformation to the normal cone of $D \subset X$ and delete the proper transform of $X \times \{0\}$. Denote its parameter by t. The fibers over 0 and over $t \neq 0$ are N and X, respectively. Give t weight -1 and fix the original X coordinate. In normal coordinates $n = y/t$ the action at $t = 0$ is scalar of weight one. The associated family of threefolds thus lies over $E = \mathrm{Tot}_B(L^{-1})$. Its restriction to the zero section is the preceding Borel family. Over the complement of that section the original X coordinate trivializes the family.

The test extends over a neighborhood of the marked cycle in this family. Choose adapted local coordinates (x, y_1, y_2) near p with $D = \{y_1 = y_2 = 0\}$ and take the transverse surface $x = x(p)$. For a nearby cycle its three intersections with this surface near p, counted with multiplicity, give a continuous unordered triple. In the deformation coordinates use the

product of their values under $\ell(y/t)$. The divided coordinates extend over $t = 0$, and the product has weight three. It restricts there to F and vanishes on the marked triple in every fiber. On smooth embedded curves transverse to the surface the evaluation is holomorphic.

Choose a small section $v \in H^0(B, L^3)$ with three simple zeros. Choose a small smooth section τ of L^{-1} with a single transverse zero b_* disjoint from those of v. The metric dual of a holomorphic section of L with a simple zero gives such a section of local index -1. On the zero section first deform F to $F - v$. Keep this translated test fixed and move the parameter over the graphs $t = s\tau(b)$, $0 \le s \le 1$.

These intersections have compact support through both homotopies. Use a closed normal disk of small radius over D. At $t = 0$ the maximum normal radius on the ith curve is a positive constant times $|a|^{1/3}$. Hence $|F|$ is bounded below away from zero where a cycle reaches the boundary of this disk, uniformly over B in unitary frames and over the finitely many families. Choose $\|v\|$ smaller than this bound. If $t \neq 0$, the disk maps into an $O(|t|)$ tube about D in the original target. For $|t|$ sufficiently small its only degree-three cycle is $3[D]$ by Proposition 10.2. Within the closed disk, a common closed taming form in each local smooth trivialization bounds the masses by the fixed homology class $3[D]$. Relative cycle compactness on a slightly larger disk therefore includes any limit across $t = 0$ [DIW26, Theorem 2.11]. These choices can be made uniform on the compact base. In particular the compactly supported cohomology class on cycles is transported in the smooth colimit [Wue26b, §2]. For the algebraic pairing use the relative virtual class over the smooth algebraic base E and intersect it with the Thom class of the oriented parameter graph and the test. The inverse images of the compact supports are proper at each fixed k. The homotopic graphs give equal pairings. At the zero section the refined fiber restriction is the virtual class of the Borel family by base change [BF97, Proposition 7.2].

Intersections on the final graph. Over each of the three zeros of v, the parameter τ is nonzero. The only supported cycle is the isolated triple in X. Its local intersection is the simple zero of $-v$, of positive index. It gives $\Phi(J_3)$ in the smooth colimit and $G_k(X, 3D)$ in the relative virtual pairing, since the family is a product there.

Over b_* there is one solution $a = v(b_*)/c_i \neq 0$ on each normal family. The local relative Hilbert scheme is smooth with coordinates (b, a) and maps to $(b, 0) \in E$. Indeed the only fixed-fiber tangent direction is the variation of a, and the t direction surjects onto the one-dimensional fixed-fiber obstruction space by Proposition 10.2. The normal curves themselves supply the smooth two-dimensional family in $t = 0$. On this chart the graph equation is $-\tau(b) = 0$, and the remaining equation has nonzero complex derivative c_i in a. Their intersection sign is -1. The graph has the orientation of the vertical complex line in E, and the test has its complex orientation. Interchanging their normal factors preserves orientation.

The Hilbert chart here satisfies the hypotheses of Proposition 7.1. In particular the motion of t surjects onto the fixed-curve obstruction and the graph is transverse before testing the amplitude. The shear argument of Lemma 7.2 is local along the embedded image. At the finitely many supporting points choose disjoint smooth submersion coordinates away from the evaluation and ramification fibers. These coordinates extend over a smaller parameter neighborhood by smooth local trivialization. The required normal variations are compactly supported there; a projection from the whole threefold to the original curve is unnecessary. On the cycle space the lemma gives $-\Phi(x_{H,1})$ for each such image. The proper degree-one map stack over the same chart accounts for the virtual contribution in every source genus, equal by (11) to $-[u^{2k-2}]w^{H-1}$. The embedded image has trivial automorphisms as a map to the threefold; contracted components of maps to it retain their ordinary stack multiplicities.

Summing the two sets of local intersections before the scalar normalization gives

$$3\Phi(J_3) - r_h\Phi(x_{H,1}), \qquad 3G_k(X, 3D) - r_h[u^{2k-2}]w^{H-1}.$$

After division by three these are respectively $\Phi(x_{h,3})$ and the coefficient of u^{2k-2} in $S^{\mathrm{sc}}_{h,3}$ computed above. This proves both identities. □

Let Φ denote the change from complex families to smooth families of almost-complex threefolds in the Chern-number-zero universal algebra. We recall the detection results used here [Wue26b, §§5–7]. In the complex algebra a primitive class annihilated by every proper division of cycles belongs to the finite rational span of the $x_{j,1}$. Their images under Φ are linearly independent. The latter assertion uses proper pullback to connected stable maps of increasing arithmetic genus. In genus below j that pullback is zero for $x_{j,1}$, and in genus j the regular embedding gives a nonzero class. Neither assertion requires an evaluation by a virtual fundamental class for arbitrary analytic families.

Proof of the universal identity in Theorem 9.2. Let $J_0 = x_{h,0} = 1$. On the tube provided by Proposition 10.2, the inverse image of the supported point $d[D]$, $1 \le d \le 3$, under cycle addition is the union of the pairs $(a[D], b[D])$ for $a+b = d$. The preimage under multiplication by m is the divided point if $m \mid d$, and is empty otherwise. The maps pull back unit cochains, so there are no stabilizer factors. Thus the J_d and the scalar $x_{h,d}$ have the same coproduct and division laws through degree three. In particular

$$\Delta J_d = \sum_{a+b=d} J_a \otimes J_b, \qquad \rho_m J_d = \begin{cases} J_{d/m} & m \mid d, \\ 0 & m \nmid d. \end{cases}$$

For the scalar classes these formulas follow from [Par25, §4] in Chern number zero.

For $d = 1, 2$ deformation to the normal bundle preserves the isolated section $d[D]$. Indeed a sequence of additional cycles would, by normal dilation and compactness as in §10, give an off-section cycle of degree at most two in the linear normal model. The entire degree-d cycle space of that model is a point, and the empty tuple represents $x_{h,d}$. Consequently $\Phi(J_d) = \Phi(x_{h,d})$. At the first degree of a possible difference the displayed operations make $J_d - x_{h,d}$ primitive and killed by proper divisions. Detection therefore gives $J_d = x_{h,d}$ successively in degrees one and two. For $d = 3$ subtract also $(r_h/3)x_{H,1}$. This correction is primitive and killed by all proper divisions. The difference lies in the same multiplicity-one span, and has zero smooth image by Proposition 11.1. This proves (16). The numerical identity is the separate algebraic pairing in that proposition. □

In the compact example, change just the cubic coefficient of the genus-three scalar factor to J_3. Its logarithm gains $(8/3)x_{10,1}$. The new exponent at 3α in genus ten is therefore $c_{3\alpha,10} - (4720/3 + 8/3)c_{\alpha,3} = c_{3\alpha,10} - 1576c_{\alpha,3}$. Thus the isolated local unit accounts for the fractional part even though the complete change from equal to opposite weights also has an integral part.

In genus five the quotient by dilation contains pencils. A single cubic evaluation no longer suffices; two independent point products give the least tuple in Theorem 8.1. Theorem D.1 in Appendix D proves the comparison in this case, including the virtual multiplicity of a cuspidal normalization.

12. Composite degrees and the contributions of multiple images

Write $p^{\epsilon}_{h,d} = [q^d] \log F^{\epsilon}_h(q)$ for a universal connected class and $S^{\epsilon}_{h,d}$ for its numerical connected residue, $\epsilon = \mathrm{sc}, \mathrm{ad}$. The power-locus bound in Theorem 2.1 excludes cycles that are proper

integral multiples of another entire cycle. A cycle can still have an individually repeated component. We now retain the double of an entire cycle and determine its contribution. In the formulas below j may have either sign.

Theorem 12.1. *Suppose $m \geq 2$, $|j| \leq n_m$ and $|j| > n_{m/e}$ for every divisor $e \geq 3$ of m. For even m write $P_{m/2,j} = \sum_h b_h x_{h,1}$, interpreting a weight outside the Laurent range as zero. For odd m set $b_h = 0$. Then there is a finite polynomial R such that*

$$P_{m,j} = \sum_h b_h p^{\mathrm{sc}}_{h,2} + \iota(R), \qquad R = \sum_k N_{k,m,j} u^{2k-2} - \sum_h b_h S^{\mathrm{sc}}_{h,2}.$$

The coefficient of w^{h-1} in R vanishes if $h < 1 + m(g-1) + |j|$. In particular this computes every nonconstant quartic weight in every core genus and the sextic weights $|j| > g - 1$.

Proof. On the projective base for the jth coefficient the translated spectral tests avoid eth powers for each $e \geq 3$ by the codimension computation of Section 5. Avoidance persists in a collar of the compact incidence. If m is even, all proper multiples in degree $m/2$ are similarly excluded, so Theorem 2.1 supplies the b_h when $m/2 > 1$. In degree one the only weight is zero. Proper division by two sends $P_{m,j}$ to $P_{m/2,j}$ and $p^{\mathrm{sc}}_{h,2}$ to $x_{h,1}$. All remaining proper divisions vanish on both sides. The logarithms are primitive. It follows from multiplicity-one detection that the remainder has the form $\iota(R)$ for a unique finite Laurent polynomial with exponents at least -1.

Fix a bound K on source genus. In the separation argument of Theorem 6.1 keep every tuple of distinct components with normalization genus sum at most K, including the root of a double whose own degree-two covers have minimum genus above K. At each step order by total reduced degree, the genus sum and then minus the number of distinct images. Lemma 6.3 makes the first remaining stratum compact after the neighborhoods already treated have been excised. The shear and jet argument makes its normalized images embedded and pairwise disjoint. A connected support is therefore a simple embedded curve or a double of one. The equations on a reduced Hilbert chart are those of its *weighted* cycle. Its transverse solutions are finite.

Before excising a doubled solution, fix its reduced curve D' and vary the normal bundle. Here $\det N_{D'} = K_{D'}$, after trivializing a constant line. At an unramified fiber away from the tests choose a framed chart, a uniformizer t, a linear form ℓ separating all sheets y_i, and the monic polynomial $S(T) = \mathrm{Norm}(T - \ell(y))$ of their union on the chart. For a constant area bivector choose $v = \iota_\ell \Omega$, so $\ell(v) = 0$. The transition

$$y \longmapsto y + vS(\ell(y)) \sum_{a=1}^{B} \sum_{l=0}^{r-1} c_{al} t^{-a} \ell(y)^l$$

is an automorphism of the punctured chart with determinant one. It fixes each sheet. Its first normal derivative on sheet i is given by $v \otimes \ell$ times $S'(\ell(y_i))$ times the displayed polynomial. The Vandermonde determinant is a unit modulo t^B and three generic forms span $\mathfrak{sl}_2$ in this way. For large B these principal parts map onto $H^1(D', \mathrm{End}_0 N_{D'})$; using the union of disjoint sheet values makes the jets independent for several marked curves. The same framed parameter bundles and base cutoffs as in Lemma 6.2 realize these jets in one algebraic target family, with a smooth section over the compact parameter base.

A good normal bundle is accessible even from an unstable one. For an ample enough line Q, the initial bundle and any prescribed good bundle of this determinant both admit quotients Q (a general section of the globally generated N^*Q has no zeros). They belong to one irreducible extension space $\mathrm{Ext}^1(Q, K_{D'}Q^{-1})$. The open set of stable two-rigid bundles there is dense

by [BP06, Theorem 2.2]. The preceding Kodaira–Spencer map is onto and deformations of fixed-determinant bundles on a curve have no second obstructions. Thus the parameter family is smooth over the bundle stack locally and can realize an arbitrarily small change into this open. The regular cut and its sign persist. Each root then has acyclic normal bundle and extends uniquely on a relative Hilbert chart. Any other nearby cycle of multiplicity at most two, recentered and dilated to unit normal height, would limit to a compact off-section cycle in the normal bundle. The disk mass bound follows from the fixed projected class and a common closed taming form. The normalization of a positive component would supply a nonzero normal section on a cover of degree at most two. Two-rigidity excludes this case. This proves isolation on the full cycle space, uniformly on a smaller parameter neighborhood. The separation induction can now proceed with these compact neighborhoods excised.

Over the isolated chart the doubled point is the scalar class with empty evaluation tuple. Its connected pullback tests give those of $p^{\mathrm{sc}}_{h,2}$: the product in the logarithmic subtraction vanishes on connected domains. The local intersection degree for the divided insertion is the same sign, since doubling maps the reduced section homeomorphically to the doubled section. Testing the divided insertion successively in genera through K identifies its normalized signed totals with b_h. The same triangular test on the residual simple intersections identifies their totals with $[w^{h-1}]R$ through K.

For the virtual pairing the difference of tangent complexes at a map f to the moving reduced curve is $\mathrm{Cone}(q^*Rp_*N \to R\pi_*f^*N)$, with the notation of Proposition 7.1. The first term vanishes by acyclicity. Two-rigidity and cohomology and base change make the second a bundle of rank $2(k-2h+1)$ in degree one on the whole proper stack of degree-two maps, including its nodal boundary. Its top Chern class pushes to the scalar degree-two connected residue times the base class: on a single fiber the equality follows by equivariant localization, where only the top Chern number survives in virtual dimension zero. Simple intersections contribute w^{h-1} by the same proposition. The fixed-genus pairings thus give the asserted equality for arbitrary K. A nonzero double coefficient has $h \geq 1 + (m/2)(g-1) + |j|$; its covers start at $1 + m(g-1) + 2|j|$. Together with the spectral genus bound for the original cut, this proves the support assertion. □

12.1. **The exceptional sextic coefficient.** For a genus-two core $n_6 = 15$. At $j = 1$ the compact base is $\mathbb{P}^{16} \times \mathbb{P}^{14}$. A cube of a trace-free quadratic $T^2 + b$ has coefficients in orders three through six equal to $0, a_2^2/3, 0, a_2^3/27$ where $a_2 = 3b$. The independent sections enforcing these four conditions on the second factor have ranks $2, 3, 4, 5$, total 14, and positive top Chern number. They cut transversely for general translations and avoid the further condition $a_2 = 0$. For general M the sole divisor of M^2 has two distinct points, so each reduced quadratic is smooth of genus four. Properness of multiplication of cycles and openness near a smooth embedded cycle show that throughout a small collar every pure triple in the cut has such a root. Pure sixth powers are absent.

Repeat the preceding proof, retaining doubles and triples separately. At a tripled root, first vary the normal bundle to the general stable locus on the genus-four curve. Theorem 9.2 shows that 288 cyclic families remain in the normal bundle. At a fresh unramified fiber the same interpolation with $S(\ell(y))^2$ fixes the first normal jets while prescribing quadratic terms on every sheet. In particular it realizes the determinant-one shears of Proposition 10.2 simultaneously at the finitely many curves. A general quadratic change now isolates each tripled root while preserving the reduced transverse cut. These changes precede excision. The full relative germ near the multiple is its moving section by the uniform isolation in that proposition. Thus no remaining connected support through the chosen bound is lost.

The two proper divisions give

$$\rho_2 P_{6,1} = 30x_{5,1} - 5x_{6,1} + \tfrac{4}{9}x_{7,1}, \qquad \rho_3 P_{6,1} = \tfrac{1}{2}x_{4,1}.$$

Indeed, subtract $P_{3,3} = x_{7,1}/18$ from one quarter of the genus-two cubic row, and apply the quadratic row in (2). Let $b_5 = 30, b_6 = -5, b_7 = 4/9$ and put the other b_h equal to zero. The two divided tests measure the root signs separately without automorphism factors. Fix $K \geq 13$ and apply the connected pullback tests up to genus K. The undivided test at each tripled root uses $J_3 = x_{4,3} + 96x_{13,1}$, and its fixed-target test has the same extra $96w^{12}$. Consequently

$$P_{6,1} = \tfrac{1}{2}p^{\mathrm{sc}}_{4,3} + \sum_h b_h p^{\mathrm{sc}}_{h,2} + \iota\left([z]S_{2,6}(z) - \tfrac{1}{2}S^{\mathrm{sc}}_{4,3} - \sum_h b_h S^{\mathrm{sc}}_{h,2}\right).$$

Here $[z]$ means Laurent weight one; the remainder in parentheses includes $48w^{12}$ in addition to the normalized simple-image counts. This use of the nonlinear germ is essential: tests on a moving reduced chart need not isolate the cubic amplitude in the linear bundle, even when they do detect the amplitude at a fixed base point. Indeed the derivative in the moving base coordinates at the root is invertible. The implicit function theorem solves the test equations as the amplitude moves.

12.2. **Conversion to opposite-weight factors.** Define $t_{g,d,h}$ by $F^{\mathrm{sc}}_g(q) = \prod_{d,h} F^{\mathrm{ad}}_h(q^d)^{t_{g,d,h}}$. Let $D_{g,d} = S^{\mathrm{sc}}_{g,d} - S^{\mathrm{ad}}_{g,d}$. Subtracting the proper divisors in the logarithm gives

$$\sum_h t_{g,4,h} w^{h-1} = D_{g,4} - \sum_h t_{g,2,h} S^{\mathrm{ad}}_{h,2},$$

$$\sum_h t_{2,6,h} w^{h-1} = D_{2,6} - 2S^{\mathrm{ad}}_{4,3} - 120S^{\mathrm{ad}}_{5,2} + 20S^{\mathrm{ad}}_{6,2} - 2S^{\mathrm{ad}}_{7,2}.$$

The quartic formula follows since the sum over the doubled images in Theorem 12.1 is the entire quadratic change. For the sextic the positive odd weights three and one contribute $1/18$ and $4/9$, respectively, to the genus-seven double, giving $4(1/18 + 4/9) = 2$. The other doubles and the triple come from weight one alone. The remaining proper factors convert scalar lower logarithms into opposite-weight logarithms by Theorem 2.1.

For a genus-two core the quartic row in genera six through eleven is

$$(6860, -1470, 1302, -317, 40, -2).$$

Its first positive Laurent coefficient starts at $1715w^5$. The sextic row in genera eight through twenty-two is

h	8	9	10	11	12
$t_{2,6,h}$	34606024	61350344	125323022	−36417600	62126672
h	13	14	15	16	17
$t_{2,6,h}$	−41880636	22703802	−9395788	2975620	−715462
h	18	19	20	21	22
$t_{2,6,h}$	128364	−16652	1476	−80	2

For reproducibility, in each degree d form the disconnected partition trace Z_d of [BP08, Sections 4–8] and recover $S_d = Z_d - d^{-1}\sum_{i=1}^{d-1} iS_iZ_{d-i}$. With $r = e^{iu}$ this calculation takes place over $\mathbb{Q}(r)$. Multiply the transposition matrix by the unit and its iterates, express the other partition operators in the resulting cyclic basis, and form the handle and inverse caps as in Appendix B. At $z = -1$ use the hook sum, where the transposition operator need not be

cyclic. For odd weights $1, 3, \dots, 2l-1$ the columns $z^j + z^{-j}$ are $y = z + z^{-1}$ times polynomials of successive degrees in y^2. Interpolate $S_d(z) - S_d(-z)$ at $z = 1, \dots, l$, with $l = 3$ for the quartic and $l = 8$ for the sextic. An additional pair of weights verifies each row. Substitution $w = 2 - r - r^{-1}$ gives the displayed polynomials. For instance the sole nonintegral coefficient of $D_{2,6}$ is $376348226/3$ at w^9. The cubic subtraction there is $563480/3$ and the sum of the double terms to be subtracted is -61440, giving 125323022.

In a compact class $m\alpha$ with α primitive in the free quotient, substitution of the factors gives

$$c_{m\alpha,h} - e_{m\alpha,h} = \sum_{d|m,\ d>1} \sum_{d\gamma = m\alpha} \sum_{g} e_{\gamma,g} t_{g,d,h}$$

over effective integral roots. For example, at $(m, h) = (6, 8)$ one gets

$$192 \sum_{2\delta = 6\alpha} c_{\delta,4} + 14448 \sum_{3\delta = 6\alpha} c_{\delta,3} + 34606024 \sum_{6\gamma = 6\alpha} c_{\gamma,2}.$$

Indeed the further balanced double of a cubic change starts at w^8 and the further triple of a quadratic change starts at w^9. A genus at least three at a sixth root contributes no term of degree seven in w, by Riemann–Hurwitz. For $m = 4$ the corresponding identity at output genus six is $24 \sum_{2\delta = 4\alpha} c_{\delta,3} + 6860 \sum_{4\gamma = 4\alpha} c_{\gamma,2}$. Here $J_2 = 24w^5 - 2w^6$ and the further double of a genus-four correction starts at w^6. No formula here identifies the inner modes on arbitrary higher power loci.

Appendix A. A compact hypersurface containing a Hirzebruch surface

We specify the compact example by its fan. Let $N = \mathbb{Z}^4$, and let $v_0, \dots, v_{10}$ be the rows of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -1 & 0 \\ -4 & -2 & 0 & -1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

The maximal cones of Σ are generated by the following four-tuples of rays:

$$\begin{gathered} (0,1,3,6),\ (0,1,3,7),\ (0,1,4,5),\ (0,1,4,7),\ (0,1,5,6), \\ (0,2,4,5),\ (0,2,4,7),\ (0,2,5,6),\ (0,2,6,7),\ (0,3,6,7), \\ (1,3,6,10),\ (1,3,7,8),\ (1,3,8,10),\ (1,4,5,10), \\ (1,4,7,8),\ (1,4,8,10),\ (1,5,6,10),\ (2,4,5,9), \\ (2,4,7,9),\ (2,5,6,9),\ (2,6,7,9),\ (3,6,7,9), \\ (3,6,9,10),\ (3,7,8,9),\ (3,8,9,10),\ (4,5,9,10), \\ (4,7,8,9),\ (4,8,9,10),\ (5,6,9,10). \end{gathered} \tag{24}$$

Write V_Σ for the associated toric fourfold and D_i for its invariant divisors. Let Y be a general anticanonical hypersurface, nondegenerate with respect to its Newton polytope, with nonzero vertex coefficients. This is the resolution chamber used in [Wue26a, Section 4].

A.1. **The smooth projective hypersurface.** The polytope $P = \mathrm{conv}(v_0, \dots, v_8)$ is reflexive, and (24) is a subdivision of its boundary fan. Each maximal cone has determinant ± 1; the subdivision is complete. For an explicit strictly convex support function take divisor coefficients

$$A_i = 1009(10000 + h_i) + (i+1), \qquad h = (10, 11, 3, 9, 2, 7, 1, 4, 8, -5, -3). \tag{25}$$

For each maximal cone σ, solve $\langle m_\sigma, v_i\rangle = -A_i$ for $i \in \sigma$. The off-cone values $\langle m_\sigma, v_i\rangle + A_i$ are all positive, with minimum 5047. This proves projectivity. Repeating the same calculation with all coefficients equal to one gives integral slopes and nonnegative values. Thus the anticanonical divisor is basepoint-free and the fan is a crepant refinement of the reflexive fan. Bertini and adjunction give smoothness and $K_Y \simeq \mathcal{O}_Y$. The anticanonical exact sequence, toric structure-sheaf vanishing, and Serre duality give $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$.

The four vertices v_0, v_1, v_3, v_4 of P form the standard basis of N. In particular, the lattice N'' generated by the lattice points on vertices and edges of P is all of N. The Brauer-group formula of [BK05, Corollary 3.9] therefore gives

$$\mathrm{Br}(Y) \simeq \mathrm{Hom}\big(\Lambda^2 N/(N \wedge N''), \mathbb{Q}/\mathbb{Z}\big) = 0.$$

For this Calabi–Yau threefold the Brauer group is the torsion in $H^3(Y;\mathbb{Z})$. The universal coefficient theorem then implies

$$\mathrm{Tors}\, H_2(Y;\mathbb{Z}) = 0. \tag{26}$$

This uses the full lattice, rather than a rational cohomology rank.

A.2. **Nef divisors and the contracted locus.** The divisor D_{10} misses Y: its dual Newton face is a vertex whose coefficient is nonzero. Define integral divisors $H_1, \dots, H_6$ on Y by the columns of

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 4 & 5 & 5 \\ 0 & 1 & 2 & 4 & 4 & 5 \\ 0 & -1 & -1 & -3 & -3 & -3 \\ 0 & -1 & -1 & -2 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix} \tag{27}$$

in the ordered divisor list $(D_2, D_5, D_6, D_7, D_8, D_9)|_Y$. Each has a nef lift to V_Σ: use the displayed coefficients, zero on the other rays except 10, and add coefficient zero on D_{10} for H_1, H_2, H_3, or coefficient one for H_4, H_5, H_6. Solving the support-function equalities as above gives nonnegative values of the support-function inequalities for every cone and every ray. Since $D_{10}|_Y = 0$, all six H_i are nef on Y. Toric linear equivalence identifies H_4 with $D_0|_Y$.

The support function of D_0 defines a semiample contraction. Its positive-dimensional orbit fibers have the following minimal cones in their closures:

$$(8), \quad (9), \quad (10), \quad (2,4), \quad (6,7). \tag{28}$$

Here is a direct finite way to obtain the list. Form $P_0 = \{m : \langle m, v_i\rangle + \delta_{i0} \geq 0\}$ and, for each cone σ, impose its equalities to get the face $F_\sigma \subset P_0$. The orbit fiber is positive-dimensional exactly when $4 - \dim F_\sigma > \dim \sigma$; taking minimal cones in this set gives (28). Consequently every curve of H_4-degree zero on Y lies in

$$E \cup F \cup C_1 \cup C_2, \qquad E = D_9|_Y, \quad F = D_8|_Y, \tag{29}$$

where C_1, C_2 are the intersections with the orbit closures of $67, 24$. Their restricted Newton polytopes are the primitive segments joining $(0,1,-1,-1)$ to $(-1,2,-1,1)$ and $(-1,2,1,-1)$, respectively. Each binomial cuts the dense torus in a translate of $\mathbb{C}^*$. Basepoint-freeness excludes boundary components for a general equation, and nondegeneracy makes the closures smooth; hence both curves are $\mathbb{P}^1$. They miss both surfaces: there are no cones containing 678 or 248, and the Newton faces at $679, 249$ are singletons. There is no cone containing 2467, so the two curves are disjoint.

Lemma A.1. *The surfaces are $E \simeq \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$ and $F \simeq \mathbb{F}_1$. Their intersection is the negative section s on F, and the strict transform of the line through p_1, p_2 on E.*

Proof. The anticanonical face at ray 9 is the primitive edge from $(0,1,-1,-1)$ to $(1,-1,-1,-1)$. Its binomial equation fixes a nonzero torus parameter. Rays $2, 7$ are excluded on E because their intersections have singleton Newton faces; ray 10 is absent. The remaining rays $3,4,5,6,8$ project to

$$(1,0), \quad (0,1), \quad (-1,0), \quad (0,-1), \quad (1,1),$$

with cones $38, 84, 45, 56, 63$. This is the fan of the plane blown up at two distinct torus fixed points.

For F, the star of ray 8 away from D_{10} has cones $137, 147, 379, 479$. Quotient by the fiber ray 7 gives the base rays $(1,0), (0,1), (-1,-1), (-1,0)$, the fan of $\mathbb{F}_1$. The Newton vertex $(1,-1,-1,-1)$ has exponents $(0,0,0,1,0,0)$ on rays $1,3,4,7,9,10$; all other vertices have fiber exponent zero. The equation is monic linear in that fiber coordinate and its zero locus is a graph over $\mathbb{F}_1$. The common divisor in these two surface fans is the stated curve. □

Let e_1, e_2 denote the exceptional curves on E, and f a ruling fiber of F. Restriction of (27) to the surface fans and to the two orbit curves gives

(30)

	H_1	H_2	H_3	H_4	H_5	H_6
s	0	1	0	0	0	0
f	0	0	1	0	0	0
e_1	1	0	0	0	1	1
e_2	0	0	0	0	0	1
C_1	0	0	0	0	1	0
C_2	1	0	0	0	0	0.

These pairings can be computed with the usual toric divisor relations: the linear relations are $\sum_i \langle m, v_i\rangle D_i = 0$, and a divisor product vanishes when its rays span no cone. The monic and binomial equations above identify the actual surfaces on which the restrictions are taken.

Proposition A.2. *For every positive integer a, the class $\alpha = as + (a+1)f$ is primitive in $H_2(Y;\mathbb{Z})$. Every effective cycle of class $m\alpha$, $m \geq 1$, is supported on F. The same holds for every effective additive summand of α.*

Proof. By (30), $(H_3 - H_2)\cdot\alpha = (a+1) - a = 1$, which proves primitivity. A cycle in class $m\alpha$ has zero H_1, H_4, H_5, H_6 degrees. Nefness forces the same for each component. The H_4 contraction confines it to (29); H_1, H_5 exclude C_1, C_2. The effective cone of E is generated by e_1, e_2, s. The H_1, H_6 degrees exclude any e_1 or e_2 coefficient, so a remaining component in E is supported on $s = E \cap F$. All components therefore lie in F.

Every effective summand of α has the same four vanishing degrees, so the argument applies to it as well. On F, the H_2, H_3 pairings recover the coefficients of s, f. Hence a decomposition of α into effective classes corresponds to an actual effective decomposition of $as + (a+1)f$ on F. □

Appendix B. The scalar handle in the partition basis

We use the multiplication operator of [BP08, Appendix, equations (29) and (31)], the cap of its Lemma 6.3, and the starred normalization of its Section 6.4.1. Put $r = e^{iu}$. Initially keep general nonzero weights t_1, t_2, and put

$$k = t_1 t_2, \qquad a = -(t_1+t_2)\frac{r-1}{r+1}, \qquad b = -3(t_1+t_2)\frac{r^2-1}{r^2+r+1}.$$

In the partition basis $1 = e_{111}$, $x = e_{21}$, $y = e_3$, multiplication by x is

$$M = \begin{pmatrix} 0 & -3k & 0 \\ 1 & a & -2k \\ 0 & 3 & b \end{pmatrix}. \tag{31}$$

Since $M1 = x$ and $M^2 1 = -3k + ax + 3y$, the unit is cyclic. Consequently

$$\begin{aligned} L_1 = I, \qquad L_x = M, \qquad L_y = \frac{M^2 - aM + 3kI}{3}, \\ \mathcal{A} = \mathbb{Q}(t_1, t_2, r)[x] / \big((x-b)(x^2 - ax + 3k) + 6kx\big). \end{aligned} \tag{32}$$

Indeed $My = -2kx + by$ gives the stated relation. Multiplication also gives

$$y^2 = 2k^2 - \frac{2kb}{3}x + \frac{b^2 - ab - 3k}{3}y.$$

The starred Frobenius pairing in this basis is

$$D = \operatorname{diag}\left(-\frac{1}{6k^3}, \frac{1}{2k^2}, -\frac{1}{3k}\right). \tag{33}$$

In general its partition entry is $1/(\mathfrak{z}(\lambda)(-k)^{\ell(\lambda)})$, where $\mathfrak{z}(\lambda) = \prod_j j^{m_j} m_j!$ and m_j is the multiplicity of j in λ. The Appendix Fock pairing differs by $(-1)^d$ in degree d; its multiplication-operator convention has the same sign. Thus (31) is the multiplication matrix in the starred normalization, and the handle operator is

$$G = -6k^3 I + 2k^2 M^2 - 3k L_y^2. \tag{34}$$

This is multiplication by the Euler element, the sum of each basis element times its dual. In the quotient algebra it equals

$$-18k^3 + 2k^2(a+b)x + (9k^2 + kab - kb^2)y.$$

Raising the level-$(-1,0)$ cap covector with (33) and removing its half-integral phase gives the vector

$$\widetilde{\eta}_\lambda = (-1)^d t_1^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (1 - r^{-\lambda_j})^{-1}, \qquad d = 3. \tag{35}$$

Let $\widetilde{A} = \sum_\lambda \widetilde{\eta}_\lambda L_\lambda$ and let $\widetilde{\widetilde{A}}$ interchange t_1, t_2. If A^* is the unscaled starred level operator, then $\widetilde{A} = r^{d/2} A^*$. The balanced-level operator is therefore

$$B = r^3 G \widetilde{A}^{-1} \widetilde{\widetilde{A}}^{-1}, \qquad [q^3] F_g = \operatorname{tr}(B^{g-1}) \quad (t_1 = t_2 = 1). \tag{36}$$

All operators commute because they act by multiplication. The closed starred normalization has exponent $d(2 - 2g + (g-1) + (g-1)) = 0$, so no further phase multiplies the ordinary GW coefficient in (36).

Proposition B.1. *At scalar weights the characteristic polynomial of B is*

$$\begin{aligned}\det(TI-B) &= T^3 - w^3(w^2+50w+81)T^2 \\ &\quad + \frac{4}{3}w^6(w-6)(w^2+54w-243)T \\ &\quad - \frac{4}{3}w^9(w-4)(w-3)^2(w^2+54w-243).\end{aligned} \tag{37}$$

In particular,

$$\begin{aligned}[q^3]F_2 &= 81w^3+50w^4+w^5, \\ [q^3]F_3 &= 2673w^6+9612w^7+2534w^8+\frac{292}{3}w^9+w^{10}.\end{aligned} \tag{38}$$

Proof. Substitute $t_1=t_2=1$ into (31)–(36) and reduce using $r^2-(2-w)r+1=0$. The three elementary symmetric functions of B are respectively

$$w^3(w^2+50w+81), \qquad \frac{4}{3}w^6(w-6)(w^2+54w-243),$$
$$\frac{4}{3}w^9(w-4)(w-3)^2(w^2+54w-243).$$

These give (37); the trace and the identity $\operatorname{tr}(B^2)=(\operatorname{tr}B)^2-2\sigma_2(B)$ give (38). The calculation can also be made entirely in (32): invert $\widetilde{\eta}^2$, multiply by r^3 times the Euler element, and take the resultant of its characteristic equation with the monic cubic defining $\mathcal{A}$. This uses the same printed algebraic data and avoids a matrix inverse. □

At antidiagonal weights the same construction has eigenvalues U,U,V, where

$$U=w^3(4-w)(3-w)^2, \qquad V=w^3(3-w)^2. \tag{39}$$

These are the three hook-length contributions of [BP08, Corollary 7.3]. The leading scalar coefficient $2673=6^4+3^4+6^4$ also agrees with the unramified degree-three cover count for a genus-three base.

B.1. **A comparison in degree five.** For a genus-two core, the first prime beyond the cubic gives

$$\begin{aligned}D_{2,5}(w) = w^6\big(&444088+142232w+340864w^2-156542w^3+75396w^4 \\ &-23737w^5+5054w^6-696w^7+56w^8-2w^9\big).\end{aligned}$$

Here the subscripts denote genus and covering degree as in (1). For example, at 5α in output genus seven the scalar exponent is the elementary exponent minus 444088 times the sum of the genus-two elementary exponents at its effective fifth roots. Larger core genera are excluded by Theorem 2.1.

For the calculation let Z_a denote the disconnected degree-a residue and L_a the coefficient of its logarithm. Apply $aL_a = aZ_a - \sum_{b=1}^{a-1} bL_bZ_{a-b}$ at each pair of weights. For a genus-two core Z_a is the trace of the balanced handle of [BP08, §§5–8]. In degrees 1 through 5 the transposition operator is cyclic on the unit over $\mathbb{Q}(r)$, $r=\exp(iu)$, at both weight specializations. Using the powers of that operator on the unit as basis, express each partition class as a polynomial in the operator. Multiply the handle and the inverse level operators modulo its characteristic polynomial and take the trace. At antidiagonal weights one may equivalently sum $\prod_{b\in\lambda}(2-r^{\ell(b)}-r^{-\ell(b)})$ over partitions, with $\ell(b)$ the hook length. Subtract the two values of L_5 and substitute $w=2-r-r^{-1}$. These are rational function identities over $\mathbb{Q}$.

Appendix C. Individual coefficients on a split genus-three curve

Let L be general of degree two on a smooth curve C of genus three, and let $M = K_C L^{-1}$. For this appendix $g = m = 3$ unless otherwise specified. The split bundle is useful for distinguishing the individual weight coefficients. The eight lines in Theorem 9.1 instead occur on a general stable bundle with the same determinant.

C.1. **A proper incidence space for a weight coefficient.** For arbitrary g, m put $a = k - 1 - m(g-1)$ and let $P_A = R\pi_* f^* A$ on the proper stack of connected genus-k maps to C of degree m. Localization gives

$$N_{k,m,j} = \int_{[\overline{\mathcal{M}}_k(C,m)]^{\mathrm{vir}}} c_{a-j}(-P_L) c_{a+j}(-P_M).$$

For $1 \le j \le a$ represent P_M as $[U \to V]$. A global presentation is obtained by the dual evaluation into a sufficiently positive relatively generated line on the universal curve tensored with its relative section-space dual: tensor by f^*M, take the quotient and push forward. Source flatness, relative Serre vanishing and the sequence give two locally free terms. On the projectivization of lines $\mathbb{P}(U)$ the tautological map to V cuts out the proper stack Y of sections up to nonzero scalar. Let α_Y be its refined zero-section class on the pullback of the stable-map virtual class. With $\xi = c_1(\mathcal{O}(1))$ the projective-bundle formula gives

$$N_{k,m,j} = \int_{\alpha_Y} \xi^{j-1} c_{a-j}(-P_L).$$

Indeed the pushforward of $\xi^{j-1} c_{\mathrm{rk}V}(V(1))$ is $c_{a+j}(V-U)$, since $\mathrm{rk}V - \mathrm{rk}U = a$.

At a genus-three cubic the first possible source genus of mode $j > 0$ is $7 + j$ by Theorem 2.1. Here the fixed-genus integral is also the leading multiplicity-one coefficient. For $j = 1, \dots, 6$ these values are

$$3612, \quad 1228, \quad 506/3, \quad 21/2, \quad 8/27, \quad 1/324.$$

To recover them directly, at $z = 1, \dots, 7$ evaluate $\operatorname{tr} B_3(z)^2 - w^2 \operatorname{tr} B_2(z)^2 + w^6/3$ in the matrices of Appendix B and their degree-two analogues. A symmetric Laurent polynomial through exponent six is a polynomial of degree at most six in $z + z^{-1}$. Interpolation at these seven distinct values, followed by extraction at $w^{6+j} z^j$, gives the list. Alternatively interpolate the odd and even parts separately at pairs $\pm z$. On one normal axis the six coefficient weights have product $2^2 3^4 = 324$, so for $j = 3, 4, 5, 6$ the corresponding degrees before this division are $54648, 3402, 96, 1$. Mode five has no contribution through genus eleven.

In genus eight the degree 3612 is integral on an automorphism-free space of smooth ramified covers. For a general M no connected étale double or triple admits a section of its pullback. Indeed there are finitely many such covers, and their trace-free bundles are semistable of degree zero and rank at most two, being trivialized by a finite étale cover. After a general degree-two twist there are no sections by [Ray82, Proposition 1.6.2]. A nonzero height on a connected stable map must be nonzero on a nonconstant component. A degree-three component then has normalization genus at least eight, while a double plus a degree-one component has genus sum at least nine. Thus at genus eight the source is smooth. A cyclic triple with total ramification two would have one branch value, contrary to the abelian monodromy relation. The transitive monodromy is S_3 and its centralizer is trivial. The deformation complex of a finite smooth-source map to C is its torsion ramification cokernel, so the Hurwitz chart is smooth. The degree comes from an integral refined Euler class and permits excess.

At genus ten it matters whether one extracts u^{18} or w^9. For positive weights one and three the change of variable gives the following decomposition. Each entry is a contribution to $[w^9]$.

source genus	$j=1$	$j=3$	4 times the sum
8	$3612(161/720)$	0	$48461/15$
9	$-2391(2/3)$	0	-6376
10	$60659/60$	$506/3$	$23593/5$
total	$674/3$	$506/3$	$4720/3$

Indeed $u = 2\arcsin(\sqrt{w}/2)$ gives the indicated source multipliers; the raw numbers are obtained from the same handle polynomial by $w = 2 - 2\cos u$. Opposite weights exchange under inversion, explaining the factor four. The genus-ten row is integral at three after summing the modes. The pole at three in the final column enters through the genus-eight term.

The cancellation of denominators at fixed genus ten is also visible in the restrictions of the two integrands, although it does not imply descent of their sum to the coarse incidence space. Consider an order-three automorphism of a point of Y, and put $t = c_1(\chi)$ for a generating character. If it acts nontrivially on the section line, then in $A^2(B\mu_3) = (\mathbb{Z}/3)t^2$ one has

$$c_2(-P_L) = -t^2, \qquad \xi^2 = t^2.$$

Indeed invariance of the fiber multiset under cubic dilation forces a cyclic spectral triple, with normalization of genus nine or ten (as classified below). In genus ten the two summands of $-R\Gamma(f^*L)$ of nontrivial character give $\chi + 2\chi^2$ up to inversion: the covering eigenlines have degrees $-1, -2$. In genus nine their degrees are both -1, since for local exponents one and two the square eigensheaf is normalized by division by a local parameter at the second point. Their contributions are then $\chi + \chi^2$, together with one additional character from the nodal or contracted part. In fact for the normalization of all positive components the correction to $-R\Gamma$ is the sum of the sections of the finite sheaves $(f_{\mathrm{norm},*}\mathcal{O})/f_*\mathcal{O}$ and $R^1 f_*\mathcal{O}$, tensored with L. In either case the pairwise products of the character exponents sum to -1 modulo three.

When the automorphism fixes the section its ξ^2 restricts to zero, and it fixes the positive normalizations pointwise: a nonzero section parametrizes its spectral image birationally in prime degree, and the only possible additional positive component has degree one. For c positive components of genera h_i the finite-sheaf correction has dimension $10 - \sum h_i + c - 1$, with at least $c - 1$ trivial characters coming from independent constants. A nonzero second Chern restriction thus requires $\sum h_i \le 8$. The preceding genus bounds leave a single genus-eight component. Such a point exists since its virtual degree is 3612. Attach to it the contracted genus-two tail $y^2 = x^6 - 1$ at $x = 0$; the action $x \mapsto \zeta x$ fixes the attachment and has characters χ, χ^2 on the two differentials $dx/y, x\,dx/y$. It gives $c_2(-P_L) = 2t^2 \neq 0$ and $\xi^2 = 0$. This excludes descent of $c_2(-P_L) + \xi^2$ as a three-local operational Chow class. The restriction to a stabilizer does not by itself compute a stratum's virtual degree.

C.2. **The support and multiplicity of the third mode.** In genus ten two independent quadratic coefficient tests on the projective section have class $4\xi^2$. Their common zero on Y has spectral polynomial $y^3 = b$ with $0 \neq b \in H^0(M^3)$, up to scale. Indeed the trace vanishes and there are two independent coordinates on $H^0(M^2)$. A meromorphic cube root of b would be holomorphic, contrary to $H^0(M) = 0$. The sole nonconstant component is the normalization of the integral spectral cubic. Its genus is $7 + r$ with r the number of distinct zeros whose multiplicities are nonzero modulo three. Thus $r \le 3$; the case $r = 2$ gives an actual genus-ten map on attaching a contracted elliptic tail.

Put $A = M^3$. For every p, $h^0(A(-3p)) = 1$: a higher dimension requires $A = K_C(3p - q)$, a locus of dimension at most two. Two triple zeros would require $A \in 3W_2$, also excluded. The sections with a triple zero therefore form a curve in $|A|$ with normalization C, and the determinant of the order-two jet bundle gives its degree $3 \deg A + 3 \deg K_C = 30$. On this curve a genus drop occurs on divisors $3p + 2q + s$. The weighted Abel map has degree $3!(3 \cdot 2)^2 = 216$: the tangent columns scale holomorphic volume by $3 \cdot 2$, hence real volume by its square, relative to the ordinary degree-six Abel map. General A avoids branch and collision loci. The other possible divisors consist of three double zeros, one for each of the 64 square roots B of A, all with a unique reduced effective divisor. Their cubic images have three cusps and smooth normalizations of genus ten.

Each isolated three-cusp polynomial contributes $2/3$ to the integral of ξ^2. To see the virtual multiplicity, take $B^2 = M^3$ and a section s with its three simple zeros p_i. The smooth normalization has $t^3 = s$ in B/M and $y = t^2$. Fix the amplitude of $b_0 = s^2$ by a linear form $\lambda(b_0) = 1$. This removes the cyclic stabilizer on a framing cover of degree three. Vary both quadratic tests by $q = \epsilon^2 Q$, with $Q \in H^0(M^2)$ nonzero at the p_i; M^2 is generated there for a general M. In coordinates at these points the equations are $F_i = y^3 + \epsilon^2 Q_i(x)y - x^2 - \beta_i(x)$ where $b = b_0 + \beta$ and $\beta \in \ker \lambda$. Evaluation of $\ker \lambda$ at the three points is invertible. Thus $F_i = F_{i,x} = 0$ determine β and the three x_i as functions of the three y_i. The residual equations after $y_i = \epsilon z_i$ are $3z_i^2 + Q_i(x_i) = 0$. There are eight simple solutions for small ϵ, and no others since the z_i are bounded. Each singular point is nodal by its Hessian. The parametric Morse lemma and cubic Weierstrass preparation bound the local delta invariant by one. Total delta three is necessary, so all three nodes must occur, and each resulting normalization is the unique lift of its curve.

At these immersed maps an infinitesimal solution fixing the quadratic tests has $\dot\beta$ zero at all three nodes by differentiating the image equation, hence identically zero. The normal sheaf of an immersion is a line, so a normal section vanishing on the smooth image is zero. These are eight reduced intersections on the framing cover. Near their limit the Hurwitz base is smooth of dimension six, and in $\mathbb{P}(U)$ the rank of the defining section plus the two quadratic tests equals the ambient dimension. Conservation of the isolated refined intersection degree gives length eight also at the limit. Divide first by three for the framing, then by four for the class of the tests. This gives $2/3$. Consequently the moving degree-thirty family together with its boundary contributes $506/3 - 64(2/3) = 126$. This computation leaves the separate degrees over the smooth part and the 216 limiting values undetermined.

C.3. **Axis support after the lower coefficient tests.** Consider the lower elementary symmetric coefficients of each projection in a balanced split bundle. Impose vanishing of all orders less than m on both axes. The pure-axis families are the vector spaces of sections of L^m and M^m, parametrizing the monic spectral divisors. For general L, the following table gives exactly the degrees in which these two spaces, with their origins identified, exhaust the cut:

g	covering degrees with only axis cycles
2	$1, 2, 3, 4, 5, 6, 7, 9$
3	$1, 2, 3, 4, 5, 6$
$4, 5$	$1, 2, 3$
$g \geq 6$	$1, 2.$

Here generality is for each fixed m. The subsequent constructions of mixed cycles work for every L in the asserted existence ranges. This criterion concerns the separate projected

equations; a fixed finite subgroup action specifies how the two root sets must be matched. Put $K = K_C$ and $M = K/L$ in the proof.

C.3.1. *Reduction to a connected root cover.* Away from the zero divisors of b and c each projection has m distinct simple values. A fiber of a mixed cycle therefore defines a bijection of the two root torsors, compatible with continuation along the punctured curve. Choose base roots and label the torsors by $\mathbb{Z}/m$. Each monodromy acts by a translation on either set. The two translation representations have the same kernel, because a permutation of either set determines the permutation of the other via the bijection. Thus the common image is cyclic of order $d \mid m$. Its orbits on each set have size d. A single orbit of the matching gives a connected component over the punctured curve, and its closure gives an irreducible degree-d cycle. Both of its projected d-th powers are single valued, extending holomorphically over the missing points by boundedness. The two projected lower coefficients vanish on that degree-d cycle. On this orbit choose the generator acting on the first coordinate by a primitive d-th root ζ. It acts on the other by ζ^k for a unit k mod d. Equivalently

$$v = \phi u^k, \qquad \phi \in H^0_{\mathrm{mer}}(C, M/L^k)$$

on this component. Here k is any positive representative of the slope. The relation follows by invariance of v/u^k under the cyclic action on the function field. The case $d = 1$ requires a nonzero global section of L, excluded for general L.

Conversely every mixed cut cycle of degree d gives one of degree m whenever $d \mid m$. Take the sum of its rotations under $(u, v) \mapsto (\zeta_m^j u, \zeta_m^j v)$ for $0 \le j < m/d$. At a general fiber the translated sets partition the roots of $u^m = b_d^{m/d}$ and of $v^m = c_d^{m/d}$. This argument needs only the two projected polynomials. In prime degree, disconnectedness of a mixed cycle would force $d = 1$, which explains the irreducibility assertion.

For genus three the slopes 1 and -1 are excluded for generic L. For slope 1 the curve lies in a saturated line subbundle $J \subset L \oplus M$ mapping nontrivially to both factors. The nonzero invariant d-th power of its fiber coordinate gives a section of J^d. A positive $\deg J$ would force $L = M$ or $L/M \in C - C$, both excluded for general L. Hence J is torsion of order dividing d and L belongs to one of finitely many translates of W_2, also excluded. For slope -1 the invariant uv is a nonzero canonical differential t. Thus $\operatorname{div}(b_d)$ is a degree-$2d$ effective subdivisor of $d \operatorname{div}(t)$. Its possible line bundles range over a closed set of dimension at most two, by the finite-subdivisor incidence over $\mathbb{P}H^0(K)$, whereas $L \mapsto L^d$ is finite surjective onto a Picard variety of dimension three.

All unit slopes for $d \mid m$, with $m = 1, 2, 3, 4, 6$, are covered by these cases. When either b or c vanishes the cycle lies on the corresponding axis and is determined by the other monic polynomial, including its multiplicities.

C.3.2. *Exclusion of the remaining slopes at degree five.* Interchanging the two coordinates replaces k by its inverse, so the only additional slope to consider is $k = 2$. Write $P = (\phi)_\infty$ and $Q = (\phi)_0$ with disjoint support. Then $\deg P - \deg Q = 2$ and $\mathcal{O}(P - Q) = L^2/M$. Since the latter line of degree two is general as L varies, $\deg P \ge 3$. Holomorphicity of $v = \phi u^2$ implies

$$(b) \;\ge\; \sum_x \left\lceil \tfrac{5}{2} \operatorname{ord}_x P \right\rceil x.$$

The left side has degree ten, giving $\deg P \le 4$. If $\deg P = 4$, equality forces $P = 2D$ and $(b) = 5D$ with $\deg D = 2$. Thus L is a torsion translate of an effective degree-two line, excluded for general L.

If $\deg P = 3$ then $Q = q$ is a point. If P is reduced, $(b) = 3P + x$ for some point x. Using $\mathcal{O}(P) = (L^2/M)(q)$ gives $M^3/L = \mathcal{O}(3q + x)$. The target line of degree four is general because $L \mapsto K^3/L^4$ is finite surjective. The locus of divisors $3q + x$ has dimension at most two, so this is impossible for general L.

Otherwise $P = 2p + r$, allowing $p = r$. The inequality gives $(b) = 5p + 3r + S$ with an effective divisor $S = s_1 + s_2$. This also holds for $p = r$, where the minimum required vanishing is eight. Eliminating L from the two identities

$$L^3/K = \mathcal{O}(2p + r - q), \qquad L^5 = \mathcal{O}(5p + 3r + s_1 + s_2)$$

gives

$$K^5 = \mathcal{O}(5p + 4r + 5q + 3s_1 + 3s_2).$$

The fiber of this weighted Abel map has dimension at most two. Stratify by the coincidences of its five marked points. At n distinct points the differential has rank equal to the rank of their canonical evaluations since all combined weights are positive. Five distinct points impose rank three, since no canonical differential has five zeros. Four distinct points impose rank at least two, since a fiber of the genus-three canonical morphism contains at most two distinct points. Indeed in the hyperelliptic case that map is two-to-one onto a conic and otherwise it is the plane quartic embedding. For $n \leq 3$ the rank is at least one by base-point freeness. On each stratum the tangent dimension of the fiber is consequently at most two. The points determine L^5 and hence finitely many L. Properness of the weighted Abel incidence shows that its image in the Picard variety of L lies in a closed subset of dimension at most two. This excludes the remaining slope for genus three.

C.3.3. *Construction of mixed cycles.* Fix a positive integer k coprime to m and a nonzero meromorphic section ϕ of M/L^k. If a nonzero section $b \in H^0(L^m)$ satisfies

$$k(b) \;\geq\; m(\phi)_\infty,$$

the graph $v = \phi u^k$ on the normalization of the spectral cycle $u^m = b$ gives a compact mixed cycle in $\mathrm{Tot}(L \oplus M)$. Indeed the coordinate v extends holomorphically at every point of the normalization by the valuation inequality. Projection onto the u coordinate identifies the generic image cycle with the original spectral cycle. Since k is a unit, over a general fiber the m values of v are exactly the distinct roots of $v^m = \phi^m b^k$. This proves all the cut equations even when the normalization is disconnected.

For odd $m \geq 9$, take $k = 2$, a point $q \in C$, and $P \in |(L^2/M)(q)|$. The degree of this last line is three, so such a divisor exists. The meromorphic section with divisor $q - P$ has pole divisor bounded by P. Let $a = (m + 1)/2$. Since $\deg L^m(-aP) = (m - 3)/2 \geq 3$, Riemann–Roch gives a nonzero b in this space. It satisfies the required inequality.

For $m = 7$ use the same construction with $a = 4$. The line in which to find b after imposing $4P$ is

$$L^7(-4P) = (M^4/L)(-4q).$$

The degree-six bundle M^4/L has four independent sections. Its complete linear series has a point q with a section vanishing to order at least four. For instance the Wronskian is a nonzero section of $(M^4/L)^4 \otimes K^6$ in characteristic zero, of positive degree. Its zeros are precisely the points where vanishing orders differ from $0, 1, 2, 3$, forcing some vanishing order at least four. Thus this choice of q supplies b. Existence of P still follows from its degree.

For $m = 8, 10$, use $k = 3$, $P \in |L^3/M|$ and $\mathrm{div}(\phi) = -P$. The degree of P is four. The line $L^m(-\lceil m/3 \rceil P)$ has degree four for both values of m and therefore a nonzero section. For $m = 12$, choose an effective degree-four divisor D with $\mathcal{O}(2D) = L^5/M$: square roots exist

on the complex Picard variety and every degree-four line on C is effective. With $k = 5$ and $\operatorname{div}(\phi) = -2D$, any nonzero $b \in H^0(L^{12}(-5D))$ satisfies $5(b) \geq 24D$, since $25D \geq 24D$. The residual degree is four.

Every even integer at least eight either has an odd divisor at least seven or is divisible by $8, 10$ or 12. For the last assertion its odd part, when below seven, is $1, 3$ or 5. The constructions and the degree-multiplication observation finish the theorem.

C.3.4. *Dependence on the genus.* The genus-three threshold depends on the degree and genus together. For example mixed solutions of degree five already exist for every L if $g \geq 4$. With $k = 2$ choose $P \in |(L^2/M)(q)|$ as above, now of degree g. The line $L^5(-3P)$ identifies with $(M^3/L)(-3q)$ of degree $2g - 5$. For $g \geq 5$ this is effective by degree. For $g = 4$ the degree-six line M^3/L has at least three sections by Riemann–Roch. If it has more than three sections every q works. Otherwise its three-dimensional complete series has positive Wronskian degree, and a ramification point supplies the order-three zero. In either case the valuation condition constructs the cycle. For a general L its normalization cover is connected since five is prime.

Genus two. For a genus-two curve and general $L \in \operatorname{Pic}^1 C$, the degrees with axis-only support are exactly $1, 2, 3, 4, 5, 6, 7, 9$. In particular the first prime admitting mixed cycles here is eleven.

For slope 1, a saturated common line of positive degree would force $L = M$. A common degree-zero line carrying a nonzero d-th power is a d-torsion translate of the trivial line, and would put L in a finite union of translates of the effective degree-one locus. For slope -1 the canonical subdivisor argument applies with parameter dimension one. For the remaining slopes at $m = 5, 7, 9$ the connected orbit reduction still applies. Lower-degree divisors of these m have only the excluded slopes. In each case use the inverse slope as well, by interchanging the two normal lines.

For slope 2 in any odd degree, write P for the pole divisor and Q for the zero divisor of ϕ. Here $\deg P - \deg Q = 1$, and degree one for P would force L^2/M effective, excluded generically. Thus $\deg P \geq 2$. The valuation bound $2(b) \geq mP$ forces $\deg P = 2$ and equality throughout. Since m is odd, $P = 2p$ and $(b) = mp$, excluded up to m-torsion. This handles both remaining unit slopes for $m = 5$.

At $m = 7$ one must also consider slope 3 and its inverse 5. Now $\deg P - \deg Q = 2$, and $3(b) \geq 7P$ bounds $\deg P \leq 3$. Degree three forces $P = 3p$ and $(b) = 7p$ by equality and coprimality. At degree two $Q = 0$ and $\mathcal{O}(P) = L^3/M$ is general of degree two. Its only effective representative is generically reduced, since divisors $2p$ vary in dimension one. Thus $(b) \geq 3P$, forcing the degree-one line $M^3/L^2 = L^7(-3P)$ to be effective, again excluded for general L.

At $m = 9$ the additional inverse pair comprises slopes $4, 7$. For $k = 4$ we have $\deg P - \deg Q = 3$ and $\deg P \leq 4$. Degree four forces $P = 4p$ and $(b) = 9p$. At degree three, $Q = 0$ and $\mathcal{O}(P) = L^4/M$. If P is reduced, $(b) = 3P$ so the degree-zero line M^3/L^3 must be trivial. If $P = 3p$ then L^4/M belongs to a locus of dimension one. If $P = 2p + r$ with $p \neq r$, write $(b) = 5p + 3r + x$. Elimination gives

$$K^9 = \mathcal{O}(7p + 6r + 5x).$$

The corresponding fiber of the weighted Abel map has dimension at most one. Three distinct points impose canonical evaluation rank two since a genus-two differential has degree two, and two or fewer distinct points impose rank at least one. These points determine L^9, so the excluded L again form a subset of dimension at most one. This proves the axis-only assertions.

For the complementary degrees the valuation construction uses that every degree-two line on C is effective. For $m \geq 8$ with $3 \nmid m$ take $k = 3$, $\operatorname{div}(\phi) = -P$ with $P \in |L^3/M|$ of degree two. The residual degree $m - 2\lceil m/3 \rceil$ is at least two (including 8 and 10). For odd $m \geq 15$ divisible by three, take $k = 4$, $\operatorname{div}(\phi) = -P$, $P \in |L^4/M|$ of degree three. The residual degree $m - 3\lceil m/4 \rceil$ is at least two: the first cases are $15, 21$, and it holds for all $m \geq 17$. For $m = 12, 18$ use $k = 5$ and $\operatorname{div}(\phi) = -2D$ where $\mathcal{O}(2D) = L^5/M$ and $\deg D = 2$. Imposing $(b) \geq \lceil 2m/5 \rceil D$ leaves degree two in both cases. An even multiple of three is divisible by twelve or is twice an odd multiple of three. In the latter case the odd half is $3, 9$ or at least 15. Degree multiplication completes the classification.

All higher genera. For $g = 4, 5$ the only axis-only degrees are $1, 2, 3$. For every $g \geq 6$ they are $1, 2$. Each exclusion holds for general L, while each existence statement below holds for every L.

Write $a = g - 1$. To construct a slope-one solution, choose $J \in \operatorname{Pic}^j C$ with J^m, L/J, and M/J all effective. Nonzero sections of the last two bundles map the line J into N. The map is injective on the generic fiber. The image of the compact spectral cycle $w^m = b_J$, for a nonzero $b_J \in H^0(J^m)$, therefore has both projected m-th powers nonzero and satisfies the lower coefficient cuts. Simultaneous vanishing of the two line maps at an exceptional point of the base leaves the proper image of the spectral curve well defined.

In $\operatorname{Pic}^j C$ the three effectivity loci are an isogeny pullback of W_{mj} and two translated inversions of W_{a-j}. By the Poincaré formula $[W_e] = \theta^{g-e}/(g-e)!$ for $0 \leq e < g$ and $W_e = \operatorname{Pic}^e C$ for $e \geq g$. Their classes thus have nonzero product whenever

$$\max(g - mj, 0) + 2(j + 1) \leq g, \qquad 0 \leq j \leq a.$$

Indeed the isogeny pullback multiplies the polarization class by m^2, and every power of the principal polarization of exponent at most g is nonzero. The nonzero product has support in the triple intersection, which is therefore nonempty, even for the linked translates specified by L and K/L. With $j = 1$ the inequality holds for $g \geq 4$, $m \geq 4$. With $j = 2$ it holds for $g \geq 6$, $m = 3$. [Deb95, Introduction]

In degree two the only unit slope is -1 mod 2 and the canonical subdivisor argument applies for every g. For $m = 3$, $g = 4, 5$, a remaining mixed cycle needs slope 1 and a saturated common line J as before. If $j = \deg J > 0$, the ratio L/M lies in $W_{a-j} - W_{a-j}$. For general L this requires $2(a - j) \geq g$, hence $j = 1$ in these genera. Degree zero is excluded by the finite torsion translates. The possible J with J^3 effective range over a locus of dimension at most three. For each fixed J the sum of the divisors of the two nonzero maps to L, M belongs to $|K/J^2|$, of dimension $g - 4 + h^0(J^2)$. Each divisor in this system has finitely many bipartitions of the prescribed degrees and each choice determines L. Where $h^0(J^2) = 0$ the dimension of this incidence is at most $g - 1$. Where $h^0(J^2) = 1$ the possible J lie in the finite pullback of W_2, restoring the same bound. If $h^0(J^2) = 2$ then C is hyperelliptic and J^2 is its unique degree-two pencil, so J ranges over a finite set. No higher value occurs. These constructible strata therefore miss general L and complete the classification.

For a given order $m \geq 2$, the antidiagonal action of μ_m on a nonzero fiber point is free. An invariant multiset of length m meeting the complement of the origin is a single orbit. A mixed orbit would have slope -1 and is excluded for general L by the canonical-divisor argument above. Thus in degree m the finite subgroup has only the two pure axes for general L, including degrees where the lower projected equations alone admit mixed curves.

In the genus-three cubic the intermediate quadratic cut is consequently the union $E_+ \cup_B E_-$ of two vector bundles of rank four over $B = \mathbb{P}^i \times \mathbb{P}^{12-i}$. On the first branch the product of the eight cubic Thom classes restricts, with support in the zero section, to $(3y)^4 \operatorname{Th}(E_+)$,

and on the second to $(3x)^4\,\mathrm{Th}(E_-)$, where x, y are the hyperplanes. The Thom class on the moving axis comes from its own fiber coordinates; the other four tests vanish identically on that axis. The two restrictions divided by 3^4 agree on B, with value $3^4x^4y^4$. Mayer–Vietoris with support in B identifies the degree-sixteen classes on the union with the kernel of the difference of restrictions (the preceding $H^{15}(B)$ is zero). Thus their common class on the union is divisible by 3^4. Cup product with the four quadratic Thom classes, supported on the union, maps this divisibility into the cohomology of the ambient cycle space with support in B. Formally this is the module action on $i^!\mathbb{Z}$ for the closed inclusion i of the union, followed by composition of supports. The original denominator 2^43^8 consequently bounds the 3-part of every weight coefficient by 3^4 after cancellation. The outer coefficient $x_{13,1}/324$ in Lemma 5.2 attains this bound.

C.4. **The genus of a mixed integral cubic.** On the same genus-three base a stronger statement about irreducible curves holds even before imposing any coefficient tests. For general L, every cubic projecting nontrivially to both L and M has normalization genus at most nine.

Let F denote its smooth normalization, $h = g(F)$ and $T = \ker(f_*\mathcal{O}_F \to \mathcal{O}_C)$, of degree $7-h$. The two trace-free coordinates give $N^* \to T$, $N = L \oplus M$. If its generic rank is one, the fiber values belong to a common line of degree at most zero, since $L \neq M$ and $L/M \notin C - C$. Its spectral divisor in this line has arithmetic genus at most seven by adjunction. Otherwise the quotient has length $11 - h$. Set $A = L^3/K_C$ and use a hat for the complementary line $\widehat{P} = K_C/P$. The four summands of $V = \mathrm{Sym}^3 N \otimes K_C^{-1} = A \oplus L \oplus M \oplus \widehat{A}$ are acyclic for general L. In genus eleven the cubic specifying the cover would be a section of V by the triple-cover correspondence [Mir85, Sections 3–4]. In genus ten its Tschirnhausen bundle is $W = \ker(N \to k(p))$ at one point. The binary cubic is now a section of $V(p)$ uniquely determined by its principal part, which is pure cube in the kernel direction. An axis direction would give a generic pure cube of zero discriminant, impossible for a separable field extension.

For an acyclic degree-two line P the residue-one Szegő kernel in $(P \boxtimes \widehat{P})(\Delta)$ exists uniquely by the diagonal sequence and Künneth [BB03]. In local complementary frames its scalar expansion is $(z-w)^{-1} + a_P(w) + b_P(w)(z-w) + \cdots$. Interchanging the two arguments gives $a_{\widehat{P}} = -a_P$, $b_{\widehat{P}} = b_P - a_P'$. In frames X of L and $Y = dz/X$ of M write a mixed direction as $X + qY$, $q \neq 0$. The four coefficients of the unique meromorphic binary cubic with principal part $(X+qY)^3$ are the kernels $s_A, 3qs_L, 3q^2s_M, q^3s_{\widehat{A}}$, with the common determinant frame suppressed. Set $\ell = X + qY$. The coefficient of ℓY^2 has constant term $3q^2(a_A - 3a_L)$; that of Y^3 has constant term $-2q^3(a_A - 3a_L)$ and first-order term $-q^3(a_A - 3a_L)'$. Since W is generated locally by $\ell, (z-w)Y$, membership in $\mathrm{Sym}^3 W \otimes K_C^{-1}(p)$ forces a multiple zero of $\eta = (a_A - 3a_L)dz$ at p. This is a global differential: the changes in $a_P dz$ are $-d\log\lambda_P + \frac{1}{2}d\log\phi'$ for a frame change λ_P and coordinate change ϕ, and $\lambda_A = \lambda_L^3/\phi'$.

For completeness the general differential so obtained has only simple zeros, including on a hyperelliptic base. On the smooth locus of $\Theta = W_2$, write $L_0 = \mathcal{O}(x+y)$ with unique generators v_0, w_0 of $L_0, \widehat{L}_0$. The product v_0w_0 annihilates the tangent of Θ by differentiating the Abel map. The Gauss map is generically finite since every differential has only finitely many degree-two subdivisors; its image is therefore dense. A general such product has four simple zeros. The map $P : L \mapsto L^3/K_C$ does not preserve Θ. If it did, multiplication by three on tangents would leave the Gauss map invariant and force $\deg(P|_\Theta) = 1$. The projection formula on the surface Θ gives instead 9^2 using the action on the polarization. We may thus choose L_0 with $P(L_0)$ acyclic. On an algebraic transverse arc through L_0, multiply the

normalized kernel by the smallest power t^e giving a regular specialization. Here $e > 0$, since $h^0(L_0(w)) = 1$ for general w and no section has the requisite simple pole. The nonzero limit has zero residue and is therefore a nonzero multiple of $v_0 \boxtimes w_0$. The kernel for $P(L_0)$ stays regular by acyclicity. On taking finite parts along the diagonal $t^e\eta$ consequently tends to a nonzero multiple of $v_0 w_0$. Algebraicity of the kernels on the acyclic locus and closedness of the multiple-zero locus prove the assertion for general L.

In particular a reduced integral cubic on an axis of normalization genus at least ten has no nearby mixed cubics. Indeed nearby cycles are integral of multiplicity one by properness of Chow addition, and a fixed normalization genus below ten in a convergent sequence would give a limiting stable map with a positive component birational to the central curve, which requires genus at least ten. The two axes are closed and meet only in the zero-section multiple.

Appendix D. Cubic pencils in genus five

Fix a smooth genus-five curve D. A genus-five cubic can move in a pencil even after quotienting by dilation. At a cusp its normalization has one fewer handle than the general member. Both genera occur in the universal comparison.

Theorem D.1. *Let N be a general stable bundle of rank two and determinant K_D. In* $\operatorname{Tot} N$ *the reduced space of degree-three cycles is the union of* 6480 *lines and* 32 *planes meeting only at* $3[D]$*. Dilation has weight three on their parameters. A general determinant-one quadratic shear along D isolates* $3[D]$ *in a threefold X. For its unit J_3,*

$$J_3 - x_{5,3} = 2160x_{16,1} + \tfrac{1280}{3}x_{18,1} - \tfrac{32}{9}x_{19,1}.$$

The difference of the connected fixed-target degree-three series in the same order is $2160w^{15} + (1280/3)w^{17} - (32/9)w^{18}$.

D.1. **The pencils and their limits.** By Section 8 all nonzero cubics are $y^3 = s$ on saturated sublines of degrees one and two. For degree one Corollary 8.2 gives 6480, with unique reduced divisors by the dimension argument in Section 9. Each smooth cubic has genus 16. There are 2^5 maximal degree-two sublines, reduced for general N [Gil11, Section 4]. One may fix the determinant here since tensoring by a line reaches every determinant of the same degree. At a degree-two subline S the projective extensions of $Q = K_D S^{-1}$ by S have dimension seven, so a proper closed locus of S has image of dimension at most eleven in the twelve-dimensional moduli of bundles. Thus S^3 is a general degree-six line. Its two sections form a base-point-free pencil: specialness requires $S^3 = K_D(-B_2)$, a base point requires $S^3 = K_D(p - B_3)$, and a triple zero or two double zeros require $S^3 = \mathcal{O}(3p + B_3)$ or $\mathcal{O}(2p + 2q + B_2)$, where B_i is effective of degree i. Each locus has dimension at most four. The pencil therefore has twenty simple branch values by Riemann–Hurwitz. A general cubic has six totally ramified points and genus 19. Each exceptional member has one cusp $x^2 = y^3$ and four other simple zeros, so its normalization has genus 18. Saturation and the spectral equation recover (S, s) from the cycle; two general evaluations bound all coefficients on each of the finitely many spaces. Hence the bijection from their union is proper and gives its topology.

For the embedded cubic F in $S \subset N$ the boundary in its normal sequence is multiplication by $3y^2 e$, where $e \in H^1(SQ^{-1})$ is the extension class. For a degree-two S the nonzero connecting block is $H^0(Q) \to H^1(S)$, invertible by acyclicity of N. Here $H^0(S^2) = 0$ and $H^1(S^3) = 0$. The finite flat algebra $\mathcal{O} \oplus S^{-1} \oplus S^{-2}$ also applies to the embedded cusp. Thus throughout the punctured plane

$$H^0(N_{F/\operatorname{Tot} N}) = H^0(S^3), \qquad H^1(N_{F/\operatorname{Tot} N}) = H^1(QS^{-2}) = H^0(S^3)^*.$$

For a degree-one line R the two additional connecting blocks are dual to the pairing $(a,b)\mapsto\langle e, sab\rangle$ on $H^0(K_DR^{-2})\times H^0(K_DR^{-3})$. Both spaces have dimension two. Two general points off $\operatorname{div} s$ give independent evaluations on both pencils, so a sum of their weighted evaluations makes this form nondegenerate. Bad forms occupy a proper subset of each extension fiber and the finite incidence excludes them for general N. The remaining tangent and obstruction each have dimension one.

Choose a determinant-one shear as in Section 10. Its quadratic term on S^2, projected to Q, gives a fixed functional in $H^0(S^3)^*$ over the entire punctured plane. On the normalization of a cusp over p the covering algebra has last eigenline $S^{-2}(p)$; its obstruction is the restriction of the functional to $H^0(S^3(-p))$. This is the line generated by the cusp section. Principal parts span $H^1(QS^{-2})$, so finitely many open conditions give a shear nonzero on all 640 cusp lines, all 32 plane obstruction spaces and all 6480 isolated-line obstructions. At a smooth embedded limit the relative Hilbert tangent space now has precisely the coefficient directions. The actual coefficient space has that dimension and hence fills its regular germ in the central target.

At a cuspidal limit the Hilbert tangent test alone would miss arcs whose generic curves have smaller arithmetic genus. Suppose such a noncentral arc exists. Curve selection in a projective Chow space and properness of stable maps extend the normalizations over a DVR, after finite base change. Write $t=cu^k+O(u^{k+1})$ for the target parameter. The sole positive special component is birational to the cusp; its normalization has the above eigenline decomposition. Since $H^0(K_DS^{-3}(p))=0$ by Riemann–Roch and $h^0(S^3(-p))=1$, and $H^0(K_DS^{-2})=0$ by 2-rigidity and Riemann–Roch, all sections of the pulled-back Q on that component come from D. Matching gives the same assertion on contracted pieces. Over $A=\mathbb{C}[[u]]/(u^k)$ all sections of g^*Q for the map $g:\mathcal{C}_A\to D_A$ still come from D_A: subtract the lift of each successive special-fiber section and use source flatness. The target is linear over A. The normal height in g^*N has zero trace since $H^0(D,N)=0$. This trace is defined even with contracted components: $Rg_*\mathcal{O}_{\mathcal{C}_A}$ is perfect by the regular graph into $\mathcal{C}_A\times_A D_A$ and proper flat perfect projection. Its rank on D_A is three. Hence a pulled-back quotient height has trace three times its base value, so the quotient height vanishes modulo u^k. Modulo u^{k+1} its first possible coefficient would have to trivialize the projected quadratic cocycle times c on the special source. Its restriction to the normalization is nonzero in $H^1(QS^{-2}(p))$, a contradiction. This excludes noncentral arcs. Rescaling any sequence of displaced cubic cycles to a fixed height, as in Proposition 10.2, would give one of the smooth or cuspidal limits just excluded. Thus the sheared triple is isolated.

D.2. **The normalization of one cusp.** Near a nonzero cuspidal member the connected genus-18 map is its smooth normalization, with trivial stabilizer. Any node, positive-genus contracted component or graph loop would raise the arithmetic genus. A contracted rational tree with one attachment would be unstable. On an Artin base the map to D stays finite flat. The same trace and projected obstruction, now applied to consecutive powers of the maximal ideal, force $t=0$ and the height into S. Its characteristic equation is $y^3=s_A$ by $H^0(S)=H^0(S^2)=0$. In local coordinates near the double zero the equation is $X^2=Y^3+b$, with b the critical value and a the independent parameter along the cusp ray. On the map the parametric Morse lemma gives $Y=z^2+r$, with r in the maximal ideal and the coordinate sign fixed on the central fiber. The equality $X(z)^2=X(-z)^2$ and regularity of $X(z)-X(-z)$ (reduce first to $2z^3$) give $X=zh(z^2)$. Then

$$b=-r^3,\qquad h(v)^2=v^2+3rv+3r^2.$$

Weierstrass preparation forces $h = v + d$ with $2d = 3r$ and $d^2 = 3r^2$. Hence $r^2 = b = 0$. Conversely $(Y, X) = (z^2 + r, z^3 + 3rz/2)$ for any $r^2 = 0$ gives the required local normalization family. Its finite free rank-three algebra agrees with the spectral algebra off the marked disk by reduction and Nakayama. The two height functions determine the gluing on the punctured disk uniquely. Thus the completed map ring in a fiber frame is $\mathbb{C}[[a, r]]/(r^2)$. With two further smooth base directions this is an lci of the absolute virtual dimension three. Its two-term obstruction theory has the same rank as the lci cotangent complex. At any residue field the degree-zero map is an isomorphism and the degree-minus-one map is surjective; equality of ranks and derived Nakayama identify the two complexes. The local virtual multiplicity is two.

D.3. **Moving the two evaluations.** Work over $B = \mathbb{P}^2$ with tautological dual $M = \mathcal{O}(1)$. Two general fiber products give an injective map on each line, an isomorphism on every plane and a Thom product representing $9x_{5,3}$. Translate by two general small cubic forms in M^3. Move the shear parameter graph to the metric dual τ of a linear section. Its zero line T has oriented normal index -1. The translations have nine common zeros off T, accounting for $9J_3$. Their ratio on T has degree three, unramified over the finitely many line and cusp directions. Each isolated line gives three negative genus-16 intersections. Over each plane there remains a compact family above the whole of T, with three incidences per cusp direction. At each of these the two translations cut the amplitude and the ratio transversely, so the genus-18 virtual degree is -2. Compactness of this homotopy follows from the uniform lower bound on the pair of linear evaluations on the unit link of the finite union of planes and lines, together with isolation on the small noncentral fibers and the disk mass estimate.

To extend the cusp calculation to all source genera we deform each whole pencil, since a genus-19 image can approach a cusp. Add a second quadratic parameter spanning the two-dimensional embedded obstruction with the first at every plane. On the graph take a second small metric-dual linear section, cut off away from T near the nine marked points. Its zero on T is a single general point. At this point the two anti-complex normal factors have positive product, yielding one positive smooth genus-19 intersection per plane. For control of the other images take a relative projective completion. The uniform mass bound bounds their ample degrees on the disk. Nearby cycles at a nonzero plane member are integral with multiplicity one, by proper Chow addition. In bounded degree their arithmetic genera are bounded by the generic plane-projection genus. A Hilbert limit with fundamental cycle C surjects onto $\mathcal{O}_C$ with a finite-length kernel, giving $p_a(\text{generic}) \le p_a(C)$ with equality only for the reduced Hilbert limit. Stable-map limits give the opposite lower bound on geometric genus by the normalization of C. Thus near smooth members the arithmetic genus equals 19 and the reduced Hilbert germ forces both parameters to vanish. Near a cusp away from the common parameter zero the same argument excludes arithmetic genus 19, while geometric genus is at least 18. Every remaining small-parameter image there is therefore smooth embedded of genus 18. Compactness gives uniform neighborhoods.

At these images the shears at fresh unramified sheets of Lemma 6.2 make the Hilbert chart regular; one may use powers two through four of the separating covector, whose values are nonzero and distinct, to preserve the first jet at the core. The new three columns interpolate on the three sheets by the Vandermonde determinant. A small smooth change of graph and translations gives transverse isolated curves. Excision in source genus 18 preserves the degree -2 in each cusp box. Proposition 7.1 now assigns the entire series w^{17} to each signed genus-18 image, including all contracted components. The same signed multiplicity-one units arise by proper connected pullback. The complete unnormalized cut is consequently

$$9J_3 - 3 \cdot 6480x_{16,1} - 2 \cdot 60 \cdot 32x_{18,1} + 32x_{19,1}.$$

Dividing by nine gives the two identities in Theorem D.1. For the complex class, the two lower degrees agree by the empty-tuple argument; hence the difference after subtraction is primitive and killed by proper division. Its smooth image vanishes by the signed calculation, so it vanishes by multiplicity-one detection. Independently the handle recurrence gives $9D_4 \equiv 3w^{17} + 5w^{18} \pmod 9$, consistent with the displayed signs.

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