

UNIFORM HERMITIAN–EINSTEIN SPECTRA FOR BIPARTITE TWO-STEP EXTENSIONS

BERND JOHANNES WUEBBEN

ABSTRACT. We determine every small eigenvalue of the Dolbeault Laplacian on trace-free endomorphisms at a Hermitian–Einstein metric near an intersection of slope-stability walls. The bundle is a fixed extension of one direct sum of stable bundles by another on a compact Kähler manifold, with pairwise nonisomorphic factors of equal slope at the reference form and a connected graph of nonzero extension components. In the stable chamber, the small spectrum is asymptotic to the nonzero spectrum of a weighted Laplacian on this graph. Its vertex masses are the factor ranks, and its edge weights balance the slope differences against the harmonic extension data. The relative error is of order the total edge weight, uniformly as positive stability margins vanish at arbitrarily different rates, and the remaining spectrum has a uniform gap. The proof uses curvature identities across graph cuts to control the nonlinear correction at each vanishing scale. As consequences, we obtain a relative approximation of the metric and a spanning-tree formula for the leading vanishing factor of the zeta-regularized determinant. For trees the leading spectrum is determined by ranks and slopes; cycles can retain dependence on the harmonic extension data.

1. INTRODUCTION

A strictly semistable bundle may have a graded object with more endomorphisms than the original bundle. As a stable polarization approaches the corresponding wall, these additional endomorphisms appear analytically as almost-holomorphic endomorphisms with small Dolbeault energy. The question is then quantitative: how many small eigenvalues occur, and how fast does each one vanish?

At an intersection of walls, there need not be one degeneration scale. One stability margin may be much smaller than all the others. An absolute spectral error, even one of higher order in the total distance to the wall, may then be larger than the smallest eigenvalue. We seek relative estimates for every small eigenvalue, with no lower bound on the ratios between the positive margins.

The bundles considered here are two-step extensions. Their stable factors form the vertices of a bipartite graph, and a nonzero extension component gives an arrow from a quotient factor to a subbundle factor. The effective operator is a weighted graph Laplacian. Its vertex masses are the factor ranks; its edge conductances are obtained by balancing the harmonic norms of the extension components against the slope defects. The main point is to justify this operator uniformly for actual Hermitian–Einstein metrics as any collection of positive cut margins becomes small relative to the others.

1.1. The setting and the effective operator. Let X be a compact connected Kähler manifold of complex dimension $n \geq 1$, and let ω_s , $s \in U \subset \mathbb{R}^a$, be a smooth family of

Date: October 27, 2026.

2020 Mathematics Subject Classification. 53C07 (primary); 53C55, 58J50, 05C50 (secondary).

Key words and phrases. Hermitian–Einstein metric, slope stability, Dolbeault Laplacian, vector bundle extension, weighted graph Laplacian, stability wall.

Kähler forms with reference parameter $s = 0$. Fix holomorphic vector bundles $F_1, \dots, F_m$, $m \geq 2$, which are pairwise nonisomorphic, stable, and of equal slope at ω_0 . Choose a partition $I \sqcup J = \{1, \dots, m\}$ and a fixed extension

$$(1) \quad 0 \longrightarrow S = \bigoplus_{i \in I} F_i \longrightarrow E \longrightarrow U_0 = \bigoplus_{j \in J} F_j \longrightarrow 0.$$

We use *two-step* to refer to this extension of a direct sum by a direct sum. In particular, arrows are not composable.

The support graph Γ has vertex set $\{1, \dots, m\}$ and an edge $e : j \rightarrow i$ exactly when the corresponding class in $\text{Ext}^1(F_j, F_i)$ is nonzero. There is one edge for each nonzero matrix block, not for each basis vector of that Ext group. Write $\Gamma(0)$ and $\Gamma(1)$ for its vertex and edge sets, and $s(e)$, $t(e)$ for the source and target of an edge. We assume that the underlying undirected graph is connected. A cycle will always mean an undirected cycle in this bipartite graph.

Write $r_v = \text{rk } F_v$, $R = \sum_v r_v$, and

$$(2) \quad \begin{aligned} V_s &= \int_X \frac{\omega_s^n}{n!}, & \deg_s F &= \int_X c_1(F) \frac{\omega_s^{n-1}}{(n-1)!}, \\ \kappa_v(s) &= \frac{2\pi \deg_s F_v}{V_s r_v}, & \kappa_E(s) &= \frac{1}{R} \sum_v r_v \kappa_v(s). \end{aligned}$$

All stability notions are slope stability: $\mu_s(F) = \deg_s(F)/\text{rk } F$, and stability requires a strict slope inequality for every coherent subsheaf of positive rank smaller than the bundle rank. The centered slope demands are

$$(3) \quad b_v(s) = -r_v(\kappa_v(s) - \kappa_E(s)), \quad \sum_v b_v(s) = 0.$$

If D is the incidence matrix with column $D_e = e_i - e_j$ for $e : j \rightarrow i$, define

$$\mathcal{C} = D\mathbb{R}_{\geq 0}^{\Gamma(1)}, \quad \mathcal{C}^\circ = D\mathbb{R}_{> 0}^{\Gamma(1)}.$$

Connectedness makes $\mathcal{C}^\circ$ the interior of $\mathcal{C}$ in the zero-sum vertex hyperplane. We prove the relevant cone characterization below. A vertex subset C is an ideal if no arrow leaves it; its stability margin is $b_C = \sum_{v \in C} b_v$. The proper nonempty ideal margins are precisely the inequalities which will characterize this cone.

Continue normalized factor Hermitian–Einstein metrics $h_{v,s}$ smoothly from the reference parameter, and let $\beta_{e,s}$ be the representative of the fixed nonzero extension component which is harmonic for ω_s and the metrics $h_{t(e),s}, h_{s(e),s}$. Set

$$(4) \quad c_e(s) = \frac{\|\beta_{e,s}\|_{L_s^2}^2}{V_s} > 0.$$

The continuation and the fixed-bundle harmonic identifications are included in the construction below. After shrinking U , all coefficients have a common positive lower bound.

For $b \in \mathcal{C}^\circ$, there is a unique positive edge vector

$$(5) \quad k_e(s, b) = c_e(s) \exp((D^T x_0)_e), \quad Dk = b, \quad \sum_v r_v(x_0)_v = 0.$$

The potential x_0 is normalized here by rank-weighted mean zero; the edge vector does not depend on that normalization. The effective operator is

$$(6) \quad A(k) = M^{-1/2} D \text{diag}(k) D^T M^{-1/2}, \quad M = \text{diag}(r_1, \dots, r_m).$$

It has one zero eigenvalue. Denote its positive eigenvalues by $\nu_1(k) \leq \dots \leq \nu_{m-1}(k)$, with multiplicity. Equivalently, these are the eigenvalues for the energy and mass

$$(7) \quad q_k(x) = \sum_e k_e |x_{t(e)} - x_{s(e)}|^2, \quad \|x\|_M^2 = \sum_v r_v |x_v|^2, \quad \sum_v r_v x_v = 0.$$

1.2. Uniform relative asymptotics. For a Hermitian metric h on E , our elliptic operator is

$$\Delta_{h,s} = \bar{\partial}_{\text{End } E}^{*h, \omega_s} \bar{\partial}_{\text{End } E} \quad \text{on } \text{End}_0 E.$$

Here $\text{End}_0 E$ denotes pointwise trace-free complex endomorphisms, and spectral multiplicities are complex. This operator is not the full Yang–Mills Hessian.

Theorem 1.1. *There are a neighborhood of 0 and constants $C, \gamma > 0$, depending on the fixed data above, with the following properties. The fixed bundle E is stable at s if and only if $b(s) \in \mathcal{C}^\circ$. In that chamber let $h_{\text{HE},s}$ be its Hermitian–Einstein metric and put*

$$(8) \quad \sigma(s) = \sum_e k_e(s, b(s)) = \sum_{i \in I} b_i(s) > 0.$$

Its endomorphism Dolbeault spectrum satisfies

$$(9) \quad \left| \frac{\lambda_j(h_{\text{HE},s})}{\nu_j(k(s, b(s)))} - 1 \right| \leq C\sigma(s) \quad (1 \leq j < m), \quad \lambda_m(h_{\text{HE},s}) \geq \gamma.$$

Thus there are exactly $m - 1$ eigenvalues tending to zero. The constants are independent of all ratios between positive conductances, positive ideal margins, or small eigenvalues. In particular, the demand direction may approach any face or intersection of faces of $\mathcal{C}$ at arbitrary relative rates.

The theorem concerns one fixed extension. It does not give constants uniform in extension classes tending to zero, changing support graphs, or increasing numbers of vertices. It allows a finite-dimensional family of forms, including tangential parameter directions; it does not assume $|s| = O(\sigma(s))$. If the chosen parameter family has no stable points, the chamber assertions are empty.

The construction also gives a metric approximation. Let Φ_s identify the fixed extension with its amplitude-one harmonic presentation, let $h_{0,s} = \bigoplus_v h_{v,s}$, and set

$$g_0 = \text{diag}(e^{(x_0)_v/2} \text{id}_{F_v}), \quad h_{\text{mod},s} = \Phi_s^* g_0^* h_{0,s}.$$

With the common determinant normalization, we obtain

$$(10) \quad e^{-C\sigma} h_{\text{mod},s} \leq h_{\text{HE},s} \leq e^{C\sigma} h_{\text{mod},s}.$$

The same Φ_s occurs in both metrics. The estimate remains valid when the diagonal weights diverge; it does not freeze the harmonic splitting while those weights become singular.

1.3. Why cuts enter the proof. The complement of the parallel block scalars has a uniform elliptic gap. Solving the curvature equation there leaves a finite equation

$$(11) \quad -b + D(c_e t_e^2)_e + \mathcal{R}(s, t) = 0.$$

The usual expansion gives $\mathcal{R} = O(|t|^4)$. That bound alone is insufficient. For example, on a three-vertex path with conductances q, q^3 , the zero-sum forcing $(0, -q^2, q^2)$ has size $O(q^2)$, but its inverse potential has oscillation q^{-1} . The weakest cut sees an error larger than its conductance.

The geometric equation has more structure. If all amplitudes crossing a vertex subset C vanish, the holomorphic bundle and its uniquely normalized complementary metric split

across that cut. Integrated curvature trace then gives an exact degree identity. A sign symmetry removes the first crossing derivative. Together these imply

$$(12) \quad \left| \sum_{v \in C} \mathcal{R}_v(s, t) \right| \leq C|t|^2 \sum_{e \in \partial C} t_e^2 \quad \text{for every cut } C.$$

This is a relative estimate at every cut, rather than only at the scale of the total error. An elementary oscillation estimate for a graph Laplacian converts it into a uniformly small vertex correction. A finite-dimensional fixed-point argument then solves the full Hermitian–Einstein equation. Harmonic cancellation and min–max give the individual relative spectral estimates.

1.4. Trees, cycles, and the wider programme. On a tree, deleting an edge gives a cut which determines that edge’s conductance from b . Thus ranks and slope demands determine the entire leading spectrum. On a cyclic graph, the equations $Dk = b$ leave additional information: the products of harmonic coefficients around cycles. This information can affect the leading spectrum even when ranks, slopes, and the chamber agree.

For example, consider a four-cycle with unit vertex masses and demand $b = \rho(1, 1, -1, -1)$, $\rho > 0$. If $\theta = c_{11}c_{22}/(c_{12}c_{21})$, its balanced conductance matrix is

$$(k_{ij}) = \rho \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}, \quad a = \frac{\sqrt{\theta}}{1 + \sqrt{\theta}}.$$

The three positive graph eigenvalues, before ordering, are $2\rho a$, $2\rho(1-a)$, 2ρ . The cycle ratio therefore changes the leading spectral coefficients. We give the geometric examples and the general cycle identities after the proof.

The local wall-crossing and convergence theory of Clarke–Tipler [CT24], the quantitative continuity theorem of Deloche [Del25], and the rescaling methods in Sektnan–Tipler [ST24] provide important analytic precedents. The last work concerns an adiabatic pullback problem with its own hypotheses; we do not import that theorem for arbitrary two-step supports. The star case of the present spectral question was treated in [Wue26a]. The contribution here is the extension to connected non-star bipartite supports, including cycles, with an actual metric construction uniform through all positive chamber ratios.

These results are one part of a programme relating stability chambers to quantitative analytic degeneration. The companion geometric construction supplies genuine toric tangent bundles with nonsimplicial chambers. We explain their stationary application below, with the actual factors, slope coordinates, and harmonic data specified. Metric-response and flow conclusions require their own arguments and play no role in the proof of the present theorem.

2. THE FINITE BALANCING PROBLEM

We first isolate the finite problem. This identifies the chamber, the data retained by cycles, and the estimate needed when the graph gap becomes small. The arguments apply to any fixed positive conductances; their constants do not depend on conductance ratios.

2.1. The positive incidence cone. A vertex subset C is an *ideal* for the extension orientation if an arrow whose source belongs to C also has its target in C . Thus no arrow leaves an ideal. Put

$$W = \{b \in \mathbb{R}^m : \mathbf{1}^T b = 0\}, \quad b_C = \sum_{v \in C} b_v.$$

Lemma 2.1. *For a connected bipartite extension graph,*

$$D\mathbb{R}_{\geq 0}^{\Gamma(1)} = \{b \in W : b_C \geq 0 \text{ for every nonempty proper ideal } C\},$$

and its interior in W is

$$D\mathbb{R}_{> 0}^{\Gamma(1)} = \{b \in W : b_C > 0 \text{ for every nonempty proper ideal } C\}.$$

Proof. If $b = Da$ with $a \geq 0$, summing over an ideal cancels internal edges and leaves the incoming weights. This proves necessity. For the converse, the dual cone consists of vertex potentials h such that $D^T h \geq 0$. Every upper level set of such an h is an ideal. Order its distinct values and express h as a constant plus nonnegative multiples of the indicators of its proper upper level sets. Since $b \in W$, the constant contributes zero, and the ideal inequalities imply $h \cdot b \geq 0$. Separation from the closed finitely generated cone $D\mathbb{R}_{\geq 0}^{\Gamma(1)}$ proves the closed-cone assertion.

Connectedness makes every proper cut nonempty. Hence $b^0 = D\mathbf{1}$ has $b_C^0 > 0$ for every proper ideal. If all ideal sums of b are positive, choose $\varepsilon > 0$ small enough that $b - \varepsilon b^0$ still has nonnegative ideal sums. Write it as Da' , $a' \geq 0$, and obtain $b = D(a' + \varepsilon \mathbf{1})$ with a strictly positive edge lift. Conversely every strictly positive lift gives strict ideal sums. The incidence matrix of a connected graph has rank $m - 1$, so the positive image is open in W . Conversely, if $b_C = 0$ for some nonempty proper ideal, then $b - \varepsilon b^0$ leaves the closed cone for every $\varepsilon > 0$, so b is not interior. This proves the interior statement. $\square$

2.2. Existence of the balanced weights. Let $c_e > 0$, $b \in \mathcal{C}^\circ$, and choose $a^0 > 0$ with $Da^0 = b$. On the rank-centered hyperplane

$$H_M = \{x \in \mathbb{R}^m : \sum_v r_v x_v = 0\}$$

consider

$$(13) \quad \phi_{c,b}(x) = \sum_e c_e e^{(D^T x)_e} - b \cdot x = \sum_e (c_e e^{(D^T x)_e} - a_e^0 (D^T x)_e).$$

Each scalar summand is bounded below and tends to infinity as its argument tends to either end of the real line. Since D^T is injective on H_M , the function is coercive there. Its Hessian is

$$D \operatorname{diag}(c_e e^{(D^T x)_e}) D^T,$$

which is positive on every nonconstant direction. It therefore has a unique minimizer x_0 . Its critical equation is $Dk = b$: the gradient belongs to W , while the Euclidean annihilator of H_M is spanned by (r_v) , whose intersection with W is zero. This proves (5).

The minimizer need not stay bounded near a chamber face. Nevertheless, in the bipartite orientation,

$$(14) \quad \sum_e k_e = \sum_{i \in I} b_i = \sigma, \quad t_e^0 = \sqrt{k_e/c_e}, \quad \sum_e (t_e^0)^2 \leq c_{\min}^{-1} \sigma.$$

Thus the harmonic amplitudes, rather than the original vertex potential, remain uniformly small. This observation allows a fixed elliptic neighborhood to serve the entire chamber.

2.3. An inverse estimate measured on cuts. For any nonempty proper vertex subset C , let ∂C consist of all edges with exactly one endpoint in C . For positive conductances k , define

$$\text{cap}_k(C) = \sum_{e \in \partial C} k_e, \quad \|q\|_{\text{cut},k} = \max_{\emptyset \neq C \neq \Gamma(0)} \frac{|q_C|}{\text{cap}_k(C)} \quad (q \in W).$$

This is a norm on W : the triangle inequality is immediate, and singleton cuts detect every coordinate. Connectedness ensures that every denominator is positive.

Lemma 2.2. *Let $L_k = D \text{diag}(k) D^T$. For any real vertex potential y ,*

$$(15) \quad \|L_k y\|_{\text{cut},k} \leq \text{osc}(y) \leq (m-1) \|L_k y\|_{\text{cut},k}.$$

If y has zero mean with respect to any positive vertex masses, then also

$$\|y\|_\infty \leq (m-1) \|L_k y\|_{\text{cut},k}.$$

Proof. For every cut, internal edges cancel and give the orientation-independent identity

$$(L_k y)_C = \sum_{\substack{uv \in \partial C \\ u \in C, v \notin C}} k_{uv} (y_u - y_v).$$

Taking absolute values proves the first inequality.

Order the vertex values as $y_{v_1} \leq \dots \leq y_{v_m}$. For $1 \leq j < m$, take $C_j = \{v_{j+1}, \dots, v_m\}$. Every potential difference directed from C_j to its complement is at least $y_{v_{j+1}} - y_{v_j}$. Hence

$$(y_{v_{j+1}} - y_{v_j}) \text{cap}_k(C_j) \leq (L_k y)_{C_j} \leq \|L_k y\|_{\text{cut},k} \text{cap}_k(C_j).$$

Divide by the positive capacity and sum over j . Repeated values give zero gaps and cause no difficulty. A positively weighted mean lies between the minimum and maximum, so a zero-mean potential has sup norm at most its oscillation. $\square$

The constant $m-1$ is sharp uniformly over graphs. On the unit-weight path with vertices $0, \dots, m-1$, take $y_j = j - (m-1)/2$. Then $L_k y = (-1, 0, \dots, 0, 1)$, the cut norm is one, and $\text{osc}(y) = m-1$. The estimate does not bound the ordinary spectral inverse uniformly. It controls a forcing whose size is measured against every cut capacity, exactly as in (12).

2.4. Stability under coefficient changes.

Proposition 2.3. *Let $k = c \exp(D^T x)$ and $k' = c' \exp(D^T x')$ be balanced at the same demand $b \in C^\circ$. If $\max_e |\log(c'_e/c_e)| \leq \eta$, then*

$$(16) \quad \text{osc}(x' - x) \leq (m-1)\eta, \quad \left| \log \frac{k'_e}{k_e} \right| \leq m\eta \quad \text{for every } e.$$

Proof. Put $y = x' - x$ and $h_e = \log(c'_e/c_e)$, so $k'_e/k_e = \exp(h_e + (D^T y)_e)$. Suppose a consecutive gap between sorted values of y exceeds η , and let C contain the vertices above it. An edge entering C has positive exponent, hence $k'_e > k_e$; an edge leaving C has negative exponent, hence $k'_e < k_e$. Each crossing edge contributes positively to $(D(k' - k))_C$. At least one crosses, contradicting $D(k' - k) = 0$. Every consecutive gap is at most η , proving the oscillation bound. Adding h_e proves the logarithmic edge-ratio bound. $\square$

In particular, multiplying the coefficients by a vertex scaling, $c'_e = c_e \exp((D^T a)_e)$, does not change the balanced weights: shift the potential by $-a$ and renormalize. Thus scalar normalizations of the factor Hermitian–Einstein metrics do not alter the effective spectrum. For general smooth coefficient variation, $\log(c(s)/c(0)) = O(|s|)$, so freezing the reference

coefficients costs relative $O(|s|)$ on every edge. This term cannot be replaced by $O(\sigma)$ unless the parameter family supplies the additional estimate $|s| = O(\sigma)$.

3. THE CURVATURE EQUATION AND ITS CUT IDENTITIES

The finite estimate becomes useful only after a corresponding estimate has been proved for the geometric error. We now construct the complementary metric and derive that error estimate. The construction is local near the graded bundle; none of its constants involves a positive stability margin.

3.1. A smooth harmonic presentation. Choose normalized Hermitian–Einstein metrics on the factors at $s = 0$. The classical correspondence supplies these initial metrics; we use its compact Kähler form, as recalled in [ST24, Theorem 8]. They continue smoothly for small s . To see this directly, first solve the scalar Poisson equation for normalized determinant metrics. With determinant fixed, the linearization of the trace-free Hermitian–Einstein equation at the initial factor metric is the positive endomorphism Dolbeault Laplacian on trace-free Hermitian sections. Here we use the conjugated curvature map $u \mapsto e^{u/2}(i\Lambda_{\omega_s} F_{h_v e^u})e^{-u/2} - \kappa_v(s) \text{id}$, as in (19) below, so the target is a fixed Hermitian section space. Stability makes its kernel zero. The implicit-function theorem in $H^{\ell+2} \rightarrow H^\ell$, $\ell > n$, followed by elliptic regularity, gives the continued metrics $h_{v,s}$. The factors are simple as holomorphic bundles, so the Hermitian–Einstein decomposition also shows that they remain stable. We use the curvature and decomposition conventions of [Kob87, §§4.2, 5.8].

The cohomology groups of the fixed holomorphic bundles $\text{Hom}(F_j, F_i)$ have constant dimension. Their harmonic projections, taken for ω_s and the continued factor metrics, therefore depend smoothly on the parameter: the elliptic coefficients vary smoothly and the kernel dimension is fixed. Applying these projections to the fixed extension classes gives $\beta_{e,s}$, smooth in every spatial norm. Their norms remain positive because each class is nonzero.

On the smooth direct sum define

$$(17) \quad \bar{\partial}_{s,t} = \bar{\partial}_0 + N_{s,t}, \quad N_{s,t} = \sum_e t_e \beta_{e,s}, \quad t \in \mathbb{R}^{\Gamma(1)}.$$

These operators are all integrable. Indeed $\bar{\partial}_0 N_{s,t} = 0$, and any product of two off-diagonal blocks would require an intermediate factor to lie both in I and in J . In particular $N_{s,t} \wedge N_{s,t} = 0$.

The amplitude-one operator is a presentation of the fixed extension. For precision, start with a fixed smooth splitting and representatives $\beta_{e,\text{ref}}$, with zero blocks on every pair without an edge. Such blocks are exact and can be removed by a triangular change of the smooth splitting. The difference $\beta_{e,s} - \beta_{e,\text{ref}}$ is exact. Choose smooth primitives in the relevant Hom bundles, using the degree-zero Green operator. Its kernel is zero at the reference form because the distinct equal-slope stable factors have no cross-Hom sections; it stays uniformly invertible nearby. The resulting triangular change of splitting has determinant one. Since products of its off-diagonal entries vanish, it identifies the two Dolbeault operators exactly. Denote the identification from E to the amplitude-one presentation by Φ_s .

Arbitrary positive amplitudes need not represent E on a cyclic support. Those on its diagonal orbit do: if $g = \text{diag}(e^{x_v/2} \text{id}_{F_v})$, then

$$(18) \quad g \bar{\partial}_{s,1} g^{-1} = \bar{\partial}_{s,t}, \quad t_e = \exp((D^T x)_e/2).$$

We will use precisely these amplitudes when solving the remaining finite equation. Allowing all small t in the present section is a device for obtaining identities on cuts.

3.2. The uniformly invertible complement. Let $h_{0,s} = \bigoplus_v h_{v,s}$. In the real space of pointwise trace-free $h_{0,s}$ -Hermitian endomorphisms, set

$$\mathcal{H} = \left\{ \text{diag}(x_v \text{id}_{F_v}) : x_v \in \mathbb{R}, \sum_v r_v x_v = 0 \right\}.$$

Let P_s be the L_s^2 -orthogonal projection onto $\mathcal{H}$ and $Q_s = 1 - P_s$. For a Hermitian u with $P_s u = 0$ and $\text{tr } u = 0$, write the curvature in the fixed Hermitian structure:

$$(19) \quad \widehat{K}(s, t, u) = e^{u/2} (i\Lambda_{\omega_s} F_{\bar{\partial}_{s,t}, h_{0,s} e^u}) e^{-u/2}.$$

The expression is $h_{0,s}$ -Hermitian. The determinant metric is unchanged by u , and the determinant holomorphic structure is independent of t . Consequently

$$\text{tr } \widehat{K}(s, t, u) = R\kappa_E(s) \quad \text{pointwise.}$$

Lemma 3.1. *For small (s, t) , the equation*

$$(20) \quad Q_s(\widehat{K}(s, t, u) - \kappa_E(s) \text{id}) = 0, \quad P_s u = 0,$$

has a unique small trace-free Hermitian solution $u(s, t)$. It is smooth in the parameters and spatial variables, and for every fixed nonnegative integer a ,

$$(21) \quad \|u(s, t)\|_{C^a} \leq C_a |t|^2.$$

For a constant unitary vertex scaling $g = \text{diag}(e^{i\theta_v} \text{id}_{F_v})$, put $(g \cdot t)_e = e^{i(\theta_{t(e)} - \theta_{s(e)})} t_e$. Then $u(s, g \cdot t) = g u(s, t) g^{-1}$ whenever $g \cdot t$ is real. In particular, equivariance holds for all vertex signs and for every unitary vertex scaling fixing t .

Proof. We give the linearization because its convention will matter. At $t = u = 0$, the split curvature $K_{0,s}$ has entries $\kappa_v(s) \text{id}_{F_v}$. Variation of the unconjugated curvature in the metric coordinate $h = h_{0,s} e^u$ gives $\Delta_{\partial,s} u$. Variation of the two conjugating factors adds $[u, K_{0,s}]/2$. The Kähler identity on endomorphism-valued zero-forms is

$$\Delta_{\partial,s} - \Delta_{\bar{\partial},s} = [K_{0,s}, \cdot].$$

Thus the real Hermitian linearization is

$$(22) \quad L_s = \frac{1}{2}(\Delta_{\bar{\partial},s} + \Delta_{\partial,s}).$$

It is self-adjoint, nonnegative, and annihilates $\mathcal{H}$. At $s = 0$ the curvature is globally central, so $L_0 = \Delta_{\bar{\partial},0}$. Stability and equal-slope Hom vanishing show that its trace-free Hermitian kernel is exactly $\mathcal{H}$. The same kernel description holds for nearby s : if $L_s u = 0$, positivity gives $\bar{\partial} u = \partial u = 0$. Thus u is a holomorphic endomorphism of the fixed split bundle, and the same Hom vanishing makes it block scalar.

Use Sobolev indices $H^{\ell+2} \rightarrow H^\ell$, $\ell > n$. Smooth block-diagonal isometries identify the varying Hermitian bundles while fixing the block scalars and preserving trace. The maps $Q_s|_{Q_0}$ then identify the complementary spaces for small s . In these fixed spaces, $Q_s L_s$ converges to the invertible complementary operator $Q_0 L_0$. Elliptic estimates and the Neumann-series criterion give a uniform inverse. The real parameter-dependent implicit-function theorem produces the unique small solution.

At $t = 0$, the split curvature is block scalar, so $u(s, 0) = 0$. Its first variation in an amplitude has off-diagonal curvature entry $-\bar{\partial}_{\text{Hom}}^* \beta_{e,s}$ and its Hermitian adjoint. Both vanish by harmonicity at every nearby s . Differentiating the complementary equation gives $u_{t_e}(s, 0) = 0$. Taylor's theorem proves the quadratic estimate. Higher Sobolev indices and uniqueness, or elliptic bootstrapping, give the assertion in every fixed spatial norm. The

constant unitary vertex action carries t to $g \cdot t$, fixes $\mathcal{H}$ pointwise, and preserves the equations and normalizations. Uniqueness proves equivariance whenever $g \cdot t$ is again real. $\square$

The self-adjoint operator in (22) is a metric-coordinate linearization. A complex gauge parameter $\exp(x)$ instead changes the diagonal metric by $\exp(2x)$. This explains the factor two in some gauge-coordinate formulas; it does not change our positive Dolbeault spectral convention.

3.3. The reduced curvature equation. Put $p_v = \text{id}_{F_v} - (r_v/R) \text{id}$, interpreted as a block endomorphism, and define the real vector

$$(23) \quad F_v(s, t) = V_s^{-1} \left\langle p_v, \widehat{K}(s, t, u(s, t)) - \kappa_E(s) \text{id} \right\rangle_{L_s^2}.$$

It has zero total sum. The p_v span $\mathcal{H}$, so (20) and $F = 0$ together are exactly the full Hermitian–Einstein equation.

Lemma 3.2. *There is a smooth zero-sum remainder $\mathcal{R}(s, t)$ such that*

$$(24) \quad F(s, t) = -b(s) + D(c_e(s)t_e^2)_e + \mathcal{R}(s, t).$$

Its amplitude derivatives of orders zero through three vanish at $t = 0$, for every nearby s . In particular

$$(25) \quad \|D_t^2 \mathcal{R}(s, t)\| \leq C|t|^2$$

uniformly on a smaller neighborhood.

Proof. The constant term is $r_v(\kappa_v - \kappa_E) = -b_v$. The first variations vanish: the first-order curvature term is off-block-diagonal and pairs to zero with every p_v , while $u_{t_e}(s, 0) = 0$ by the preceding lemma. The quadratic curvature term of $t_e \beta_{e,s}$ pairs positively with the target block and negatively with the source block, with coefficient $\|\beta_{e,s}\|^2/V_s = c_e(s)$. Distinct matrix blocks have no mixed diagonal trace contribution. The quadratic metric correction contributes $L_s u_2$; its pairing with every p_v is zero because L_s is self-adjoint and $L_s p_v = 0$.

Multiplication by -1 on every factor in I sends t to $-t$ and fixes all p_v . Equivariance gives $F(s, -t) = F(s, t)$, so the third-order amplitude terms vanish. Joint Taylor expansion, with uniformly bounded fourth derivatives, now gives (25). This argument requires no relation between $|s|$ and $|t|$. $\square$

Separate evenness in each edge amplitude need not hold on a cycle. The symmetry which survives for every cut is sufficient, and we use it next.

3.4. Exact identities at a cut.

Proposition 3.3. *For every vertex subset C , the remainder satisfies*

$$(26) \quad |\mathcal{R}_C(s, t)| \leq C|t|^2 \sum_{e \in \partial C} t_e^2, \quad \mathcal{R}_C = \sum_{v \in C} \mathcal{R}_v.$$

One constant works for all subsets and all nearby parameters.

Proof. Separate $t = (a, z)$, where z consists of the crossing amplitudes and a of the internal ones. At $z = 0$, the holomorphic bundle splits into the factors in C and those in its complement. The scalar circle on the entire C -summand fixes the Dolbeault operator, the split metric, and the complementary equation. Uniqueness in Lemma 3.1 forces $u(s, a, 0)$ to commute with this circle, so the corrected metric also splits.

Integrated curvature trace on that summand is its topological degree, whether or not the summand is Hermitian–Einstein. Since conjugation by $e^{u/2}$ preserves the trace of the split block, Chern–Weil gives

$$(27) \quad F_C(s, a, 0) = \sum_{v \in C} r_v(\kappa_v - \kappa_E) = -b_C(s).$$

The incidence term in (24) also sums to zero on C when $z = 0$. Therefore $\mathcal{R}_C(s, a, 0) = 0$. This is why arbitrary subsets, rather than only ideals, can be used for the error estimate.

Now multiply the factors in C by -1 , leaving the others unchanged. This reverses precisely the crossing amplitudes. Equivariance and invariance of p_v give $\mathcal{R}_C(s, a, -z) = \mathcal{R}_C(s, a, z)$, so its first z -derivative vanishes at $z = 0$. Taylor expansion in z alone, using (25) along $(a, \theta z)$, yields

$$|\mathcal{R}_C(s, a, z)| \leq C(|a|^2 + |z|^2)|z|^2.$$

There are only finitely many cuts. Taking the largest constant proves the proposition. $\square$

The estimate controls the error relative to the size of every cut. It asserts neither divisibility by an individual cyclic edge amplitude nor a uniform ordinary inverse for the graph Laplacian. The next section uses precisely this distinction to complete the nonlinear construction.

4. THE HERMITIAN–EINSTEIN METRIC AND THE STABLE CHAMBER

We now solve the finite curvature equation on the diagonal orbit of the fixed extension. The balancing potential may diverge, but its amplitudes are small. The correction to that potential is uniformly small because the curvature error is small relative to every cut.

4.1. A nonlinear correction without a lower spectral gap. Fix an interior parameter s , abbreviate $k = k(s, b(s))$, and put

$$L_k = D \operatorname{diag}(k) D^T, \quad t_e^0 = \sqrt{k_e/c_e(s)}.$$

We regard L_k as an isomorphism from the rank-centered hyperplane H_M to the zero-sum demand space W . For $y \in H_M$, define

$$(28) \quad t_e(y) = t_e^0 \exp((D^T y)_e/2).$$

All these amplitudes lie on the diagonal orbit of (1).

Proposition 4.1. *For all sufficiently small interior parameters, the full curvature equation has a solution with*

$$(29) \quad \|y_s\|_\infty \leq A\sigma, \quad \tilde{t} = t(y_s), \quad \tilde{k}_e = c_e(s)\tilde{t}_e^2 = k_e(1 + O(\sigma)), \quad \|u(s, \tilde{t})\|_{C^a} \leq C_a\sigma.$$

The comparison is relative separately on every edge, with constants independent of all positive conductance ratios.

Proof. Since $Dk = b$, equation $F(s, t(y)) = 0$ is equivalent to

$$(30) \quad y = \Psi_s(y) := -L_k^{-1} \left\{ D(k_e(e^{(D^T y)_e} - 1 - (D^T y)_e))_e + \mathcal{R}(s, t(y)) \right\}.$$

The inverse exists at each interior point. We estimate this specific forcing in the cut norm, rather than bound the inverse in an ordinary operator norm.

Suppose $\|y\|_\infty \leq a \leq 1$. Then $|(D^T y)_e| \leq 2a$, and Taylor's inequality gives

$$|e^z - 1 - z| \leq \frac{1}{2}e^{|z|}|z|^2.$$

Only crossing edges contribute to a cut sum of the incidence term, so its cut norm is at most $2e^{2a}a^2$. Moreover

$$|t(y)|^2 \leq c_{\min}^{-1} e^{2a} \sigma, \quad \sum_{e \in \partial C} t_e(y)^2 \leq c_{\min}^{-1} e^{2a} \text{cap}_k(C).$$

Proposition 3.3 therefore gives

$$|\mathcal{R}_C(s, t(y))| \leq C c_{\min}^{-2} e^{4a} \sigma \text{cap}_k(C).$$

Both terms in braces have zero total sum. Lemma 2.2 yields a constant C_0 with

$$(31) \quad \|\Psi_s(y)\|_{\infty} \leq C_0(a^2 + \sigma) \quad \text{if } \|y\|_{\infty} \leq a \leq 1.$$

Choose $A \geq 2 \max(1, C_0)$, set $a = A\sigma$, and shrink the parameter neighborhood so that the amplitudes stay in the complementary construction and $C_0(A^2\sigma^2 + \sigma) \leq A\sigma$. The continuous map Ψ_s sends the compact convex ball $\{y \in H_M : \|y\|_{\infty} \leq A\sigma\}$ to itself. Brouwer's theorem gives a fixed point y_s .

This fixed point solves $F = 0$, and the complementary equation already holds. Thus the corrected metric is exactly Hermitian–Einstein. Finally

$$\tilde{k}_e = k_e \exp((D^T y_s)_e), \quad |(D^T y_s)_e| \leq 2A\sigma,$$

which proves the separate relative edge bounds. The amplitude bound and Lemma 3.1 give the last estimate in (29). $\square$

Brouwer's argument gives a solution and its bounds at every interior parameter. It makes no assertion about differentiating a selected fixed point uniformly near the chamber boundary. Such a derivative estimate would require a further argument; it is not needed for the stationary spectral theorem.

4.2. Simplicity and exact stability.

Lemma 4.2. *The fixed extension E is simple.*

Proof. The reference stable factors are distinct and of equal slope, so all cross-Hom groups vanish. For an endomorphism of E , its composition $S \rightarrow E \rightarrow U_0$ is therefore zero, and it preserves S . The induced actions on S and U_0 are scalars on each stable factor. Compatibility with a nonzero extension component forces the scalars at its endpoints to agree. Connectedness makes every scalar equal. After subtracting that common scalar, a remaining endomorphism would factor through $U_0 \rightarrow S$, whose Hom group is also zero. $\square$

The Hermitian–Einstein decomposition theorem splits an Hermitian–Einstein bundle into stable summands of the same slope; see [Kob87, Theorem 5.8.3]. Simplicity excludes a nontrivial splitting. Consequently the metric constructed in Proposition 4.1 makes E stable.

Conversely, each nonempty proper ideal C defines a holomorphic subbundle E_C of the harmonic presentation. Indeed the smooth sum of its factors is preserved by the Dolbeault operator precisely because no arrow leaves C . Transport by Φ_s makes it a subbundle of the original fixed E , and

$$\kappa(E_C) - \kappa_E = -\frac{b_C}{r_C}, \quad r_C = \sum_{v \in C} r_v.$$

Stability forces $b_C > 0$ for every such ideal. Lemma 2.1 then implies $b \in \mathcal{C}^\circ$. This proves both directions of the chamber assertion without classifying arbitrary subsheaves: ideal subbundles give necessity, and the actual metric construction gives sufficiency.

4.3. The metric approximation. Normalize x_0 and y_s by rank-weighted mean zero, and put

$$g_y = \text{diag}(e^{(y_s)_v/2} \text{id}_{F_v}), \quad g = g_y g_0.$$

Equation (18) identifies the corrected Dolbeault operator with the fixed extension, so its metric is

$$h_{\text{HE},s} = \Phi_s^* g^*(h_{0,s} e^{u(s,\tilde{t})}).$$

Both diagonal gauges have determinant one. The triangular identification also has determinant one, and $\text{tr } u = 0$. Thus the constructed metric and $h_{\text{mod},s}$ have the same determinant $\prod_v \det h_{v,s}$. On the simple bundle E , an Hermitian–Einstein metric for ω_s is unique up to a positive scalar [Kob87, Proposition 6.3.37(a),(d)]. The determinant normalization fixes that scalar, so the constructed metric is the uniquely normalized $h_{\text{HE},s}$.

The C^0 estimate on u gives $e^{-C\sigma} h_{0,s} \leq h_{0,s} e^u \leq e^{C\sigma} h_{0,s}$. Pull back first by g_y ; the bound $\|y_s\|_\infty \leq A\sigma$ gives the same type of comparison with $h_{0,s}$. Pullback by the common $g_0 \Phi_s$ preserves Hermitian-form inequalities, even when g_0 is unbounded. This proves (10).

The estimates are made in a specified harmonic presentation. They should not be interpreted as connection estimates in an arbitrarily fixed unbounded gauge.

5. THE COMPLETE SMALL SPECTRUM

The remaining step is spectral. We first work with the split metric and the corrected amplitudes. Harmonicity makes the coupling to the elliptic complement small relative to each scalar's own energy. This is the spectral counterpart of the cut-relative estimate.

5.1. The split kernel and its graph energy. For fixed nearby s , let $D_0 = \bar{\partial}_{\text{End}(\oplus F_v)}$ and write the endomorphism Dolbeault operator at amplitudes t as

$$D_t = D_0 + \mathcal{A}_t, \quad \mathcal{A}_t = [N_{s,t}, \cdot].$$

In this section the Hilbert space is complex, with the split endomorphism metric and the base form ω_s . The kernel of D_0 on trace-free sections is

$$\mathcal{H}_{\mathbb{C}} = \left\{ \text{diag}(x_v \text{id}_{F_v}) : x_v \in \mathbb{C}, \sum_v r_v x_v = 0 \right\}.$$

It has complex dimension $m - 1$. The factor metrics are smooth, and the holomorphic Hom groups are fixed, so the split positive gap on $\mathcal{H}_{\mathbb{C}}^\perp$ is uniform for nearby s . Let P and Q be the two orthogonal projections in this complex decomposition.

For $x \in \mathcal{H}_{\mathbb{C}}$, harmonicity gives the exact cancellation

$$(32) \quad D_0^* \mathcal{A}_t x = 0.$$

Distinct matrix blocks are pointwise orthogonal in the split endomorphism metric. Hence, if $k_e(t) = c_e(s) t_e^2$,

$$(33) \quad \|D_t x\|^2 = V_s \sum_e k_e(t) |x_{t(e)} - x_{s(e)}|^2, \quad \|x\|^2 = V_s \sum_v r_v |x_v|^2.$$

The compression $G = P D_t^* D_t P$ therefore has positive eigenvalues exactly $\nu_j(k(t))$.

5.2. Relative control of the complementary coupling. Set $\varepsilon = |t|$ and write the non-negative self-adjoint operator, in quadratic-form blocks, as

$$D_t^* D_t = \begin{pmatrix} G & B^* \\ B & C \end{pmatrix} \quad \text{on } \mathcal{H}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}}^{\perp}.$$

Multiplication by $\mathcal{A}_t$ has norm at most $C_1 \varepsilon$. If the split gap is $\gamma_0 > 0$, then for $z \in QH^1$,

$$\|D_t z\| \geq \|D_0 z\| - C_1 \varepsilon \|z\| \geq (\sqrt{\gamma_0} - C_1 \varepsilon) \|z\|.$$

Thus $C \geq \gamma_1 > 0$ for all small amplitudes.

The cancellation (32) also gives

$$D_t^* D_t x = \mathcal{A}_t^* \mathcal{A}_t x \quad (x \in \mathcal{H}_{\mathbb{C}}),$$

and therefore

$$(34) \quad \|Bx\|^2 \leq C_2 \varepsilon^2 \|D_t x\|^2 = C_2 \varepsilon^2 \langle x, Gx \rangle.$$

The right side contains the graph energy of x , not the ordinary norm of x . It remains useful when that energy is arbitrarily small compared with $\varepsilon^2 \|x\|^2$.

For $0 \leq \lambda \leq \gamma_1/2$, eliminate the complementary block. Its Schur operator on $\mathcal{H}_{\mathbb{C}}$ is

$$S_{\lambda} = G - \lambda - B^*(C - \lambda)^{-1}B.$$

The inverse on the complement is uniformly bounded, so (34) implies, as quadratic forms,

$$(35) \quad (1 - C_3 \varepsilon^2)G - \lambda \leq S_{\lambda} \leq G - \lambda.$$

The finite space consists of smooth sections, so B is a bounded map into L^2 ; the complementary inverse maps into the operator domain. Completing the square in the closed quadratic form justifies Schur elimination and preserves the number of negative directions. Explicitly, for $x \in \mathcal{H}_{\mathbb{C}}$ and $z \in QH^1$, put $z_0 = -(C - \lambda)^{-1}Bx$. Then

$$\langle x + z, (D_t^* D_t - \lambda)(x + z) \rangle = \langle x, S_{\lambda} x \rangle + \|(C - \lambda)^{1/2}(z - z_0)\|^2,$$

where the left side is interpreted as a closed quadratic form. Since $C - \lambda \geq \gamma_1/2$, the number of eigenvalues of $D_t^* D_t$ strictly below λ equals the number of strictly negative eigenvalues of S_{λ} , including when λ itself is an eigenvalue.

Proposition 5.1. *For all sufficiently small positive amplitudes on the connected support, exactly $m - 1$ eigenvalues lie below a fixed positive threshold, and*

$$(36) \quad (1 - C\varepsilon^2)\nu_j(k(t)) \leq \lambda_j(h_{0,s}, \bar{\partial}_{s,t}) \leq \nu_j(k(t)), \quad 1 \leq j < m.$$

The next eigenvalue has a uniform positive lower bound.

Proof. Restriction of the Rayleigh quotient to P gives the upper bound by min-max. Since all conductances are $O(\varepsilon^2)$, these $m - 1$ eigenvalues lie below $\gamma_1/2$ for small amplitudes. At that threshold, the positive complementary block leaves at most $m - 1$ negative directions, so there are no additional low eigenvalues.

For the lower bound, use the first inequality in (35). At $\lambda = (1 - C_3 \varepsilon^2)\nu_j(k(t))$, its right-hand comparison has at most $j - 1$ strictly negative eigenvalues. The same is true of S_{λ} and therefore of the full operator below λ . This proves the lower bound for each ordered eigenvalue. Repeated graph eigenvalues and arbitrarily small ratios do not change the argument. $\square$

5.3. Transfer to the actual metric. Apply the proposition to $t = \tilde{t}$ from Proposition 4.1. Its squared total amplitude is $O(\sigma)$, and its conductances satisfy $\tilde{k}_e = k_e(1 + O(\sigma))$ separately on every edge. Consequently the graph forms are bounded by $(1 \pm C\sigma)q_k$. Min-max compares every ordered graph eigenvalue by the same relative factors.

The metric correction also preserves relative information. Changing $h_{0,s}$ to $h_{0,s}e^u$, with $\|u\|_{C^0} \leq C\sigma$, changes the endomorphism norms in degrees zero and one by factors $e^{O(\sigma)}$. The Dolbeault operator and its trace-free domain remain fixed. Every Rayleigh quotient, and hence every eigenvalue, is multiplied by bounds of that same form. No spatial derivative of the metric is needed in this norm comparison.

Finally $g\Phi_s$ is a holomorphic isomorphism from the fixed extension to the amplitude- $\tilde{t}$ bundle, and is an isometry for the pulled-back metric. Its possibly divergent weights cause no further spectral error. Combining these comparisons gives (9), including the uniform next gap. Together with Section 4, this completes the proof of Theorem 1.1.

The proof uses a relative estimate at each stage: cut forcing, corrected edges, scalar energy, and metric norms. Replacing any one of those estimates by an error measured only in the total scale would lose control of the weakest modes.

6. TREES, CYCLES, AND THE SMALL-SPECTRUM PRODUCT

The spectral theorem separates two questions. The graph determines the form of the effective operator; balancing determines its weights. We now describe when slope data alone determine those weights and when extension data remain necessary.

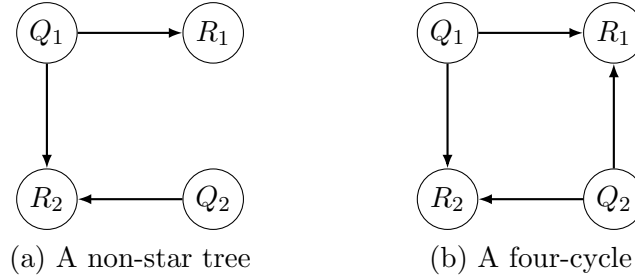


FIGURE 1. Two-step supports. Every arrow starts at a quotient factor Q_j and ends at a subbundle factor R_i . The undirected path in (a) is not a star; the cycle in (b) has no composable arrows.

6.1. Trees have cut-determined weights. Suppose the undirected support is a tree. Deleting an edge e leaves two components; let C_e be the one containing its target. Summing $Dk = b$ over this component gives

$$(37) \quad k_e = b_{C_e}.$$

Thus the balanced conductances are determined by the demands. The harmonic coefficients affect the balancing potential and the model metric, but not the leading spectrum.

Conversely, the $m - 1$ edge columns of the incidence matrix of a tree form a basis of the zero-sum vertex space. A demand lies in the strict stable cone exactly when all the cut values in (37) are positive. Theorem 1.1 therefore supplies a completely slope-determined operator on that cone, with no restriction on the ratios of those cut values. This includes the non-star path in Figure 1(a). It does not include a higher-step filtration with composable arrows.

6.2. Cycles retain harmonic orbit data. For a connected graph, the cycle space is $\ker D$, of dimension $|\Gamma(1)| - m + 1$. Pairing $\log k - \log c = D^T x_0$ with an integral cycle vector a gives

$$(38) \quad \prod_e k_e^{a_e} = \prod_e c_e^{a_e}.$$

For a bipartite cycle the nonzero coefficients of a alternate between 1 and -1 . A basis of such identities, together with $Dk = b$, characterizes the balanced positive vector: if the cycle identities hold, then $\log k - \log c$ lies in $(\ker D)^\perp = \text{im } D^T$, and uniqueness of balancing applies.

The right side of (38) is invariant under scalar changes of the factor metrics, because those multiply c by a vertex scaling. It records harmonic information about the extension orbit. It is not determined by the incidence cone.

For the four-cycle with unit masses and $b = \rho(1, 1, -1, -1)$, the row and column sums force

$$k = \rho \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}.$$

Writing $\theta = c_{11}c_{22}/(c_{12}c_{21})$, the cycle identity gives

$$\frac{a^2}{(1-a)^2} = \theta, \quad a = \frac{\sqrt{\theta}}{1 + \sqrt{\theta}}.$$

In vertex order (R_1, R_2, Q_1, Q_2) , the constant vector is the null vector. The other three vectors

$$(1, -1, -1, 1), \quad (1, -1, 1, -1), \quad (1, 1, -1, -1)$$

have eigenvalues, respectively,

$$(39) \quad 2\rho a, \quad 2\rho(1-a), \quad 2\rho.$$

Changing θ changes the leading coefficients with the same masses and demands. Harmonic coefficients cannot therefore be discarded from the general cyclic theorem.

The geometric family below has an additional reference symmetry which fixes its cycle ratio. That simplification is a property of the chosen geometric and metric data, not of the chamber alone.

6.3. Freezing the coefficients. Let $k^{(0)}(b(s))$ denote balancing at the actual demand $b(s)$ with the reference coefficients $c(0)$. Smoothness and Proposition 2.3 give separate edge comparisons with relative error $O(|s|)$. Min-max and Theorem 1.1 yield

$$(40) \quad \lambda_j(h_{\text{HE},s}) = \nu_j(k^{(0)}(b(s)))(1 + O(\sigma(s) + |s|)), \quad 1 \leq j < m.$$

The constants remain independent of positive demand ratios. An $O(\sigma)$ formula with frozen coefficients requires an additional geometric reason for $|s| = O(\sigma)$. We will verify such a reason in the tangent example.

6.4. A spanning-tree expression. For a connected graph let

$$\mathcal{T}_\Gamma(k) = \sum_{\substack{T \subset \Gamma \\ T \text{ a spanning tree}}} \prod_{e \in T} k_e$$

be its weighted spanning-tree polynomial. The following finite identity is classical; we give the proof to make the rank normalization explicit. For a primary account of the more general all-minors identity, see [NZ13].

Proposition 6.1. *The positive eigenvalues of (6) satisfy*

$$(41) \quad \prod_{j=1}^{m-1} \nu_j(k) = \frac{\sum_v r_v}{\prod_v r_v} \mathcal{T}_\Gamma(k).$$

Consequently the actual Hermitian–Einstein small cluster obeys

$$(42) \quad \prod_{j=1}^{m-1} \lambda_j(h_{\text{HE},s}) = \frac{R}{\prod_v r_v} \mathcal{T}_\Gamma(k(s, b(s))) (1 + O(\sigma(s))).$$

Proof. Delete one row i of the incidence matrix. Cauchy–Binet expresses the corresponding principal cofactor of L_k as

$$\sum_{\substack{T \subset \Gamma(1) \\ |T|=m-1}} \det(D_{\hat{i},T})^2 \prod_{e \in T} k_e.$$

The reduced incidence determinant is zero unless T is a spanning tree, and is 1 or -1 for a tree. The zero case follows from a cycle dependence or disconnectedness; the tree case follows by successively removing leaves. Thus every such cofactor is $\mathcal{T}_\Gamma(k)$.

The matrix $A(k) = M^{-1/2} L_k M^{-1/2}$ has null vector $z = (\sqrt{r_v})_v$, with $\|z\|^2 = R$. Its adjugate is

$$\text{adj } A = \frac{\prod_{j < m} \nu_j}{R} z z^*.$$

The i -th diagonal cofactor is therefore $(r_i/R) \prod_{j < m} \nu_j$. On the other hand it equals $\mathcal{T}_\Gamma(k) / \prod_{v \neq i} r_v$. Equating these expressions proves (41). The individual relative estimates give (42), since the number of factors is fixed. $\square$

For a tree the polynomial is a single monomial, and (37) expresses every factor as a cut margin. For the four-cycle at $\theta = 1$, each edge has weight $\rho/2$; its four spanning trees give $\mathcal{T}_\Gamma = \rho^3/2$. The unit-mass positive eigenvalue product is $2\rho^3$, as in (39). With masses $(1, 2, 3, 5)$ and $\rho = 1$, the product instead equals $11/60$. The rank factor in (41) is essential.

6.5. The full zeta-regularized determinant. The small-spectrum product also determines the leading vanishing factor of the full elliptic determinant. Its remaining factor is the determinant of the graded bundle at the current polarization. For a positive elliptic operator B , write $\det_\zeta B = \exp(-\zeta'_B(0))$, where ζ_B is the meromorphic continuation of its spectral zeta function. A prime on the determinant means that zero eigenvalues are omitted.

Corollary 6.2. *Let $\Delta_{\text{gr},s}$ be the endomorphism Dolbeault Laplacian of $\bigoplus_v F_v$, with metric $h_{0,s}$ and Kähler form ω_s , on complex pointwise trace-free endomorphisms. Under the hypotheses of Theorem 1.1,*

$$(43) \quad \det_\zeta \Delta_{h_{\text{HE},s},s} = \det'_\zeta \Delta_{\text{gr},s} \frac{R}{\prod_v r_v} \mathcal{T}_\Gamma(k(s, b(s))) (1 + O(\sigma(s))).$$

The error is uniform through all positive conductance ratios. The graded determinant is a smooth positive function of s .

Proof. Use the small real amplitudes of (17) and the corrected metric $h_{0,s} e^{u(s,t)}$ from Lemma 3.1. Their endomorphism Dolbeault Laplacians $\Delta(s,t)$ form a smooth family of nonnegative self-adjoint elliptic differential operators of order two. Smooth fiber isometries and volume-density factors identify the varying L^2 spaces with one fixed space, preserving the pointwise trace-free subbundle. The common operator domain is then H^2 .

At $t = 0$, the kernel consists of the $m - 1$ complex block scalars for every nearby s . Indeed the split holomorphic bundle is fixed, and its cross-Hom groups vanish. A fixed small contour about zero therefore separates this cluster from the remaining spectrum throughout a smaller product neighborhood. The resolvent contour integral defines a smooth rank- $m - 1$ orthogonal projection $\Pi(s, t)$; elliptic regularity makes it a smooth family of smoothing operators. The complementary spectrum has a common positive lower bound. Set

$$B(s, t) = \Delta(s, t) + \Pi(s, t).$$

This is a smooth, uniformly positive elliptic family of order two, with unchanged principal symbol and common spectral cut along the negative real axis. The parameter-dependent complex-power construction gives a smooth function $f(s, t) = \log \det_{\zeta} B(s, t)$. For this regularity statement, apply the degree-zero, trivially graded case of [Sco07, §2, Lemma 0.3]: the order is constant, the kernel is zero, and the same spectral cut works throughout. The resolvent estimates there hold in every parameter C^l norm on relatively compact sets.

Let g be -1 on the factors in I and $+1$ on those in J . It carries t to $-t$. Lemma 3.1 shows that it is an isometry between the corresponding corrected bundles. Its action on endomorphisms unitarily intertwines $\Delta(s, t)$ and $\Delta(s, -t)$, and hence also $B(s, t)$ and $B(s, -t)$. Thus $f(s, -t) = f(s, t)$. Taylor expansion on a smaller product neighborhood gives, uniformly in s ,

$$f(s, t) - f(s, 0) = O(|t|^2), \quad \det_{\zeta} B(s, 0) = \det'_{\zeta} \Delta_{\text{gr}, s}.$$

The second equality replaces the $m - 1$ zero modes by ones. It also proves the asserted smoothness of the graded determinant.

Now evaluate at the amplitudes $\tilde{t}$ supplied by Proposition 4.1, for which $|\tilde{t}|^2 = O(\sigma)$. The bundle identification and pulled-back metric preserve the Dolbeault spectrum, so $\Delta(s, \tilde{t})$ has exactly the spectrum of the actual Hermitian–Einstein operator. No parameter derivative of the selected finite correction is needed. Simplicity makes this trace-free operator invertible. Changing only its small cluster changes the zeta function by the entire function

$$\zeta_{\Delta}(z) - \zeta_B(z) = \sum_{j=1}^{m-1} (\lambda_j^{-z} - (1 + \lambda_j)^{-z}).$$

Differentiation at zero consequently gives the exact identity

$$\det_{\zeta} \Delta = \det_{\zeta} B \prod_{j=1}^{m-1} \frac{\lambda_j}{1 + \lambda_j}.$$

The graph trace bounds every ν_j by $2\sigma / \min_v r_v$, so Theorem 1.1 gives $\lambda_j = O(\sigma)$. Hence $\prod_{j < m} (1 + \lambda_j) = 1 + O(\sigma)$. Combining the quadratic determinant estimate with (42) proves (43). $\square$

The graded factor is retained at the current s . Freezing it and the harmonic coefficients at zero gives relative error $O(\sigma + |s|)$, as in (40). Formula (43) does not evaluate that global graded determinant or compute analytic torsion.

7. A TANGENT BUNDLE WITH THREE INDEPENDENTLY SEPARATED SCALES

We finish with a genuine tangent bundle. The example has a four-cycle support, full local slope coordinates, and a square-cone stable chamber. A reference symmetry removes its cycle

coefficient from the leading spectrum. This lets us realize three distinct spectral orders on one fixed bundle and a smooth family of ample forms.

7.1. The fourfold and its tangent sequence. Let

$$B = \mathbb{P}^1 \times \mathbb{P}^1, \quad L = \mathcal{O}_B(1, -1), \quad X = \mathbb{P}_B(\mathcal{O}_B \oplus L) \times_B \mathbb{P}_B(\mathcal{O}_B \oplus L),$$

where projectivization parametrizes one-dimensional quotients. Write $\pi : X \rightarrow B$, let h_1, h_2 be the pulled-back hyperplane classes, and let ξ_1, ξ_2 be the tautological classes. Set

$$\ell = h_1 - h_2, \quad \eta_i = \xi_i - \ell/2.$$

This is a smooth projective toric fourfold of Picard rank four. For an explicit description, in a lattice basis e_1, e_2, f_1, f_2 its ray pairs are

$$\{e_1, -e_1\}, \quad \{e_2, -e_2\}, \quad \{f_1, -f_1 + e_1 + e_2\}, \quad \{f_2, -f_2 - e_1 - e_2\}.$$

Here e_i is the fibre direction of the i -th projective bundle and f_j the j -th base direction. The divisors of the pair $\{e_i, -e_i\}$ have classes ξ_i and $\xi_i - \ell$; both divisors in the j -th base pair have class h_j . A maximal cone chooses one ray from each pair; every resulting basis is unimodular.

For a real divisor $H = x\eta_1 + y\eta_2 + uh_1 + vh_2$, the ample inequalities are

$$(44) \quad x, y > 0, \quad u, v > (x + y)/2.$$

Here is a direct verification. The divisor polytope has inequalities

$$\begin{aligned} -x/2 \leq t_1 \leq x/2, & \quad -y/2 \leq t_2 \leq y/2, \\ -u/2 \leq q_1 \leq u/2 + t_1 + t_2, & \quad -v/2 \leq q_2 \leq v/2 - t_1 - t_2. \end{aligned}$$

Under (44), all intervals have positive length and the normal fan is precisely the displayed fan. Conversely, the invariant fiber curves have degrees x, y , and the minimum degrees of base curves over the four fiber sections are $u - (x + y)/2, v - (x + y)/2$. Positivity on these curves forces (44).

The tangent sequence of the smooth fiber product is

$$(45) \quad 0 \longrightarrow R_1 \oplus R_2 \longrightarrow T_X \longrightarrow Q_1 \oplus Q_2 \longrightarrow 0,$$

where

$$R_i = \mathcal{O}_X(2\xi_i) \otimes \pi^* L^{-1}, \quad Q_j = \pi^* \mathcal{O}_B(2h_j).$$

Their first Chern classes $2\eta_1, 2\eta_2, 2h_1, 2h_2$ form a basis of $N^1(X)_{\mathbb{R}}$, so the line bundles are distinct.

Every component of (45) is nonzero. Indeed, trivialize L with transition functions g_{ab} . An affine projective coordinate transforms as $z_{i,b} = g_{ab}^{\pm 1} z_{i,a}$. Differentiation gives the off-diagonal tangent cocycle

$$\pm d \log(g_{ab}) \otimes z_i \partial_{z_i}.$$

The Cartan field $z_i \partial_{z_i}$ spans the $\mathcal{O}_B$ -summand in $\pi_* R_i = L^{-1} \oplus \mathcal{O}_B \oplus L$; higher direct images vanish because R_i has nonnegative degree on each fiber factor. The j -th base component therefore has, in the corresponding direct summand

$$H^1(B, \mathcal{O}_B(-2h_j)) \subset H^1(X, R_i \otimes Q_j^{-1}),$$

the nonzero coefficient 1 or -1 of $c_1(L)$. This proves complete four-cycle support for the actual tangent extension, rather than for an abstract graph realization.

7.2. An actual ample equal-slope class. The intersection ring has relations

$$h_1^2 = h_2^2 = 0, \quad \eta_i^2 = \ell^2/4 = -h_1 h_2/2, \quad \int_X \eta_1 \eta_2 h_1 h_2 = 1.$$

Thus, for $H_r = \eta_1 + \eta_2 + r(h_1 + h_2)$,

$$(46) \quad H_r^4 = 24r^2 - 4, \quad c_1(R_i)H_r^3 = 12r^2 - 4, \quad c_1(Q_j)H_r^3 = 12r.$$

Equality of the four line slopes occurs at

$$(47) \quad r_* = \frac{3 + \sqrt{21}}{6} > 1, \quad H_0 = \eta_1 + \eta_2 + r_*(h_1 + h_2).$$

This is a real ample class, not an integral polarization. At the wall,

$$H_0^4 = 16 + 4\sqrt{21}, \quad c_1(R_i)H_0^3 = c_1(Q_j)H_0^3 = 6 + 2\sqrt{21}.$$

All four line factors are stable and equal-slope there. They meet every hypothesis of Theorem 1.1.

The anticanonical class is $-K_X = 2(\eta_1 + \eta_2 + h_1 + h_2)$. It is nef and big, but is on the boundary in (44), so X is weak Fano and not Fano. No Fano or Kähler–Einstein assertion is used in the example.

We need a full chart of demands, not just the one-parameter classes H_r . Let C_v be the four factor Chern classes and $K_v = C_v - \frac{1}{4} \sum_w C_w$. The K_v span the three-dimensional primitive hyperplane at H_0 , because the C_v are independent and $K_v H_0^3 = 0$. The real ample Hodge index theorem makes the pairing $A, B \mapsto ABH_0^2$ negative definite on that hyperplane; see [GRT16, Theorem 6.4 and Corollary 6.5]. It follows that the centered slope derivative has rank three and kernel $\mathbb{R}H_0$.

Volume changes nontrivially along that kernel: $dV(H_0)[H_0] = 4V_0 > 0$. The analytic inverse-function theorem on the hypersurface $V(H) = V_0$ therefore gives a full local chart

$$(48) \quad H = H(s), \quad b(H(s)) = s, \quad \sum_v s_v = 0, \quad V(H(s)) = V_0.$$

Choose a smooth Kähler representative family with $\|\omega_s - \omega_0\|_{C^a} = O_a(|s|)$, by taking bounded closed representatives of divisor coordinates and shrinking the chart. Such control is a choice of representatives, not a consequence of fixed volume alone.

In vertex order (R_1, R_2, Q_1, Q_2) , put

$$x_i = b_{R_i}, \quad y_j = -b_{Q_j}.$$

The positive incidence cone is exactly

$$(49) \quad x_1, x_2, y_1, y_2 > 0, \quad x_1 + x_2 = y_1 + y_2 = \sigma.$$

Necessity follows by summing incoming and outgoing weights; the positive lift $k_{ij} = x_i y_j / \sigma$ proves sufficiency. Theorem 1.1 identifies this with the actual local stable chamber of T_X . Its closure is the semistable locus: closed limits of stable polarizations satisfy each subsheaf slope inequality, while a negative coordinate destabilizes via a target line or the kernel of a source quotient. The section $\sigma = 1$ of the closed demand cone is a square, so it is nonsimplicial. The full polarization space has the additional free volume coordinate.

7.3. A symmetry which determines the leading weights. The involution exchanging the two projective factors preserves H_0 . Choose an invariant Kähler representative ω_0 by averaging. Choose compatible factor Hermitian–Einstein metrics: exchange the two R_i metrics, and take invariant metrics on Q_j . The tangent sequence is natural under this involution, which exchanges its two Cartan extension classes for each fixed j . Pullback is unitary on the coefficient Dolbeault complexes. It commutes with the differential and its adjoint, so uniqueness of harmonic representatives gives

$$c_{1j}(0) = c_{2j}(0) = q_j > 0.$$

Scalar changes of the factor metrics allow the equivalent form $c_{ij}(0) = a_i q_j$, without changing any cycle ratio. The reference cycle ratio is therefore one.

Only the reference form was required to be invariant. The full chart contains $x_1 \neq x_2$; its current coefficient array need not have rank one. Smoothness gives $\log(c_{ij}(s)/c_{ij}(0)) = O(|s|)$. On the stable part of (48), $\|s\|_1 = 2\sigma$, so this error is $O(\sigma)$. Balancing the reference coefficients yields

$$(50) \quad k_{ij}^0 = \frac{x_i y_j}{\sigma}, \quad L^0 = \begin{pmatrix} \text{diag}(x) & -xy^T/\sigma \\ -yx^T/\sigma & \text{diag}(y) \end{pmatrix}.$$

Proposition 2.3 and Theorem 1.1 imply

$$\lambda_j(h_{\text{HE},s}) = \nu_j(L^0(s))(1 + O(\sigma)), \quad 1 \leq j \leq 3, \quad \lambda_4 \geq \gamma > 0,$$

uniformly through all positive facet ratios. The symmetry fixes the cycle data, not the absolute q_j . For example, the diagonal metric still uses

$$w_{R_i} = x_i/a_i, \quad w_{Q_j} = \sigma q_j/y_j, \quad W_v = w_v / \left(\prod_w w_w \right)^{1/4}.$$

The model $\Phi_s^*(\bigoplus_v W_v h_{v,s})$ has the same determinant as the actual metric and satisfies (10). Thus the spectral simplification does not discard the harmonic information needed to describe metrics.

7.4. Three spectral orders on the fixed tangent bundle. For $0 < \varepsilon \ll 1$, prescribe the demands

$$(51) \quad x_1 = \varepsilon^3, \quad x_2 = \varepsilon, \quad y_1 = \varepsilon^2, \quad y_2 = \varepsilon - \varepsilon^2 + \varepsilon^3.$$

They lie in (49) and are realized by the actual ample chart (48). Let $\nu_1 \leq \nu_2 \leq \nu_3$ be the positive eigenvalues of (50). Their elementary symmetric functions satisfy

$$(52) \quad e_1 = 2\varepsilon + O(\varepsilon^3), \quad e_2 = 3\varepsilon^3(1 + O(\varepsilon)), \quad e_3 = 4\varepsilon^6(1 + O(\varepsilon)).$$

For direct verification on the balance hyperplane $x_1 + x_2 = y_1 + y_2 = \sigma$, if $S_x = x_1^2 + x_2^2$, $S_y = y_1^2 + y_2^2$, then

$$e_1 = 2\sigma, \quad e_2 = \frac{4\sigma^2 - S_x - S_y - 2S_x S_y / \sigma^2}{2}, \quad e_3 = \frac{4x_1 x_2 y_1 y_2}{\sigma}.$$

The last formula follows either by a principal cofactor or from Proposition 6.1.

These formulas separate the roots without selecting eigensections. Since $\nu_3 \geq e_1/3$ and $e_2 \geq \nu_3(\nu_1 + \nu_2)$, we have $\nu_1 + \nu_2 = O(\varepsilon^2)$, hence $\nu_3 = 2\varepsilon(1 + O(\varepsilon))$. Next

$$\nu_1 \nu_2 = 2\varepsilon^5(1 + O(\varepsilon)), \quad \nu_1 + \nu_2 = \frac{3}{2}\varepsilon^2(1 + O(\varepsilon)).$$

Their product divided by the square of their sum is $O(\varepsilon)$, so the smaller root is negligible in the sum. Therefore

$$\nu_1 = \frac{4}{3}\varepsilon^3(1 + O(\varepsilon)), \quad \nu_2 = \frac{3}{2}\varepsilon^2(1 + O(\varepsilon)), \quad \nu_3 = 2\varepsilon(1 + O(\varepsilon)).$$

The actual relative error is also $O(\sigma) = O(\varepsilon)$. Taking $\varepsilon = t^2$ gives a smooth approach of actual ample forms on which

$$(53) \quad \lambda_1 = \frac{4}{3}t^6(1 + O(t^2)), \quad \lambda_2 = \frac{3}{2}t^4(1 + O(t^2)), \quad \lambda_3 = 2t^2(1 + O(t^2)), \quad \lambda_4 \geq \gamma.$$

The trace-free L^2 inverse norm consequently grows as $\frac{3}{4}t^{-6}(1 + O(t^2))$. These are endomorphism energies and a linear inverse norm; nonlinear metric-response and actual-flow conclusions require separate arguments.

The same line factors also give a non-star tree example on this fourfold. Retain three of the four nonzero tangent extension components and set the fourth to zero. The resulting holomorphic extension remains integrable and has the path support in Figure 1(a). It need not be the tangent bundle, but the full demand chart and Theorem 1.1 apply to it. Its leading weights are the three cut sums in (37). For the path $Q_1 \rightarrow R_1$, $Q_1 \rightarrow R_2$, $Q_2 \rightarrow R_2$, these are x_1 , $y_1 - x_1$, and y_2 . Its stable chamber is therefore the subcone $y_1 > x_1$ of (49).

The higher-dimensional products $\mathbb{P}_B(\mathcal{O} \oplus \mathcal{O}(1^k, -1^k)) \times_B \mathbb{P}_B(\mathcal{O} \oplus \mathcal{O}(1^k, -1^k))$, with $B = (\mathbb{P}^1)^{2k}$, lead to the broader geometric realization problem. The fourfold already shows how the same actual bundle can carry chamber, extension, and complete small-spectrum descriptions. Its construction uses no metric-response or dynamical theorem as a premise.

8. SCOPE AND SUPPORTING CALCULATIONS

The analytic mechanism proved here has three ingredients: a finite parallel algebra at the graded bundle, a uniformly invertible elliptic complement, and curvature identities which control the remaining error at each weak cut. In the distinct-factor two-step case, the parallel algebra is the scalar vertex space and every amplitude assignment is integrable. This is the setting in which the graph description has been justified throughout the whole positive chamber.

Repeated stable factors replace the scalar parallel space by a matrix algebra. Composable arrows require an integrability equation, and singular graded sheaves require a different analytic setting. The present theorem does not resolve those problems. The full determinant asymptotic in Corollary 6.2 has the same two-step scope; its positive prefactor retains the graded bundle's global spectral data.

The tree and cyclic examples distinguish the information needed for a quantitative theory. On trees, ranks and slope demands determine the leading spectrum. Cycles can retain harmonic orbit data, while actual model metrics retain harmonic coefficients even when a symmetry simplifies the spectrum. These distinctions remain necessary when the spectral construction is combined with geometric realizability or with metric and flow estimates. The latter require their own parameter and dynamical arguments.

All finite data used above can be checked directly from the printed matrices and intersection relations. The accompanying exact-arithmetic programs also recompute the path counterexample, the four-cycle eigenvectors, the nonunit-rank product, the fourfold intersections and the three-scale symmetric coefficients.¹ Their records use rational arithmetic and

¹The sources are `figures/code/compute06_graphs.py` and `figures/code/compute07_tangent.py`; they require only the Python standard library. The support diagram is given by `figures/code/fig06_supports.tex`.

the exact field $\mathbb{Q}(\sqrt{21})$. They provide reproducible finite calculations, not numerical solutions of the Hermitian–Einstein equation. The metric existence, relative spectral estimates and geometric parameter identifications are established by the proofs above.

REFERENCES

- [CT24] A. Clarke and C. Tipler, *Semi-stability and local wall-crossing for Hermitian Yang–Mills connections*, J. Symplectic Geom. **22** (2024), no. 3, 673–694; arXiv:2304.05245.
- [Del25] R. Delloque, *Continuity of HYM connections with respect to metric variations*, J. Lond. Math. Soc. (2) **112** (2025), no. 1, e70219; arXiv:2403.16814v4.
- [GRT16] D. Greb, J. Ross, and M. Toma, *Variation of Gieseker moduli spaces via quiver GIT*, Geom. Topol. **20** (2016), no. 3, 1539–1610; arXiv:1409.7564v2.
- [Kob87] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Publications of the Mathematical Society of Japan **15**, Kanō Memorial Lectures **5**, Iwanami Shoten and Princeton Univ. Press, 1987.
- [NZ13] A. Netser Zernik, *Taylor expansion proof of the matrix tree theorem—Part II*, arXiv:1308.2160v1 (2013).
- [Sco07] S. Scott, *Zeta forms and the local family index theorem*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 1925–1957. doi:10.1090/S0002-9947-06-04321-2.
- [ST24] L. M. Sektnan and C. Tipler, *Hermitian Yang–Mills connections on pullback bundles*, Calc. Var. Partial Differential Equations **63** (2024), no. 1, Paper No. 13, 63 pp.; arXiv:2006.06453v2.
- [Wue26a] B. J. Wuebben, *Small Dolbeault eigenvalues near stability walls*, preprint, 2026; Zenodo:23682045.

BERND JOHANNES WUEBBEN, NEW YORK, NY, wuebben@gmail.com