

UNIVERSAL CURVE COUNTING UNDER CONIFOLD TRANSITIONS

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ABSTRACT. For projective conifold transitions of Calabi–Yau threefolds, we lift the numerical degeneration formula to Pardon’s rational universal curve-counting algebra. We assume a smoothing with smooth algebraic total space and torsion-free second homology at both endpoints. After removing the exceptional local-curve factors, effective lifts to the common blow-up give a support bound that makes specialization finite in every bounded target degree. We carry out this specialization completely in the primitive line class of a plane in the smoothing of a compact four-node transition. The source calculation uses an Euler-characteristic bound for pure curves on two intersecting rational surfaces, allowing arbitrary multiple exceptional components. Together with a finite genus-zero mirror calculation, this bound determines all 37 candidate universal coefficients: eight are nonzero, and their sum agrees with an independent stable-pair calculation on a plane in the smoothing. We also determine the smoothing coefficient in twice the line class. If the resolution has integral product exponents, then so does the smoothing.

1. INTRODUCTION

Curve-counting invariants of a Calabi–Yau threefold change under a conifold transition by summing over curve classes on the small resolution. The numerical formula holds in every genus, but its interpretation in a universal coefficient algebra requires a single support bound valid before any numerical evaluation. Such a bound must also account for the whole exceptional contribution and specify the completion in which specialization is performed.

We combine Pardon’s universal curve-counting algebra with the numerical conifold formula to obtain this interpretation, and then compute every source contribution in one nonexceptional class of a compact example. The geometric step in the computation is an Euler-characteristic bound for pure curves, including nonreduced exceptional components. It makes a finite genus-zero calculation sufficient to determine the full universal coefficients.

For the first result, let $Y \rightarrow X_0 \rightsquigarrow S$ be a projective conifold transition with trivial canonical bundles, torsion-free second integral homology, and a smoothing whose total space is a smooth algebraic fourfold. Fix the common-complement marking, giving $\pi : H_2(Y; \mathbb{Z}) \rightarrow H_2(S; \mathbb{Z})$. Pardon’s local generators $x_{g,m}$ form the rational polynomial coefficient algebra H , and $f_g(z) = 1 + \sum_{m \geq 1} x_{g,m} z^m$ is the local series. Write $\mathcal{Z}_V$ for the universal effective-cycle series of a threefold V . Theorem 3.2 proves

$$\mathcal{Z}_S = \mathrm{Sp}_\pi \left(\frac{\mathcal{Z}_Y}{\prod_i f_0(Q^{[C_i]})} \right),$$

where C_i are the exceptional curves. Every coefficient is a finite sum of polynomials in H . The proof uses an ample divisor on the common blow-up to bound all effective lifts in bounded target degree. Empty-contact degeneration then controls universal logarithmic support. We

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apply the total-space-class form of degeneration to handle projected classes which might be ineffective on the smoothing.

The compact application uses a four-node toric hypersurface transition, specified in Section 4 and Appendix A. Its smoothing contains a plane whose primitive line class is $\beta = (1, 1, 0)$. In a fixed integral source basis, $\pi(n_1, \dots, n_6) = (n_2, n_3, n_4)$. There are 37 possible normalized contributions over β under our finite support bound. Section 5 proves that every pure curve in the three bidegrees needed for this calculation has Euler characteristic at least one, even after adding arbitrary multiple exceptional components. The proof identifies the contracted locus as two rational surfaces and two isolated rational curves, then bounds intersection lengths with multiple $(-1, -1)$ -curves. This removes all positive-genus primitive generators from the relevant universal logarithm. Section 6 consequently determines every one of the 37 coefficients, of which eight are nonzero. Their sum is $3x_{0,1}$.

An independent stable-pair calculation on the smoothing plane gives

$$U_{S,\beta} = 3x_{0,1}, \quad U_{S,2\beta} = 3x_{0,2} + 3x_{0,1}^2 - 6x_{0,1}.$$

These are identities in H . The Euler-characteristic arguments, rather than agreement of finitely many GW invariants, exclude the remaining primitive generators.

Pardon's structure theorem and connected evaluations are the algebraic inputs throughout [Par25]. The all-genus numerical conifold formula and its finite class sums are already established by Lee–Lin–Wang [LLW18, Proposition 2.1]. Lin–Wang give disconnected GW and PT transition formulas with exceptional factors [LW25, Proposition 3.5], and a transfer theorem for stationary descendent correspondence. Our unmarked statement does not extend that descendent theorem. Its content is the coefficientwise finite identity in the universal algebra, together with the complete source calculation just described. The values 3 and -6 on the plane are familiar local-curve-counting values; their role here is to determine universal polynomials and to check the compact specialization independently.

The product exponents obey a finite sum formula, Corollary 3.3, so integrality on the resolution implies integrality on the smoothing. The hypothesis is substantive: the scalar equivariant local generators do not give integer exponents for every projective Calabi–Yau threefold. The companion paper [Wue26b] constructs a counterexample on the same resolution Y_+ used here. Section 7 explains the distinction from the integer elementary expansion of Ionel–Parker and records the scope of the transfer statement. The specialization theorem and the compact calculation use only the rational universal algebra.

2. UNIVERSAL SERIES AND RECONSTRUCTION

We use the Chern-zero, virtual-dimension-zero part of Pardon's semi-Fano universal curve-counting algebra [Par25, Theorem 1.2]. With rational coefficients it is

$$H = \mathbb{Q}[x_{g,m} : g \geq 0, m \geq 1], \quad I = (x_{g,m}), \quad f_g(z) = \sum_{m \geq 0} x_{g,m} z^m, \quad x_{g,0} = 1.$$

Each element of H is a polynomial; there is no completion in genus. The coproduct and division operations are algebra homomorphisms with

$$(1) \quad \Delta x_{g,m} = \sum_{a+b=m} x_{g,a} \otimes x_{g,b}, \quad \rho_d x_{g,m} = \begin{cases} x_{g,m/d} & d \mid m, \\ 0 & d \nmid m. \end{cases}$$

These are the Chern-zero specializations of [Par25, Definitions 3.9, 3.12, Lemmas 4.12–4.13 and Theorem 7.6]; the specialization also uses vanishing in negative virtual dimension.

Let Γ be an additive submonoid of a finite-rank torsion-free lattice, equipped with an integral degree positive off zero and having finite sublevel sets. The notation $H[[\Gamma]]$ means completion in this degree. A series $A = \sum A_\gamma Q^\gamma$, with $A_0 = 1$ and $A_\gamma \in I$ for $\gamma \neq 0$, is called compatible if

$$(2) \quad \Delta A_\gamma = \sum_{\alpha+\eta=\gamma} A_\alpha \otimes A_\eta, \quad \rho_d A_\gamma = A_{\gamma/d}.$$

The last coefficient is zero when γ/d does not belong to Γ . This condition expresses compatibility with addition and integer multiplication of cycles.

For a smooth projective Calabi–Yau threefold V with torsion-free $H_2(V; \mathbb{Z})$, let Γ_V be its monoid of effective integral curve classes. An ample divisor gives the required degree. To see finiteness, represent a basis of the dual real homology space by smooth two-forms. Their bounded norms bound every coordinate of an effective cycle by a constant times its Kähler area. Bounded area thus bounds its integral class. The cycle locus $\mathcal{C}_\gamma(V)$ is compact and open and closed in the cycle space. Its characteristic function defines

$$U_{V,\gamma} = (V, 1_{\mathcal{C}_\gamma(V)}) \in H, \quad \mathcal{Z}_V = \sum_{\gamma \in \Gamma_V} U_{V,\gamma} Q^\gamma.$$

Pulling back the characteristic function under cycle addition gives $\sum_{\alpha+\eta=\gamma} 1_\alpha \boxtimes 1_\eta$, and pullback under multiplication by d gives $1_{\gamma/d}$. Thus $\mathcal{Z}_V$ is compatible. The constant term represents the empty cycle.

Write $\lambda_h = \text{GW}_{h,0}$ for connected, unmarked GW evaluation. Pardon’s triangularity result [Par25, Lemma 7.7] gives

$$(3) \quad \lambda_h(I^2) = 0, \quad \lambda_h(x_{g,1}) = 0 \ (g > h), \quad \lambda_h(x_{h,1}) = 1.$$

For $\gamma \neq 0$, $\lambda_h(U_{V,\gamma}) = N_{h,\gamma}^V$. Contracted components attached to a nonconstant connected stable map are included; wholly contracted connected components are excluded from the corresponding disconnected series.

Lemma 2.1. *Compatible series are determined by their connected evaluations in all classes and genera. Each has a unique factorization*

$$(4) \quad A = \prod_{\gamma \neq 0, g \geq 0} f_g(Q^\gamma)^{e_{\gamma,g}}, \quad e_{\gamma,g} \in \mathbb{Q},$$

with finitely many nonzero exponents at each class.

Proof. Put $p_{g,m} = [z^m] \log f_g(z)$. These are polynomial coordinates on H , and are primitive. A primitive polynomial in these coordinates is additive, hence linear over $\mathbb{Q}$. Moreover $\rho_d p_{g,m} = p_{g,m/d}$ if $d \mid m$, and zero otherwise. By applying ρ_M at the largest multiplicity in a finite linear combination,

$$(5) \quad \text{Prim}(H) \cap \bigcap_{d>1} \ker \rho_d = \bigoplus_{g \geq 0} \mathbb{Q} x_{g,1}.$$

At a least-degree class where two compatible series differ, the difference is primitive and killed by all ρ_d , $d > 1$. Its least nonzero genus coefficient is detected by (3). This proves reconstruction.

For factorization, each coefficient of $L = \log A$ is primitive. Write $L_\gamma = \sum a_{\gamma;g,m} p_{g,m}$. Comparing the coefficient of $p_{g,1}$ in $\rho_m L_\gamma = L_{\gamma/m}$ gives $a_{\gamma;g,m} = a_{\gamma/m;g,1}$. Thus

$$L_\gamma = \sum_{m: \gamma/m \in \Gamma} \sum_g e_{\gamma/m,g} p_{g,m}, \quad e_{\gamma,g} = a_{\gamma;g,1}.$$

This proves both existence and uniqueness. Polynomiality and bounded degree make every displayed coefficient a finite sum. $\square$

We also use the following support consequence. If $F \subset \Gamma \setminus \{0\}$ is closed under positive integer multiplication, and $\lambda_h((\log A)_\gamma) = 0$ for every h and $\gamma \notin F$, then $\text{Supp log } A \subset F$. Indeed, every proper divisor of a class outside F is outside F . Induction on degree, followed by (5) and (3), proves the assertion. Since λ_h kills I^2 , it is equivalent to test A_γ .

Lemma 2.2. *For every $g \geq 0$,*

$$\text{PT}(x_{g,1}) = z^{1-g}(1 + O(z)).$$

If $R \in \text{Prim}(H) \cap \bigcap_{d>1} \ker \rho_d$ and $\text{PT}(R) \in z\mathbb{Q}[[z]]$, then $R = ([z] \text{PT}(R))x_{0,1}$.

Proof. Choose a genus- g curve C and a degree- $g-1$ line bundle L with $H^0(L) = H^1(L) = 0$; for $g = 0$ use $\mathcal{O}(-1)$. The bundle $L \oplus K_C L^{-1}$ has no cohomology and determinant K_C . Every compact degree-one cycle in its total space is the graph of a section, hence is C . Pardon's definition of $x_{g,1}$ can therefore be taken with no cut-down sections. Purity forces a curve scheme with this cycle to be C itself. The minimal stable pair is $\mathcal{O} \rightarrow \mathcal{O}_C$, at Euler characteristic $1 - g$; its moduli space is a smooth point, with virtual count $+1$. This proves the leading term. By (5), R is a finite sum of $x_{g,1}$. Its largest genus would contribute a unique lowest power of z . All positive genera therefore vanish under the stated lower bound. $\square$

3. SUPPORT AND SPECIALIZATION

Let $\mathcal{X} \rightarrow B$ be a flat projective morphism from a smooth complex algebraic fourfold to a smooth curve, with central fiber X_0 having only ordinary double points. Suppose a nearby fiber S and a projective small resolution $p : Y \rightarrow X_0$ are smooth Calabi–Yau threefolds with trivial canonical bundles and torsion-free second integral homology. Fix a marking of the common complement of the node neighborhoods and a relatively ample line bundle, restricting to H_0 and H_S . Write C_i for the exceptional curves and set

$$W = \text{Bl}_{\coprod_i C_i} Y, \quad \phi : W \rightarrow Y, \quad q = p\phi, \quad E = \sum_i E_i, \quad j = \text{PD}_W^{-1} \phi^* \text{PD}_Y.$$

Here j is an integral homology map, not the strict transform of a chosen curve. Effective means represented by an effective integral one-cycle, not merely belonging to a closed real cone.

Proposition 3.1. *The marking gives an integral surjection*

$$\pi : H_2(Y; \mathbb{Z}) \longrightarrow H_2(S; \mathbb{Z}), \quad \ker \pi = \langle [C_i] \rangle.$$

The set

$$(6) \quad F = \{\gamma \neq 0 : j\gamma \text{ is effective on } W\}$$

is contained in Γ_Y , is additive after adjoining zero, contains no class in $\ker \pi$, and has finitely many elements of bounded $H_S \cdot \pi\gamma$ degree. The monoid $\Gamma_T = \langle \Gamma_S \cup \pi(F) \rangle$ has positive, locally finite degree $H_S \cdot (-)$.

Proof. Let U be the common compact complement and $L = \coprod_i (S^2 \times S^3)$ its boundary. The local resolution pieces retract to S^2 , with boundary inclusion inducing an isomorphism on H_2 ; the smoothing pieces retract to S^3 . If $a : H_2(L) \rightarrow H_2(U)$ is inclusion, integral Mayer–Vietoris gives

$$H_2(Y) = (H_2(U) \oplus \mathbb{Z}^r) / \{(a(v), -v) : v \in \mathbb{Z}^r\} \simeq H_2(U), \quad H_2(S) = H_2(U) / a(\mathbb{Z}^r).$$

This proves the asserted quotient, including its integral meaning.

Locally a node is the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with polarization $\mathcal{O}(1, 1)$. Blowing up its vertex gives $\text{Tot}(\mathcal{O}(-1, -1))$, as does blowing up the zero section of either small resolution. These identifications glue, so $W = \text{Bl}_{\text{Sing } X_0} X_0$. In particular $-E$ is q -ample, and for some fixed integer $N > 0$ the divisor

$$A = Nq^*H_0 - E$$

is ample. The projection formula gives $\phi_* j\gamma = \gamma$. Restriction of $\phi^* \text{PD}_Y \gamma$ to E_i factors through $H^4(C_i) = 0$, hence $E_i \cdot j\gamma = 0$. The marked polarization agrees on U , and therefore

$$(7) \quad A \cdot j\gamma = NH_S \cdot \pi\gamma.$$

For $\gamma \in F$ the left side is positive. Pushforward proves effectivity on Y , and sums of effective lifts prove additivity. Bounded projected degree bounds ample degree on W . The homology boundedness argument of Section 2, and injectivity of j , prove finiteness. Finally, the generators of Γ_T of degree at most D form a finite set, and a sum of degree at most D uses at most D nonzero generators. This proves the final assertion. $\square$

The integral blow-up decomposition also gives

$$(8) \quad \text{im } j = \bigcap_i \ker(E_i \cdot -).$$

Indeed $H_2(W) = jH_2(Y) \oplus \bigoplus_i \mathbb{Z}a_i$, where a_i is a fiber of $E_i \rightarrow C_i$ and $E_i \cdot a_j = -\delta_{ij}$.

Theorem 3.2. *Under the hypotheses of this section, put*

$$\mathcal{E}_Y = \prod_i f_0(Q^{[C_i]}), \quad A_Y = \mathcal{Z}_Y / \mathcal{E}_Y.$$

Then $\text{Supp}(A_Y - 1) \subset F$. Coefficientwise summation along π is finite in bounded target degree, and

$$(9) \quad \mathcal{Z}_S = \text{Sp}_\pi A_Y \quad \text{in } H[\Gamma_T].$$

Here $\mathcal{Z}_S$ is extended by zero outside Γ_S , and $[T^\beta] \text{Sp}_\pi A_Y = \sum_{\pi\gamma=\beta} (A_Y)_\gamma$.

Proof. We first remove the entire exceptional contribution. The divisor p^*H_0 is nef and vanishes precisely on contracted curves, so $\Gamma_Y \cap \ker \pi$ is a face of Γ_Y , consisting of sums $\sum m_i [C_i]$ with $m_i \geq 0$. Restriction to this face commutes with products, inverses and logarithms. In the resolved local node $V = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)^{\oplus 2})$, the only compact degree- m cycle is $m[\mathbb{P}^1]$: a compact curve maps to a point in the affine node and must lie in the exceptional fiber. In Pardon's Definition 4.7, one can take $N = 0$ and no cut-down sections, giving $(V, 1_{m[\mathbb{P}^1]}) = x_{0,m}$. Locality and the product operation for the disjoint node neighborhoods give

$$(Y, 1_{\sum m_i C_i}) = \prod_i x_{0,m_i}.$$

At fixed class there are finitely many multiplicity vectors, bounded by an ample degree on Y . Summing over them shows that the face restriction of $\mathcal{Z}_Y$ is exactly $\mathcal{E}_Y$, even when the exceptional classes satisfy relations. Thus A_Y restricts to 1 on this face.

Next deform Y to the normal cone of $\coprod C_i$. Its central fiber is $W \cup_E \coprod_i \mathbb{P}_{C_i}(N_{C_i/Y} \oplus \mathcal{O})$. For total relative class δ on W , contact order μ and number of relative markings ρ , the virtual dimension is

$$(10) \quad c_1(W) \cdot \delta + \rho - \mu = \rho - 2\mu,$$

since $K_W = E$. This holds also for disconnected W parts by summing over vertices. If contacts are nonempty, $\mu \geq \rho > 0$, and the virtual class vanishes. A surviving connected noncontracted term lies entirely on W ; a term on a projective bundle pushes to a multiple of $[C_i]$. Equation (8) therefore gives

$$(11) \quad N_{h,\gamma}^Y = N_{h,j\gamma|\emptyset}^{(W,E)} \quad (\pi\gamma \neq 0).$$

This is the unmarked resolution comparison used by Lee–Lin–Wang [LLW18, Section 2.1]. In particular all genera vanish when $\pi\gamma \neq 0$ and $\gamma \notin F$. For $L_Y = \log A_Y$, connected evaluation at a noncontracted class equals $N_{h,\gamma}^Y$; at a contracted nonzero class L_Y itself is zero. The support consequence of Lemma 2.1 gives $\text{Supp } L_Y \subset F$, and additivity of $F \cup \{0\}$ gives $\text{Supp}(A_Y - 1) \subset F$.

Proposition 3.1 now licenses all sums in $\text{Sp}_\pi A_Y$, including products. Coproduct commutes with the finite sums. For division, only $\gamma = d\eta$ contributes, and torsion-freeness of the target gives

$$\rho_d \sum_{\pi\gamma=\beta} (A_Y)_\gamma = \sum_{\pi\eta=\beta/d} (A_Y)_\eta.$$

Thus the specialization is compatible. The zero extension of $\mathcal{Z}_S$ is compatible as well: sums and positive multiples of effective classes remain effective.

It remains to compare connected evaluations also at classes of Γ_T which need not be effective on S . Semistable reduction of the smoothing gives a smooth total space $\mathcal{V}$ with central fiber $W \cup_E \coprod_i Q_i$, where Q_i is a smooth quadric threefold and E_i is its hyperplane section [LW25, Section 3.1]. If $i : S \rightarrow \mathcal{V}$ and $i_0 : W \rightarrow \mathcal{V}$ are inclusions, the common-complement marking gives

$$(12) \quad i_{0*}j\gamma = i_*\pi\gamma.$$

Indeed a representative in U has zero exceptional intersections on W , hence represents $j\gamma$ by (8); its transport through the smooth complement supplies the required cylinder. Node monodromy is trivial on $H_2(S)$, and the invariant cycle theorem gives rational injectivity of i_* , as in [LW25, Lemmas 2.18 and 3.4]. Its integral kernel is torsion and therefore zero here.

For every $\beta \in \Gamma_T$, the class $i_*\beta$ is effective in $\mathcal{V}$: this holds for generators from S , and for generators from F by (12), then for their sums. Apply the degeneration formula graded by this total-space class [LW25, Theorem 2.16], retaining connected dual graphs. This grading restricts the virtual-cycle formula to open-and-closed components of the map moduli space, so its smoothing side may be empty. It is zero when β is ineffective on S ; injectivity of i_* otherwise identifies its sole class with β . Nonempty contacts again vanish by (10). A nonzero empty-contact curve on Q_i is impossible because E_i is ample. Surviving classes on W are exactly $j\gamma$ with $\gamma \in F$ and $\pi\gamma = \beta$. Consequently, for all $0 \neq \beta \in \Gamma_T$ and every h ,

$$N_{h,\beta}^S = \sum_{\substack{\gamma \in F \\ \pi\gamma=\beta}} N_{h,j\gamma|\emptyset}^{(W,E)} = \sum_{\substack{\gamma \in F \\ \pi\gamma=\beta}} N_{h,\gamma}^Y.$$

These are precisely the connected evaluations of the two compatible series in (9). Lemma 2.1 proves their equality. In particular no preservation of effectivity under smoothing was assumed. $\square$

Corollary 3.3. *The rational product exponents satisfy*

$$e_{\beta,g}(S) = \sum_{\substack{\gamma \in F \\ \pi\gamma=\beta}} e_{\gamma,g}^{\text{red}}(Y).$$

In particular an integral product expansion on Y implies one on S .

Proof. Take logarithms in (9) and compare the $p_{g,1}$ coefficients in Lemma 2.1. Normalization changes only exceptional exponents, so the reduced and original exponents agree at every noncontracted class. All sums are finite. $\square$

There is a useful chamber independence in this formulation. If Y_- and Y_+ are two projective small resolutions of the same nodal threefold, their blow-ups are the same W . The maps $j_{\pm}$ have common image (8), so $\Phi = \phi_{+*}j_-$ is an integral isomorphism satisfying $j_+\Phi = j_-$ and $\Phi(F_-) = F_+$. Their normalized series agree under this identification: (11) identifies all connected evaluations in the common positive completion, and reconstruction applies. A flopped exceptional class changes sign; this statement does not identify the unnormalized effective expansions.

4. A COMPACT TRANSITION AND ITS FINITE LIFTS

We use the four-node transition denoted X_9 in [Wue26c]. For clarity, the numerical label only identifies the toric example specified below. Its source fan, divisor basis and smoothing fan are printed in Appendix A. The general anticanonical hypersurfaces are taken nondegenerate with respect to their Newton polytopes, with nonzero vertex coefficients.

4.1. The geometric hypotheses and class marking. The nodal model X'_9 is projective, normal, has trivial dualizing sheaf and $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$. Its four nodes admit a projective small resolution Y_- . We use the nodal model and the integral divisor-evaluation injection of [Wue26c, Proposition 3.5, Lemma 9.7 and Section 11.1]. The source Y_+ used for the explicit calculation differs from Y_- by one flop: the original square diagonals 23, 24 are replaced by 67, 24. The relevant square circuits are $v_2 + v_3 = v_6 + v_7$ and $v_2 + v_4 = v_5 + v_7$. The pentagon with vertices v_3, v_4, v_5, v_6, v_8 is subdivided at its interior ray v_9 in both resolutions. Our ray indexing has the pentagon center at 9 and the divisor missing the hypersurface at 10; these two indices are interchanged in [Wue26c].

Let $H_1, \dots, H_6$ be the integral nef basis on Y_+ given in Appendix A, and let $b_1, \dots, b_6$ be the dual curve basis. Identify the full integral lattices of Y_- and Y_+ by the common blow-up, as at the end of Section 3. The four exceptional classes in these coordinates are

	q_1	q_2	q_3	q_4
Y_-	b_6	$-b_5$	$b_1 + b_5 + b_6$	b_1
Y_+	b_6	b_5	$b_1 + b_5 + b_6$	b_1

Their relation on Y_- is $(1, -1, -1, 1)$. More intrinsically, the four divisor-evaluation columns on Y_- are

$$e_6 + e_8 - e_3 - e_9, \quad e_6 + e_7 - e_2 - e_3, \quad e_5 + e_8 - e_4 - e_9, \quad e_5 + e_7 - e_2 - e_4.$$

On the first three columns, rows 3, 4, 9 have determinant -1 . Thus their span is primitive in the ambient integer lattice and in the full curve lattice, which injects into it. This proves that the source and its exceptional quotient are torsion-free; it is more than an ambient rational rank calculation.

The nowhere-zero exceptional relation supplies a global first-order smoothing nonzero at every node. The abstract deformation space is smooth by the nodal case of Friedman's unobstructedness theorem [Fri26, Theorem 1.6]: the singularities are rational, Du Bois and local complete intersections, and the projective resolution satisfies the $\partial\bar{\partial}$ -lemma. To obtain an algebraic family of the kind required in Theorem 3.2, choose a sufficiently high ample embedding of X'_9 . The vanishing of $H^2(\mathcal{O})$ lifts its line bundle, and the vanishing of $H^1(L^m)$ for

large m lifts a projective frame. Thus embedded deformations map smoothly to abstract deformations. Equivalently one can use the Hilbert-scheme construction in [FMR25, Theorem 4.1]. Its local sheaf-generation condition follows from a tangent nonzero at each node; surjectivity onto all local smoothing parameters simultaneously is not needed. Choose a smooth algebraic curve in this Hilbert scheme with the specified tangent. The restricted universal family is projective, and at each node has equation $\sum_{j=1}^4 z_j^2 = s_i(t)$ with $s'_i(0) \neq 0$. Its total fourfold is smooth. Relative duality and base change extend the nowhere-vanishing canonical section near the central fiber, so the nearby fibers have trivial canonical bundle.

The toric smoothing model in Appendix A belongs to this smooth deformation component, by [Wue26c, Theorem 9.25]. Fix transport along a path in its smooth-fiber locus, and choose the marking for which the smoothing divisors K_1, K_2, K_3 descend from H_2, H_3, H_4 . Then the full integral class map is

$$(13) \quad \pi(n_1, \dots, n_6) = (n_2, n_3, n_4).$$

The exceptional subgroup is $\mathbb{Z}b_1 \oplus \mathbb{Z}b_5 \oplus \mathbb{Z}b_6$. These class identifications retain the chosen transport path.

4.2. An explicit bound on the lifts. Blow up the ambient pair cones 24, 67, 49, 39, appending the sums of their rays as rays 11, 12, 13, 14. These four ambient centers meet the hypersurface transversely in its four disjoint exceptional curves. Their possible mutual intersections are outside the hypersurface. The resulting smooth 47-cone fan therefore contains $W = \text{Bl}_{\coprod C_i} Y_+$ as the strict transform. The pulled-back original anticanonical equation defines this strict transform, since none of the ambient centers is contained in the hypersurface. Its divisor coefficients are one on the eleven original rays and two on the four new rays; its Newton polytope is unchanged. Intersecting its ambient Mori cone with zero degree on the four exceptional divisors and on D_{10} gives the necessary inequalities

$$(14) \quad \begin{aligned} n_a &\geq 0, & n_6 &\geq n_1, n_5, & n_1 &\leq n_2 + n_3, & n_1 &\leq n_2 + n_5, \\ n_6 &\leq n_2 + n_3 + n_5, & n_1 + n_6 &\leq 2n_2 + n_3 + n_5. \end{aligned}$$

These follow from the 94 invariant walls of the subdivided fan.

One further inequality is needed. On the common blow-up the divisor

$$\mathcal{D} = \phi^*(2H_2 + 3H_3 + H_4 - H_6) - \sum_i E_i$$

is nef. To check this, its coefficients on rays $0, \dots, 14$ are

$$d = (0, 0, 2, 0, 0, 7, 7, -5, -5, 3, 0, 1, 1, 2, 2).$$

On the single open toric neighborhood $\tilde{P} \setminus D_{10}$ of W , the characters in Table 1 give global sections of $2\mathcal{D}$. For each cone σ maximal among those whose orbits can meet W , the displayed character m satisfies

$$\langle m, v_i \rangle + 2d_i \geq 0 \quad (i \neq 10), \quad \langle m, v_i \rangle + 2d_i = 0 \quad (i \in \sigma).$$

Here the Newton faces are those of the pulled-back original equation. There are 99 cones with nonsingleton faces, each a face of one of the 41 listed cones. The remaining orbits have singleton Newton faces and miss the hypersurface. The sections are global on the same open set and generate along W ; hence $2\mathcal{D}|_W$ is globally generated. Pairing with $j\gamma$ gives

$$(15) \quad n_6 \leq 2n_2 + 3n_3 + n_4.$$

Let $\beta = (1, 1, 0)$ on the smoothing. By Theorem 3.2, all possible nonzero normalized source coefficients over $d\beta$ have class $(a, d, 0, b, c)$, where

$$(16) \quad 0 \leq a \leq 2d, \quad b \geq \max(0, a - d), \quad \max(a, b) \leq c \leq \min(5d, 2d + b, 3d + b - a).$$

There are 37 integer solutions for $d = 1$, and 183 for $d = 2$. For $d = 1$ they are listed compactly by the permitted values of c :

$a \backslash b$	0	1	2	3	4	5
0	0:2	1:3	2:4	3:5	4:5	5
1	1:2	1:3	2:4	3:5	4:5	5
2	—	2	2:3	3:4	4:5	5.

These lists are necessary-condition supersets; effectivity and nonvanishing are not asserted for each entry. The same inequalities give nine candidates in bidegree $(n_2, n_3) = (1, 0)$ and ten in bidegree $(0, 1)$, with $n_4 = 0$.

4.3. The plane on the smoothing.

Proposition 4.1. *The smoothing S contains a plane $D \simeq \mathbb{P}^2$ with $[D] = -2K_1 - K_2 + K_3$ and normal bundle $\mathcal{O}(-3)$. Its line class is primitive and equals $\beta = (1, 1, 0)$. Every effective cycle of class $d\beta$, $d > 0$, is supported on D .*

Proof. Use the smoothing rays r_i of Appendix A and write T_i for their divisors. The open set away from $T_4 \cup T_5$ has cones 1236, 0236, 0123, all unimodular, and ray relation $r_0 + r_1 - 3r_2 + 2r_3 + r_6 = 0$. It is $\text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-3) \oplus \mathcal{O}(2))$, with fiber coordinates u, v , and $T_2 = \{u = 0\}$. The anticanonical equation on $u = 0$ is $av + q_2(x_0, x_1, x_6) = 0$. Its Newton face consists of the six quadratics and the vertex monomial v , so $a \neq 0$. There is no cone containing 24; the only intersection of T_2 with T_5 is the fixed point of the cone 01256, missed by the hypersurface. Thus all of $D = S \cap T_2$ lies in this open set and is the graph of $v = -q_2/a$ over the plane.

The charge matrix gives $T_0 = K_1$, $T_4 = K_3$, $T_5 = K_2 - K_1$, and $T_2 = -2K_1 - K_2 + K_3$. Hence $K_1|_D = K_2|_D = h$ and $K_3|_D = 0$, proving the class and normal bundle statements. Subdivide 01256 by either $\{1256, 0256, 0125\}$ or $\{0156, 0126\}$. Both modifications are isomorphisms near S . Convex support functions give T_0, T_4 nef on the first and T_5, T_4 nef on the second. Thus every effective class (ℓ, m, n) on S satisfies $\ell \geq 0$, $m - \ell \geq 0$, $n \geq 0$. Every component of a cycle in $d\beta$ has class $(a, a, 0)$, $a > 0$, and intersects D in $-3a$. An effective Cartier divisor has nonnegative degree on a curve outside it, so all components lie in D . Finally $K_1 \cdot \beta = 1$ proves primitivity. $\square$

5. AN EULER-CHARACTERISTIC BOUND ON THE RESOLUTION

We now prove the geometric bound which permits a complete universal calculation from the genus-zero data. Continue with the resolution Y_+ and the integral nef divisors of Section 4.

Theorem 5.1. *Every pure one-dimensional Cohen–Macaulay subscheme $Z \subset Y_+$ with*

$$H_4[Z] = 0, \quad (H_2[Z], H_3[Z]) \in \{(1, 0), (0, 1), (1, 1)\}$$

satisfies $\chi(\mathcal{O}_Z) \geq 1$. No connectedness assumption or bound on the multiplicities of exceptional components is required.

5.1. The contracted surfaces. The toric divisor D_0 restricts to H_4 . Its convex integral support function defines a semiample contraction. The minimal cones whose orbits have positive-dimensional fibers are

$$(17) \quad (8), (9), (10), (2, 4), (6, 7).$$

This is a finite face calculation from the fan in Appendix A: form $P_0 = \{m : \langle m, v_i \rangle + \delta_{i0} \geq 0\}$. For a cone σ , impose its equalities to obtain F_σ . The image orbit cone has dimension $4 - \dim F_\sigma$; the orbit fiber is positive-dimensional exactly when this exceeds $\dim \sigma$. Taking minimal cones gives (17).

The divisor D_{10} misses the general hypersurface, since its dual Newton face is a vertex with nonzero coefficient. Thus every H_4 -degree-zero curve is supported on

$$E \cup F \cup C_1 \cup C_2, \quad E = D_9|_{Y_+}, \quad F = D_8|_{Y_+},$$

where C_1, C_2 come from the orbit closures 67, 24. There are no cones containing 678 or 248, and the Newton faces at 679, 249 are singletons. Hence the C_i miss both surfaces. The absence of a cone containing 2467 makes them disjoint.

We need the actual surfaces, rather than only their intersection forms:

$$(18) \quad E \simeq \text{Bl}_{p_1, p_2} \mathbb{P}^2, \quad F \simeq \mathbb{F}_1.$$

For E , the anticanonical face is the primitive edge with endpoints $(0, 1, -1, -1)$ and $(1, -1, -1, -1)$. Its binomial fixes a nonzero torus parameter. Since D_{10} is absent, rays 2, 7 are also excluded on E . The remaining rays 3, 4, 5, 6, 8 project respectively to

$$(1, 0), \quad (0, 1), \quad (-1, 0), \quad (0, -1), \quad (1, 1),$$

with cones 38, 84, 45, 56, 63, the fan of the plane blown up at two distinct torus fixed points. For F , the star away from D_{10} has cones 137, 147, 379, 479. Quotient by the fiber ray 7 gives base rays $(1, 0), (0, 1), (-1, -1), (-1, 0)$, the fan of $\mathbb{F}_1$. The Newton vertex $(1, -1, -1, -1)$ has exponents $(0, 0, 0, 1, 0, 0)$ on rays 1, 3, 4, 7, 9, 10; all other vertices have fiber exponent zero. The equation is therefore monic linear in the fiber coordinate, and its zero locus is a graph over $\mathbb{F}_1$. This proves (18).

Write h for the pullback of a plane line and e_1, e_2 for the exceptional curves of E . The two surfaces intersect transversely in $s = h - e_1 - e_2$, which is the negative section on F . The curves e_i meet s transversely at distinct points q_i . All four curves e_1, e_2, C_1, C_2 are pairwise disjoint $(-1, -1)$ -curves. For e_i the normal sequence has subbundle and quotient $\mathcal{O}(-1)$, since $N_{E/Y_+} = K_E$; it splits by $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. For C_i this is the local node model. The integral class identities are

$$(19) \quad \begin{aligned} s &= b_2, & f &= b_3, & e_1 &= b_1 + b_5 + b_6, & e_2 &= b_6, \\ h &= b_1 + b_2 + b_5 + 2b_6, & [C_1] &= b_5, & [C_2] &= b_1. \end{aligned}$$

Thus the bidegree (H_2, H_3) of a curve in E is $(h \cdot C, 0)$, and that of $bs + af$ on F is (b, a) . The only irreducible curves of bidegree zero in the contracted locus are the four just listed.

5.2. Multiple exceptional components.

Lemma 5.2. *Let M be a pure Cohen–Macaulay curve supported with multiplicity m on a smooth rational $(-1, -1)$ -curve C in a smooth threefold. Then $\chi(\mathcal{O}_M) \geq 2m - 1$. If a reduced curve B near $p \in C$ lies in a smooth surface transverse to C , then $\text{length}_p(M \cap B) \leq m$.*

Proof. The finite conormal filtration has quotients $G_k = I_C^k \mathcal{O}_M / I_C^{k+1} \mathcal{O}_M$, each a quotient of $\mathcal{O}_C(k)^{\oplus(k+1)}$. If its generic rank is r_k , its torsion-free quotient has degree at least kr_k , because its twist by $-k$ is globally generated. Torsion adds nonnegative length, so $\chi(G_k) \geq (k+1)r_k$. Since $G_0 = \mathcal{O}_C$ and $\sum r_k = m$, summation gives $\chi(\mathcal{O}_M) \geq 1 + 2(m-1)$.

If t is an equation of the transverse surface, the completed local ring of M is finite over $\mathbb{C}[[t]]$, and is torsion-free because M is Cohen–Macaulay and t is a parameter. It is free of rank m , so its quotient by t has length m . Since t vanishes on B , the asserted intersection length is at most m . The argument applies to several branches together whenever they lie in the same transverse smooth surface. $\square$

5.3. Positive-degree components and attachments.

Proof of Theorem 5.1. A pure curve has no embedded points. Isolated primary decomposition therefore expresses Z as $B \cup M_1 \cup \cdots \cup M_k$, where each M_i is a pure multiple structure on one of the four disjoint exceptional curves. Every positive-bidegree component has generic multiplicity one by nefness and the stated bidegrees. Its primary ideal is the contraction of its generic localization, of length one, so that primary component is reduced. Their union is B .

The possibilities for B are

$$\begin{array}{l|l} (1,0) & \text{a strict transform } L \text{ of a plane line, including } s, \\ (0,1) & \text{a ruling fiber } f \text{ of } F, \\ (1,1) & \text{an irreducible member of } |s+f| \text{ on } F, \text{ or } L \cup f. \end{array}$$

Indeed a degree-one curve under $E \rightarrow \mathbb{P}^2$ is a line, and the effective cone of $\mathbb{P}_1$ is generated by s, f . All the listed irreducible curves are smooth rational. An irreducible member of $|s+f|$ misses s , and thus misses E and the e_i .

A line of class h misses the e_i and meets s once away from q_1, q_2 . A line of class $h - e_i$ meets e_i once away from q_i and misses s . The line s meets both e_i at the q_i . A ruling fiber meets s once, hence at most one e_i . All these intersections are transverse. Thus $L \cup f$ is either disjoint, with $\chi(\mathcal{O}_B) = 2$, or has one transverse node, with $\chi(\mathcal{O}_B) = 1$.

Only one configuration permits two distinct attachments to an exceptional curve: L has class $h - e_i$ and f passes through q_i . Then L and f are disjoint and $\chi(\mathcal{O}_B) = 2$; at most one e_i has two attachments. If instead $L = s$ and f passes through q_i , both branches lie in the smooth surface F transverse to e_i . Lemma 5.2 bounds their combined intersection length by m_i , rather than $2m_i$.

The union exact sequence now gives

$$\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_B) + \sum_i (\chi(\mathcal{O}_{M_i}) - \text{length}(B \cap M_i)).$$

A multiple component with zero or one attachment contributes at least zero. A component with two distinct attachments contributes at least -1 ; there is at most one, and it occurs only when $\chi(\mathcal{O}_B) = 2$. The result follows in every case. $\square$

For a stable pair $\mathcal{O}_{Y_+} \rightarrow \mathcal{F}$, the image is $\mathcal{O}_Z$ for a pure curve Z , and the cokernel is zero-dimensional. The theorem therefore implies $\chi(\mathcal{F}) \geq 1$. Every stable-pair series in the three bidegrees starts in positive Euler characteristic, including those with arbitrarily large exceptional multiplicities.

6. COMPLETE UNIVERSAL COEFFICIENTS

6.1. The source genus-zero input. We first calculate connected genus-zero invariants on Y_+ without using the smoothing answer. The charge matrix and triple intersections are printed in Appendix A. The relevant mirror theorem is [CCIT19, Theorem 25 and Corollary 28]. Removing ray 10 gives a projective simplicial 24-cone fan and a smooth canonical toric Deligne–Mumford stack. Its nonunimodular maximal cones are 1369, 1389, 1459, 1489, 1569, all of index two. Every nonsmooth orbit has a singleton Newton face and misses the chosen hypersurface. The anticanonical support function is integral and convex: the line bundle is nef, Cartier on the coarse space, and pulled back to the stack. On a genus-zero orbicurve it has trivial stabilizer action and nonnegative degree on each component, hence vanishing first cohomology. It is therefore convex in the cited theorem. Its transverse zero locus is the ordinary smooth threefold Y_+ , all twisted-sector classes restrict to zero, and the Calabi–Yau asymptotics give $J = I/W_0$. The 29-cone refinement changes only the missed locus. The charge lattice is saturated and the ambient Mori cone is the nonnegative orthant in these coordinates. These checks justify using the untwisted Gamma series for the actual source invariants.

For the charge matrix Q put $Q_{a*} = \sum_i Q_{ai}$ and

$$c(n + \rho) = \frac{\Gamma(1 + \sum_a (n_a + \rho_a) Q_{a*})}{\prod_i \Gamma(1 + \sum_a (n_a + \rho_a) Q_{ai})}.$$

Let $W_0, C_a, \hat{C}_{ab}$ be the sums of the scalar, first and second derivatives at $\rho = 0$ of $c(n + \rho)/c(\rho)$ times z^n . This quotient removes the universal zeta constants from the derivatives. Let a_n denote the coefficients of the genus-zero dilogarithm expansion, indexed by the curve class n . The mirror coordinates and prepotential equations are

$$(20) \quad q_a = z_a \exp(C_a/W_0), \quad \sum_{n>0} n_a a_n \operatorname{Li}_2(q^n) = \frac{1}{2} \sum_{b,c} \kappa_{abc} \left(\frac{\hat{C}_{bc}}{W_0} - \frac{C_b C_c}{W_0^2} \right).$$

These are the Frobenius formulas in [DKMMR24, Section 4.4 and Appendix A]. Derivatives at reciprocal-Gamma zeros are retained.

All calculations are exact in

$$\mathbb{Q}[z_1, \dots, z_6]/(z_1^3, z_2^2, z_3^2, z_4, z_5^6, z_6^6).$$

Its 432 monomials contain all the candidates in the three relevant bidegrees and every componentwise smaller exponent. Products, inversion and mirror substitution cannot bring an omitted exponent back into this set. Solving all six equations (20) and extracting the dilogarithm coefficients gives the following complete nonzero table, where $(a, b, c) = (n_1, n_5, n_6)$:

$$(21) \quad \begin{array}{c|ccc} (a, b, c) & (n_2, n_3) = (1, 0) & (0, 1) & (1, 1) \\ \hline (0, 0, 0) & 1 & -2 & 3 \\ (0, 0, 1) & -2 & 1 & -2 \\ (1, 1, 1) & -2 & 1 & -2 \\ (1, 1, 2) & 3 & 0 & 4. \end{array}$$

All other entries of these three sectors in the finite quotient vanish. Every class in the table is indivisible, so $a_n = N_{0,n}^Y$. The finite support bounds ensure that this calculation covers every normalized contribution in those sectors.

6.2. From the numerical table to universal coefficients. Put $A = \mathcal{Z}_{Y_+}/\mathcal{E}_{Y_+}$ and $L = \log A$, and work on the face $n_4 = 0$. Normalization makes the entire bidegree- $(0, 0)$ sector equal to one. Each nonempty exceptional local coefficient has PT realization in $z\mathbb{Q}[[z]]$ by Lemma 5.2; the same is true of each nonconstant coefficient of its inverse series, since the inverse uses finite products at fixed source degree. Theorem 5.1 thus puts every coefficient of A in the three stated bidegrees in $z\mathbb{Q}[[z]]$ after PT evaluation. The same holds for L : its only additional lower-bidegree products have bidegrees $(1, 0)$ and $(0, 1)$ and start at z^2 .

Each L_γ is primitive and, by indivisibility, killed by ρ_d for $d > 1$. Lemma 2.2 therefore implies

$$(22) \quad L_\gamma = N_{0,\gamma}^{Y_+} x_{0,1}$$

in all three sectors. This is the point at which the arbitrary exceptional multiplicities in Theorem 5.1 are essential. The numerical table alone would not exclude positive-genus primitive generators. Higher-genus GW evaluations of $x_{0,1}$ can still be nonzero.

Set $x = x_{0,1}$, $r = Q^{b_2}$, $s = Q^{b_3}$, $u = Q^{b_1+b_5}$, $v = Q^{b_6}$. Equations (21)–(22) give, through these bidegrees,

$$L = x(rP_{10} + sP_{01} + rsP_{11}) \pmod{(r^2, s^2)},$$

where

$$P_{10} = 1 - 2v - 2uv + 3uv^2, \quad P_{01} = -2 + v + uv, \quad P_{11} = 3 - 2v - 2uv + 4uv^2.$$

Thus $[rs]A = xP_{11} + x^2P_{10}P_{01}$. Exactly eight of the 37 candidates have nonzero coefficients:

$$(23) \quad \begin{array}{c|c} \gamma & (A)_\gamma \\ \hline (0, 1, 1, 0, 0, 0) & 3x - 2x^2 \\ (0, 1, 1, 0, 0, 1) & -2x + 5x^2 \\ (0, 1, 1, 0, 0, 2) & -2x^2 \\ (1, 1, 1, 0, 1, 1) & -2x + 5x^2 \\ (1, 1, 1, 0, 1, 2) & 4x - 10x^2 \\ (1, 1, 1, 0, 1, 3) & 3x^2 \\ (2, 1, 1, 0, 2, 2) & -2x^2 \\ (2, 1, 1, 0, 2, 3) & 3x^2. \end{array}$$

The remaining 29 coefficients vanish. Four of the nonzero universal coefficients have zero connected genus-zero value; they arise from the lower-bidegree product. Since $P_{10}(1, 1) = P_{01}(1, 1) = 0$ and $P_{11}(1, 1) = 3$, finite specialization gives

$$(24) \quad \sum_{\pi\gamma=\beta} (A)_\gamma = 3x_{0,1}.$$

The common-blow-up comparison transfers the same normalized table to Y_- . No smoothing invariant was used in its derivation.

6.3. Independent calculation on the plane.

Proposition 6.1. *For the smoothing plane of Proposition 4.1,*

$$U_{S,\beta} = 3x_{0,1}, \quad U_{S,2\beta} = 3x_{0,2} + 3x_{0,1}^2 - 6x_{0,1}.$$

Proof. All cycles in these classes lie on D . In degree one, purity forces the curve scheme to be a reduced line. In degree two a generically reduced curve is a plane conic. A pure curve Z with cycle $2L$ satisfies $I_L^2 \subset I_Z \subset I_L$: the image of I_L^2 vanishes generically and hence vanishes by purity. The rank-one torsion-free sheaf I_L/I_Z is $\mathcal{O}_L(a)$. The normal sequence splits as $N_{L/S} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(-3)$, so there is a surjection $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(3) \rightarrow \mathcal{O}_L(a)$. It forces $a \geq -1$,

whence $\chi(\mathcal{O}_Z) = a + 2 \geq 1$. At equality the $\mathcal{O}(3)$ subbundle maps to zero, so the double line is scheme-theoretically in D . Thus every stable pair in degrees one or two has Euler characteristic at least one, and at equality is a line or conic with its structure sheaf.

Scheme-theoretically the minimal pair spaces are $P_1(S, \beta) = (\mathbb{P}^2)^*$ and $P_1(S, 2\beta) = \mathbb{P}^5$. Indeed surjectivity of the pair section follows by Nakayama, giving the Hilbert functor. For a degree- k plane divisor C , $k = 1, 2$, including a double line, the normal sequence is

$$0 \rightarrow \mathcal{O}_C(k) \rightarrow N_{C/S} \rightarrow \mathcal{O}_C(-3) \rightarrow 0.$$

The last sheaf has no sections and $h^0(\mathcal{O}_C(k))$ is 2 or 5. Thus the ambient Hilbert scheme has exactly the tangent dimension of its smooth closed plane-curve subscheme, which exhausts its points; its local rings are regular and the immersion is an isomorphism. The symmetric obstruction theory gives counts $(-1)^2 e(\mathbb{P}^2) = 3$ and $(-1)^5 e(\mathbb{P}^5) = -6$ [PT10, p. 3].

The primitive extremal class β has no nontrivial effective splitting. Compatibility and Lemma 2.2 give $U_{S,\beta} = 3x_{0,1}$. In degree two, compatibility makes $R = U_{S,2\beta} - 3x_{0,2} - 3x_{0,1}^2$ primitive and killed by every ρ_d , $d > 1$: the removed coproduct cross terms total $(3 + 6)x_{0,1} \otimes x_{0,1}$. Lemma 5.2 gives $\text{PT}(x_{0,2}) \in z^3 \mathbb{Q}[[z]]$, while $\text{PT}(x_{0,1}^2) \in z^2 \mathbb{Q}[[z]]$. Consequently $\text{PT}(R) = -6z + O(z^2)$ with no nonpositive powers, and Lemma 2.2 gives $R = -6x_{0,1}$. $\square$

This verifies (24) using different enumerative input. The two target coefficients are full universal polynomials, not genus truncations. The source coefficients over 2β are not calculated here; the 183-class bound alone makes no assertion about their values.

7. PRODUCT EXPONENTS AND LOCAL-CURVE CONVENTIONS

The factorization of Lemma 2.1 is rational. The compatible series $f_0(Q)^{1/2}$ shows that the formal axioms alone cannot imply integral exponents. For projective Calabi–Yau threefolds, integer exponents in this particular product expansion are the content of [Par25, Conjecture 1.4].

The local series and the elementary series of Ionel–Parker [IP18] arise from different torus actions. The distinction is visible in an exact degree-two calculation. Let $F_g(q, u) = 1 + \sum_{d \geq 1} \text{GW}(x_{g,d}) q^d$, where q records curve degree and u is the GW variable. Pardon’s Proposition 4.8 identifies this with the residue theory for scalar fiberwise scaling. Let $E_g(q, u)$ be the exponential of the connected elementary series of [IP18, Proposition 3.3], given by the antidiagonal specialization

$$E_g(q, u) = \sum_{\mu} q^{|\mu|} \prod_{\square \in \mu} (2 \sin(h(\square)u/2))^{2g-2}.$$

Both use balanced level $(g-1, g-1)$. Put $s = 2 \sin(u/2)$. The scalar formula of Bryan–Pandharipande [BP08, Section 8], and the two partitions of two in the hook formula, respectively give

$$[q]F_g = [q]E_g = s^{2g-2},$$

$$[q^2]F_g = s^{4g-4}((4-2s)^{g-1} + (4+2s)^{g-1}), \quad [q^2]E_g = 2s^{4g-4}(4-s^2)^{g-1}.$$

In particular

$$(25) \quad [q^2](\log F_2 - \log E_2) = 2s^6, \quad F_2(q, u) = E_2(q, u)E_4(q^2, u)^2 \pmod{q^3}.$$

Thus direct identification fails, although this first correction has an integer exponent. More generally, for $g \geq 2$ binomial expansion gives

$$F_g(q, u) = E_g(q, u) \prod_{j=1}^{g-1} E_{2g-1+j}(q^2, u)^{a_{g,j}} \pmod{q^3},$$

$$a_{g,j} = 2 \cdot 4^{g-1-j} \left(\binom{g-1}{2j} - (-1)^j \binom{g-1}{j} \right) \in \mathbb{Z},$$

with out-of-range binomial coefficients zero. The degree-two discrepancy vanishes for $g = 0, 1$.

At degree three the comparison has a fractional correction. For a primitive class α , write $c_{\alpha,g}$ for the integer exponents in the elementary expansion of [IP18, Theorem 8.1], and $e_{\alpha,g}$ for the scalar-local exponents of (4). The companion paper [Wue26b] proves

$$e_{3\alpha,10} = c_{3\alpha,10} - \frac{4720}{3} c_{\alpha,3}.$$

On the resolution Y_+ of Section 4, the primitive class $\alpha = 3b_2 + 4b_3$ has $c_{\alpha,3} = -14$. Consequently $e_{3\alpha,10} \in \mathbb{Z} + 2/3$. This disproves the integer-exponent assertion of [Par25, Conjecture 1.4] for its scalar generators, even with torsion-free second homology. It does not contradict the integer elementary expansion or the distinct integral geometric generators in [Par25, Theorem 1.2].

Corollary 3.3 describes how the rational exponents specialize. Its integrality consequence applies whenever the source exponents are integral; the counterexample shows that this condition cannot be inferred from projectivity and the Calabi–Yau hypothesis. None of the support arguments or universal coefficient calculations above requires it.

APPENDIX A. TORIC AND ENUMERATIVE DATA

The source rays, numbered from zero, are the rows of the first matrix below. The columns of N give $H_1, \dots, H_6$ in the divisor basis $(D_2, D_5, D_6, D_7, D_8, D_9)$; $\det N = -1$. These coordinates and the period conventions agree with [Wue26a, Appendix A].

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -1 & 0 \\ -4 & -2 & 0 & -1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 4 & 5 & 5 \\ 0 & 1 & 2 & 4 & 4 & 5 \\ 0 & -1 & -1 & -3 & -3 & -3 \\ 0 & -1 & -1 & -2 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

The maximal source cones are the following tuples of ray indices:

$$\begin{aligned}
& (0, 1, 3, 6), \quad (0, 1, 3, 7), \quad (0, 1, 4, 5), \quad (0, 1, 4, 7) \\
& (0, 1, 5, 6), \quad (0, 2, 4, 5), \quad (0, 2, 4, 7), \quad (0, 2, 5, 6) \\
& (0, 2, 6, 7), \quad (0, 3, 6, 7), \quad (1, 3, 6, 10), \quad (1, 3, 7, 8) \\
& (1, 3, 8, 10), \quad (1, 4, 5, 10), \quad (1, 4, 7, 8), \quad (1, 4, 8, 10) \\
& (1, 5, 6, 10), \quad (2, 4, 5, 9), \quad (2, 4, 7, 9), \quad (2, 5, 6, 9) \\
& (2, 6, 7, 9), \quad (3, 6, 7, 9), \quad (3, 6, 9, 10), \quad (3, 7, 8, 9) \\
& (3, 8, 9, 10), \quad (4, 5, 9, 10), \quad (4, 7, 8, 9), \quad (4, 8, 9, 10) \\
& (5, 6, 9, 10).
\end{aligned}$$

The fan is regular for heights $(10, 11, 3, 9, 2, 7, 1, 4, 8, -5, -3)_i + (i + 1)/1009$. The charge matrix, with columns $0, \dots, 9$, is

$$Q = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

The symmetric cubic $\kappa_{abc} = H_a H_b H_c$ is specified by the following entries with $a \leq b \leq c$. Indices here run from one to six.

abc	κ_{abc}	abc	κ_{abc}	abc	κ_{abc}
111	0	156	20	335	31
112	0	166	24	336	35
113	0	222	0	344	49
114	0	223	2	345	57
115	0	224	3	346	65
116	0	225	4	355	65
122	0	226	4	356	75
123	2	233	6	366	85
124	3	234	11	444	83
125	4	235	13	445	101
126	4	236	15	446	115
133	4	244	17	455	117
134	8	245	22	456	136
135	8	246	25	466	155
136	10	255	26	555	133
144	14	256	30	556	156
145	16	266	34	566	180
146	19	333	14	666	204
155	16	334	27		

Table 1 lists sections of $2\mathcal{D}$ grouped by their character. Parenthesized tuples denote cones. The 41 listed cones are maximal among those with nonsingleton Newton face for the pulled-back original equation. Their faces comprise all 99 cones whose orbits may meet the hypersurface in the subdivided fan. The inequalities in Section 4 can be checked directly against its fifteen rays.

Table 1: Characters generating $2\mathcal{D}$ along the hypersurface.

Character	Cones
(0, 0, 0, 10)	(0, 1, 3), (0, 3, 7), (1, 3, 7), (1, 3, 8), (3, 7, 8)
(0, 0, 10, 0)	(0, 1, 4), (0, 1, 7), (0, 4, 7), (1, 4, 7), (1, 4, 8), (1, 7, 8), (4, 7, 8)
(0, 0, 14, 14)	(0, 1, 5), (0, 1, 6), (1, 5, 6)
(0, 4, 6, 2)	(0, 2, 5), (2, 5, 11)
(0, 4, 6, 6)	(0, 2, 6), (0, 5, 6), (2, 5, 6)
(0, 4, 2, 4)	(0, 2, 7), (2, 7, 12)
(0, 0, 0, 14)	(0, 3, 6)
(0, 2, 10, 0)	(0, 4, 5), (4, 5, 11)
(2, 0, 0, 6)	(1, 3, 6), (3, 6, 12), (3, 6, 14), (3, 8, 14)
(0, 0, 14, 0)	(1, 4, 5)
(0, 4, 2, 6)	(2, 6, 12)
(0, 4, 4, 2)	(2, 7, 11)
(0, 2, 0, 8)	(3, 7, 12)
(2, 0, 6, 0)	(4, 5, 13), (4, 8, 13)
(0, 2, 8, 0)	(4, 7, 11)
(3, 0, 2, 2)	(5, 6, 9), (5, 9, 13), (6, 9, 14), (8, 9, 13), (8, 9, 14)

The smoothing rays are the rows of

$$\begin{pmatrix} -1 & -3 & -1 & 0 \\ -1 & -3 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix}.$$

Their maximal cones are 1456, 1346, 0456, 0346, 0145, 0134, 1236, 0236, 0123, 01256. The last cone is nonsimplicial. The divisor charge matrix in the basis K_1, K_2, K_3 is

$$\begin{pmatrix} 1 & 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

Reproducibility. The source code, exact fan, monomial-section, period and coefficient data, and reproduction instructions are available at <https://github.com/bwuebben/universal-curve-counting>. All script paths below are relative to the repository root. From that directory, the source numerical calculation is reproduced by

```
sage -python checks/x9-source-genus-zero-box.py
--source-root vendor/global-higgs-branch
--output /tmp/x9-source-genus-zero.json
```

The source files used for the charge and cubic inputs are pinned by SHA-256. The finite quotient has 431 positive monomials. All six equations in (20) have zero remainder; the

computation also checks 1440 Gamma contiguity identities. These arithmetic checks are separate from the mirror-theorem hypotheses in Section 6.

The scripts `checks/x9_lift_bound.py` and `checks/x9_contracted_support.py`, run with Sage, reproduce the finite lift and contracted-locus calculations. The standard-library scripts `checks/x9_plane_geometry.py` and `checks/x9_universal_line.py` reproduce the smoothing fan calculation and the 37-row universal table. The stored exact outputs are in `evidence/`. The symbolic two-by-two matrix comparison for (25) is `checks/local_series_comparison.py`, using SymPy.

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