A COUNTEREXAMPLE TO INTEGRAL PRODUCT EXPONENTS IN UNIVERSAL CURVE COUNTING

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Abstract. We construct a smooth projective Calabi–Yau threefold with torsion-free second homology whose product expansion in Pardon's scalar equivariant local-curve generators has a nonintegral exponent. The genus-ten exponent at three times a primitive class has fractional part $2/3$, disproving the integer-exponent assertion for these generators. The obstruction comes from an exact degree-three calculation in the local Gromov–Witten Frobenius algebra. Comparison with the integer elementary expansion of Ionel–Parker gives a necessary divisibility condition on the primitive genus-three count. We realize a count violating this condition on a first Hirzebruch surface in a compact toric hypersurface. A filtration by powers of the surface ideal controls transverse thickenings and disconnected products; the stable-pair moduli space at minimal Euler characteristic is a projective space of dimension thirteen, with virtual count -14. We also prove integrality in classes of divisibility at most two. The counterexample concerns the scalar local factors; the integer elementary expansion is an input to its proof.

1. Introduction

Pardon's universal curve-counting algebra expresses the enumeration of effective cycles on a projective Calabi–Yau threefold in terms of local curves [Par25]. Its rational generators $x_{g,m,0}$ are defined by the equivariant theory of degree-m cycles in a rank-two bundle over a genus-g curve, with scalar multiplication on the fibers. Write $x_{g,m} = x_{g,m,0}$ and

$$f_g(q) = 1 + \sum_{m\geq 1} x_{g,m} q^m.$$

The coproduct and cycle-division operations imply a unique rational factorization of the universal cycle series,

$$\mathcal{Z}_X = \prod_{\beta\neq 0,\, g\geq 0} f_g(Q^\beta)^{e_{\beta,g}(X)}, \qquad e_{\beta,g}(X) \in \mathbb{Q}, \tag{1}$$

when $H_2(X;\mathbb{Z})$ is torsion-free. Each class has only finitely many nonzero exponents. Conjecture 1.4 of [Par25] asserts that the exponents in this factorization are integers. We give a counterexample.

The compact threefold used below is a general anticanonical hypersurface in the smooth projective toric fourfold specified in Section 4. It contains a surface $F \simeq \mathbb{F}_1$. Let s be its negative section and f a ruling fiber.

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Theorem 1.1. *The threefold Y of Section 4 is a smooth projective Calabi–Yau threefold with torsion-free $H_2(Y;\mathbb{Z})$. The class $\alpha = 3s + 4f$ is primitive, and its scalar-local product exponents satisfy*

$$e_{\alpha,3}(Y) = -14, \qquad e_{3\alpha,10}(Y) \in \mathbb{Z} + \frac{2}{3}.$$

The arithmetic mechanism is a general comparison formula. Denote by $c_{\beta,g}(X)$ the integer elementary exponents in the expansion of Ionel–Parker [IP18, Theorem 8.1]. Their elementary local series uses the antidiagonal action on a balanced local curve [IP18, Proposition 3.3]. These integers differ from the scalar exponents in (1) and, in general, from the usual Gopakumar–Vafa numbers. We prove in Proposition 3.2 that every primitive class α satisfies

$$e_{3\alpha,10}(X) = c_{3\alpha,10}(X) - \frac{4720}{3} c_{\alpha,3}(X). \tag{2}$$

Thus integrality of the exponent on the left requires $3 \mid c_{\alpha,3}(X)$. For our class, $c_{\alpha,3}(Y) = -14$. The unknown integer $c_{3\alpha,10}(Y)$ need not be calculated.

The proof has two parts. First, the Frobenius algebra of Bryan–Pandharipande [BP08] determines the scalar local series in degree three. We give its multiplication matrix, pairing, cap and characteristic polynomial, retaining the signs and phases distinguishing scalar from antidiagonal residues. The connected comparison has a $4720/3$ coefficient in the genus-ten term. A leading-order estimate excludes contributions from all higher local genera.

Second, nef divisors on the compact hypersurface force every cycle in a multiple of α, and every effective summand of α, onto F. Cycle support alone does not exclude transverse nonreduced curve schemes. A filtration by the powers of the ideal of F proves that a stable pair in class α has Euler characteristic at least -2. At equality its curve is a divisor in $|3s + 4f| \simeq \mathbb{P}^{13}$. We identify this minimal moduli space scheme-theoretically, including its nonreduced divisors, and obtain virtual count -14. The same filtration shows that lower products in the logarithm have nonnegative Euler characteristic. The primitive genus-three coefficient is therefore exactly -14.

The geometry also occurs in the conifold specialization calculation of [Wue26b]. Here the needed fan, nef divisors, contracted locus and surface identifications are supplied directly. No transition formula is used to prove Theorem 1.1. The universal algebra and local TQFT are imported from [Par25, BP08]; the integer elementary expansion is imported from [IP18]. The result concerns the proposed integer product expansion in the specified scalar generators. Pardon's integral geometric-generator description [Par25, Theorem 1.2] and ordinary numerical Gopakumar–Vafa integrality are compatible with the counterexample.

2. Universal exponents and the two local series

Throughout, a Calabi–Yau threefold is smooth and projective over $\mathbb{C}$, with trivial canonical bundle and $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$. Integral second homology is assumed torsion-free. All completions in curve classes use degree against an ample divisor. Effective integral classes of bounded degree form a finite set: norms of a finite collection of smooth two-forms bound homology coordinates by Kähler area. All exponents in nonzero classes of nonpositive ample degree are understood to be zero.

2.1. Rational factorization.

The Chern-zero part of Pardon's rational algebra is

$$H = \mathbb{Q}[x_{g,m} : g \geq 0,\ m \geq 1], \qquad I = (x_{g,m}).$$

Each coefficient is a polynomial; H is not completed in genus. For the monoid Γ_X of effective integral curve classes, write

$$\mathcal{Z}_X = \sum_{\beta\in\Gamma_X} U_{X,\beta}Q^\beta, \qquad U_{X,0}=1.$$

Here $U_{X,\beta}$ is the universal class of the compact open-and-closed locus of cycles of class β. Addition and integer multiplication of cycles give

$$\begin{aligned} \Delta x_{g,m} &= \sum_{a+b=m} x_{g,a}\otimes x_{g,b}, \qquad x_{g,0}=1,\\ \rho_d x_{g,m} &= \begin{cases} x_{g,m/d}, & d\mid m,\\ 0, & d\nmid m,\end{cases} \qquad (3)\\ \Delta U_{X,\beta} &= \sum_{\eta+\gamma=\beta} U_{X,\eta}\otimes U_{X,\gamma}, \qquad \rho_d U_{X,\beta}=U_{X,\beta/d}. \end{aligned}$$

A coefficient at a class outside Γ_X is zero. These are [Par25, Definitions 3.9 and 3.12, Lemmas 4.12–4.13 and Theorem 7.6] in Chern number zero.

Lemma 2.1. *The series $\mathcal{Z}_X$ has the unique factorization (1), with rational exponents and finitely many nonzero genera at each effective class.*

Proof. Put $p_{g,m}=[q^m]\log f_g(q)$. These are polynomial coordinates on H, primitive for Δ, with $\rho_d p_{g,m}=p_{g,m/d}$ when $d\mid m$ and zero otherwise. A primitive polynomial in these coordinates is additive, hence linear over $\mathbb{Q}$. By (3), each coefficient of $L=\log\mathcal{Z}_X$ is therefore a finite sum $L_\beta=\sum a_{\beta;g,m}p_{g,m}$. Comparing the coefficient of $p_{g,1}$ in $\rho_m L_\beta = L_{\beta/m}$ gives $a_{\beta;g,m}=a_{\beta/m;g,1}$. Taking $e_{\beta,g}=a_{\beta;g,1}$ yields

$$L_\beta = \sum_{m:\,\beta/m\in\Gamma_X}\ \sum_g e_{\beta/m,g}p_{g,m}.$$

Every sum is finite, by polynomiality and bounded degree. This proves existence, uniqueness and the stated genus support. □

2.2. **Scalar and elementary evaluations.** Let u be the GW variable and q the local curve-degree variable. Set

$$r=e^{iu}, \qquad w=2-r-r^{-1}=(2\sin(u/2))^2, \qquad F_g(q,u)=\mathrm{GW}(f_g(q)), \quad K_g=\log F_g. \tag{4}$$

By [Par25, Section 4.2 and Proposition 4.8], F_g is the residue theory for scalar fiberwise multiplication, which we compute at torus weights $(t_1,t_2)=(1,1)$. We choose the balanced splitting of degrees $(g-1,g-1)$.

The elementary series is

$$E_g(q,u)=\sum_\lambda q^{|\lambda|}\prod_{\square\in\lambda}(2\sin(h(\square)u/2))^{2g-2}, \qquad J_g=\log E_g, \tag{5}$$

where λ runs over partitions and $h(\square)$ is hook length. It is the balanced antidiagonal specialization [IP18, Proposition 3.3]. Ionel–Parker's Theorem 8.1 gives integers $c_{\beta,g}$ such that

$$\sum_{\beta,g} e_{\beta,g}K_g(Q^\beta,u)=\mathrm{GW}_X^{\mathrm{conn}}=\sum_{\beta,g}c_{\beta,g}J_g(Q^\beta,u). \tag{6}$$

We use the standard projective comparison of algebraic and symplectic GW invariants [Sie99, Theorem 0.1]; the symplectic invariants in [IP18] may be interpreted in the semipositive

setting of its Remark 1.1. Both connected theories omit class zero, and the disconnected theories omit wholly contracted connected components. Contracted components attached to a nonconstant connected map remain included. The logarithm thus gives the connected series in (6) without a degree-zero factor.

The right side of (6) is interpreted in the area-and-genus completion of [IP18]. We do not assume finite genus support for its integers. Each coefficient in $Q^\beta u^{2h-2}$ used below is a finite expression. In a fixed class, the summands come from integer divisors of that class, rather than all additive decompositions.

The degree-one terms satisfy

$$[q]K_g = [q]J_g = w^{g-1}. \tag{7}$$

Since $w = u^2 + O(u^4)$, these powers form a topological basis of $u^{-2}\mathbb{Q}[[u^2]]$. For every primitive effective class α,

$$e_{\alpha,g} = c_{\alpha,g} \in \mathbb{Z}. \tag{8}$$

This comparison remains valid for an ineffective primitive class, with both sequences zero. More generally the elementary coefficients at positive-area ineffective classes vanish by induction on degree: none of their positive integral divisors can be effective, and (7) detects the remaining coefficients.

Under the local GW/PT correspondence of [Par25, Corollary 4.9], the PT variable is $z = -r$, so

$$w = \frac{(1+z)^2}{z}, \qquad \mathrm{PT}(x_{g,1}) = z^{1-g}(1+z)^{2g-2}. \tag{9}$$

Its lowest Laurent coefficient is $+1$. The geometric PT series uses $z^{\chi(\mathcal{F})}$ with no additional sign in its coefficients.

Lemma 2.2. *We have $F_0 = E_0$ and $F_1 = E_1$. For $g \geq 2$ and $d \geq 1$,*

$$[q^d]K_g,\ [q^d]J_g \in u^{2d(g-1)}\mathbb{Q}[[u^2]], \qquad [q^d](K_g - J_g) \in u^{2d(g-1)+2}\mathbb{Q}[[u^2]].$$

Proof. For genus zero the balanced bundle is $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Each summand has zero H^0 on a positive-degree connected stable map, including attached contracted components. The obstruction rank equals the virtual dimension of the proper map space to $\mathbb{P}^1$; only the ordinary top Chern class contributes. The residue is independent of the weights. For genus one, the level-$(0,0)$ formula [BP08, Theorem 5.2] gives the number of partitions of d for the degree-d coefficient at either action.

For $g \geq 2$, the balanced residue formula [BP08, Section 2.2] has the form

$$u^{2d(g-1)} \sum_{\substack{b_1,b_2 \geq 0 \\ b_1+b_2 \text{ even}}} u^{b_1+b_2}(t_1/t_2)^{(b_2-b_1)/2} I^{(d,g)}_{b_1,b_2},$$

where the intersection numbers $I^{(d,g)}_{b_1,b_2}$ do not depend on the weights. The leading term $b_1 = b_2 = 0$ is the same for both actions; their difference has two additional powers of u. The claim passes to logarithms: every product of coefficients in curve degree d has total degree d, and a difference of products contains at least one such difference. □

3. The degree-three local calculation

3.1. **Degree two and its integral correction.** Put $s_0 = 2\sin(u/2)$, so $w = s_0^2$. The scalar degree-two formula [BP08, Section 8] and the two partitions of two give

$$[q^2]F_g = s_0^{4g-4}\big((4-2s_0)^{g-1} + (4+2s_0)^{g-1}\big), \qquad [q^2]E_g = 2s_0^{4g-4}(4-s_0^2)^{g-1}.$$

The degree-one terms agree, so these also give their logarithmic difference. For $g \geq 2$, binomial expansion yields

$$\begin{aligned} [q^2](K_g - J_g) &= \sum_{j=1}^{g-1} a_{g,j} w^{2g-2+j}, \\ a_{g,j} &= 2\,4^{g-1-j}\left(\binom{g-1}{2j} - (-1)^j\binom{g-1}{j}\right) \in \mathbb{Z}. \end{aligned} \tag{10}$$

Out-of-range binomial coefficients are zero. The first two rows are $2w^3$ for $g = 2$ and $24w^5 - 2w^6$ for $g = 3$.

3.2. **The Frobenius algebra in degree three.** We use the multiplication operator of [BP08, Appendix, equations (29) and (31)], the cap of its Lemma 6.3, and the starred normalization of its Section 6.4.1. Initially keep general nonzero weights t_1, t_2, and put

$$k = t_1 t_2, \qquad a = -(t_1+t_2)\frac{r-1}{r+1}, \qquad b = -3(t_1+t_2)\frac{r^2-1}{r^2+r+1}.$$

In the partition basis $1 = e_{111}$, $x = e_{21}$, $y = e_3$, multiplication by x is

$$M = \begin{pmatrix} 0 & -3k & 0 \\ 1 & a & -2k \\ 0 & 3 & b \end{pmatrix}. \tag{11}$$

Since $M1 = x$ and $M^2 1 = -3k + ax + 3y$, the unit is cyclic. Consequently

$$\begin{aligned} L_1 = I, \qquad L_x = M, \qquad L_y &= \frac{M^2 - aM + 3kI}{3}, \\ \mathcal{A} &= \mathbb{Q}(t_1,t_2,r)[x]/\big((x-b)(x^2-ax+3k)+6kx\big). \end{aligned} \tag{12}$$

Indeed $My = -2kx + by$ gives the stated relation. Multiplication also gives

$$y^2 = 2k^2 - \frac{2kb}{3}x + \frac{b^2 - ab - 3k}{3}y.$$

The starred Frobenius pairing in this basis is

$$D = \operatorname{diag}\left(-\frac{1}{6k^3}, \frac{1}{2k^2}, -\frac{1}{3k}\right). \tag{13}$$

In general its partition entry is $1/(\mathfrak{z}(\lambda)(-k)^{\ell(\lambda)})$, where $\mathfrak{z}(\lambda) = \prod_j j^{m_j} m_j!$ and m_j is the multiplicity of j in λ. The Appendix Fock pairing differs by $(-1)^d$ in degree d; its multiplication-operator convention has the same sign. Thus (11) is the multiplication matrix in the starred normalization, and the handle operator is

$$G = -6k^3 I + 2k^2 M^2 - 3kL_y^2. \tag{14}$$

This is multiplication by the Euler element, the sum of each basis element times its dual. In the quotient algebra it equals

$$-18k^3 + 2k^2(a+b)x + (9k^2 + kab - kb^2)y.$$

Raising the level-$(-1,0)$ cap covector with (13) and removing its half-integral phase gives the vector

$$\widetilde{\eta}_\lambda = (-1)^d t_1^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} (1 - r^{-\lambda_j})^{-1}, \qquad d = 3. \tag{15}$$

Let $\widetilde{A} = \sum_\lambda \widetilde{\eta}_\lambda L_\lambda$ and let $\widetilde{\bar{A}}$ interchange t_1, t_2. If A^* is the unscaled starred level operator, then $\widetilde{A} = r^{d/2} A^*$. The balanced-level operator is therefore

$$B = r^3 G \widetilde{A}^{-1} \widetilde{\bar{A}}^{-1}, \qquad [q^3] F_g = \operatorname{tr}(B^{g-1}) \quad (t_1 = t_2 = 1). \tag{16}$$

All operators commute because they act by multiplication. The closed starred normalization has exponent $d(2 - 2g + (g-1) + (g-1)) = 0$, so no further phase multiplies the ordinary GW coefficient in (16).

Proposition 3.1. *At scalar weights the characteristic polynomial of B is*

$$\begin{aligned} \det(TI - B) = {} & T^3 - w^3(w^2 + 50w + 81)T^2 \\ & + \frac{4}{3} w^6 (w-6)(w^2 + 54w - 243) T \\ & - \frac{4}{3} w^9 (w-4)(w-3)^2 (w^2 + 54w - 243). \end{aligned} \tag{17}$$

In particular,

$$\begin{aligned} [q^3] F_2 &= 81w^3 + 50w^4 + w^5, \\ [q^3] F_3 &= 2673w^6 + 9612w^7 + 2534w^8 + \frac{292}{3} w^9 + w^{10}. \end{aligned} \tag{18}$$

Proof. Substitute $t_1 = t_2 = 1$ into (11)–(16) and reduce using $r^2 - (2-w)r + 1 = 0$. The three elementary symmetric functions of B are respectively

$$w^3(w^2 + 50w + 81), \qquad \frac{4}{3} w^6 (w-6)(w^2 + 54w - 243),$$
$$\frac{4}{3} w^9 (w-4)(w-3)^2 (w^2 + 54w - 243).$$

These give (17); the trace and the identity $\operatorname{tr}(B^2) = (\operatorname{tr} B)^2 - 2\sigma_2(B)$ give (18). The calculation can also be made entirely in (12): invert $\widetilde{\eta}^2$, multiply by r^3 times the Euler element, and take the resultant of its characteristic equation with the monic cubic defining $\mathcal{A}$. This uses the same printed algebraic data and avoids a matrix inverse. □

At antidiagonal weights the same construction has eigenvalues U, U, V, where

$$U = w^3(4-w)(3-w)^2, \qquad V = w^3(3-w)^2. \tag{19}$$

These are exactly the three partition contributions in (5). The leading scalar coefficient $2673 = 6^4 + 3^4 + 6^4$ also agrees with the unramified degree-three cover count for a genus-three base.

One consequence already concerns the local PT generator:

$$[z^{-9}] \operatorname{PT}(x_{3,3}) = 20 + \frac{292}{3} = \frac{352}{3}. \tag{20}$$

Indeed, after (9), only w^{10} and w^9 in (18) contribute. This is an equivariant residue of a rational universal generator, not an invariant of a proper stable-pair space. A compact counterexample requires the geometric argument below.

3.3. **The connected obstruction.** Since the degree-one terms agree,

$$[q^3](K_g - J_g) = [q^3](F_g - E_g) - w^{g-1}[q^2](F_g - E_g). \tag{21}$$

Use $[q^3]E_g = 2U^{g-1} + V^{g-1}$, (10), and (18). Exact subtraction gives

$$\begin{aligned} [q^3](K_2 - J_2) &= 120w^4 - 20w^5 + 2w^6, \\ [q^3](K_3 - J_3) &= 14448w^7 - 1136w^8 + \frac{4720}{3}w^9 \\ &\qquad - 332w^{10} + 40w^{11} - 2w^{12}. \end{aligned} \tag{22}$$

Proposition 3.2. *For every primitive class α on a smooth projective Calabi–Yau threefold with torsion-free second homology,*

$$e_{3\alpha,10} = c_{3\alpha,10} - \frac{4720}{3}c_{\alpha,3}.$$

In particular $e_{3\alpha,10}$ is integral if and only if $c_{\alpha,3}$ is divisible by three.

Proof. Primitivity and torsion-freeness imply that the only positive integral divisors of 3α are α and 3α. Subtract the two sides of (6) at this class and use (8). The degree-one terms at 3α give

$$\sum_h (c_{3\alpha,h} - e_{3\alpha,h})w^{h-1} = \sum_g c_{\alpha,g}[q^3](K_g - J_g).$$

By Lemma 2.2, the terms for $g = 0, 1$ vanish, and those for $g \geq 4$ have order at least u^{20}, hence w^{10}. The genus-two term in (22) ends at w^6. The coefficient of w^9 is therefore exactly $(4720/3)c_{\alpha,3}$. The change of coordinate $w = u^2 + O(u^4)$ is formal; no numerical evaluation of an infinite genus sum is involved. Since $c_{3\alpha,10} \in \mathbb{Z}$ and $4720 \equiv 1 \pmod 3$, the last assertion follows. □

Corollary 3.3. *If a nonzero class has divisibility one or two in integral second homology, all its scalar-local exponents are integers.*

Proof. The primitive case is (8). For $\beta = 2\alpha$ with α primitive, the degree-two comparison (10) gives

$$e_{2\alpha,h} = c_{2\alpha,h} - \sum_{\substack{g\geq 2,\ 1\leq j\leq g-1 \\ 2g-1+j=h}} a_{g,j}c_{\alpha,g}.$$

Every summand is integral, and the sum is finite for each h. The same formula applies when α is ineffective, with its primitive exponents zero. □

4. A compact hypersurface containing a Hirzebruch surface

We specify the compact example by its fan. Let $N = \mathbb{Z}^4$, and let $v_0, \dots, v_{10}$ be the rows of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -2 & -1 & 0 \\ -4 & -2 & 0 & -1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

The maximal cones of Σ are generated by the following four-tuples of rays:

$$\begin{gathered} (0,1,3,6),\ (0,1,3,7),\ (0,1,4,5),\ (0,1,4,7),\ (0,1,5,6), \\ (0,2,4,5),\ (0,2,4,7),\ (0,2,5,6),\ (0,2,6,7),\ (0,3,6,7), \\ (1,3,6,10),\ (1,3,7,8),\ (1,3,8,10),\ (1,4,5,10), \\ (1,4,7,8),\ (1,4,8,10),\ (1,5,6,10),\ (2,4,5,9), \\ (2,4,7,9),\ (2,5,6,9),\ (2,6,7,9),\ (3,6,7,9), \\ (3,6,9,10),\ (3,7,8,9),\ (3,8,9,10),\ (4,5,9,10), \\ (4,7,8,9),\ (4,8,9,10),\ (5,6,9,10). \end{gathered} \tag{24}$$

Write V_Σ for the associated toric fourfold and D_i for its invariant divisors. Let Y be a general anticanonical hypersurface, nondegenerate with respect to its Newton polytope, with nonzero vertex coefficients. This is the resolution chamber used in [Wue26a, Wue26b]; the construction below uses only the displayed toric data.

4.1. **The smooth projective hypersurface.** The polytope $P = \operatorname{conv}(v_0, \dots, v_8)$ is reflexive, and (24) is a subdivision of its boundary fan. Each maximal cone has determinant ± 1; the subdivision is complete. For an explicit strictly convex support function take divisor coefficients

$$A_i = 1009(10000 + h_i) + (i+1), \qquad h = (10, 11, 3, 9, 2, 7, 1, 4, 8, -5, -3). \tag{25}$$

For each maximal cone σ, solve $\langle m_\sigma, v_i \rangle = -A_i$ for $i \in \sigma$. The off-cone values $\langle m_\sigma, v_i \rangle + A_i$ are all positive, with minimum 5047. This proves projectivity. Repeating the same calculation with all coefficients equal to one gives integral slopes and nonnegative values. Thus the anticanonical divisor is basepoint-free and the fan is a crepant refinement of the reflexive fan. Bertini and adjunction give smoothness and $K_Y \simeq \mathcal{O}_Y$. The anticanonical exact sequence, toric structure-sheaf vanishing, and Serre duality give $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$.

The four vertices v_0, v_1, v_3, v_4 of P form the standard basis of N. In particular, the lattice N'' generated by the lattice points on vertices and edges of P is all of N. The Brauer-group formula of [BK05, Corollary 3.9] therefore gives

$$\operatorname{Br}(Y) \simeq \operatorname{Hom}\bigl(\Lambda^2 N/(N \wedge N''), \mathbb{Q}/\mathbb{Z}\bigr) = 0.$$

For this Calabi–Yau threefold the Brauer group is the torsion in $H^3(Y;\mathbb{Z})$. The universal coefficient theorem then implies

$$\operatorname{Tors} H_2(Y;\mathbb{Z}) = 0. \tag{26}$$

This uses the full lattice, rather than a rational cohomology rank.

4.2. **Nef divisors and the contracted locus.** The divisor D_{10} misses Y: its dual Newton face is a vertex whose coefficient is nonzero. Define integral divisors $H_1, \dots, H_6$ on Y by the columns of

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 4 & 5 & 5 \\ 0 & 1 & 2 & 4 & 4 & 5 \\ 0 & -1 & -1 & -3 & -3 & -3 \\ 0 & -1 & -1 & -2 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix} \tag{27}$$

in the ordered divisor list $(D_2, D_5, D_6, D_7, D_8, D_9)|_Y$. Each has a nef lift to V_Σ: use the displayed coefficients, zero on the other rays except 10, and add coefficient zero on D_{10} for H_1, H_2, H_3, or coefficient one for H_4, H_5, H_6. Solving the support-function equalities as above gives nonnegative values of the support-function inequalities for every cone and every ray. Since $D_{10}|_Y = 0$, all six H_i are nef on Y. Toric linear equivalence identifies H_4 with $D_0|_Y$.

The support function of D_0 defines a semiample contraction. Its positive-dimensional orbit fibers have the following minimal cones in their closures:

$$(8), \quad (9), \quad (10), \quad (2,4), \quad (6,7). \tag{28}$$

Here is a direct finite way to obtain the list. Form $P_0 = \{m : \langle m, v_i\rangle + \delta_{i0} \geq 0\}$ and, for each cone σ, impose its equalities to get the face $F_\sigma \subset P_0$. The orbit fiber is positive-dimensional exactly when $4 - \dim F_\sigma > \dim \sigma$; taking minimal cones in this set gives (28). Consequently every curve of H_4-degree zero on Y lies in

$$E \cup F \cup C_1 \cup C_2, \qquad E = D_9|_Y, \quad F = D_8|_Y, \tag{29}$$

where C_1, C_2 are the intersections with the orbit closures of $67, 24$. Their restricted Newton polytopes are the primitive segments joining $(0,1,-1,-1)$ to $(-1,2,-1,1)$ and $(-1,2,1,-1)$, respectively. Each binomial cuts the dense torus in a translate of $\mathbb{C}^*$. Basepoint-freeness excludes boundary components for a general equation, and nondegeneracy makes the closures smooth; hence both curves are $\mathbb{P}^1$. They miss both surfaces: there are no cones containing 678 or 248, and the Newton faces at $679, 249$ are singletons. There is no cone containing 2467, so the two curves are disjoint.

Lemma 4.1. *The surfaces are $E \simeq \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$ and $F \simeq \mathbb{F}_1$. Their intersection is the negative section s on F, and the strict transform of the line through p_1, p_2 on E.*

Proof. The anticanonical face at ray 9 is the primitive edge from $(0,1,-1,-1)$ to $(1,-1,-1,-1)$. Its binomial equation fixes a nonzero torus parameter. Rays $2, 7$ are excluded on E because their intersections have singleton Newton faces; ray 10 is absent. The remaining rays $3, 4, 5, 6, 8$ project to

$$(1,0), \quad (0,1), \quad (-1,0), \quad (0,-1), \quad (1,1),$$

with cones $38, 84, 45, 56, 63$. This is the fan of the plane blown up at two distinct torus fixed points.

For F, the star of ray 8 away from D_{10} has cones $137, 147, 379, 479$. Quotient by the fiber ray 7 gives the base rays $(1,0),(0,1),(-1,-1),(-1,0)$, the fan of $\mathbb{F}_1$. The Newton vertex $(1,-1,-1,-1)$ has exponents $(0,0,0,1,0,0)$ on rays $1,3,4,7,9,10$; all other vertices have fiber exponent zero. The equation is monic linear in that fiber coordinate and its zero locus is a graph over $\mathbb{F}_1$. The common divisor in these two surface fans is the stated curve. □

Let e_1, e_2 denote the exceptional curves on E, and f a ruling fiber of F. Restriction of (27) to the surface fans and to the two orbit curves gives

(30)

	H_1	H_2	H_3	H_4	H_5	H_6
s	0	1	0	0	0	0
f	0	0	1	0	0	0
e_1	1	0	0	0	1	1
e_2	0	0	0	0	0	1
C_1	0	0	0	0	1	0
C_2	1	0	0	0	0	0.

These pairings can be computed with the usual toric divisor relations: the linear relations are $\sum_i \langle m, v_i\rangle D_i = 0$, and a divisor product vanishes when its rays span no cone. The monic and binomial equations above identify the actual surfaces on which the restrictions are taken.

Proposition 4.2. *The class $\alpha = 3s + 4f$ is primitive in $H_2(Y;\mathbb{Z})$. Every effective cycle of class $m\alpha$, $m \geq 1$, is supported on F. The same holds for every effective additive summand of α.*

Proof. By (30), $(H_3 - H_2)\cdot\alpha = 4 - 3 = 1$, which proves primitivity. A cycle in class $m\alpha$ has zero H_1, H_4, H_5, H_6 degrees. Nefness forces the same for each component. The H_4 contraction confines it to (29); H_1, H_5 exclude C_1, C_2. The effective cone of E is generated by e_1, e_2, s. The H_1, H_6 degrees exclude any e_1 or e_2 coefficient, so a remaining component in E is supported on $s = E \cap F$. All components therefore lie in F.

Every effective summand of α has the same four vanishing degrees, so the argument applies to it as well. On F, the H_2, H_3 pairings recover the coefficients of s, f. Hence a decomposition of α into effective classes corresponds to an actual effective decomposition of $3s+4f$ on F. □

5. The primitive genus-three count

Put $D = 3s + 4f$ on $F \simeq \mathbb{F}_1$. The intersection numbers are

$$s^2 = -1, \quad sf = 1, \quad f^2 = 0, \quad K_F = -2s - 3f, \qquad D^2 = 15, \quad K_F D = -11.$$

Thus $p_a(D) = 3$. In the presentation $F = \mathrm{Bl}_p \mathbb{P}^2$, $D = 4H - E_p$; consequently

$$h^0(F, \mathcal{O}_F(D)) = 5 + 4 + 3 + 2 = 14, \qquad |D| \simeq \mathbb{P}^{13}. \tag{31}$$

We must control all stable pairs in class α, including curves with nonreduced structure transverse to F.

5.1. **A surface inequality.** For $a, b \geq 0$, the Euler characteristic of a divisor of class $as+bf$ is

$$\phi(a,b) = \frac{a(a+1)}{2} + b(1-a), \qquad \phi(3,4) = -2. \tag{32}$$

This follows from the surface Riemann–Roch formula $\chi(\mathcal{O}_C) = -C(C + K_F)/2$.

Lemma 5.1. *If $\sum_i (a_i, b_i) = (3,4)$ is a sum of at least two nonzero pairs of nonnegative integers, then $\sum_i \phi(a_i, b_i) \geq 0$.*

Proof. Partition the positive a_i's. If the partition is (3), let $t \geq 1$ be the total b-coefficient on the other terms. The sum is $-2+3t \geq 1$. For the partition $(2,1)$, write b_A, b_B for the two b-coefficients and t for the remaining total. The sum is $4 - b_A + t = b_B + 2t \geq 0$. For $(1,1,1)$ the sum is $3 + t \geq 3$. These are all possibilities. □

5.2. **Transverse thickenings.**

Proposition 5.2. *Every pure one-dimensional Cohen–Macaulay subscheme $Z \subset Y$ of class α satisfies $\chi(\mathcal{O}_Z) \geq -2$. Equality forces Z to be a divisor in $|D|$ on F, with its scheme structure contained in F.*

Proof. The cycle of Z is supported on F by Proposition 4.2. The filtration by powers of its ideal I_F is finite. Since $N_{F/Y} = K_F$, its graded quotients have the form

$$G_j = I_F^j \mathcal{O}_Z / I_F^{j+1} \mathcal{O}_Z \simeq K_F^{-j} \otimes (\mathcal{O}_F / J_j) \tag{33}$$

for ideal sheaves $J_j \subset \mathcal{O}_F$. On the smooth surface, the divisorial hull of J_j is $\mathcal{O}_F(-D_j)$ for an effective divisor D_j. The quotient $\mathcal{O}_F(-D_j)/J_j$ has finite length $\ell_j \geq 0$. Additivity of Euler characteristic gives

$$\chi(G_j) = \chi(\mathcal{O}_{D_j}) - jK_F D_j + \ell_j, \qquad \sum_j [D_j] = [D]. \tag{34}$$

The last equality holds in the divisor-class group of F, since its two coefficients are detected by H_2, H_3. Each twist $-jK_F D_j$ is nonnegative because $-K_F$ is ample.

If at least two D_j are nonzero, Lemma 5.1 shows $\chi(\mathcal{O}_Z) \geq 0$. Otherwise there is exactly one nonzero divisorial layer. At the generic point of every component, a nonzero higher layer requires a nonzero zeroth layer, so the unique divisorial layer must be the zeroth one. Then $I_F \mathcal{O}_Z$ is supported at finitely many points. As a subsheaf of the pure sheaf $\mathcal{O}_Z$, it is zero. Thus Z is contained in F scheme-theoretically. A pure curve on a smooth surface is an effective Cartier divisor, so $Z \in |D|$ and $\chi(\mathcal{O}_Z) = -2$. This also excludes a transverse thickening confined to finitely many points, which cycle support alone would not detect. □

For a stable pair $\mathcal{O}_Y \to \mathcal{F}$, its image is $\mathcal{O}_Z$ for a pure curve and its cokernel is zero-dimensional. The exact sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{F} \longrightarrow Q \longrightarrow 0$$

therefore gives $\chi(\mathcal{F}) \geq -2$. At equality, $Q = 0$ and $Z \in |D|$.

5.3. **The minimal moduli scheme.**

Proposition 5.3. *The stable-pair moduli space $P_{-2}(Y, \alpha)$ is scheme-theoretically $\mathbb{P}^{13}$. Its virtual count is -14, and $P_n(Y, \alpha)$ is empty for $n < -2$.*

Proof. The set-theoretic assertions follow from Proposition 5.2. Every pair section at Euler characteristic -2 is surjective. Fiberwise surjectivity and Nakayama's lemma identify the pair functor with the corresponding Hilbert functor. The surface divisors give a closed immersion $|D| \hookrightarrow P_{-2}(Y, \alpha)$ which exhausts the closed points.

For every $C \in |D|$, including nonreduced divisors, the normal sequence for the regular embeddings $C \subset F \subset Y$ is

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow N_{C/Y} \longrightarrow K_F|_C \longrightarrow 0. \tag{35}$$

We have $H^0(F, K_F) = 0$ and

$$H^1(F, K_F - D) = H^1(F, D)^* = 0,$$

by Serre duality and Kodaira vanishing, since $D - K_F = 5s + 7f$ is ample. Restriction of K_F to C therefore has no sections. Rationality of F and (31) give $h^0(\mathcal{O}_C(D)) = 13$. By (35) the ambient Hilbert tangent space has dimension thirteen at every point of the closed $\mathbb{P}^{13}$. Its local dimension is at least thirteen, so its local rings are regular of dimension thirteen. The closed immersion is consequently an isomorphism.

The Calabi–Yau stable-pair obstruction theory is symmetric. On a smooth proper moduli space of dimension r its virtual degree is $(-1)^r \chi_{\mathrm{top}}$ [PT10, p. 3]. Hence

$$P_{-2,\alpha} = (-1)^{13} \chi_{\mathrm{top}}(\mathbb{P}^{13}) = -14. \tag{36}$$

□

5.4. **Passing to the connected coefficient.** Taking the logarithm of the geometric PT series adds products over at least two nonzero effective summands of α to its coefficient at α. Each constituent cycle lies in F. Collect the divisorial layers (33) of all constituent curves. They give at least two nonzero effective terms with total class D. Lemma 5.1 and (34) show that the sum of their Euler characteristics is nonnegative. The twists, zero-dimensional torsion and pair cokernels only increase this sum. Thus none of these lower products contributes a negative power of the PT variable z.

It follows that the connected coefficient has no term below z^{-2} and has coefficient -14 at z^{-2}. By primitivity, Lemma 2.1 and (9) give exactly

$$[Q^\alpha] \log \mathrm{PT}(\mathcal{Z}_Y) = \sum_{g \geq 0} e_{\alpha,g} z^{1-g} (1+z)^{2g-2}. \tag{37}$$

This sum has finite genus support. Its largest nonzero genus contributes a unique lowest Laurent power, with coefficient one. Consequently

$$e_{\alpha,g} = 0 \ (g > 3), \qquad e_{\alpha,3} = c_{\alpha,3} = -14. \tag{38}$$

6. The nonintegral compact exponent

Proof of Theorem 1.1. Section 4 constructs a smooth projective Calabi–Yau threefold with torsion-free second homology and proves primitivity of $\alpha = 3s+4f$. The stable-pair argument of Section 5 gives (38). Proposition 3.2 now yields

$$e_{3\alpha,10} = c_{3\alpha,10} - \frac{4720}{3}(-14) = c_{3\alpha,10} + \frac{66080}{3}.$$

The first term is an integer by Ionel–Parker's elementary expansion, while $66080/3 = 22026 + 2/3$. This proves the theorem. □

The class α has divisor-degree vector $(0,3,4,0,0,0)$ in (30); the nonintegral exponent occurs at $(0,9,12,0,0,0)$ and local genus ten. Corollary 3.3 shows that divisibility three is the first possible divisibility for such a counterexample. We do not claim that genus ten is the smallest possible local genus.

The rational factorization remains well-defined and unique. Its arithmetic depends on the chosen local factors: the integer elementary expansion and the integral geometric-generator algebra are different structures. Formula (2) isolates one condition that an integral scalar expansion would impose on compact curve counts, and the surface calculation shows that this condition fails.

Appendix A. Exact calculations

All local formulas in Section 3 are rational identities in r, or polynomial identities after the substitution $w = 2 - r - r^{-1}$. They require no Taylor truncation or numerical approximation. The multiplication, pairing and cap data needed to reproduce them are printed in (11)–(16). As one numerical specialization in exact arithmetic, at $r = 2$ the characteristic coefficients in decreasing powers of T are

$$1, \quad \frac{225}{32}, \quad \frac{14027}{384}, \quad \frac{158613}{4096}, \qquad \operatorname{tr}(B^2) = -\frac{72557}{3072}.$$

The general characteristic identity, rather than this single specialization, is used in the proof.

The source code, exact calculation outputs and reproduction instructions are available at `https://github.com/bwuebben/universal-curve-counting`. All script paths below are relative to the repository root. The code gives two implementations: matrix reconstruction in `checks/local_curve_degree_three.py` and quotient-algebra inversion with a resultant in `checks/degree_three_quotient_check.py`. Both use SymPy 1.14.0. The first also checks degrees one and two and the antidiagonal eigenvalues (19). The script `checks/integrality_counterexample.py` checks the connected subtraction, the Laurent coefficient (20), and all 56 unordered nontrivial effective decompositions of $(3, 4)$. Their minimum sum in Lemma 5.1 is zero. The lemma's three-case argument proves the inequality without this enumeration.

The toric data are printed in (23)–(27). The script `checks/integrality_counterexample_geometry.py` checks the strict support function, anticanonical support, nef lifts and lattice-basis test. It reads the same ray and cone lists from `checks/x9_lift_bound.py`. The latter runs with SageMath and checks completeness and smoothness of the fan. The contracted-face and surface calculations are reproduced by `checks/x9_contracted_support.py`, also using SageMath. The integer degree table (30) can be recovered from the displayed divisors and surface fans.

These finite computations concern the local algebra and toric data. The pure-curve filtration, moduli-scheme argument and virtual sign are the geometric steps establishing that these data yield a compact counterexample.

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