UNRAMIFIED COVERS AND WALL CROSSING FOR LOCAL GROMOV–WITTEN INVARIANTS

BERND JOHANNES WUEBBEN

Abstract. Let two independent unramified double covers of an embedded curve acquire normal deformations along transverse walls. We construct such a family over every curve of genus at least two with a complex rank-two normal bundle of degree twice the genus minus two. The alternating difference of the local Gromov–Witten invariants in its four chambers equals the local invariant on the common fourfold cover, in every source genus. At each fixed covering-degree bound one neighborhood suffices for all genera. On a finite principal unramified cover we realize simultaneous crossings for pairwise inequivalent nontrivial irreducible representations of real type, provided their associated flat bundles together embed in the underlying real normal bundle. Fixed vectors determine the signs, and the result is a sum of local invariants on intermediate covers. Along a path of these crossings the determinant signs transport the initial series on the covers in all genera. For arbitrary rigid endpoints, the signs at both endpoints determine the change in the least source genus, the genus of an unramified cover of the given degree.

Date: November 24, 2026
This version: November 24, 2026.

2020 *Mathematics Subject Classification.* 53D45 (primary); 14N35, 32Q65, 57R91 (secondary).

Key words and phrases. Local Gromov–Witten invariants, unramified covers, real Cauchy–Riemann operators, wall crossing, Burnside ring.

Contents

1. Two double covers and four local counts 3
2. Two double covers and their common fourfold cover 4
3. Local elementary factors and the least source genus 5
4. Families with prescribed kernels on an unramified cover 8
5. The orientation on an exact monodromy stratum 11
6. The multiplicity of a common character cover 13
7. Transport on the tower of finite covers 15
8. Exact prime-power denominators at the least genus 18
9. The elliptic suspension and its projective example 20
10. The wall associated to a real character 29
11. The partition function and the highest changing genus 33
12. Kernels on finite covers 35
13. Reflection and rotation groups 39
14. Spin structures and the signs of covers 43
15. Bifurcation of multisections in every degree 45
16. Integral multiplicities from finite covers 51
Appendix A. A congruence and individual signs in degree four 56
Appendix B. Denominators at fixed genus excess 58
Appendix C. Horizontal leaves and their finite monodromy 60
Appendix D. Isolation for maximal symplectic holonomy 64
Appendix E. The normal operator of a maximal surface representation 67
Appendix F. The signed character formula for horizontal curves 71
References 75

1. Two double covers and four local counts

Take two independent unramified double covers of a smooth curve C. Each can acquire a normal deformation while C itself stays rigid. Suppose the first deformation occurs on one axis of a parameter plane and the second on the other. Crossing the first axis changes the local curve count. How much does that change depend on which side of the second axis we cross? The answer is a local invariant on the connected fourfold cover formed by taking the fiber product of the two doubles.

We hold the curve C and its complex rank-two normal bundle N fixed, where C has genus $g \geq 2$ and $\deg N = 2g - 2$. The parameter varies a real Cauchy–Riemann operator D on N. A section in the kernel of its pullback to a cover is an infinitesimal normal deformation there. At a parameter where the pullback operator has zero kernel on every connected stable map of positive degree at most M, the zero section has a local Gromov–Witten contribution in each degree $d \leq M$ and each arithmetic source genus h. Write it as $L^C_{d,h}$. It counts connected stable maps to C with the Euler class of their normal obstruction bundle. The covers on which we prescribe a kernel are unramified. The stable maps in the count can have branch points, nodes, and contracted components.

Let Z be the common fourfold cover. Theorem 2.1 constructs a family with the two prescribed walls. Label its chambers by the signs of the two parameters. For $d \leq M$, in a sufficiently small rectangle about the crossing, one has

$$L^C_{d,h}(++) - L^C_{d,h}(-+) - L^C_{d,h}(+-) + L^C_{d,h}(--) = L^Z_{d/4,h}(++).$$

The right side is zero unless four divides d. Its operator is the pullback of the positive-corner operator. A map changes sign across an axis precisely when it lifts to the corresponding double. In the alternating difference only maps lifting to both doubles survive, each with multiplier four. Such a map has exactly four lifts to Z with its domain fixed. This accounts for the coefficient one on the right, including the automorphism weights of the maps. For example, over a curve of genus two the first common lift has degree four and genus five. Its contribution to the difference in that genus is its determinant sign.

From signs on covers to an equality of invariants. Parker computed the determinant signs carried by the irreducible real representations of a covering group [Par13, §1.2]. Bryan and Pandharipande stratified the space of maps to a curve by the intermediate covers to which they lift [BP01, §2.2]. Wendl's super-rigidity theorem and his analysis of nonabelian bifurcations, and Doan and Walpuski's study of the double-cover wall, describe how normal kernels arise [Wen23, §2.4], [DW25, Theorem 1.3]. To obtain the displayed equality one must realize both walls without other kernels intervening, then compare the two normal Euler classes over the entire compactified space of maps.

There is a compactness issue even for the two doubles. A bound on the degree does not bound the source genus, so a separate neighborhood for each moduli space would give less than the formula asserts. A normal section has a graph whose area is bounded by its covering degree. A sequence of unexpected sections therefore has a limit as a cycle. The sheets of its support are separated. Lifting to the nearest sheet and averaging the sections there produces a kernel on a fixed intermediate cover. The prescribed kernels account for all such limits (Section 4). For the Euler class comparison we fix the monodromy image of each map. Parallel kernel vectors match across a node exactly when their values on its two branches agree. Removing these vectors leaves a continuous reduced obstruction bundle across the wall. Its orientation comparison supplies the determinant sign in the integral (Section 5).

The same argument applies on a finite principal unramified G-cover. The kernels may be prescribed by pairwise inequivalent nontrivial irreducible real representations of real type, provided their associated flat bundles together embed in the underlying real bundle of N. On a map with monodromy image H, the sign of each crossing is given by the parity of the dimension of its H-fixed subspace. Writing these signs in the basis of transitive G-sets expresses the mixed difference as a sum over intermediate covers (Theorem 4.1). Section 3 proves the corresponding identity for the local factors in the universal cycle algebra, using the integral generators of [Wue26e, Theorem 1.1]. In the two-double example the mixed connected count in its least source genus has exact denominator p^j in degree $4p^j$, for every prime p and every $j \geq 1$ (Theorem 8.1).

Endpoints and examples. At the minimum source genus the Riemann–Hurwitz formula leaves only unramified maps. At two arbitrary rigid endpoints, the determinant signs at both ends therefore give the change in that genus. Equivalently, multiplying each initial covering sign by its endpoint ratio gives its final sign. In higher genus we prove a transport formula along paths made up of the realized unramified crossings and rigid segments. Its initial data are the full series on the intermediate covers (Theorem 7.1). Thus a path calculation can start with an independently computed chamber. For positive maximal normal holonomy, Appendices D–F identify the full series when the orthogonal bundle is orientable and compute the finite-cover signs in both cases. The absolute all-genus calculation for a nonorientable orthogonal bundle is given in [Wue26f, Theorem 1.1].

For an elliptic base the suspension of a symplectic return map gives a separate description. Its horizontal tori are the finite orbits of the return map. When the finite orbits are isolated as a set in the sense of Section 9, the fixed-point indices of the iterates recover the local cycle exponents by Möbius inversion. A projective example has sixteen torsion-distinct sections with the same double class. In higher genus one generic path between specified spin operators forces a bifurcation in every covering degree $m \geq 2$ (Theorem 15.1). Along a subsequence, the domains converge to an infinite cover carrying a nonzero bounded solution of the limiting operator (Proposition 15.2).

2. Two double covers and their common fourfold cover

Let C be a closed smooth complex curve of genus $g \geq 2$, and let N have complex rank two and degree $2g-2$. For a real Cauchy–Riemann operator D with Dolbeault symbol on N, let D_u be its pullback to $u : X \to C$. On a nodal domain the sections on the normalization have matching values. We call D *rigid through degree* M if D_u is injective for every connected stable map of degree between one and M. At such a parameter write

$$L^C_{d,k}(D) = \int_{[\overline{\mathcal{M}}_k(C,d)]^{\mathrm{vir}}} e(\operatorname{coker} D_u), \qquad L^C_d(D;v) = \sum_k L^C_{d,k}(D) v^{2k-2}. \tag{1}$$

We orient the normal complex by continuation from complex-linear operators. Its real cokernel has rank $4(k-1) - 4d(g-1)$, the virtual real dimension of the map space. The fixed-target comparison theorem identifies the integral with the contribution of the isolated zero section in the homogeneous normal bundle [Zin11, Theorem 1.2]. For its balanced product we write $P_C(D;q)$; its construction and degreewise uniqueness are given below. Set $H_d = 1 + d(g-1)$. The sign of an invertible operator on an unramified cover T will be written $\epsilon_T(D)$.

Choose distinct nonzero classes $\alpha_1, \alpha_2 \in H^1(C;\mathbb{F}_2)$. Write Y_i for the double classified by α_i. Since the characters are independent, $Z = Y_1 \times_C Y_2$ is connected. Each corner of a

two-parameter family will be denoted by the signs of its two parameters. In the following displays a count at nonintegral degree is zero.

Theorem 2.1. *There is a family on (C, N) for which, at every finite cutoff M, a sufficiently small rectangle consists of four rigid open quadrants and the two crossing axes. For $d \le M$ and every k, the connected difference satisfies*

$$L_{++} - L_{-+} - L_{+-} + L_{--} = L^{Z}_{d/4,k}(D_{++}|_Z).$$

Here $L_\epsilon = L^{C}_{d,k}(D_\epsilon)$. The quotient of the corresponding balanced products is

$$\frac{P_{++}P_{--}}{P_{-+}P_{+-}} = P_Z(D_{++}|_Z; q^4) \pmod{q^{M+1}}.$$

For $M \ge 4$, the first differing exponent has label four, elementary genus $4g-3$, and value $\epsilon_Z(D_{++})$.

The geometric reason for the factor one is worth spelling out. Embed the sum W of the two flat sign lines in $N_{\mathbb{R}}$. Indeed, $W \oplus W$ is oriented with $w_2 = w_1(W)^2 = 0$; the oriented rank-four bundle $N_{\mathbb{R}}$ also has $w_2 = 0$ and is isomorphic to it on a surface. Prescribe the parallel vectors in these copies to be central kernel sections. Pair a variation of the normal operator with the first twisted cokernel while killing the second prescribed section, and do the converse for the other parameter. The construction in §4 makes these the only kernels apart from their pullbacks and restrictions. At a stable map each sign reverses exactly when its character pulls back trivially, including around loops through nodes. On taking the alternating sum, a map sees a multiplier four if it lifts to both doubles and zero otherwise. A specified map in the surviving locus has four lifts to Z with its domain fixed. Representable finite étale pushforward divides its upstairs integral by four, giving the formula. The orientation and compactness statements needed for this computation are proved in §§4–5.

More explicitly the increments from left to right on the two opposite edges are the positive-side counts in degree $d/2$ on Y_1, with the second sign positive or negative respectively. Subtracting them is the sign-wall calculation for $Z \to Y_1$. Subtracting the other pair instead uses $Z \to Y_2$ and gives the identical answer. The oriented sum along the entire boundary is zero. At degree four and genus $4g-3$ the only contributing common lift downstairs is Z itself, of deck order four. Its four alternating masses give its sign. In the product this also follows from the signed degree-one contribution on Z, which is the full series $\epsilon_Z(D_{++})(2\sin(v/2))^{2(4g-3)-2}$.

3. Local elementary factors and the least source genus

Let S be the rational algebra of almost-complex cycle families of [Wue26b, §2]. Its relations include open excision and transport of a compactly supported insertion over an interval. Multiplication uses disjoint union of targets. Write H for the Chern-number-zero, virtual-dimension-zero part of Pardon's rational algebra of complex one-cycles [Par25], and $\Phi : H \to S$ for the comparison. We use A_h for the balanced antidiagonal series of genus h in H. It has constant term one. The series $E_h = \Phi(A_h)$ is the contribution of a neighborhood of an elementary curve of Ionel and Parker [Wue26b, §3]. Its coefficient of degree one is $\Phi(x_{h,1})$, where $x_{h,1}$ is Pardon's genus-h class of a single curve of multiplicity one.

Let C be a smooth embedded almost-complex curve of genus $g \ge 2$ in a six-dimensional almost-complex germ with $c_1(TX)[C] = 0$. Assume a relatively compact tube around C contains no other compact irreducible curve and that the normal Cauchy–Riemann operator of C is invertible on every connected finite étale cover, including C. These conditions hold for the germs of Section 14. Choose a smaller disk tube. For $m > 0$ let P_m be the class

in S of the characteristic function of the point $m[C]$ in its degree-m cycle space, and set $P = 1 + \sum_{m>0} P_m q^m$. The variable q labels $m[C]$ by m under the retraction of the tube to C. An inclusion of two such tubes preserves these classes by excision. Write $h_d = 1 + d(g-1)$, and let $C_k(h)$ be the groupoid cardinality of connected unramified degree-k covers of a genus-h curve. Each isomorphism class contributes the reciprocal of its deck group order. In particular $C_1(h) = 1$.

Proposition 3.1. *For every such germ there are unique integers $n_{d,h}$, of finite support in h at each d, such that*

$$P = \prod_{d\geq 1,\ h\geq 0} E_h(q^d)^{n_{d,h}}.$$

They vanish when $h < h_d$. If c_m is the sum of the local determinant signs divided by deck orders over the connected degree-m covers, then $n_d = n_{d,h_d}$ satisfies

$$c_m = \sum_{d|m} n_d \, C_{m/d}(h_d). \tag{2}$$

In particular this determines a full sequence of exponents of the unique balanced product lifting P to $H[[q]]$.

Proof. We first realize the neighborhood in a closed symplectic six-manifold of vanishing first Chern class. In a product of two symplectic two-tori take one horizontal torus and $g-1$ distinct vertical tori. Smooth their $g-1$ positive transverse intersections. Locally the smoothing of the coordinate axes is $zw = \delta$. In a fixed outer annulus of each axis its graph may be cut off to the zero graph while remaining symplectic, since the perturbation there is C^1-small as $\delta \to 0$. The annuli remove a disk from each incident torus and glue the two boundary circles. The intersection graph is a tree, so the resulting embedded symplectic surface has genus g. Regard it as a surface in the product with a third two-torus, all three areas chosen integral. Its class A is primitive, since intersection with a generic vertical torus in the first two factors pairs to one with the four-dimensional surface class. Use the pullback of that pairing to the six-manifold. In particular $d \mapsto dA$ is injective.

Identify the marked surface with this surface by an orientation-preserving diffeomorphism. The symplectic normal bundle has first Chern number $2g-2$. Choose a complex complement to TC in the original tangent bundle along the curve and a Hermitian metric on that complement. A unitary normal identification with the model exists by the classification of complex bundles over a surface. A tubular embedding with this prescribed normal derivative carries the model's almost-complex structure on the zero section to an almost-complex structure compatible with the ambient symplectic form there. Any positive complex structure on the oriented tangent plane is compatible with its area form. Extend the compatible structure along the section to an ambient one by extending a metric and using polar decomposition. On a small tube the prescribed structure is tame and is uniformly close to the extension, with equality along the section. Cut off their difference in a chart of almost-complex structures to obtain a global tame structure agreeing with the given germ on a smaller tube. This construction uses the local almost-complex structure itself, including its normal derivative.

Fix an energy bound. Take a smaller disk V with closure in the isolating tube. For every sufficiently close global tame structure, there is no nonempty cycle contained in $\overline{V}$ whose support meets ∂V in this energy range. A violating sequence would have a geometrically convergent subsequence by [DIW26, Theorem 2.11]. Its limit at the prescribed structure would be supported on C and would meet ∂V, because supports converge in the Hausdorff topology. Hence, in every positive integral class, containment in V selects a compact open and closed

insertion over a sufficiently short closed path to a generic tame structure. At the first endpoint this is precisely the coefficient of P. At the second endpoint the connected supports selected by V form a union of finitely many clusters, after making the cluster partition subordinate to this open and closed subset [DIW26, Propositions 3.18–3.21]. All their classes are positive multiples of A since V retracts onto C and the symplectic area is positive.

Apply the geometric excision and cluster recursion of [Wue26b, §4] to these selected clusters. This use of the recursion is local: at the generic endpoint one includes the exterior as an additional color to separate the supports, and retains the factors of the selected colors. Each recursive correction is a cluster in a proper multiple of its predecessor's class, by cluster isotopy [DIW26, Theorem 3.25] and its cycle-space version in [Wue26b, §4]. Thus the remaining energy bound divided by the core energy decreases at each recursion. Its leaves give an exact finite product of elementary factors and inverses in S at the given cutoff. This part of the cluster argument uses a closed symplectic ambient manifold and a bounded energy. The projective and coherent-cohomology hypotheses of the compact complex product theorem enter when one starts with the entire cycle series of a complex target and lifts that identity to H.

At the first q-degree where two such products differ, the difference is a sum of the degree-one classes $\Phi(x_{h,1})$ with coefficients the differences of exponents. These classes are rationally independent by the finite-genus test of [Wue26b, Theorem 5.1]. Consequently the exponents stabilize as the cutoff increases, are intrinsic to the germ, and the product of the $A_h(q^d)$ with these exponents gives the stated lift. The uniqueness asserted here is among balanced products with finite genus support at each degree.

Apply connected, fixed-target Gromov–Witten counts to the same geometric decompositions, as in [Wue26b, §7]. Compact open localization, deformation of these insertions, and additivity on a partition hold by [DIW26, Lemma 3.8 and Proposition 3.10]. On disjoint targets the connected contributions add for normalized factors, since the domain has an image in exactly one target and the other insertion is evaluated on the empty cycle. These facts identify the local connected series with the sum of the elementary connected series with the very same integers $n_{d,h}$. This argument applies to the given geometric relations on fixed targets. It requires no enumerative pairing on arbitrary analytic parameter families.

For the original germ the count in degree m vanishes below h_m by Riemann–Hurwitz, and at equality every stable map is a smooth unramified cover: the arithmetic genus bound rules out multiple nonconstant components and positive-genus ghosts, and a contracted rational tree would have an unstable leaf. The tangent deformation complex of an unramified map cancels against source deformations via df. The remaining operator is the pulled-back normal operator, invertible by hypothesis. Thus its determinant sign divided by the deck order gives the local degree, and the sum is c_m. For a balanced elementary core of genus $h \geq 2$, the same argument gives vanishing below $1 + k(h-1)$ in degree k. At equality both spin lines coincide, so every cover has sign $+1$ and the count is $C_k(h)$. In degree one an elementary genus-h factor starts at source genus h with coefficient one for every $h \geq 0$ [IP18, Proposition 3.3]. Now induct on m. An elementary factor from $d < m$ enters degree m only for $d \mid m$, and by induction it has genus $h \geq h_d \geq 2$. Thus its least possible source genus in degree m is $1 + (m/d)(h-1) \geq h_m$. Comparing the degree-one terms of the remaining genus factors in order of increasing genus gives $n_{m,h} = 0$ for $h < h_m$. At equality the contributions from proper divisors occur only for $h = h_d$. They have the values just computed, which proves (2). □

Corollary 3.2. *For a homogeneous normal operator rigid through degree M, the same construction gives the insertion of each pure multiple through M and a unique balanced lift*

$$P_T(D;q) = \prod_{j=1}^{M} \prod_{s} A_s(q^j)^{n^T_{j,s}(D)} \pmod{q^{M+1}}.$$

Connected fixed-target integrals on these compact insertions distinguish the truncated products.

Proof. A compact irreducible curve in the tube projects with positive degree, since a fiber is a complex vector space. Its normalization gives a kernel graph. Hence rigidity excludes all other curves through M, which suffices for the finite-energy argument above. The connected integrals respect open excision and continuation of the selected compact clusters. In their product, a connected domain in a disjoint union lies in one target. If two lifts first differ at label j, the connected difference at that label is a finite sum $\sum_s a_s(2\sin(v/2))^{2s-2}$. These functions have distinct lowest powers, hence all a_s vanish if the integrals agree. □

4. Families with prescribed kernels on an unramified cover

Let C be a fixed smooth compact complex curve of genus $g \geq 2$ and N a complex plane bundle of degree $2g-2$. Fix a connected unramified principal G-cover $Z \to C$ for a finite group G. Choose distinct, nontrivial, irreducible real representations $V_1, \dots, V_r$, up to isomorphism, with $\mathrm{End}_G(V_i) = \mathbb{R}$. Write $V = \bigoplus_i V_i$. We may take its action to be faithful: if K is its kernel, use the principal G/K-cover $Z/K \to C$. The associated bundles on C are unchanged. Inflation carries each orbit $[(G/K)/(H/K)]$ to $[G/H]$, with $K \subset H$; both give the same intermediate cover Z/H. For any subgroup J the fixed space is $V^{JK/K} = V^J$. Hence the orbit formula and its marks pull back to the original group. Suppose the associated real bundle $E = Z \times_G V$ admits a fiberwise real monomorphism in N. This forces $\dim_{\mathbb{R}} V \leq 4$. Orthogonal metrics identify each representation with its dual. The crossing parameters below vary real Cauchy–Riemann operators on the *same* curve and normal bundle.

For a subgroup $H \leq G$ put $T_H = Z/H$ and $a_H = [G : H]$. At a regular operator through degree M, $L^C_{d,h}$ is the connected fixed-target local Gromov–Witten degree in covering degree $d \leq M$ and arithmetic source genus h. Write P_C for the integral balanced lift of the local insertion through M, normalized by constant coefficient one. On T_H use the pulled-back operator and the induced cutoff $\lfloor M/a_H \rfloor$. The balanced factors and their degreewise uniqueness are established in Proposition 3.1. Their integral classes are identified by the basis theorem of [Wue26e, Theorem 1.1].

Theorem 4.1. *There is a smooth family $D_{\mathbf{t}}$ for $\mathbf{t}$ near $0 \in \mathbb{R}^r$ in which the r twisted kernel lines cross independently. For every finite M there is a box about zero on whose open orthants all pullbacks to positive-degree connected stable maps of degree at most M are injective, without restriction on source genus. On a coordinate face the entire kernel on such a map is the flat fixed subspace of the summands whose coordinates vanish. One may use the same disk tube for the marked curve at all regular parameters in this box. The compact local insertion there is its pure multiple in each degree.*

Let I be a nonempty subset of the coordinates. Fix signs outside I. Use $+$ on all coordinates of I to designate the reference corner $*$. For $J \subset I$, write $*J$ for this corner with the signs indexed by J reversed. The values at points of a given open orthant agree throughout the box. Set

$$B_I = \prod_{i \in I} \chi^G(S(V_i)) = \sum_{(H)} b_{I,H}[G/H] \quad \text{in } A(G).$$

Here $S(V_i)$ denotes the unit sphere, and $A(G)$ is the Grothendieck ring of finite G-sets with Cartesian product. The equivariant Euler characteristic uses cell orbits counted with their alternating signs. Then

$$\begin{aligned}\sum_{J\subset I}(-1)^{|J|}L^{C}_{d,h}(*J) &= \sum_{(H):\, a_H\mid d} b_{I,H}L^{T_H}_{d/a_H,h}(*),\\ \prod_{J\subset I}P_C(*J;q)^{(-1)^{|J|}} &= \prod_{(H)}P_{T_H}(*;q^{a_H})^{b_{I,H}} \pmod{q^{M+1}}.\end{aligned} \tag{3}$$

Shrinking the box as M grows defines one series for each orthant as a germ in the degree completion. An individual nonzero parameter realizes the coefficients through the cutoff of its box.

4.1. **Simultaneous twisted kernels.** Write $E_i = Z \times_G V_i$. At the center we prescribe parallel vectors in each E_i as solutions via a monomorphism $S : E \to N_{\mathbb{R}}$. Start with any reference operator D_*. For a local parallel section v of E, prescribe a real zeroth-order correction on the image by $Q(Sv) = -D_*(Sv)$ and extend it along a complement. This defines a global pair (D, S) with $D^E S = 0$. The section and operator may both vary. We give the varying-section argument because four real columns can span the whole normal fiber.

A chart of pairs chooses a nearby monomorphism and freely prescribes the operator correction on a complementary subbundle. For any disk of regular values, a pair with an infinite-order Petri jet at a chosen inner point is dense. Perturb the operator on the disk by the unrestricted jet theorem. For each parallel column take a close solution of the new equation on a smaller disk using the small-disk Cauchy transform. Interpolate each solution and the operator back to the original data with cutoffs. The interpolated monomorphism S' is still fiberwise injective; correcting the interpolated operator D' by $-(D'^E S')J$, where $JS' = 1$, restores all prescribed equations and leaves the inner disk intact. These choices approximate the initial pair in any specified finite set of smooth seminorms [Wen23, Definition 5.1 and Corollary 5.10].

Consider an additional twisted solution u for a finite-monodromy irreducible real local system U with s distinct effective punctures. Use its orbifold extension at the branch values. If $U = E_i$ restrict to u orthogonal to the one prescribed intertwiner S_i. On a common connected normal cover the columns of S give one copy of every V_i. The columns of a nonzero u form an irreducible subrepresentation. An intersection with their direct sum would put this whole subrepresentation inside it. Its projection to each inequivalent summand would vanish, and in its own type the intertwiner would be a real multiple of S_i. This contradicts the complement condition. Thus the full list of columns is real linearly independent, including as germs by unique continuation.

At a Petri point away from the branch values the joint section equations for (S, u) have surjective derivative when the operator is allowed to vary. In fact an annihilator solves the local adjoint equations and unrestricted variations of the operator give the identity $\sum s_j \otimes \alpha_j + \sum u_l \otimes \eta_l = 0$ on the disk. Independence of input germs and the Petri condition force the adjoint germs to vanish. Unique continuation on the normal covers removes the annihilator. The Fredholm section derivatives have closed images of finite codimension. Projecting onto the second equation on the tangent space where the first equations are satisfied is therefore surjective. In a distinguished type one may adjust $\dot{u}$ in the S_i direction to maintain orthogonality to the moving prescribed intertwiner.

For this orbifold system the real index is $-2\sum_x \operatorname{codim} U^{I_x}$, where I_x is local inertia. The projectivized incidence index after moving branch values is at most $2s-1-2\sum_x \operatorname{codim} U^{I_x} \leq$

-1. Removing the prescribed direction in a distinguished type decreases the index further. On each compact block of branch configurations avoid the branch values with a common disk. Normalized solutions over that block form a compact set by elliptic estimates on its smooth family of normal covers. At a pair having a Petri point there, finitely many smooth tangent perturbations of the pair suffice for surjectivity near all of these solutions. The chart of pairs integrates these variations. Sard's theorem on the resulting negative-relative-index incidence removes the extraneous solutions; their absence over the block is open by the same compactness. A countable exhaustion over configurations and finite monodromy systems gives a pair (D_0, S) with no other irreducible kernels. This uses the orbifold index and removability as in [DW23b, §2.8].

For every smooth finite cover $u : X \to C$, with any branching, passage to a common normal cover and descent now give $\ker u^* D_0 = S(V^{K_u})$ in parallel frames. Here $K_u \leq G$ is the monodromy image of $u^* Z$, defined up to conjugacy. A contracted component of a stable map has constant normal sections. Every connected contracted subcurve meets a positive component. At every node fiberwise injectivity of S identifies matching values with matching parallel vectors. Thus the same result holds for the full connected nodal domain, including its graph loops.

4.2. **Transverse parameters and control on a whole face.** The twisted operator for each E_i has index zero by Riemann–Roch and has kernel and cokernel of real dimension one. Choose a variation Q_i killing every other S_j and pairing nontrivially on S_i with its twisted adjoint. On a disk where the adjoint is nonzero, injectivity of S permits independent choices on all the real columns, hence this prescription. Put $D_{\mathbf{t}} = D_0 + \sum_i t_i Q_i$. In the ith twisted kernel equation the real scalar after Lyapunov–Schmidt reduction is t_i times a nonvanishing smooth function: it vanishes identically on $t_i = 0$ since S_i survives there, and its normal derivative at zero is nonzero. On Z the other isotypic blocks stay invertible. Indeed the V_i blocks already account for the full cokernel at zero by their index calculation, equivariance and the index zero on Z. Exactness of taking H-invariants gives the base kernel on every fixed quotient T_H for small parameters:

$$\ker D_{\mathbf{t}}|_{T_H} \simeq \bigoplus_{i : t_i = 0} V_i^H. \tag{4}$$

These kernels are represented by the fixed columns S_i.

For completeness, (4) on finitely many quotients implies a uniform assertion on all smooth degree-bounded covers, even along the punctured faces. Otherwise choose parameters tending to the origin on one face (pass to a subsequence with the same zero coordinates F). On offending smooth connected covers u_l of fixed degree $d \leq M$, select an extra solution. Subtract a best approximation in sup norm by the finite-dimensional space K_l of parallel sections of $u_l^*(\oplus_{i \in F} E_i)$, evaluated via the pulled-back S. This space is closed and a minimizer exists by finite dimensionality. Rescale the resulting representatives so their sup norm and their distance from K_l both equal a fixed small $\rho > 0$.

The normalized graphs lie in one compact disk subbundle. In the graph almost-complex structures a common taming form there is the pullback of base area plus the exact differential of the vertical Liouville form. Their areas, with mapping multiplicities, are d times the base area. Use the compact embedding from the proof of Proposition 3.1 on a slightly larger tube. The graphs stay in the inner disk, so cycle compactness on the closed ambient six-torus gives a connected central limit support in the Hausdorff topology meeting the radius-ρ sphere. Each simple component is a graph over a positive-degree cover. By the central classification,

its image is the embedded orbit graph of some $v \in V$, on the target T_H with $H = \mathrm{Stab}(v)$. Distinct G-orbits have disjoint compact graph images by fiberwise injectivity. The finite union in the limiting support is connected and therefore consists of one such graph, with $v \neq 0$. This is bounded-area cycle compactness, independent of genus [DIW26, Theorem 2.11].

The sheets of this fixed graph have positive minimum separation whenever there is more than one sheet. Taking the nearest sheet gives a holomorphic lift $a_l : X_l \to T_H$ of u_l and the sections are uniformly close to $\sigma_v \circ a_l$. For a single sheet just use u_l. Average each section over the sheets of a_l. The average solves $D_{\mathbf{t}_l}|_{T_H}$ by real linearity and extends over branch points by boundedness, the weak equation and elliptic regularity. The averages converge uniformly to σ_v. By (4) they lie in one fixed closed subspace, so $v \in \oplus_{i \in F} V_i^H$. The pulled-back approximation $\sigma_v \circ a_l$ therefore belongs to K_l. Its distance from the normalized sections tends to zero, contradicting the distance ρ. Nodal matching gives the asserted classifications on connected stable maps. This proof applies to the finitely many faces at once. In open orthants no extra support can occur in the disk tube: normalization of any nonzero compact curve would give a kernel graph, and a vector space fiber contains no compact curve. The fixed tube identifies the markings at regular parameters; at the central parameter entire dilation orbits may meet its boundary.

5. The orientation on an exact monodromy stratum

Partition the proper stack of degree-d, genus-h connected stable maps to C by its exact subgroup type (K_u). For each H the lifts to T_H form a representable finite étale cover of its image, including at the nodal boundary. Its fiber over a specified map is $(G/H)^{K_u}$. Formal étaleness gives continuation of lifts; properness of the stack of maps to the finite target cover gives closed image. The subgroup marks of all transitive sets distinguish conjugacy classes, so the exact-type loci are open and closed. The horizontal virtual class pulls back, since the covering differential identifies the target tangent bundles. The normal obstruction complex and its determinant orientation pull back as well.

Fix an exact monodromy type (K). Pass to its stack of lifts $a : X \to T_K$ of representable degree $|N_G(K)/K|$. Write F_t for the base operator on T_K. At the central parameter its kernel pulls back isomorphically onto the full normal kernel on X, by the central classification. Its cokernel injects into the full cokernel. Indeed, if $(a^* F_0)s = a^* \alpha$ for a smooth form α, average the values of s on one positive normalization over a regular fiber of a. The average satisfies $F_0 \bar{s} = \alpha$. The average is bounded near a branch point. Testing on complements of shrinking disks, the boundary terms are bounded by a constant times their perimeter. The average is therefore a weak solution on the whole base, hence a smooth solution. We use the reduction of the normal complex of [BS24, Appendix B], on the lifted stratum. Stabilize F_t by a fixed space E of smooth forms mapping onto its cokernel at zero. At a lift a, stabilize $a^* F_t$ by the pullbacks of these forms and by additional equivariant forms in a space W supported away from the nodes. In a chart of the universal curve the resulting two-term complexes and the pullback between them have the form

$$[P \longrightarrow E] \;\longrightarrow\; [P' \longrightarrow E \oplus W].$$

Here P and P' consist of solutions of the stabilized equations, and the arrows of the complexes record their form parameters. Pullback on P is injective, since each domain has a component surjecting onto T. The quotient complex is $[P'/P \to W]$. The linear gluing of [BS24, (B.5)–(B.10)] gives continuous bundles and maps also at nodal domains. In degree one over T, marking preimages of auxiliary regular target points stabilizes the component mapping isomorphically to T. The contracted components are already stable with their attaching points.

The same construction then applies in that degree. The auxiliary markings are temporarily ordered in the chart and forgotten after the stabilization comparison.

By the cohomology sequence, the kernel isomorphism and cokernel injection just proved imply that $P'/P \to W$ is injective at the center. This is open, and its cokernel is a bundle there. Under common enlargements of the stabilizations these cokernels agree, giving the reduced cokernel bundle on a neighborhood of time zero on the compact lifted substack. Its orientation is specified by the exact determinant isomorphism

$$\det(a^* F_t) \simeq \det\big((a^* F_t)_{\mathrm{red}}\big) \otimes \det F_t$$

and the absolute orientations of the full and base operators [BS24, (6.8)–(6.13)]. At a nearby regular time the base complex is acyclic. The full and reduced cokernels then identify, and their orientations differ by the determinant sign $\epsilon(F_t)$. Thus the Euler numbers of the full cokernel on the two sides agree up to the ratio of the determinant signs on T.

The construction works over the compact lifted stratum at fixed (d, k) and for finitely many parameters. Only finitely many charts are needed; surjectivity persists on shrinking their parameter neighborhoods. The determinant of F_t changes across the ith wall in exactly $\dim V_i^K$ real directions. Thus if $I_K(*)$ denotes the positive-corner downstairs integral on the exact K-stratum, its value at $*J$ is $(-1)^{\sum_{i\in J} \dim V_i^K} I_K(*)$. Division by the degree of the stack of lifts restores all automorphism weights. Although the neighborhood used in this comparison may depend on k, the Euler number is constant in each degree-bounded rigid orthant; therefore any designated corners in the box give the same comparison.

The alternating sum on the (K) locus is therefore $I_K(*) \prod_{i\in I}(1-(-1)^{\dim V_i^K})$. Each factor is the ordinary Euler characteristic of the K-fixed unit sphere, with value zero for the empty sphere. It is the K-mark of the equivariant Euler characteristic: use a finite equivariant triangulation subdivided so that a cell stabilizer fixes that cell pointwise. Multiplication of marks corresponds to Cartesian products. Summing the mark identity for B_I against the $I_K(*)$, and using the lift counts $|(G/H)^K|$, proves the connected formula in (3).

On each rigid target at each corner, compact realization and finite cluster decomposition give the isolated insertion its balanced lift. If the quotient of the two sides of the proposed identity first differs at degree d, all elementary factors in proper divisors cancel. The connected degree-d series of this quotient would be a finite sum $\sum_s b_s (2\sin(v/2))^{2s-2}$, with coefficients the different elementary exponents. The first formula makes this sum zero, and its distinct leading powers force each $b_s = 0$. This proves the second formula through M.

Across a square of walls i, j, apply (3) to each edge with the other sign fixed. On any opposite pair of edges, the quotient of the two corrections equals the product for $B_{\{i,j\}}$. To compute one order, take the i-wall transfers across the j-wall on each target T_H. The restricted H-representation $V_j|_H$ can contain repeated or trivial summands. On an upstairs map of exact monodromy $K \le H$, the same reduction on Z/K gives the sign $(-1)^{\dim V_j^K}$. Thus the upstairs change has the marks of $\mathrm{Res}^G_H \chi^G(S(V_j))$, without a separate generic crossing hypothesis upstairs. For a virtual G-set b, inducing its restriction from H to G gives $[G/H]b$. Extended linearly over the orbits of the i-wall transfer, this computes the product $B_{\{i,j\}}$. Reversing the order gives the same product. Geometrically a lift to both quotient targets is a lift to their fiber product, whose components are the transitive summands of the Cartesian product of the two G-sets. Thus both orders give the same oriented transfer, including each common lift. The compact cluster realization uses the same primitive homology label for the marked support at all corners. This proves Theorems 2.1 and 4.1.

6. The multiplicity of a common character cover

Let C be a closed Riemann surface of genus $g \geq 2$, and let N have complex rank two and degree $2g-2$. A nonzero character $\alpha \in H^1(C;\mathbb{F}_2)$ defines a flat real sign line E_α and a connected étale double cover $Y_\alpha \to C$. Fix $r \geq 1$ distinct nonzero characters $\alpha_1,\dots,\alpha_r$ spanning a space of dimension b. Put $K = \bigcap_i \ker \alpha_i$, $G = \pi_1(C)/K \simeq (\mathbb{Z}/2)^b$, and write $p : Z \to C$ for its connected regular cover. Thus Z has degree $a = 2^b$ and genus $H = 1+a(g-1)$. The fiber product of the r double covers has 2^{r-b} components isomorphic over C to Z. Indeed G acts on its fiber of 2^r points by translation through the character embedding $G \hookrightarrow (\mathbb{Z}/2)^r$.

Suppose $W = \bigoplus_i E_{\alpha_i}$ admits a real bundle monomorphism into $N_{\mathbb{R}}$. Use the family of Theorem 4.1 with this prescribed sum of kernel lines and with all the central pullback kernels determined by their monodromy invariants. Let D_ϵ be an operator in the local chamber with signs $\epsilon = (\epsilon_1,\dots,\epsilon_r) \in \{+,-\}^r$ of the corresponding wall coordinates. For any additive quantity T put $\Delta T = \sum_\epsilon (-1)^{\#\{i:\epsilon_i=-\}} T(D_\epsilon)$. Use D_+ for the all-positive choice. Write $L^C_{d,k}(D)$ for the connected fixed-target local degree in projection degree d and arithmetic source genus k. Its domain includes nodal curves and contracted components. Write $P^C(D;q)$ for the balanced product of the isolated curve germ and $n^C_{d,h}(D)$ for its exponent at degree label d and elementary genus h.

6.1. **The connected transfer and the geometric product.** For each finite degree bound M, use corners in the sufficiently small coordinate neighborhood for that bound. In every arithmetic source genus, for $1 \leq d \leq M$,

$$\Delta L^C_{d,k} = \begin{cases} 2^{r-b} L^Z_{d/a,k}(p^*D_+), & a \mid d, \\ 0, & a \nmid d. \end{cases}$$

In the complex cycle algebra the corresponding identity is

$$\prod_\epsilon P^C(D_\epsilon;q)^{(-1)^{\#\{i:\epsilon_i=-\}}} = P^Z(p^*D_+;q^a)^{2^{r-b}} \pmod{q^{M+1}}.$$

The upstairs cutoff is $\lfloor M/a \rfloor$. Consequently $\Delta n^C_{d,h}$ vanishes off multiples of a and is $2^{r-b} n^Z_{d/a,h}(p^*D_+)$ on those multiples.

To verify the coefficient, fix the monodromy image of a connected stable map $u : X \to C$ in G. Write $I(u) = \{i : u^*E_{\alpha_i} \text{ is trivial as a real local system}\}$. This includes the matching conditions around loops passing through nodes. Across wall i the determinant orientation of the normal contribution changes sign precisely for $i \in I(u)$. The quotient obstruction bundle after removing the pulled-back central kernel and cokernel extends across the wall. Thus the alternating sum of the oriented contributions on the stratum is its all-positive value multiplied by $\prod_i (1-(-1)^{\mathbf{1}_{i\in I(u)}})$. It vanishes unless every character is trivial on the source, when the multiplier is 2^r. This is the character case of the obstruction comparison in Theorem 4.1.

The surviving open-and-closed substack is the image of composition from connected maps to Z. A specified lift of an object preserves its domain and gives a section of the pulled-back finite étale torsor. Hence the map from the stack of lifts is representable, finite étale of degree a on its image. Since $TZ = p^*TC$, the ordinary stable-map obstruction theory pulls back, as does the oriented normal family. Dividing its Euler integral by a gives the downstairs all-positive value on this substack. This proves the factor $2^r/a$, including when source automorphisms permute the lifts. Connected counts add on the products obtained from compact open cluster excision. At the first degree of disagreement between two balanced products, the remaining

connected difference is a finite sum of the degree-one series $(2\sin(v/2))^{2h-2}$, whose lowest orders distinguish h. This gives the product formula from the same fixed-target comparison.

The cutoff applies to the neighborhoods used to select the corners. In a sufficiently small neighborhood of the prescribed central data, the punctured coordinate chambers are rigid through M in every source genus. This uses classification of the central kernel graphs, their disjoint embedded orbit images, and the nearest-sheet and averaging argument. The finite set of operators on the orbit covers is invertible off the relevant coordinate walls. Compactness then excludes additional kernel graphs in bounded degree. On each fixed stable-map stack the obstruction comparison may first use a smaller neighborhood, followed by continuation in the rigid chambers. Isolation of the four or more corner points in unrelated germs would give no control of this continuation. A common neighborhood for unbounded M requires an additional hypothesis.

6.2. **Direct count at the least source genus.** In positive degree d the least possible genus of a connected map to C is $h_d = 1 + d(g-1)$. If its positive normalized components have degrees d_j, each contributes at least $1 + d_j(g-1)$ to the arithmetic genus by Riemann–Hurwitz. Equality at h_d requires one positive component, no ramification and no positive-genus contracted component. A nonempty contracted rational tree attached to it would have an unstable leaf. Thus the count at h_d is over smooth connected étale covers, with the determinant sign divided by the order of the deck group.

At $d = a$, exactly one isomorphism class of connected covers lifts to every Y_i. Its index-a subgroup is contained in K, which also has index a, hence it is K. This is $p : Z \to C$, with deck group of order a. Write $\eta = \operatorname{sgn}(p^* D_+)$. The r sign lines are distinct one-dimensional real summands on this cover. Crossing the ith wall flips its sign there. Its signed mass at the corner ϵ is $a^{-1}\eta(-1)^{\#\{i:\epsilon_i=-\}}$. Every other étale cover misses one character and cancels when the corners differing in that coordinate are paired. The mixed connected count at (a, H) is therefore $2^{r-b}\eta$, directly from automorphism orders and the real determinant. In degree ae the same argument identifies the least-genus mixed count with 2^{r-b} times the signed groupoid count of connected étale degree-e maps to Z.

At label a the entire mixed connected series in the variable v is $2^{r-b}\eta(2\sin(v/2))^{2H-2}$. Indeed the upstream label is one, and the local cluster of an invertible embedded curve has a single degree-one factor of its own genus and sign. All lower mixed exponents vanish, so the first nonzero mixed elementary exponent is

$$\Delta n^C_{a,H} = 2^{r-b}\eta.$$

It is the only mixed elementary exponent at label a. For the square on a genus-two base this gives a single exponent of magnitude one at label four and genus five.

6.3. **Which multiplicities occur in a real rank-four normal bundle?** Write $\alpha\beta$ for the cup product in $H^2(C;\mathbb{F}_2)$. Every degree-one class squares to zero on an oriented closed surface. The underlying oriented real bundle $N_{\mathbb{R}}$ has $w_2 = c_1(N) \bmod 2 = 0$. Oriented real bundles of rank at least three on a surface are classified by w_2, by obstruction theory using $\pi_1(SO(n)) = \mathbb{Z}/2$. In the following examples this identifies the constructed oriented rank-four bundle with $N_{\mathbb{R}}$ and supplies the monomorphism required in Theorem 4.1.

- For any two distinct nonzero characters the span has $b = 2$. Put $W = E_{\alpha_1} \oplus E_{\alpha_2}$. The bundle $W \oplus W$ is oriented and $w_2(W \oplus W) = w_1(W)^2 = 0$, so W embeds. The square has multiplier one.
- For three characters take $\alpha, \beta, \alpha + \beta$ with α, β independent and $\alpha\beta = 0$. Then $w_1(W) = 0$ and $w_2(W) = \alpha\beta = 0$. Taking $W \oplus \mathbb{R}$ gives the embedding. In a

symplectic basis a_i, b_i of $H^1(C;\mathbb{F}_2)$ one can take $\alpha = a_1$, $\beta = a_2$ for every $g \geq 2$. The eight corners give a first mixed coefficient 2η at label four and genus $4g-3$.

- For four characters take $\alpha, \beta, \gamma, \alpha+\beta+\gamma$, with α, β, γ independent. Then $w_1(W) = 0$ and $w_2(W) = \alpha\beta+\alpha\gamma+\beta\gamma$. The choice (a_1, a_2, b_1+b_2) makes the three terms evaluate to $0, 1, 1$, giving $w_2(W) = 0$. Here W itself is the required real rank-four bundle. The sixteen corners give a first mixed coefficient 2η at label eight and genus $8g-7$.

The factor two is the largest possible power in the transfer for distinct prescribed sign characters in this rank. Indeed $r \leq 4$ and $r \leq 2^b - 1$. For $r = 4$, the monomorphism is an isomorphism, so $w_1(W) = \sum_i \alpha_i = 0$ also forces $b \leq 3$. These inequalities give $r - b \leq 1$, attained by both dependent examples.

7. Transport on the tower of finite covers

Fix the curve and the identification of its normal bundle. For a rigid operator through degree M put $\mathbf{P}(D) = (P_T(D|_T; q))_T$ for all connected unramified covers T of degree $a \leq M$, each retained through degree $\lfloor M/a \rfloor$. The degree-one coefficient on T is $\epsilon_T(D) x_{g(T),1}$.

Theorem 7.1. *For two endpoint operators D_0, D_1 rigid through degree M on every connected stable-map space there is a unique integral Burnside ring unit through the cutoff whose marks are the ratios $\epsilon_T(D_1)/\epsilon_T(D_0)$ on all visible covers. Its action on the entire assignment $\mathbf{P}(D_0)$ gives the least-source-genus contribution at every label of $\mathbf{P}(D_1)$ on every such cover. It gives the full assignment, in every source genus, if a connecting path through the cutoff consists of rigid segments and finitely many of the realized unramified wall comparisons of Theorem 4.1. All finite concatenations of such comparisons compose, including after changing the finite monodromy group. With the curve marked, their transport depends only on the endpoints. A homotopy may leave this class if the two boundary paths being compared each satisfy the hypotheses.*

The theorem determines the transformation between chambers from finite étale determinants and the *initial* insertion on every cover. The finite determinants give relative signs, and the geometry of the initial chamber supplies the rest of its series. This qualification matters even if every crossing in a particular path is unramified.

7.1. **The action and its filtration.** Write $\Pi = \pi_1(C)$. A finite quotient G of Π corresponds to a connected principal cover Z. Write $T_K = Z/K$ for $K \leq G$. We may choose G so that Z dominates every connected cover of degree at most M: intersect the cores of all such subgroups of Π. Finite generation makes this a finite intersection, as degree-d covers are obtained from homomorphisms to S_d. We may refine the resulting quotient further, for example to include the finitely many groups on a path. A subgroup of G is visible if $[G : K] \leq M$. An assignment consists of conjugacy-invariant series P_K with constant coefficient one, defined on visible subgroups through degree $\lfloor M/[G : K] \rfloor$.

For a finite G-set X, restrict it to K and take one point x of each K-orbit. Put

$$(T_X\mathbf{P})_K(q) = \prod_{x \in K\backslash X} P_{K_x}(q^{[K:K_x]}), \tag{5}$$

using factor one for an invisible stabilizer. The retained exponent multiplies the relative projection degree, so the total degree on C is preserved. Disjoint unions of sets give products and virtual differences give quotients. Each T_b, $b \in A(G)$, is an endomorphism of the abelian group of assignments. With the group law written multiplicatively,

$$T_{b+c}\mathbf{P} = (T_b\mathbf{P})(T_c\mathbf{P}), \qquad T_{bc} = T_b \circ T_c, \qquad T_1 = \mathrm{id}.$$

Indeed a K-orbit of pairs in $X \times Y$ is determined by a K-orbit of x and a K_x-orbit of y; its degree is $[K : K_x][K_x : (K_x)_y]$. This proves composition for actual sets and hence virtual sets. The operation preserves integral coefficients and balanced products with integral exponents by (5). For a single orbit it pulls the insertions from the appropriate components of an actual fiber product of covering curves.

The span of the orbits $[G/H]$ with $[G : H] > M$ is an ideal $I_{>M}$: a point stabilizer in any product is contained in a conjugate of H. Such orbits act trivially in the additive sense ($T_b\mathbf{P} = 1$). Their marks vanish on visible subgroups. Conversely, if the marks of a virtual set vanish on all visible subgroups, descend in order of subgroup size. At K its as-yet undetermined orbit coefficient is multiplied by the nonzero diagonal $|N_G(K)/K|$, and orbits for subgroups of smaller size have zero K-mark. This proves that the kernel is exactly $I_{>M}$. Application of T_b multiplies the degree-one coefficient at K by the mark $m_K(b)$, since only K-fixed points of the restricted virtual set contribute. Thus its action on the degree-one part of one geometric assignment recovers all visible marks. Write $A_{\le M}(G) = A(G)/I_{>M}$.

7.2. **The unit from the endpoint operators.** Parker's finite-group generalized sign already assigns each real isotypic crossing the fixed-subgroup signs used here, multiplies them, and treats change of the finite covering quotient. His algebra uses actual stabilizers with multiplication by intersection. Formula (6) expresses the relative signs by Burnside marks through a degree cutoff. Formula (7) uses them on the complete stable-map contributions over intermediate covers [Par13].

Take A_t on Z to be the pullback of any path from D_0 to D_1. The path in this paragraph need not be rigid. The Fredholm complexes decompose into real isotypic parts; write U for an irreducible real G-representation and take the operator on its multiplicity space over $\mathrm{End}_G(U)$. For each visible K, the operator on T_K is the K-fixed part by unramified descent. Every type with $U^K \neq 0$ for such a K therefore has an invertible multiplicity operator at both endpoints, even if $|G| > M$ and the full operator upstairs is singular there. Its real index at the endpoints is zero. The space of allowed operators on C is affine, so the relative determinant parity of this block between the endpoints is independent of the chosen path. Concretely, pull its real determinant line over that affine space and compare the canonical trivializations at the invertible endpoints using continuous orientation transport. On complex and quaternionic multiplicity spaces this sign is always positive. The complex orientation persists throughout the path and agrees with the canonical trivialization at both invertible ends.

For a real-type U visible in this sense, let $e_U \in \{0, 1\}$ record this sign. The element $v_U = 1 - \chi^G(S(U))$ has mark $(-1)^{\dim U^K}$ at each subgroup, so $v_U^2 = 1$. Set

$$u_{01} = \prod_{U \text{ visible of real type}} v_U^{e_U} \quad \text{in } A_{\le M}(G). \tag{6}$$

On the K-fixed operator the real-type multiplicity block occurs in $\dim U^K$ copies. Ratios of endpoint determinant signs multiply in this direct sum. Other blocks preserve orientation. Hence $m_K(u_{01}) = \epsilon_{T_K}(D_1)/\epsilon_{T_K}(D_0)$. In particular $u_{01}^2 = 1$, $u_{02} = u_{12}u_{01}$ for three endpoints, and uniqueness follows from the preceding mark calculation. This construction is valid even if the path between regular endpoints encounters ramified kernels. It uses only determinants on the finite unramified covers.

For complete quotients dominating the degree cutoff, refinement merely reindexes visible covers. More explicitly, under a surjection $G' \to G$ of such quotients every visible subgroup of G' is the inverse image of one in G. The marks determine the same truncated unit there. For endpoints rigid in every degree there is also a unit on each finite quotient with *all* its

marks, by (6) without truncation. Under $G' \to G$ the fixed points of the kernel give a ring map $A(G') \to A(G)$; its mark at K is the upstairs mark at the inverse-image subgroup. The units are compatible under these maps. Thus they define a unit in the completed Burnside ring of almost finite continuous $\widehat{\Pi}$-sets, which can be evaluated at each bounded degree using the quotients above. This is the fixed-point convention of Theorem 16.1 and [DS88].

7.3. **Least-genus comparison for every pair of regular endpoints.** For a cover T_K and relative degree e, the least possible source genus is $1 + e(g(T_K) - 1)$. Only connected unramified covers occur there, including in the stable-map compactification (Section 8). Because G dominates the whole cutoff, each such map is represented by a covering $T_L \to T_K$ for some $L \leq K$. Its sign at the new endpoint equals its old sign times $m_L(u_{01})$. Its deck weight over T_K is $1/|N_K(L)/L|$. Alternatively, the number of its lifts when a transitive component of a restricted G-set is used in (5) is the corresponding L-mark. Composition is representable finite étale and its pushforward weights lifts with exactly these cardinalities. The degree substitution gives the same least source genus at either target by Riemann–Hurwitz. Summing over covers, (5) multiplies each old signed mass by $m_L(u_{01})$. Additivity in the logarithm of the local series yields precisely the new least-genus connected coefficient.

This gives the corresponding equality of all least-genus elementary exponents by proper-divisor induction in Proposition 3.1. In particular the full change of these exponents, for arbitrary regular endpoints, can be recovered from the signs on the finite covers without an analysis of ramified strata along an intervening path.

7.4. **Full transfer on the unramified domain.** On a rigid segment all assignments are constant by excision and continuation in the fixed marking. On one simultaneous crossing apply Theorem 4.1 on every visible target. On a refined target the restrictions of the wall representations may contain equivalent, trivial or non-real-type summands. The proof uses the exact monodromy strata and the pullback of the downstairs reduced obstruction complex. It gives the same mark $(-1)^{\dim V_i^L}$ on a stable map of refined monodromy L on reversing one crossing coordinate, regardless of the decomposition of the restricted representation. The subgroup L indexes a cover through which the map lifts, so its index in G divides the total degree and is visible. The center classification on that cover is precisely the pullback-invariant description of Theorem 4.1. No regularity of the entire refined Galois cover in degree $|G| > M$ is needed. The connected transfer therefore holds on T_K with its restricted Burnside element. The elementary product identity follows from the first differing degree and the fixed-target genus test on T_K.

In particular write $B_i = \chi^G(S(V_i))$, inflating from the actual wall group if needed. Changing one coordinate from positive to negative gives $\mathbf{P}_- = T_{1-B_i}\mathbf{P}_+$ by (5) and the single-coordinate transfer. This equality is symmetric since $(1 - B_i)^2 = 1$. Its effect on the degree-one marks identifies the multiplier with the endpoint determinant unit (6). Composing over a path proves

$$\mathbf{P}(D_1) = T_{u_{01}}\mathbf{P}(D_0). \tag{7}$$

Two paths using different covers can be compared in their common refinement with the full degree cutoff. The unit of a closed marked path of this kind has every visible mark equal to one and is the identity. Thus full numerical and universal corrections along these paths satisfy the same relations, even for several walls on one support. These are equalities in the integral balanced lattice; the operations do not require an enumerative evaluation on arbitrary analytic families.

For general regular endpoints the quotient

$$\mathbf{R}_{01} = \mathbf{P}(D_1)/(T_{u_{01}}\mathbf{P}(D_0)), \qquad \mathbf{R}_{02} = \mathbf{R}_{12}\,T_{u_{12}}\mathbf{R}_{01}$$

has integral elementary exponents and vanishing least-genus exponents on every visible target. It equals one on the unramified domain of (7).

The conditional character of the higher-genus conclusion can be seen on the standard representation of S_3 at a simply branched cubic of a genus-two curve with two branch values. A connected unramified subgroup of the closed surface pulls back to a subgroup of the punctured surface group containing all conjugates of the peripheral loops. Its image contains the normal closure of a transposition, hence all of S_3. The standard plane has no fixed vector there. Thus if this plane supplies the full central kernel, every unramified sign through any fixed cutoff persists across the crossing. In the ramified example of [Wue26d, Theorems 1.1–1.2], the genus-five degree-three exponent changes by two in absolute value; the genus-thirteen degree-seven exponent changes by one. These last two numerical values require its ramified deformation calculation. The proof of the transport theorem above is independent of that example.

8. Exact prime-power denominators at the least genus

Let r be the number of distinct characters and b the dimension of their span over $\mathbb{F}_2$. Their joint cover $Z \to C$ has degree $a = 2^b$. Write $m = 2^{r-b}$. For the r-fold difference of the connected fixed-target contributions in degree d at source genus $h = 1 + d(g(C) - 1)$, write δ_d. Use the alternating sum with coefficient $+1$ at the all-positive corner. Let ϵ be the absolute determinant sign of the normal operator on Z at that corner; this sign is constant as a corner germ.

Theorem 8.1. *For each prime p and integer $j \geq 1$,*

$$\frac{p^j}{m}\,\delta_{ap^j} \in \mathbb{Z}, \qquad \frac{p^j}{m}\,\delta_{ap^j} \equiv \epsilon \pmod{p}, \qquad \operatorname{den}(\delta_{ap^j}) = \frac{p^j}{\gcd(p^j, m)}. \tag{8}$$

In particular every such projected degree has a nonzero mixed contribution in its least possible source genus. In degree four at the square of two independent double characters the mixed elementary exponent is also nonzero, by Theorem 2.1. At the higher labels in (8) the assertion concerns the connected invariant, to which lower elementary factors can contribute.

8.1. **The integral series of unramified covers.** For a regular normal germ on a curve T of genus $H \geq 2$ put $h_e(T) = 1 + e(H - 1)$ and

$$c_e(T) = L^T_{e,h_e(T)} = \sum_{[f:Y\to T]} \frac{\epsilon(f^*D_T)}{|\operatorname{Aut}_T f|},$$

the sum over connected unramified degree-e covers. Indeed Riemann–Hurwitz and the genus of the dual graph force one smooth unramified positive component at the minimum genus. To see this with possibly disconnected positive components after normalization, each additional positive component adds one to the sum of their genera. A contracted positive-genus component or a loop also raises arithmetic genus, and an unmarked tree of contracted rational components attached to one smooth component is unstable. The horizontal obstruction theory at an unramified cover is regular. Thus only the displayed smooth covers occur, with their normal determinant signs and stack deck weights. See Proposition 3.1.

Let $C_e(H)$ be the unweighted-by-sign groupoid cardinality of the same connected covers. The exponential $U_H(q) = \exp(\sum_{e\geq 1} C_e(H)q^e)$ counts disconnected unramified covers. It

belongs to $1 + q\mathbb{Z}[[q]]$. In fact the degree-n coefficient is $|\operatorname{Hom}(\pi_1 T, S_n)|/n!$; the surface character formula and the hook-length formula give

$$[q^n]U_H = \sum_{\lambda \vdash n} \prod_{v \in \lambda} h(v)^{2H-2},$$

where $h(v)$ is the hook length of the box v. The empty partition contributes one. The unsigned integrality and the congruences for the pointed counts $eC_e(H)$ are already in [BP01, Appendix C, Lemmas C1–C4]. The same unsigned series is the least-genus part of the elementary local curve formula used in Proposition 3.1. The following argument incorporates the real normal determinant signs and applies the result to the mixed wall contribution.

Fix a cutoff E. Write n_l for the integral exponent at label l and its minimum genus $h_l(T)$ in the balanced factorization through E. No smaller genus occurs. Proposition 3.1 gives $c_e(T) = \sum_{l|e} n_l C_{e/l}(h_l(T))$ for $e \le E$. Its proof is degreewise and applies to the truncated rigid products of Corollary 3.2. Consequently

$$\exp\left(\sum_{e=1}^{E} c_e(T) q^e\right) = \prod_{l=1}^{E} U_{h_l(T)}(q^l)^{n_l} \pmod{q^{E+1}}$$

has integral coefficients. Every such series has unique integer exponents b_l in a product of $(1-q^l)^{-b_l}$ through E: divide out the factors in increasing order using the current integer coefficient at the next degree. Taking the logarithmic derivative gives the integer

$$G_e = e c_e(T) = \sum_{l|e} l b_l, \qquad b_1 = \epsilon(D_T).$$

More generally $G_{np^j} \equiv G_{np^{j-1}} \pmod{p^j}$ if $p \nmid n$ and $np^j \le E$. For $n = 1$ this implies $G_{p^j} \equiv \epsilon(D_T) \pmod{p}$. Apply the formula on $T = Z$ with $E = p^j$. The character-wall transfer gives $\delta_{ap^j} = m c_{p^j}(Z)$ since the least genera upstairs and downstairs agree in these degrees. This proves (8) at each cutoff, hence for the orthant germs. The prime-power congruence for one isolated curve follows already from its integral least-genus factorization; its occurrence in the mixed geometric change uses the simultaneous wall theorem.

8.2. **Independent check from oriented regular covers.** One can obtain the residue modulo p without an integral factorization. For $e = p^j$ the deck group of a connected cover acts freely on any fiber, so $e/|\operatorname{Aut} f|$ is integral. In $e c_e(T)$ all terms except the regular covers vanish modulo p. A regular cover has deck group Q of order p^j. If $p = 2$ every determinant sign is 1 modulo 2. For odd p the determinant sign on a regular cover equals the sign of the base operator. Indeed squaring permutes Q, and every nontrivial irreducible complex character χ has Frobenius–Schur indicator $|Q|^{-1}\sum_x \chi(x^2) = 0$. The nontrivial characters thus occur in distinct conjugate pairs $\chi, \bar{\chi}$. On each real pair the endomorphism $I = i(e_\chi - e_{\bar{\chi}})$, where the e_χ are the central idempotents in the complex group algebra, is real and squares to -1. It commutes with the pulled-back operator all along a path to a complex-linear reference. Its determinant orientation is consequently constant along the path.

At the reference compare I with the reference complex structure J on the normal bundle. In the J-complex character decomposition of the pair, $I = J$ on the χ block and $I = -J$ on its conjugate block. The latter has complex index zero: by unramified descent it consists of $\dim \chi$ copies of the reference operator twisted by the dual flat character representation, whose degree is zero as a flat factor. Riemann–Roch and $\deg N_T = 2g(T) - 2$ give index zero. Reversing complex structure on that block therefore changes determinant orientation by $(-1)^{\dim_{\mathbb{C}} \ker + \dim_{\mathbb{C}} \operatorname{coker}} = 1$. This compares the sign conventions also at a possibly singular

complex-linear reference. The invariant summand is the base operator, proving the sign assertion.

Finally the number of regular covers is the number of normal subgroups of index p^j in the finitely generated surface group Π. Intersect those finitely many kernels. The quotient P is a finite p-group, and all normal subgroups of P of that index correspond to the kernels counted. There is at least one since Π surjects onto $\mathbb{Z}/p^j$. The number is $1 \pmod p$. Here is the elementary counting argument. In a finite p-group the number of subgroups of a specified order p^k dividing its order is $1 \pmod p$. For $k = 0$ this is immediate. Otherwise choose a central order-p subgroup A; the subgroups containing A are counted by induction in P/A. For each subgroup of order p^{k+1} that arises as BA with $|B| = p^k$ and $A \cap B = 1$, the choices of complement to the central A in $B \times A$ are the graphs of $\mathrm{Hom}(B, A)$. Its order is divisible by p because B is a nontrivial p-group. This proves the congruence for all subgroups. Conjugation then shows it for normal subgroups, since every nonfixed orbit has size a multiple of p. Summing the determinant signs of the regular covers gives the same residue as (8).

Example 8.2. Take two independent double covers of a genus-two curve. Their common cover has degree four and genus five. Put $\epsilon = \epsilon_Z(D_{++})$. The first mixed connected coefficient is ϵ in genus five. At degree eight the least genus is nine. The coefficient there is

$$\frac{1}{2} \sum_{0 \neq \lambda \in H^1(Z;\mathbb{F}_2)} \epsilon(D_{++}|_{Z_\lambda}).$$

There are $2^{10} - 1 = 1023$ summands, all signs. Its reduced denominator is two for every such square. At degree twelve the least genus is thirteen. Each regular cyclic triple of Z has the base sign ϵ, and there are $(3^{10} - 1)/2$ such triples. Each other connected triple has deck order one. Thus the degree-twelve coefficient is congruent to $\epsilon/3$ modulo integers, in agreement with Theorem 8.1. These are connected coefficients; their higher-label elementary exponents also involve the subtractions in (2).

9. The elliptic suspension and its projective example

Let f be a smooth symplectomorphism of standard $\mathbb{R}^4$. Take the quotient of $\mathbb{R}^4 \times \mathbb{R}^2$ by

$$(x, s, t) \longmapsto (f(x), s + 1, t), \qquad (x, s, t) \longmapsto (x, s, t + 1).$$

Give the horizontal planes the complex structure of the base elliptic curve and the vertical planes any compatible almost-complex structure. Section 9.1 shows that every compact irreducible pseudoholomorphic curve is the horizontal leaf of a finite orbit. Let $U \Subset \mathbb{R}^4$ be open. Assume that any finite orbit contained in $\bar{U}$ lies in U. For $k \geq 1$ set

$$U_k = \bigcap_{j=0}^{k-1} f^{-j} U, \qquad I_k = \deg(\mathrm{id} - f^k, U_k, 0).$$

These are the local fixed point indices of the iterates, restricted to orbits staying in U. The integrality of their Möbius transforms is the subject of Dold's iteration congruences [Dol83].

Let H denote the Chern-number-zero, virtual-dimension-zero part of Pardon's rational algebra of semi-Fano complex one-cycles [Par25]. Here semi-Fano means that every irreducible component has nonnegative Chern number. All components at the complex targets below have Chern number zero. Use the comparison $\Phi : H \to S$ to the algebra defined using smooth almost-complex families [Wue26b, §2]. A compact open set of cycles in a target defines an element of S by its characteristic function. Take the series $P_U(q)$ of these classes for the cycles supported on orbits in U, graded by projection degree over the elliptic base, with constant

term one. The compactness needed for this definition holds separately in each degree. For a nontorsion degree-zero line L on an elliptic curve C, let $F_1(q)$ be the series in H of the point insertions at $m[C]$ in the total space of $L \oplus L^{-1}$, including the empty cycle. These are the scalar local classes in genus one [Par25, Definition 4.7]. Put $E(q) = \Phi(F_1(q))$.

Theorem 9.1. *With the isolation assumption above,*

$$P_U(q) = \prod_{d \geq 1} E(q^d)^{b_d}, \qquad b_d = \frac{1}{d} \sum_{k|d} \mu(d/k) I_k \in \mathbb{Z}.$$

Equality of two such series is equivalent to equality of all their local iterate indices. Through degree K the formula requires isolation only for orbits of period at most K.

Ionel and Parker related the numerical Gromov series of symplectic mapping tori to the Lefschetz zeta function of the return map [IP99, Theorems 0.1 and 4.1]. In real dimension six their formula uses the genus-one numerical invariants. Dold's indices already determine the product expansion of the local dynamical zeta function. Theorem 9.1 identifies their contribution to the cycle classes themselves, before an enumerative evaluation. The step that makes this identification possible is the calculation at one orbit. A saddle-center bifurcation expresses the positive hyperbolic contribution as the inverse of the elliptic one. At period doubling an additional embedded torus accounts for a factor in twice the primitive class. Ionel and Parker describe the corresponding relations for numerical generating functions and their connection with Taubes's torus counts [IP99, Remark 3.4]. Here compact continuation gives the relations for the classes represented by the characteristic functions of the entire cycle loci. They determine the contribution of every four-dimensional symplectic return whose relevant iterates are nondegenerate. For each degree bound a small Hamiltonian perturbation then treats arbitrary degeneracies in the isolated set.

9.1. **Compact curves in a suspension.** The product horizontal distribution on the covering space descends to the suspension. A metric on its vertical symplectic bundle gives a compatible vertical complex structure by polar decomposition. Choices of compatible structures form a contractible space and can be made smoothly along a path of symplectomorphisms. Use the direct sum of the vertical and horizontal structures throughout. The standard symplectic form of the fiber descends to a closed two-form Ω whose kernel is the horizontal plane. It is pulled back from the five-dimensional mapping torus over the s circle. That mapping torus has the homotopy type of a circle, since its fiber is contractible. In particular Ω is exact.

Lemma 9.2. *The compact irreducible pseudoholomorphic curves are the suspended periodic orbits. A curve from an orbit of least period j is an embedded elliptic curve of degree j over the base and has Chern number zero.*

Proof. On the normalization of a compact irreducible curve the integral of Ω is zero. In a holomorphic local coordinate its area density is the squared norm of the vertical part of the first derivative, using the compatible vertical metric. The vertical derivative vanishes everywhere. The normalization therefore maps to a single horizontal leaf, where projection to the base is locally biholomorphic. A nonconstant map from a compact connected curve to that leaf is open and has closed image, by lifting it in the flat foliation charts. Its image is the whole leaf. A compact leaf meets a transverse fiber in one finite orbit, and conversely a finite orbit of length j gives the embedded quotient with periods j, i in the base coordinate $s + it$. The tangent line to the leaf is trivial. Its vertical normal symplectic bundle is pulled back from the s circle, so its first Chern class vanishes. □

In a compact-fiber suspension the same proof identifies all the curves of zero Ω-area, since the area density stays nonnegative. For a cycle in the local suspension, intersect with the fiber over the origin of the elliptic base. This gives an f-invariant multiset of length equal to the degree of the cycle. Conversely, an invariant multiset is a union of finite orbits with constant multiplicity on each orbit, so determines a unique cycle. This correspondence allows several formerly distinct orbits to specialize to the same multiple leaf. In a flat target chart the invariant points give constant horizontal sheets. Convergence of the finite multiset thus gives Hausdorff convergence of the supports and weak convergence of their currents, the topology used in [Wue26b, §2]. Finitely many strips cover the base and the same argument works when their gluing map varies.

For a fixed degree m consider all the invariant multisets in $\mathrm{Sym}^m \bar{U}$. They form a closed subset of this compact space. Every constituent orbit has period at most m, so boundary isolation puts the entire multiset inside U. Its continuous parametrization of the cycles is a homeomorphism onto its image by compactness and the Hausdorff property. This image is open in the degree-m cycle space. Indeed, for a retained cycle all its intersection points with the transverse fiber lie in U. Nearby supports avoid the closed complement on that fiber, by compactness of the original support and upper Hausdorff continuity. The degree itself is locally constant, by testing the currents against the base area form with a cutoff equal to one near the compact support. Over a compact parameter interval with boundary isolation the argument applies in the relative symmetric power and gives a compact open family. Its end characteristic functions agree in S by the interval relation [Wue26b, §2].

For disjoint neighborhoods of finitely many closed leaf supports, choose disjoint open tubes in the suspension containing those supports. For each distribution of the total degree among the tubes, external product followed by open extension represents the characteristic function of the corresponding cycles. Summing over all distributions gives the product of the local series. A single orbit of least period j has deck gluing f^j in its tubular germ over the elliptic curve of first period j. In terms of the original base degree its series is obtained from the local one by the substitution $q \mapsto q^j$.

9.2. **The contribution of a nondegenerate orbit.** Let x have least period d under the symplectomorphism f, and put $T = D_x f^d$. A transverse disc at x, followed for d periods, gives the germ of the primitive leaf as a suspension of f^d over an elliptic curve with first period d. We use q for the class of this leaf. In the grading by degree over the original base its contribution is obtained by substituting q^d for q. Our deck transformation advances the first base coordinate by one and acts on the fiber by f. Thus positive parallel transport has holonomy f^{-1}. The periodic orbits are the same, and $\det(I - T^{-j}) = \det(I - T^j)$ since T is symplectic and its real dimension is even.

Proposition 9.3. *Suppose* $\det(I - T^j) \neq 0$ *for* $1 \leq j \leq K$, *and put* $\sigma_j = \operatorname{sign} \det(I - T^j)$ *in this range. For* $K \geq 2$ *the local series of the primitive leaf is*

$$E(q)^{\sigma_1} E(q^2)^{(\sigma_2 - \sigma_1)/2} \pmod{q^{K+1}}. \tag{9}$$

The sign σ_j *is* σ_1 *for odd* j *and* σ_2 *for even* j. *For* $K = 1$ *the series is* $E(q)^{\sigma_1}$ *modulo* q^2. *If every iterate is nondegenerate,* (9) *holds as an identity of formal series.*

For a sum of two symplectic planes write $P_{A,B}$ for its local series, using $e, +, -$ for elliptic, positive hyperbolic and negative hyperbolic blocks. Fix the second block B with its first K iterates nondegenerate. A saddle-center bifurcation and a period doubling give, respectively,

modulo q^{K+1},

$$P_{+,B}(q)P_{e,B}(q) = 1, \qquad P_{-,B}(q)P_{e,B^2}(q^2) = P_{e,B}(q). \tag{10}$$

Each elliptic symbol refers to the rotation at its indicated endpoint. The two points exchanged at period doubling give one leaf of base degree two. Its return on the second plane is B^2, which explains both the single factor and the substitution in the second identity.

Proof. Work in a Darboux chart centered at the return point and denote the primitive return germ by g. For $0 < \lambda \leq 1$ the maps $g_\lambda(y) = \lambda^{-1} g(\lambda y)$ are symplectic and extend smoothly to $g_0 = T$. For each $j \leq K$ their jth iterates have the form $T^j y + O(\|y\|^2)$ uniformly in λ. Since the finitely many $I - T^j$ are invertible, there is a common ball in which zero is the only fixed point of these iterates. To realize the family of germs in a suspension, the isotopy $T^{-1} g_\lambda$ has a Hamiltonian on a smaller ball, obtained by integrating the closed one-form of its generating vector field. A cutoff of this Hamiltonian, followed by composition with T, extends the family to $\mathbb{R}^4$ while preserving it on a neighborhood of zero. The characteristic insertion of each multiple through degree K has compact support along the central leaf over the parameter interval. Continuation identifies its class with that of the linear suspension. The same argument identifies the insertions along any path of linear maps satisfying the stated nondegeneracy conditions.

First suppose T is a sum of two real symplectic planes. A planar block is elliptic if it is conjugate to a rotation, and positive or negative hyperbolic if its eigenvalues are distinct positive or negative reciprocal reals, respectively. Blocks of either hyperbolic type can be joined within that type by symplectic conjugation and variation of the real eigenvalues. Such paths avoid all resonances. We retain the angle of each elliptic block until its contribution has been determined.

For two elliptic blocks choose a complex structure making both rotations unitary. The suspension is the total space of two flat holomorphic lines on the elliptic base. Each power of either line of order $1, \ldots, K$ is nontrivial, since a holomorphically trivial unitary flat line has trivial holonomy. For a cycle of degree $m \leq K$, the elementary symmetric coefficients of its fiber projection to a line L are holomorphic sections of L^j for $1 \leq j \leq m$. This follows across branch points from the characteristic polynomial on the finite direct image of the normalization, using the cycle multiplicities. All these coefficients vanish. Thus the degree-m cycle is the m-fold zero section. In the elliptic Picard variety the complement of the torsion points of orders at most K is path connected. Move the two lines there independently to a nontorsion pair L, L^{-1}. At every stage the same multiple zero sections give compact point insertions. At the inverse pair these are precisely the point insertions defining F_1. It follows that $P_{e,e} = E$, for either angular orientation and every base modulus [Par25, Definition 4.7], [Wue26b, §3].

Fix a second planar block B. On the first plane consider the time-δ Hamiltonian map of

$$H_a(u, v) = \tfrac{1}{2} v^2 + \tfrac{1}{3} u^3 - au.$$

For small $a < 0$ it has no equilibrium in a disc about the origin. For $a > 0$ the critical points at $u = \sqrt{a}, -\sqrt{a}$, $v = 0$, give an elliptic and a positive hyperbolic fixed point. For a second family take negation composed with the time-δ map of

$$J_a(u, v) = \tfrac{1}{2} v^2 + \tfrac{1}{4} u^4 - \tfrac{a}{2} u^2.$$

Negation commutes with this flow. At $a < 0$ the origin is elliptic. At $a > 0$ it is negative hyperbolic and the two minima at $u = \pm\sqrt{a}$ are exchanged. They give one orbit of period

two. The second iterate at a minimum is the time-2δ elliptic linearized flow on the first plane and acts by B^2 on the second.

For both families take a Hamiltonian cutoff equal to one on a larger disc, even under negation in the second family. The resulting vector fields have a common Lipschitz bound L_0 for a in a compact interval about zero. A nonconstant periodic trajectory of period τ has $\tau L_0 \geq 1$. Indeed, its velocity has mean zero, and the difference of the velocities at two times is bounded by $L_0\tau$ times the maximum speed. Evaluate the mean difference at a point of maximum speed. Choose $2K\delta L_0 < 1$. In the first family any point fixed by the jth iterate is fixed by the time-$j\delta$ flow. In the second it is fixed by the time-$2j\delta$ flow. Hence the only periodic points through the cutoff in the smaller disc are the equilibria just described, including the degenerate one at $a = 0$. The second block contributes only its origin. Choose the regular endpoint parameters to avoid the finitely many forbidden elliptic resonances. All retained orbits lie in the interior of the disc throughout. Their invariant multisets in each degree form a compact family, so the interval relation applies to the whole insertion across either collision.

Excision into the distinct orbit neighborhoods at the endpoints now gives (10). In the second identity the squared block is needed only through degree $\lfloor K/2 \rfloor$. At $K = 1$ its factor is omitted.

First use (10) with B elliptic. The two-elliptic calculation at each endpoint gives $P_{+,e} = E^{-1}$ and $P_{-,e} = E/E(q^2)$ for every admissible second angle. Interchanging the planes now determines $P_{e,B}$ for either hyperbolic type and every admissible first angle. Applying (10) once more yields

$$P_{+,+} = E, \qquad P_{+,-} = E(q^2)/E, \qquad P_{-,-} = E.$$

For a planar elliptic block the determinant signs are positive at every allowed power. A positive hyperbolic block has negative sign at every power, and a negative hyperbolic block has positive sign at odd powers and negative sign at even powers. Multiplying the two signs proves the proposition for the split cases.

For general T, take a small coordinate ball in $\mathrm{Sp}(4, \mathbb{R})$ on which the first K determinants remain nonzero. This ball contains a matrix of simple spectrum. Indeed the discriminant of the characteristic polynomial is a real analytic function on the connected symplectic group and is nonzero on a diagonal matrix with distinct reciprocal pairs. A path in the ball to a simple-spectrum matrix preserves the local insertion and every relevant sign. In the complexification, the symplectic form pairs eigenlines only when their eigenvalues are reciprocal. A real reciprocal pair or a conjugate pair on the unit circle therefore determines a real symplectic plane, reducing these cases to the calculation above.

The remaining spectrum consists of a nonreal quartet $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ off the unit circle. The real stable and unstable planes are Lagrangian by invariance of the symplectic form under forward and backward iteration, respectively. In dual bases of the two planes the matrix has the form $\mathrm{diag}(A, A^{-\mathsf{T}})$. Choose the stable basis to give $A = rR_\theta$, with $0 < r < 1$ and R_θ a rotation, and change the unstable basis dually. Deforming θ to zero keeps the spectrum off the unit circle. The endpoint splits into two positive hyperbolic planes with eigenvalues r, r^{-1} on each. Its contribution is E, and all its determinant signs are positive. This proves (9) through the prescribed cutoff for every T. Open excision identifies the coefficients obtained in smaller neighborhoods as K increases, giving the formal identity when all iterates are nondegenerate. □

9.3. **Fixed point indices of the iterates.** Fix $K \geq 1$ and assume that every orbit of period at most K contained in $\bar{U}$ lies in U. This condition persists under a sufficiently small

perturbation of f in the compact-open topology. Indeed, from a sequence of perturbed orbits in $\bar{U}$ meeting ∂U, take a subsequence of one fixed period and choose a boundary point on each orbit. Compactness gives a limiting orbit of f in $\bar{U}$ with a point on ∂U.

For a path f_t satisfying this condition, put

$$U_{k,t} = \bigcap_{i=0}^{k-1} f_t^{-i} U, \qquad 1 \le k \le K.$$

A zero of $\mathrm{id} - f_t^k$ on $\partial U_{k,t}$ would give a forbidden boundary orbit. The retained zero sets have compact union in the product of the parameter interval with $\bar{U}$. Indeed, one may replace U by $\bar{U}$ in their definition and then use the isolation assumption. At any t_0 choose an open set $W \Subset U_{k,t_0}$ containing the retained zeros. For nearby t we have $\bar{W} \subset U_{k,t}$ by continuity, and W still contains every retained zero by compactness. If there is no zero at t_0, this remains true on a short interval. Excision and homotopy invariance of the Brouwer degree on W show that I_k is constant along the path. The series P_U is constant through degree K by the compact family of invariant multisets in §9.1.

Lemma 9.4. *There is an arbitrarily small Hamiltonian perturbation g of f for which every orbit in U of least period $j \le K$ is nondegenerate at every iterate of order $aj \le K$.*

Proof. We successively postcompose the map with compactly supported Hamiltonian diffeomorphisms, proceeding in order of least period $j = 1, \dots, K$. Suppose that the orbits of smaller least period already have the required property. There are finitely many of them, by compactness and nondegeneracy. Around their points choose disjoint inverse function neighborhoods for the finitely many iterate equations of order at most K. For an iterate fixing a point, its unique continued zero is the continuation of the primitive periodic point. For every other iterate the neighborhood can be chosen disjoint from its fixed points. These properties persist under small C^1 perturbations.

Write F for the current map. Outside smaller neighborhoods of the reserved points, the set of $x \in \bar{U}$ with $F^j x = x$ and orbit in $\bar{U}$ is compact and consists of points of exact period j. At each such x take a Darboux ball disjoint from the other points of its orbit. Four smooth Hamiltonians supported in this ball can be chosen with Hamiltonian vectors spanning $T_x\mathbb{R}^4$ at x. For the postcomposition of F with their flows, the derivatives of the jth iterate in these four time parameters receive a contribution only at the last return to x. They therefore span $T_x\mathbb{R}^4$. This rank condition persists on a neighborhood of x.

Finitely many such neighborhoods cover the compact set. Compose all their flows in one family with independent parameters. At zero parameters the four columns belonging to each chosen ball retain the rank just computed, even if different balls overlap. The parameter-dependent iterate is consequently transverse to the diagonal near all the relevant zeros. Compactness and boundary isolation place all nearby retained zeros in these neighborhoods or in the reserved ones. By parametric transversality an arbitrarily small choice gives transversality at each of the remaining fixed points of the jth iterate. They now form a finite set.

Select one point on each of the new orbits and choose disjoint Darboux balls there that avoid every other orbit point of period at most j in U. A quadratic Hamiltonian centered at the chosen point and cut off in its ball fixes the orbit and multiplies the linear return on the left by a prescribed symplectic matrix near the identity: take the Hamiltonian logarithm of this matrix as the infinitesimal linear map. For each $1 \le a \le \lfloor K/j \rfloor$ the equation $\det(1 - T^a) = 0$ defines a proper real analytic subset of the connected symplectic group. We can therefore

perturb the primitive return T arbitrarily little so that all these determinants are nonzero. The other points of the orbit have conjugate return matrices. The previously transverse fixed points of the iterates of order at most j persist uniquely, and compactness excludes new ones for a sufficiently small perturbation. This completes the induction. All the composing flows can be taken small enough to give a path from f to g preserving boundary isolation through K. □

Proof of Theorem 9.1. Use the perturbation of Lemma 9.4. For an orbit of g of least period $j \le K$, let T be its primitive return matrix and $s = \operatorname{sign}\det(1-T)$. If $2j \le K$, put $t = \operatorname{sign}\det(1-T^2)$. The sign at the ath power of T is s for odd a and t for even a, as follows from the planar blocks and the symplectic deformations in Proposition 9.3. The j points of the orbit thus contribute js to I_k for odd k/j, jt for even k/j, and zero if j does not divide k.

By Proposition 9.3 the orbit contributes

$$E(q^j)^s\, E(q^{2j})^{(t-s)/2}$$

through degree K, omitting the second factor when $2j > K$. Excision into disjoint orbit tubes multiplies these factors. Write e_d for the resulting exponent of $E(q^d)$ for $d \le K$. Each e_d is an integer. In the divisor sum $\sum_{d|k} d e_d$ the first factor of this orbit contributes js at each multiple of j. Its second adds $j(t-s)$ at the even multiples. These are exactly the local index contributions above. Hence

$$I_k = \sum_{d|k} d e_d, \qquad 1 \le k \le K.$$

Möbius inversion gives the stated formula for $b_d = e_d$. Both the series and the indices agree for f and g through this order. The exponents thus depend only on the original indices, and the products for successive K agree in the overlapping degrees.

Finally write $E(q) = 1 + cq + O(q^2)$. The class $c = \Phi(x_{1,1})$ is nonzero by multiplicity-one detection [Wue26b, Theorem 5.1]. If two exponent sequences first differ at d, their products first differ in degree d by the difference of the two exponents times c. This is nonzero over $\mathbb{Q}$. Equality of the series is therefore equivalent to equality of the exponents and hence, by the divisor sum, to equality of the iterate indices. □

In particular, when the retained orbits split among disjoint open sets, each orbit lying wholly in one of them, index excision gives the product of the corresponding series. The moving-domain argument also applies along any path with boundary isolation. It allows permutation of individual orbits while preserving the series graded by degree over the base.

Under an isolated deformation the retained periodic orbits can be permuted. The index formula recovers exactly the unlabelled iterate indices. In the following example the labels in integral homology also distinguish the individual section orbits.

9.4. **A projective quotient with sixteen section classes.** Let A be the product of two elliptic curves with a translation-invariant product Kähler form ω_A. Set

$$E' = \mathbb{C}/(2\mathbb{Z} + i\mathbb{Z}), \qquad B = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \qquad X = (A \times E')/\langle (x,z) \longmapsto (-x, z+1)\rangle.$$

The projection $p : X \to B$ is an abelian surface bundle. The involution acts freely and preserves a nowhere-vanishing holomorphic three-form, so X is smooth and projective with K_X trivial. Its invariant one-forms come from E', and its invariant two-forms come from A. In particular $h^{1,0}(X) = h^{2,0}(X) = 1$. The Hodge vanishing assumptions of the compact product theorem of [Wue26b] fail for this quotient.

Write Ω for the closed semipositive form on X induced by ω_A. Let Γ^0 consist of the integral homology classes of effective one-cycles on X that pair to zero with Ω, including the class of the empty cycle. This is a face of the effective monoid since Ω pairs nonnegatively with every irreducible curve. For $\gamma \in \Gamma^0$ let $U_{X,\gamma} \in H$ be the class of the characteristic function of the class-γ locus in the cycle space. With $\Lambda = H_1(A;\mathbb{Z})$, identify $A[2]$ with $\Lambda/2\Lambda$ by $v \mapsto v/2 \bmod \Lambda$. Denote by β_v the class of the section through this point and by $F_1(q)$ the scalar local series of an elliptic curve.

Theorem 9.5. *The sixteen classes β_v are distinct and generate Γ^0. They satisfy $2\beta_v = 2\beta_0$ for every v. In the monoid algebra over H completed by degree over B one has*

$$\sum_{\gamma\in\Gamma^0} U_{X,\gamma} Q^\gamma = F_1(Q^{2\beta_0})^{-8} \prod_{v\in\Lambda/2\Lambda} F_1(Q^{\beta_v}). \tag{11}$$

Proof. Write $z = s + it$. As a smooth manifold X is $M \times S^1_t$, where M is the mapping torus of negation on A. On its universal cover a lattice translation T_v acts on the fiber and the first period acts by $R(x,s) = (-x, s+1)$. The relation $RT_vR^{-1} = T_{-v}$ gives

$$H_1(M;\mathbb{Z}) = (\Lambda/2\Lambda) \oplus \mathbb{Z}.$$

The positively oriented s-loop based at $v/2$ lifts to the deck transformation T_vR, of class $(v,1)$. Taking its cross product with $[S^1_t]$ gives β_v with its complex orientation. This cross product identifies $H_1(M;\mathbb{Z})$ with a direct summand of $H_2(X;\mathbb{Z})$ by the integral Künneth formula. It follows that the sixteen labels are distinct and that their doubles coincide.

On the normalization of an irreducible curve the pullback of Ω is pointwise nonnegative. If its integral vanishes, the differential has zero vertical component, so the curve is a compact horizontal leaf, as in §9.1. At the complex structure on X these leaves correspond to the one- and two-element orbits of negation. A two-element orbit traverses two s-periods, with deck transformation R^2 and zero lattice translation. The class of its leaf is therefore $2\beta_0$. This proves the assertion about the monoid. Its generators all have base degree one. There are finitely many labels of bounded base degree, and any ample degree on X gives the same completion on this face since the degrees of its finitely many generators have positive minimum and finite maximum.

Deformation of the return. Choose period-one real coordinates (u_1, v_1, u_2, v_2) in which ω_A is a positive constant multiple of $du_j \wedge dv_j$ on the jth plane. Let φ^τ be the Hamiltonian flow of

$$G(u_1,v_1,u_2,v_2) = \sum_{j=1}^{2} \big(\cos(2\pi u_j) + \cos(2\pi v_j)\big).$$

It commutes with negation. Put $f_\tau = (-1)\circ\varphi^\tau$ and suspend f_τ as in §9.1, with descended vertical form Ω_τ. Choose compatible vertical complex structures smoothly in the parameter, starting with the product complex structure, and retain the flat horizontal complex line throughout. Under the resulting identification of the underlying manifolds, $[\Omega_\tau] = [\Omega]$. Indeed, choose a smooth function h on the fundamental strip $0 \le s \le 1$, identically zero near 0 and one near 1. The map from the initial suspension to the one at τ given on this strip by $(x,s,t) \mapsto (\varphi^{\tau h(s)}x, s, t)$ respects the seams. The change of vertical form is $\pm\tau h'(s)\, ds \wedge dG$. It has a primitive $\mp\tau h'(s) G\, ds$ vanishing near both seams. This identification fixes the sixteen section tori since the points of $A[2]$ are equilibria.

For any parameter, every component in a cycle of a face class is horizontal, by the nonnegativity of its vertical area. At a fixed base degree m, intersecting with the fiber over $0 \in B$ identifies the horizontal cycles with the f_τ-invariant elements of $\mathrm{Sym}^m A$. The

inverse takes the union of the corresponding horizontal sheets with multiplicity. In flat charts convergence of such multisets gives Hausdorff convergence of supports and weak convergence of the integration currents. Over a compact parameter interval the invariant multisets form a closed subset of the product of that interval with $\mathrm{Sym}^m A$ and are therefore compact.

The integral class is locally constant on this family, including its torsion component. To see this also in the ambient relative cycle space, take disjoint tubular neighborhoods of the distinct leaf tori supporting a given cycle. All sufficiently nearby supports lie in these tubes. In each tube a closed cycle has integral class a multiple of the core since the tube retracts onto the torus. Pair with the pullback of an area form of integral one on B. Its value on the core is the positive degree of that leaf. The integral in each tube is continuous at the given cycle, by a cutoff equal to one near the part of its support in that tube. Being a multiple of the core degree, it is locally constant. The argument persists with the parameter under the smooth identification just constructed. Thus the locus of each face class is compact and open over the interval. Its characteristic function yields the endpoint relation in S.

Fix a positive integer K. For all sufficiently small $\tau > 0$, every point of f_τ of period at most K is one of the sixteen equilibria. Indeed $f_\tau^j x = x$ implies $\varphi^{2j\tau} x = x$. Write V for the lifted periodic Hamiltonian vector field on $\mathbb{R}^4$. Bounded speed ensures that any closed trajectory on A of sufficiently short period lifts to a closed trajectory in $\mathbb{R}^4$. If V has Lipschitz constant L and such a lift has period T, its velocity has zero average. At a point of maximum speed M_0, subtracting this average and bounding the difference of the two velocities gives $M_0 \le LTM_0$. Taking $2K\tau$ smaller than both short-time bounds excludes every nonconstant trajectory in question.

At an equilibrium, the return derivative on either plane is minus the exponential of the linear Hamiltonian flow. A definite Hessian on that plane gives an elliptic block, and an indefinite Hessian gives a negative hyperbolic block. We may choose τ in the preceding interval to give irrational rotation angles divided by π on all the elliptic blocks, by omitting countably many values. For the full return derivative D_v at $v/2$ we have

$$\operatorname{sign}\det(1 - D_v) = +1, \qquad \operatorname{sign}\det(1 - D_v^2) = (-1)^{\mathrm{ind}_{v/2}(G)}.$$

Indeed each negative hyperbolic plane reverses precisely the second sign and has Morse index one. Each elliptic plane has even Morse index. There are eight odd-index equilibria since each cosine coordinate independently contributes zero or one negative Hessian direction. The local return formula of Proposition 9.3 therefore gives $E(Q^{\beta_v})$ at the even-index points and $E(Q^{\beta_v})/E(Q^{2\beta_v})$ at the odd-index points, where $E = \Phi(F_1)$. All cycles through degree K at this endpoint are supported on the sixteen disjoint tori. Excision into disjoint tubes makes their local series multiply, retaining the integral label of each summand. The endpoint relation and $2\beta_v = 2\beta_0$ now give (11) after Φ, through degree K. Each cutoff compares with the same complex endpoint, so equality in S holds in every degree.

Lifting the identity. Let V_γ denote the coefficient of either side of (11) before applying Φ. The coproduct Δ induced by addition and pullback ρ_d under multiplication of cycles give

$$\Delta V_\gamma = \sum_{\eta+\xi=\gamma} V_\eta \otimes V_\xi, \qquad \rho_d V_\gamma = \sum_{d\eta=\gamma} V_\eta.$$

For the class loci these identities follow from their inverse images. Nonnegative vertical area puts every effective summand and effective root of γ in Γ^0. In particular every section label is a root of the common double. For the product the identities follow directly on the nontorsion inverse pair defining F_1. Its entire cycle locus of degree m is the point $m[C]$. Under addition its preimage consists of the ordered pairs of multiples whose degrees sum to m.

Under multiplication by d the preimage is empty unless $d \mid m$, when it is the point $(m/d)[C]$. Substitution $Q^{\eta} \mapsto Q^{d\eta}$ is a ring homomorphism even when several monomials have the same image, and preserves inverses of series with constant term one.

If there were a first base degree of disagreement, the difference of the coefficients at any such label would be primitive for Δ, since the summands of smaller positive degree agree. Every proper division would kill this difference, since all roots in its division formula have smaller base degree. By the primitive-division lemma of [Wue26b, Lemma 7.1] the difference is a finite linear combination of the multiplicity-one scalar classes $x_{h,1}$. Their images under Φ are linearly independent by [Wue26b, Theorem 5.1]. The identity in S just proved makes this difference zero, a contradiction. □

After forgetting the torsion labels, the exponents at degrees one and two are 16 and -8. Their divisor sums give fixed point indices 16 on the odd iterates and 0 on the even iterates, in agreement with the Lefschetz numbers of negation on A and the iteration formula of Dold [Dol83]. For comparison, put $\mathcal{E}(q) = \prod_{j \geq 1} (1 - q^j)^{-1}$, the numerical elliptic partition. The Lefschetz zeta function here is $\zeta(q) = (1 + q)^8/(1 - q)^8$, whence

$$\mathcal{E}(q)^{16} \mathcal{E}(q^2)^{-8} = \prod_{j \geq 1} \zeta(q^j).$$

Ionel and Parker use a Möbius-normalized Gromov series in their mapping-torus formula. In real dimension six it concerns the genus-one numerical series. Their distinguished-section hypothesis requires $\det(1 - f_*|_{H_1(A)}) = \pm 1$, whereas negation on A has determinant $\det(1 - f_*|_{H_1(A)}) = 16$ [IP99, Theorems 0.1 and 4.1]. Equation (11) retains all sixteen integral section labels before numerical specialization.

10. The wall associated to a real character

Fix a nontrivial sign character χ and let $f : Y \to C$ be its connected double cover. Let $E_{\mathbb{R}}$ be the flat real line on C with holonomy χ and let $E = E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Its parallel unit frames differ by signs and give E a conjugation. Set

$$N = E \oplus (K_C \otimes E).$$

Choose a conformal metric $\rho(z)|dz|^2$ on C. In a holomorphic coordinate and a parallel sign frame write a section as $(u, p\,dz)$. For $-1 \leq t \leq 1$ define D_t by subtracting the right-hand sides of

$$\partial_{\bar{z}} u = \bar{p}, \qquad \partial_{\bar{z}} p = \rho(z)(t \operatorname{Re} u - i \operatorname{Im} u) \tag{12}$$

from the left-hand sides. In the second equation both sides transform by the squared modulus of the coordinate Jacobian. The first equation is an equality of $(0,1)$-forms. All entries transform by the frame sign, so D_t is a global real Cauchy–Riemann operator with fixed Dolbeault symbol. We write $B_t = f^* D_t$.

Theorem 10.1. *For every such f there is a path D_t, $-1 \leq t \leq 1$, with the following properties.*

1. *The operator on C is invertible near zero. At zero the operator on Y has a transverse real kernel line on which the deck involution acts by -1. Up to real scaling its graph is the only simple nonzero normal graph in any covering degree.*
2. *Every D_t with $t > 0$ is rigid in all degrees. For each $M \geq 1$ there exists $0 < \delta_M \leq 1$ such that D_t is rigid through degree M for $-\delta_M \leq t < 0$. For $-\delta_M \leq t_- < 0 < t_+ \leq 1$,*

$d \le M$, and all k,

$$L^C_{d,k}(D_{t_+}) - L^C_{d,k}(D_{t_-}) = \begin{cases} L^Y_{e,k}(B_1), & d = 2e, \\ 0, & d \text{ odd}. \end{cases} \tag{13}$$

(3) *There exists $T > 0$ such that the parameters at which some finite unramified cover carries a nonzero kernel are dense in $(-T, 0)$. In particular no open subinterval of $(-T, 0)$ is rigid in every degree.*

Proposition 10.2. *At $t > 0$ the pullback of D_t by every connected stable map of positive degree is injective. At $t = 0$ its kernel on a stable map $a : X \to C$ consists of the parallel real sections of $a^*E_{\mathbb{R}}$ in the first summand. It has dimension one precisely when a lifts to Y. The base operator D_0 is invertible. Up to real dilation the unique simple nonzero kernel graph at zero is the embedded double cover Y. The odd kernel of B_0 crosses transversely, with positive determinant sign on the side $t > 0$ and negative sign immediately before zero.*

Proof. On a nonconstant component of the normalization use a coordinate w. Write $a' = d(z \circ a)/dw$. The first pulled-back equation gives $\partial_w \bar{u} = a'p$. Differentiating and using the second gives

$$\partial_{\bar{w}} \partial_w \bar{u} = |a'|^2 (\rho \circ a)(t \operatorname{Re} u - i \operatorname{Im} u). \tag{14}$$

Both identities hold at critical points. For $s = \operatorname{Im} u$ the equation is $\partial_{\bar{w}} \partial_w s = |a'|^2 (\rho \circ a) s$. Pairing with s and integrating by parts gives $s = 0$ for all t. The integral is defined for the flat real line because the square of a frame sign is one. The real part satisfies the same equation with the potential multiplied by t. It therefore vanishes when $t > 0$ and is parallel when $t = 0$. The first equation forces $p = 0$ off the finite critical set and hence everywhere in either case. On a contracted normalization the kernel sections are holomorphic maps to a fixed complex vector space and are constant. Matching at the nodes gives the asserted global kernel, including the monodromy condition around cycles in the dual graph. Conversely, a nonzero parallel section of $a^*E_{\mathbb{R}}$ selects a unit sign over every point of X, which is precisely a lift of a to the unit-frame double Y. For any $r > 0$ the two values $\pm r$ in a local frame form a single embedded copy of Y in N. Every nonzero kernel graph has this image for its value of r. On C itself χ is nontrivial, so the kernel vanishes. Riemann–Roch and $\deg N = 2g - 2$ give real index zero.

Trivialize $f^*E_{\mathbb{R}}$ by its tautological frame. The kernel of B_0 is generated by $\xi = (1, 0)$, negated by the deck involution. The cokernel has real dimension one by Riemann–Roch on Y. Use the isomorphism $f^*K_C = K_Y$ in the second summand. The functional on the target of B_0 obtained by integrating the real part of the second coefficient against $dx\,dy$ annihilates the derivative term by Stokes' theorem and the purely imaginary zeroth-order term pointwise. On $\dot{B}_0\xi$ its value is minus the area of Y for the pulled-back metric. Thus the derivative induces an isomorphism from the kernel to the cokernel, and the determinant sign changes.

At $t = 1$ both zeroth-order terms are off-diagonal and conjugate-linear. Scaling them to zero gives a complex-linear homotopy for the complex structure $(i, -i)$ on the two summands. At its split Dolbeault end, conjugating the second kernel and cokernel changes the usual complex orientation of the real determinant by $(-1)^{\operatorname{ind}_{\mathbb{C}} \bar{\partial}_{K_Y}} = (-1)^{g(Y)-1} = 1$. The invertible B_1 thus has positive sign, which persists for $0 < t \le 1$ by injectivity. □

10.1. **Isolation in bounded degree.** The isolation in bounded degree follows from a more general property of an embedded unramified double. For a section ξ of f^*N, let Γ_r be the image in N of the graph of $r\xi$ over Y.

Proposition 10.3. *Let D_t be a smooth path of real Cauchy–Riemann operators with fixed symbol on a complex bundle $N \to C$ and let $M \geq 1$. Suppose $B_t = f^*D_t$ is invertible for sufficiently small $t \neq 0$, and that $\ker B_0$ contains a section ξ odd under the deck involution whose graph over Y is embedded in N. Assume that for every holomorphic map to C of positive degree at most M from a smooth connected curve, allowing arbitrary branching, the image of each nonzero kernel graph at zero is some Γ_r with $r \neq 0$. Then there exists $\delta_M > 0$ such that a^*D_t is injective on every connected stable map a of degree between one and M whenever $0 < |t| \leq \delta_M$.*

Proof. If a kernel section is nonzero on a nodal domain, constancy and matching on contracted components imply that it is nonzero on some positive-degree normalization. Suppose therefore that $t_n \neq 0$ tends to zero and there are smooth connected covers $a_n : X_n \to C$ of degrees at most M with nonzero kernel sections s_n. After passing to a subsequence their degrees have one value d.

The graphs are pseudoholomorphic for the almost complex structures J_{t_n} of the linear normal equation. In a bundle chart the projection is holomorphic, the vertical structure is the given complex structure and the horizontal-to-vertical part is given by the zeroth-order term of the operator. Fix a Hermitian metric and a unitary connection on N. On a disk bundle there is a common taming form Ω obtained by adding the differential of the fiberwise Liouville form, extended by the connection, to the pullback of an area form on C. Along the zero section all the structures split, so a fixed small disk bundle suffices for t near zero. By real homogeneity we may rescale each s_n to have the same positive maximum norm η, with the closed η-disk inside this bundle. The image cycle, counted with mapping multiplicity, has Ω-area $d\,\mathrm{area}(C)$ by Stokes' theorem. Its mass in a smooth family of Hermitian metrics is therefore bounded.

The compactness theorem for pseudoholomorphic cycles [DIW26, Definition 2.6 and Theorem 2.11] gives a subsequence converging as currents and in the Hausdorff topology of supports to a J_0-holomorphic cycle. The compact disk containing all supports allows application of the theorem on the open taming neighborhood. The limiting support is connected and contains a point of height η. Each of its irreducible curves has positive projection degree since the complex vector fibers contain no compact complex curves. Integrating the pullback of the normalized area form, cut off outside the disk containing all supports, shows that the sum of those degrees with multiplicity is d. Each simple parametrization therefore solves the central equation over a cover of degree at most M. By hypothesis its image is the zero section or some Γ_r. Embeddedness and oddness give $\xi(y) \neq 0$ and $\xi(\iota y) = -\xi(y)$, where ι is the deck involution. Consequently these images are pairwise disjoint after identifying r with $-r$. There are only finitely many components. The connected limit meeting height η must be supported on a single Γ_r, $r > 0$, possibly with multiplicity.

The two values of $r\xi$ in each fiber have distance at least $2r \min_Y |\xi| > 0$. Hausdorff convergence and compactness imply that $s_n(x)$ is uniformly close to one of the two values *over* $a_n(x)$: in a local trivialization the two sheets vary continuously with the base point. Choosing the nearer sheet thus defines a continuous lift $b_n : X_n \to Y$ with

$$fb_n = a_n, \qquad \sup_{x \in X_n} |s_n(x) - r\xi(b_n(x))| \longrightarrow 0.$$

The lift is holomorphic, since f is locally biholomorphic. At regular values of b_n average s_n over its sheets to obtain a section σ_n of f^*N on Y. Every local inverse sheet solves $B_{t_n} s = 0$ there, so their average does also by real linearity. This average is bounded at every branch value. Integrating against test forms away from disks of radius δ about the branch values gives

boundary errors of order $O(\delta)$, so the extension is a distributional solution and is smooth by elliptic regularity. The displayed estimate implies $\|\sigma_n - r\xi\|_\infty \to 0$. In particular σ_n is a nonzero element of $\ker B_{t_n}$ for large n, contrary to invertibility. □

10.2. **Singular parameters on cyclic covers.** For the path (12), all the hypotheses hold for each M by Proposition 10.2. The interval of isolation cannot be chosen uniformly in M.

Proposition 10.4. *There exists $0 < T \leq 1$ such that the parameters $t \in [-T, 0)$ at which D_t has a kernel on a connected unramified cover of C are dense in $[-T, 0]$. In particular there are connected unramified covers of degrees $2n$ with singular parameters $-\lambda_n$, where $0 < \lambda_n \leq A/n^2$ for all sufficiently large n.*

Proof. Represent a primitive class in $H^1(Y;\mathbb{Z})$ by a smooth map $\theta : Y \to \mathbb{R}/\mathbb{Z}$, and write $\alpha = d\theta$ for the associated closed real one-form with integral periods. For $0 \leq s \leq 1/2$ use the flat unitary connection $\nabla_s = d - 2\pi i s\alpha$ on the trivial smooth complex line on Y. Its holonomy around a loop γ is $\exp(2\pi i s \int_\gamma \alpha)$. Let $\lambda(s)$ be the lowest eigenvalue of $\frac{1}{4}\nabla_s^*\nabla_s$ for the pullback of the metric on C. This normalization gives $-\rho^{-1}\partial_{\bar w}\partial_w$ in a parallel frame for a metric $\rho|dw|^2$. The common form domain is $H^1(Y;\mathbb{C})$. By the Rayleigh principle $2\sqrt{\lambda(s)}$ is the infimum of $\|\nabla_s u\|_{L^2}$ over the unit L^2 sphere in this domain. The triangle inequality gives

$$|\sqrt{\lambda(s)} - \sqrt{\lambda(s')}| \leq \pi\|\alpha\|_\infty |s - s'|.$$

Thus λ is continuous, with $\lambda(0) = 0$. For $0 < s \leq 1/2$ the holonomy is nontrivial by primitivity. A zero eigenfunction would be parallel, so $\lambda(s) > 0$.

For rational $s = p/n$ in lowest terms take the connected cyclic n-fold cover $Y_n \to Y$ classified by $[\alpha]$ modulo n. There θ lifts to $\widetilde\theta : Y_n \to \mathbb{R}/n\mathbb{Z}$ and $\exp(2\pi i s\widetilde\theta)$ is a global parallel frame for the pulled-back connection. Express a pulled-back lowest eigensection in this frame and choose a nonzero real or imaginary part u. It is a real scalar eigenfunction on Y_n with eigenvalue $\lambda(s)$. In the parallel sign frame induced from Y, take u as the first entry of a section over the composite $a : Y_n \to C$, and use the isomorphism $a^*K_C \cong K_{Y_n}$ to take ∂u as the second entry. The first equation of (12) holds by construction. The second follows from (14) at $t = -\lambda(s)$, since a' is everywhere nonzero.

Set $T = \min\{1, \lambda(1/2)\}$. By continuity the values of λ on rationals in $(0, 1/2]$ are dense in its image on $[0, 1/2]$, which contains $[0, T]$. This gives the asserted density by restricting to $0 < \lambda(s) < T$ and taking closures. Finally the constant trial section at $s = 1/n$ gives

$$0 < \lambda_n = \lambda(1/n) \leq \frac{\pi^2\|\alpha\|^2_{L^2(Y)}}{n^2\,\mathrm{area}(Y)}.$$

The resulting singular parameters lie on covers of C of degree $2n$, and belong to the parameter interval for large n. □

For sufficiently large n, the first negative singularity on the specified cyclic cover has even multiplicity. Denote by β the harmonic representative of the chosen primitive integral class on Y.

Corollary 10.5. *For all sufficiently large n the negative singular parameter closest to zero on the specified cover $Y_n \to C$ satisfies*

$$t_n = -\frac{\pi^2\|\beta\|^2_{L^2(Y)}}{\mathrm{area}(Y)}\, n^{-2} + O(n^{-4}).$$

The normal kernel on this map has real dimension two. Its determinant sign agrees on the two sides of t_n sufficiently close to t_n.

Proof. Use a unitary gauge to replace α by β in the flat connection. For completeness we recall the bottom-eigenvalue calculation for twisted Laplacians [GI21, Remark 2.3 and Lemmas 4.4–4.5]. On the fixed Sobolev domain $H^2(Y;\mathbb{C})$ put $H(s) = \nabla_s^*\nabla_s/4$. The simple eigenvalue zero at $s = 0$ continues to a simple real analytic eigenvalue $\lambda(s)$: integration of the resolvent on a small circle about zero gives an analytic rank-one projection. Conjugation makes the eigenvalue even. Since $d^*\beta = 0$, $H'(0)1 = 0$, and the quadratic term in $H(s)1$ is $\pi^2 s^2|\beta|^2$. Differentiating the eigenvalue equation at zero and pairing with the normalized constant therefore gives

$$\lambda(s) = \frac{\pi^2\|\beta\|^2_{L^2(Y)}}{\operatorname{area}(Y)}s^2 + O(s^4).$$

In particular λ is strictly increasing immediately to the right of zero. The remaining spectrum stays away from zero there. The spectrum on Y_n is the union, with multiplicities, of the spectra on the cyclic character spaces $s = j/n$, which are the spectra of $H(j/n)$ on Y. The bottom eigenvalue is periodic in s by integral holonomy, and bounded below away from zero on compact subintervals of $(0, 1)$. Thus for large n its first positive value on Y_n is $\lambda(1/n)$, attained only by the two conjugate characters $j = 1, n-1$. The real scalar eigenspace has dimension two.

On the unramified map $Y_n \to C$, equation (14) and the reconstruction by ∂u identify the real normal kernel at $t_n = -\lambda(1/n)$ with this eigenspace. Its crossing map to the cokernel is injective: if the derivative applied to a kernel vector with first entry u lay in the range, the first component of a preimage would have real part w satisfying $(\Delta_{Y_n}/4 - \lambda(1/n))w = u$, with Δ_{Y_n} the nonnegative scalar Laplacian. Self-adjointness gives $\int_{Y_n} u^2 = 0$. The normal operator has index zero, hence its crossing map is an isomorphism of real two-dimensional spaces. The two sign changes cancel in its determinant. □

Completion of the proof of Theorem 10.1. The central sign line and its pullbacks in Proposition 10.2 satisfy the central kernel hypothesis of the oriented comparison in Section 5. Its reversal has multiplier two on the liftable stratum, whose stack of lifts has degree two. The interval in Proposition 10.3 permits continuation to the stated endpoints. The singular parameters on the other side are given by Proposition 10.4. □

11. The partition function and the highest changing genus

Write $h = g(Y)$, $b = h - 1 = 2(g-1)$, $y = e^{iv}$ and $z_j = 2 - y^j - y^{-j}$, so that $z = z_1 = (2\sin(v/2))^2$. For a box $a = (i, j)$ in a partition λ, let $\ell(a)$ be its hook length and put $c(\lambda) = \sum_{(i,j)\in\lambda}(j - i)$. For $p \geq 1$ define

$$Z_p(Q, y) = \sum_{\lambda}(-1)^{p|\lambda|}Q^{|\lambda|}y^{pc(\lambda)}\prod_{a\in\lambda} z_{\ell(a)}^p. \tag{15}$$

The empty partition supplies the constant term.

Proposition 11.1. *For the character path of Theorem 10.1 and each $e \geq 1$ the connected contribution on Y is*

$$L_e^Y(B_1; v) = [Q^e]\log Z_b(Q, y).$$

Put $c = g - 1$. On C the positive and negative endpoints have disconnected series

$$Z_c(q, y), \qquad Z_c(q, y)\, Z_b(q^2, y)^{-1} \pmod{q^{M+1}}$$

at the endpoints through degree M specified in that theorem.

Proof. The flat frame upstairs identifies the two lines with $L_1 = \mathcal{O}_Y$ and $L_2 = K_Y$. At $t = 1$, the order-zero term in B_1 sends each summand to the other by a conjugate-linear map. Scale this term by $\tau \in [0, 1]$. For every τ the resulting real operator commutes with the circle of weights $(1, -1)$, hence with the complex structure $J' = (i, -i)$ on sections and on bundle-valued forms.

Let π and a be the universal curve and map over $\mathcal{M} = \overline{\mathcal{M}}_k(Y, e)$ and set $W_i = -R\pi_* a^* L_i$ in complex K-theory. Put $s = k - 1 - be$. Riemann–Roch gives $r_1 = \operatorname{rk} W_1 = k - 1$, $r_2 = \operatorname{rk} W_2 = k - 1 - 2be$ and $r_1 + r_2 = 2s = \dim^{\mathrm{vir}}_{\mathbb{C}} \mathcal{M}$. At $\tau = 1$ the pullback operators are injective. Their cokernels form a bundle V with

$$[V, J'] = W_1 + \overline{W_2}. \tag{16}$$

Indeed, the entire homotopy consists of complex-linear Fredholm maps for J', including the matching conditions at the nodes. In a nodal chart one stabilizes the family by forms on the thick part, taking their complex span for J' and their translates under the finite stabilizer. Enlarging the stabilization preserves the index. This gives a single family index over $\mathcal{M} \times [0, 1]$ by the nodal family construction of [Zin11, §2.4]. At $\tau = 0$ its negative is the sum of the first Dolbeault cokernel index and the complex conjugate of the second. The Dolbeault resolution in families of nodal curves identifies these indices with the classes in (16) [Sie98, §4].

At $\tau = 0$ the change from the ordinary complex determinant orientation to the J' orientation conjugates both cohomology groups of the second line. The relative sign is $(-1)^{h^0(a^*L_2)+h^1(a^*L_2)} = (-1)^{r_2}$. Transporting both orientations along the homotopy gives

$$e_{\det}(V) = (-1)^{r_2} c_{2s}(W_1 + \overline{W_2}) = [u^0]\, e_u(W_1) e_{-u}(W_2).$$

Here $e_u(W) = u^{\operatorname{rk} W} \sum_{j \geq 0} u^{-j} c_j(W)$. The expression is valid also for negative virtual ranks. For $e = 1$ and $k = h$, the identity map has a zero-dimensional cokernel and the sign is $(-1)^b = 1$.

Only the coefficient of u^0 has the cohomological degree of the virtual class. Pairing with this class gives the antidiagonal residue integral of [BP08, §2.2], using the comparison of virtual classes for the projective target Y [Sie99, Theorem 0.1]. The line degrees $(0, 2b)$ sum to $2h - 2$, so the genus-variable shift in the residue convention is zero. Exponentiation of the connected integrals counts unordered collections of positive-degree connected components with their automorphism weights. Corollary 7.2 of [BP08] gives (15) for the result. Transposing the summation partition reverses the content and preserves the hooks, so either order of the two levels gives this expression. The same argument on C uses the lines E and $K_C \otimes E$ of degrees $(0, 2c)$. Theorem 10.1 now gives the negative-side series by subtraction in the connected logarithm. □

Write $n_{e,l}$ for the exponent at (e, l) of $P^H(B_1)$ for the character path of Theorem 10.1. Thus $n_{e,l}$ is the jump of the downstairs exponent in degree $2e$ and genus l. For the balanced factor A_l, denote its evaluated logarithm by

$$G_l(Q, y) = \log\left(\sum_{\lambda} Q^{|\lambda|} \prod_{a \in \lambda} z_{\ell(a)}^{l-1} \right).$$

This is the balanced specialization [BP08, Corollary 7.3]. In particular $[Q]G_l = z^{l-1}$. Finite genus support at each degree in the geometric factorization gives the coefficientwise identity

$$\log Z_b(Q, y) = \sum_{r \geq 1} \sum_{l \geq 0} n_{r,l} G_l(Q^r, y). \tag{17}$$

Theorem 11.2. *For the character path, in projection degree $2e$ the highest genus of a changing elementary exponent is $1+be^2$, and its change is $+1$.*

Proof. For a partition of e, let P_R and P_C count unordered pairs of boxes sharing a row and a column, respectively. Counting arms and legs gives $\sum_a \ell(a) = e+P_R+P_C$ and $c(\lambda) = P_R-P_C$. The highest power of y contributed by λ in (15) is therefore $b(e+2P_R)$. Its maximum be^2 occurs only in the single row and has coefficient 1. Products of lower-degree terms in the logarithm have strictly smaller highest power, since $\sum_i e_i^2 < e^2$ for every decomposition of e into two or more positive summands.

First observe that $n_{r,l} = 0$ for $l < 1+br$. Riemann–Hurwitz on the positive-degree components of a connected map of degree r to Y gives this lower bound on its arithmetic genus. Consequently $[Q^r]\log Z_b$ vanishes to order at least $2br$ in v. Subtract proper-divisor contributions in (17) in order of increasing r. A preceding factor at degree $d \mid r$ with $l-1 \geq bd$ contributes a term of order at least $2(l-1)r/d \geq 2br$, by the hook formula for G_l. The remaining finite sum $\sum_l n_{r,l} z^{l-1}$ has terms of distinct initial orders, proving the lower bound by induction.

For $l \geq 1$ and $j \geq 1$, the highest Laurent power in $[Q^j]G_l$ is at most $(l-1)j(j+1)/2$. Indeed $P_R + P_C \leq \binom{j}{2}$, and products in the logarithm satisfy the same bound by superadditivity. Suppose inductively that $l-1 \leq bd^2$ for every nonzero exponent at a smaller degree d. At degree e each proper divisor d, with $j = e/d > 1$, contributes powers no higher than

$$bd^2\, j(j+1)/2 < be^2.$$

The term y^{be^2} with coefficient 1 thus survives subtraction of every proper-divisor term. In the remaining polynomial $\sum_l n_{e,l} z^{l-1}$ the top Laurent coefficient of z^{be^2} is $(-1)^{be^2} = 1$ since b is even. This proves both assertions, including the starting degree $e = 1$. □

11.1. **The quartic correction.** The upstairs degree-one logarithm is z^b, so the sole new factor in projection degree two is $A_{b+1}(q^2)$. The partitions of size two have hooks $1, 2$ and total contents ± 1. At upstairs degree two the quadratic logarithm term $-z^{2b}/2$ is common to $\log Z_b$ and G_{b+1}. Subtracting the latter gives

$$\sum_l n_{2,l} z^{l-1} = z^{2b}(4-z)^b\big(y^b + y^{-b} - 2\big), \tag{18}$$

since $z_2 = z(4-z)$. The last parenthesis is a polynomial in z with linear term $-b^2 z$, as follows by expansion at $v = 0$. Thus the first changing elementary exponent in projection degree four occurs in genus $2b+2$, with change $-b^2 4^b$.

For $g = 2$ one has $b = 2$ and $y^2 + y^{-2} - 2 = -z(4-z)$. Formula (18) becomes

$$-z^5(4-z)^3 = -64z^5 + 48z^6 - 12z^7 + z^8.$$

The four jumps occur in genera six through nine. There is already a connected contribution in genus five. Indeed the leading coefficient of $[Q^2]\log Z_b$ at z^{2b} is $2\cdot 4^b - \frac{1}{2} = (2^{2h}-1)/2$, which is $63/2$ at $h = 3$. It counts the connected unramified doubles of Y with weight $1/2$ each and is entirely accounted for by $A_{b+1}(q^2)$.

12. Kernels on finite covers

Let $p : Z \to C$ be a connected unramified principal G-cover, where $G = D_n$ is the group of order $2n$ acting by the symmetries of a regular n-gon, $n \geq 3$. Write V for its real reflection representation and $\mathcal{V} = Z \times_G V$ for the associated flat Euclidean plane. Rotation through $2\pi/n$ has no real invariant line, so V is irreducible. Moreover $\mathrm{End}_G(V) = \mathbb{R}$. An endomorphism

commuting with this rotation is complex multiplication in the oriented plane, and commutation with a reflection makes its multiplier real.

For any real Euclidean local system $\mathcal{W}$, denote by $D^{\mathcal{W}}$ the twist of D by $\mathcal{W}^*$. Its sections are real homomorphisms $\mathcal{W} \to N$, and on a flat frame the equation $D^{\mathcal{W}}S = 0$ means that D kills every column of S. The Euclidean form identifies the dual with $\mathcal{W}$. A transverse crossing on $\mathcal{V}$ at t_0 means that $\ker D^{\mathcal{V}}_{t_0}$ is a real line and that differentiation in t induces an isomorphism from this line to the cokernel. Both spaces have the same dimension by the index formula for an unramified twist. For a map $u : X \to C$ from a connected curve, let $K_u \leq G$ be the image of $\pi_1(X)$ under the monodromy of u^*Z. The choice of a lift of the base point identifies parallel sections of $u^*\mathcal{V}$ with V^{K_u}.

Theorem 12.1. *For a residual set of smooth paths of real Cauchy–Riemann operators on N with fixed symbol, every interior parameter where $D^{\mathcal{V}}_t$ has a kernel is a transverse crossing. At any such parameter let S generate that kernel and put $D = D_t$. The map $S : \mathcal{V} \to N$ is injective as a real map on every fiber. For every smooth connected finite holomorphic cover $u : X \to C$, with arbitrary branching,*

$$\ker u^*D = (u^*S)\big(H^0_{\mathrm{flat}}(X, u^*\mathcal{V})\big) \simeq V^{K_u}. \tag{19}$$

The path may have prescribed endpoints. Conversely, for every $p : Z \to C$ as above there exists a smooth path with a transverse crossing satisfying these conclusions.

We first work at a prescribed section S and later prove the generic evaluation and the existence assertions. Suppose $D^{\mathcal{V}}S = 0$ and that on a disk the two columns s_1, s_2 of S in a flat frame form a complex basis of N. Choose a closed subdisk in this disk with nonempty interior. Let $\mathcal{A}_S$ be the space of smooth real zeroth-order variations supported in the closed subdisk that annihilate both columns. This closed real subspace of the space of smooth bundle maps has its usual Fréchet topology.

12.1. **Perturbations preserving two solutions.** Write $F = \Lambda^{0,1}T^*C \otimes_{\mathbb{C}} N$, and let $F^\dagger$ denote its real dual tensored with densities. The infinite-order Petri condition at x means that a real tensor of local solution and adjoint-solution germs is zero whenever its pointwise product in $N \otimes_{\mathbb{R}} F^\dagger$ vanishes to infinite order at x. Arbitrarily small unrestricted smooth real zeroth-order variations in any neighborhood of x attain this condition for the Cauchy–Riemann symbol [Wen23, Definition 5.1 and Corollary 5.10]. The formal-jet version is given in [DW23b, Definition 1.6.3 and Theorems 1.6.17, 2.5.3].

The condition at any point in the interior of the subdisk is dense also in $D + \mathcal{A}_S$. To see this, start with an unrestricted nearby operator D' having the condition at x. On a small disk about x choose solutions s'_j of $D's'_j = 0$ close to s_j. In a local frame $D' = \bar\partial + M'$ with M' real linear. If T is the Cauchy transform on a disk of radius r, then $\|T\|_{L^\infty \to L^\infty} \leq cr$. Thus $w_j = -T(D's_j + M'w_j)$ has a solution for small r, and $s'_j = s_j + w_j$ has the required equation. The sup norm of w_j tends to zero as D' approaches D. Interior elliptic estimates on a smaller disk give closeness in any prescribed finite number of smooth seminorms. The complex gauge U sending s_j to s'_j is then close to the identity. The conjugate $U^{-1}D'U$ has the same symbol and kills s_1, s_2. The Petri condition is invariant under this conjugation using the output action on F and its contragredient on $F^\dagger$. Multiply $U^{-1}D'U - D$ by a cutoff supported inside the subdisk and equal to one near x. This proves the claimed density at D and, by the same construction, at every point of $D + \mathcal{A}_S$.

Consider an irreducible real orthogonal local system $\mathcal{W}$ of finite monodromy on C minus finitely many points. Retain only punctures with nontrivial inertia on $\mathcal{W}$. At such a puncture

use the orbifold disk whose stabilizer has the order of the inertia action. This gives a Fredholm operator $D^{\mathcal{W}}$ on orbifold sections. Equivalently, one can take equivariant sections on the smooth compactification of the finite normal cover defined by the monodromy. The two descriptions identify kernels also after passage to a common normal cover. A bounded invariant solution descends off the branch values and extends over them by removability for the Cauchy–Riemann equation [DW23b, Propositions 2.7.3 and 2.8.3].

Let w be a nonzero twisted solution with columns $w_1, \dots, w_r$ in a real orthonormal flat frame of $\mathcal{W}$. If $\mathcal{W}$ is inequivalent to $\mathcal{V}$, these columns together with s_1, s_2 are linearly independent as real global solutions on a common connected normal cover. Indeed w induces a nonzero intertwiner from the irreducible fiber representation to the real solution space. It is injective, and its image has zero intersection with the inequivalent irreducible plane given by S. For $\mathcal{W} = \mathcal{V}$ the same independence holds for every nonzero w orthogonal to S in L^2. In this case two copies with a common nonzero vector would coincide, and the two intertwiners with that image would differ by a real scalar since $\mathrm{End}_G(V) = \mathbb{R}$. In either case strong unique continuation implies independence of the infinite jets of these solutions at any regular point.

Choose a disk of flat frames in the variation region disjoint from the branch values, and a point x there at which D satisfies the Petri condition. We claim that the equation on nonzero sections, linearized in sections and variations from $\mathcal{A}_S$, is surjective at w. For $\mathcal{W} = \mathcal{V}$ use the L^2-complement of S as section space. Omitting S leaves the image of the section derivative unchanged. An element of the annihilator consequently solves the twisted adjoint equation. Write its columns on the disk as v_i in dual output densities. Testing against all local real variations that kill s_1, s_2 gives the pointwise identity

$$\sum_{i=1}^{r} w_i \otimes v_i = s_1 \otimes \alpha_1 + s_2 \otimes \alpha_2 \tag{20}$$

for smooth dual densities α_1, α_2. This is the annihilator of the subspace of real maps vanishing on $\mathrm{span}_{\mathbb{R}}(s_1, s_2)$. Both α_j satisfy the local adjoint equation. Indeed, test (20) against $[D, \Phi]$ for a compactly supported complex endomorphism Φ of N, with its induced action on F. This commutator is zeroth order. Integration of the left side gives $\sum_i \int \langle v_i, D(\Phi w_i)\rangle = 0$. The corresponding expression on the right is $\sum_j \int \langle \alpha_j, D(\Phi s_j)\rangle$. The two test sections Φs_1 and Φs_2 can be prescribed independently since s_1, s_2 are a complex basis. Thus $D^\dagger \alpha_j = 0$. Taking infinite jets in (20), the Petri condition and independence of the input jets force every v_i to vanish to infinite order. Adjoint unique continuation on the connected punctured normal cover gives a zero annihilator. The section derivative is Fredholm, so the asserted surjectivity follows.

12.2. **Exclusion of further kernels.** For each ramification type and representation, the distinct positions of the effective punctures give local coordinates on its marked configuration space. Write s for their number and $c_x = \dim_{\mathbb{R}}(\mathcal{W}_x / \mathcal{W}_x^{I_x})$ for the codimension of the invariants of local inertia on a nearby regular fiber. The orbifold index formula [DW23b, Proposition 2.8.6] gives

$$\mathrm{ind}_{\mathbb{R}}\, D^{\mathcal{W}} = (\dim_{\mathbb{R}} \mathcal{W})\, \mathrm{ind}_{\mathbb{R}}\, D - (\mathrm{rank}_{\mathbb{C}}\, N) \sum_x c_x = -2 \sum_x c_x.$$

In the incidence of projective kernel lines the moving punctures add $2s$ parameters and passing to lines removes one. Thus the projection index is $2s - 1 - 2\sum_x c_x \le -1$. For $\mathcal{W} = \mathcal{V}$ the twist is unramified and fixed. Restriction to the complement of S lowers its index by one more.

We spell out how to apply this count in the smooth constrained parameter space. The finite monodromy groups, irreducible representations up to equivalence, and homomorphisms from a punctured surface group to a finite group give countably many discrete choices. The compactified normal covers vary smoothly on a chart of distinct branch positions by using the local maps $z \mapsto z^e$ at the branches. Exhaust the configuration charts by countably many compact subsets, chosen small enough that each subset misses some open disk in the interior of the prescribed variation region. Here it is the branch divisors that must miss the disk. The unramified system $\mathcal{V}$ is a separate discrete choice.

Over each compact parameter subset the absence of a nonzero solution (a complementary solution for $\mathcal{V}$) is open. Indeed L^2-normalized solutions satisfy uniform elliptic estimates on the smooth family of compact normal covers. To prove density, first attain the Petri condition at a single point on the chosen open disk. At that operator the normalized zeros over the compact parameter subset form a compact set. By the surjectivity just proved, finitely many allowed smooth operator variations suffice for surjectivity near all these zeros. Shrinking their common parameter ball ensures that all nearby zeros over the compact subset lie in those neighborhoods. In section Sobolev completions the implicit function theorem now gives finite-dimensional projective incidences over this ball, of dimension strictly less than that of the ball by the index count. Sard's theorem, applied on open chart neighborhoods, gives arbitrarily small parameters with no zero over the compact subset. Local normalization of kernel lines is legitimate since the scaling direction maps to zero in the differential. Intersecting these open dense conditions gives the asserted residual set in $D + \mathcal{A}_S$.

To deduce (19), pass to a finite normal cover dominating Z and the given X. Every irreducible summand in its kernel supplies a twisted solution for the corresponding monodromy system. Ineffective inertia can be omitted by descent and removability. The only possible summand is V pulled back from G. Its multiplicity is one, since all such intertwiners descend to Z and the complement of S in $\ker D^{\mathcal{V}}$ is zero. Descent through the sheet stabilizer of X identifies $\ker u^* D$ with the invariants of its image K_u on this single copy. Deleting preimages of the finite branch set does not affect that image, since the lower cover $Z \to C$ is unramified.

12.3. **Generic paths and realization.** It remains to obtain the complex basis disk and fiberwise real injectivity on generic paths. With p fixed, consider the equation $D^{\mathcal{V}} S = 0$ at a nonzero S, varying D without constraints in this paragraph. On Z two flat basis vectors map to independent global real solutions by irreducibility. Their values are real independent on an open dense subset. Indeed, on an open disk where two values are proportional and one is nonzero, the ratio is a smooth real function a. The product rule gives $(\bar{\partial} a) s_j = 0$, so a is constant there, contradicting global independence by unique continuation. Thus, on some disk away from a specified point, variations of D can prescribe the two inhomogeneous columns independently. An annihilating adjoint vanishes on that disk and hence everywhere. This proves universal transversality.

Evaluation of both columns at any marked point is submersive on this incidence. To check the augmented equation, a distributional adjoint annihilating the variations satisfies the homogeneous adjoint equation off the mark and vanishes there by the same argument. Its support is the mark, so it has finite order. Prescribe arbitrary column values there for a test section and arrange that its image under $D^{\mathcal{V}}$ vanishes to higher order than the distribution. Such jets follow recursively from the solved $\bar{\partial}$ equation in a coordinate, with the pure holomorphic Taylor coefficients free. Cutoff Taylor polynomials realize them. Pairing with the augmented equation first kills the evaluation multiplier and then, by elliptic regularity, the point-supported adjoint.

For fixed p, the projection of the projectivized kernel incidence to operators has index -1. Adding a path parameter therefore gives regular isolated crossings on a generic path. A higher-dimensional kernel would give a positive-dimensional projective fiber. At a regular isolated crossing the time derivative maps the kernel line isomorphically to the cokernel. The two evaluated real four-vectors have rank one in real codimension three and rank zero in codimension eight. Moving the marked point adds only two dimensions, so neither stratum occurs at any crossing on a generic path. At one fixed point the locus of complex rank one has real codimension two and the rank-zero locus has codimension eight. Thus the columns also form a complex basis on a disk at each crossing. All these rank strata are invariant under nonzero real scaling, so the marked incidence calculation passes to projective lines. Parametric transversality can be applied with finitely many smooth variations on compact exhaustions of the normalized solution charts as above, with cutoffs in the interior of the time interval.

Mark a transverse crossing and choose a local linear normalization of its generator. Variation of the section in a complement to its kernel maps isomorphically onto the image of $D^{\mathcal{V}}$, and the time variation spans the remaining cokernel. The implicit function theorem therefore continues the marked crossing uniquely over an open path chart. One may use a finite regularity completion and restrict to smooth paths by elliptic regularity. On this chart, absence of the additional twisted kernels over any one of the compact configuration subsets is open. For the distinguished system the complement to the continued line varies continuously. Density follows by perturbing at the marked crossing while keeping its particular S fixed, as proved above, then multiplying the perturbation by a smooth time cutoff equal to one near that crossing. The center and its generator still solve the equation and hence are their own continuations. Such crossing charts admit a countable cover, and their maps to path space are local homeomorphisms. The exceptional subsets are therefore meagre in path space. This proves the generic assertion simultaneously at all interior crossings.

Finally start with an arbitrary connected principal D_n-cover $Z \to C$. Choose a smooth real bundle map $S : \mathcal{V} \to N$. Transversality to the real rank strata, of codimension three and eight, gives a choice injective everywhere. At a fixed point we may at the same time avoid the deficient complex rank strata. Choose any real Cauchy–Riemann operator D_*. For a local flat v prescribe the real zeroth-order correction by

$$P(Sv) = -D_*(Sv).$$

Flat transition functions make this a smooth map on the subbundle $S(\mathcal{V})$, and a real bundle complement extends it to N. Then $(D_* + P)^{\mathcal{V}} S = 0$. The constrained perturbation proved above gives an operator D_0 with this same S and precisely the kernels (19). Its twisted cokernel is a line. On the complex basis disk an unrestricted real variation Q can prescribe both columns of QS independently. A nonzero twisted adjoint cannot vanish on the disk, so some such Q pairs nontrivially with it. The path $D_0 + tQ$ has the required transverse crossing.

13. Reflection and rotation groups

For $n \geq 3$ write $G = D_n$ for the symmetry group of a regular n-gon, of order $2n$, and V for its real representation on the plane. Fix a connected principal unramified G-cover $p : Z \to C$. It determines the flat plane bundle $\mathcal{V} = Z \times_G V$. The twist $D^{\mathcal{V}}$ acts in a flat real frame by D on each component. Equivalently its kernel is $\mathrm{Hom}_G(V, \ker p^* D)$, using the invariant Euclidean metric on V. Its real Fredholm index is zero. Choose one subgroup H_i generated

by a reflection in each conjugacy class of such subgroups. Set

$$f_i : Y_i = Z/H_i \longrightarrow C, \qquad B_{i,t} = f_i^* D_t, \qquad A_t = p^* D_t.$$

For odd n there is one quotient Y_i; for even n there are two. Each has degree n over C.

We say a path crosses the *wall for* $\mathcal{V}$ transversely at zero if $\ker D_0^{\mathcal{V}}$ is a real line and the derivative along the path induces an isomorphism from this kernel to its cokernel. Generic means residual in the smooth topology on paths of fixed-symbol operators. We consider crossings in the interior of paths of base-invertible operators; endpoints of such paths can be held fixed. In the following theorem a term whose indicated degree is not a positive integer is taken to be zero.

Theorem 13.1. *At a transverse crossing of the wall for $\mathcal{V}$ on a generic path, for every finite M there is $\delta_M > 0$ such that D_t is rigid through M on $0 < |t| \le \delta_M$. For $-\delta_M \le t_- < 0 < t_+ \le \delta_M$, $1 \le d \le M$ and every k, one has*

$$L^C_{d,k}(D_{t_+}) - L^C_{d,k}(D_{t_-}) = \sum_i \frac{2}{m_i} L^{Y_i}_{d/n,k}(B_{i,t_+}) - L^Z_{d/(2n),k}(A_{t_+}), \qquad m_i = |N_G(H_i)/H_i|. \tag{21}$$

Each m_i equals one for odd n and two for even n. For every $p : Z \to C$ as above and every N of the specified degree there is a path with such a crossing satisfying all the conclusions.

Proof. Theorem 12.1 gives central kernels on all smooth covers and matching on nodal domains. The proof of bounded-degree isolation in Section 4 uses only this classification, fiberwise injectivity and transversality on the prescribed summand, so applies to these paths too. The unit circle of the reflection plane has, as a G-cell decomposition, two vertex orbits and one orbit of open edges, obtained by subdividing along the reflection axes. Its class is $\sum_i (2/m_i)[G/H_i] - [G/1]$. The proof of Theorem 4.1 now gives the asserted signed integrals, including the regular-cover subtraction. In particular their balanced products obey

$$\frac{P^+_C(q)}{P^-_C(q)} = \frac{\prod_i P^+_{Y_i}(q^n)^{2/m_i}}{P^+_Z(q^{2n})} \pmod{q^{M+1}} \tag{22}$$

by Corollary 3.2. □

Corollary 13.2. *Suppose n is odd and $M \ge 2n$. Write $Y = Y_1$, $B_t = f_1^* D_t$, $h = g(Y)$, $H_d = 1 + d(g-1)$ and $\eta = \epsilon(B_{t_+})$. Let $\lambda_0 \in H^1(Y;\mathbb{F}_2)$ classify the connected unramified double $Z \to Y$. Let N_- count the $\lambda \in H^1(Y;\mathbb{F}_2) \setminus \{0, \lambda_0\}$ for which the twist B_0^λ by the associated real sign line has negative determinant sign. These twists are invertible. With Δ denoting the value at t_+ minus the value at t_-, one has*

$$\Delta n^C_{n,H_n} = 2\eta, \qquad \Delta n^C_{2n,H_{2n}} = -\eta(1 + 2N_-). \tag{23}$$

In every degree strictly below n all exponent changes vanish.

Proof. From (22), nonmultiples of n have zero change. The degree-one contribution on Y at genus h is η, giving the first assertion. To compute the next multiple, all genus-H_{2n} maps in degree two over Y are connected unramified doubles. Indeed Riemann–Hurwitz on the positive normalizations and the arithmetic genus formula force one such component, with no self-nodes or contracted decorations. Each double has weight $1/2$. The degree-one contribution over Z in this genus is $\epsilon(A_{t_+})$. It cancels the contribution in twice the upstairs degree-two count from the distinguished double λ_0.

Identify the reflection subgroup H with $\mathbb{F}_2$. On $\pi_1(Y)$ the monodromy of $Z \to Y$ is the character λ_0. Thus the monodromy image in G of the double of Y specified by $\lambda \neq 0$ is

conjugate to $\lambda_0(\ker\lambda) \le H$. It is trivial precisely when $\ker\lambda \subseteq \ker\lambda_0$. Both kernels have index two, so this happens precisely for $\lambda = \lambda_0$. Every other double has reflection image. The central kernel on that double is a single line from Y by Theorem 12.1. The odd twist on the double is thus injective on Y, and invertible by index zero. Its sign is constant near the crossing. The full determinant sign at t_+ is $\eta\,\epsilon(B_0^\lambda)$ by splitting under the deck involution with its complex reference orientation. Formula (21) now gives

$$\Delta L^C_{2n,H_{2n}} = \eta(2^{2h} - 2 - 2N_-).$$

In the product only the changing degree-n factor can contribute from a proper divisor. Only its genus-h term enters at $H_{2n} = 2h-1$, since twice a class of genus $s > h$ first contributes in genus at least $2s-1$. The exponent at genus h changes by 2η. A balanced elementary genus-h curve has connected degree-two coefficient $(2^{2h}-1)/2$ in the minimum genus $2h-1 = H_{2n}$. This follows either by expanding its partition formula [BP08, Corollaries 7.2 and 7.3] or by counting the unramified doubles with complex orientation. Subtracting that coefficient times 2η gives (23). In using the minimum genus here, the same lower bound holds for the unchanging and changing endpoint exponents: connected degree-d maps to a genus-g target have genus at least H_d, and the connected elementary series on its j-fold multiple starts in genus at least $1 + j(s-1)$ for $s \ge 2$. Induction on the label and then genus gives the exponent bound (Proposition 3.1). □

Corollary 13.3. *Fix C and N as above. For every odd $n \ge 3$ there are two operators on N, rigid on all smooth covers, whose geometric local products coincide through degree $n-1$ and differ in degree n. They can be chosen arbitrarily close to the same transverse crossing. Their degree-$2n$ exponents at genus H_{2n} also differ.*

Proof. In the surface group presentation send two of the a-generators to a rotation and a reflection generating D_n, all the b-generators to the identity, and the remaining a-generators to the identity. This defines a surjection and hence a connected unramified D_n-cover. Apply the realization in Theorem 12.1 and take $M = 2n$. Choose two arbitrarily close regular endpoints as in Theorem 13.1.

Rigidity through M is open here. Otherwise normalized kernel graphs for operators approaching a rigid endpoint would have a bounded-area limit of degree at most M meeting positive height, as in Proposition 10.3. Any positive-height component of that limit contradicts rigidity. In a small convex neighborhood of each endpoint the connected Euler numbers therefore remain constant through M, in every genus. All-cover rigidity is dense in the unrestricted operator space. One can see this directly from Section 12: omit the prescribed columns s_1, s_2 from the Petri test and use unrestricted local zeroth-order perturbations. For each irreducible finite system the remaining projective incidence has index at most -1 after moving its effective branch values. The same compact exhaustions and Baire argument exclude them all, including the trivial system and the prescribed plane. Choose such an operator in each endpoint neighborhood. The full local products exist by the compatible finite-energy constructions at every cutoff. Their truncations through $2n$ agree with those of the chosen endpoints by the connected comparison and the uniqueness argument of Theorem 4.1. Corollary 13.2 gives the claim. □

13.1. **Two mixed representations.** The sign character and standard real plane of S_3 have zero mixed class. The sphere of the sign line is G/A_3 and the plane contributes $2[G/C_2] - [G/1]$. The first Cartesian product with G/C_2 is regular and the product with $G/1$ is twice regular. Thus both single-wall transfers can be present while their mixed difference vanishes in every degree.

For the order-eight group $G = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1} \rangle$ take its standard plane and the sign line with $r \mapsto -1$, $s \mapsto 1$. The plane sphere contributes $[G/\langle s \rangle]+[G/\langle rs \rangle]-[G/1]$. The sign kernel contains the first reflection and misses the second. Multiplying gives $2[G/\langle s \rangle] - [G/1]$. Both examples are realized on every genus-g curve as above: choose a surjection of the surface group through a free group on two generators. The associated rank-three real bundle pulls back from a graph. Adding its determinant line produces an oriented rank-four bundle pulled back from that graph, hence trivial. The oriented real bundle underlying N is also trivial on the surface since $c_1(N)$ is even. This gives the required monomorphism.

13.2. **The tetrahedral, octahedral and icosahedral walls.** Take $W = \mathbb{R}^3$ and the rotation group of a regular tetrahedron, cube or icosahedron. Each standard real representation is faithful irreducible of real type: two nonparallel rotation axes of rotations of order greater than two preclude an invariant real line, and irreducibility in odd real dimension forces real commutant. The complement of the associated rank-three bundle in $N_{\mathbb{R}}$, when an embedding exists, is a real line. The representation is oriented, so this complement must be orientable, hence trivial. Consequently a monomorphism exists exactly when $w_2(E) = 0$. Indeed $N_{\mathbb{R}}$ is oriented with $w_2 = c_1(N) \bmod 2 = 0$, and oriented bundles of rank at least three over a surface are classified by w_2. This condition is attained on every base curve of genus at least two: take the surface group quotient onto a free group on two generators and follow it by a generating pair for the rotation group. Its flat bundle pulls back from a graph, on which H^2 vanishes.

Radially project the barycentric subdivision of the boundary of the solid onto S^2. There are three vertex orbits (face center, edge midpoint, original vertex), three open edge orbits and two open triangle orbits. Edge interiors and triangle interiors have trivial stabilizer in the rotation group. The vertex stabilizer orders are the three entries in the table. Thus the equivariant Euler characteristic has one term for each vertex orbit and minus one regular orbit. In the table P_j denotes the positive-side geometric product on the indicated degree-j quotient, evaluated with q replaced by q^j.

Rotation group	Stabilizer orders	Quotient of the two downstairs products
A_4	$2, 3, 3$	$P_6 P_4^2 / P_{12}$
S_4	$2, 3, 4$	$P_{12} P_8 P_6 / P_{24}$
A_5	$2, 3, 5$	$P_{30} P_{20} P_{12} / P_{60}$

For A_4 the two order-three vertex stabilizers are conjugate; their quotient targets give the same operator up to pullback equivalence over C. For S_4 the order-two stabilizer is the edge-midpoint half turn; on the four body diagonals it acts as a transposition. It is distinct from the subgroup generated by the square of the rotation of order four. The table proves that at minimum genus the first exponent change has magnitude two in tetrahedral degree four, and magnitude one in octahedral degree six and icosahedral degree twelve. The change in the tetrahedral exponent at degree six and its minimum genus also has magnitude one: no positive multiple of the degree-four term enters degree six. These are changes of geometric elementary exponents through a finite cutoff. For endpoints rigid in every degree, fix a cutoff containing the labels in question. Finite-degree rigidity is open by the normalized graph-cycle argument. Unrestricted all-cover rigidity is residual by the same projective index argument with no prescribed solution. Perturb each end within its small convex rigidity neighborhood.

As in Proposition 3.1, the full products exist and their exponents through the chosen cutoff persist.

These conclusions use fiberwise monomorphism. At a rank-r input the first failure of real injectivity in a rank-four output has real codimension $5-r$. The usual marked evaluation argument on a two-dimensional base excludes these defects automatically for a generic transverse rank-two crossing. In rank three the defect has codimension two, and in rank four it has codimension one. The construction gives realized crossing families and (3) throughout their stated domain. Noninjective evaluations and ramified principal covers allow different orbit graphs to meet and require a separate wall calculation.

14. Spin structures and the signs of covers

Let C be a closed Riemann surface of genus $g \geq 2$, and write K_C for its canonical bundle. A theta characteristic is a holomorphic line bundle L with a specified isomorphism $L^{\otimes 2} \simeq K_C$. Given two theta characteristics L_1, L_2, put $E = L_1 \oplus L_2$. The two square maps define a nondegenerate holomorphic symmetric pairing $\gamma : E \otimes E \to K_C$, with the summands orthogonal. Fix a Hermitian metric h on E. Applying its conjugate-linear identification $E^* \to E$ to $\gamma(s, \cdot)$ and conjugating the one-form factor gives a conjugate-linear map $R : E \to \Lambda^{0,1} T^*C \otimes E$. Our endpoint operator is

$$D = \bar\partial_E + R.$$

A real Cauchy–Riemann operator with the same symbol differs from D by a real bundle homomorphism of order zero.

Give the total space of E the almost-complex structure for which the projection is holomorphic, each fiber has its complex linear structure, and the graph equation over a holomorphic map $f : \Sigma \to C$ is $D_f s = 0$. Here D_f is the pullback real Cauchy–Riemann operator, with df applied to the differential factor of R. In holomorphic bundle coordinates its horizontal lift of $\partial_{\bar z}$ has vertical $(1,0)$ part $-R_{\bar z}(v)$ at (z, v). The ambient first Chern number on the zero section is $2 - 2g + \deg E = 0$.

Lemma 14.1. *For $D = \bar\partial_E + R$, the zero section is the only compact irreducible curve in the total space. With metrics pulled back from C, there exists $b > 0$ such that on every finite étale cover f the pulled-back operator D_f has index zero and satisfies*

$$\|D_f s\|_{L^2} \geq b \|s\|_{L^2}.$$

The same inequality, with $b/2$, holds simultaneously on all such covers after a zeroth-order perturbation of pointwise operator norm less than $b/2$.

Proof. For a compact curve projecting nonconstantly to C, let $f : \Sigma \to C$ be the projection of its normalization and let s be its vertical coordinate. The form $\alpha = df^* \gamma(s, s)$ on Σ is globally defined, where df^* denotes pullback on the one-form factor. Holomorphicity of γ and symmetry give $\bar\partial \alpha = 2\, df^* \gamma(\bar\partial s, s)$, with exterior multiplication in the differential factors. In coordinates u on Σ, z on C, and $\gamma = G\, dz$, the equation for the graph therefore gives

$$\bar\partial \alpha = -2\, |G(s, \cdot)|^2_{h^*}\, |\partial_u f|^2\, d\bar u \wedge du.$$

Stokes' theorem implies $s = 0$ on the unramified locus and hence on Σ. A compact curve in a single complex vector space is constant.

For an étale cover and an arbitrary smooth section, the mixed L^2 inner product of $\bar\partial s$ with $R_f s$ is a fixed constant multiple of $\int_\Sigma \bar\partial \alpha$. Indeed, in a conformal coordinate the factor in the cotangent metric cancels the factor in the volume form. Thus for every real a,

$$\|\bar\partial s + a R_f s\|_2^2 = \|\bar\partial s\|_2^2 + a^2 \|R_f s\|_2^2. \tag{24}$$

The least singular value of R has a positive minimum b on C and retains that bound under étale pullback. Riemann–Hurwitz gives $g(\Sigma) - 1 = (\deg f)(g-1)$ on each connected cover. Riemann–Roch then gives complex Dolbeault index zero for f^*E, and hence real index zero for D_f. The perturbation bound follows from the triangle inequality. □

Let P_f be the real permutation local system on the sheets of f. The evaluation of $w_1(P_f)$ on a loop is the parity of its monodromy permutation. Write $\ell_w = L_2 L_1^{-1}$ for the flat sign line determined by $w \in H^1(C; \mathbb{Z}/2)$, using its specified square trivialization, and set

$$\epsilon = (-1)^{h^0(C,L_1)+h^0(C,L_2)}.$$

We orient the determinant line of a real Cauchy–Riemann operator by continuation of the complex orientation through the affine operator space. At an invertible operator its comparison with the canonical orientation of the determinant of the zero spaces is a sign.

Proposition 14.2. *For any finite étale cover f of degree m, the determinant sign of D_f is*

$$\sigma(f) = \epsilon^m (-1)^{\langle w \smile w_1(P_f), [C] \rangle}. \tag{25}$$

For each $w \neq 0$ there are pairs with this w and either value of ϵ. A pair of identical theta characteristics gives $w = 0$ and $\epsilon = 1$. These pairs can all be realized on the same smooth complex rank-two bundle over C.

Proof. First follow $\bar\partial + aR_f$ from $a = 0$ to $a = 1$ on the source. Identify the complex cokernel of $\bar\partial$ with $H^0(f^*E)^*$ by Serre duality and the pulled-back symmetric pairing, using $f^*K_C = K_\Sigma$. The map from the kernel to the cokernel induced by the derivative R_f is a conjugate-linear isomorphism. Indeed, its Serre pairing on two holomorphic sections is a nonzero constant times the integrated Hermitian product of their images under the symmetric isomorphism. The corresponding real determinant has sign $(-1)^{h^0(f^*E)}$. Splitting off complex complements gives a finite-dimensional reduced map $a[R_f] + O(a^2)$ with that sign for small $a > 0$. Equation (24) extends it to $a = 1$. This is the two-line version of the spin determinant computation of Lee and Parker [LP13, Theorem 3.1].

Let q_L be the spin quadratic refinement of the mod-two intersection form on $H_1(C; \mathbb{Z}/2)$. Spin parity equals its Arf invariant, and twisting by w adds intersection with $x = \mathrm{PD}(w)$ to q_L. The Arf formula in a symplectic basis therefore gives

$$h^0(L\ell_w) + h^0(L) = q_L(x) \pmod 2$$

[Ati71, Joh80]. For $w \neq 0$, choose a simple nonseparating circle c representing x. Its preimage consists of disjoint circles c_i covering c with degrees n_i. For a simple circle the quadratic refinement is one plus the evaluation of the spin double cover on the tangentially framed circle [Joh80, Theorems 1A and 3A]. The tangent framing pulls back along the degree-n_i cover. Hence

$$q_{f^*L}([c_i]) = n_i q_L(x) + n_i - 1 \pmod 2.$$

The union represents $\mathrm{PD}(f^*w)$ and all pairwise intersection terms vanish. Summing and using $\sum_i n_i = m$ gives $m q_L(x)$ plus the parity of the permutation around c. This proves (25) for $L = L_1$. For $w = 0$ the two parities on the cover coincide. On a disconnected cover the determinants multiply.

Twisting both theta characteristics by a sign line whose class evaluates nontrivially on x reverses $q_{L_1}(x)$ and preserves w. This realizes both signs for $w \neq 0$. All the resulting bundles have rank two and degree $2g - 2$, which determine their smooth complex bundle isomorphism class on a surface. After any such identification the operators are joined by

a smooth path of zeroth-order perturbations. Complex bundle isomorphisms preserve the determinant convention. □

The sign formula also gives the determinant of the twist D^U by any flat real orthogonal local system U of rank r. The pairing on U and the argument for (24) give invertibility. For a theta characteristic L the orthogonal mod-two index formula is

$$h^0(L \otimes U_{\mathbb{C}}) \equiv r h^0(L) + q_L(\mathrm{PD}(w_1(U))) + \langle w_2(U), [C] \rangle \pmod 2$$

[BS18, Lemma 5.8]. Adding this equality for $L = L_1, L_2$ cancels the second Stiefel–Whitney terms. Thus

$$\operatorname{sign}(D^U) = \epsilon^r (-1)^{\langle w \smile w_1(U), [C] \rangle}. \tag{26}$$

Finally, for a connected stable map of covering degree $m > 0$ in an endpoint tube, let d_i be the positive degrees of its nonconstant normalized components. Each has genus at least $1 + d_i(g-1)$. The arithmetic genus is the sum of the normalized genera and the first Betti number of the dual graph. Its minimum $1 + m(g-1)$ therefore requires one nonconstant component, étale over C, and a tree of components of which every contracted one is rational. Any nonempty contracted subtree has an unstable leaf for unmarked maps. The maps at the minimum are precisely the connected étale covers. The derivative df identifies the tangent deformation complex in the target curve with the source deformation complex with its complex orientation. The remaining normal operator D_f is invertible. The regular local contribution of this map in the fixed target is consequently $\sigma(f)/|\operatorname{Aut}_C(f)|$.

15. Bifurcation of multisections in every degree

Fix a closed Riemann surface C of genus $g \geq 2$ and a smooth complex plane bundle N of degree $2g-2$. We work with a fixed Dolbeault operator $\bar\partial_N$ on N. A smooth fiber map $B_t(x,v) \in \Lambda^{0,1}_x C \otimes N_x$, defined near the zero section with $B_t(x,0) = 0$, determines an almost complex structure with holomorphic projection to C and complex affine fibers. For an étale cover $f : Y \to C$, the graph of $s \in \Gamma(f^*N)$ is holomorphic precisely when

$$\bar\partial_{f^*N} s + f^* B_t(\cdot, s) = 0. \tag{27}$$

Here pullback in the form factor uses df. In a holomorphic frame, the horizontal $(0,1)$ lift of $\partial_{\bar z}$ has vertical $(1,0)$ component $-B_{t,\bar z}$. This gives the asserted structure and equation. Over a compact interval these structures are tamed on a common tube about the zero section. Indeed the differential of the fiberwise radial primitive, extended by a unitary horizontal distribution, restricts to the standard symplectic form on each fiber. Adding a sufficiently large multiple of base area gives tameness on the zero section and hence on a tube. The normal operator of the core is $D_t = \bar\partial_N + d_v B_t(\cdot, 0)$. Every smooth path of real Cauchy–Riemann operators on N with this symbol arises in this way.

Take as D_0, D_1 two of the spin models of Section 14 with core signs $+1$, and with difference characters 0 and $w \neq 0$ respectively. One can arrange positive parity for a fixed nonzero w by twisting the two spin lines simultaneously by a class pairing nontrivially with w if necessary. The two smooth complex bundles are isomorphic since they have equal rank and degree. Fix a complex isomorphism with N for each. Let a_1 be a simple nonseparating circle representing the Poincaré dual of w modulo two. Complete it to surface generators a_i, b_i with relation $\prod_i [a_i, b_i] = 1$. For each $m \geq 2$ let $f_m : Y_m \to C$ be the connected étale cover on m letters defined by

$$a_1 \longmapsto (1\ 2), \qquad a_2 \longmapsto (1\ 2\ \cdots\ m), \qquad b_i, a_j \longmapsto 1 \quad (j \geq 3 \text{ for } a_j). \tag{28}$$

These permutations generate S_m, and Y_m has genus $h_m = 1+m(g-1)$. Write $d_m = |\operatorname{Aut}_C(f_m)|$, so $d_2 = 2$ and $d_m = 1$ for $m \geq 3$.

Theorem 15.1. *For every smooth path from D_0 to D_1, the fiberwise linear graph equations have primitive holomorphic multisections with projection f_m arbitrarily close to the zero section at some parameter, for every $m \geq 2$. Here primitive means that the graph map is simple and its image projects to C with degree m.*

For a residual choice of path of normal operators, each f_m has an odd finite number of transverse crossings in its trace-free summand. These crossings occur at different parameters for different m and the base operator is invertible at each one. Fix such a path of first jets. For a residual choice of the higher fiber jets in (27), each crossing has a branch of primitive multisections. In a neighborhood of the multiple cover the change, from the left to the right, in their signed count is

$$\frac{\sigma(f_m)^- - \sigma(f_m)^+}{d_m}, \tag{29}$$

where $\sigma(f_m)^\pm$ are the determinant signs of $f_m^* D_t$ on the adjacent sides. Multiplying each change by the sign of the base operator at the crossing and summing along the path gives $2/d_m$.

One may instead choose the nonlinear terms in the space of fiberwise odd maps. A residual choice in this space gives a birth or a death at every one of the stated crossings. There are two nearby simple supports on the occupied side for $m \geq 3$ and one for $m = 2$.

The count in (29) concerns the specified projection cover at source genus h_m. Indeed the nonconstant part of an arbitrary map of projection degree m has arithmetic genus at least h_m by Riemann–Hurwitz. Equality requires a single étale component. A contracted genus-zero tree attached to it has an unstable leaf, and a cycle in the dual graph would increase the genus. The topological type of the resulting étale projection is discrete. For a single elementary wall in the space of compatible almost complex structures on a closed symplectic Calabi–Yau threefold, Bai and Swaminathan proved the local signed change $\pm 2/|\operatorname{Aut}(f)|$, including the standard symmetric-group wall. They also give unramified dihedral examples in every odd degree [BS24, Theorems 1.9 and 4.12, Examples 4.3–4.4]. Here the endpoint signs force walls of every degree for one path in the space of graph equations. We prove the local assertion in that space and retain the orientation of the change.

The trace-free summand. Pushing the pullback normal operator down to C amounts to twisting D_t by the real permutation local system of f_m. That local system is $\mathbb{R} \oplus W_m$, where W_m is the sum-zero standard representation of S_m pulled back by (28). Both summands have index zero. This follows by deforming to a complex Dolbeault operator and using that a complexified flat real local system has degree zero. The complex orientations give

$$\operatorname{sgn}(D_t^{W_m}) = \operatorname{sgn}(f_m^* D_t) \operatorname{sgn}(D_t) \tag{30}$$

when both operators on the right are invertible. The endpoint formula of Section 14 gives $+1$ and -1 on the left, since the monodromy of a_1 is odd. Thus some $D_t^{W_m}$ has a nonzero kernel. For the linear graph equation any nonzero real multiple of such a kernel element is a solution on Y_m with zero fiberwise trace.

A section on Y_m pulls back to a section on the normal closure invariant under S_{m-1}. If two distinct local sheets agree on an open disk, the corresponding translates agree identically on the normal closure. Indeed their difference satisfies a real linear Cauchy–Riemann equation, obtained by integrating the fiber derivative along the segment joining them in a common tube. Unique continuation applies. The stabilizer would then properly contain S_{m-1}, which is a

maximal subgroup, so all sheets would agree identically. It follows that every section which does not descend has pairwise distinct values over a dense open set. Its graph map is simple and immersive because its projection is étale. A nonzero trace-free section cannot descend. Positive multiples tending to zero give distinct supports, as can also be seen from their distinct maximum fiber norms. The deck orders stated above follow from $\mathrm{Aut}_C(f_m) = N_{S_m}(S_{m-1})/S_{m-1}$.

Choice of the path. We use equivariant Brill–Noether transversality for real Cauchy–Riemann operators [Wen23, DW23b]. Here are the precise hypotheses for this application. The affine parameter space of normal operators allows every real zeroth-order bundle map as an infinitesimal variation, including variations supported in any given coordinate disk, with arbitrary prescribed finite jet at an interior point. The real Cauchy–Riemann symbol satisfies Wendl's condition [DW23b, Theorem 2.5.3]. Off a set of infinite codimension the operator then satisfies the equivariant Petri condition for any specified finite list of inequivalent irreducible local systems [DW23b, Proposition 1.6.6 and Theorem 1.6.17]. The operator and its formal adjoint satisfy strong unique continuation by the Cauchy–Riemann symbol and the zeroth-order similarity principle (the adjoint has the conjugate symbol). In these statements one may take the smooth Banach perturbations with prescribed jets used for differential operators in [DW23b, Sections 2.2–2.3]. Away from this exceptional set, the joint kernel stratum for the trivial system and any finite selection of W_m has real codimension $\sum_i k_i^2$, where k_i is the real nullity of the ith twist [DW23b, Theorem 1.3.5 and Proposition 1.3.11]. Each standard representation is absolutely irreducible and of real type. The different ranks distinguish the W_m, and W_2 is distinguished from the trivial line by its nontrivial monodromy.

Relative transversality for paths now gives transverse intersections with the corank-one strata and avoidance of the higher-codimension strata and of the exceptional set [DW23b, Proposition 1.B.5]. For clarity, variations of a path supported in a short interior parameter interval realize the operator variations just described at its center. Thus evaluation of paths at an interior point is a submersion in these perturbation spaces. One obtains the same residual assertion for smooth paths by the usual local perturbation argument: for each finite list the required property is open on a compact interval, by Fredholm reduction and endpoint invertibility, and dense in each smooth neighborhood using small smooth Banach perturbations there. To see openness one can express the condition directly as simple transverse crossings with no simultaneous nullities; this expression requires no condition on jets away from the crossing set. Intersect the resulting open dense sets over all finite initial lists. For each W_m its crossing set is closed in the compact interval and each of its points is isolated, so it is finite. Its parity is odd by (30). For any prescribed degree bound this gives only finitely many of the selected crossings.

Nonlinear graphs and their signs. Fix one of the crossings t_* for a selected $f = f_m$. Let ξ and η span the real kernel and adjoint kernel of $f^*D_{t_*}$ and choose L^2 splittings. The crossing pairing $\langle \eta, f^*\dot{D}_{t_*}\xi\rangle_{L^2}$ is nonzero. Lyapunov–Schmidt reduction of (27) on $W^{1,r}(Y_m, f^*N)$, $r > 2$, solves the image equation uniquely for the complementary part of $s = a\xi + z(t,a)$. Here $z(t,0) = 0$ and $z_a(t_*,0) = 0$. Elliptic regularity gives smooth sections. The remaining real equation is $aG(t,a) = 0$. Its derivative $G_t(t_*,0)$ is the crossing pairing up to the nonzero normalization of the cokernel coordinate. Thus all nearby nonzero solutions lie on a smooth branch $t = T(a)$ through $T(0) = t_*$. Their nonzero kernel projection rules out descent, since base sections are orthogonal to ξ in the fiberwise permutation splitting.

We verify that a higher fiber jet can give $T(a) - t_*$ a first nonzero Taylor coefficient, keeping the normal path fixed. Put $A = \dot{D}_{t_*}$, a real zeroth-order map. For some base point x the sum

$\sum_i u_i$ over its sheets is nonzero, where $u_i = \langle \eta_i, A(x)\xi_i\rangle_{\mathbb{R}}$ in the pulled-back metrics. This follows by integration of the crossing pairing. Let $r_1, \dots, r_k$ be the distinct positive values of $|\xi_i|^2$ at x. The matrix $(r_j^l)_{1\le l,j\le k}$ is invertible. Since $u_i = 0$ when $\xi_i = 0$, there is an l with $1 \le l \le m$ such that $\sum_i |\xi_i|^{2l} u_i \neq 0$. Add to B a small real multiple of

$$\chi(t,x)|v|^{2l} A(x)v, \tag{31}$$

where a nonnegative cutoff localizes the pairing to a disk on which this sum keeps its strict sign, and equals a base cutoff at t_*. This changes the coefficient of a^{2l+1} in the reduced equation at t_* by a nonzero multiple of the perturbation, keeping lower coefficients fixed. Indeed substitution of $s = a\xi + O(a^2)$ gives the displayed pairing at that order. The induced change in z starts at the same order, where its linear image under $f^* D_{t_*}$ has zero cokernel projection. If lower coefficients vanish, this changes the coefficient of a^{2l} in $T(a) - t_*$ by a nonzero multiple as well, because $G_t(t_*, 0) \neq 0$. Finite order at most $2m$ is therefore an open dense condition at this crossing, including in the fiberwise odd subspace. The nonlinearities with a prescribed first jet form an affine Fréchet space on a closed tube; intersect these conditions over the countably many crossings. One may fix the germs at both endpoints by using parameter cutoffs in (31).

Finite order implies that $a = 0$ is the only solution at t_* in a small interval of kernel coordinates. The branch has no critical point on either punctured half of a sufficiently small such interval. On regular parameter slices to either side all solutions there are isolated and regular. Brouwer degree of the reduced real map is constant on these slices. The sign of the complementary invertible block supplies the same orientation factor on both sides. The core solution contributes $\sigma(f)^{\pm}$, so subtraction yields the numerator in (29). For étale projections the tangential map $T_{Y_m} \to f^* T_C$ is a complex isomorphism and cancels the source deformation factor with positive orientation. The section sign is consequently the determinant sign for the graph as a map with variable domain. The deck group acts freely on the non-descending sections with stabilizer S_{m-1} on the normal closure, and its orbits parameterize their images. Deck changes preserve the complex determinant orientation. Division by d_m therefore gives the signed count of distinct simple curves.

In an odd family the splittings may be used unchanged under $s \mapsto -s$. Uniqueness in the image equation gives $T(-a) = T(a)$. Its first nonconstant term has even order, so one side has two solutions with opposite kernel coordinates, and the other has none. In degree two the nontrivial deck transformation acts by -1 on the kernel. Its action exchanges the solutions, giving one simple support. In higher degree the deck group is trivial and the two supports are distinct. Finally at a selected crossing the sign of D_t is constant. Multiplying (29) by this sign gives the left relative sign minus the right one divided by d_m, by (30). The relative sign stays constant across all other parameters at which $D_t^{W_m}$ is invertible, including base-only crossings. The sum is the difference between its endpoint values divided by d_m. This proves Theorem 15.1.

A limiting operator on an infinite cover. The infinitely many crossings have an analytic accumulation point even though each individual projection cover has only finitely many. Give C a fixed Hermitian metric and give the pullback bundles and covers their lifted metrics.

Proposition 15.2. *For the generic paths of Theorem 15.1, there is an interior accumulation parameter t_∞ of the selected crossing times and a connected infinite étale cover of C on which the pulled-back D_{t_∞} has a nonzero bounded smooth solution. On this cover the bottom of the spectrum of the Friedrichs extension of $D_{t_\infty}^* D_{t_\infty}$ is zero.*

Proof. At each spin endpoint write $D = \bar\partial + R$, with R the antilinear mass defined by the nondegenerate holomorphic symmetric form $\gamma : N \to K_C N^*$ and the metric, as in Section 14. On any finite étale cover the integrated real cross term of $\bar\partial s$ and Rs vanishes: up to a fixed area normalization it is the real part of the integral of $\bar\partial(\gamma(s,s))$, where the canonical factor is pulled through the covering differential. Hence

$$\|Ds\|_2^2 = \|\bar\partial s\|_2^2 + \|Rs\|_2^2 \geq c^2 \|s\|_2^2$$

for the positive least singular value bound c of R on C. This bound is independent of the étale cover. An endpoint complex bundle identification only changes the equivalent norms by constants from the base. Along a fixed-symbol path the difference from the endpoint is zeroth order and its uniform norm lifts unchanged. There are therefore collars of both endpoints free of these crossings in all degrees. Choose $m \to \infty$ and one crossing t_m for each m. A subsequence converges to an interior parameter t_∞.

Take ξ_m in the trace-free kernel on Y_m, normalized to have maximum fiber norm one. Choose a point in the fiber of a fixed base point within uniformly bounded distance of a maximum point, by lifting a shortest path on C. The based covers have a subsequence converging on bounded sets to an infinite cover, as can be seen directly from (28). Label the sheets cyclically, with the chosen point labelled zero. The second generator acts by unit shift and the first interchanges two consecutive labels. After passage to a subsequence this interchange either occurs at fixed integer labels or leaves every bounded set, in which case the first generator acts trivially in the limit. In either case the limiting sheet set is $\mathbb{Z}$ and the second generator acts by translation, so the cover is connected and infinite. More explicitly, words of any bounded length have eventually the same action at the distinguished sheet as on this labelled limit. The resulting covering charts on each compact part identify the lifted operator coefficients there, up to their smooth dependence on t_m. The genus and the compact base are fixed. Interior elliptic estimates on successively larger compact sets and a diagonal subsequence give smooth local convergence of ξ_m to a solution ξ with $|\xi| \leq 1$. The points of maximum remain in a bounded set of the based covers, so $|\xi|$ attains one somewhere.

The limiting cover has area growth $O(R)$ on radius-R balls. To see this, use a fixed finite triangulation of C. Its translates at each sheet have uniformly bounded area; after choosing a spanning tree, crossing any edge moves the sheet by at most a fixed number of places on $\mathbb{Z}$, since its monodromy is a fixed word in the surface generators. A metric ball of radius R reaches only $O(1+R)$ sheets: lifting a fixed finite cover by evenly covered balls gives an edge word of length $O(1+R)$ for any path in that ball of length at most R. Let χ_R equal one on the radius-R ball and vanish outside the radius-$2R$ ball, with weak gradient bounded by $2/R$. The Leibniz rule gives

$$\|D_{t_\infty}(\chi_R \xi)\|_2^2 = O(R^{-1}), \qquad \liminf_{R\to\infty} \|\chi_R \xi\|_2^2 > 0.$$

The cutoffs belong to the form domain by compact support and smooth approximation. Their Rayleigh quotients tend to zero, proving the spectral assertion. The construction gives boundedness of ξ; its square integrability on the limiting cover is a separate question. □

There is also a spectral statement on the fixed base. On the Hilbert direct sum over $m \geq 2$ of the two L^2 spaces of the twist W_m (sections and $(0,1)$-forms), take the self-adjoint direct sum of the operators

$$\mathcal{A}_{m,t} = \begin{pmatrix} 0 & (D_t^{W_m})^* \\ D_t^{W_m} & 0 \end{pmatrix}.$$

The direct sum has its natural graph domain. At a selected t_m put an L^2-unit kernel section into the section component of its own summand. For a subsequence $t_m \to t_\infty$ these vectors are orthonormal, and the norm of their image under the direct sum at t_∞ tends to zero: it is bounded by the uniform zeroth-order norm of $D_{t_m} - D_{t_\infty}$ on C. Weyl's criterion puts zero in the essential spectrum of this direct sum at t_∞. At the two endpoints the preceding uniform estimate applies to every summand. Each twist has index zero, so its inverse and the adjoint inverse have uniformly bounded norms. Thus the direct sum has a spectral gap about zero at both endpoints.

Corollary 15.3. *For every $g \geq 2$ a closed symplectic six-torus admits a smooth path of tame almost-complex structures fixing an embedded genus-g curve in a primitive integral class A, with the bifurcations of Theorem 15.1 in all the classes mA, $m \geq 2$. One may realize the fiberwise odd families and their signed changes.*

Proof. Use the same smoothed symplectic surface and the same complex normal identification for the whole path. All the graph structures agree as complex structures along the zero section. Compactness of the parameter interval gives a single tube on which they lie in the chosen chart about the ambient compatible extension and are tame. Apply the cutoff construction there, leaving them unchanged on a common smaller tube. For every m the multisections occur arbitrarily close to the zero section, so they persist in this one compact realization. Their integral homology classes are mA by the tubular retraction. □

15.1. **Reduction to the normal operator.** Let C be a closed embedded pseudoholomorphic curve of genus $g \geq 2$ in an almost-complex six-manifold (X, J) with $c_1(TX)[C] = 0$. Write D for the normal Cauchy–Riemann operator on $N = TX|_C/TC$. For an integer $M \geq 1$, consider the condition

(32)
$$\ker(f^*D) = 0 \quad \text{for every connected holomorphic cover } f : Y \to C \text{ of degree at most } M.$$

Here Y is smooth and closed, ramification is allowed, and the identity is included. Write J_D for the homogeneous almost-complex structure on N associated with D. Its projection to C is holomorphic, its fibers have their vector-space complex structures, and a lift over f is holomorphic exactly when its section of f^*N lies in $\ker(f^*D)$ [DW21, Lemma 3.6].

Proposition 15.4. *Under (32) there is a disk tube about C in which the only cycle of degree m is $m[C]$ for $1 \leq m \leq M$. Its point class $P_m(J)$ in S equals $P_m(J_D)$. The tube can be chosen uniformly over a smooth compact interval of marked germs satisfying (32) throughout, and these classes are constant along the interval. If the condition holds in every degree, the entire series of the germ has the same unique lift respecting the cycle operations as that of its normal model. This lift is constant under such deformations.*

Proof. Use the marked surface to identify the cores of the family J_t. Choose complex complements along the cores and identify the normal bundles with a fixed smooth complex bundle N over the interval. In the resulting tubular charts put $s_r(x, v) = (x, rv)$ and $J_{t,r} = s_r^* J_t$ for $0 < r \leq 1$. In bundle coordinates the tangent-to-normal block is $r^{-1}B_t(x, rv)$ with $B_t(x, 0) = 0$. It extends smoothly in all variables as

$$\int_0^1 \partial_v B_t(x, urv)[v]\, du.$$

The normal-to-tangent block is multiplied by r and the diagonal blocks are evaluated at (x, rv). The limit $J_{t,0}$ is J_{D_t} by the normal rescaling formula [DW21, Lemma 3.10]. For a change F

of tubular charts, $s_r^{-1} F s_r$ extends at zero to the induced map on the quotient normal bundles. Indeed, its base component tends to the map on the cores and its vertical component tends to the vertical derivative of F applied to v. Thus the identification is independent of the complement and is natural under complex normal isomorphisms.

Realize a common disk of this compact family on the symplectic six-torus of Proposition 3.1, with $A = [C]$ primitive. Indeed, in the symplectic splitting along the core the vertical complex structure is fixed, and each induced complex structure on TC is compatible with its area form. The extension and cutoff in that proposition therefore give a smooth family tamed by one form ω, agreeing with $J_{t,r}$ on a common closed disk of some radius $2a > 0$.

If there were no uniform isolating radius through degree M, there would be J_{t_i,r_i}-holomorphic cycles Z_i of degrees $1 \le m_i \le M$ with positive maximum fiber norms $b_i \to 0$. Pass to a subsequence with $m_i = m$ and $t_i \to t$. Put $u_i = b_i/a$. Then $(s_{u_i}^{-1})_* Z_i$ is holomorphic for $J_{t_i,r_i u_i}$, also when $r_i = 0$ by homogeneity. It lies in the closed disk of radius a and meets its boundary. Dilation preserves the class mA by the tubular retraction, so its symplectic area is $m\langle [\omega], A \rangle$. Geometric cycle compactness gives a limit for $J_{t,0}$ in the same class, with maximum norm a by Hausdorff convergence [DIW26, Theorem 2.11 and Remark 2.13]. This is the rescaling argument of [DW21, §6].

Normalize an irreducible component of the limit off the zero section. Its projection f is nonconstant, since a complex vector space contains no compact curve. All components thus have positive projection degree, counted with their multiplicities to give m. Our component gives a nonzero element of $\ker(f^* D_t)$ on a smooth cover of degree at most m [DW21, Proposition 3.12]. This contradicts (32). The argument uses cycle compactness without imposing a bound on the genus.

In class mA for $1 \le m \le M$, containment of the support in a smaller open disk selects just the section $m[C]$. It is open by Hausdorff continuity and compact over the closed parameter rectangle. Its characteristic class therefore transports along either parameter by the interval relation [Wue26b, §2]. Open excision identifies the classes obtained using smaller tubes. When (32) holds for every M, the linear model has only the zero section as a compact irreducible curve, and each unramified pullback of D is invertible by index zero. Proposition 3.1 gives its balanced lift, and Proposition 16.5 gives uniqueness among lifts respecting the cycle operations. The coefficientwise equalities just proved give the same lift for the original germ and for every member of the interval. □

16. Integral multiplicities from finite covers

In this section take the two-spin germ of Proposition 14.2, with w the difference of its spin structures and ϵ its core sign. Let $\Gamma = \pi_1(C)$ and let $\sigma(H)$ denote the normal determinant sign on the connected cover corresponding to a finite-index subgroup $H \le \Gamma$. Write x for a simple nonseparating loop representing the Poincaré dual of w when $w \neq 0$, and take $x = 1$ when $w = 0$. The covering sign is

$$\sigma(H) = \epsilon^{[\Gamma:H]} \operatorname{sgn}(x \mid \Gamma/H). \tag{33}$$

For a finite group Q, the Burnside ring $A(Q)$ is the Grothendieck ring of finite Q-sets. Its mark m_H at a subgroup $H \le Q$ counts the H-fixed points. Marks on virtual sets take integer values. For the profinite completion $\widehat{\Gamma}$ we use the completed Burnside ring

$$\widehat{A}(\Gamma) = \varprojlim_N A(\Gamma/N),$$

where N ranges over finite-index normal subgroups of Γ. The map associated with a quotient $Q' \to Q$ takes fixed points of its kernel. A transitive orbit survives precisely when its stabilizer contains the kernel. Thus an element has an integer coefficient $b_{[H]}$ for each conjugacy class of finite-index subgroups. It can be read in any quotient through which the action on Γ/H factors. This is the Burnside ring of almost finite sets of Dress and Siebeneicher [DS88].

Let c_m denote the signed connected count of étale covers in degree m, with the usual inverse deck orders. Let $C_m(h)$ be the same count with all signs positive for a base of genus h. Recall $h_d = 1 + d(g-1)$ and the exponents n_d in (2).

Theorem 16.1. *The signs (33) are the marks of a unit in $\widehat{A}(\Gamma)$. Its orbit coefficients satisfy*

$$n_d = \sum_{\substack{[H] \\ [\Gamma:H]=d}} b_{[H]}. \tag{34}$$

Suppose $w \neq 0$ and p is an odd prime. Let M_p be the number of connected degree-p covers with a specified nonzero permutation sign character in $H^1(C;\mathbb{F}_2)$. This number is independent of the chosen nonzero character, and

$$n_p = -\epsilon\, 2^{2g} M_p, \qquad |n_p| \geq 2p!\,(p!k_p)^{g-2}, \tag{35}$$

where k_p is the number of partitions of p. For $w = 0$ one has $n_p = 0$ for every odd prime p.

Proof. Fix a quotient Q of Γ and denote by T the subgroup generated by the image of x. Write $\epsilon = (-1)^a$ with $a \in \{0,1\}$. Consider the real permutation representation

$$V_Q = \mathbb{R}[Q/T] \oplus \mathbb{R}[Q]^{\oplus(1-a)}.$$

The reduced equivariant Euler characteristic of the one-point compactification S^{V_Q} is an element $u_Q \in A(Q)$ with mark $(-1)^{\dim V_Q^H}$ at H. Indeed, after equivariant triangulation and subdivision, each fixed-set Euler characteristic is the alternating simplex count on the corresponding fixed sphere. Subtracting the fixed point at infinity gives the reduced Euler characteristic. This construction of units from real representations is classical [tD09, Proposition 5.8.2].

If $d = [Q : H]$, then $\dim \mathbb{R}[Q]^H = d$, and $\dim \mathbb{R}[Q/T]^H = |H\backslash Q/T|$ is the number k of cycles of x on Q/H. The sign of this permutation is $(-1)^{d-k}$. Thus the marks of u_Q are precisely (33). Marks distinguish elements of $A(Q)$ by triangularity on transitive orbits. In particular $u_Q^2 = 1$. Under a further quotient the mark of the kernel-fixed set at a subgroup downstairs is the old mark at its inverse image. Formula (33) is identical on the two coset sets, so the u_Q give the asserted unit.

For completeness let μ denote the Möbius function on the subgroup interval $[H,\Gamma]$. This interval is finite, since it can be computed in the quotient by the normal core of H. Inverting the marks gives

$$b_{[H]} = \frac{1}{|N_\Gamma(H)/H|} \sum_{H \leq K \leq \Gamma} \mu(H,K)\, \sigma(K). \tag{36}$$

The sum is over subgroups themselves. The numerator counts virtual points of stabilizer equal to H. A single orbit of type Γ/H contains $|N_\Gamma(H)/H|$ such points.

The number of subgroups of index m in the genus-h surface group is $mC_m(h)$, by conjugacy and the deck order. Hence mc_m is the sum of $\sigma(K)$ over all subgroups K of index m in Γ. One orbit Γ/H of size d contributes one for each pair (K,z) with $z \in (\Gamma/H)^K$. For each of

the d choices of z, its stabilizer is a surface group of genus h_d. The subgroup K must have index m/d therein. It follows that

$$c_m = \sum_{d \mid m} \left(\sum_{\substack{[H] \\ [\Gamma : H] = d}} b_{[H]} \right) C_{m/d}(h_d).$$

All sums are finite since a finitely generated group has finitely many subgroups of each finite index. Comparison with (2) gives (34) by induction, using $C_1(h) = 1$.

Now $[\Gamma : H] = p$ implies that H is maximal. Its deck group acts freely on a fiber of p points. If the group is nontrivial, it acts as a regular cyclic group of order p. The monodromy then lies in the centralizer of a p-cycle, all of whose permutations are even. Consequently any cover on which x acts oddly has trivial deck group. Formula (36) gives $b_{[H]} = -2\epsilon$ for each such cover and zero for the others. In particular all coefficients vanish in degree p when $w = 0$.

If $w \neq 0$, the mapping class group acts transitively on nonzero mod-two cohomology classes and carries covers to covers with the transformed sign character. Hence each specified nonzero character occurs on the same number M_p of covers. Exactly 2^{2g-1} characters evaluate to one on x, which gives the equality in (35).

To prove the lower bound, take standard surface generators a_i, b_i with $a_1 = x$. Send a_1 to any transposition and a_2 to any p-cycle in S_p. Their images generate S_p. Indeed the translates of the transposition under the cycle are the edges of a connected graph on $\mathbb{Z}/p$, and the edge transpositions of a connected graph generate the symmetric group. Send each of b_1, b_2 to an arbitrary element centralizing the image of the corresponding a_i. Each of these two handles gives $p!$ choices by the conjugacy class formula. On each of the remaining $g - 2$ handles choose any commuting pair. There are $p!k_p$ such pairs. The surface relation holds, and every resulting monodromy is surjective. Relabelling acts freely on these homomorphisms since the center of S_p is trivial. Dividing their number by $p!$ gives at least $p!(p!k_p)^{g-2}$ covers on which x acts oddly. Each contributes -2ϵ. □

Example 16.2 (Degree three)**.** For a nonzero character α consider the one-dimensional Γ-module $\mathbb{F}_{3,\alpha}$ on which $s \in \Gamma$ acts by $(-1)^{\alpha(s)}$. Identify S_3 with the affine group of $\mathbb{F}_3$. Specifying monodromy with sign α amounts to specifying its translation part, a cocycle in $Z^1(\Gamma; \mathbb{F}_{3,\alpha})$. The image fixes a point if and only if the cocycle is a coboundary. Every intransitive action on three letters has a fixed point. Conjugation by translations changes a cocycle by a coboundary, and conjugation by reflection negates its class. The nonzero classes therefore give transitive covers in pairs. The local system has no invariants and is self-dual, so H^0 and H^2 vanish and $\dim H^1 = 2g - 2$ by the Euler characteristic. We obtain, for $w \neq 0$,

$$M_3 = \frac{3^{2g-2} - 1}{2}, \qquad n_3 = -\epsilon\, 2^{2g-1}(3^{2g-2} - 1).$$

In genus two the exponent is -64ϵ.

Remark 16.3 (Comparison with Parker's signs). Parker records determinant signs on all subgroup-fixed subspaces of the deck group in the integral algebra on subgroups whose product is intersection [Par13, Definitions 1.1–1.2 and Lemma 1.3]. The map from $A(Q)$ to that algebra sends a finite Q-set X to $\sum_{z \in X} [Q_z]$, where Q_z is its point stabilizer. Intersection computes the stabilizer in a product, and the marks recover the virtual set. Under this injection u_Q gives Parker's sign element on the regular Q-cover, since both give the same determinant on every fixed subspace. Parker's local contribution of an individual cover uses

its own deck group. For instance a cover of odd prime degree on which x acts oddly has trivial deck group and local sign $-\epsilon$. Its orbit coefficient in the regular closure in (36) is -2ϵ.

Corollary 16.4. *Let C be any isolated genus-g curve satisfying Proposition 3.1, and let ϵ be its normal determinant sign. At every odd prime p its elementary exponent at $(p, 1+p(g-1))$ satisfies*

$$n_p = -2\epsilon\,\#\{[f] : \deg f = p,\ f \text{ connected étale},\ \sigma_D(f) = -\epsilon\}.$$

In particular $-\epsilon n_p$ is an even nonnegative integer. Every connected cover with odd-order monodromy has sign ϵ.

Proof. For the absolute signs use the coincident-spin operator of Proposition 14.2 as reference. Its two identical determinant signs multiply to $+1$ on every unramified cover. The relative unit from Theorem 7.1 with this initial operator therefore has the prescribed absolute marks.

For p prime, (2) subtracts the unsigned degree-p count times ϵ from the signed count. An index-p cover with deck transformations is regular cyclic, as in the proof of Theorem 16.1. On a regular cyclic p-cover the cardinality of a finite C_p-set is congruent modulo p to its number of fixed points. This holds for virtual sets as well. Since p is odd, the two signs of the unit $u_{C_p}(D)$ agree. Every surviving term in the difference thus has trivial deck group, proving the formula.

More generally, on an odd-order monodromy group the trivial representation is the only irreducible representation of real type. Indeed squaring permutes the elements of such a group, so the Frobenius–Schur indicator of each nontrivial irreducible complex character vanishes by character orthogonality. In the construction of Theorem 7.1 only the trivial representation can contribute a sign, and its fixed space has dimension one at every subgroup. The sign is therefore the base sign. □

The full product on an isolating interval. There is a separate all-genus conclusion when a neighborhood isolates the entire family. We state it with the compactness hypothesis required for the interval relation. Say that a normalized series $R(q) = 1 + \sum_{m>0} R_m q^m$ in $H[[q]]$ respects the cycle operations if addition and multiplication of cycles induce

$$\Delta R_m = \sum_{i+j=m} R_i \otimes R_j, \qquad \rho_k R_m = \begin{cases} R_{m/k} & k \mid m, \\ 0 & k \nmid m. \end{cases}$$

Balanced products and their integral powers have these properties [Wue26b, §7]. Substitution $q \mapsto q^r$ preserves them: for the division operation on the substituted series this follows directly by asking whether kr divides the original label m. The coefficient at label m on either side is zero in the other cases.

Proposition 16.5. *The balanced lift in Proposition 3.1 is the unique normalized lift respecting the cycle operations. Suppose an interval of tame almost-complex neighborhoods carries an additive positive integral degree label. At each end let the selected connected supports be finitely many pairwise disjoint curves satisfying Proposition 3.1, of positive labels r_i at the left end and s_j at the right end. Assume, at each finite degree cutoff, that the characteristic insertion of the cycles in the selected neighborhood is compact open and closed over the interval; it selects all sums of multiples of these supports at the endpoints, up to the cutoff. Then in $H[[q]]$ one has the full equality*

$$\prod_i \widetilde{P}_i(q^{r_i}) = \prod_j \widetilde{P}_j(q^{s_j}).$$

Proof. For two lifts with the given operations, a first nonzero difference is primitive by the coproduct formula and is killed by all ρ_k for $k > 1$. It is a finite rational linear combination of $x_{h,1}$ by [Wue26b, Lemma 7.1]. Its image in S vanishes, which contradicts [Wue26b, Theorem 5.1]. For the interval the family insertion transports the entire selected series in S at each cutoff. At the ends disjoint open excision factors the insertion into the product of the local insertions, where the substitutions retain the positive labels. Apply the first-difference argument to the two lifted products in each degree. The same proof works after concatenation, or on either boundary path of a compactly isolated two-parameter family. □

One sufficient form of the interval hypothesis is a closed tame ambient family with a relatively compact spatial tube and a fixed energy bound for each degree cutoff, in which no selected cycle can reach the tube boundary while remaining in its closure. Bounded-energy cycle compactness and Hausdorff convergence give the required compact insertion as in Proposition 3.1. The fixed-cover bifurcation theorem gives no such all-genus spatial isolation along its full path. Indeed, in its linear realization a nonzero solution can be scaled all the way to the boundary.

A universal correction on the specified cover. For a crossing of Theorem 15.1 fix $f : Y_m \to C$ and pull back the tube to Y_m. In this tube the cores of the graph equation over f become embedded sections. The unit inclusion of an isolated such section of degree one defines an insertion even when its higher multiples have not been isolated. Write $x_{h,1}$ for the degree-one genus-h class as in Section 3.

Corollary 16.6. *In the pulled-back tube, the change of the sum of the degree-one classes of the nonzero graphs at a crossing is* $(\sigma(f)^- - \sigma(f)^+)x_{h_m,1}$ *in* H. *Counting these graphs modulo reparametrization over* C *divides this change by* $|\operatorname{Aut}_C(f)|$.

Proof. A compact effective cycle of projection degree one in the pulled-back tube has precisely one irreducible component, of multiplicity one. A vertical component in an affine fiber is impossible, and the normalized degree-one projection is an isomorphism. Hence the entire degree-one cycle is the graph of a smooth section of Y_m. Near the crossing the finite-order reduction in Section 15 isolates these sections. More explicitly, bounded sections whose sup norms tend to zero satisfy the fixed-cover semilinear equation with a right hand side tending uniformly to zero. The elliptic estimate for the fixed Dolbeault operator then gives convergence to zero in $W^{1,r}$ for $r > 2$. This places them in the reduction chart. Conversely its solutions converge smoothly, by elliptic regularity. A small tube about zero and then a sufficiently short interval isolate the selected graphs in degree one. At the central parameter there is no other small solution, and compactness of bounded sections follows by the same estimate and elliptic bootstrapping.

For an isolated regular embedded graph of genus h its degree-one insertion equals $\operatorname{sign}(D)\Phi(x_{h,1})$ in S. Indeed work at the energy of the primitive core in the compact realization of Proposition 3.1. At the original endpoint the degree-one core is unique in an isolating tube and is regular. One can choose the nearby generic structure fixing it as in [DIW26, proof of Proposition 3.54]. In the short transport it is the unique nearby curve of its class, by the openness of the embedded multiplicity-one locus [DIW26, Proposition 2.21] and the parameterized implicit function theorem. The cluster isotopy to the elementary endpoint has only corrections in proper multiples, all above this cutoff. It transports the leading factor with the orientation sign of the core. The unique lift of this degree-one class compatible with the operations is the same signed $x_{h,1}$, by the primitive argument of Proposition 16.5 at degree one. Apply this fact to each regular graph on the two adjacent slices and the signed

count of Theorem 15.1. Each nonzero simple support with projection f has exactly $|\operatorname{Aut}_C(f)|$ parametrizations in the fixed-cover chart. This proves both statements. □

This equality retains the universal class of the embedded graph on the specified cover. Recovering the entire degree-m coefficient of the original, unmarked tube would also require the other projection types.

Appendix A. A congruence and individual signs in degree four

Let C be a genus-g curve, $g \geq 2$, with normal bundle of rank two and degree $2g-2$. Let D be regular on all connected étale covers of degrees $1, 2, 4$. Write ϵ for its base determinant sign and n_d for the coefficient of the balanced exponent in degree d at genus $1 + d(g-1)$. For operators alone use the equivalent triangular extraction from the signed étale cover counts. For an isolated germ admitting the geometric factorization, these are actual product exponents. The distinction matters here since signs of four-sheeted unramified covers do not account for higher-genus stable maps.

Theorem. The primitive quartic coefficient obeys the sharp congruence

$$n_4 \equiv \binom{-\epsilon n_2}{2} \pmod 2. \tag{37}$$

On a genus-two curve the path between the positive repeated-spin and distinct-spin endpoints must reverse the twist determinant on each of 480 different S_4 quartics and on the faithful plane of each of 96 dihedral quotients. Each of the latter quotients supplies two inequivalent quartic covers singular together at the plane crossing. A primitive S_4 crossing can change n_4 by two while keeping every connected étale cover sign in degrees at most three fixed. Thus even that whole collection of lower-degree signs does not determine n_4 modulo four. These statements about crossings concern fixed finite covers and their linear operators. The two full quartic endpoint signs in a dihedral pair include their respective intervening double characters.

A.1. **Inverting the étale counts.** Set $\Gamma = \pi_1(C)$. For a binary character $a \neq 0$ let p_a be the determinant sign of the operator twisted by its sign line, and K the number of negative p_a. At the least source genus only connected étale covers occur. If a connected cover corresponds to $U \leq \Gamma$, its sign is the product of the twist signs of the constituents of the real permutation system $\mathbb{R}[\Gamma/U]$, counted with multiplicity. In particular the sign of the double for a is ϵp_a. Möbius inversion of the stabilizer counts assigns the individual orbit coefficient

$$\frac{1}{|N_\Gamma(U)/U|} \sum_{U \leq J \leq \Gamma} \mu(U, J) \operatorname{sign}\left(D^{\mathbb{R}[\Gamma/J]}\right).$$

Summing over conjugacy classes of index d gives n_d: reinducing the virtual transitive orbits counts the towers of connected covers with their inverse deck orders, exactly the proper-divisor terms of the triangular relation. This inversion uses only the interval above the particular subgroup. For degree two it gives $n_2 = -\epsilon K$. The underlying determinant marks and their compatibility with finite quotients are proved in Theorem 7.1. The formula also follows by transitive-set inversion on any finite quotient through which the covers in question factor, taking regular endpoints on that quotient after a small perturbation preserving the required signs.

For a cyclic quartic, the new plane is of complex type and has positive sign. Its orbit coefficient vanishes. For a Klein four quotient the three binary signs p, q, r give coefficient $\epsilon(pqr - p - q - r + 2)/4$, equal to ϵ precisely when at least two signs are negative. Every pair of negative characters spans one binary plane. A plane with three negative signs is counted three

times by this pair count and gives only one orbit coefficient. Thus all Klein four contributions sum to $\epsilon(\binom{K}{2} - 2T)$, where T counts the planes with three negative signs.

For a dihedral quotient of order eight the two reflection-stabilizer actions have deck order two. Each contains the base, its own intervening binary character (a or b), and the common faithful plane of sign s. Their combined coefficient is $\epsilon(s-1)(p_a + p_b)/2$. For a primitive A_4 or S_4 cover the new constituent is the faithful three-dimensional augmentation representation of sign s. There is no intermediate and the deck order is one, giving $\epsilon(s-1)$. This exhausts transitive subgroups of S_4. Both the dihedral pair and the primitive term are even; reduction modulo two gives (37).

To speak of individual crossings use the finite common normal closure of the selected degree-at-most-four covers. In the unrestricted fixed-symbol space the equivariant Brill–Noether theorem gives generic avoidance of simultaneous inequivalent singular twists, and a simple transverse real-type block at each crossing. The local variations realize arbitrary zeroth-order tensors and their jets on a disk, with unique continuation for the equation and adjoint. Each twist has index zero, and the singular sets for complex- and quaternionic-type blocks have real codimension at least two. These are the equivariant incidence theorems of [DW23b]. Covers involving the same singular constituent can of course be singular simultaneously.

A.2. **Which quartics cross between positive spin endpoints?** Let $w \neq 0$ be the difference of the two spin characters and let $\ell(a) = \langle w \smile a, [C] \rangle$. At the repeated endpoint every twist is positive. At the distinct endpoint the sign on a real twist V is $(-1)^{\ell(w_1(V))}$ by the mod-two index formula in Proposition 14.2. There are 2^{2g-1} binary twists with negative sign. No binary plane has three negatives. For a dihedral plane the determinant is $a + b$ for the two reflection normalizers, hence the combined dihedral endpoint term vanishes. The A_4 augmentation determinant is trivial. The S_4 augmentation determinant is its ordinary permutation sign.

Fix a nonzero permutation sign with $\ell(a) = 1$. Surjective S_3 resolvents of this sign correspond to the nonzero classes of $H^1(C; \mathbb{F}_{3,a})$ modulo negation, since S_3 is an extension of the sign group by its additive three-element module. Invariants and degree-two cohomology vanish, so the number of resolvents is $(3^{2g-2} - 1)/2$. The extension $S_4 = V \rtimes S_3$ with $V = \mathbb{F}_2^2$ describes lifts of a specified resolvent by classes of $H^1(C; V)$. The S_3-action is irreducible and symplectic, so this space has dimension $4g - 4$. A nonsurjective lift must be a complement. Every complement is conjugate to the split one since $H^1(S_3; V) = 0$, by restriction to the normal subgroup of order three, which has no invariants on V. The centralizer of the surjective resolvent in S_3 is trivial, leaving $2^{4g-4} - 1$ surjective lifts modulo conjugacy. Therefore exactly

$$P = 2^{2g-2}(3^{2g-2} - 1)(2^{4g-4} - 1)$$

different S_4 quartics reverse their twist sign. Each has a singularity along every such path. On a generic path the corresponding faithful twists have distinct crossing parameters. The signed primitive change at a single such crossing has magnitude two and no other quartic affected by that constituent: the kernel of its monodromy representation determines the faithful quotient, and the S_3 point stabilizers of that quotient are conjugate. This proves sharpness of (37). Summing the endpoint coefficients gives $n_4 = \binom{2^{2g-1}}{2} - 2P$. At genus two the decomposition is $28 - 960 = -932$.

For the dihedral count, the central extension of the two binary reflection coordinates has class $b \smile a$. A pair a, b of independent characters lifts exactly when $\langle b \smile a, [C] \rangle = 0$. Then the lifts form a torsor for the 2^{2g} central characters. They are all surjective since the extension has no complement. The four inner changes over the fixed ordered coordinates act freely,

and interchanging the two reflection coordinates identifies the orders. Thus each admissible unordered pair determines 2^{2g-2} distinct quotient kernels. To reverse the plane sign choose a with $\ell(a) = 1$ and $b \neq 0$ with $\ell(b) = \langle a \smile b, [C] \rangle = 0$. These two linear conditions are independent since $a \neq w$. The resulting number of dihedral planes forced to cross is $2^{4g-3}(2^{2g-2} - 1)$. At genus two it is 96.

For the two-spin example with $g = 2$, $\epsilon = 1$, and nonzero spin difference, the first four least-genus exponents are $1, -8, -64, -932$. Here is a direct evaluation of the last entry. The disconnected signed cover series through degree four has coefficients $1, 1, 0, 9, 144$, since only the self-conjugate partitions contribute and the hook product for $(2, 1)$ is three and that for $(2, 2)$ is twelve. Its connected degree-four coefficient is $539/4$. The unsigned disconnected genus-two series has coefficients $1, 1, 8, 81, 1424$: at four the five squared hook products are $576, 64, 144, 64, 576$. Taking its logarithm gives $C_4(2) = 5275/4$; also $C_2(3) = 63/2$. Hence

$$n_4 = 539/4 - C_4(2) + 8C_2(3) = -932.$$

Appendix B. Denominators at fixed genus excess

Let C be an isolated smooth embedded index-zero curve of genus $g \geq 2$ whose normal operator is regular on all finite étale covers. Equivalently for the following argument, take a core in the all-cover super-rigid domain of Proposition 15.4 and use its cutoff-dependent isolated insertions. Let $L_{m,k}$ be the connected, fixed-target local Gromov–Witten degree at covering degree m and arithmetic source genus k. Write $h_m = 1 + m(g - 1)$. On super-rigid curves these numbers are the absolute obstruction Euler numbers of Section 3; the denominator argument uses the geometric elementary factorization, not a family-level numerical evaluation on analytic targets.

Theorem. Put $T_0 = 1$ and let $T_{2r} = \prod_{p-1 \leq 2r} p^{\lfloor 2r/(p-1) \rfloor}$, the product over prime p. For every $r \geq 0$,

$$\gcd(m, r)\, T_{2r}\, L_{m,h_m+r} \in \mathbb{Z}.$$

For $r > 0$ this gives the uniform bound rT_{2r} on denominators in the sense of divisibility. The only possible denominator primes are at most $2r + 1$, independent of the covering degree, deck groups and genus of the core. When $r = 1$ the sharper bound is $12/\gcd(12, g - 1)$, and the residue modulo integers is determined by the signs on the étale covers of degrees dividing m. The common genus-independent bounds are optimal for $1 \leq r \leq 4$.

Proof. Use the elementary connected series $G_a = \log Z_a$ for a core of genus $a \geq 2$, with the notation

$$y = qt^{2a-2},$$

$$Z_a(y, t) = \sum_{\lambda} y^{|\lambda|} H_\lambda^{2a-2} \prod_{b \in \lambda} \left(\frac{2(1 - \cos(\ell_b t))}{(\ell_b t)^2} \right)^{a-1}, \qquad H_\lambda = \prod_{b \in \lambda} \ell_b.$$

Here ℓ_b is the hook length of box b, the empty partition contributes one, and y, t are treated as independent formal variables. This is Ionel–Parker's elementary series after removal of its minimal power of t in each degree. The coefficient $[y^k t^{2s}] \log Z_a$ is the connected coefficient of covering degree k at source genus $1 + k(a - 1) + s$ [IP18] [Wue26b].

Let $\mathcal{R}$ be the subring of $\mathbb{Q}[[y, t^2]]$ in which a coefficient of t^{2s} has p-denominator exponent at most $\lfloor 2s/(p-1) \rfloor$ for every prime p, coefficientwise in y. It is a ring because these bounds are superadditive in s. The coefficient of t^{2j} in $2(1-\cos t)/t^2$ is $2(-1)^j/(2j+2)!$. For odd p the sum of the base-p digits of the positive even integer $2j + 2$ is even, hence at least two. Legendre's formula gives $v_p((2j+2)!) \leq 2j/(p-1)$. For $p = 2$, $v_2((2j+2)!) - 1 = 2j + 1 - s_2(2j+2) \leq 2j$.

Thus Z_a lies in $\mathcal{R}$, also after scaling the variable in each factor by the integral hook length. It has y-constant one and therefore has an inverse in $\mathcal{R}$ by y-adic geometric expansion. Put $u = t^2$ and let $b_{k,s} = [y^k u^s] \log Z_a$. Both $y\partial_y Z_a/Z_a$ and $u\partial_u Z_a/Z_a$ lie in $\mathcal{R}$. Their coefficients give

$$kT_{2s}b_{k,s} \in \mathbb{Z}, \qquad sT_{2s}b_{k,s} \in \mathbb{Z}.$$

Bézout's identity gives the multiplier $\gcd(k,s)T_{2s}$. This includes $s = 0$, where it recovers the denominator bound k for unsigned connected unramified covering counts.

Let $n_{d,a}$ be the integral elementary exponents of the geometric local product. They vanish for $a < h_d$. At degree m, set $k = m/d$. An exponent with $a = h_d + j$ contributes to L_{m,h_m+r} with relative excess $s = r - kj$. Negative s contributes zero. For $s \geq 0$ we have $\gcd(k,s) = \gcd(k,r) \mid \gcd(m,r)$ and $T_{2s} \mid T_{2r}$. Summing the elementary terms with their integer exponents proves the theorem's divisibility, including when $r = 0$.

The first positive excess. Put

$$Z_a^0(y) = \sum_\lambda H_\lambda^{2a-2} y^{|\lambda|}, \qquad V_a(y) = \sum_\lambda H_\lambda^{2a-2} \Big(\sum_{b \in \lambda} \ell_b^2 \Big) y^{|\lambda|}.$$

The t^2 coefficient of the elementary logarithm is $-(a-1)V_a(y)/(12Z_a^0(y))$. If $r = 1$ and $s = 0$, the equality $kj = 1$ gives only $k = j = 1$, contributing the integer n_{m,h_m+1}. Writing $n_d = n_{d,h_d}$ therefore gives

$$L_{m,h_m+1} \equiv -\frac{g-1}{12} \sum_{d \mid m} d\, n_d\, [y^{m/d}] \frac{V_{h_d}(y)}{Z_{h_d}^0(y)} \pmod{\mathbb{Z}}.$$

The quotient has integral coefficients. Each n_d is determined by the equivariant determinant signs on the finite covers by Theorem 7.1. This proves the stated sharper denominator and gives a computable residue in a source genus containing branched maps and ghosts.

In genus two the equal-spin core has $n_1 = 1$, $n_2 = 0$, while the same-parity distinct-spin core has $n_1 = 1$, $n_2 = -8$. Both base signs are positive. At $m = 2$ the first exponent supplies $-13/4$ to the genus-four count at both endpoints. The difference due to n_2 is $(-8)(-1/6) = 4/3$. Thus the change of the genus-four Euler degree is $1/3$ modulo $\mathbb{Z}$. For the distinct-spin core the residue is $1/12$; its reduced denominator is exactly 12. In particular the genus-independent bound at $r = 1$ is sharp. The first-excess congruence determines this residue modulo integers. The genus-four integral itself also depends on the degree-two branched contribution.

At excess zero there is an exact statement in the other direction. For every prime p, including 2, the connected local degree L_{p,h_p} has reduced denominator p. Indeed, all nonnormal index-p covers have deck order one. The regular cyclic covers correspond to the kernels of nonzero homomorphisms $\pi_1(C) \to \mathbb{Z}/p$, modulo the $p-1$ automorphisms of the target. Their number is $1 + p + \cdots + p^{2g-1}$, congruent to one modulo p. For odd p each cyclic cover has sign ϵ by the equivariant determinant signs of Section 7; for $p = 2$ both signs are one modulo two. Hence pL_{p,h_p} is prime to p. The sharp denominator 12 one genus higher and the unbounded prime denominators at excess zero concern ordinary connected local degrees, distinct from both the denominators for scalar equivariant weights and the integral balanced exponents of Proposition 3.1.

B.1. **Optimal common denominators through excess four.** On the equal-spin elementary genus-two curve the entire connected local series is G_2 by the geometric elementary comparison. Thus individual coefficients of its hook logarithm are realized geometric values. The following entries, together with the divisibility theorem above, give both the upper and lower common denominators across all degrees and all curves in this domain.

Excess r	Covering degrees	Exact elementary values in those degrees	Least common denominator rT_{2r}
1	1	$-1/12$	12
2	1, 2	$1/360$, $19/32$	1440
3	1, 3	$-1/20160$, $-8278/567$	181440
4	1, 2, 4	$1/1814400$, $211/48384$, $233871691/193536$	14515200

These rational numbers can be obtained directly as follows. Let $A_k = [y^k]Z_2$ with each sine squared expanded up to t^8. Then $b_{k,r}$ is the t^{2r} coefficient of A_1 for $k = 1$, of $A_2 - A_1^2/2$ for $k = 2$, of $A_3 - A_1A_2 + A_1^3/3$ for $k = 3$, and of $A_4 - A_1A_3 - A_2^2/2 + A_1^2A_2 - A_1^4/4$ for $k = 4$. The least common multiples of their reduced denominators attain the upper bounds.

Bryan and Pandharipande studied the étale contribution and the contribution of covers with two branch points in local algebraic curve theory. Their Lemmas C3–C4 include the unsigned index-p congruence at excess zero. The degree-one sine formula already gives the sharp first-excess denominator at fixed base genus. Here the input is the integral balanced product for the almost-complex curve germ, including cases with orientation-reversing covers. The bidegree divisibility, the sign-dependent residue, and the witnesses at positive excess use the elementary factors in that geometric product [BP01].

The prime support of the uniform bound is sharp at *every* positive excess. The degree-one coefficient on the genus-two elementary core is $2(-1)^r/(2r+2)!$. Its reduced denominator contains every prime at most $2r+1$, including 2 since $r \geq 1$. Together with the upper bound, this characterizes the union of possible denominator primes exactly in each excess [Pan99].

Appendix C. Horizontal leaves and their finite monodromy

We construct compact symplectic six-manifolds with integral first Chern class zero for which an entire face of the universal cycle series can be computed in every covering degree. The geometric factors have unbounded genera in a single compact example. In the corresponding local models, passing to all finite covers of the base recovers exactly the monodromy of any finite collection of sheets, and characterizes the integer multiplicities by subgroup congruences. We use the smooth universal cycle group S of the geometric product formula of [Wue26b]. Write $E_h = \Phi(A_h)$, where A_h denotes the balanced antidiagonal series. Its coefficient of order one is $\Phi(x_{h,1}) \neq 0$ [Wue26b].

C.1. **The local model in every genus.** Let $C = \Gamma\backslash\mathbb{H}$ be a closed Riemann surface of genus $g \geq 2$. Choose a lift of its uniformizing representation to $\mathrm{SL}_2(\mathbb{R})$. Such lifts correspond to spin structures: the obstruction to lifting the oriented tangent circle bundle to its spin double cover is the Euler characteristic modulo two, which vanishes. Let V be its flat real plane bundle in the standard representation. Choose the sign of its invariant area form so that the oriented plane has Euler number $g - 1$. Indeed the projective-line circle bundle of V is the ideal-boundary bundle of the hyperbolic surface. It is identified with the unit tangent circle bundle by sending a direction to the endpoint of its geodesic. Passing from oriented unit vectors in V to their unoriented lines doubles the Euler class up to orientation. Thus its absolute Euler number is $g - 1$. Choose a compatible complex structure on V varying

smoothly over C. On the flat bundle $V \oplus V$ take the product vertical complex structure and take the base complex structure on the horizontal distribution. The sum of the two constant vertical area forms in flat coordinates is a global closed form α. It is nonnegative on every complex tangent line and vanishes precisely on horizontal lines. In this vector bundle α is exact by the radial Liouville primitive. Every compact nonconstant holomorphic curve is consequently horizontal.

This argument is unchanged on a finite cover of the base, where each of the two normal complex lines has degree $g(T) - 1$ on the covering curve T, and their direct sum has degree $2g(T) - 2$. For a branched holomorphic cover $f : D \to C$, the graph equation for a normal section is the pullback of the real Cauchy–Riemann operator defined by the flat connection and the chosen fiber complex structure. Its kernel consists of parallel sections: the same exact area form on the pullback vanishes on the entire holomorphic graph. The image of $\pi_1(D)$ has finite index, since f lifts to the associated connected covering surface, where its image is both compact and open. Any finite-index subgroup of Γ contains a nonzero power of a hyperbolic element, which fixes no nonzero vector in the standard linear representation. Hence every pullback operator is injective. The two summands have the *same* real operator, including its orientation data. Furthermore a finite orbit in the linear fiber is just the origin, by applying a power of the same hyperbolic element. Thus the core is the sole irreducible curve in this model. The doubled elementary-neighborhood theorem of [Wue26b] identifies the actual characteristic insertion of its m-fold core with $[q^m]E_g$ for every m. This application uses the full universal elementary-neighborhood result.

For any finite Γ-set Ω form the flat suspension with fiber $\Omega \times (\mathbb{R}^2 \oplus \mathbb{R}^2)$, acting by permutation on the first factor and by the doubled representation on the second. This is a union of vector-bundle neighborhoods over the connected covers associated with the orbits. The reduced curves in it are precisely the zero sections of those covers. If O is an orbit of cardinality d_O, its curve D_O has genus $1 + d_O(g - 1)$. Retaining all actual homology labels, the support series is

$$\prod_{O \in \Gamma \backslash \Omega} E_{1+d_O(g-1)}(Q^{[D_O]}).$$

Indeed a cycle in any fixed total degree is a choice of nonnegative multiples of the disjoint zero sections. It gives the product of their open compact characteristic insertions by excision and disjoint union. No finiteness in source genus is used to identify this cycle space. In this class of repeated normal operators it is the geometry of the single zero section and all its multiples that fixes the factor attached to a transitive support.

C.2. **Detecting the support under finite base change.** Retain only the projection degree in the displayed product. The resulting series determines the multiset of cardinalities of the orbits. For two different such multisets, take the smallest cardinality d at which the counts differ. The difference of the coefficients of q^d is that nonzero integer difference times $\Phi(x_{1+d(g-1),1})$. Contributions from smaller orbits agree and the class just displayed is nonzero. Conversely equal cardinality multisets plainly give the same projected series.

For a finite-index subgroup $K \leq \Gamma$, pull the neighborhood back to the corresponding connected unramified cover C_K. Its sheet set is the restriction of the Γ-action to K. Its coefficient of projection degree one over C_K is

$$|\Omega^K|\, \Phi(x_{g(C_K),1}).$$

Thus the collection of these coefficients for all finite covers determines the entire isomorphism class of Ω as a finite Γ-set. To see this directly for two candidates take a common finite

quotient G through which both actions factor. Every subgroup $H \leq G$ is tested by its inverse image in Γ. If $m_H = |\Omega^H|$, Möbius inversion in the finite subgroup lattice gives the number of points with stabilizer *equal* to H as

$$t_H = \sum_{H \leq J \leq G} \mu(H, J) m_J.$$

These points occur in groups of $[N_G(H) : H]$ in each orbit isomorphic to G/H. For conjugacy-invariant integer data m_H, the conditions $t_H \geq 0$ and $[N_G(H) : H] \mid t_H$ are also sufficient. Take $t_H/[N_G(H) : H]$ disjoint copies of G/H for a representative of each conjugacy class and invert the formula. Dropping the inequalities gives the Grothendieck group, the additive group of the Burnside ring of finite G-sets. Under restriction these constructions are ordinary base change of finite covers; induction from K to Γ is composition of the corresponding covers. Fiber product of covers gives multiplication of the marks. Every finite set used in this inversion is geometrically realized by the preceding suspension, including when the union of neighborhoods is disconnected. The integrality conditions here come from stabilizers of actual sheets.

One series over C alone does lose information. Take two distinct index-two subgroups K_1, K_2 of the surface group. They are normal and are not conjugate. The two transitive two-sheet models give identical projected series $E_{2g-1}(q^2)$. On the cover for K_1 the first has two degree-one leaves while the second has none. This distinguishes geometric support data even when every coefficient in the original projected series agrees.

C.3. **A compact symplectic Calabi–Yau application.** Take a rational quaternion division algebra split at the real place, a maximal order Λ and a torsion-free principal congruence subgroup Γ of the norm-one units, of level divisible by an integer at least three chosen to eliminate torsion. The arithmetic Fuchsian quotient $C = \Gamma\backslash\mathbb{H}$ is compact. Under a real splitting, the four-dimensional real algebra is $M_2(\mathbb{R})$ and left multiplication is the direct sum of the standard representations on its two columns. The order is a rank-four lattice preserved by this action. Put

$$X = (\mathbb{H} \times (M_2(\mathbb{R})/\Lambda))/\Gamma.$$

The smooth torus bundle has a zero section C_0. Give its horizontal and vertical distributions the almost-complex structures above. The form α descends to the torus bundle. If ω_C is an area form on C, then $\alpha + \pi^*\omega_C$ is a compatible symplectic form. The first Chern class vanishes *integrally* since the complex tangent splitting is the pullback of $TC \oplus V \oplus V$ and its first Chern number on C is $2 - 2g + 2(g - 1) = 0$. These statements use the chosen almost-complex structure; the resulting class formula is in S. The classical quaternionic abelian surface construction supplies the lattice and the compact hyperbolic base [Voi21].

A pseudoholomorphic irreducible curve with α-area zero is horizontal by pointwise nonnegativity on its normalization. It lies in one horizontal leaf and covers that leaf, so its reduced image is a compact leaf of a finite orbit. One can see the last implication by lifting base paths in the leaf: the normalization maps nonconstantly into its covering surface, and is open and compact there. Conversely each finite orbit gives a compact embedded holomorphic leaf. The finite-orbit points in the torus are precisely its torsion subgroup. An orbit of size at most n is fixed by $\gamma^{n!}$ for a single chosen hyperbolic $\gamma \in \Gamma$. The matrix $\gamma^{n!} - 1$ is invertible on the real four-space and integral in Λ, so its kernel on the torus is a finite torsion group. Every torsion point lies in a finite invariant torsion subgroup of the torus. In particular only finitely many leaves occur in bounded projection degree.

A leaf associated with a point p of stabilizer Γ_p has an embedded neighborhood of the zero section in the local model over C_p. The map from its flat vector bundle to X sends v to $p+v$ in the torus. It is a local diffeomorphism along the compact section, embeds that section, and restricts to an embedding on a small tubular neighborhood by compactness. It preserves the horizontal and vertical complex structures. It follows that for every integral homology class β with $\langle\alpha,\beta\rangle = 0$ the universal insertion of the entire cycle space in that class has generating series

$$\sum_{\langle\alpha,\beta\rangle=0} [Z(X,J;\beta)]\, Q^\beta = \prod_{O\in\Gamma\backslash(M_2(\mathbb{R})/\Lambda)_{\mathrm{tors}}} E_{1+|O|(g-1)}(Q^{[D_O]}).$$

On the left the bracket denotes the image in S of the unit of that compact open-and-closed class locus, including the empty cycle. Homology labels of integral cycles are locally constant in the cycle topology, as in [Wue26b]; alternatively each contributing cycle has disjoint tubular neighborhoods with only its prescribed leaf and multiples. For the latter argument sufficiently small perturbations of the cycle keep the same component degrees in those tubes. Compactness in each class here also follows directly from the finite-orbit bound. Completion by projection degree on this face makes every coefficient finite and does not depend on the choice of area form on C. This face has infinitely many leaves with unbounded positive genus.

C.4. **Homology under multiplication and an infinite arithmetic family.** Let $P_n : X \to X$ be multiplication by n in the torus fibers. For any orbit in $X[n]$ its leaf of degree d maps with degree d to C_0. In particular the actual class $[D_O]$ satisfies $P_{n*}[D_O] = d[C_0]$. One also has the integral bound

$$n^3\big([D_O] - d[C_0]\big) = 0.$$

Indeed $\delta = [D_O] - d[C_0]$ projects to zero on $H_2(C;\mathbb{Z})$, hence is in filtration step F_1 of the homological Serre spectral sequence of the torus bundle. The edge quotient F_2/F_1 injects into $H_2(C;\mathbb{Z})$; the zero section splits the projection. On F_1/F_0 multiplication by n acts by n, and on F_0 by n^2, since those are subquotients of the terms with torus fiber degrees one and two. Thus $(P_{n*} - n^2)(P_{n*} - n)\delta = 0$. Since $P_{n*}\delta = 0$, the bound follows. It does not assert equality of torsion-distinct labels at the exceptional primes.

At any prime p away from the quaternion discriminant and the congruence level, reduction of Γ acts on $X[p] \simeq M_2(\mathbb{F}_p)$ by the full group $\mathrm{SL}_2(\mathbb{F}_p)$ on the left. This follows by strong approximation for the norm-one group at the split real place, with the prescribed congruence condition away from p and the chosen residue at p. The maximal local order is $M_2(\mathbb{Z}_p)$ and the reduction of its determinant-one group is surjective [Voi21].

The zero matrix gives the section. Rank-one matrices have $p+1$ orbits, indexed by their row line, each of size $d_1 = p^2 - 1$. After fixing one generator of that row line the column is any nonzero vector, on which SL_2 acts transitively. Invertible matrices have $p-1$ orbits, indexed by determinant, each of size $d_2 = p(p^2-1)$. The sizes add to p^4 after including orbit multiplicities and the zero matrix. The union of these leaves is exactly the preimage of the zero section under P_p. It is compact and has a product of the elementary contributions of its disjoint leaves in every multiple. For every such prime the genuine homology classes group by degree. Here is the integral check. Set $U = \mathbb{F}_p^2$. The reduced fiber lattice is $U \oplus U$ for the full group $\mathrm{SL}_2(\mathbb{F}_p)$, so it has no invariants. Thus $H^1(C;\Lambda)$ has no p-torsion by the long exact sequence for multiplication by p, and surface Poincaré duality gives the same assertion for $H_1(C;\Lambda)$. The coinvariants $(\bigwedge^2 \Lambda)_\Gamma$ have dimension three after reduction modulo p. Indeed the wedge of the direct sum decomposes as two determinant lines and $U \otimes U$; the latter has one-dimensional coinvariants since U is symplectically self-dual and the centralizer of the two elementary transvections consists of scalars. This works also in characteristic two. Over

the reals the coinvariants have dimension at least three from the two invariant column area forms and the invariant alternating form pairing the two columns by their determinant. Their integral rank is therefore three and there is no p-torsion. In the homological Serre sequence the only possible incoming differential to this fiber-degree-two term is from $H_2(C;\Lambda) = \Lambda^\Gamma = 0$ by Poincaré duality. The $(1,1)$ term is untouched and the zero section splits the base term. Hence $H_2(X;\mathbb{Z})$ has no p-torsion at every good prime. Applying the preceding n^3 bound for $n = p$ proves the claim. With $q = Q^{[C_0]}$, the series supported on that preimage, in the actual integral-class completion, is then

$$E_g(q)\, E_{1+d_1(g-1)}(q^{d_1})^{p+1}\, E_{1+d_2(g-1)}(q^{d_2})^{p-1}.$$

At every good prime the same expression holds after retaining just the projection degree. For orders divisible by a prime at the level or discriminant the individual orbit product retains all integral labels. The formula concerns the indicated finite preimage inside the horizontal face; other torsion orders also contribute to the entire face.

There is also a formula for the entire part of the horizontal face supported on points of order prime to the level and discriminant. At a good prime and exact order p^k, set $R = \mathbb{Z}/p^k$ and $u_k = p^{2k} - p^{2k-2}$. The exact-order points are the matrices over R with a unit entry. Put $t = \min(v_p(\det), k)$, with $t = k$ for zero determinant. Every orbit with this value has degree $u_k p^{k-t}$. The number of these orbits is

$$\begin{cases} \varphi(p^k), & t = 0, \\ \varphi(p^{k-t})(p^t + p^{t-1}), & 0 < t < k, \\ p^k + p^{k-1}, & t = k. \end{cases}$$

Indeed, if the first column is primitive, normalize it to the first standard basis vector by $\mathrm{SL}_2(R)$. There are u_k primitive columns. The other column becomes $(a, \det)$, and the stabilizer adds multiples of det to a. This gives p^t choices of its residue and p^{k-t} points per full fiber of that orbit. If the first column is nonprimitive, normalize the second one instead. Now $p \mid a$ and $p \mid \det$, giving p^{t-1} more residues when $t > 0$. Counting determinants proves the formula. For any composite good order n, strong approximation and the Chinese remainder theorem give the product of the prime-power actions on the product of the exact-order point sets. Choose one row of this formula at each prime dividing n and multiply the orbit degrees and the orbit counts separately. An orbit of resulting degree d contributes $E_{1+d(g-1)}(q^d)$; its exponent is the resulting count. The true label is always $d[C_0]$ by the integral calculation at the primes dividing n. The product over exact orders is locally finite by the single-hyperbolic-element argument above.

The construction identifies the geometric integral structure and the information lost on forgetting sheets in this class of higher-genus suspensions. Its local normal operators are doubled and injective on all covers. A higher-genus wall law at failure of that injectivity, and a complex-algebra lift of general compact symplectic face identities, require different arguments.

Appendix D. Isolation for maximal symplectic holonomy

Let C be a closed Riemann surface of genus $g \geq 2$. Consider a germ, along a marked horizontal section, of a flat fibration with symplectic four-dimensional fibers. Its local transition maps are symplectomorphisms independent of the base coordinate. On the horizontal distribution lift the complex structure of C, and on the vertical distribution take any smooth compatible complex structure. The derivative of holonomy at the marked point gives a

symplectic representation $\rho : \pi_1(C) \to \mathrm{Sp}(4, \mathbb{R})$ with associated flat bundle W. Assume the Calabi–Yau condition along the section,

$$c_1(W)[C] = 2g - 2.$$

In this normalization it is the positive maximal value for a surface-group symplectic representation of rank four. We prove that one neighborhood isolates the marked curve in *all* multiplicities, that its full series depends only on the deformation class of the linear holonomy, and that different linear holonomies can already be distinguished by the oriented count of the underlying curve. This statement concerns the smooth universal cycle algebra S.

D.1. **Uniform isolation and deformation to the linear representation.** There is no nonzero vector with bounded orbit under ρ. Suppose there is such a vector. On its span under all translates the representation has compact closure: choose finitely many translates as a basis; every group element and its inverse are uniformly bounded on them. If the restricted symplectic form on this invariant subspace has a nonzero radical, the radical is a nonzero flat isotropic subbundle I. The quotient $I^{\perp}/I$ is a flat symplectic bundle of real rank at most two. Topologically the remaining symplectic summand is $I \oplus I^{\vee}$ and its first Chern class vanishes on C. With a metric it is the complexification of a real bundle, whose first Chern class is two-torsion; $H^2(C; \mathbb{Z})$ is torsion-free. Milnor's inequality bounds the absolute Euler number of the remaining rank-two flat bundle by $g - 1$. If the restricted form has zero radical, the bounded subrepresentation is itself symplectic with compact holonomy, hence admits a flat unitary metric and has first Chern class zero on C. The symplectic orthogonal is again of rank at most two. Both alternatives contradict the displayed Chern number. The same proof applies to every finite-index subgroup by multiplicativity of the Chern number and of $g - 1$ [Mil58].

Choose a norm on a transverse Darboux chart. For each unit vector v there is a holonomy word w with $\|\rho(w)v\| > 2$. The strict inequalities and compactness of the sphere give a finite set of such words covering the sphere. In a compact smooth parameter family of marked germs with this Chern number one may do the same over the entire unit-sphere bundle in finitely many parameter charts. All chosen words are defined on a common smaller transverse ball. Their nonlinear maps differ from their derivatives by $o(\|x\|)$ uniformly. For every sufficiently small nonzero x at least one word therefore takes it to norm greater than $3\|x\|/2$. A nonzero finite invariant set contained in the ball is impossible by applying this at its point of maximum norm. Shrinking the tube about the entire marked section ensures that any horizontal curve contained in it determines such a full finite invariant set. In fact the same finite-word argument rules out any nonzero full orbit trapped in a strictly smaller ball by iteration.

The closed form given by the fiber symplectic form in the flat charts vanishes on the marked section. It is exact on the tube since that tube retracts onto the section. It is nonnegative on complex lines and vanishes precisely on horizontal lines. Any compact irreducible holomorphic curve in the tube is therefore horizontal and must be the marked curve. For every finite or branched cover of C, a solution of the pullback linear normal equation has a holomorphic graph in the flat vector bundle. Its vertical area vanishes by the radial primitive, so the solution is parallel. Its monodromy image has finite index by compactness and openness of the lifted covering map, hence there are no nonzero solutions. The original normal operator is the one from ρ with the chosen compatible structure along the section. Its Fredholm index on the identity cover is zero, so this marked map is regular.

In transverse Darboux charts conjugate every transition F by positive real dilations, using $F_t(v) = t^{-1}F(tv)$ for $0 < t \leq 1$ and $F_0 = dF(0)$. These are symplectic local maps

satisfying the same cocycle identities; they extend smoothly at zero on a common smaller tube. The pulled-back vertical complex structures likewise extend with value at the zero section. Equivalently these charts are the rescaling of the given smooth fibration germ along its marked section. The finite-word argument and the exact-area argument apply uniformly throughout the parameter interval. In every degree m the multiple section is therefore a compact open family of cycles, and its characteristic class is constant by the compact prism relation. Interpolating two compatible vertical structures gives the same result. A piecewise smooth path of actual maximal representations also gives a family of linear flat bundles and the identical compact prism. Conjugation changes the transversal by an isomorphism. Thus to each such deformation class one can assign the universal series of the marked support. The statements about consistency along paths and loops here concern a *persistent marked leaf* with this degree condition. Birth and loss of horizontal sections in an unmarked affine deformation are different questions. The proof requires no uniform genus bound on moduli spaces of stable maps.

For the double of a Fuchsian spin lift the value in every multiplicity is E_g by the elementary-neighborhood theorem, as in Section C. More generally, if the two identical summands are mixed by an oriented flat rotation representation on their multiplicity space, one may deform the rotation character of $\pi_1(C)$ to the trivial character in its torus of characters. The Chern number is unchanged along this path and the entire series is again E_g. The component decomposition of maximal symplectic representations is nontrivial; the preceding arguments do not identify the series for every component [Got01] [GM04].

D.2. **An oriented count distinguishing two maximal models.** Take the sum of two spin lifts of the same uniformizing representation, with theta characteristics θ_1, θ_2 on C. The Chern number, underlying smooth symplectic normal-bundle type and geometric support all agree with those of a doubled model. The determinant orientation of the regular core map is

$$(-1)^{h^0(\theta_1)+h^0(\theta_2)}.$$

For one lift let $z \in \mathbb{H}$, $y = \operatorname{Im} z$, and write (a, b) for real flat fiber coordinates. Give the plane its positive-degree complex structure using the complex coordinate $s = (a - \bar{z}b)/y$. Under a matrix $\left(\begin{smallmatrix} A & B \\ c & d \end{smallmatrix}\right)$ this transforms by $s \mapsto (cz + d)s$, so $s^2 dz$ is an ordinary differential and s is a spin coordinate. In this frame the normal graph operator and an interpolation to the holomorphic spin operator are

$$D_t s = \bar{\partial} s + t\, \frac{i\bar{s}}{2y}\, d\bar{z}, \qquad 0 \le t \le 1.$$

For $t = 1$ the formula follows by differentiating the coordinate of a constant flat vector and using the Leibniz rule for a real Cauchy–Riemann operator. If $D_t s = 0$, then $d(s^2 dz)$ is $2t|s|^2\, dx \wedge dy/y$ upstairs, hence defines a nonnegative density on C. Stokes forces $s = 0$ for $t > 0$. The real Fredholm index is zero, so all these operators are invertible.

At zero the kernel is $H^0(\theta)$ and the cokernel is $H^1(\theta) = H^0(\theta)^\vee$. The first derivative between these spaces pairs s with w by integration of $w\bar{s}/y$ times a nonzero constant against oriented coordinate area. This is the antilinear isomorphism of a positive Hermitian form up to that constant. In the canonical complex orientations on kernel and cokernel its real determinant has sign $(-1)^{h^0(\theta)}$. In a kernel-complement splitting the other Fredholm block is invertible and has the complex orientation at zero; the reduced map is t times this isomorphism plus $O(t^2)$. This computes the canonical orientation sign at small positive t, and invertibility up to $t = 1$ preserves it. Signs multiply for the two summands. Changing auxiliary compatible complex structures in the flat linear model preserves invertibility by the exact-area argument.

Even and odd theta characteristics both exist in every genus $g \geq 2$. Choosing opposite parities gives a negative regular mark; choosing the same lift twice gives a positive one. Their degree-one characteristic insertions are *different elements of* S. If they agreed, proper pullback at source genus g would leave just their two regular embeddings, by the minimum genus and the unstable-leaf argument. The compact witness construction of [Wue26b] transports these points to constant parameter balls in one family. Its oriented local functional gives their respective determinant signs, contradicting the asserted equality. The regular map has trivial stabilizer in both cases. This distinguishes the full local series without assuming a universal numerical evaluation, and without identifying the negative series with an inverse elementary series. The familiar spin parity enters through an orientation of an actual stable map [Wue26b] [Ati71].

For opposite parities the quotient θ_2/θ_1 is a nontrivial flat two-torsion line. On its connected unramified double cover, the two spin lifts become identical. The entire pulled-back series is therefore E_{2g-1}. The original negative count and this positive elementary pullback give a concrete case in which the covering operation carries information beyond the scalar count on the base. The finite-cover signs and the resulting least-genus coefficients are computed in Appendix F.

Appendix E. The normal operator of a maximal surface representation

Let C be a closed Riemann surface of genus $g \geq 2$. Let a flat symplectic four-dimensional fiber germ have a marked horizontal section over C, and give its flat horizontal and vertical distributions the split almost-complex structures. Assume the normal symplectic bundle has first Chern number $2g-2$. Write $P_\rho(Q)$ for the series of compact open insertions of the multiples of the section in the smooth universal cycle algebra S, with constant term one. Here ρ denotes the derivative holonomy. A nonlinear germ deforms to this linear model without losing isolation in any degree. This follows from uniform escape of nonzero transverse orbits and exactness of the vertical form on a tube.

Fix a theta characteristic $\theta^2 = K_C$. In the harmonic reduction of ρ, write V for its positive-degree holomorphic rank-two bundle and $\gamma : V \longrightarrow K_C V^*$ for its symmetric lower Higgs block. Maximality makes γ an isomorphism. The resulting orthogonal rank-two bundle $F = V \otimes \theta^{-1}$, with γ untwisted, is the orthogonal bundle underlying its Cayley partner. Write $w = w_1(F)$ and $e = \langle w_2(F), [C] \rangle$. If $w = 0$, the two isotropic lines split F holomorphically as $N \oplus N^{-1}$. Put $c = |\deg N|$. The reduction, the component classification and these characteristic classes are those of Bradlow–García-Prada–Gothen; the results here concern the resulting classes of local curve counts [BGG12].

Theorem. The connected genus-g degree-one test of P_ρ is $(-1)^{h^0(C,V)}$. On the components above it equals

$$\begin{cases} (-1)^c, & w = 0, \\ (-1)^{q_\theta(w)+e}, & w \neq 0, \end{cases} \qquad q_\theta(a) = h^0(\theta \otimes \ell_a) - h^0(\theta) \pmod 2,$$

where ℓ_a is the flat sign line of $a \in H^1(C; \mathbb{F}_2)$. When $w = 0$, the full series in S equals the image of the holomorphic antidiagonal local series for a pair of line bundles of degrees $g-1+c$ and $g-1-c$ with product K_C. When $w \neq 0$, its pullback *as a neighborhood* to its connected orientation double cover has full series E_{2g-1}, the balanced elementary factor there. More generally the genus-$2g-1$ degree-one test on the connected double cover classified by $0 \neq a \in H^1(C; \mathbb{F}_2)$ is $(-1)^{\langle a \smile w, [C] \rangle}$. These last signs recover w. For $w \neq 0$ the downstairs sign then recovers e once θ is fixed.

The expression of $h^0(\theta \otimes F)$ modulo two in terms of w_1, w_2 is the rank-two case of the established orthogonal spin index formula. The geometric identification here is with the determinant of the suspended core and its universal degree-one test [BS18].

The sign gives actual distinctions among series in S. For example, two copies of one uniformizing spin lift give positive degree-one sign. The direct sum of lifts belonging to opposite spin parities has negative degree-one sign. At its least source genus equality of the two degree-one insertions would identify two regular marks with opposite determinant orientations. The multiplicity-one detection theorem [Wue26b, Theorem 5.1] excludes this equality. No identification of the full negative-sign series with the inverse of the positive one follows from that test [Wue26b].

E.1. **The real normal operator from the harmonic reduction.** Every maximal representation here is completely reducible. Indeed the elementary Euler bound excludes any nonzero invariant isotropic subspace; the radical of any invariant subspace would be such a subspace. The orthogonal complement therefore splits off each invariant subspace. Nonabelian Hodge theory supplies a harmonic reduction to U(2). Its compatible vertical complex structure may replace the original one along a path of compatible linear structures: the flat fiber form still makes all compact curves in the tube horizontal. The associated rank-two bundle V has $\deg V = c_1(W)[C] = 2g - 2$. In a unitary frame the complexification of the real standard representation is $V \oplus \bar{V}$, where the Hermitian metric identifies $\bar{V}$ with V^*. A real vector with first coordinate s has second coordinate $\bar{s}$. In these coordinates the complex flat connection is $d_A + \Phi + \Phi^*$ with A unitary and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$

In particular the $(0,1)$ operator on the first coordinate of a real section is

$$Ds = \bar{\partial}_V s + R_h(s), \qquad R_h(s) = h^\sharp\big(\gamma(s, \cdot)\big).$$

Here $h^\sharp : V^* \to V$ is the conjugate-linear metric identification; it conjugates the differential on the canonical factor as well. To check this independently in a flat real connection, its Cartan decomposition in unitary frames has a unitary part A and a symmetric complex-antilinear part $B \circ \mathrm{conj}$. Write the first-row coefficient of the latter as $P\,dz + R\,d\bar{z}$. The conjugate row has $(1,0)$ term $\bar{R}\,dz$, namely γ. Symmetry gives $R = \overline{\gamma_z}$ in this unitary frame. The $(0,1)$ projection on the vertical complex coordinate is exactly $\bar{\partial}_A s + R\bar{s}\,d\bar{z}$. This avoids a dependence on a choice of scalar in the identification of the standard symplectic form with its block matrix. The Hermitian sign and V convention can be checked on a uniformizing plane, where it gives the degree-$g-1$ spin line [BGG12].

The graph equation for a smooth section of the flat bundle is $Ds = 0$. More generally a graph over a holomorphic map $f : D' \to C$ satisfies the pullback equation (in the zeroth-order term the base differential is pulled back). Conversely the holomorphic structure on V and any smooth conjugate-linear $R : V \to \Lambda^{0,1}C \otimes V$ determine a fiberwise-linear almost-complex structure on its total space with holomorphic projection and standard complex fibers. In bundle coordinates its $(0,1)$ directions are the vertical $(0,1)$ directions and a lift of $\partial_{\bar{z}}$ whose vertical $(1,0)$ part is $-R_{\bar{z}}(s)$. This is intrinsic because a change of holomorphic bundle frame gives the Dolbeault transformation rule. Graphs satisfy $\bar{\partial}s + R(s) = 0$. The harmonic flat structure corresponds to the displayed R_h. These structures are tamed on a common small disk bundle in any compact parameter family: use a closed form consisting of a sufficiently positive base area form and a fiberwise symplectic form obtained by differentiating the metric radial primitive with a unitary connection. On the zero section the horizontal-vertical splitting

is complex; tameness persists by compactness on a tube. Its first Chern number stays zero on the section and hence throughout the tube.

For any Hermitian metric h and $t > 0$ the structure with equation $D_t = \bar\partial_V + tR_h$ has only the compact reduced core. Indeed on the normalization of a compact curve projecting nonconstantly by f, its fiber coordinate is a section $s \in \Gamma(f^*V)$. The symmetric form paired with s, s and then pulled through df gives a globally defined $(1,0)$-form. In a local coordinate u on the normalization, with $\gamma = G\,dz$ in unitary frames, its $\bar\partial$ is

$$-2t\,\|G(f)s\|^2\,|f_u|^2\,d\bar u \wedge du.$$

To compute in unitary frames use the metric compatibility of A and $\bar\partial_A \gamma = 0$; equivalently differentiate the holomorphic symmetric form and contract both copies of $\bar\partial_{f^*V} s$. In nonunitary holomorphic frames the same expression includes the positive inverse Hermitian metric in the norm. Stokes' theorem forces s to vanish on the unramified open set and then everywhere, since G is invertible. A compact curve in a single complex affine fiber is constant. In particular the only degree-m cycle is $m[C]$ for all positive t, including through an arbitrary smooth homotopy of the metric. This argument uses actual cycles and every branched normalization.

E.2. **Determinant signs.** For the identity map $C \to C$, the complex-linear operator $D_0 = \bar\partial_V$ has index zero. Its kernel is $H^0(V)$ and its complex cokernel is $H^1(V)$. By the isomorphism γ and Serre duality the cokernel is the dual of $H^0(V)$. The first-order map from kernel to cokernel for D_t is induced by R_h. Pairing its value at s with the dual section $\gamma(r,\cdot)$ gives, up to a common nonzero scalar from the area convention, the integral of the Hermitian inner product of $\gamma(r,\cdot)$ and $\gamma(s,\cdot)$. This is a nondegenerate conjugate-linear identification. Splitting kernel and cokernel off by complex complements gives a finite-dimensional reduced operator $t[R_h] + O(t^2)$; the complementary block remains invertible with its complex orientation. The real determinant of a conjugate-linear isomorphism of complex d-planes has sign $(-1)^d$. The Stokes argument makes D_t invertible for every $t > 0$, so the computed sign persists at $t = 1$. This is the determinant sign of the isolated regular graph, since varying the source cancels the tangent operator of the degree-one map to C. Its automorphism group is trivial and the source of genus g leaves no contracted handles or nodes. The construction generalizes the Lee–Parker real spectral-flow computation for one spin line [LP13] [DW23a].

If $w = 0$ the oriented orthogonal group in complex rank two is $\mathbb{C}^\times$ acting in reciprocal characters. Hence $V = \theta N \oplus \theta N^{-1}$. By Serre duality the two zeroth cohomology dimensions are $h^0(\theta N)$ and $h^1(\theta N)$. Their sum modulo two is $\chi(\theta N) = \deg N$. This computes the signs on the balanced component, the exceptional oriented components $1 \le c \le 2g-3$, and the Hitchin components $c = 2g-2$, with the standard integer convention of the classification. In particular all Hitchin components have positive degree-one test; individual odd c have negative test.

For $w \neq 0$ use a split orthogonal representative $\ell_b \oplus \ell_{w+b}$ with $\langle b \smile w, [C]\rangle = e$. One exists for either e by nondegeneracy of the surface cup product. These examples are the tensor products of a fixed uniformizing lift for θ with two real sign characters. Their Cayley invariants are $w_1 = w$ and $w_2 = b \smile (w + b)$ by the Whitney formula and $b^2 = 0$. They represent the components labeled by those invariants. Alternatively within flat $\mathrm{O}(2)$ twists, after fixing the determinant, representations are circle cocycles with action by this determinant. The exponential sequence of twisted coefficients shows that the obstruction distinguishing components is the twisted Euler class in $H^2(C;\mathbb{Z}_w) = \mathbb{Z}/2$; equal classes differ by a cocycle lifting to a real cocycle and can be joined by scaling the lift, up to a rotation coboundary. Its mod-two reduction is w_2 [BGG12].

On this split model $h^0(V) = h^0(\theta\ell_b) + h^0(\theta\ell_{w+b})$. The Riemann–Mumford identity $q_\theta(b + w) = q_\theta(b) + q_\theta(w) + \langle b \smile w, [C]\rangle$ proves the sign formula. Negative and positive signs occur even at fixed nonzero w. The sign persists on a representation component by the geometric continuation above, or directly by continuity of the invertible normal operator. The same calculation gives the stated two-spin test without invoking the component classification [Mum71].

E.3. **All multiples for the oriented Cayley bundle.** Write $L_+ = \theta N$, $L_- = \theta N^{-1}$, so $L_+L_- = K_C$. In this splitting γ pairs the two lines and vanishes on each individual line. Move the harmonic metric through Hermitian metrics to one making the lines orthogonal, using $\bar\partial_V + R_h$ throughout. Stokes keeps every multiple isolated; at this stage no circle equivariance is needed. For the final metric R_h sends the conjugate of each line to the other. Hence the circle acting on (L_+, L_-) with weights $(+1, -1)$ commutes with $D_t = \bar\partial_V + tR_h$ for every $0 \le t \le 1$. The endpoint at $t = 0$ is the ordinary holomorphic total space of $L_+ \oplus L_-$. At every $t > 0$ there are only the core cycles.

For completeness fix m and choose enough base evaluations of the elementary symmetric polynomials of the two coordinate multisets that their common zero in the integrable cycle space is $m[C]$. Each order-i coefficient is a holomorphic section of L_+^i or L_-^i, by the characteristic polynomial on the finite direct image of the normalization with cycle multiplicities. More than $m \max(0, \deg L_+, \deg L_-)$ distinct points suffice. The resulting test weights are $+i$ and $-i$. Let n be the total number of scalar tests. On the finite circle Borel stage $\mathbb{P}^n$, take their Thom product divided by the product of their signed weights. It defines the complex antidiagonal degree-m insertion for these two line bundles. The original balanced construction carries over with this explicit degree bound. It depends only on the unordered degrees and genus, by continuation in connected Picard varieties and in the moduli of smooth genus-g curves, using sufficiently many moving evaluations locally and tuple stabilization on overlaps. One may exchange the labels by reindexing and reversing the circle generator; in the Borel quotient complex conjugation on $\mathbb{P}^n$ has degree $(-1)^n$, which cancels reversal of the n signed weights. Equivalently fix the larger degree as the positive label throughout [Wue26c].

In a small invariant disk over the whole interval the relative cycle space is the union of its integrable endpoint and the interval of core cycles. Both are closed subspaces by the compact-cycle topology. The map which fixes that endpoint and collapses the core interval onto $m[C]$ is continuous by pasting and equivariant. Its inverse image of a compact subset is compact, being the union of that subset and a compact interval if the core belongs to it. On the finite Borel stage the analogous proper retraction pulls back the normalized Thom insertion with compact support. The invariant metric gives the associated disk bundle; its integrable endpoint may equally be identified, by open excision at the compact supported core locus, with the holomorphic Borel family of the full split bundle. On $t > 0$ the restriction is the ordinary Euler class divided by the signed weights, namely h^n with $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Its pushforward is the unit of the isolated multiple at a point. The smooth-family interval relation therefore identifies each original coefficient with the image of the corresponding complex antidiagonal coefficient. The homotopy from the original harmonic metric preceded this equivariant interval, so its failure to preserve the circle is immaterial [Wue26b].

In particular $c = 0$ gives E_g as in Section C. For $c > 0$ the result is a geometric comparison with the *unbalanced* antidiagonal construction. Its algebraic numerical degree-two residue is the balanced one times $\cos(cu)$ by the Bryan–Pandharipande local-curve formula. Distinguishing the full smooth classes through a numerical evaluation requires a justified comparison of

that evaluation in its stated domain. The geometric identification just proved uses no such evaluation [Wue26c].

E.4. **Covers and the limits of the comparison.** For $w \neq 0$, pullback to the connected orientation double cover $p : \widehat{C} \to C$ splits $p^*F = N \oplus N^{-1}$. Its deck involution exchanges the two isotropic lines, so $\deg N = -\deg N$ and $\deg N = 0$. The pullback theta characteristic squares to $K_{\widehat{C}}$ since p is étale. The pulled-back harmonic reduction solves the harmonic equations, and the preceding proof gives the balanced factor E_{2g-1}. This statement concerns the neighborhood on the covering base. A circle rotating the two ordered lines is sent to its inverse by the deck involution. The argument does not descend that circle construction or determine the full downstairs series on the components with $w \neq 0$.

For an arbitrary connected double cover p_a the normal determinant sign can be computed at the split sign representative when $w \neq 0$. Since $p_{a*}\mathcal{O} = \mathcal{O} \oplus \ell_a$, its parity exponent is the sum of the four parities for spin lines indexed by $b, w+b, b+a, w+b+a$. The quadratic identity shows that this sum is $\langle a \smile w, [C]\rangle$. If $w = 0$, the oriented split integer doubles under pullback and again gives the asserted sign. Thus the full collection of connected double-cover degree-one tests recovers w by nondegeneracy of the intersection form. Within the nonoriented components these tests together with the downstairs degree-one test recover the two component labels relative to θ. These are tests on a system of neighborhoods under base change. They make no assertion that a single downstairs series distinguishes every component.

Corollary for the unmarked base. Among the components with $w \neq 0$, the unmarked downstairs series has precisely two values. Its degree-one sign determines which of the two occurs. Indeed, for each fixed nonzero w the component classification gives two components, and the two signs distinguish them. An orientation-preserving diffeomorphism pulls the flat target and its section isomorphically to the corresponding neighborhood over the pulled-back base. Changing that complex structure back along a path preserves the full unmarked series: the exact vertical form and isolation argument are unchanged, and the tangent first Chern number is constant. Such diffeomorphisms act transitively on nonzero mod-two cohomology classes of a closed oriented surface (symplectic changes of homology basis realize this transitivity). They preserve the sign of the regular core. Consequently all the components of a given sign are equivalent for this unmarked series, and opposite signs give different values. This does not give the higher coefficients of either value. The earlier detection of w uses the indexed collection of covers of a marked base.

Appendix F. The signed character formula for horizontal curves

Let C be a marked closed Riemann surface of genus $g \geq 2$. Take the flat rank-four normal model given by the sum of two lifts of the same uniformizing representation. Write θ_1, θ_2 for their theta characteristics, $e \in H^1(C; \mathbb{F}_2)$ for the flat two-torsion line $\theta_2\theta_1^{-1}$, and

$$\epsilon = (-1)^{h^0(\theta_1)+h^0(\theta_2)}.$$

For any finite degree-m étale cover $f : D \to C$, let P_f be the real permutation local system on its sheets. Its first Stiefel–Whitney class $w_1(P_f)$ is the sign of the monodromy permutation. The determinant orientation on the map to the marked core has sign

$$\sigma(f) = \epsilon^m (-1)^{\langle e \smile w_1(P_f), [C]\rangle}. \tag{38}$$

For a disconnected cover the orientation is the product of its component orientations. The formula determines the connected, fixed-target local Gromov–Witten degrees at minimum

covering genus in every degree. For a fixed marked base, the regular degree-one signs of the pulled-back germs on $2g$ double covers together with the original sign determine (38) for every cover.

F.1. **The fixed-target degree.** The local threefold has a neighborhood in which the unique reduced compact curve is the core. For a connected stable map of degree m to this curve, each positive-degree normalized component of degree m_i has genus at least $1 + m_i(g-1)$. The arithmetic genus is the sum of the normalized component genera and the first Betti number of the dual graph. Equality with $1 + m(g-1)$ therefore requires a single nonconstant component, étale over C, with a tree dual graph and no positive-genus contracted components. A nonempty contracted rational tree would have an unstable leaf. Thus all maps in this degree and minimum source genus are precisely the smooth connected étale covers. Such maps form a finite groupoid. In the disconnected convention each connected component must have positive degree, and total Euler characteristic $m(2-2g)$ forces equality on every connected component. See Section D.

For an étale cover the source deformation complex cancels the tangent summand $T_D \simeq f^*T_C$ with its complex orientation. Each real normal line operator is injective by the exact-area calculation on a pullback and has real index zero since $\deg f^*\theta_i = g(D) - 1$. Thus there is no obstruction. Its determinant sign is $(-1)^{h^0(f^*\theta_i)}$. Indeed the étale domain has its pulled-back uniformization, so the path to the spin Dolbeault operator of Section D applies on D. At the Dolbeault endpoint the derivative from its complex kernel to its cokernel is the conjugate-linear isomorphism given, up to a nonzero complex scalar, by the positive Hermitian pairing. Its real determinant has the stated sign. The two normal signs multiply. Deck transformations act on the determinant of the regular point with its canonical zero-dimensional orientation. Its fixed-target local GW degree is consequently $\sigma(f)/|\operatorname{Aut}_C(f)|$. Adding positively oriented complex transverse tests in independent parameter directions multiplies the orientation by $+1$. For smooth tests of either orientation, the additional factor is their ordinary local degree. The local curve orientation here is the product of two spin parities. The usual spin Hurwitz number weights a cover by the parity of one spin structure [LP12].

Along a path of marked maximal symplectic holonomies the restriction to any one finite-index surface subgroup remains maximal. Its fixed-fiber normal operator stays injective and has index zero. Its sign therefore stays constant even if the intervening model has no splitting into line operators. The degrees computed here are chamber invariants for paths containing one of the stated models.

F.2. **Transfer of the quadratic refinement.** Let $q_\theta : H_1(C;\mathbb{F}_2) \to \mathbb{F}_2$ be the spin quadratic refinement of the intersection pairing, and write $x = \mathrm{PD}(e)$. Spin parity is the Arf invariant, and twisting θ by e adds the linear form of intersection with x to q_θ. The symplectic-basis formula for the Arf invariant gives

$$h^0(\theta \otimes e) + h^0(\theta) \equiv q_\theta(x) \pmod 2.$$

These conventions give $q = 0$ on a simple circle with bounding induced spin structure. For a simple oriented circle c, $q_\theta([c])$ is one plus the evaluation of the spin double cover on the tangentially framed circle [Joh80] [Ati71].

Suppose $e \neq 0$. Represent x by a simple nonseparating circle c, for example by lifting x to a primitive integral homology class and choosing a corresponding simple representative. Its inverse image represents $\mathrm{PD}(f^*e)$ and is a union of disjoint circles c_i of degrees ℓ_i. With pulled-back metrics the tangent framing on c_i covers the tangent framing on c with degree ℓ_i.

Consequently

$$q_{f^*\theta}([c_i]) = 1 + \ell_i(q_\theta(x) + 1) = \ell_i q_\theta(x) + \ell_i - 1 \pmod 2.$$

Intersection terms vanish on the disjoint lifts. The sum of $\ell_i - 1$ is m minus the number of cycles of the monodromy around c, hence is the exponent of its permutation sign. Applying the preceding parity identity on the source with $\theta = f^*\theta_1$ proves (38). For $e = 0$ both spin structures agree, and both sides equal $+1$. This uses only transfer of an unramified embedded circle. The more general parity problem for odd branched coverings involves the lifting of the full covering monodromy [EOP08].

F.3. **The character sum in every degree.** Let c_m be the sum of $\sigma(f)/|\operatorname{Aut}_C(f)|$ over connected covers of degree m. Let b_m be the corresponding sum over all covers, with $b_0 = 1$. On a union of r equal connected components the automorphism order is the product of the r deck group orders times $r!$. Permuting the factors of index zero introduces no determinant sign. The decomposition into connected components gives

$$B(t) = 1 + \sum_{m\geq 1} b_m t^m = \exp\left(\sum_{m\geq 1} c_m t^m\right). \tag{39}$$

For a partition $\lambda \vdash m$, let d_λ be the dimension of its complex irreducible representation of S_m, and λ' its transpose. Then

$$b_m = \epsilon^m \sum_{\substack{\lambda\vdash m \\ e=0 \text{ or } \lambda=\lambda'}} (m!/d_\lambda)^{2g-2}, \qquad m \geq 1. \tag{40}$$

In particular (39)–(40) compute each c_m by one formal logarithm. Each b_m is an integer since $m!/d_\lambda$ is the hook-length product of λ.

Here is the character calculation including its normalization. Label one fiber of the cover, set $G = S_m$, $n = |G|$, and choose the first surface generator to be the circle c when $e \neq 0$. The signs after removing ϵ^m are $\chi(a_1)$, where $\chi = \operatorname{sgn}$ for $e \neq 0$ and $\chi = 1$ for $e = 0$. A tuple $(a_i, b_i)_{i=1}^g$ defines a labeled cover exactly when $\prod_i [a_i, b_i] = 1$. Dividing the weighted tuple count by n gives the required automorphism-weighted count, by orbit–stabilizer under relabeling.

In the complex group algebra put

$$K_\chi = \sum_{a,b\in G} \chi(a)\,[a, b], \qquad [a, b] = aba^{-1}b^{-1}.$$

This is central. On an irreducible unitary representation V of dimension d and character ψ, the sum over b conjugating a^{-1} is $n\overline{\psi(a)}I/d$. Taking the trace and dividing by d hence gives the scalar of K_χ as

$$\frac{n}{d^2}\sum_a \chi(a)|\psi(a)|^2 = \left(\frac{n}{d}\right)^2 \mathbf{1}_{V\otimes\chi\simeq V}.$$

The identity coefficient of $K_\chi K_1^{g-1}$ is its regular trace divided by n. Dividing once more by n for relabeling, the contribution of V is $(n/d)^{2g-2}$ precisely when $V \otimes \chi \simeq V$. The sign character sends V_λ to $V_{\lambda'}$, proving (40). The untwisted calculation is the ordinary surface-group character formula [Klu25].

In degree one, $b_1 = c_1 = \epsilon$. In degree two there is no self-conjugate partition. Thus if $e \neq 0$, then $b_2 = 0$ and $c_2 = -1/2$. Directly, the connected double covers correspond to the nonzero characters $a \in H^1(C; \mathbb{F}_2)$ and have deck order two. The sum of $(-1)^{\int_C e\smile a}$ over those

characters is -1. For $e = 0$ the corresponding counts are $b_2 = 2^{2g-1}$ and $c_2 = (2^{2g} - 1)/2$. In degree three the only self-conjugate partition is $(2, 1)$, of dimension two. Thus for $e \neq 0$ we get $b_3 = \epsilon^3 3^{2g-2}$ and $c_3 = \epsilon^3(3^{2g-2} + 1/3)$. As a direct finite check, at $g = 2$, with the overall factor ϵ^m removed, enumeration of the quadruples in S_m^4 with $[a_1, b_1][a_2, b_2] = 1$ and weight $\operatorname{sgn}(a_1)$ gives disconnected counts $1, 0, 9$ and transitive counts $1, -1/2, 28/3$ for $m = 1, 2, 3$ after division by $m!$.

F.4. **Which marked chamber data are detected.** For the connected double cover classified by $a \neq 0$, the orientation of the embedded degree-one core in the pulled-back threefold is $(-1)^{\langle e \smile a, [C] \rangle}$ by (38). There is no deck denominator for that embedded map to its own target. By nondegeneracy of the mod-two intersection form, the signs on the covers for a basis of $H^1(C; \mathbb{F}_2)$ recover e. The mark on the original base gives ϵ. Equality of degree-one universal insertions on each of these corresponding marked targets forces equality of their signs by the regular-point detection argument in Section D. These finitely many measurements suffice for all individual étale signs and all the fixed-target degrees (39)–(40). The unlabelled sequence (40) uses only ϵ and whether $e = 0$.

On a marked genus-g base there are $1 + 2(2^{2g} - 1)$ different realizable sign profiles of this form. At $e = 0$ only $\epsilon = +1$ occurs. For $e \neq 0$, twisting both theta characteristics by a class a with $\int_C e \smile a = 1$ reverses ϵ and preserves e. Distinct profiles obstruct a deformation through marked maximal germs. This conclusion distinguishes finitely tested chambers and leaves possible additional invariants within a sign profile. The degrees c_m concern the minimum source genus in a fixed target. The full multiplicity-m universal insertion also includes information from other source genera and parametrized targets.

F.5. **All positive maximal flat holonomies.** Let $\rho : \pi_1(C) \to \mathrm{Sp}(4, \mathbb{R})$ have positive maximal Chern number $2g - 2$, possibly arising from the linearization of a nonlinear flat germ along a marked section. Fix a reference theta characteristic θ. Let F be its orthogonal Cayley bundle, $w = w_1(F)$, and ϵ_ρ the determinant sign of the embedded core. Then (38) holds with (e, ϵ) replaced by (w, ϵ_ρ) for every étale cover. The disconnected degrees are given by (40) with all partitions when $w = 0$ and only the self-conjugate partitions when $w \neq 0$, with overall factor ϵ_ρ^m. The logarithm (39) gives the connected degrees. See Section E.

For $w \neq 0$, the representation component contains a pair of uniformizing spin lifts whose difference character is w. For each fixed connected étale f, restriction of the deformation path to its finite-index surface subgroup stays maximal. All normal operators on that cover are invertible throughout the path, by exactness of the vertical area form and index zero. Thus the sign computed on the mixed-spin representative carries over. For $w = 0$ the oriented splitting in Section E has parameter $c = |\deg N|$, with $\epsilon_\rho = (-1)^c$. On a degree-m cover the two oriented lines pull back and the absolute degree becomes mc. The sign is $(-1)^{mc} = \epsilon_\rho^m$, as required. Deforming a nonlinear flat germ to its derivative preserves the core, its unramified normal operators and their orientations. Products give the disconnected case.

One can also apply the rank-two orthogonal spin index on every connected source. The parity exponent on the base is $q_\theta(\mathrm{PD}(w)) + \int_C w_2(F)$, which for an oriented F is its oriented plane degree modulo two. On the degree-m source the second term multiplies by m. The first term changes by the circle-transfer calculation above, giving again the sign $\epsilon_\rho^m (-1)^{\int_C w \smile w_1(P_f)}$. This check also works when $w \neq 0$ downstairs and $f^* w = 0$ [BS18].

For all positive maximal germs, the $2g$ double-cover signs recover w and the base test supplies ϵ_ρ. These 2^{2g+1} possible profiles are all realized. The nonzero w profiles were constructed above, and the oriented components with $c = 0, 1$ give the two signs at $w = 0$.

The unlabelled numerical degrees have precisely four possibilities at fixed g: c_1 determines ϵ_ρ, and c_2 distinguishes $w = 0$ from $w \neq 0$. These numerical distinctions use ordinary regular fixed-target degrees. The classification of full unmarked universal series on the nonoriented components in Section E uses an additional geometric and mapping-class argument.

References

[Ati71] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 47–62, doi:10.24033/asens.1205.

[BS24] S. Bai and M. Swaminathan, *Bifurcations of embedded curves and toward an extension of Taubes's Gromov invariant to Calabi–Yau 3-folds*, Duke Math. J. **173** (2024), 2947–3057, doi:10.1215/00127094-2023-0070. Numbering as in arXiv:2106.01206v4.

[BS18] D. Baraglia and L. P. Schaposnik, *Monodromy of rank 2 twisted Hitchin systems and real character varieties*, Trans. Amer. Math. Soc. **370** (2018), 5491–5534, doi:10.1090/tran/7144.

[BGG12] S. B. Bradlow, O. García-Prada and P. B. Gothen, *Deformations of maximal representations in* $\mathrm{Sp}(4,\mathbb{R})$, Q. J. Math. **63** (2012), 795–843, arXiv:0903.5496.

[BP01] J. Bryan and R. Pandharipande, *BPS states of curves in Calabi–Yau 3-folds*, Geom. Topol. **5** (2001), 287–318, doi:10.2140/gt.2001.5.287.

[BP08] J. Bryan and R. Pandharipande, *The local Gromov–Witten theory of curves*, J. Amer. Math. Soc. **21** (2008), 101–136, doi:10.1090/S0894-0347-06-00545-5.

[Dol83] A. Dold, *Fixed point indices of iterated maps*, Invent. Math. **74** (1983), 419–435, doi:10.1007/BF01394243.

[DIW26] A. Doan, E.-N. Ionel and T. Walpuski, *The Gopakumar–Vafa finiteness conjecture*, Ann. of Math. (2) **204** (2026), 819–871, doi:10.4007/annals.2026.204.3.1. Numbering as in arXiv:2103.08221v2.

[DW23a] A. Doan and T. Walpuski, *Counting embedded curves in symplectic 6-manifolds*, Comment. Math. Helv. **98** (2023), 693–769, doi:10.4171/CMH/556.

[DW21] A. Doan and T. Walpuski, *Castelnuovo's bound and rigidity in almost complex geometry*, Adv. Math. **379** (2021), 107550, doi:10.1016/j.aim.2020.107550.

[DW23b] A. Doan and T. Walpuski, *Equivariant Brill–Noether theory for elliptic operators and superrigidity of J-holomorphic maps*, Forum Math. Sigma **11** (2023), e3, doi:10.1017/fms.2022.104.

[DW25] A. Doan and T. Walpuski, *Chambered invariants of real Cauchy–Riemann operators*, arXiv:2410.21057v3 (2025).

[DS88] A. W. M. Dress and C. Siebeneicher, *The Burnside ring of profinite groups and the Witt vector construction*, Adv. Math. **70** (1988), 87–132, doi:10.1016/0001-8708(88)90052-7.

[EOP08] A. Eskin, A. Okounkov and R. Pandharipande, *The theta characteristic of a branched covering*, Adv. Math. **217** (2008), 873–888, arXiv:math/0312186.

[GM04] O. García-Prada and I. Mundet i Riera, *Representations of the fundamental group of a closed oriented surface in* $\mathrm{Sp}(4,\mathbb{R})$, Topology **43** (2004), 831–855, arXiv:math/0304027.

[GI21] X. Geng and G. Iyer, *Long time asymptotics of heat kernels and Brownian winding numbers on manifolds with boundary*, ALEA Lat. Am. J. Probab. Math. Stat. **18** (2021), 1297–1323, arXiv:1804.00368.

[Got01] P. B. Gothen, *Components of spaces of representations and stable triples*, Topology **40** (2001), 823–850, arXiv:math/9904114.

[IP99] E.-N. Ionel and T. H. Parker, *Gromov invariants and symplectic maps*, Math. Ann. **314** (1999), 127–158, doi:10.1007/s002080050289. Theorem numbering follows arXiv:dg-ga/9703013.

[IP18] E.-N. Ionel and T. H. Parker, *The Gopakumar–Vafa formula for symplectic manifolds*, Ann. of Math. (2) **187** (2018), 1–64, arXiv:1306.1516.

[Joh80] D. Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. (2) **22** (1980), 365–373, doi:10.1112/jlms/s2-22.2.365.

[Klu25] M. R. Klug, *Counting homomorphisms from surface groups to finite groups*, Canad. Math. Bull. **68** (2025), 141–153, doi:10.4153/S0008439524000420.

[LP13] J. Lee and T. H. Parker, *Spin Hurwitz numbers and the Gromov–Witten invariants of Kähler surfaces*, Comm. Anal. Geom. **21** (2013), 1015–1060, doi:10.4310/CAG.2013.v21.n5.a6.

[LP12] J. Lee and T. H. Parker, *Recursion formulas for spin Hurwitz numbers*, arXiv:1212.1825.

[Mil58] J. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223, publisher edition.

[Mum71] D. Mumford, *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 181–192, doi:10.24033/asens.1209.

[Pan99] R. Pandharipande, *Hodge integrals and degenerate contributions*, Comm. Math. Phys. **208** (1999), 489–506, doi:10.1007/s002200050766.

[Par25] J. Pardon, *Universally counting curves in Calabi–Yau threefolds*, arXiv:2308.02948v3 (2025).

[Par13] B. Parker, *Integral counts of pseudo-holomorphic curves*, arXiv:1309.0585 (2013).

[Sie98] B. Siebert, *Gromov–Witten invariants of general symplectic manifolds*, arXiv:dg-ga/9608005v2 (1998).

[Sie99] B. Siebert, *Algebraic and symplectic Gromov–Witten invariants coincide*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1743–1795, arXiv:math/9804108v1.

[tD09] T. tom Dieck, *Representation theory*, Göttingen lecture notes (2009), Proposition 5.8.2.

[Voi21] J. Voight, *Quaternion algebras*, Graduate Texts in Mathematics 288, Springer, Cham, 2021, author edition.

[Wen23] C. Wendl, *Transversality and super-rigidity for multiply covered holomorphic curves*, Ann. of Math. (2) **198** (2023), 93–230, doi:10.4007/annals.2023.198.1.2.

[Wue26b] B. J. Wuebben, *A geometric product formula for universal curve counting on Calabi–Yau threefolds*, 23 November 2026, doi:10.5281/zenodo.24455518.

[Wue26c] B. J. Wuebben, *Antidiagonal local curves and integral exponents in universal curve counting*, 11 November 2026, revised 22 November 2026, doi:10.5281/zenodo.24432700.

[Wue26d] B. J. Wuebben, *Branched triple covers and wall crossing for local curve counts*, 24 November 2026, doi:10.5281/zenodo.24510974.

[Wue26e] B. J. Wuebben, *Integral generators for curve counting on threefolds*, 24 November 2026, doi:10.5281/zenodo.24510962.

[Wue26f] B. J. Wuebben, *Spin structures and local Gromov–Witten invariants of curves*, 24 November 2026, doi:10.5281/zenodo.24510976.

[Zin11] A. Zinger, *A comparison theorem for Gromov–Witten invariants in the symplectic category*, Adv. Math. **228** (2011), 535–574, doi:10.1016/j.aim.2011.05.021.