

L^2 METRICS ON HERMITIAN–YANG–MILLS MODULI SPACES NEAR A STABILITY WALL

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ABSTRACT. We prove a uniform relative comparison between the harmonic L^2 metric on a local Hermitian–Yang–Mills moduli space and the quotient metric of its finite-dimensional harmonic deformation model. The result concerns a rank-four family on the product of three projective lines, near a split sum of four line bundles at a stability wall. The model retains the full analytic obstruction and describes a neighborhood of the corresponding Hermitian–Einstein classes. On its smooth complex pieces, the relative metric error is quadratic in the coefficient radius. Both the comparison constant and the allowed radius are independent of the distance to the wall, even when the infinitesimal orbit map and the derivative of the obstruction lose rank. The proof estimates the ambient deformation norm before taking the quotient, avoiding inversion of either degenerating map. We also identify the transition from this wall model to the harmonic model at a reducible rank-three-plus-line endpoint, with quantitative domains of validity. These comparisons extend to intrinsic lengths and their completions; in an explicit family of stable extensions they give a joint asymptotic for the intrinsic distance to the reducible bundle.

1. INTRODUCTION

A finite deformation space can describe nearby bundles without giving a useful approximation to their metric geometry. Near a reducible bundle, the orbit directions lose rank. Near a stability wall, the harmonic spaces used to define the deformation chart also change. An additive error in a tangent norm may then be larger than the norm being measured. The problem is to retain the complete integrability equations and obtain a relative estimate that survives both degenerations.

We solve this problem in a concrete threefold example. The family is small enough for its endpoint deformation complex to be computed, but large enough that a selected extension family omits most of the actual deformations. At the wall we use one fixed harmonic coefficient space. The complete obstruction and the corrected finite mean-curvature equation define the geometric domain. On this domain an independently specified finite-dimensional quotient form approximates the actual harmonic L^2 metric, with relative error quadratic in the coefficient radius. The constants do not deteriorate when singular values of the infinitesimal orbit map tend to zero or stabilizers change.

1.1. The geometric setting and the principal estimate. Let $X = (\mathbb{P}^1)^3$ and consider the split bundle with line factors

$$(1) \quad \mathcal{O}(-2, -2, 4), \quad \mathcal{O}(1, -2, 1), \quad \mathcal{O}(-2, 1, 1), \quad \mathcal{O}.$$

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Denote these lines, in order, by F_1, F_2, F_3, F_4 . At the equal product polarization their slopes agree. A one-parameter family ω_l , specified in (5) with $k = 3l/5$, has centered slope differences $l(1, -2/5, 0, -3/5)$. For $l > 0$ there is a reducible polystable bundle $E_l = W \oplus F_3$: the rank-three factor W is a nonsplit chain through F_1, F_2, F_4 , including the quadratic term required by integrability. Its balanced adjacent squared norms are $l, 3l/5$. The actual HE connections tend smoothly to the split connection as $l \rightarrow 0$.

Let V be the 83-dimensional split harmonic degree-one space. The fixed Hodge contraction gives a holomorphic reconstruction $a(x)$ and a five-component analytic obstruction $\kappa(x)$ by

$$a(x) = \iota x - K(a(x)^2), \quad \kappa(x) = P_2(a(x)^2).$$

Here ι , K and P_2 are the harmonic inclusion, split Hodge homotopy and degree-two harmonic projection. These equations retain every higher term. The effective diagonal group $T = (\mathbb{C}^*)^4/\mathbb{C}^*$ acts on V ; its compact form is T_c and its infinitesimal orbit map is C_x . Solving the complementary metric equation produces $H_x = h_0 e^{u(l,x)}$, with $u = O(|x|^2)$, and a remaining three-component real curvature equation $m_l(x) = 0$. On a fixed small ball set

$$\mathcal{N}_l(r) = \{x : |x| < r, \kappa(x) = 0, m_l(x) = 0\}/T_c.$$

Section 9 identifies this quotient with an open neighborhood of actual HE classes, quantified in fixed connection norms. It constructs smooth locally closed complex pieces on which tangent metrics are defined.

The finite-dimensional quotient metric is

$$g_{\text{fin},l,x}([v]) = \inf_{\xi \in \mathfrak{t}} \|v - C_x \xi\|_l^2,$$

where the coefficient norm is the split harmonic norm with its explicit polarization weights in (85). The actual metric g_{HE} is the volume-normalized harmonic Kodaira–Spencer norm of the reconstructed bundle. Theorem 9.2 proves, on every specified piece and for all $0 \leq l \leq l_0$,

$$(2) \quad (1 - Cr^2)g_{\text{fin},l} \leq g_{\text{HE},l} \leq (1 + Cr^2)g_{\text{fin},l}.$$

The constant and the permitted radius are independent of l , of positive singular values of C_x and of $D\kappa_x$, with the coefficient and parallel-endomorphism norms specified below. The exact Kodaira–Spencer kernel is the complex orbit tangent. Compatible lifted tangents converge across the indicated singular pieces; the limiting tangent must belong to the limiting piece. Intrinsic lengths on the same local path domain compare by $\sqrt{1 \pm Cr^2}$, and their abstract metric completions agree. These statements do not assign a metric to every coarse Zariski tangent direction.

1.2. Why the full chart is necessary. At $l > 0$, the actual endpoint cohomology has dimensions

$$h^q(\text{End}_0 E_l) = (1, 77, 1, 0),$$

whereas at the split wall it has dimensions $(3, 83, 5, 0)$. Section 3 gives an explicit convergent one-equation endpoint base. Its quadratic term has rank six, but the higher terms need not vanish in those coordinates. The residual scalar acts with character dimensions $(41, 16, 20)$ in weights $0, -1, +1$. The four-arrow extension family on blocks 12, 13, 24, 34, defined in (8)–(9) and called the diamond family below, has only eighteen endpoint tangent directions. We exhibit a 35-dimensional polystable family transverse to those directions, as well as regular stable quotient charts of complex dimension 75 in the full germ. These dimension statements locate the missing geometry; they do not substitute for the complete obstruction equation.

The endpoint Dolbeault complex explains the change at the wall. Its positive low eigenvalue counts in degrees $0, 1, 2, 3$ are $(2, 6, 4, 0)$, all of order l , with explicit leading constants. The

corresponding endpoint Hodge homotopy has norm $O(l^{-1/2})$. The fixed split contraction keeps these directions finite and has a uniform complementary inverse. At $l > 0$, two gauge normalizations and four equation eliminations recover the 77-coordinate base on a ball of radius $c\sqrt{l}$. Theorem 7.1 then constructs an analytic comparison with the actual endpoint harmonic chart on a ball of radius cl , with bounded coordinate derivatives and an exact identity of the two scalar defining equations up to a nonvanishing factor.

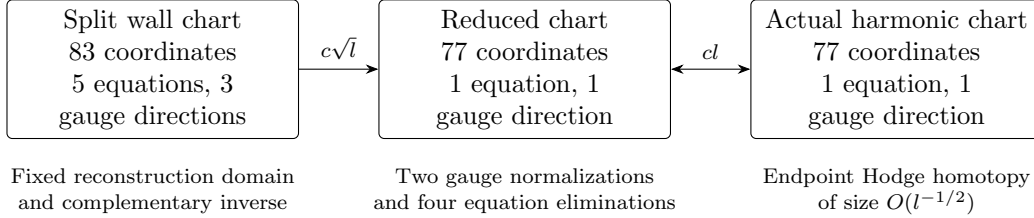


FIGURE 1. Three descriptions of the same positive-parameter deformation germ. The arrow labels are sufficient radii about the endpoint for the indicated operations. The left-hand chart itself has a fixed domain as $l \rightarrow 0$. The scalar equations are identified analytically, including higher terms.

The deformation equations used below are summarized in Table 1. The selected coefficients describe the four-arrow subfamily; the other four rows describe complete local deformations. In the wall and endpoint harmonic models, the real curvature equation selects the corrected HE representatives. The selected family is first balanced quadratically and then corrected. Neither normalization introduces another holomorphic chart.

Presentation	Coordinate	Equation and contraction	Section
Selected extensions	$z = (A, C, B, D)$	Five quadratic equations $\Omega = 0$; current split Hodge primitive	2
Endpoint Čech model	$t \in \mathbb{C}^{77}$	Scalar $F(t) = 0$; explicit Čech contraction	3
Endpoint harmonic model	$t \in V_*$	Scalar $f_*(t) = 0$; actual endpoint Hodge contraction	4
Split wall model	$x \in V \simeq \mathbb{C}^{83}$	Five equations $\kappa(x) = 0$; fixed split Hodge contraction	6
Reduced wall model	$v \in V_l \simeq \mathbb{C}^{77}$	Scalar $f_l(v) = 0$; two gauge and four equation eliminations	7

TABLE 1. The coordinate presentations and their defining equations. The symbol t is local to the two endpoint constructions. In Section 7, the actual harmonic obstruction is denoted F_e to distinguish it from the reduced wall equation f_l .

1.3. The estimate and an explicit boundary distance. The metric proof separates two operations. First eliminate the infinite-dimensional gauge complement, where a fixed Poincaré gap is available. Harmonic orthogonality makes the harmonic part of the reconstruction derivative orthogonal to its correction. Its angle with the complementary gauge range is $O(|x|)$, so orthogonal projection loses only $O(|x|^2)$ of the squared norm. The corrected HE metric changes it by another relative $O(|x|^2)$. The result is a positive Hermitian form on the

entire coefficient space, uniformly close to the harmonic one. Apply this estimate to every $v - C_x \xi$ and only then minimize over ξ . No inverse of the finite orbit operator is required. Section 8 states this argument with explicit hypotheses for general compact stabilizers, so it can also be used in matrix multiplicity problems.

To turn that estimate into the geometric theorem one must still prove coverage, compact-orbit uniqueness and existence of smooth quotient pieces. We do so using the full holomorphic slice, the normalization $P_0 \log H = 0$, and a closed adapted two-form that tames the coefficient complex structure. We use its moment identity without assuming that it is a Kähler form. The same positivity proves that finite intrinsic lengths separate distinct quotient points.

The selected diamond family provides a calculable consequence. An orbit parameter $M \rightarrow \infty$ approaches $W \oplus F_3$ through regular stable bundles. One transverse squared tangent norm is $(l - \delta)/(M^2 + 2)$ to relative $O(l)$, where $\delta \sim \sqrt{3/5} l/M$. A different coordinate, $r = M^{-1/2}$, extends the balanced family smoothly to the endpoint. Its finite horizontal speed has squared leading value $2l\sqrt{3/5}$. Section 11 proves the actual selected-family distance formula

$$(3) \quad d_{\text{HE}}(E(M), E_*) = \sqrt{2l\sqrt{3/5}} M^{-1/2} (1 + O(M^{-1} + l)).$$

The error is uniform in unrelated limits $l \rightarrow 0$, $M \rightarrow \infty$. At fixed l the leading coefficient is known to relative $O(l)$. The distance uses paths in the selected local family, not shortcuts through the larger complete moduli germ.

1.4. Relation to earlier work. Buchdahl–Schumacher construct equivariant analytic Kuranishi models and prove the local polystability correspondence near a fixed HE bundle [BuS20]. Their results supply the qualitative fixed-endpoint framework. Clarke–Tipler [CT24] and Deloche [Del25] develop local wall-crossing and continuity with respect to the background metric using complementary reduction and an adapted finite curvature equation. We use these mechanisms and prove the fixed-domain estimates and chart transitions required by the present full threefold deformation space.

The harmonic moduli metric is the standard Weil–Petersson metric of [BiS16, Section 4.1], divided here by the base volume. Its positivity is on effective Kodaira–Spencer directions. The relation to the real unitary-connection norm has the explicit factor two stated in Section 4.

The order of quotient elimination and the basic relative minimization principle already appear in [Wue26b], which proves relative L^2 metric estimates across reducible Yang–Mills surface strata. That paper also treats local intrinsic completions. The present contribution is the higher-dimensional integrable geometry: a complete convergent obstruction equation with explicitly computed higher-order terms in the chosen coordinates, a uniform wall chart, its quantitative transition to actual endpoint harmonic coordinates, and the realization of the relative estimate on the full local HE quotient. The stationary construction for the selected three-step family is taken from [Wue26a] with its scope restated in Section 2. Complete deformation and quotient arguments are proved here.

Manetti’s formal contraction theorem [Man99] is used only at the formal step of the explicit endpoint calculation; convergence and analytic comparison are proved separately. The comparison gauges also identify the harmonic tangent metrics, connecting the deformation calculation to the relative metric theorem.

2. THE FAMILY AND ITS REDUCIBLE ENDPOINT

Our example is a bundle on a threefold. It has a small family of three-step presentations, a larger complete deformation space, and a reducible endpoint at which both the automorphism group and the obstruction rank change. These are distinct phenomena. We first describe

the smaller family explicitly; its relation with the full moduli problem will then be visible in coordinates.

2.1. Fixed geometry and conventions. Let $X = (\mathbb{P}^1)^3$, with Fubini–Study forms ω_i of integral one, and put

$$(4) \quad F_1 = \mathcal{O}(-2, -2, 4), \quad F_2 = \mathcal{O}(1, -2, 1), \quad F_3 = \mathcal{O}(-2, 1, 1), \quad F_4 = \mathcal{O}, \quad E_0 = \bigoplus_i F_i.$$

Give the lines their product metrics and E_0 their orthogonal sum h_0 . Write d for the split Dolbeault differential, on the bundle or its endomorphisms as indicated by the argument. Block ij means $\text{Hom}(F_j, F_i)$. The four factors have equal slope for $\omega_0 = \omega_1 + \omega_2 + \omega_3$.

For small real k, l define

$$(5) \quad \omega_{k,l} = h_1\omega_1 + h_2\omega_2 + \omega_3, \quad h_1 = \left(1 + \frac{k+5l}{24\pi}\right)^{-1}, \quad h_2 = \left(1 + \frac{5k+l}{24\pi}\right)^{-1}.$$

The volume is $h_1 h_2$. Every L^2 inner product below is divided by this volume. Hermitian products are linear in their second argument; the associated real metric is their real part. The centered slope differences, with $\kappa_i = 2\pi \deg(F_i)/\text{Vol}$ and $\kappa_E = \frac{1}{4} \sum_i \kappa_i$, are

$$(6) \quad b_i = \kappa_E - \kappa_i, \quad b = \frac{1}{8}(5(k+l), 3k-5l, -5k+3l, -3(k+l)).$$

Our full wall chart uses the path $k = 3l/5$, $0 \leq l \leq l_0$. We write ω_l for this path. Then $b = l(1, -2/5, 0, -3/5)$. A separate selected-family result retains all small $k, l > 0$.

Write $(s_i : t_i)$ for homogeneous coordinates and $(s : t) = (s_3 : t_3)$. Let $S_j = H^0(\mathbb{P}_3^1, \mathcal{O}(j))$ and $U_i = H^0(\mathbb{P}_i^1, \mathcal{O}(1))$. Choose $e_{0,i} = t_i, e_{1,i} = s_i$, a class $\theta_i \in H^1(\mathcal{O}(-2))$, and its Serre-dual harmonic classes $\eta_i^0, \eta_i^1 \in H^1(\mathcal{O}(-3))$, so that $\eta_i^j e_{m,i}$ has harmonic projection $\delta_{jm} \theta_i$. All factors are pulled back to X and constant tensor factors are suppressed. The selected harmonic coordinates are

$$(7) \quad \begin{aligned} A &= \eta_1^0 a_0 + \eta_1^1 a_1 \in V_{12} = U_1^* \otimes S_3, & C &= \eta_2^0 c_0 + \eta_2^1 c_1 \in V_{13} = U_2^* \otimes S_3, \\ B &= (e_{0,1} b_0 + e_{1,1} b_1) \theta_2 \in V_{24} = U_1 \otimes S_1, & D &= \theta_1 (e_{0,2} d_0 + e_{1,2} d_1) \in V_{34} = U_2 \otimes S_1. \end{aligned}$$

Thus A, C are pairs of cubics, B, D pairs of linears, and the four dimensions are 8, 8, 4, 4.

2.2. The nonlinear relation and the stationary input. Let $K_h = d_h^* G_h$ be the split Hodge homotopy. Define

$$(8) \quad N_h(z) = AE_{12} + CE_{13} + BE_{24} + DE_{34} + \Gamma_h(z)E_{14}, \quad \Gamma_h = -K_h(A \wedge B + C \wedge D).$$

Here $z = (A, C, B, D)$; each E_{ij} records the block only. The endpoint line $F_1 F_4^{-1}$ has $H^1 = 0$ and $H^2 = \mathbb{C}(\theta_1 \wedge \theta_2) \otimes S_4$. There are no products of length three. Hodge decomposition therefore gives

$$(9) \quad (d + N_h(z))^2 = (\theta_1 \wedge \theta_2) \Omega(z) E_{14}, \quad \Omega = a_0 b_0 + a_1 b_1 - c_0 d_0 - c_1 d_1.$$

The minus sign orders the two differential legs by projective-line factor. The endpoint primitive is part of the bundle, even when a path's harmonic product vanishes. These identities hold on the entire coefficient space, and $d_h^* N_h = 0$ there.

The effective torus $T = (\mathbb{C}^*)^4 / \mathbb{C}^*$ acts by g_i/g_j on block ij ; $T_c = (S^1)^4 / S^1$ is its compact form. On zero-sum vectors $\xi \in \mathbb{C}^4$, its infinitesimal action is

$$(10) \quad C_z \xi = ((\xi_1 - \xi_2)A, (\xi_1 - \xi_3)C, (\xi_2 - \xi_4)B, (\xi_3 - \xi_4)D).$$

For $w = (\|A\|_h^2, \|C\|_h^2, \|B\|_h^2, \|D\|_h^2)$, write $\mu_h(z) = Dw$, where

$$(11) \quad D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

We call z quadratically balanced if $\mu_h(z) = b$. This is twice the usual quadratic moment map for the real part of the harmonic Hermitian form, with its level multiplied by the same factor. It changes neither the level set nor the quotient metric convention.

We use the following stationary result of [Wue26a]. On the ambient selected ball, the complementary curvature equation has a unique small solution $u(h, z)$ perpendicular to real constant diagonals, with $\|u\|_{C^m} \leq C_m \|z\|_h^2$. The remaining equation is

$$(12) \quad F(h, z) = -b + \mu_h(z) + R(h, z), \quad |R_S(h, z)| \leq C \|z\|_h^2 \sum_{e \text{ crossing } S} \|z_e\|_h^2.$$

Here $R_S = \sum_{i \in S} R_i$. Define the cut energy $q_{\partial S} = \sum_{e \text{ crossing } S} \|z_e\|_h^2$. The bound holds for every vertex subset and every ambient coefficient point, whether or not its obstruction vanishes. It also holds after a diagonal derivative: if $\dot{z}_{ij} = (Y_i - Y_j)z_{ij}/2$, then

$$(13) \quad |(D_Y R[Y])_S| \leq C \|z\|_h^2 \sum_{e \text{ crossing } S} \|z_e\|_h^2 \|Y\|_\infty.$$

For connected integrable support admitting positive balance, the actual fixed-determinant HE metric satisfies $e^{-C_\sigma h_{\text{bal}}} \leq H \leq e^{C_\sigma h_{\text{bal}}}$, where σ is total balanced energy. The constants do not depend on positive edge ratios. The same ambient zero-form argument identifies holomorphic endomorphisms with diagonal constants on support components. These are the precise stationary inputs used below; quotient tangent norms require the separate argument of Section 8.

2.3. A sequence with a reducible limit. Choose normalized cubic monomials p, q proportional to s^3, t^3 and a normalized linear monomial ℓ proportional to s . The normalizations include their harmonic factors in (7). The two path products have equal normalization, since polarization weights cancel around the diamond. For $M \geq 0$ take

$$(14) \quad A = (p, Mp + q), \quad C = (p, 0), \quad B = D = (\ell, 0).$$

Both products in (9) equal $p\ell$. Variations of a_0 span ℓS_3 , while a variation of b_1 in the complementary linear direction supplies a nonzero t^4 term. Thus $D\Omega$ has rank five at every finite M . The invariant norm ratio is $\rho = M^2 + 2$.

On $k = 3l/5$, the positive balanced representatives are

$$(15) \quad \begin{aligned} A^0 &= \sqrt{\frac{l - \delta}{M^2 + 2}}(p, Mp + q), & C^0 &= \sqrt{\delta}(p, 0), \\ B^0 &= \sqrt{3l/5 - \delta}(\ell, 0), & D^0 &= \sqrt{\delta}(\ell, 0), \end{aligned}$$

where the unique root in $0 < \delta < 3l/5$ satisfies

$$(16) \quad (M^2 + 2)\delta^2 = (l - \delta)(3l/5 - \delta).$$

The cycle identity verifies both the relation and the diagonal orbit. As $M \rightarrow \infty$, $M\delta/l \rightarrow \sqrt{3/5}$, and the limit is

$$(17) \quad A_* = (0, \sqrt{l}p), \quad B_* = (\sqrt{3l/5}\ell, 0), \quad C_* = D_* = 0.$$

Its bundle splits as $E_* = W \oplus F_3$, where W has graded lines F_1, F_2, F_4 , both adjacent classes nonzero, and the retained quadratic endpoint primitive.

Proposition 2.1. *For every sufficiently small $l > 0$, W is stable at ω_l and has the same slope as F_3 . The endpoint is polystable, $\text{Aut}(E_*) = (\mathbb{C}^*)^2$, and it has an HE metric H_* with $e^{-Cl}h_0 \leq H_* \leq e^{Cl}h_0$ in the balanced presentation. Fixing the determinant on each summand fixes its metric scales.*

Proof. On support $\{12, 24\}$ the complementary solution and connection split off F_3 , by equivariance and uniqueness. Its curvature component is exactly $-b_3 = 0$. On $\{1, 2, 4\}$ the positive weights are $l, 3l/5$, with demand $(l, -2l/5, -3l/5)$. Restrict diagonal logarithms to $Y_3 = 0$, $Y_1 + Y_2 + Y_4 = 0$. The cut argument in (12) corrects this balance by $O(l)$: on a sphere $\|Y\|_\infty = Al$, sort its three values and use a cut across a largest gap. The moment increment pairs positively with the cut by at least cAl times its positive cut energy; the residual is at most Cl times the same energy. For fixed large A and small l there is no boundary zero. The quadratic moment map has positive radial pairing, hence degree one; the homotopy to F has the same degree. This constructs the metric with the asserted bound.

The ambient zero-form estimate gives just one constant endomorphism on each connected component. Thus W is simple. An HE bundle is an orthogonal sum of stable equal-slope factors, so simplicity makes W stable; see [Kob87, Theorem 5.8.3]. The third demand is zero, proving equality of slopes. The same endomorphism calculation gives the stated automorphisms. The independent positive scalars are parallel and do not change the Chern connection; separate determinant normalizations remove them. $\square$

2.4. The selected slice and its five pieces. For fixed $l > 0$, fix the nonzero s^3 coefficient of a_1 and s coefficient of b_0 by two independent torus characters, to values α, β . The residual $H = \mathbb{C}^*$ acts as $(C, D) \mapsto (\lambda^{-1}C, \lambda D)$. Put $L = b_0 = \beta s + \beta_1 t$. The pair (L, t) is a basis. Write

$$\begin{aligned} a_1 &= (\alpha/\beta^3)L^3 + \gamma_1 L^2 t + \gamma_2 L t^2 + x t^3, & b_1 &= \eta L + y t, \\ c_i &= P_i(L, t) + u_i t^3, & d_i &= e_i L + v_i t \quad (i = 0, 1), \end{aligned}$$

where P_i uses $L^3, L^2 t, L t^2$ only. Division by L solves four equations for a_0 and leaves

$$(18) \quad xy = u_0 v_0 + u_1 v_1, \quad a_0 = -\frac{a_1 b_1 - c_0 d_0 - c_1 d_1 - (xy - u_0 v_0 - u_1 v_1)t^4}{L}.$$

The numerator is identically divisible. Thus the slice is $\mathbb{C}^4 \times \{xy = u_0 v_0 + u_1 v_1\} \subset \mathbb{C}^4 \times \mathbb{C}^2 \times \mathbb{C}^8 \times \mathbb{C}^4$, of dimension 17, with 18-dimensional tangent at its origin. The four extra neutral variables are $\beta_1, \gamma_1, \gamma_2, \eta$. The six displayed quadratic variables vanish precisely on its singular locus, of dimension 12; equivalently $D\Omega$ then has rank four.

Let $W_{jm} = C_j D_m$ be the 8 by 4 invariant matrix. Its relations are the 2 by 2 minors, since invariant monomials pair negative and positive weights. The quotient germ is

$$(19) \quad \mathbb{C}^4 \times \{(x, y, W) : \text{rk } W \leq 1, \quad xy = W_{u_0 v_0} + W_{u_1 v_1}\}.$$

The closed orbits have both C, D nonzero or both zero. A one-sided orbit has the neutral point in its closure. A closed orbit meets the quadratic moment level uniquely modulo T_C : in full support its weights are $(l - \delta, \delta, 3l/5 - \delta, \delta)$, where the invariant ratio uniquely determines δ ; on the two-edge support, the two characters set the norms to l and $3l/5$. These representatives remain local. Indeed residual scaling gives $\|C\| = \|D\| = \sqrt{\|W\|}$, so $C, D \rightarrow 0$ as $W \rightarrow 0$, and the positive scales on the nonzero A, B edges tend to one.

Put $Z_6 = (x, y, u_0, u_1, v_0, v_1)$. The following are smooth complex pieces of this selected quotient:

Closed representative	Dimension	Stabilizer	rk $D\Omega$
$C, D \neq 0, Z_6 \neq 0$	16	1	5
$C, D \neq 0, Z_6 = 0$	11	1	4
$C = D = 0, x \neq 0, y = 0$	5	$\mathbb{C}^*$	5
$C = D = 0, x = 0, y \neq 0$	5	$\mathbb{C}^*$	5
$C = D = 0, x = y = 0$	4	$\mathbb{C}^*$	4

For the second row the charged dimensions are 6 and 2, giving $6 + 2 - 1 + 4 = 11$; the other dimensions follow from (18). The first two rows are stable. In the last three the bundle is $W_z \oplus F_3$, with W_z stable by the proof of Proposition 2.1. The two five-dimensional branches meet on the four-dimensional piece.

These points are actual isomorphism classes. On full support the reference-slope socle is F_1 and the socle of its quotient is $F_2 \oplus F_3$: cross-line Hom groups vanish and the two adjacent extensions on each side are nonzero. Every isomorphism preserves this filtration and induces four scalar maps; it acts by exactly the torus characters on the adjacent classes. Conversely those characters give an isomorphism, including the endpoint primitive. This is the orbit classification in [Wue26a, Section 6]. On the reducible locus the stable decomposition distinguishes W_z from F_3 , and its reference-slope socle filtration distinguishes F_1, F_2, F_4 . The same argument gives exactly the two active characters. A stable bundle cannot be isomorphic to a reducible one. The complete deformation problem is substantially larger than these five pieces, as we now show.

3. THE COMPLETE DEFORMATION EQUATION AT THE ENDPOINT

The selected slice has eighteen infinitesimal directions at E_* . The full fixed-determinant space has seventy-seven. We compute the missing directions and give a convergent equation that retains their interactions. The quadratic term is particularly simple, but it does not replace the complete equation.

3.1. A finite complex computing the endpoint cohomology. For this calculation fix $l > 0$ and rescale the two nonzero extension classes by a constant diagonal, so they are $A = \eta_1^1 s^3$ and $B = e_{0,1} \theta_2 s$. This is a holomorphic normalization; its metric bounds are allowed to depend on l . On the three affine factors use coordinates $\xi = s_1/t_1$, $\eta = s_2/t_2$, $\zeta = s/t$. The one-factor Čech complex for $\mathcal{O}(d)$ is

$$(20) \quad C^0(d) = \mathbb{C}[\xi] \oplus \xi^d \mathbb{C}[\xi^{-1}], \quad C^1(d) = \mathbb{C}[\xi, \xi^{-1}], \quad d(f_0, f_1) = f_1 - f_0.$$

For the ordered cup product, use the first-chart component in degree $(0, 1)$, the second-chart component in degree $(1, 0)$, and componentwise multiplication in degree $(0, 0)$. Products of two degree-one cochains on one factor vanish. Take the tensor product over the three factors, with Koszul signs and matrix composition. Products pair the appropriate Hom bundles; a single line-bundle complex is not an algebra by itself.

The inclusion i sends a degree-zero cohomology monomial to both charts and a degree-one monomial to the intersection. The projection p reads the second-chart coefficients $0 \leq e \leq d$

in degree zero and retains $d < e < 0$ in degree one. On a degree-one monomial set

$$(21) \quad h(\xi^e) = \begin{cases} (-\xi^e, 0), & e \geq 0, \\ (0, \xi^e), & e < 0 \text{ and } e \leq d, \\ 0, & d < e < 0. \end{cases}$$

It is zero in degree zero. Direct substitution gives $dh + hd = 1 - ip$, $hi = ph = h^2 = 0$, and $pi = 1$. On the tensor product use $h_1 \otimes 1 \otimes 1 + i_1 p_1 \otimes h_2 \otimes 1 + i_1 p_1 \otimes i_2 p_2 \otimes h_3$, with tensor signs. These formulas also recover projective-line cohomology and Künneth; see [Stacks, Tags 01XT and 0BED].

In this complex the endpoint is

$$(22) \quad A = i(\xi^{-2}\zeta^3 E_{12}), \quad B = i(\eta^{-1}\zeta E_{24}), \quad \Gamma = -h(AB), \quad N = A + B + \Gamma.$$

Here $p(AB) = 0$, so $d\Gamma = -AB$ and $dN + N^2 = 0$. For an explicit descent check, put $A_0 = \xi^{-2}\zeta^3$, $B_0 = \eta^{-1}\zeta$. The correction on a ξ edge is $1 + A_0 E_{12}$. On an η edge it is $1 + B_0 E_{24}$ in the first ξ chart and $1 + B_0 E_{24} - A_0 B_0 E_{14}$ in the second. The two square products both equal $1 + A_0 E_{12} + B_0 E_{24}$. The third-coordinate twists are 3, 1, 4. Thus these invertible triangular transitions glue the endpoint bundle. Its central lift is unique up to gauge because $H^1(F_1) = 0$, so it agrees with (8). For endomorphisms the edge differential $Gu_1 - u_0 G$ becomes the usual transition-conjugated differential after multiplication by G^{-1} ; the square identity gives the same identification in higher degrees.

Set $d_N = d + \delta$, where $\delta u = Nu - (-1)^{\deg u} uN$. The transferred operators are

$$(23) \quad I = (1 + h\delta)^{-1}i, \quad P = p(1 + \delta h)^{-1}, \quad K' = h(1 + \delta h)^{-1}, \quad \partial = p\delta I.$$

Every inverse here is a finite sum. Give the four factors heights $(2, 1, 1, 0)$. Multiplication by an adjacent endpoint arrow raises matrix-block height by one and multiplication by Γ raises it by two; the Hom-block heights range from -2 to 2 . The homotopy preserves blocks. Expanding the finite sums using $dh + hd = 1 - ip$ and $d_N^2 = 0$ gives

$$d_N I = I\partial, \quad P d_N = \partial P, \quad d_N K' + K' d_N = 1 - IP, \quad PI = 1.$$

Thus this finite complex computes all endpoint cohomology, including blocks outside the original filtration.

The split cohomology is

Hom block	h^1	h^2
12, 13, 21, 31, 23, 32	8 each	0
24, 34	4 each	0
41	27	0
14	0	5

Every off-diagonal H^0 and every H^3 vanish; four diagonal constants remain. The split dimensions are $(4, 83, 5, 0)$. The image of ∂_0 is $\text{span}(A, B)$, of rank two. On the adjacent blocks,

$$(24) \quad \partial_1(\dot{A}, \dot{C}, \dot{B}, \dot{D}) = \zeta \dot{a}_0 + \zeta^3 \dot{b}_1 \in S_4.$$

Its image is sS_3 , of dimension four. There is only one additional arrow on a path from an adjacent input to block 14, so this formula is exact, rather than a first-order truncation of the transfer.

The omitted blocks give zero to every order. Indeed the residual scalar on F_3 has nonzero character on 13, 23, 31, 32, 34, whereas the target 14 is neutral. Rotate the third coordinate,

compensating on the lines by weights $(4, 1, 0, 0)$. A block- ij monomial of exponent e has weight $e + u_j - u_i$. The background and all transfer operators preserve this weight. The remaining omitted blocks 21 and 41 have strictly positive weights, respectively 1, 2 and 1, 2, 3; the target weights are $-4, -3, -2, -1, 0$. Their images must vanish. Consequently

$$(25) \quad (h^0, h^1, h^2, h^3)(\text{End}_0 E_*) = (1, 77, 1, 0).$$

Removing the scalar only changes degree zero, since $H^{>0}(\mathcal{O}_X) = 0$.

The residual effective scalar acts on H^1 with decomposition

$$(26) \quad V_* = V_0 \oplus V_- \oplus V_+, \quad (\dim V_0, \dim V_-, \dim V_+) = (41, 16, 20).$$

The neutral blocks 12, 21, 24, 41 have 47 coordinates before removing two gauge and four equation directions. Blocks 13, 23 have weight -1 and dimensions $8+8$; blocks 31, 32, 34 have weight $+1$ and dimensions $8+8+4$. The selected tangent injects with character dimensions $(6, 8, 4)$. Its complementary 59 directions are represented by corrected classes from blocks

$$(27) \quad V_{21} \oplus V_{41} \oplus V_{23} \oplus V_{31} \oplus V_{32}, \quad \dim = (8, 27, 8, 8, 8).$$

Their actual cocycles are obtained with I , not by using uncorrected split forms.

3.2. An explicit convergent scalar equation. We now specify every coefficient of the full obstruction, including its higher terms. In the transferred degree-one complex retain all split monomials except the four a_0 monomials and the two background monomials A, B . Replace the two b_1 monomials by

$$(28) \quad \xi \eta^{-1} \zeta^j E_{24} - \xi^{-1} \zeta^{j+2} E_{12}, \quad j = 0, 1.$$

These 77 vectors define a cohomology inclusion j_1 . In degree two use $j_2(1) = \xi^{-1} \eta^{-1} E_{14}$. A second homotopy on the finite complex is zero except for

$$(29) \quad k(A) = -E_{11}, \quad k(B) = E_{44}, \quad k(\xi^{-1} \eta^{-1} \zeta^{j+1} E_{14}) = \xi^{-1} \zeta^j E_{12} \quad (0 \leq j \leq 3).$$

In degree zero retain the constant on $\{1, 2, 4\}$ and the constant on $\{3\}$. If π reads cohomology in these complements, then $\partial k + k \partial = 1 - j \pi$. Hence

$$(30) \quad J = Ij, \quad \Pi = \pi P, \quad K = K' + IkP$$

satisfy $d_N K + K d_N = 1 - J \Pi$, $K^2 = KJ = \Pi K = 0$, and $\Pi J = 1$. All these operators are fixed at the endpoint.

For $t \in V_*$ define homogeneous cochains and scalar polynomials by

$$(31) \quad a_1(t) = Jt, \quad c_n(t) = \sum_{i=1}^{n-1} a_i(t) a_{n-i}(t), \quad a_n(t) = -K c_n(t), \quad F_n(t) = \Pi c_n(t) \quad (n \geq 2).$$

In degree two Π is the scalar coefficient of $\xi^{-1} \eta^{-1} E_{14}$. Each coefficient in this recursion is obtained by finite Laurent operations with rational coefficients.

Theorem 3.1. *The series $F = \sum_{n \geq 2} F_n$ converges near zero in V_* . Its zero germ is a fixed-determinant semiuniversal deformation base of E_* , of dimension 76. It is invariant under the scalar action with weights $0, -1, +1$ in (26). In the specified raw coordinates its quadratic term is*

$$(32) \quad F_2 = xy - u_0 v_0 - u_1 v_1,$$

of rank six and radical dimension 71. The selected family is its intersection with the linear 18-dimensional selected coordinate space. The complete base contains a transverse 35-dimensional family of polystable bundles and has regular stable quotient charts of dimension 75.

Proof. We first prove convergence and analytic semiuniversality, then compute the quadratic term and the selected restriction. The final part constructs the transverse polystable family and the regular stable charts. For convergence complete the cochain spaces in the weighted Laurent norm

$$\left\| \sum_e c_e \xi^{e_1} \eta^{e_2} \zeta^{e_3} \right\|_R = \sum_e |c_e| R^{|e_1|+|e_2|+|e_3|}, \quad R > 1.$$

The chart exponent conditions are those of (20). These series are holomorphic on the corresponding discs and exterior discs, with annular intersections. Cup multiplication is bounded because $R^{|e+f|} \leq R^{|e|} R^{|f|}$. The monomial projections and homotopies, multiplication by N , finite transfer sums and finite-dimensional second contraction are bounded. If M bounds K composed with multiplication, the scalar majorant $v = u + Mv^2$, $u = \|Jt\|$, converges for $4Mu < 1$. It majorizes (31) and proves convergence of

$$(33) \quad a(t) = Jt - K(a(t)^2), \quad F(t) = \Pi(a(t)^2).$$

For $c = d_N a + a^2$ the contraction and Bianchi identities give

$$c = JF + K(ca - ac).$$

Smallness makes the last operator have norm less than one. Thus $F = 0$ is equivalent to the full Maurer–Cartan equation. The perturbed transitions remain invertible in a norm controlling their sup norms on smaller charts, and therefore glue actual holomorphic bundles.

Gauge fixing in degree zero uses parameters in $\text{im } K$ to impose $Ka = 0$. Its derivative there is $\pm Kd_N = \pm 1$, so the analytic implicit-function theorem gives the slice. The remaining coordinate has identity Kodaira–Spencer differential on H^1 . The formal contraction argument supplies a hull, as in [Man99, Proposition 4.6 and Theorem 4.7]; convergence has just been proved separately. For analytic completeness one may compare with the full Dolbeault Kuranishi construction at the fixed endpoint. Its Green contraction and complex gauge fixing are bounded between Sobolev spaces of sufficiently high order. Passing from transitions to smooth Dolbeault operators and then fixing this gauge gives a holomorphic map of the two zero germs, with identity derivative on H^1 and an isomorphism of formal hulls. Extend its coordinate functions to the ambient analytic germs. The inverse-function theorem makes this an ambient coordinate change. The completed defining ideals agree; faithful flatness of completion for a Noetherian local ring [Stacks, Tag 00MC] implies that their analytic ideals agree. The local rings of convergent power series are Noetherian. This proves analytic semiuniversality, including the defining ideal rather than just the reduced zero set.

The determinant imposes no new equation: $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ and $\text{Pic}(X)$ is discrete. Locally identify the determinant by taking logarithms of its transition ratios and solving the scalar Čech cocycle; normalize by a local fourth root. This removes the scalar automorphism only. All contraction operators respect the residual action, so F is invariant.

The raw coordinates in (32) are the coefficients of

Coordinate	Transferred cohomology vector
x	$\xi^{-2}E_{12}$
y	$\xi\eta^{-1}E_{24} - \xi^{-1}\zeta^2E_{12}$
u_0, u_1	$\eta^{-1}E_{13}, \eta^{-2}E_{13}$
v_0, v_1	$\xi^{-1}E_{34}, \xi^{-1}\eta E_{34}$

Substitution in $\Pi((Jt)^2)$ gives the three pairs with coefficients $1, -1, -1$. There are no other quadratic pairs. One can see the vanishing on omitted directions directly from the weights: the target has third-coordinate weight -4 . Omitted neutral directions have positive weights, while selected neutral weights are at least -3 . The omitted negative-character block 23 has weight -1 ; omitted positive-character blocks 31, 32 have weights $2, 3$ and 1 . The selected negative and positive weights are respectively $-4, \dots, -1$ and $0, 1$. No pair involving an omitted direction has both residual weight zero and third-coordinate weight -4 .

All contractions preserve the selected subalgebra of constant diagonals and blocks 12, 13, 24, 34, 14. Its gauge condition fixes the two original nonzero coefficients, and $H^1(F_1) = 0$ fixes the central lift up to gauge. Consequently zero omitted coordinates give precisely the selected family, and conversely that family can be normalized inside this subalgebra. Its adapted coordinates give (18); the raw recursive coordinates agree to first order and may differ at higher order.

On $V_{21} \oplus V_{41}$ every coordinate has positive third weight, so no monomial can contribute to the obstruction of weight -4 . Thus F vanishes there to all orders and analytically. Reconstruction uses only the W blocks, giving $W_t \oplus F_3$. At fixed l , the trace-free HE linearization on stable W is invertible. The implicit-function theorem gives nearby HE metrics; openness of simplicity makes W_t stable. Its slope is unchanged, hence these are polystable bundles. Their Kodaira–Spencer map is injective on this 35-dimensional space, transverse to the selected tangent. The stable W slice identifies distinct sufficiently near parameters with distinct bundles.

Finally the quadratic initial term of F cannot factor into two linear forms, because its rank is six. Therefore the convergent germ F is irreducible and its zero set has dimension 76. The regular stable selected points (14) approach the endpoint; their nonzero selected equation derivative makes $dF \neq 0$. Stability is open at fixed polarization. The residual scalar action is free there, and fixing one nonzero charged coordinate gives a local 75-dimensional stable quotient chart. The complementary gauge linearization is $Kd_a = 1 + K[a, \cdot]$ on $\text{im } K$, still invertible on a small ball, so no further infinitesimal gauge direction is lost. $\square$

3.3. Higher terms that the quadratic equation does not record. For a concrete restriction of the recursion, retain the following five raw parameters:

$$x : \xi^{-2}E_{12}, \quad y : \xi\eta^{-1}E_{24} - \xi^{-1}\zeta^2E_{12}, \quad r : \xi^{-2}\zeta E_{12}, \quad b : \eta^{-1}E_{24}, \quad c : \xi^3\zeta^{-2}E_{21}.$$

Apply I to these vectors to obtain their endpoint cocycles. The last parameter is outside the selected family. Equations (21)–(31) give

$$(34) \quad F_2 = xy, \quad F_3 = -r(by + cx), \quad F_4 = -b^3y + bcr^2.$$

For example, the cubic is $-\Pi(a_1K(a_1^2) + K(a_1^2)a_1)$; the quartic is

$$\Pi(a_1K(a_1K(a_1^2) + K(a_1^2)a_1) + K(a_1K(a_1^2) + K(a_1^2)a_1)a_1 + K(a_1^2)K(a_1^2)).$$

The printed monomial rules determine every coefficient of these expressions. They illustrate a higher interaction with an omitted direction. They do not prove an intrinsically nonquadratic

singularity: solving the critical x, y pair gives $x = br + O(3)$, $y = cr + O(3)$, and the residual quartic cancels, $bcr^2 - (br)(cr) = 0$. The complete convergent equation, rather than a proposed analytic normal form, is the deformation model used in the metric arguments.

4. THE COMPLETE METRIC GERM AT FIXED POLARIZATION

Fix $l > 0$ sufficiently small. The endpoint $E_* = W \oplus F_3$ has an HE metric $H_* = H_W \oplus H_3$, with the determinant of each summand fixed separately. We first construct the metric germ using its actual harmonic representatives. Constants in this section may depend on l . This dependence will be resolved by a different chart in Sections 5–6.

4.1. Harmonic coordinates and the remaining scalar equation. Let D be the Dolbeault differential on $\text{End}_0 E_*$ and let Π_q, G_q be its harmonic projections and Green operators. The calculation of Section 3 identifies

$$V_* = \mathcal{H}_D^1 = V_0 \oplus V_- \oplus V_+, \quad (\dim V_0, \dim V_-, \dim V_+) = (41, 16, 20), \quad \dim \mathcal{H}_D^2 = 1.$$

The effective group $H = \mathbb{C}^*$ acts with weights $0, -1, +1$. Its Hermitian trace-free generator and infinitesimal orbit map are

$$(35) \quad T_* = \text{diag}(-\tfrac{1}{4}I_W, \tfrac{3}{4}I_{F_3}), \quad \|T_*\|^2 = \tfrac{3}{4}, \quad C_t \xi = \xi(0, -t_-, t_+).$$

These three subspaces are orthogonal in the volume-normalized harmonic norm. Relative rescaling of H_W, H_3 changes the cross-summand norms; our fixed choice is part of the coordinates.

For an integer $s \geq 6$, solve in H^s

$$(36) \quad a_*(t) = t - D^*G_2(a_*(t) \wedge a_*(t)), \quad f_*(t) = \Pi_2(a_*(t) \wedge a_*(t)).$$

Sobolev multiplication and the order-minus-one homotopy give a convergent analytic solution on a small ball. Its correction $b_* = a_* - t$ satisfies

$$(37) \quad \Pi_1 b_* = D^* b_* = 0, \quad \|b_*(t)\|_{H^s} \leq C_l |t|^2, \quad \|D_t b_* v\|_{H^s} \leq C_l |t| |v|.$$

The curvature identity and Bianchi identity imply that $D + a_*(t)$ is integrable exactly when $f_*(t) = 0$: the remaining curvature satisfies an equation $c = \Pi_2 c + K(ca_* - a_*c)$, whose last operator has norm less than one. The construction commutes with H . The analytic comparison in Section 3 identifies this scalar equation with its complete Čech equation, up to an invertible analytic factor and an equivariant analytic change of coordinates. In particular no higher obstruction terms are removed in passing to f_* .

Let P_* be the real orthogonal projection onto $\mathbb{R}T_*$ and $Q_* = 1 - P_*$. On Q_* the endpoint operator D^*D has a positive gap γ_l . For every small t , including nonintegrable ones, form the compatible unitary connection from $D + a_*(t)$ and a metric $H_* e^u$, transported by $e^{u/2}$ to H_* . Write its centered mean curvature as $\mathcal{M}_*(u, t)$. At zero its u derivative is D^*D ; its t derivative is zero by $D^*a_* = 0$. The real analytic implicit-function theorem gives

$$(38) \quad Q_* \mathcal{M}_*(u_*(t), t) = 0, \quad P_* u_* = 0, \quad \|u_*(t)\|_{H^{s+1}} \leq C_l |t|^2.$$

Define

$$m_*(t) = \langle \mathcal{M}_*(u_*(t), t), T_* \rangle.$$

On $f_*^{-1}(0)$ the metric $H(t) = H_* e^{u_*(t)}$ is HE exactly when $m_*(t) = 0$.

4.2. Correction relative to charged energy. The neutral coordinates may be much larger than the charged ones. An absolute Taylor estimate for m_* is therefore insufficient. Put $\rho_*(t) = |t_-|^2 + |t_+|^2$ and $m_2(t) = |t_+|^2 - |t_-|^2$.

Lemma 4.1. *On an ambient ball of radius r ,*

$$(39) \quad |m_* - m_2| \leq C_l r \rho_*, \quad |E(m_*) - 2\rho_*| \leq C_l r \rho_*, \quad E = (0, -t_-, t_+).$$

In particular these estimates do not require a lower bound for ρ_/r^2 .*

Proof. The quadratic part of m_* is the sum of block norms multiplied by target-minus-source eigenvalues of T_* . This is exactly m_2 . The linear curvature of b_* has zero T_* projection, and D^*Du_* does also, so there is no other quadratic term. On the neutral subspace the reconstructed connection and its corrected metric preserve $W \oplus F_3$. The integrated partial trace on each summand equals its degree. Their slopes coincide, hence $m_*(t_0, 0) = 0$, including when $f_*(t_0, 0) \neq 0$. This uses a compatible unitary connection, not a nonexistent holomorphic structure off the zero set. Circle invariance makes the first derivative in charged directions zero at every neutral point.

Thus $R = m_* - m_2$ vanishes to second order in the charged variables, and its full quadratic jet at zero vanishes. On the radius- r ball its charged second derivative is $O_l(r)$. Taylor's integral formula in those variables gives $|R| \leq C_l r \rho_*$. Differentiating the same formula along E gives the second bound. All derivatives are taken before restriction to the obstruction locus. $\square$

Proposition 4.2. *Every sufficiently small closed H orbit in $f_*^{-1}(0)$ has a representative with $m_* = 0$, unique up to the compact circle within the local construction. The closed orbits are exactly those with both t_-, t_+ nonzero and those with both zero. In the first case, after flat balance $|t_-| = |t_+|$, the further positive gauge has logarithm $O_l(r)$. These orbits give stable bundles. The neutral orbits give the nearby polystable decompositions $W_t \oplus F_3$.*

Proof. For weights $0, -1, +1$, an orbit with both charged components nonzero is closed, as is a neutral point. A one-sided orbit has the neutral point in its closure. These assertions follow directly by considering $\lambda \rightarrow 0, \infty$; a sequence with neither behavior has a convergent subsequence in $\mathbb{C}^*$.

If $a = |t_-| > 0$ and $b = |t_+| > 0$, the real gauge $s_0 = \frac{1}{2} \log(a/b)$ makes both norms equal to $\sqrt{ab}$. It decreases their sum of squares, so stays in the coordinate ball. Starting there, Lemma 4.1 gives a residual $O_l(r)\rho_*$ and a derivative at least ρ_* on a sufficiently small interval. The intermediate value theorem gives a unique zero with $|s| \leq C_l r$. Monotonicity gives local uniqueness. On the neutral subspace m_* is identically zero; on a one-sided orbit its sign is that of m_2 , by the same relative estimate.

The local polystability correspondence for deformations of an HE bundle identifies closed automorphism orbits with polystable bundles [BuS20, Theorems 5.4 and 8.1]. We use it at the fixed endpoint and fixed polarization. The HE metrics have also been constructed directly above. A holomorphic endomorphism in this small slice is determined by its parallel part: applying D^* to its endomorphism equation and absorbing the small coefficient terms forces its complementary part to vanish. Equivariance then identifies the endomorphism algebra with the stabilizer of t . A charged closed orbit has no effective stabilizer, so its polystable bundle is stable. A neutral deformation preserves the two summands; the rank-three one remains stable after shrinking the neighborhood. This also follows from its unchanged scalar endomorphism algebra and its constructed HE metric. $\square$

The analytic quotient retains the full equation f_* . Its invariant coordinates are t_0 and the 16 by 20 matrix $t_- \otimes t_+$, subject to the 2 by 2 minors and to the descended analytic obstruction ideal. To see that this description is analytic, expand an invariant convergent power series in charged monomials. Every invariant monomial pairs the same number of negative and positive variables, hence factors into entries of that matrix. On a smaller polydisc the absolute majorant remains convergent. This argument does not replace the obstruction by its quadratic term.

4.3. The harmonic quotient metric. For t with $f_*(t) = m_*(t) = 0$ and $v \in \ker Df_{*,t}$, set

$$(40) \quad g_{*,\text{fin}}([v]) = \inf_{\xi \in \mathbb{C}} |v - C_t \xi|^2,$$

$$(41) \quad g_{*,\text{HE}}([v]) = \inf_{\zeta} \|D_t a_* v - (D + [a_*, \cdot])\zeta\|_{H(t),l}^2.$$

The latter is the volume-normalized harmonic Kodaira–Spencer norm. It is Vol_l^{-1} times the Weil–Petersson convention of [BiS16, Section 4.1]. A real unitary-connection variation has squared norm twice its Hermitian $(0,1)$ norm; all lengths here use $\text{Re } g_{\text{HE}}$.

Theorem 4.3. *On every smooth complex piece of this local polystable quotient,*

$$(42) \quad (1 - C_l r^2) g_{*,\text{fin}} \leq g_{*,\text{HE}} \leq (1 + C_l r^2) g_{*,\text{fin}}.$$

The exact Kodaira–Spencer kernel is $\text{im } C_t$. The constants are independent of the positive charged energy and of nonzero singular values of Df_ . The assertion includes the complete 75-dimensional regular stable charts and the neutral polystable subfamilies constructed in Section 3.*

Proof. Apply Theorem 8.1 with the endpoint as reference, $\mathfrak{g} = \mathbb{C}T_*$, and $q = |t|^2$. Equations (37) give harmonic orthogonality of the derivative correction. Equivariance gives $(D + [a_*, \cdot])\xi = -D_t a_* C_t \xi$. The fixed endpoint Poincaré estimate gives complementary coercivity; (38) gives the metric bound. All hypotheses are now verified, with constants depending only on the fixed endpoint. No inverse of C_t occurs. At a charged point the finite form can also be written

$$|v|^2 - \frac{|\langle v, (0, -t_-, t_+) \rangle|^2}{\rho_*};$$

on the neutral piece there is no subtraction. The two expressions refer to their respective tangent spaces, not to arbitrary vectors in a common coarse Zariski tangent space. $\square$

This theorem alone does not control $l \rightarrow 0$. The reference gap and the harmonic reconstruction constants may deteriorate. The next section computes that deterioration, and the following two sections construct a chart in which the elliptic gap remains fixed.

5. THE ENDPOINT DEFORMATION COMPLEX AT THE WALL

The endpoint harmonic space has dimension 77, whereas the split harmonic space has dimension 83. The difference comes from two gauge directions and four equation directions that disappear from cohomology for $l > 0$. Their eigenvalues all tend to zero with l . We compute them before choosing coordinates for the wall neighborhood.

5.1. A smooth normalization of the endpoint. Put $\tau = \sqrt{l}$, and retain the fixed split Hodge contraction (ι, P, K) at ω_0 . Write

$$(43) \quad A^o = \eta_1^1 s^3 E_{12}, \quad B^o = e_{0,1} \theta_2 s E_{24}, \quad a_l = \|A^o\|_l^{-1}, \quad b_l = \sqrt{3/5} \|B^o\|_l^{-1}, \quad \alpha_l = a_l A^o + b_l B^o.$$

Here a_l, b_l are positive normalizing constants; the subscript on a polynomial coordinate such as $\dot{a}_0$ has a different meaning. Product harmonic representatives are independent of the positive polarization coefficients. Indeed the product Laplacian is the sum of commuting factor Laplacians with positive scalar multipliers. Consequently a_l, b_l are smooth in l .

An integrable fixed-reference presentation of the endpoint is

$$(44) \quad N_\tau = \tau \alpha_l - \tau^2 K(\alpha_l^2), \quad D_\tau = d + [N_\tau, \cdot].$$

There is only the composable pair 12, 24, its harmonic product is zero, and no longer product occurs. Thus $dN_\tau + N_\tau^2 = 0$. Replacing K by the current homotopy K_l changes the central entry by an exact form. Since $K_l - K = O(l)$ and the product has size τ^2 , the gauge $1 + \phi_\tau E_{14}$ identifying the two presentations has $\phi_\tau = O(\tau^4)$ in every fixed smooth norm. The sign is fixed by $g(d + N)g^{-1}$; explicitly $d\phi_\tau$ is the old central entry minus the new one.

The endpoint HE metric can be chosen smoothly in τ , with logarithm $O(\tau^2)$ relative to h_0 . To prove the needed regularity, use the complementary metric solution of (12) and vary a diagonal Y with $Y_3 = 0$, $Y_1 + Y_2 + Y_4 = 0$. Its reduced equation is

$$-\tau^2(1, -2/5, 0, -3/5) + \mu_l(e^Y \tau \alpha_l) + R(l, e^Y \tau \alpha_l) = 0.$$

The remainder is $O(\tau^4)$, smoothly in τ, Y . Taylor's integral formula divides the equation smoothly by τ^2 . At zero its Y derivative is positive definite on the indicated two-dimensional space, with quadratic form a positive multiple of

$$|Y_1 - Y_2|^2 + \frac{3}{5}|Y_2 - Y_4|^2.$$

The ordinary implicit-function theorem gives $Y(\tau) = O(\tau^2)$. The third curvature component is identically zero. Together with the complementary solution and the central gauge, this proves the claim in every fixed smooth order. Separate determinant normalization on W, F_3 removes the remaining constant rescaling. Transport by the metric isometry to h_0 changes the differential only by $O(\tau^2)$, so the actual endpoint differential is $d + \tau[\alpha_0, \cdot] + O(\tau^2)$.

5.2. A bounded finite complex. Let $\delta_\tau = [N_\tau, \cdot]$. Heights $(2, 1, 1, 0)$ on the four factors make δ_τ strictly increase block height, whereas K, P preserve it. Hence

$$(45) \quad I_\tau = (1 + K\delta_\tau)^{-1}\iota, \quad P_\tau = P(1 + \delta_\tau K)^{-1}, \quad K_\tau = K(1 + \delta_\tau K)^{-1}, \quad \partial_\tau = P\delta_\tau I_\tau$$

are finite Neumann sums. Expanding $dK + Kd = 1 - \iota P$ gives

$$(46) \quad \begin{aligned} D_\tau I_\tau &= I_\tau \partial_\tau, & P_\tau D_\tau &= \partial_\tau P_\tau, & P_\tau I_\tau &= 1, \\ D_\tau K_\tau + K_\tau D_\tau &= 1 - I_\tau P_\tau. \end{aligned}$$

All these maps are uniformly bounded in fixed Sobolev orders, and K_τ gains one derivative. They retain the small finite differential rather than inverting it.

The fixed trace-free harmonic dimensions are $(3, 83, 5, 0)$ by the cohomology table in Section 3. On these spaces the exact nonzero maps are

$$(47) \quad \begin{aligned} \partial_\tau \xi &= \tau a_l(\xi_2 - \xi_1)A^o + \tau b_l(\xi_4 - \xi_2)B^o, \\ \partial_\tau(\dot{A}, \dot{C}, \dot{B}, \dot{D}, \dots) &= \tau(b_l s \dot{a}_0 + a_l s^3 \dot{b}_1) \quad \text{in } S_4. \end{aligned}$$

The first formula follows because $K\delta_\tau\xi = 0$ and the quadratic central term has zero harmonic projection. For the selected adjacent blocks only one further arrow can lead to the target 14, which gives the second formula without higher terms.

There are no omitted columns in the second map. Blocks incident to vertex 3 have nonzero character under its residual scalar, whereas 14 is neutral. For the remaining blocks 21, 41, rotate the third projective coordinate and compensate by line weights $(4, 1, 0, 0)$. A monomial of third exponent e in block ij has weight $e + u_j - u_i$. The weights of 21 and 41 are respectively $\{1, 2\}$ and $\{1, 2, 3\}$, while the target weights are $\{-4, -3, -2, -1, 0\}$. The endpoint and the contraction commute with this circle, so every term in (45) preserves weights. This proves the all-orders vanishing.

For $\tau > 0$, the ranks are two and four, with degree-one image $sS_3 \subset S_4$. Their composite is zero: the first map changes a_1, b_0 , not a_0, b_1 . The cohomology is therefore exactly $(1, 77, 1, 0)$. Let d_1 denote (47) with the factor τ removed and $l = 0$.

Proposition 5.1. *The actual endpoint Dolbeault Laplacians in degrees 0, 1, 2, 3 have smooth low spectral subspaces of dimensions $(3, 83, 5, 0)$ on $0 \leq \tau < \tau_0$. Their complementary Green operators are uniformly bounded $H^m \rightarrow H^{m+2}$ for each fixed m . For $\tau > 0$ the low positive eigenvalue counts are $(2, 6, 4, 0)$; every such eigenvalue satisfies*

$$(48) \quad \lambda_j(l)/l = c_j + O(\sqrt{l}),$$

where the positive constants in degree zero are $(8 - \sqrt{19})/5, (8 + \sqrt{19})/5$, those in degree two are $6/25, 12/25, 28/25, 64/25$, and those in degree one are their union. The zero multiplicities are exactly $(1, 77, 1, 0)$.

Proof. Use the smooth endpoint normalization and the smooth form and density isometries to put all operators on fixed L^2 spaces with common H^2 domains. Their difference from the split Laplacian is $O(\tau) : H^2 \rightarrow L^2$. Choose a contour surrounding zero below the split positive spectrum. The resolvent Neumann series converges there, so the Riesz projections $\Pi_{\tau,q}$ are smooth and have ranks $(3, 83, 5, 0)$. The complements have one positive uniform gap. That gap and elliptic regularity give their asserted Green estimates.

The differential commutes with the successive Laplacians. Polar normalization of $\Pi_{\tau,q}$ identifies the low spaces isometrically with the fixed harmonic spaces. In those identifications the finite differential is $\tau d_1 + O(\tau^2)$: derivatives of the identifications multiplied by d vanish after harmonic projection. Thus the low Laplacian is $\tau^2(d_1^*d_1 + d_1d_1^*) + O(\tau^3)$. Finite-dimensional min-max proves (48). The exact cohomology computed above identifies all remaining low modes as zero, so no smaller positive eigenvalues are hidden in the error term.

For the constants, in the coordinate $z = s/t$ use

$$\theta = \frac{d\bar{z}}{(1 + |z|^2)^2}, \quad \eta^0 = \frac{2d\bar{z}}{(1 + |z|^2)^3}, \quad \eta^1 = \frac{2\bar{z}d\bar{z}}{(1 + |z|^2)^3}.$$

The Fubini–Study normalization gives $|d\bar{z}|^2 = 2\pi(1 + |z|^2)^2$. A beta integral yields

$$\|s^j t^{d-j}\|^2 = ((d+1)\binom{d}{j})^{-1}, \quad \|\theta\|^2 = 2\pi, \quad \|\eta^j\|^2 = 4\pi.$$

Consequently $\|A^o\|^2 = \pi$, $\|B^o\|^2 = \pi/2$, $a_0 = \pi^{-1/2}$ and $b_0 = \sqrt{6/(5\pi)}$. The zero-form operator is the three-vertex path of weights $1, 3/5$ plus the isolated vertex. Its two positive eigenvalues are the stated ones. In degree two the monomials $s^j t^{4-j}$ are orthogonal, and $d_1 d_1^*$ has entries

$$\frac{4}{5\binom{4}{j}} \left(\frac{6}{5} \binom{3}{j-1} + 2\mathbf{1}_{j \geq 3} \right), \quad 1 \leq j \leq 4.$$

They are the four printed rational constants. Hodge decomposition of the finite complex gives their union with the two degree-zero constants in degree one. $\square$

The actual endpoint homotopy $D_\tau^* G_\tau$ therefore has norm at most C/τ from H^m to H^{m+1} , while its harmonic projections remain uniformly bounded. On the low spaces this follows from the singular values of the differential, and on their complements from the uniform elliptic bounds. In contrast, the homotopy K_τ in (45) is uniformly bounded because it does not contract those finite directions. This distinction explains both the larger wall chart and the shrinking domain of its transition to the endpoint harmonic chart.

6. A COMPLETE DEFORMATION CHART WITH A FIXED DOMAIN

We now keep all split harmonic directions. The resulting chart has 83 coordinates, five obstruction components and three effective diagonal gauge directions. Its elliptic inverse and its domain remain fixed as $l \rightarrow 0$. Kuranishi reconstruction and complementary metric reduction are standard local tools; the point here is to retain the entire collapsing finite complex of Proposition 5.1.

6.1. Reconstruction and exact integrability. Let $V = \mathcal{H}_d^1$, $O = \mathcal{H}_d^2$, with dimensions 83 and 5. All Hodge operators in this subsection use the fixed split reference. There are holomorphic maps on a fixed ball $B_{r_0} \subset V$

$$(49) \quad a(x) = \iota x - K(a(x) \wedge a(x)), \quad \kappa(x) = P_2(a(x) \wedge a(x)).$$

To specify their convergence, let I_s bound harmonic inclusion into H^s , $s \geq 6$, and let M dominate the bilinear norms of $K(\cdot \wedge \cdot)$ in H^s and H^{s-1} . Choose $8MI_s r_0 < 1/2$. The first map in (49) contracts the H^s ball of radius $2I_s|x|$. Equivalently,

$$a_1 = \iota x, \quad a_n = -K \sum_{j=1}^{n-1} a_j \wedge a_{n-j} \quad (n \geq 2)$$

has a convergent Catalan majorant. This specifies all higher terms. The trace of a square of an endomorphism-valued one-form is zero, so the trace-free condition is preserved.

Write $J_x = D_x a$. Differentiation gives

$$(50) \quad J_x v = \iota v - K(J_x v \wedge a + a \wedge J_x v).$$

The inverse on the right has norm at most two. In particular

$$(51) \quad \|a - \iota x\|_{H^s} \leq C|x|^2, \quad \|J_x v - \iota v\|_{H^s} \leq C|x||v|,$$

and second derivatives are uniformly bounded on a smaller ball. Elliptic regularity gives any fixed higher spatial order after a corresponding fixed decrease of radius.

For the full curvature $c = da + a^2$, Hodge decomposition and Bianchi give

$$(52) \quad c = \iota \kappa + K(c \wedge a - a \wedge c).$$

The last operator has norm less than one in H^{s-1} . Thus $\kappa = 0$ is equivalent to full integrability. Constant diagonals commute with the contraction, so reconstruction and obstruction are T -equivariant whenever both coordinates lie in the domain. The group need not preserve the ball.

6.2. Completeness and all isomorphisms. For any small perturbation b of d , use a complex gauge e^v , with v perpendicular to the reference harmonic zero-forms, to impose

$$d^*(e^v \cdot (d + b) - d) = 0.$$

The derivative in v is $-d^*d : H^{s+1} \rightarrow H^{s-1}$ on this fixed complement. Its inverse is bounded. The analytic implicit-function theorem gives a gauge and bounded first derivatives on one fixed neighborhood. If b is integrable, its Coulomb representative b' satisfies $b' = \iota P_1 b' - K(b'^2)$, so $b' = a(x)$. This proves completeness, including deformations that do not preserve the selected three-layer filtration.

Lemma 6.1. *Every holomorphic bundle map between two sufficiently small integrable members $d + a(x), d + a(y)$ is a constant diagonal. Consequently two such members are isomorphic exactly when $y = gxg^{-1}$ for $g \in T$. No bound on the size of the isomorphism is required.*

Proof. In the full endomorphism space decompose a holomorphic map as $g = g_0 + g_1$, where g_0 is a constant diagonal and g_1 is orthogonal to the four diagonal constants. Its equation is

$$dg_1 + a(y)g_1 - g_1a(x) + a(y)g_0 - g_0a(x) = 0.$$

Since g_0 is parallel and both reconstructed forms are co-closed, applying d^* kills the last two terms. Pairing with g_1 gives

$$\|dg_1\|^2 \leq C(\|a(x)\|_\infty + \|a(y)\|_\infty)\|dg_1\|\|g_1\|.$$

The fixed Poincaré inequality forces $g_1 = 0$ on one sufficiently small ball. Harmonic projection then gives $yg_0 = g_0x$. For an isomorphism all diagonal entries are nonzero, giving the T action after removing the common scalar. Conversely equivariance makes every such diagonal an isomorphism. The proof never estimates $\|g_0\|$. $\square$

This is the co-closed-slice mechanism of [BuS20, Proposition 4.5 and Corollary 4.6]. That source's quadratic reconstruction estimate is used for harmonic input, as in its equation (2.4); an estimate for arbitrary nonclosed one-forms is unnecessary here. The direct argument above proves the precise converse needed in this chart. Continuous gauge fixing gives local inverses on orbit spaces. This assertion retains nonclosed orbits and makes no Hausdorff claim for their unseparated quotient.

The point $x_l = \tau\alpha_l$ satisfies $a(x_l) = N_\tau$. For small l , it lies in $B_{r_0/4}$. Thus one fixed chart contains a fixed-radius neighborhood of every moving endpoint, in the fixed smooth presentation. This follows from completeness about d and the triangle inequality, not from an endpoint inverse with a collapsing gap.

6.3. The complementary HE equation. For a trace-free h_0 -Hermitian u , form the compatible unitary connection of $d + a(x)$ and h_0e^u , transported to h_0 by $e^{u/2}$. Denote it by $A(l, x, u)$ and put

$$\mathcal{M}(l, x, u) = i\Lambda_l F_{A(l, x, u)} - \kappa_E(l)I.$$

The fixed determinant connection has constant mean curvature, so this expression is pointwise trace-free. Let P_0 project onto real constant trace-free diagonals, and $Q_0 = 1 - P_0$. The normalized zero-form measure is the fixed product measure, so these projections do not vary with l .

At $x = u = 0$ the curvature is $-b(l)$, and its Q_0 projection vanishes. Its metric derivative is

$$\frac{1}{2}(\Delta_{\partial, l} + \Delta_{\bar{\partial}, l}).$$

The quadratic form is the sum of two derivative energies; its kernel on Hermitian zero-forms is exactly the constant diagonals. The fixed gap and ellipticity give a uniformly bounded inverse on $Q_0 H^{s+1} \rightarrow Q_0 H^{s-1}$. The parameter implicit-function theorem therefore gives a smooth T_c -equivariant map $u(l, x)$ on a fixed product neighborhood, with

$$(53) \quad Q_0 \mathcal{M}(l, x, u(l, x)) = 0, \quad P_0 u = 0, \quad \|u(l, x)\|_{H^{s+1}} \leq C|x|^2, \quad \|D_x u(l, x)v\|_{H^{s+1}} \leq C|x||v|.$$

Indeed $u(l, 0) = 0$, and the first x derivative vanishes because all split harmonic one-forms remain harmonic for ω_l . The Sobolev elliptic inverse gives the usual complementary reduction in [CT24, Section 3] and [Del25, Lemma 4.1 and Proposition 4.2], with a fixed reference complement containing no collapsing modes.

Define the remaining real equation and the adapted connection by

$$(54) \quad m_l(x) = P_0 \mathcal{M}(l, x, u(l, x)), \quad \Phi_l(x) = A(l, x, u(l, x)).$$

The slope term $m_l(0) = -b(l)$ is retained. The equations $\kappa(x) = 0$, $m_l(x) = 0$ are precisely the integrable HE equations in this normalization. The holomorphic family $d + a(x)$ supplies the complex geometry; Φ_l is a smooth real family.

For completeness among nearby HE pairs, first use the preceding complex gauge fixing and transport the metric to the holomorphic slice. In an h_0 -orthonormal frame, a further real constant diagonal Y imposes

$$(55) \quad P_0 \log(e^{-Y} H e^{-Y}) = 0.$$

Its derivative in Y at $H = I, Y = 0$ is -2id , so the finite-dimensional implicit-function theorem has uniform bounds. The corresponding action $e^Y x$ preserves the holomorphic slice. Uniqueness in the complementary equation now identifies the metric with $h_0 e^{u(l, e^Y x)}$.

Finally the selected 24-dimensional subspace is an exact restriction of the complete chart. Its products have only the central block 14, and further products vanish. Hence its recursion terminates as (8), with K replacing K_h , and its obstruction is precisely (9). The central gauge changing K to K_l has bounds $O(l|x|^2)$ and derivative $O(l|x|)$. It follows by applying K to the closed difference of the two primitives. The full chart itself is not restricted to this subspace and its recursion need not terminate.

7. THE TRANSITION TO ENDPOINT HARMONIC COORDINATES

At every $l > 0$ the larger chart can be reduced to the endpoint chart. The reduction has two different scales. Four finite equations can be eliminated on a ball of radius $c\sqrt{l}$; a controlled comparison with the actual harmonic chart is obtained on a ball of radius cl . The following theorem collects the two comparisons. Its proof occupies the three subsections below.

Theorem 7.1. *Let $0 < l \leq l_0$, $\tau = \sqrt{l}$, and let $x_l = \tau\alpha_l$ be the endpoint in (44). There are constants $c, c', C > 0$, independent of l , with the following properties. Fixing the two nonzero coefficients along the endpoint arrows gives an 81-dimensional holomorphic slice to the two nonstabilizing diagonal gauge directions. On this slice, four obstruction equations can be solved for four coordinates on $|v| < c\sqrt{l}$. The resulting 77-coordinate base is defined by the full scalar equation $f_l(v) = 0$ in (59).*

On $|v| < cl$, a gauge imposing the actual endpoint Coulomb condition defines an analytic coordinate change $t = t_l(v)$ to the actual endpoint harmonic space. The derivatives of this

map and its inverse are bounded by C , and its image contains $|t| < c'l$. If $F_{e,l}$ is the full endpoint harmonic obstruction, then

$$F_{e,l}(t_l(v)) = u_l(v)f_l(v), \quad u_l(v) \neq 0.$$

The maps commute with the residual scalar action and identify the reconstructed bundles on the zero sets. The displayed radii are sufficient radii for these conclusions; no optimality is asserted.

7.1. Two gauge normalizations and four equations. At $x_l = \tau\alpha_l$, differentiate (49). The exact finite transfer of Section 5 gives

$$(56) \quad D\kappa_{x_l}(\dot{x}) = \tau(b_l s \dot{a}_0 + a_l s^3 \dot{b}_1).$$

Its image is $E = sS_3$, with complement $\mathbb{C}t^4 \subset S_4$. The omitted-block columns vanish exactly by the character argument already given, rather than just to leading order in τ .

Fix the coefficient of s^3 in a_l and that of s in b_l to their endpoint values $\tau a_l, \tau b_l$. These are two independent characters of T . On $|x - x_l| < c\tau$ the coefficients remain within half their nonzero endpoint values. The required logarithms of their ratios are bounded, with derivatives at most C/τ . Their action changes x by at most $C|x - x_l|$. The remaining group is the scalar H of the isolated third factor.

The resulting affine slice has dimension 81. Split its tangent as $V_l \oplus W$, where W is the four-dimensional a_0 polynomial space and V_l is the kernel of (56) inside the slice. Thus in V_l

$$\dot{a}_0 = -(a_l/b_l)s^2\dot{b}_1.$$

The projections and coordinate identifications are uniformly bounded, since a_l/b_l stays in a compact positive interval. In particular $\dim V_l = 77$. For $x = x_l + v + w$, the four E equations are

$$(57) \quad \tau L_l w + R_l(v, w) = 0, \quad L_l w = b_l s w, \quad |R_l(v, w)| \leq C(|v| + |w|)^2.$$

Their derivative remainder is at most $C(|v| + |w|)$. These bounds follow from the uniformly bounded second derivatives of κ . The map $L_l : W \rightarrow E$ has a uniformly bounded inverse. Contraction on $|w| \leq C|v|^2/\tau$ therefore gives an analytic solution for $|v| < c\tau$, with

$$(58) \quad |w_l(v)| \leq C|v|^2/\tau, \quad \|Dw_l(v)\| \leq C|v|/\tau.$$

Choose c so that the contraction derivative is below $1/2$. The remaining equation is the full scalar function

$$(59) \quad f_l(v) = P_{E^\perp} \kappa(x_l + v + w_l(v)).$$

We identify its target with $\mathbb{C}$ by the monomial t^4 .

All choices are H -equivariant, so f_l is invariant. The neutral blocks 12, 21, 24, 41 have 47 coordinates before removal of two gauge and four equation directions, leaving 41. Blocks 13, 23 have weight -1 and dimension 16; blocks 31, 32, 34 have weight $+1$ and dimension 20. This recovers the endpoint character spaces. The fixed reference slice, the two finite normalizations and (57) give analytic completeness. The Kodaira–Spencer map is an isomorphism onto the endpoint H^1 , as follows from the exact finite complex. This proves semiuniversality; the following comparison gives the stronger coordinate assertion.

7.2. The smaller harmonic comparison domain. Write $\mathcal{H}_\tau, \mathcal{K}_\tau$ for actual endpoint harmonic projection and Hodge homotopy, in the presentation $d + N_\tau$ with its normalized HE metric. Proposition 5.1 gives, in every fixed Sobolev order,

$$(60) \quad \|\mathcal{H}_\tau\|_{H^m \rightarrow H^m} \leq C_m, \quad \|\mathcal{K}_\tau\|_{H^m \rightarrow H^{m+1}} \leq C_m/\tau.$$

Let $J_l = D_x a|_{x_l}$. The linear map

$$(61) \quad L_\tau : V_l \longrightarrow \mathcal{H}_{D_\tau}^1, \quad L_\tau v = \mathcal{H}_\tau J_l v$$

is uniformly invertible. Indeed the low-complex identifications converge to the fixed harmonic spaces. In the finite limiting complex, V_l converges to $\ker d_1$ with the two coefficients along A^o, B^o removed; those two orthogonal monomials span $\text{im } d_1$ in degree one. It is therefore the finite cohomology representative space. Projection to it and its inverse stay bounded.

For a small perturbation b of D_τ , impose the endpoint Coulomb condition by a gauge e^{v_g} , with v_g in the degree-zero image of $\mathcal{K}_\tau$:

$$\mathcal{K}_\tau(e^{v_g} \cdot (D_\tau + b) - D_\tau) = 0.$$

Its v_g derivative at zero is $-\text{id}$. By (60), the constant term is at most $C\|b\|/\tau$ and the nonlinear Lipschitz bound is at most $C(\|b\| + \|v_g\|)/\tau$. Thus on $\|b\|_{H^s} < c\tau^2$ contraction gives

$$(62) \quad \|v_g(b)\|_{H^{s+1}} \leq C\|b\|_{H^s}/\tau, \quad \|Dv_g\| \leq C/\tau.$$

This is a sufficient domain, not an optimal-radius assertion.

For $b = a(x_l + v + w_l(v)) - N_\tau$, define the new harmonic coordinate

$$t(v) = \mathcal{H}_\tau^1(e^{v_g(b)} \cdot (D_\tau + b) - D_\tau).$$

Harmonic projection kills the linear term $D_\tau v_g$. The remaining exponential terms have bounds $C(\|v_g\|\|b\| + \|v_g\|^2)$, and (58) bounds the finite elimination. Consequently

$$(63) \quad |t(v) - L_\tau v| \leq C|v|^2/\tau^2, \quad \|Dt(v) - L_\tau\| \leq C|v|/\tau^2.$$

On a sufficiently small $c\tau^2$ ball the quantitative analytic inverse-function theorem gives an inverse with uniformly bounded first derivative; the image contains a $c'\tau^2$ ball. Thus the coordinate transition and inverse have bounded derivatives on specified cl neighborhoods.

7.3. The exact analytic equation identity. Equality of zero sets would not be enough to identify the complete obstruction germ. We prove equality of its defining ideals directly. Let $e = t^4$ denote the fixed obstruction basis, and abbreviate $a = a(x_l + v + w_l(v))$. Equation (52) gives

$$(64) \quad da + a^2 = S(v)f_l(v), \quad S(v) = (1 + K \text{ad}_a)^{-1} \iota e.$$

Here ad_a is the graded commutator. Its sign follows by moving the last term of (52) to the left. This inverse is bounded on the fixed reference ball.

Set $g = e^{v_g(b)}$ and $b' = g \cdot (D_\tau + b) - D_\tau$. Its curvature is $S'f_l$, where $S' = gSg^{-1}$. On the $c\tau^2$ ball,

$$\|b'\|_{H^s} \leq C|v|/\tau, \quad |t| \leq C|v|, \quad \|S'\|_{H^{s-1}} \leq C.$$

Let $a_e(t)$ be the complete endpoint reconstruction with homotopy $\mathcal{K}_\tau$, and set

$$F_e(t) = \mathcal{H}_\tau^2(a_e(t)^2).$$

It converges for $|t| < c\tau$, by the bound C/τ on that homotopy, with $\|a_e(t)\|_{H^s} \leq C|t|$. The endpoint Coulomb condition and Hodge identity give

$$b' = t - \mathcal{K}_\tau(b'^2) + \mathcal{K}_\tau S'f_l.$$

Subtract the recursion for a_e . With $z = b' - a_e$,

$$z + \mathcal{K}_\tau(z \wedge b' + a_e \wedge z) = \mathcal{K}_\tau S' f_l.$$

The operator on the left is invertible because its perturbation has norm bounded by

$$C\tau^{-1}(\|b'\| + \|a_e\|) \leq Cc.$$

Thus $z = B(v)f_l(v)$ with analytic B and $\|B\|_{H^s} \leq C/\tau$. Harmonic projection of the curvature now yields the identity

$$(65) \quad F_e(t(v)) = u_l(v)f_l(v), \quad u_l = \mathcal{H}_\tau^2 S' - \mathcal{H}_\tau^2(B \wedge b' + a_e \wedge B).$$

Choose the endpoint obstruction coordinate along $\mathcal{H}_\tau^2 I_\tau e$. At $v = 0$ the gauge is the identity, $b' = a_e = 0$ and $S = I_\tau e$, so $u_l(0)$ is nonzero. Its norm has a uniform positive lower bound: e is perpendicular to the finite image sS_3 , and actual harmonic projection converges to the finite cohomology projection. The preceding estimates give $|u_l(v) - u_l(0)| \leq C|v|/\tau^2$. Decrease the fixed c once more to make u_l nonvanishing throughout the comparison ball.

Together with the ambient analytic inverse-function theorem, (65) identifies the defining ideals in both directions, including multiplicity. All maps commute with the residual H action. On the zero sets the displayed gauges identify the reconstructed bundles. Hence the 77-coordinate model, the actual harmonic endpoint model and the explicit Čech model of Section 3 describe the same full analytic germ. In particular its 76-dimensional base, 35-dimensional transverse polystable family and 75-dimensional regular stable quotient charts survive the wall-chart reduction. Stability in this last assertion comes from the fixed-polarization geometric argument, not from integrability alone.

8. A RELATIVE ESTIMATE BEFORE TAKING THE QUOTIENT

The analytic estimate is best proved on the whole coefficient space. One first removes the uniformly coercive gauge directions. The resulting positive form differs quadratically from the harmonic coefficient form. Only then does one minimize over the finite gauge algebra. This order preserves the estimate even when the finite orbit loses rank. We formulate the minimization principle of [Wue26b, Section 4] with the hypotheses needed for the present integrable charts and general compact stabilizers.

8.1. Hypotheses and the two forms. We state the argument for a general compact complex manifold and a fixed Hermitian bundle (E, h_0) , so that its precise analytic requirements are visible. Let d be a split Dolbeault operator on $\text{End}_0 E$. Let p range over a relatively compact finite-dimensional parameter domain of inner products on forms. Their spatial norms are uniformly equivalent. Assume that the normalized measure, the zero-form L^2 inner product and the splitting

$$L^2(\text{End}_0 E) = \mathfrak{g} \oplus QL^2$$

are fixed, where $\mathfrak{g}$ is the complex algebra of parallel d -holomorphic trace-free zero-forms. Let G be its complex gauge group and K a compact form acting by parallel h_0 -unitary automorphisms. Let V be a fixed finite-dimensional harmonic degree-one space, with inclusion ι and norm $\|v\|_p = \|\iota v\|_p$. In every parameter norm, $\langle \iota v, d\eta \rangle_p = 0$.

On one product neighborhood assume the following data. The reconstruction $a_p(x)$ and obstruction $\kappa_p(x)$ are holomorphic in x , smooth in p , and spatially regular enough for the operations below, with $a_p(x) = O(|x|)$ in C^0 . They reconstruct an integrable operator exactly on $\kappa_p^{-1}(0)$. Write

$$D_{p,x} = d + [a_p(x), \cdot], \quad J_{p,x} = D_x a_p.$$

The first derivative has the orthogonal decomposition

$$(66) \quad J_{p,x}v = \iota v + B_{p,x}v, \quad \langle \iota v, B_{p,x}v \rangle_p = 0, \quad \|B_{p,x}v\|_p \leq C|x|\|v\|_p.$$

This orthogonality, rather than a first-order bound alone, will give a quadratic error. The linear G action has infinitesimal orbit map $C_x : \mathfrak{g} \rightarrow V$. Exact reconstruction equivariance and differentiated integrability give

$$(67) \quad D_{p,x}\xi = -J_{p,x}C_x\xi, \quad D_{p,x}J_{p,x}v = 0 \quad (\kappa_p(x) = 0, v \in \ker D\kappa_{p,x}).$$

The second identity uses the full obstruction equation.

Assume the split estimate

$$(68) \quad \|\eta\|_{W^{1,2}} \leq C\|d\eta\|_p \quad (\eta \in QW^{1,2}),$$

and smooth K -equivariant metrics $H_{p,x} = h_0 e^{u(p,x)}$ with $\|u\|_\infty \leq Cq_p(x)$, where $q_p(x) = \|x\|_p^2$. The metrics are HE on the geometric solution locus. The estimate itself also makes sense off that locus. Define

$$(69) \quad g_{\text{fin},p,x}([v]) = \inf_{\xi \in \mathfrak{g}} \|v - C_x\xi\|_p^2,$$

$$(70) \quad g_{\text{harm},p,x}([v]) = \inf_{\zeta \in W^{1,2}(\text{End}_0 E)} \|J_{p,x}v - D_{p,x}\zeta\|_{H_{p,x},p}^2.$$

The brackets mean quotient by $\text{im } C_x$. At an integrable point and for $v \in \ker D\kappa$, the second expression is the harmonic Kodaira–Spencer squared norm. Its interpretation as a tangent metric requires a specified smooth piece of the geometric quotient.

Theorem 8.1. *Under these hypotheses, after decreasing the common radius,*

$$(71) \quad (1 - Cq_p(x))g_{\text{fin},p,x}([v]) \leq g_{\text{harm},p,x}([v]) \leq (1 + Cq_p(x))g_{\text{fin},p,x}([v])$$

for every ambient $v \in V$. On $\ker D\kappa$ at an integrable point, the exact kernel of $v \mapsto [J_{p,x}v]$ is $\text{im } C_x$. There is a smooth positive K -equivariant Hermitian form $\mathcal{B}_{p,x}$ on V such that

$$(72) \quad (1 - Cq_p(x))\|v\|_p^2 \leq \mathcal{B}_{p,x}(v, v) \leq (1 + Cq_p(x))\|v\|_p^2,$$

$$(73) \quad g_{\text{harm},p,x}([v]) = \inf_{\xi \in \mathfrak{g}} \mathcal{B}_{p,x}(v - C_x\xi, v - C_x\xi).$$

No lower bound for a positive singular value of C_x or $D\kappa_{p,x}$ is required. The group need not be abelian.

Proof. Suppress p, x . Multiplication by a has norm $O(|x|)$ from zero-form L^2 to one-form L^2 . Thus (68) and absorption give

$$(74) \quad \|\eta\|_{W^{1,2}} \leq C\|D\eta\|_p \quad (\eta \in QW^{1,2}).$$

In particular $D(QW^{1,2})$ is closed. Orthogonality in (66) gives $\|Jv\|_p^2 = \|v\|_p^2 + \|Bv\|_p^2$. For the angle with that range, expand

$$\langle Jv, D\eta \rangle_p = \langle Bv, d\eta \rangle_p + \langle Jv, [a, \eta] \rangle_p.$$

The omitted harmonic term vanishes exactly. Each term on the right is bounded by $C|x|\|v\|_p\|D\eta\|_p$, using (74); also $\|d\eta\|_p \leq C\|D\eta\|_p$ by the same absorption. Hence the orthogonal projection of Jv onto $D(QW^{1,2})$ has norm at most $C|x|\|v\|_p$. Removing it loses the square of that norm. We obtain, for every v ,

$$(75) \quad (1 - C|x|^2)\|v\|_p^2 \leq \inf_{\eta \in QW^{1,2}} \|Jv - D\eta\|_p^2 \leq (1 + C|x|^2)\|v\|_p^2.$$

The metric $h_0 e^u$ changes squared endomorphism norms by factors between $e^{-2\|u\|_\infty}$ and $e^{2\|u\|_\infty}$. Thus the same estimate holds in $H_{p,x}$.

Let $R_{p,x}$ be the $H_{p,x}$, p -orthogonal projection off $D_{p,x}(QW^{1,2})$ and set

$$\mathcal{B}_{p,x}(v, w) = \langle R_{p,x} J_{p,x} v, R_{p,x} J_{p,x} w \rangle_{H_{p,x}, p}.$$

This proves (72). To justify smoothness, regard $A = D_{p,x}|_{QW^{1,2}}$ as a map between fixed Hilbert spaces. Its weak normal operator $A^* A : QW^{1,2} \rightarrow (QW^{1,2})^*$ is uniformly coercive by (74). Its inverse and the range projection $A(A^* A)^{-1} A^*$ therefore depend smoothly on the parameters. All ingredients are K -equivariant. Only the fixed complementary operator has been inverted.

Every zero-form gauge is uniquely $\xi + \eta$, with $\xi \in \mathfrak{g}$ and $\eta \in QW^{1,2}$. Equation (67) changes its contribution to $J(v + C_x \xi) - D\eta$. First minimize over η , then change ξ to $-\xi$. This proves (73). Apply (72) to every $v - C_x \xi$ before taking the remaining infimum. It gives (71).

At an integrable point Jv is closed for $v \in \ker D\kappa$. Its harmonic norm vanishes precisely when its cohomology class is zero, by elliptic Hodge theory. Equations (72)–(73) identify that condition with $v \in \text{im } C_x$, a finite-dimensional closed subspace. Conversely these orbit vectors lie in $\ker D\kappa$ and are exact by (67). This proves the kernel statement. $\square$

8.2. Changes of presentation. Suppose two such charts describe the same smooth complex quotient piece, with coordinate transition $y = T(x)$ and a C^1 family of holomorphic isometries $g(x)$ between their reconstructed metrics and bundles. The existence of this transition is a geometric hypothesis. Differentiating $d_y = g d_x g^{-1}$ gives

$$(76) \quad J_y(DT_x v) = g(J_x v)g^{-1} - D_y(\dot{g}g^{-1}).$$

This formula applies to a smooth real gauge choice as well; the induced map on quotient cohomology is complex linear. The last term is exact, and conjugation preserves the norms. Minimizing over all zero-forms gives

$$(77) \quad T^* g_{\text{harm},2} = g_{\text{harm},1}.$$

The exact tangent kernels correspond. Combining the two relative estimates, with their errors below $1/2$, yields

$$(78) \quad (1 - C(q_1 + q_2))g_{\text{fin},1} \leq T^* g_{\text{fin},2} \leq (1 + C(q_1 + q_2))g_{\text{fin},1}.$$

Uniformity follows on any collection of domains with uniform hypothesis bounds. This assertion is fibrewise in polarization: a metric on a total space with varying polarization would also require its parameter derivative and an assigned norm for that input.

8.3. A transverse limit when stabilizers change. The next linear lemma specifies exactly which limiting tangents survive a change of stabilizer. We allow an arbitrary invariant transverse complement, so the same slice may be used for two different metrics.

Lemma 8.2. *Let K act unitarily on a complex vector space V , with complexified Lie algebra $\mathfrak{g}$ and orbit map $C_x \xi = \xi x$. Suppose the complex infinitesimal stabilizer of x_0 is $\mathfrak{h} = \text{Lie}(H)_\mathbb{C}$ for a compact subgroup $H \subset K$. Let N be an H -invariant complex complement to $L_0 = \text{im } C_{x_0}$. Suppose*

$$x_n = x_0 + z_n, \quad z_n \in N, \quad z_n \rightarrow 0, \quad \mathcal{B}_n \rightarrow \mathcal{B}_0,$$

where the forms are positive Hermitian and $\mathcal{B}_0$ is H -invariant. If $v_n \rightarrow v_0 \in N^H$, then

$$(79) \quad \inf_{\xi} \mathcal{B}_n(v_n - C_{x_n} \xi, v_n - C_{x_n} \xi) \longrightarrow \inf_{\xi} \mathcal{B}_0(v_0 - C_{x_0} \xi, v_0 - C_{x_0} \xi).$$

No condition on the ranks or disappearing positive singular values of C_{x_n} is imposed.

Proof. Choose an Ad_H -invariant complement $\mathfrak{a}$ to $\mathfrak{h}$. The restriction $C_{x_0}|_{\mathfrak{a}}$ is injective, so $L_n = C_{x_n}\mathfrak{a}$ has fixed dimension near the limit. Its $\mathcal{B}_n$ -orthogonal projection P_n converges to the $\mathcal{B}_0$ -projection P_0 onto L_0 . Since $N \cap L_0 = 0$, the restriction $(1 - P_0)|_N$ is injective. Consequently $\|(1 - P_n)w\| \geq c\|w\|$ for $w \in N$ and large n . This lower bound concerns the fixed transverse decomposition.

Split $N = N^H \oplus N_{\text{mov}}$ into its fixed and nontrivial H -representations. For $\xi \in \mathfrak{h}$, $C_{x_n}\xi = \xi z_n \in N_{\text{mov}}$. The operator P_0 is H -equivariant. Every unit vector of $W_n = (1 - P_n)C_{x_n}\mathfrak{h}$ can therefore be written as $(1 - P_n)w$ with $w \in N_{\text{mov}}$ and $\|w\| \leq C$; it has distance $o(1)$ from the moving representation subspace V_{mov} after replacing P_n by P_0 . In contrast, $(1 - P_n)v_n$ tends to $(1 - P_0)v_0 \in V^H$. These limiting representation spaces are $\mathcal{B}_0$ -orthogonal. The pairing with every unit vector of W_n tends uniformly to zero, so the additional projection onto W_n has norm tending to zero. Since $\text{im } C_{x_n} = L_n \oplus_{\mathcal{B}_n} W_n$, removing these two projections computes the quotient and proves the assertion. $\square$

The slice in the lemma has a local geometric meaning. The map $(\eta, z) \mapsto e^\eta(x_0 + z)$ from $\mathfrak{a} \oplus N$ has invertible derivative at zero. Its holomorphic inverse supplies a local product parametrization. Tangents and metrics must be transported by that product before applying the lemma. For the actual quotient form, one transports the bundle metric as well, then constructs the positive ambient form by the complementary projection used in the theorem. A convergent transported tangent in N^H , tangent to the limiting smooth piece, has the required limit. An arbitrary raw tangent need not have it: for a weight-one action on $\mathbb{C}$, $x_n \neq 0 \rightarrow 0$ and $v_n = 1$ have zero quotient norm before the limit and norm one at the limit.

8.4. Lengths on a common domain. On a declared domain with $q \leq r^2$, give continuous piecewise C^1 paths in the specified smooth pieces their lengths for the real parts of the two Hermitian forms. Integrating the square root of (71) gives

$$(80) \quad \sqrt{1 - Cr^2} \text{Length}_{\text{fin}}(\gamma) \leq \text{Length}_{\text{harm}}(\gamma) \leq \sqrt{1 + Cr^2} \text{Length}_{\text{fin}}(\gamma).$$

Taking infima over the same paths gives the same inequalities for intrinsic length pseudodistances. Their finite-distance components and zero-distance equivalence relations agree. After this common identification, the identity is bi-Lipschitz on each finite-distance component and extends uniquely to their metric completions, by equivalence of Cauchy sequences. Separation of geometric points and identification of a geometric boundary are additional assertions. We prove them for the present moduli domain in Sections 10 and 11.

9. THE ACTUAL LOCAL QUOTIENT AND ITS UNIFORM METRIC

The obstruction zero set alone does not identify the polystable domain at the varying polarization. We use the corrected curvature equation from Section 6 to define that domain, prove its local geometric meaning, and then apply the relative estimate.

9.1. The domain and its geometric coverage. For $0 \leq l \leq l_0$ and $r \leq r_0$ put

$$(81) \quad \mathcal{B}_l(r) = \{x \in V : |x| < r, \kappa(x) = 0, m_l(x) = 0\}, \quad \mathcal{N}_l(r) = \mathcal{B}_l(r)/T_c.$$

The radius uses the fixed harmonic norm. Each point reconstructs an HE connection $\Phi_l(x)$ and a holomorphic bundle with metric $H_x = h_0 e^{u(l,x)}$. Let A_0 be the split product Chern connection. There are fixed constants $c_0, C_0 > 0$ such that the image of $\mathcal{N}_l(r)$ contains every fixed-determinant HE class with an h_0 -unitary representative satisfying $\|A - A_0\|_{H^s} < c_0 r$. Every point in its image has a representative with $\|A - A_0\|_{H^s} < C_0 r$.

To prove these statements, the reconstruction and metric bounds give

$$(82) \quad \|\Phi_l(x) - A_0\|_{H^s} \leq C|x|, \quad \|D_x \Phi_l - \mathcal{I}\|_{V \rightarrow H^s} \leq C|x|, \quad \mathcal{I}(v) = \iota v - (\iota v)^*.$$

Conversely the fixed reference complex gauge puts a nearby HE connection in the form $d + a(x_{\text{raw}})$, with coordinate, gauge and their first derivatives bounded uniformly by the connection perturbation. Transport h_0 by this gauge and apply (55). The metric logarithm now lies in Q_0 , so complementary uniqueness gives H_x , and the HE equation gives $m_l(x) = 0$. The resulting coordinate satisfies $|x| \leq C\|A - A_0\|_{H^s}$. On a reconstructed connection the inverse reference gauge is exactly $e^{-u/2}$: its exponent lies in the fixed complement and returns the co-closed representative $d + a(x)$. The further diagonal normalization is then zero. The inverse construction is continuous and returns x . It follows, after the injectivity proof below, that the image is open in the local HE quotient and that its inverse is continuous. These are connection-neighborhood statements; they do not assert that every unbalanced holomorphic orbit intersecting a ball stays inside it when balanced.

Lemma 9.1. *Two points of $\mathcal{B}_l(r_0)$ define isomorphic bundles if and only if they are in the same T_c orbit.*

Proof. An isomorphism g is a constant diagonal by Lemma 6.1. Normalize its determinant by a constant scalar. The HE constants of source and target agree, so the Bochner identity on their Hom bundle makes g parallel. Its positive polar factor $p = (g^\dagger g)^{1/2}$ is a parallel holomorphic automorphism of the source. The same converse lemma makes p a constant diagonal in the source stabilizer, with positive real entries. Consequently $q = gp^{-1}$ is a constant diagonal isometry. The determinant metrics give $\det p = \det q = 1$.

Write $q = ke^Y$ with $k \in T_c$ and Y real trace-free diagonal. Remove k by compact equivariance. Then $H_y = e^{-Y} H_x e^{-Y}$. Since both metrics are between $e^{\pm Cr_0^2} h_0$, this forces $|Y| \leq Cr_0^2$. Both satisfy $P_0 \log H = 0$. For H this close to h_0 , the derivative of $Y \mapsto P_0 \log(e^{-Y} H e^{-Y})$ is $-2\text{id} + O(r_0^2)$ throughout that neighborhood. Its uniform local injectivity forces $Y = 0$, proving the assertion. The polar step is necessary because the original isomorphism could include an arbitrarily large stabilizer rescaling. No orbit inverse was used. $\square$

The HE implication to polystability and the parallel-Hom argument are standard; see [Kob87, Theorem 5.8.3 and Proposition 4.1.7] and [BuS20, Section 4 and Proposition 7.4]. The coordinate and metric normalizations above provide the present uniform local identification.

9.2. The adapted moment form and smooth complex pieces. On the affine space of h_0 -unitary connections take the closed form

$$\Omega_l(U, V) = 2 \text{Im} \langle U^{0,1}, V^{0,1} \rangle_l, \quad \Omega_l = \Phi_l^* \Omega_l.$$

Our Hermitian product is linear in its second argument. Integration by parts for the curvature derivative, with gauge convention $g \cdot D = g D g^{-1}$, gives for a real constant diagonal ξ

$$(83) \quad d\langle m_l, \xi \rangle_x(v) = \Omega_{l,x}(v, iC_x \xi).$$

The infinitesimal unitary action on connections is $-d_{\Phi_l(x)}(i\xi)$; the centering scalar pairs to zero. This fixes the sign and factor. By (82),

$$(84) \quad \Omega_{l,x}(v, iv) \geq c\|v\|_l^2, \quad |\Omega_{l,x}(v, w)| \leq C\|v\|_l\|w\|_l.$$

At $x = 0$ the first expression is $2\|v\|_l^2$. Thus the form is symplectic and tames the original complex structure on a small fixed ball. We do not need, or assert, that it is of type (1,1)

there. This distinction also occurs in the adapted reductions of [CT24, Section 3] and [Del25, Section 4.1].

The stabilizer of a point is determined by its support graph: a diagonal is constant on each connected component. All weights are $e_i - e_j$, so the effective stabilizer $H \subset T$ is connected. On V^H , (83) makes the $\text{Lie}(H)$ components of m_l constant, equal to $-b(l)$ in those directions. At a balanced point this says that every support component I has $\sum_{i \in I} b_i(l) = 0$. For $l > 0$ the only proper zero-sum partition is $\{1, 2, 4\} \mid \{3\}$. At $l = 0$ additional partitions occur.

Fix a balanced point with stabilizer H . Choose a spanning forest of its support and a nonzero scalar coefficient on each forest edge. In V^H fix those coefficients to their values at the point. The forest characters form a lattice basis for T/H , so this is a holomorphic slice Σ with no root choices. Keeping those coefficients nonzero fixes the stabilizer to be exactly H . Intersect Σ with the reduced analytic space $\kappa^{-1}(0)$; refine by the determinantal loci of $D\kappa$, then take regular loci and successive singular loci. Their connected regular components are smooth locally closed complex pieces. The complete analytic obstruction ideal remains the one in the deformation problem; reducing here only specifies the underlying geometric tangent pieces.

For z near the given point in such a slice, seek a real diagonal Y modulo $\text{Lie}(H)$ with $m_l(e^Y z) = 0$. Its derivative obeys

$$\langle Dm_l(C_x Y), Y \rangle = \Omega_l(C_x Y, iC_x Y) > 0 \quad (C_x Y \neq 0).$$

It is invertible on the nonstabilizing diagonal space. The ordinary implicit-function theorem gives a local balanced section, and Lemma 9.1 identifies its quotient with the complex slice piece. Differentiating the real balancing gauge adds only a complex orbit vector, so the induced tangent map is complex linear. The lifts lie in V^H . The chart radius may depend on the point; no such radius enters the uniform metric inequality below. We assert these specified smooth pieces, not a Whitney stratification or one manifold tangent space across their boundaries.

9.3. The finite-dimensional quotient metric and the actual metric. The independently specified harmonic coefficient norm is

$$(85) \quad \begin{aligned} \|v\|_l^2 = & h_1^{-1}(\|v_{12}\|_0^2 + \|v_{32}\|_0^2 + \|v_{34}\|_0^2) \\ & + h_2^{-1}(\|v_{13}\|_0^2 + \|v_{23}\|_0^2 + \|v_{24}\|_0^2) + \|v_{21}\|_0^2 + \|v_{31}\|_0^2 + \|v_{41}\|_0^2. \end{aligned}$$

In these three groups, the $(0, 1)$ -form component lies on the first, second and third projective-line factor, respectively. Their dimensions are 20, 20, 43. Set $D_x = d + [a(x), \cdot]$ and, for $x \in \mathcal{B}_l(r)$,

$$(86) \quad g_{\text{fin}, l, x}([v]) = \inf_{\xi \in \mathfrak{t}} \|v - C_x \xi\|_l^2, \quad g_{\text{HE}, l, x}([v]) = \inf_{\zeta} \|J_x v - D_x \zeta\|_{H_x, l}^2.$$

For $v \in \ker D\kappa_x$ the second form is the actual normalized harmonic Kodaira–Spencer norm.

Theorem 9.2. *There are fixed $l_0, r_0, C > 0$, with $Cr_0^2 < 1/2$, such that on every smooth complex piece of $\mathcal{N}_l(r)$, $0 \leq l \leq l_0$, $r \leq r_0$,*

$$(87) \quad (1 - Cr^2)g_{\text{fin}, l} \leq g_{\text{HE}, l} \leq (1 + Cr^2)g_{\text{fin}, l}.$$

The same inequality holds on $\ker D\kappa_x / \text{im } C_x$, and the Kodaira–Spencer kernel is exactly $\text{im } C_x$. No lower bound on a positive singular value of C_x or $D\kappa_x$ is imposed.

Proof. We verify Theorem 8.1. The zero-form L^2 inner product, normalized measure and complement to constant diagonals are fixed. The reference Poincaré estimate is uniform

in l . Reconstruction bounds follow from (51), and the metric estimate from (53). Exact equivariance gives $D_x \xi = -J_x C_x \xi$; differentiated integrability gives $D_x J_x v = 0$ for $v \in \ker D\kappa_x$.

It remains to check the orthogonality needed for a quadratic relative error. The correction $J_x v - \nu v$ is reference-orthogonal to all harmonic one-forms, by the Hodge homotopy in (50). In each harmonic Hom block, there is only one differential leg. Changing polarization multiplies that leg's inner product by a constant, while the other legs remain orthogonal to it. Thus this orthogonality persists for (85). The harmonic term also remains orthogonal to $d\eta$. All hypotheses of Theorem 8.1 are verified uniformly. Its ambient estimate is applied to every $v - C_x \xi$ before finite minimization, which proves both the relative inequality and its exact kernel. On the declared smooth pieces that kernel is precisely the removed orbit tangent, so the resulting forms are positive Hermitian metrics. $\square$

10. COMPATIBILITY, LIMITING TANGENTS AND LOCAL LENGTHS

The uniform estimate has two geometric consequences that require care at singular points. Tangent metrics have limits only for tangents compatible with the limiting piece, and a local intrinsic completion uses paths inside its declared domain. We establish these statements before the explicit selected-family calculation.

10.1. Tangents fixed by the limiting stabilizer. Suppose $(l_n, x_n) \rightarrow (l_\infty, x_\infty)$ in the balanced loci, and lifted piece tangents $v_n \rightarrow v_\infty$ converge, with v_∞ tangent to the limiting piece in the lifts of Section 9. Then both forms in (86) converge to their limiting values. This allows $l_\infty = 0$, stabilizer change and equation-rank loss.

Here is a direct proof for the present torus. The positive ambient form $\mathcal{B}_{l,x}$ in Theorem 8.1 is smooth on the fixed 83-dimensional space, including off the solution locus. Its quotient is exactly the actual harmonic norm. Split $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{h}$, where $\mathfrak{h} = \ker C_{x_\infty}$. The images $L_n = C_{x_n} \mathfrak{a}$ have fixed dimension and converge to $\text{im } C_{x_\infty}$. Their $\mathcal{B}_{l_n, x_n}$ projections P_n therefore converge. Since the group is abelian, the remaining vectors $C_{x_n} \mathfrak{h}$ belong to the nontrivial character space V_{mov} of the limiting stabilizer H . In contrast, L_∞ and v_∞ lie in V^H . These spaces are orthogonal for the limiting H -invariant form. Hence $P_n|_{V_{\text{mov}}} \rightarrow 0$. Every unit vector in $(1 - P_n)C_{x_n} \mathfrak{h}$ is consequently at distance $o(1)$ from V_{mov} , regardless of its unnormalized size. Its pairing with $(1 - P_n)v_n$ tends uniformly to zero. The additional projection thus has norm tending to zero, which proves convergence of the quotient norm. The same proof with the finite coefficient form proves its convergence. Polarization gives convergence of Hermitian pairings.

Lemma 8.2 gives the corresponding transported statement for general compact stabilizers. The limiting-tangent condition is substantive: the weight-one example following that lemma rules out continuity for arbitrary raw directions. No inverse of $D\kappa$ was used here, so a change in its rank imposes no further quantitative cutoff.

10.2. Agreement with the complete endpoint metric. The raw endpoint $x_l = \sqrt{l} \alpha_l$ need not itself solve $m_l = 0$. Normalize its actual metric, which is $h_0 e^{O(l)}$, by (55). This produces

$$(88) \quad \widehat{x}_l = e^{Y_l} x_l \in \mathcal{B}_l, \quad |Y_l| \leq Cl, \quad |\widehat{x}_l - x_l| \leq Cl^{3/2}.$$

The central gauge is included when passing between current and reference Hodge primitives. Thus the full endpoint belongs to $\mathcal{N}_l(r)$ whenever $C\sqrt{l} < r$.

We specify a common domain with the actual harmonic chart of Section 4. Let $r_*(l)$ be one of its valid radii, and choose constants A_l, U_l, C_l such that

$$\|a_*(t)\|_{H^s} \leq A_l |t|_*, \quad \|u_*(t)\|_{H^{s+1}} \leq U_l |t|_*^2,$$

and (42) holds with C_l . For fixed $l > 0$ choose $\rho > 0$ satisfying

$$(89) \quad \begin{aligned} \rho < \min(r_*(l), c'l), \quad C_l \rho^2 < 1/4, \quad A_l \rho + U_l \rho^2 < \delta, \\ r := C_1(\sqrt{l} + A_l \rho) < r_0. \end{aligned}$$

Here δ, c', C_1 are fixed constants from the reference gauge construction and the transition of Section 7. Such a ρ exists after decreasing l_0 .

Take an endpoint-balanced representative with its actual HE metric. Put its operator into the reference Coulomb slice and then apply (55); call the resulting coordinate $y_l(t)$. The reference gauge is the identity at $t = 0$ before metric normalization, and nearby it is bounded by $CA_l |t|_*$. The transported metric logarithm is bounded by $C(l + A_l |t|_* + U_l |t|_*^2)$. Thus (89) puts it inside the uniform construction, with $|y_l(t)| \leq r$. Uniqueness of the Q_0 metric correction identifies the transported metric with $H_{y_l(t)}$. The same gauge transports the bundle and the metric, so

$$(90) \quad y_l^* g_{\text{HE}} = g_{*, \text{HE}}.$$

On each smooth piece, differentiating a real gauge choice adds an exact term to the Kodaira–Spencer representative, as in (76). The induced complex tangent map is the holomorphic germ identification of Section 7. All 77 infinitesimal directions and all the stable and polystable subfamilies of Section 3 are included.

The finite metrics use different reference norms and are not identified without a comparison. Combining the two actual estimates with (90) gives

$$(91) \quad \frac{1 - C_l \rho^2}{1 + C_l r^2} g_{*, \text{fin}} \leq y_l^* g_{\text{fin}} \leq \frac{1 + C_l \rho^2}{1 - C_l r^2} g_{*, \text{fin}}.$$

The l -dependent conditions concern only translation into the smaller endpoint chart. The uniform theorem on $\mathcal{N}_l(r)$ has no such shrinking-domain restriction.

10.3. Intrinsic lengths and point separation. Fix l, r . An admissible curve in $\mathcal{N}_l(r)$ is continuous and piecewise C^1 in the specified smooth complex pieces, with local balanced coefficient lifts that are piecewise C^1 . Use the real parts of the Hermitian forms for its lengths. Restrict attention to a component connected by finite-length admissible paths.

The finite length separates distinct quotient points. Indeed let w be a real tangent to a balanced lift and minimize the finite form by writing $w = v + C_x Y + i C_x Z$, with Y, Z real diagonals and $\|v\|_l^2 = g_{\text{fin}}([w])$. Since $Dm_l(w) = 0$ and the compact action preserves m_l , pairing with Y gives

$$0 = \Omega_l(v, i C_x Y) + \Omega_l(C_x Y, i C_x Y).$$

The bounds (84) imply $\|C_x Y\|_l \leq C \|v\|_l$, and hence

$$(92) \quad \inf_Z \|w - i C_x Z\|_l \leq C \sqrt{g_{\text{fin}}([w])}.$$

The left side is the speed in the compact Euclidean quotient. Integrating bounds every admissible length from below by $C^{-1} \min_{k \in T_c} \|x - ky\|_l$ for its endpoints. Compactness of T_c makes this positive for distinct compact orbits.

Integrating (87) and taking infima over the same path class now gives genuine distances with

$$(93) \quad \sqrt{1 - Cr^2} d_{\text{fin}} \leq d_{\text{HE}} \leq \sqrt{1 + Cr^2} d_{\text{fin}}.$$

The identity extends to a bi-Lipschitz bijection of their abstract metric completions, because the Cauchy sequences and their equivalence relations coincide. Actual isometric chart transitions identify the corresponding intrinsic lengths when the same paths are used. These conclusions concern the declared local domain. They do not identify a global moduli compactification or compare a selected-family distance with shortcuts through a larger ambient family.

11. THE SELECTED FAMILY: WEAK TANGENT NORMS AND FINITE DISTANCE

The complete chart is needed for the full moduli germ. The smaller family of Section 2 has the advantage that some norms and distances can be calculated explicitly. We first keep both polarization parameters, then approach its reducible endpoint.

11.1. The two-parameter regular metric. Let $k, l > 0$ be small, $\sigma = k + l$, and take a full-support, quadratically balanced point $z \in \Omega^{-1}(0)$ where $D\Omega$ has rank five. The local quotient has complex dimension 16. Let $J_{h,z} = D_z N_h$ and $D_{h,z} = d + [N_h(z), \cdot]$. Its two forms are

$$g_{\text{sel,fin}}([v]) = \inf_{\xi} \|v - C_z \xi\|_h^2, \quad g_{\text{sel,HE}}([v]) = \inf_{\zeta} \|J_{h,z} v - D_{h,z} \zeta\|_{H_z, h}^2.$$

Here H_z is the actual HE metric in the balanced presentation, including both the diagonal and complementary corrections. The stationary input gives $\|\log(h_0^{-1} H_z)\|_{\infty} \leq C\sigma$. Since the balanced total energy is σ , Theorem 8.1 applies with $q = \sigma$ and yields

$$(94) \quad (1 - C\sigma) g_{\text{sel,fin}} \leq g_{\text{sel,HE}} \leq (1 + C\sigma) g_{\text{sel,fin}}.$$

Indeed the quadratic endpoint derivative is orthogonal to the four linear blocks, with norm $O(\sqrt{\sigma})\|v\|_h$; equivariance and the fixed complementary gap are unchanged. Thus the estimate is uniform in k/l , all extension-orbit ratios and the positive singular values of $D\Omega$ as they approach zero.

For clarity, the actual metric is smooth at each stable point by a local implicit-function argument. If $L_z = C_z^* C_z$ on real zero-sum diagonals, sorting $L_z^{-1} f$ and taking a cut across a largest gap gives

$$\|L_z^{-1} f\|_{\infty} \leq 3 \max_{\emptyset \neq S \neq \{1,2,3,4\}} \frac{|\sum_{i \in S} f_i|}{q_{\partial S}}.$$

Combining this with (13) factors the corrected finite derivative as $L_z(I + E)$, with $\|E\| \leq C\sigma$. It is invertible at every connected point. Local smooth corrections therefore exist and patch by uniqueness of the fixed-determinant HE metric. Their coordinate radii need not be uniform near a weak orbit.

The harmonic norm is also the derivative norm of actual connections. If $U(z)$ transports H_z isometrically to h_0 , differentiation gives

$$\dot{\hat{D}} = U J_{h,z} v U^{-1} - \hat{D}(\dot{U} U^{-1}).$$

Here $\dot{U}$ is the differential of the matrix exponential. The central gauge identifying N_h with the fixed-background family has its own derivative, again an exact term by (76). Harmonic projection removes these gauge derivatives exactly. Thus (94) compares the moduli tangent norms of the two presentations.

In the regular family (14), choose a normalized cubic r_3 orthogonal to p, q , and vary a_1 by εr_3 . The equation remains satisfied because $b_1 = 0$. At the balanced point its tangent has only this a_1 component, multiplied by $\sqrt{(l - \delta)/\rho}$. It is orthogonal to every torus orbit direction, and the first variations of all block norms vanish. Consequently its squared norms satisfy

$$(95) \quad g_{\text{sel,fin}}(\partial_\varepsilon) = \frac{l - \delta}{\rho}, \quad g_{\text{sel,HE}}(\partial_\varepsilon) = \frac{l - \delta}{\rho}(1 + O(l)), \quad \rho = M^2 + 2.$$

In arbitrary joint limits $l \rightarrow 0$, $M \rightarrow \infty$, the actual squared norm multiplied by ρ/l tends to one. This is a deformation norm of order l/ρ , distinct from the weak graph edge energy $\delta \sim l\sqrt{3/(5\rho)}$.

11.2. All five pieces and the full chart. On $k = 3l/5$, take a small relatively compact neighborhood of the endpoint compact orbit in the selected quotient, keeping A, B nonzero. Its five smooth pieces are those of Section 2.4, of dimensions 16, 11, 5, 5, 4. The same estimate gives on each piece

$$(96) \quad (1 - Cl)g_{\text{sel,fin}} \leq g_{\text{sel,HE}} \leq (1 + Cl)g_{\text{sel,fin}}.$$

On the reducible pieces perform the cut correction on the connected component $\{1, 2, 4\}$ and fix each summand determinant. The complementary metric splits, and the third curvature component is identically zero. The same differentiated cut argument gives smooth metrics on that piece. Constant rescaling of its two stable summands does not change the metric on tangents varying W while F_3 is fixed. Thus the piece metric is independent of this auxiliary choice. The proof of Theorem 8.1 is ambient and never inverts $D\Omega$, so it includes the rank-four pieces as well.

These are restrictions of the actual full quotient metric, with a controlled conversion of the finite forms. First change the current primitive to the reference one by the central gauge of Section 6. It changes no harmonic coordinate. Normalize the transported actual metric by (55). The resulting real diagonal G_z has $\|\log G_z\| \leq Cl$, and

$$y(z) = G_z z \in \mathcal{B}_l, \quad |y(z)|^2 \leq Cl.$$

On each piece $Dy(v) = G_z v + C_{y(z)}\eta(v)$. Because G_z intertwines the orbit maps and changes the coefficient norm by $1 + O(l)$, taking finite infima gives

$$(1 - Cl)g_{\text{sel,fin}} \leq y^* g_{\text{fin}} \leq (1 + Cl)g_{\text{sel,fin}}, \quad y^* g_{\text{HE}} = g_{\text{sel,HE}}.$$

No derivative bound for G_z divided by an orbit energy is used; its derivative contributes only the discarded orbit vector. This proves the stated restriction of the full theorem, with the quadratic primitive and all five tangent pieces retained.

The selected map into actual HE classes is continuous at the endpoint. The corrected diagonal gauges are bounded by Cl , so every sequence approaching it has a convergent subsequence of gauges. Smoothness of the complementary construction gives an endpoint HE connection. Uniqueness on the stable summands identifies all such limits up to unitary gauge; constant summand rescaling leaves that connection unchanged. Compatible limiting tangents have convergent norms by the argument of Section 10.

The selected quadratic balanced set is semialgebraic in its coefficient coordinates. Choose a bounded semialgebraic neighborhood with A, B nonzero. Semialgebraic triangulation, compatible with the five pieces, gives piecewise smooth semialgebraic paths on each connected component; see [Cos05, Proposition 1.6, Theorem 1.10 and Proposition 1.13]. They have

finite Euclidean length: one-variable monotonicity [Cos99, Theorem 2.1] partitions every coordinate into finitely many monotone pieces, and the sum of its variations bounds length. Their finite-dimensional quotient lengths are no larger. Equation (96) then compares the two intrinsic distances and their local completions by factors $\sqrt{1 \pm Cl}$, on the same selected path domain. The reconstructed analytic metric itself is not being called semialgebraic.

11.3. An explicit distance to the reducible bundle. Set $r = M^{-1/2}$, and write $\delta = lr^2\chi(r)$. Equation (16) becomes

$$(97) \quad (1 + r^4)\chi^2 + \frac{8}{5}r^2\chi - \frac{3}{5} = 0, \quad \chi(0) = c := \sqrt{3/5}, \quad \chi(r) = c - \frac{4}{5}r^2 + O(r^4).$$

The positive root is smooth near zero. Put $f(r) = \sqrt{(1 - r^2\chi(r))/(1 + 2r^4)}$. The balanced path (15) is then

$$(98) \quad \begin{aligned} A(r) &= \sqrt{l} f(r)(r^2p, p + r^2q), & C(r) &= \sqrt{l} r \sqrt{\chi(r)}(p, 0), \\ B(r) &= \sqrt{l} \sqrt{3/5 - r^2\chi(r)}(\ell, 0), & D(r) &= \sqrt{l} r \sqrt{\chi(r)}(\ell, 0). \end{aligned}$$

It extends smoothly to the endpoint at $r = 0$. In these normalized orthogonal monomial coordinates all coefficients are real. Differentiating the fixed moment equation shows $\langle z'(r), C_{z(r)}\xi \rangle = 0$ for real diagonal ξ : the real pairing is half the moment derivative, and its imaginary part vanishes because the coefficients are real. Complex linearity then gives orthogonality to the entire complex orbit. The path is therefore horizontal for the finite form.

Only C, D have a nonzero first derivative at zero. Thus

$$(99) \quad \|z'(r)\|_l^2 = 2lc(1 + O(r^2)), \quad \text{Length}_{\text{fin}}(z|_{[0,r]}) = \sqrt{2lc}r(1 + O(r^2)).$$

This is an upper bound for the local selected distance. For a matching lower bound, on the flat quadratic moment level the real tangent is orthogonal to positive real orbit directions. Minimizing over the complex orbit is therefore the same as minimizing over the compact orbit for that tangent. Finite length is bounded below by the compact Euclidean quotient distance. Every endpoint representative has $C = D = 0$, whereas these blocks at $z(r)$ have combined norm $\sqrt{2lr^2\chi(r)}$. Hence every path from the endpoint to $z(r)$ has length at least this quantity. Combining both bounds gives

$$d_{\text{sel,fin}}(z(M), z_*) = \sqrt{2l\sqrt{3/5}} M^{-1/2}(1 + O(M^{-1})).$$

The path is contained in a fixed small neighborhood after dividing coefficients by $\sqrt{l}$, for M above a constant independent of l . Thus (96) yields the joint asymptotic

$$(100) \quad d_{\text{sel,HE}}(z(M), z_*) = \sqrt{2l\sqrt{3/5}} M^{-1/2}(1 + O(M^{-1} + l)).$$

The rates $l \rightarrow 0$ and $M \rightarrow \infty$ are unrelated. At fixed l the coefficient is determined to relative $O(l)$, not exactly. Lengths in the real unitary-connection convention are $\sqrt{2}$ times these lengths. The distance in (100) is intrinsic to the selected local family; the full moduli space may admit additional paths. The formula establishes a genuine finite-distance reducible limit without assigning an unproved global distance to it.

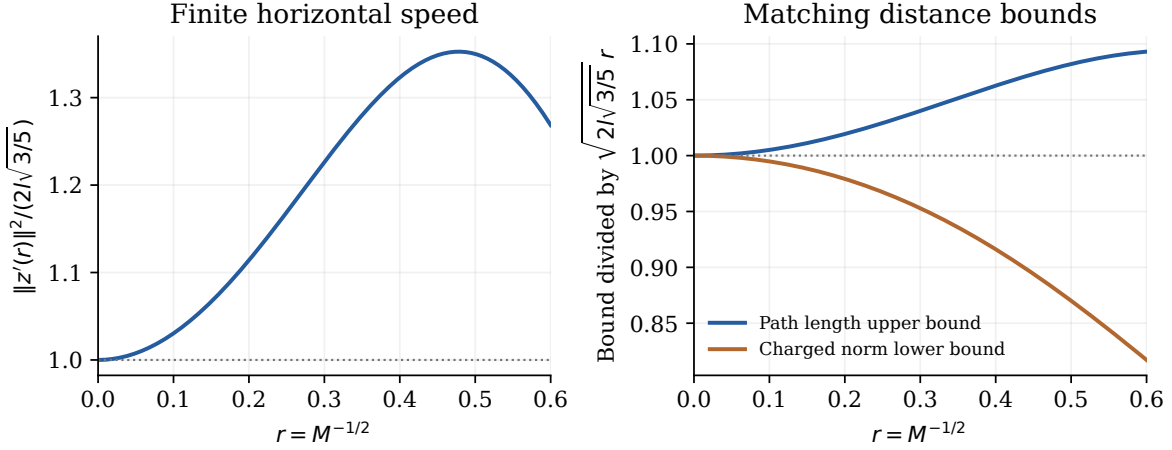


FIGURE 2. The explicit finite balanced curve, after removing its overall l scale. Left: squared horizontal speed divided by its endpoint value. Right: the integrated path-length upper bound and charged-norm lower bound, divided by the common leading distance. Both tend to one as $r \rightarrow 0$. The curves are finite-model calculations; the passage to the actual metric is the proved relative estimate, not a numerical HE computation.

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